Logics for Artificial Intelligence – AI Master

Sessions 2-3: Resolution in first-order predicate logic

Any formula in predicate logic can be translated into **Skolem Normal Form (SNF)**. In this form there is a sequence of universal quantifiers on the left, and a formula in Conjunctive Normal Form (without any quantifier) on the right. The procedure to translate any formula into SNF is the following:

- Translate the formula into **Prenex Conjunctive Normal Form** (all the quantifiers are moved to the beginning of the formula, and the part without quantifiers is written in Conjunctive Normal Form). This process has basically the following steps:

    - Replace the conditionals ($A \rightarrow B$) by their equivalent form ($\neg A \vee B$).
    - Move the negations inside the parenthesis, via deductive equivalences (De Morgan's laws, involution).
        - $\neg(A \wedge B) = (\neg A \vee \neg B)$
        - $\neg(A \vee B) = (\neg A \wedge \neg B)$
        - $\neg\neg A = A$
    - Apply distributivity and idempotence as necessary.
        - $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
        - $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
        - $(A \wedge A) = A$
        - $(A \vee A) = A$
    - Rename the quantified variables if the same variable appears several times.
    - Move the quantifiers to the leftmost part of the formula, using the fact that ($Q_xAx \wedge/\vee Q_yBy$) may be written as $Q_x\ Q_y\ (Ax \wedge/\vee By)$. In this expression Q_x and Q_y may be existential or universal quantifiers.

    The body of the formula (the part without quantifiers) is in **Conjunctive Normal Form** when it is a conjunction of clauses. A ***clause*** is a disjunction of (possibly negated) atomic predicates.

- After that, we remove the existential quantifiers sequentially, from left to right. Each existentially quantified variable is replaced by a **new Skolem function** that depends on the universally quantified variables to its left. If the existential quantifier does not have any previous universal quantifier, its variable is replaced by a **new Skolem constant**.
    - For instance, the formula $\exists x \forall y \exists z\ P(x,y,z)$ would be translated to $\forall y\ P(a,y,f(y))$, where *a* is a new Skolem constant and *f* is a new Skolem function.

    In order to allow the use of functions, we expand the definition of term given in the first session: a **term** is now a constant, a variable, or *a function applied to terms.*

Resolution is a method that can be used to try to validate an argument in predicate logic, using only the resolution rule. The basic form of this rule allows the inference of $(F \vee G)$ from the premises $(H \vee F)$ and $(\neg H \vee G)$. In order to prove that B is a logical consequence of a set of premises $A=\{A_1, A_2, \ldots A_n\}$, we have to show that the set $A \cup \{\neg B\}$ is inconsistent. The first step is to translate the formula $(A_1 \wedge A_2 \wedge \ldots \wedge A_n \wedge \neg B)$ into Skolem Normal Form. After that we remove the universal quantifiers and we have a set of clauses $\{c_1, c_2, \ldots, c_r\}$. The aim is to apply the resolution rule to these clauses until we obtain the empty clause (proving the validity of the argument) or until we have exhausted all the resolution possibilities (proving the invalidity of the argument).

We can try to apply the resolution rule to two clauses if one of them contains a predicate P and the other one contains the negation of the same predicate, ¬P. It will be possible to apply the resolution rule if these two predicates are **unifiable**, that is, if we can find a substitution (a mapping from variables to terms) that makes all the positions of the two predicates equal. If Q is the result of applying a substitution to P, Q is an **instance** of P. Given two predicates that are unifiable with different substitutions s_1 and s_2, s_1 is **more general than** s_2 if the result of applying s_2 is an instance of the result of applying s_1. Given two unifiable predicates, a unifier is **the most general unifier** if it is more general than any other unifier.

When we apply the resolution rule to two clauses, we must apply the *most general unifier* of the resolving predicates. This unifier is calculated by comparing the terms of the resolving predicates, one position at a time in a sequential fashion, composing the substitutions needed to make all the terms equal. We can compute a unifier if all the pairs of terms are **disagreement pairs**, in which one of them is a variable (*v*) and the other (*t*) is a constant, a variable or a function that does not depend on *v*. In this case, the substitution needed to equalize the terms is $\{t/v\}$. It is not possible to unify a constant with another constant or with a function, to unify different functions, or to unify a variable with a function containing the same variable.

A resolution is **linear** if, in each step, we use the clause that was obtained in the previous step. A linear resolution is **restricted** if the resolution rule is always applied on the predicate that appears in the rightmost position of the previous clause. **It can be shown that $A \vdash B$ if and only if it is possible to reach the empty clause from the clauses derived from $(A \wedge \neg B)$ by making a linear restricted resolution.**

A linear restricted resolution is **primary** if, in each step, the last clause is resolved with a clause of the initial set. A linear restricted resolution is **unitary** if, in each step, at least one of the two clauses is unitary (it contains a single predicate). **It can be shown that, if the initial set of premises A is composed only of *Horn clauses* (clauses containing at most a positive literal), then $A \vdash B$ if and only if it is possible to reach the empty clause from the clauses derived from $(A \wedge \neg B)$ by making a linear restricted primary (or unitary) resolution.**

The **support set** is the set of clauses obtained from the negation of the conclusion B. **It can be shown that, if the set of premises A is consistent, then $A \vdash B$ if and only if it is possible to reach the empty clause from the clauses derived from $(A \wedge \neg B)$ by making a linear restricted resolution that starts with a clause of the support set.**

Some important aspects to be taken into account in the resolution method:

- Use different variable names in each clause.
- If we reuse the same clause several times in the same proof, rename the variables in the clause in each use.
- We should try to use **factoring**, simplifying a clause (eliminating repetitions of the same predicate with the same terms) **after** applying the unifier of the two resolving predicates and **before** applying the resolution rule.

Resolution (with factoring) is a **complete** method; therefore, if the argument is valid, we should be able to find a proof by resolution from the premises and the negation of the conclusion.

When we apply resolution to the clauses derived from the premises and the negation of the conclusion, three situations may arise:

- We obtain the empty clause. In this case, we have shown that the argument is valid.
- We check all the possibilities of applying resolution and we do not obtain the empty clause. In this case, we have shown that the argument is invalid.
- It may also be the case that the resolution goes on and on indefinitely, without ever getting the empty clause or exhausting all the possibilities of applying resolution. In this case, we would not be able to decide in a finite time if the argument is valid or invalid. In fact, it may be shown that predicate logic is **undecidable**, in the sense that there does not exist any validation method that, in a finite time, given any set of premises and a conclusion, confirms if the argument is valid or invalid.

Basic material you should study on weeks 3 and 4:

- Textbook by Brachman and Levesque: chapters 4 and 5 (especially chapter 4).

Complementary material:

- Stanford introductory course (Genesereth and Rao): sections Propositional Resolution and Relational Logic.
- Material of First-Order Logic (Catalan): sections 2.4.6 and 2.5 (normal forms and propositional resolution) and, especially, sections 3.4 (normal forms in predicate logic), 3.5.1 (functions) and 3.6 (resolution in first-order logic). You can try the exercises associated to these sections, especially 5.3, 5.4, 5.5 and 5.8.