1) a)

(1) I

(2) H

(Pl.: $\{(1,1)\}$ SZIMM és ANTISZIMM

$\emptyset$

(3) $R \subseteq A \times A$ ekvivalencia reláció.

- Ekvivalencia osztályok uniója = A.

(<u>Def.</u>: $a \in A : [a] = \{ b \mid bRa \}$)

$$\left(\forall a \in A : aRa \Rightarrow a \in [a] \Rightarrow \Rightarrow \bigcup_{a \in A} [a] = A \right)$$

- $dmn(R) = \{ a \mid \exists b \in A : aRb \}$

R refl. $\Rightarrow \forall a \in A : aRa \Rightarrow a \in dmn(R)$
$\Rightarrow dmn(R) = A$

(3) I

(4) I

b) $\mathrm{dmn}(R) = \{x \mid \exists y : xRy\}$

$$R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 10x - 5 = y\}$$

Ha $x \in \mathbb{Z}$, akkor $y = 10x - 5 \in \mathbb{Z}$.

$\Rightarrow$ $\underline{\underline{\mathrm{dmn}(R) = \mathbb{Z}}}$

(Azon kívül: $\mathrm{rng}(R) = ?$

$$y = 10\overset{\mathbb{Z}}{x} - 5 = 5(\underbrace{2\overset{\mathbb{Z}}{x} - 1}_{\text{páratlan}})$$

$\mathrm{rng}(R) = \{n \in \mathbb{Z} : 5 \mid n \text{ és } n \text{ páratlan}\}$)

$$R^{-1}(\{-20\}) = ?$$

$$y = -20$$

$$10x - 5 = y$$

$$10x - 5 = -20 \quad /+5$$

$$10x = -15 \quad /:10$$

$$x = -1{,}5 \notin \mathbb{Z}$$

$$\Rightarrow \nexists\, x \in \mathbb{Z} : (x, -20) \in R.$$

$$\Rightarrow \underline{\underline{R^{-1}(\{-20\}) = \emptyset}}.$$

1.all

c) $R \subseteq \{1,2,3\} \times \{1,2,3\}$.

nem szimmetrikus és

nem tranzitív.

Def.: $R \subseteq A \times A$

SZIMM, ha $\forall x,y:$ $(x,y) \in R \Rightarrow (y,x) \in R$.

TRANZ, ha $\forall x,y,z:$

$(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$.

$R = \{(\overset{x}{1},\overset{y}{2}),(\overset{y}{2},\overset{z}{3})\}$

Nem szimm: $(1,2) \in R$, de $(2,1) \notin R$.

Nem tranzitív: $(1,2) \in R \wedge (2,3) \in R$,

de $(1,3) \notin R$.

2 a) $R \subseteq \mathbb{Z} \times \mathbb{Z}$

$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y - x$ nemnegatív páros szám$\}$

REFL: $\forall x \in \mathbb{Z}$: $x - x = 0$ nemnegatív páros szám

$\Rightarrow (x, x) \in R$

$\Rightarrow$ R reflexív ✓

TRANZ: Tfh. $(x, y) \in R \wedge (y, z) \in R$.

$$\left.\begin{array}{l} (x,y) \in R \Leftrightarrow y - x \geq 0 \text{ és páros} \\ (y,z) \in R \Leftrightarrow z - y \geq 0 \text{ és páros} \end{array}\right\} \Rightarrow$$

$$\Rightarrow z - x = \underbrace{\overbrace{(z - y)}^{\geq 0}}_{\text{páros}} + \underbrace{\overbrace{(y - x)}^{\geq 0}}_{\text{páros}} \geq 0 \text{ és páros}$$

$\Leftrightarrow (x, z) \in R$. $\Rightarrow$ R tranzitív. ✓

ANTISZIMM: Tfh. $(x,y) \in R$ és $(y,x) \in R$.

$(x,y) \in R \iff y - x \geq 0$ és páros

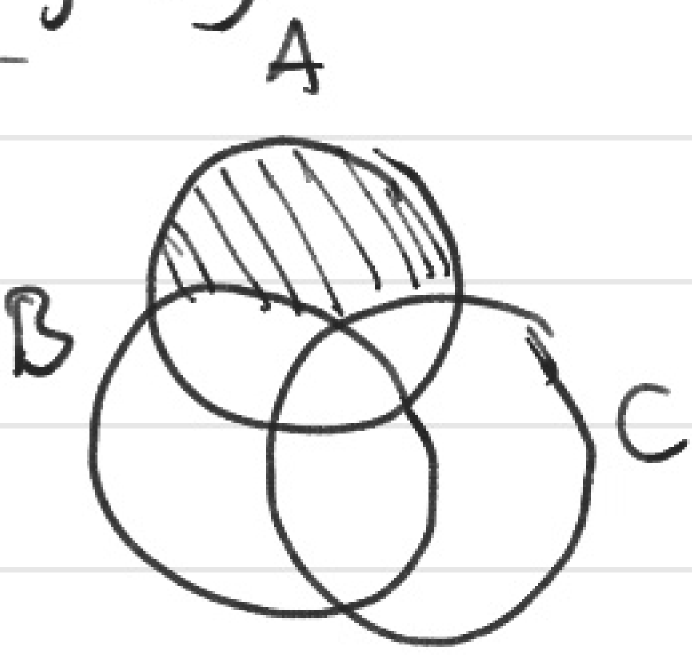
$(y,x) \in R \iff x - y \geq 0$ és páros

$$\left.\begin{aligned} y - x \geq 0 &\iff y \geq x \\ x - y \geq 0 &\iff x \geq y \end{aligned}\right\} \Rightarrow x = y.$$

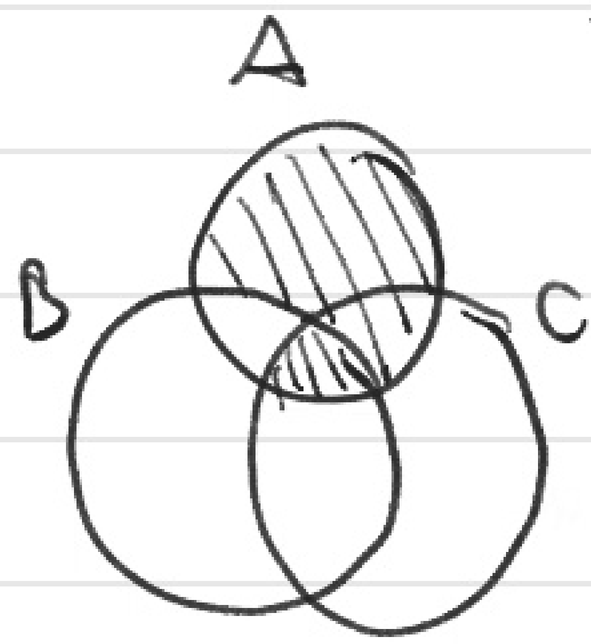
$\Rightarrow$ R antiszimmetrikus ✓

$\Rightarrow$ R <u>részbenrendezés</u> $\square$

2, b)



$(A \setminus B) \setminus C$



$A \setminus (B \setminus C)$

$\Rightarrow (A \setminus B) \setminus C = A \setminus (B \setminus C)$ pontosan akkor, ha $A \cap C = \emptyset$

$A = \{1\}, \quad B = \{2\}, \quad C = \{2\}.$

$(A \setminus B) \setminus C = (\overbrace{\{1\} \setminus \{2\}}^{\{1\}}) \setminus \{2\} = \{1\} \setminus \{2\} =$

$= \{1\}$

$A \setminus (B \setminus C) = \{1\} \setminus (\overbrace{\{2\} \setminus \{2\}}^{\emptyset}) = \{1\}$

2 c)

1. módszer : Igazságtáblázattal.

2. módszer : De Morgan

$$A \setminus (B \cup C) = A \cap (\overline{B \cup C}) =$$

$$= A \cap (\overline{B} \cap \overline{C})$$

$\cap$ asszociatív

$$(A \setminus B) \setminus C = (A \cap \overline{B}) \cap \overline{C} =$$

$$= A \cap (\overline{B} \cap \overline{C})$$

$$\Rightarrow A \setminus (B \cup C) = A \cap (\overline{B} \cap \overline{C}) \quad \square$$

3.) $R \subseteq \mathbb{R} \times \mathbb{R}$, $S \subseteq \mathbb{R} \times \mathbb{R}$

$R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid 3y + 5 = -8x\}$

$S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid 3x \geq -8y + 4\}$

$$\overset{②}{S} \circ \overset{①}{R} = \{(x,y) \mid \exists z : (x,z) \in R \wedge (z,y) \in S\}$$

$(x,y) \in S \circ R$:

$$\exists z : \begin{cases} ① \ (x,z) \in R \\ \wedge \\ ② \ (z,y) \in S \end{cases} \iff \exists z : \begin{cases} ① \ 3z + 5 = -8x \\ \wedge \\ ② \ 3z \geq -8y + 4 \end{cases}$$

① $3z + 5 = -8x$

$$3z = -8x - 5$$

Behelyettesítve ②-be

$$-8x - 5 \geq -8y + 4 \quad / +5$$

$$-8x \geq -8y + 9 \quad / : (-8)$$

$$x \leq y - \frac{9}{8}$$

$$\Rightarrow \underline{\underline{S \circ R = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y - \tfrac{9}{8}\}}}$$

$$\overset{2}{R} \circ \overset{1}{S} = \{(x,y) \mid \exists z : (x,z) \in S \wedge (z,y) \in R\}$$

$$(x,y) \in \overset{2}{R} \circ \overset{1}{S}$$

$$\Leftrightarrow \exists z : \begin{cases} 1 \ (x,z) \in S \\ \wedge \\ 2 \ (z,y) \in R \end{cases} \Leftrightarrow \exists z : \begin{cases} 1 \ 3x \geq -8z + 4 \\ \wedge \\ 2 \ 3y + 5 = -8z \end{cases}$$

2 -ből: $3y + 5 = -8z$

1 -be behelyettesítve:

$$3x \geq 3y + 5 + 4$$

$$3x \geq 3y + 9 \quad / :3$$

$$x \geq y + 3$$

$\Rightarrow$ $\underline{\underline{R \circ S = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \geq y+3\}}}$

4.) a)

$f_1 \subseteq (\mathbb{R} \setminus \{1\}) \times \mathbb{R}$

$f_1 = \{(x,y) \in (\mathbb{R} \setminus \{1\}) \times \mathbb{R} \mid (x-2)y = 1\}$.

$(x,y) \in f_1 \iff x \in \mathbb{R} \setminus \{1\}, \; y \in \mathbb{R}$

és $(x-2)y = 1$

Két eset:

① Ha $x = 2$: Ekkor

$(\overset{0}{2-2})y = 1$ egyenletnek nincs

megoldása y-ban.

$x=2$ nem áll relációban semmilyen y-nal.

② $x \neq 2$

$$(x-2)y = 1 \quad / : (x-2)$$

$$y = \frac{1}{x-2}$$

$\Rightarrow$ y értéke egyértelműen meghatározható.

$\Rightarrow$ f_1 függvény.

$f_2 \subseteq \mathbb{R} \times \mathbb{R}$

$f_2 = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^3 = y^2\}$

$x = 1$: $1^3 = y^2$

$1 = y^2 \Rightarrow$ $(1,1) \in f_2$ és $(1,-1) \in f_2$ $\Rightarrow$

$y_1 = 1$

vagy

$y_2 = -1$

$\Rightarrow$ f_2 <u>nem függvény</u>

$f_3 \subseteq \mathbb{R} \times \mathbb{R}$

$f_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y - x^{12} = -1 + 3y\}$

$x, y \in \mathbb{R}$: $(x, y) \in f_3 \Leftrightarrow$

$$y - x^{12} = -1 + 3y \quad / -y + 1$$

$$1 - x^{12} = 2y \quad / : 2$$

$$\frac{1 - x^{12}}{2} = y$$

$\Rightarrow$ Adott $x \in \mathbb{R}$ esetén y egyértelműen meghatározható. $\Rightarrow$ f_3 függvény.

4 b) $f: \mathbb{R}_0^+ \to \mathbb{R}$

$$f(x) := 2\sqrt{x+13}$$

INJ.:

Def.: $f: X \to Y$ injektív, ha

$$\forall x_1, x_2 \in X: f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

1. módon:

Mivel $f(x)$ szig. mon. növő

$\left(\forall x_1, x_2 \in \mathbb{R}_0^+ : x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\right)$

ezért injektív.

2. módszer:

Tfh. $f(x_1) = f(x_2)$ valamely $x_1, x_2 \in \mathbb{R}_0^+$

$$2\sqrt{x_1+13} = 2\sqrt{x_2+13} \quad /:2$$

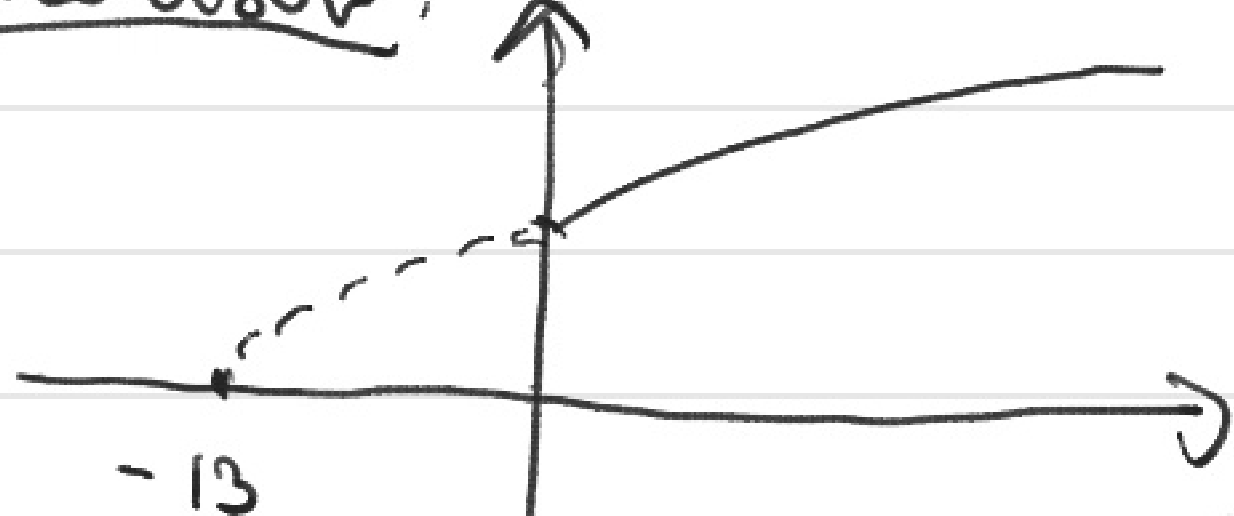
$$\sqrt{x_1+13} = \sqrt{x_2+13} \quad /^2$$

$$x_1+13 = x_2+13 \quad /-13$$

$$x_1 = x_2$$

$\Rightarrow f$ injektív.

3. módszer:



f injektív.

$\Uparrow$

$f(x_1) \neq f(x_2)$

szig. mon. nő.

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow$

SZÜRJ:

$$\mathrm{rng}(f) = \mathbb{R}$$

$$f(x) = 2\underbrace{\sqrt{\underbrace{\underbrace{x}_{\geq 0}+13}_{\geq 13}}}_{\geq \sqrt{13}} \geq 2\sqrt{13}$$

$$\Rightarrow \mathrm{rng}(f) \subseteq [2\sqrt{13}, \infty)$$

(Igazából: $\mathrm{rng}(f) = [2\sqrt{13}, \infty)$

$\Rightarrow \mathrm{rng}(f) \neq \mathbb{R} \Rightarrow$ f nem szürj.

5.)

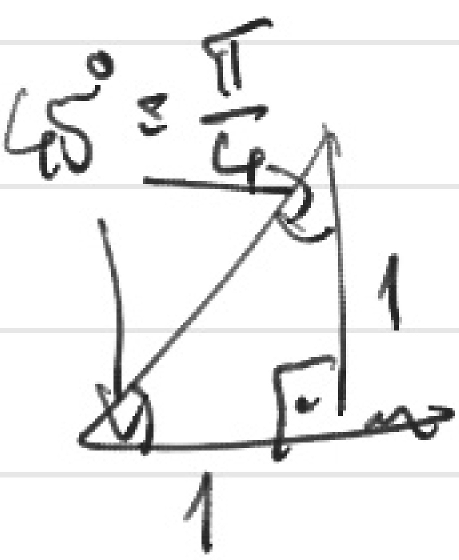
I. a trigonometrikus alakja:

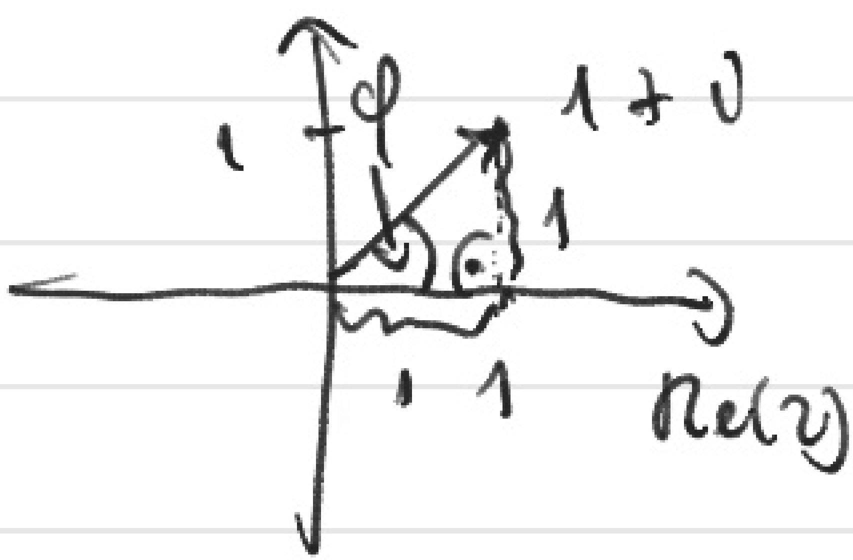
① $z_1 = 1+i$ trig. alakja

$a = 1, \; b = 1$

- $|z_1| = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$

- φ meghatározása:





$$\Rightarrow \varphi = \frac{\pi}{4}$$

$$\Rightarrow 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$$

$$(1+i)^{32} = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{32} =$$

$$= \sqrt{2}^{32}\left(\cos\frac{32\pi}{4} + i\sin\frac{32\pi}{4}\right) =$$

$$= 2^{16}\left(\cos 8\pi + i\sin 8\pi\right) =$$

$$= 2^{16}\left(\cos 0 + i\sin 0\right)$$

(2) $z_2 = -1 - \sqrt{3}\,i$ trig. alakja:

$a = -1, \quad b = -\sqrt{3}$

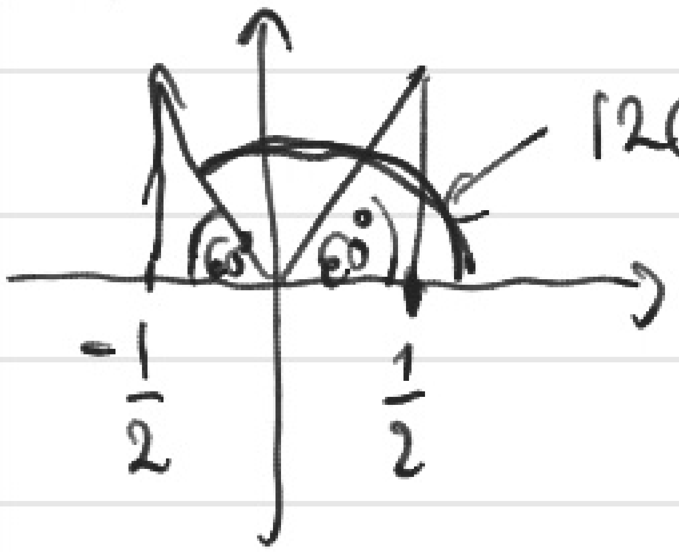
- $|z_2| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$

- φ meghatározása

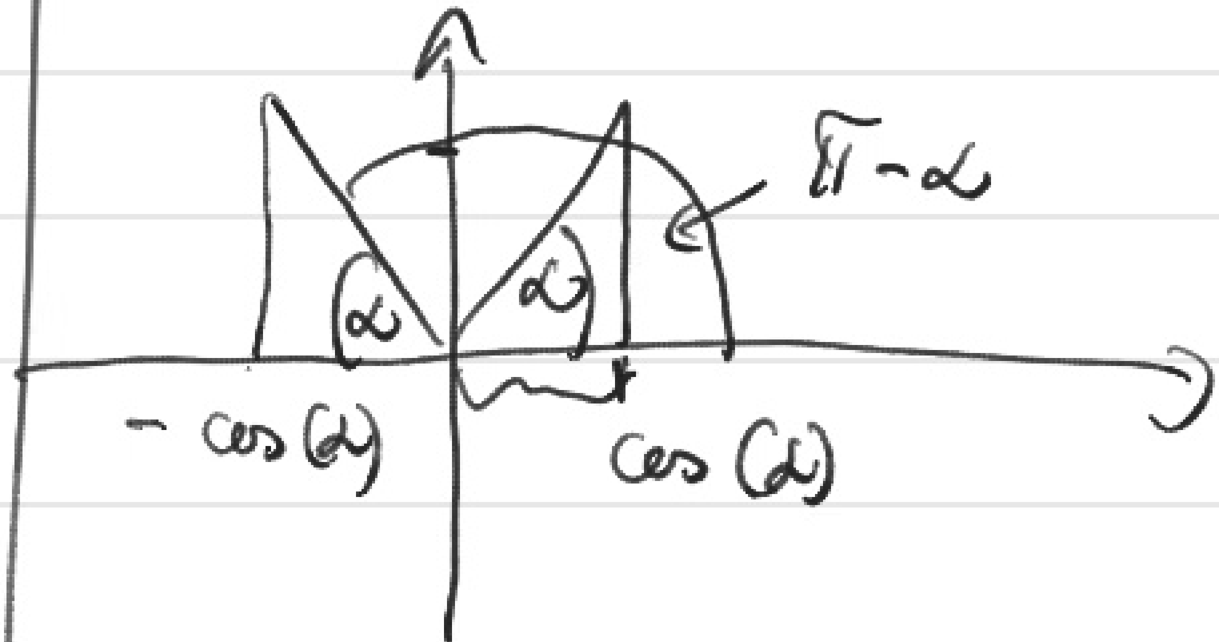
$b < 0 \Rightarrow \varphi = -\arccos\frac{a}{|z_2|} =$

$$= -\arccos\frac{-1}{2} = -\frac{2\pi}{3}$$

$$\Rightarrow -1-\sqrt{3}i = 2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$$



$$\cos 120° = \cos\frac{2\pi}{3} = -\frac{1}{2}$$



$$\cos(\pi - \alpha) = -\cos(\alpha)$$

$$(-1-\sqrt{3}\,i)^{12} = \left(2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)\right)^{12} =$$

$$= 2^{12}\left(\cos\frac{-24\pi}{3} + i\sin\frac{-24\pi}{3}\right) =$$

$$= 2^{12}\left(\cos -8\pi + i\sin -8\pi\right) =$$

$$= 2^{12}\left(\cos 0 + i\sin 0\right)$$

$$z = \frac{(1+i)^{32}}{(-1-\sqrt{3}i)^{12}} = \frac{2^{16}(\cos 0 + i\sin 0)}{2^{12}(\cos 0 + i\sin 0)} =$$

$$= \frac{2^{16}}{2^{12}}\left(\cos(0-0) + i\sin(0-0)\right) =$$

$$= 2^4(\cos 0 + i\sin 0) =$$

$$= \underline{\underline{16(\cos 0 + i\sin 0)}}$$

II. z algebarski oblik!

$$z = 2^4(\overbrace{\cos 0}^{1} + i\overbrace{\sin 0}^{0}) = 2^4 \cdot 1 = \underline{\underline{2^4}} = 16$$

$$\left(= \underline{\underline{2^4 + 0i}}\right) = \left(\underline{\underline{16 + 0i}}\right)$$

$\sqrt{\ }$ z 3. stopnia:

$$\boxed{\begin{gathered} z = |z|(\cos\varphi + i\sin\varphi) \quad n.\ \text{stopnia:} \\ w_k = \sqrt[n]{|z|}\left(\cos\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)\right) \\ k = 0, 1, \dots, n-1 \end{gathered}}$$

$$z = 16(\cos 0 + i\sin 0)$$

$$n = 3$$

$$w_k = \sqrt[3]{16}\left(\cos\left(\overset{=0}{\frac{0}{3}} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{2k\pi}{3}\right)\right)$$

$$k = 0, 1, 2$$

$$\underline{\underline{w_0 = \sqrt[3]{16}(\cos 0 + i\sin 0)}}$$

$$\underline{\underline{w_1 = \sqrt[3]{16}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}}$$

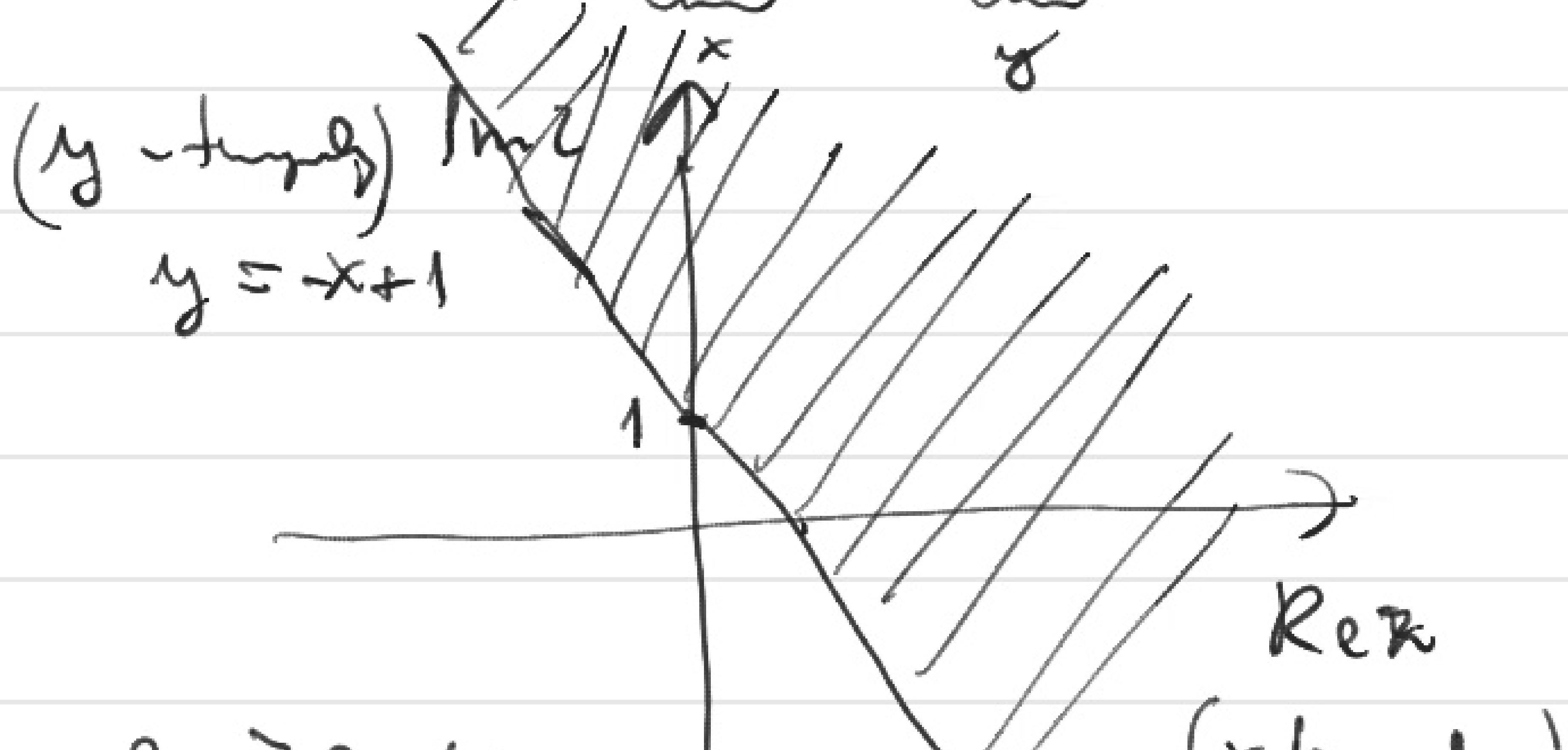
$$w_2 = \sqrt[3]{16}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

6.) a)

$$\left\{z \in \mathbb{C} \mid \overbrace{2\operatorname{Re} z + 2\operatorname{Im} z \geq 2}^{①} \wedge \overbrace{\operatorname{Im} z < 5}^{②}\right\}$$

$$2x + 2y \geq 2$$

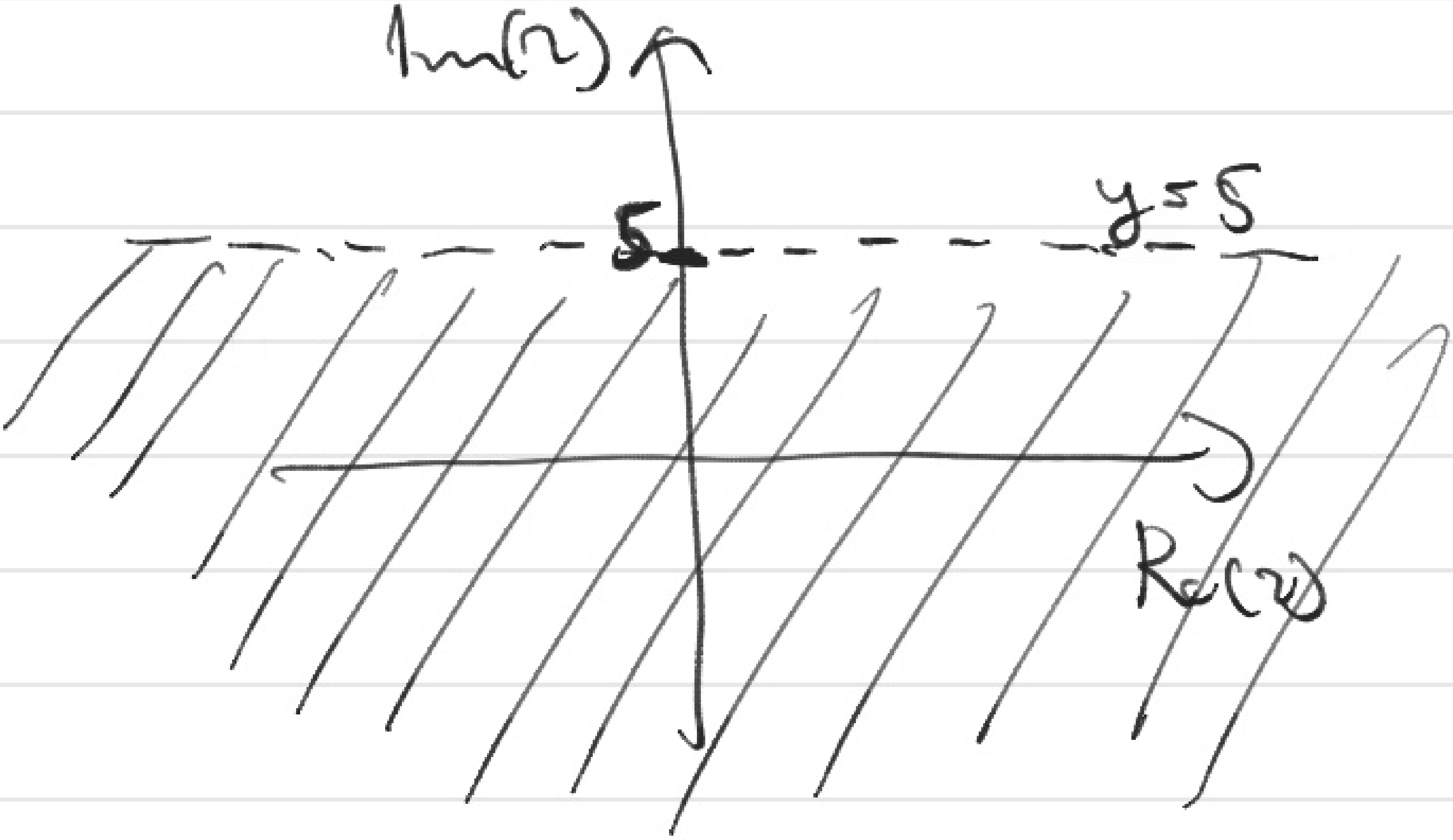
① $\{z \in \mathbb{C} \mid 2\underbrace{\operatorname{Re} z}_{x} + 2\underbrace{\operatorname{Im} z}_{y} \geq 2\}$



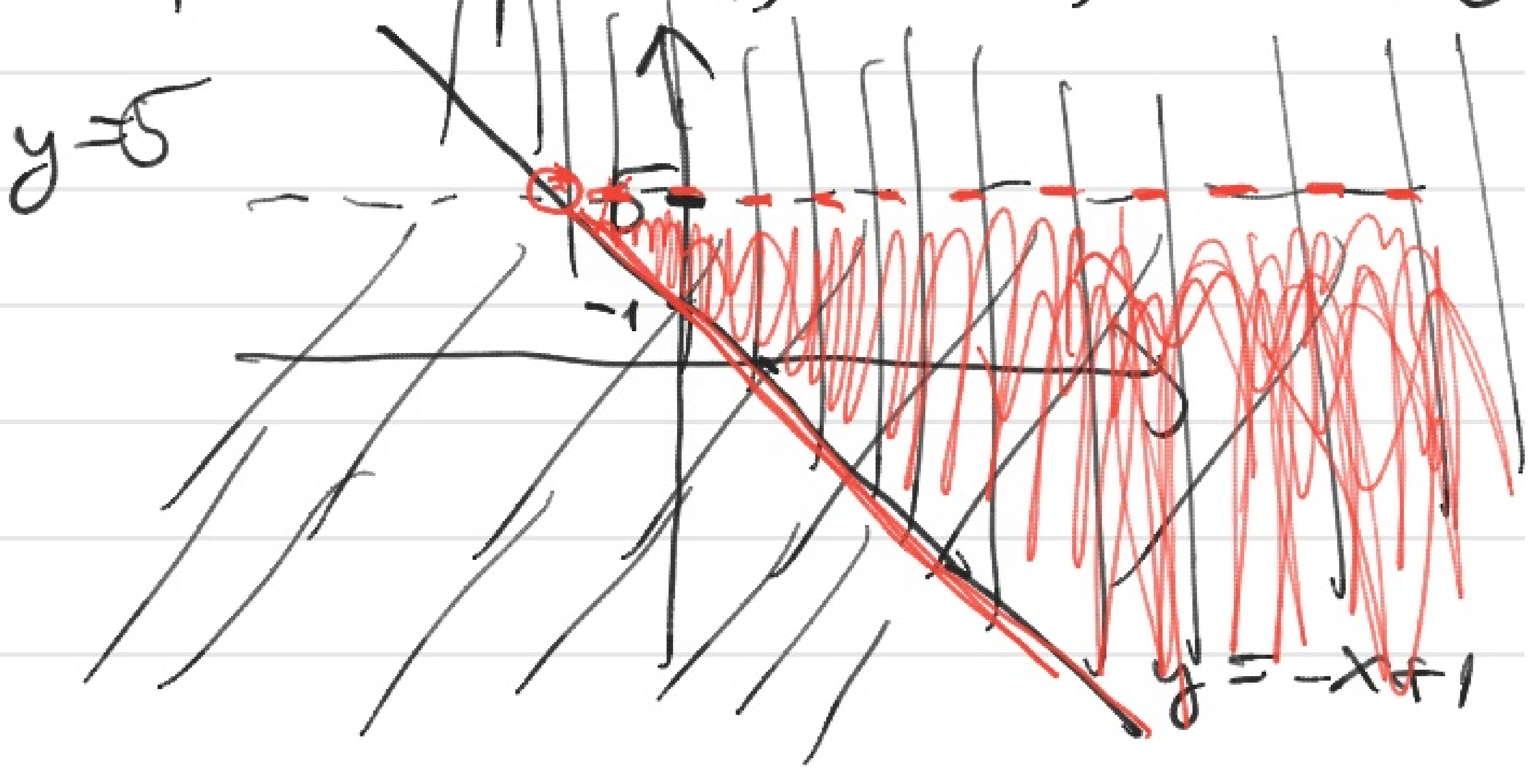
$$2x + 2y \geq 2 \quad /-2x$$
$$2y \geq -2x + 2 \quad /:2$$
$$y \overset{(=)}{\geq} -x + 1$$

$y < 5$

② $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 5\}$



$\Rightarrow \{z \in \mathbb{C} \mid 2\mathrm{Re}(z) + 2\,\mathrm{Im}(z) \geq 2 \wedge \mathrm{Im}(z) < 5\}$



6 b) $\{z \in \mathbb{C} \mid \overbrace{|z-1| \le 4}^{1} \wedge \overbrace{\operatorname{Re} z < 10}^{2}\}$

1 $\{z \in \mathbb{C} \mid |z-1| \le 4\}$

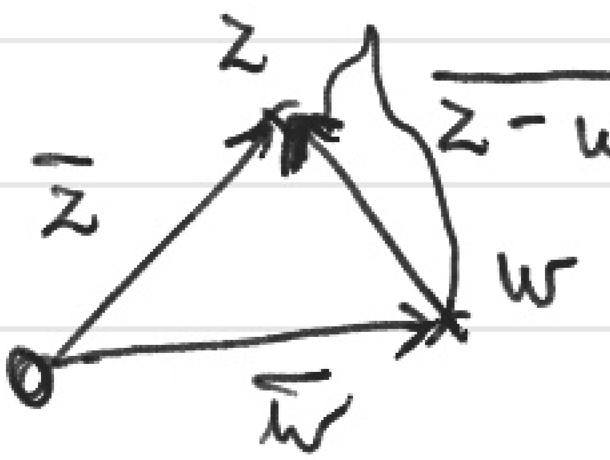
1. Módszer:

$z, w \in \mathbb{C}$:

$|z-w|$ geometriai jelentése:

z-t és w-t reprezentáló pontok távolsága

$|z-w|$



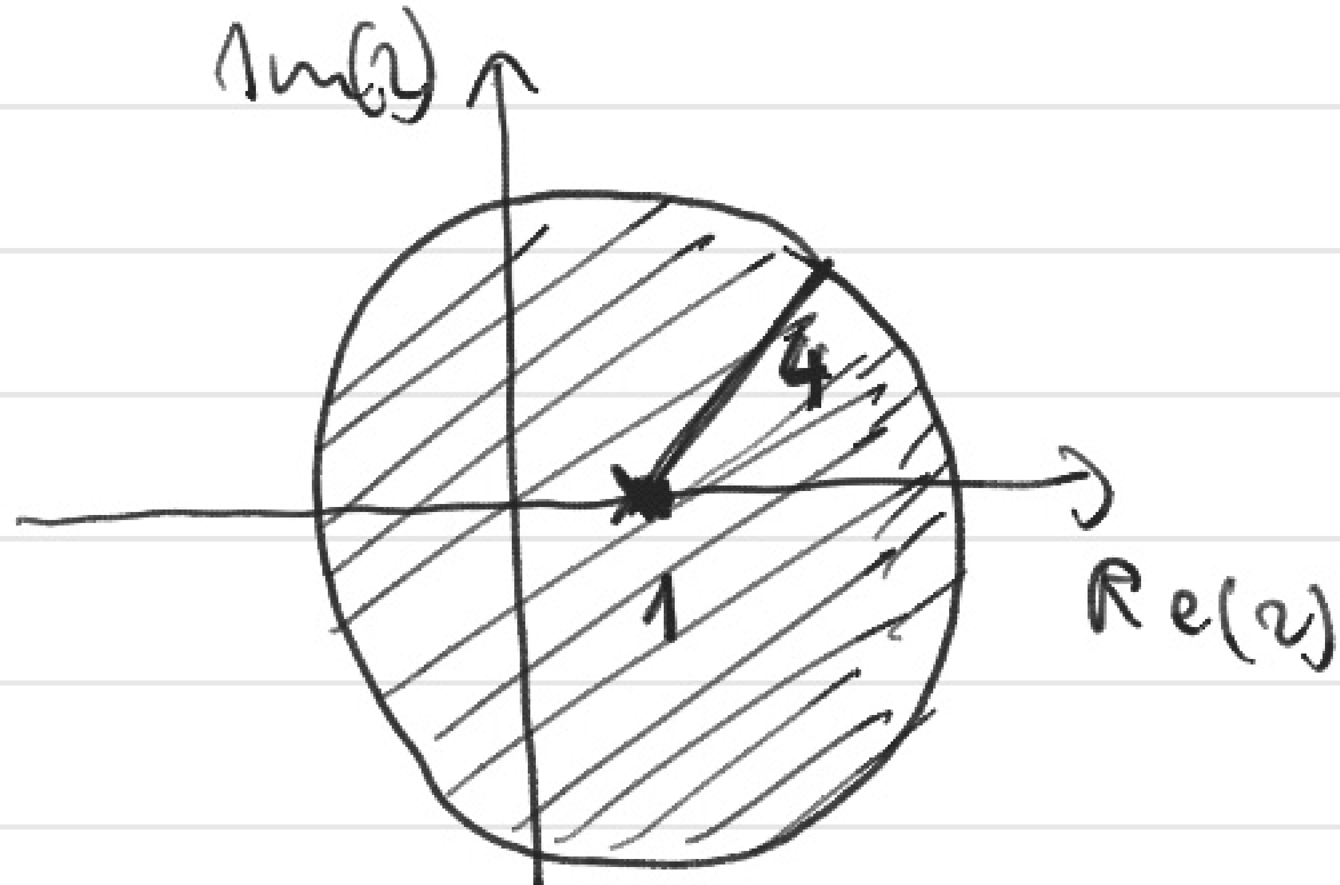
$\overline{z-w} = \overline{z} - \overline{w}$

$|z-w| = \overline{z-w}$ hossza

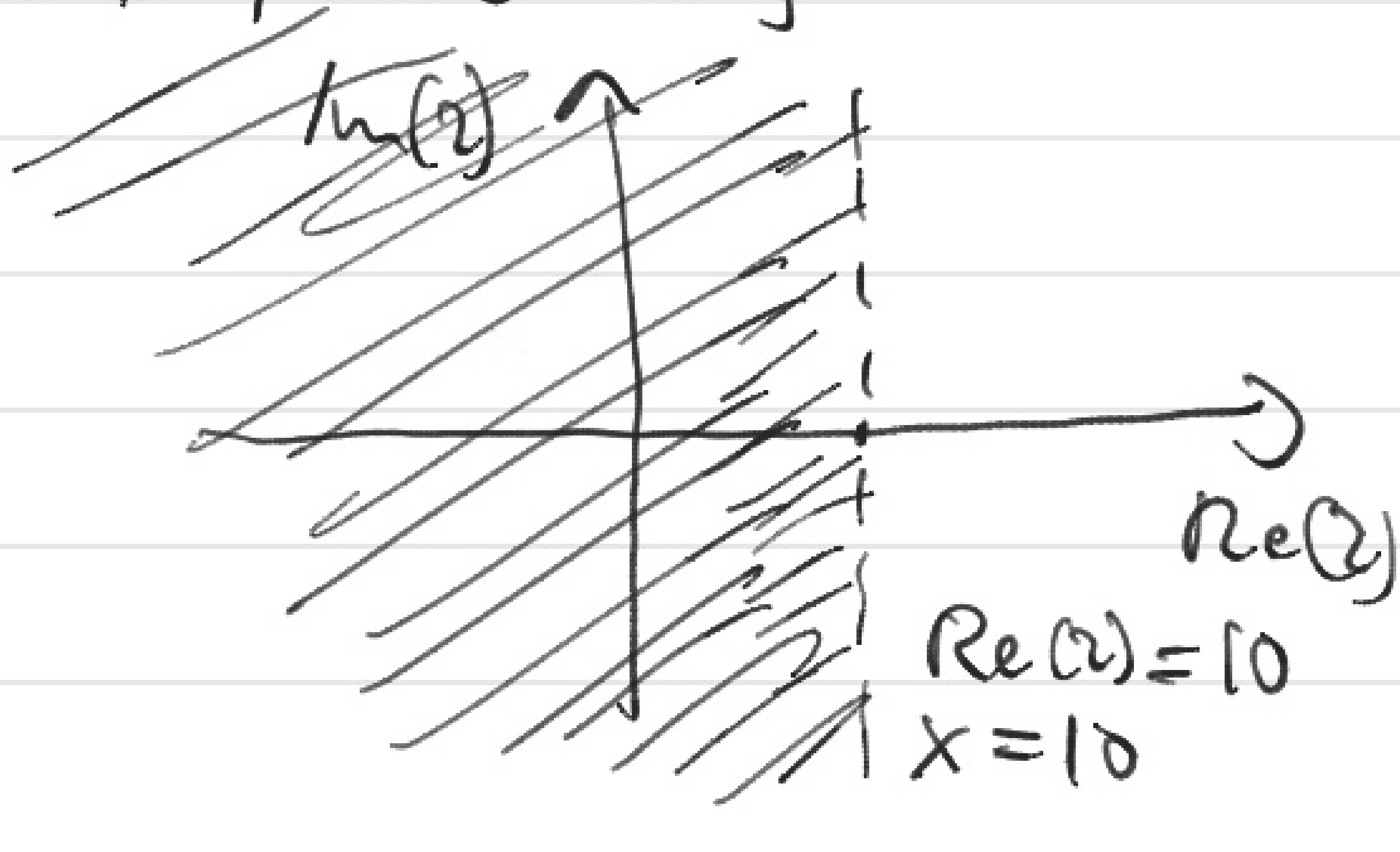
$= z$ és w pontok távolsága

$|z-1| \le 4 \iff$ z távolsága 1-től ≤ 4

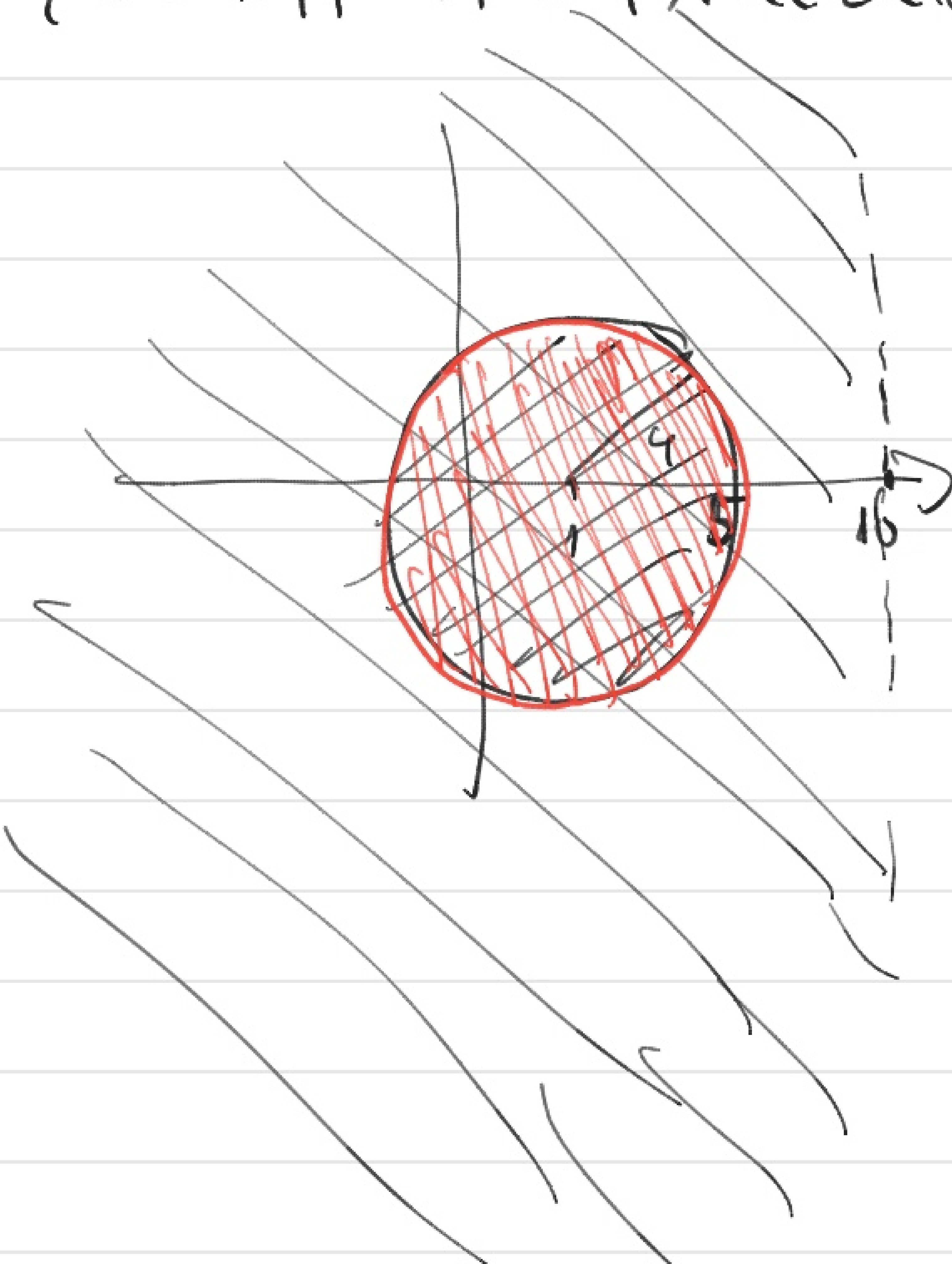
z-nek az 1-től vett távolsága



$$\text{②}\quad \{ z \in \mathbb{C} \mid \overset{x\ <10}{\mathrm{Re}\, z < 10} \}$$



$\Rightarrow \{ z \in \mathbb{C} \mid |z-1| \leq 4 \wedge \operatorname{Re} z < 10 \}$

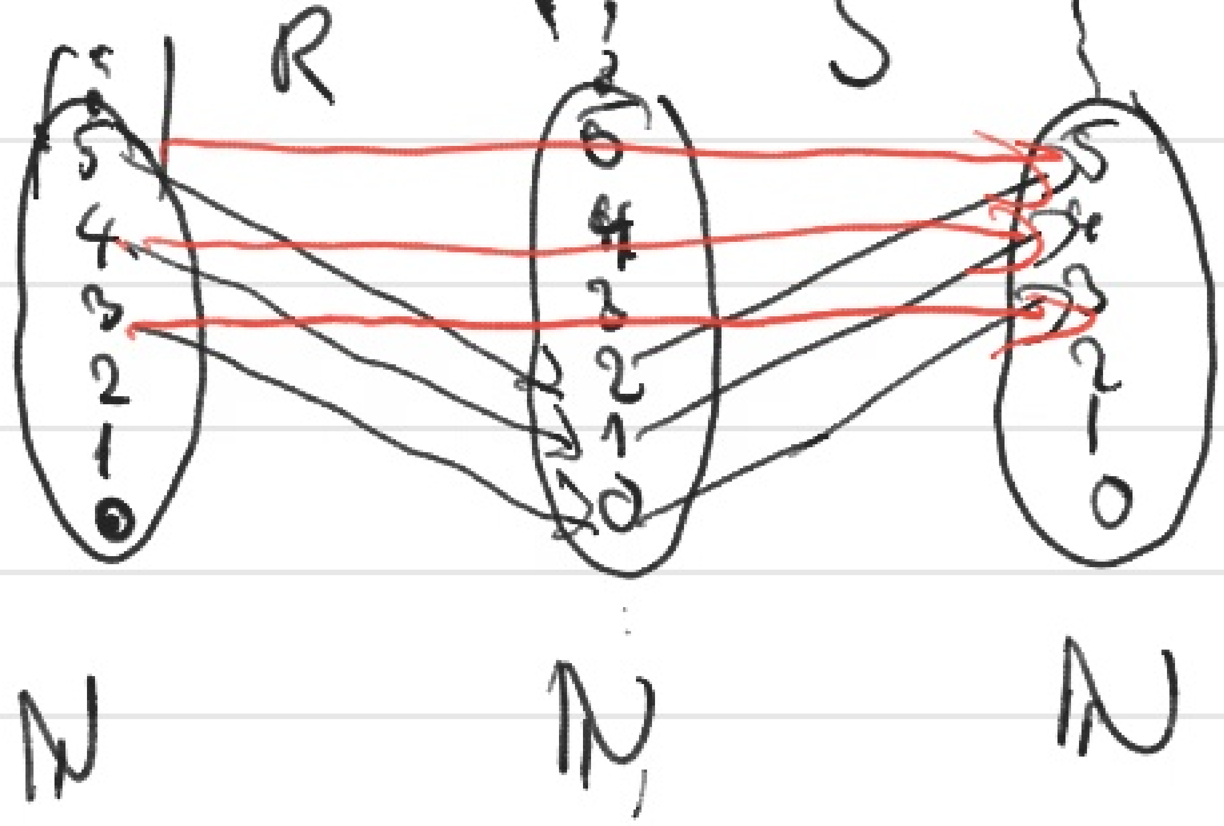


$$R = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = \frac{2}{x}\}$$

$$R = \{(x,y) \in \mathbb{N}\times\mathbb{N} : y+3 = x\}$$

$$S = \{(x,y) \in \mathbb{N}\times\mathbb{N} : x+3 = y\}$$

$$\overset{②}{S} \circ \overset{①}{R} = \{(x,x) \in \mathbb{N}\times\mathbb{N} : x \geq 3\}$$



$$S \circ R = \{(x,y) \mid \exists z : (x,z) \in R \wedge (z,y) \in S\}$$

$$(x,y) \in S \circ R$$

$$\Leftrightarrow \exists z \begin{cases} (x,z) \in R \\ (z,y) \in S \end{cases} \Leftrightarrow \exists z \in \mathbb{N} : \begin{cases} z+3 = x \\ \wedge \\ z+3 = y \end{cases}$$

$$z+3 = x$$

$$z = x-3$$

hinzufügt $\mathbb{N}$ - bül