and $r_1 = 2$, $\theta_1 = n(m_x^2 - 2\rho m_x m_y + m_y^2)/(1 - \rho^2)$ respectively. Accordingly, the non-negative form

$$\sum_1^n [(x_i - \bar{x})^2 - 2\rho(x_i - \bar{x})(y_i - \bar{y}) + (y_i - \bar{y})^2]/(1 - \rho^2)$$

has a central χ^2 distribution with $2n - 2$ degrees of freedom.

REFERENCE

[1] OSMER CARPENTER, "Note on the extension of Craig's theorem to noncentral variates," *Ann. Math. Stat.*, Vol. 21 (1950), p. 455.

A NOTE ON THE GENERATION OF RANDOM NORMAL DEVIATES[1]

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1. Introduction. Sampling experiments often require the generation of large numbers of random normal deviates. When an electronic computer is used it is desirable to arrange for the generation of such normal deviates within the machine itself rather than to rely on tables. Pseudo random numbers can be generated by a variety of methods within the machine and the purpose of this note is to give what is believed to be a new method for generating normal deviates from independent random numbers. This approach can be used on small as well as large scale computers. A detailed comparison of the utility of this approach with other known methods (such as: (1) the inverse Gaussian function of the uniform deviates, (2) Teichroew's approach, (3) a rational approximation such as that developed by Hastings, (4) the sum of a fixed number of uniform deviates and (5) rejection-type approach), has been made elsewhere [1] by one of the authors (M.M.). It is shown that the present approach not only gives higher accuracy than previous methods but also compares in speed very favourably with other methods.

2. Method. The following approach may be used to generate a pair of random deviates from the same normal distribution starting from a pair of random numbers.

Method: Let U_1, U_2 be independent random variables from the same rectangular density function on the interval (0, 1). Consider the random variables:

$$\begin{aligned} X_1 &= (-2 \log_e U_1)^{1/2} \cos 2\pi U_2 \\ X_2 &= (-2 \log_e U_1)^{1/2} \sin 2\pi U_2 \end{aligned} \tag{1}$$

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Then (X_1, X_2) will be a pair of independent random variables from the same normal distribution with mean zero, and unit variance.

Justification: From (1) (giving attention to principal values), one obtains at once the inverse relationships:

$$U_1 = e\,\frac{-(X_1^2 + X_2^2)}{2}\cdot$$

$$U_2 = -\frac{1}{2\pi}\arctan\frac{X_2}{X_1}.$$

It follows that the joint density of X_1, X_2 is

$$f(X_1, X_2) = \frac{1}{2\pi}\,e\,\frac{-(X_1 + X_2)}{2} = \frac{1}{\sqrt{2\pi}}\,e\,\frac{-X_1^2}{2}\cdot\frac{1}{\sqrt{2\pi}}\,e\,\frac{-X_2^2}{2} = f(X_1)f(X_2);$$

thus the desired conclusions, including the independence of X_1 and X_2 is obtained.

The above approach is motivated by the following considerations: the probability density of $f(X_1, X_2)$ is constant on circles, so $\Theta = \arctan X_2/X_1$ is uniformly distributed $(0, 2\pi)$. Further, the square of the length of the radius vector $r^2 = X_1^2 + X_2^2$ has a Chi-squared distribution with two degrees of freedom. If U has a rectangular density on (0, 1) then $-2\log_e U$ has a Chi-squared distribution with two degrees of freedom. Proceeding in the reverse order we arrive at (1).

3. Generalizations and other random variables. Observations from the Chi-squared distribution with $2k$ degrees of freedom can of course be generated by adding together the k terms, $\sum_{i=1}^{k}(-2\log_e U_i)$ and for Chi-squared with $2k + 1$ degrees of freedom one may add the square of a normal deviate generated by the above method. Deviates from the F-distribution and for the t-distribution are obtained by calculating the appropriate ratio of deviates generated as above. From independent random normal deviates well known methods can of course be used to generate n-dimensional normal deviates with arbitrary means and variance-covariance matrix.

4. Convenience and accuracy. The method suggested here grew out of the desire to have a way of generating normal deviates which would be reliable in the tails of the distribution. Since most computing centers have library programs to compute values of trigonometric functions, logarithms, and square roots this approach requires little additional machine program writing. The accuracy obtained depends essentially on the precision of the available library programs, whereas that of other methods cannot readily be increased.

REFERENCE

[1] M. E. Muller, "Generation of Normal Deviates," Technical Report No. 13, Statistical Techniques Research Group, Department of Mathematics, Princeton University.