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HW 2 - Skolemization

1) $\forall x \forall y ((P_x \wedge Q_y \wedge \neg \exists z (R_z \wedge S_{xy})) \rightarrow \forall r (Q_r \vee S_{yx}))$

a) Remove implication

$\forall x \forall y (\neg (P_x \wedge Q_y \wedge \neg \exists z (R_z \wedge S_{xy})) \vee \forall r (Q_r \vee S_{yx}))$

b) Apply De Morgan

$\forall x \forall y (\neg (P_x \wedge Q_y) \vee \exists z (R_z \wedge S_{xy}) \vee \forall r (Q_r \vee S_{yx}))$

$\forall x \forall y (\neg P_x \vee \neg Q_y \vee \exists z (R_z \wedge S_{xy}) \vee \forall r (Q_r \vee S_{yx}))$

c) Extract Quantifiers

$\forall x \forall y \exists z \forall r (\neg P_x \vee \neg Q_y \vee (R_z \wedge S_{xy}) \vee Q_r \vee S_{yx})$

d) Rearrange and apply distributive property iteratively to obtain PNF

$\forall x \forall y \exists z \forall r (\neg P_x \vee \neg Q_y \vee S_{yx} \vee (Q_r \vee R_z) \wedge (S_{xy} \vee Q_r))$

$\forall x \forall y \exists z \forall r (\neg P_x \vee \neg Q_y \vee (S_{yx} \vee Q_r \vee R_z) \wedge (S_{yx} \vee S_{xy} \vee Q_r))$

$\forall x \forall y \exists z \forall r (\neg P_x \vee (\neg Q_y \vee S_{yx} \vee Q_r \vee R_z) \wedge (\neg Q_y \vee S_{yx} \vee S_{xy} \vee Q_r))$

$\forall x \forall y \exists z \forall r ((\underbrace{\neg P_x \vee \neg Q_y \vee S_{yx} \vee Q_r \vee R_z}_{C_1}) \wedge (\underbrace{\neg P_x \vee \neg Q_y \vee S_{yx} \vee S_{xy} \vee Q_r}_{C_2}))$

e) Convert into Skolem Normal form.

$z = f(x,y)$

$\forall x \forall y \forall r ((\underbrace{\neg P_x \vee \neg Q_y \vee S_{yx} \vee Q_r \vee R_{f_{xy}}}_{C_1}) \wedge (\underbrace{\neg P_x \vee \neg Q_y \vee S_{yx} \vee S_{xy} \vee Q_r}_{C_2})$

2) $\exists x(\exists y(Q_y \wedge S_{yx}) \rightarrow \forall x(\neg S_{xx} \vee Q_x))$

- Remove implication

$\exists x(\neg \exists y(Q_y \wedge S_{yx}) \vee \forall x(\neg S_{xx} \vee Q_x))$

- Introduce negation inside quantifiers

$\exists x(\forall y \neg(Q_y \wedge S_{yx}) \vee \forall x(\neg S_{xx} \vee Q_x))$

- Rename the second x variable as t

$\exists x(\forall y \neg(Q_y \wedge S_{yx}) \vee \forall t(\neg S_{tt} \vee Q_t))$

- Take out all the quantifiers and apply De Morgan

$\exists x \forall y \forall t \, \underbrace{(\neg Q_y \vee \neg S_{yx} \vee \neg S_{tt} \vee Q_t)}_{C_1}$

- Convert into Skolem Normal Form substituting x by a Skolem constant a

$\forall y \forall t(\neg Q_y \vee \neg S_{ya} \vee \neg S_{tt} \vee Q_t)$

3) $\neg\exists x(\neg\exists y(Q_y \wedge S_{yx}) \vee \exists z \forall r (R_z \vee T_{xrz}))$

- Introduce the negation of the outer quantifier

$\forall x\, \neg(\neg\exists y(Q_y \wedge S_{yx}) \vee \exists z \forall r (R_z \vee T_{xrz}))$

- Apply De Morgan and cancel double negation

$\forall x(\exists y(Q_y \wedge S_{yx}) \wedge \neg\exists z \forall r (R_z \vee T_{xrz}))$

- Introduce the negation into the existential quant

$\forall x(\exists y(Q_y \wedge S_{yx}) \wedge \forall z\, \neg\forall r (R_z \vee T_{xrz}))$

- Introduce the negation into the universal quantifier

$\forall x(\exists y(Q_y \wedge S_{yx}) \wedge \forall z \exists r\, \neg(R_z \vee T_{xrz}))$

- Apply De Morgan and ~~to~~ put all the quant. outside.

$$\forall x \exists y \forall z \exists r(\underbrace{Q_y}_{c_1} \wedge \underbrace{S_{yx}}_{c_2} \wedge \underbrace{\neg R_z}_{c_3} \wedge \underbrace{\neg T_{xrz}}_{c_4})$$

- Convert into SNF defining $f = f(x)$ for y and $g = g(x, z)$ for r variable

$$\forall x \forall z(\underbrace{Q_{f_x}}_{c_1} \wedge \underbrace{S_{f_x x}}_{c_2} \wedge \underbrace{\neg R_z}_{c_3} \wedge \underbrace{\neg T_{x g_{xz} z}}_{c_4})$$

4) $\forall x((P_x \to \exists y(Q_y \wedge S_{xy})) \to \forall z S_{xz})$

- Remove internal implication

$\forall x((\neg P_x \vee \exists y(Q_y \wedge S_{xy})) \to \forall z S_{xz})$

- Remove implication

$\forall x(\neg(\neg P_x \vee \exists y(Q_y \wedge S_{xy})) \vee \forall z S_{xz})$

- Apply De Morgan

$\forall x (P_x \wedge \neg\exists y (Q_y \wedge S_{xy}) \vee \forall z\, S_{xz})$

- Introduce negation in existential quantifier

$\forall x (P_x \wedge \forall y\, \neg(Q_y \wedge S_{xy}) \vee \forall z\, S_{xz})$

- Apply De Morgan

$\forall x (P_x \wedge \forall y (\neg Q_y \vee \neg S_{xy}) \vee \forall z\, S_{xz})$

- Extract the quantifiers

$\forall x \forall y \forall z (P_x \wedge (\neg Q_y \vee \neg S_{xy}) \vee S_{xz})$

- Apply distributive property

$\forall x \forall y \forall z (\underbrace{(P_x \vee S_{xz})}_{C_1} \wedge \underbrace{(\neg Q_y \vee \neg S_{xy} \vee S_{xz})}_{C_2})$