Lista 4

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Questão 1

$$p_{X,Y}(x,y) = \begin{cases} k(2x+y) & ,\ x = 1,2;\ y = 1,2 \\ 0 & ,\ \text{c.c} \end{cases}$$

a)

$$\begin{aligned}
\sum_x \sum_y p_{X,Y}(x,y) =& 1 \\
\sum_{x=1}^{2} \sum_{y=1}^{2} k(2x+y) =& 1 \\
\sum_{y=1}^{2} k(2+y) + k(2.2+y) =& 1 \\
k(2+1) + k(2.2+1) + k(2+2) + k(2.2+2) =& 1 \\
k(3) + k(5) + k(4) + k(6) =& 1 \\
k(18) =& 1 \\
k =& \frac{1}{18}
\end{aligned}$$

b)

$$\begin{aligned}
p_X(x) =& \sum_y p_{X,Y}(x,y) \\
=& \sum_{y=1}^{2} k(2x+y) \\
=& k(2x+1) + k(2x+2) \\
=& 2kx + k + 2kx + 2k \\
=& 4kx + 3k \\
=& k(4x+3) \\
=& \frac{1}{18}(4x+3)
\end{aligned}$$

$$
\begin{aligned}
p_Y(y) = &\sum_x p_{X,Y}(x,y)\\
= &\sum_{x=1}^{2} k(2x+y)\\
=&k(2+y)+k(2.2+y)\\
=&2k+yk+4k+yk\\
=&2yk+6k\\
=&k(2y+6)\\
=&\frac{1}{18}(2y+6)
\end{aligned}
$$

c)

$$
\begin{aligned}
p_{X,Y}(x,y) &\overbrace{=}^{hip.} p_X(x).p_Y(y)\\
p_{X,Y}(x,y) &= \frac{(4x+3)}{18}\frac{(2y+6)}{18}\\
\frac{(2x+y)}{18} &\neq \frac{(4x+3)}{18}\frac{(2y+6)}{18}
\end{aligned}
$$

Questão 2

X = números de caras de A

Y = números de caras de B

- $P(A) = \frac{1}{2}$
- $P(B) = \frac{1}{4}$

a)

$\{(x,y) : x, y = \{0,1,2,3\}\}$

b)

$\{(x,y) : x, y = \{0,1,2,3\}\}$

Jogas repetidas sem importar a ordem

$$
p_x(x) = \begin{cases} \binom{3}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{3-x} & , \ x = \{0,1,2,3\} \\ 0 & , \ c.c \end{cases}
$$

$$
p_y(y) = \begin{cases} \binom{3}{y}\left(\frac{1}{4}\right)^y\left(\frac{3}{4}\right)^{3-y} & , \ y = \{0,1,2,3\} \\ 0 & , \ c.c \end{cases}
$$

c)

Pensar que existe 3 lançamentos onde **todas** vão ter algum resultado entre cara e coroa da escolha aleatória de moedas distintas (A e B):

$$\overline{1}\,\overline{2}\,\overline{3}$$

(Não importa a ordem e com repetição) A jogada não é enumerada e os resultado **cara** e **coroa** pode ser escolhido varias vezes de moedas distintas pra 3 lançamentos. No qual a probabilidade de escolher uma cara depende de qual foi a moeda escolhida

$$p_{x,y}(x,y) = \begin{cases} \binom{3}{x\ y} \left(\frac{1}{2}\right)^x \left(\frac{1}{4}\right)^y \left(\frac{3}{4}\right)^{3-y} \left(\frac{1}{2}\right)^{3-x} & , x+y \leq 3\,; x,y = \{0,1,2,3\} \\ 0 & , \ c.c \end{cases}$$

d)

$$\begin{aligned} P(X=Y) &= \sum_{y=0}^{3} P(X=y, Y=y) \\ &= \sum_{y=0}^{3} P(X=y).P(Y=y) \end{aligned}$$

$$\begin{aligned} P(X>Y) &= \sum_{y=0}^{3} P(X>y, Y=y) \\ &= \sum_{y=0}^{3} P(X>y).P(Y=y) \end{aligned}$$

$$\begin{aligned} P(X+Y\leq 4) &= \sum_{y=0}^{3} P(X\leq 4-y, Y=y) \\ &= \sum_{y=0}^{3} P(X\leq 4-y).P(Y=y) \end{aligned}$$

Questão 3

$X \sim Geo(p)$

$$Y = \min(X,M) = \begin{cases} X & ,X<M \\ M & ,X\geq M \end{cases}$$

$$
\begin{aligned}
P(Y=y) &= P(\min(X,M)=y, \overbrace{\Omega}^{(x<M)\cup(x\geq M)}) \\
&= P\overbrace{(\min(X,M)=y, X<M)}^{Impõe\ X=y} + P\overbrace{(\min(X,M)=y, X\geq M)}^{M=y} \\
&= P\overbrace{(X=y, X<M)}^{y<M} + P\overbrace{(X\geq M)}^{y=M} \\
&= \begin{cases} P(X=y) & , y=\{1,2,...,M-1\} \\ P(X\geq M) & , y=M \\ 0 & ,\ c.c \end{cases}
\end{aligned}
$$

Queremos saber sobre y, assim procuramos seus limites.

Questão 4

$$\sum_{i=1}^{6} x_i = 10$$

$$\vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$$

a)

Pensar que existe 10 lançamentos onde **todas** vão ter algum resultado i:

$$\overline{1}\,\overline{2}\,\overline{3}\cdots\overline{9}\,\overline{10}$$

(Não importa a ordem e com repetição) Pois a jogada não é enumerada e uma resultado pode ser escolhido varias vezes pra 10 lançamentos. No qual a probabilidade de escolher um resultado é de 1/10

$$p_{\vec{x}} = \begin{cases} \binom{10}{x_1,x_2,x_3,x_4,x_5,x_6} \prod_{i=1}^{6} \left(\frac{1}{6}\right)^{x_i} & , \sum_{i=1}^{6} x_i = 10;\ 0 \leq x_i \leq 10 \\ 0 & ,\ c.c \end{cases}$$

b)

$$p_{x_i} = \sum_{\substack{0\leq x_j\leq 10 \\ x_i + \sum_{j\neq i} x_j = 10 \\ i=(1,2,3,4,5,6)}} P(X_i = x_i) = \binom{10}{x_i}\left(\frac{1}{6}\right)^{x_i}\left(\frac{5}{6}\right)^{10-x_i}$$

$$X_i \sim B(10, 1/6)\ X_i = (1,2,3,4,5,6)$$

c)

Não, pois o resultado de quantas vezes saiu o valor i, depende de quantas vezes saiu os outros valores diferentes de i. O que diz respeito a probabilidade de i (p_i) ser o complementar dos valores diferentes de i, ($\sum_{j\neq i} p_j = 1 - p_i$).

Questão 5

$$\sum_{i=1}^{r} x_i = 2r$$

$$\vec{X} = (X_1, X_2, X_3, ..., \; X_{r-1}, X_r)$$

a)

$X_i = \{$caixa i com x_i bolas $\}$

Pensar que existe 2r bolas onde **todas** vão ter alguma caixa:

$$\overline{1}\,\overline{2}\,\overline{3} \cdots \overline{2r-1}\,\overline{2r}$$

(Não importa a ordem e com repetição) pois a bola não é enumerada e uma caixa podem ser escolhidas varias vezes pra 2r bolas. No qual a probabilidade de escolher uma caixa é de 1/r

$$p_{\vec{X}}(\vec{x}) = \begin{cases} \binom{2r}{x_1, x_2, \ldots, x_r} \prod_{i=1}^{r} \left(\frac{1}{r}\right)^{x_i} & , \sum_{i=1}^{r} x_i = 2r; \; 0 \leq x_i \leq 2r \\ 0 & , \; c.c \end{cases}$$

b)

$$P_{\vec{X}}(2) = \binom{2r}{2, 2, ..., 2} \left(\frac{1}{r}\right)^{2r}$$

Questão 6

a)

$$
\begin{aligned}
P(X \geq Y) = \sum_y P(X \geq Y, Y = y) &= \sum_y P(X \geq y, Y = y) \\
&= \sum_{y=0}^{N} P(X \geq y) P(y = y) \\
&= \sum_{y=0}^{N} [1 - P(X < y)] P(y = y) \\
&= \sum_{y=0}^{N} [1 - P(X \leq y - 1)] P(y = y) \\
&= \sum_{y=0}^{N} \left[1 - \frac{y - 1 - 0 + 1}{N + 1}\right] \frac{1}{N + 1} \\
&= \sum_{y=0}^{N} \left[1 - \frac{y}{N + 1}\right] \frac{1}{N + 1} \\
&= \frac{1}{N + 1} \left[\sum_{y=0}^{N} 1 - \frac{1}{N + 1} \sum_{y=0}^{N} y\right] \\
&= \frac{1}{N + 1} \left[(N + 1) - \frac{(N(N + 1))}{2(N + 1)}\right] \\
&= \left[1 - \frac{N}{2(N + 1)}\right]
\end{aligned}
$$

b)

$$
\begin{aligned}
P(X = Y) = \sum_y P(X = Y, Y = y) &= \sum_y P(X = y, Y = y) \\
ind. &= \sum_{y=0}^{N} P(X = y) P(y = y) \\
&= \sum_{y=0}^{N} \frac{1}{N + 1} \frac{1}{N + 1} \\
&= \frac{N + 1}{(N + 1)^2} \\
&= \frac{1}{(N + 1)}
\end{aligned}
$$

c)

$$
Z = \min(X, Y),\ X, Y = \{0, 1, 2, ..., N\}
$$

$Z \in \{0, 1, ..., N\}$:

$$\begin{aligned}
P(Z=z) &= P(\min(X,Y)=z) \\
&= P(\min(X,Y)=z, X<Y) + P(\min(X,Y)=z, X \geq Y) \\
&= P(X=z, X<Y) + P(Y=z, X \geq Y) \\
&= P(X=z, z<Y) + P(Y=z, X \geq z) \\
ind. &= P(X=z)P(Y>z) + P(Y=z)P(X \geq z) \\
&= \frac{1}{(N+1)}[1-P(Y \leq z)] + \frac{1}{(N+1)}[1-P(X \leq z-1)] \\
&= \frac{1}{(N+1)}\left[1-\frac{z-0+1}{N+1}\right] + \frac{1}{(N+1)}\left[1-\frac{z-1-0+1}{N+1}\right] \\
&= \frac{1}{(N+1)}\left[1-\frac{z+1}{N+1}\right] + \frac{1}{(N+1)}\left[1-\frac{z}{N+1}\right] \\
&= \frac{1}{(N+1)}\left[\frac{N+1-(z+1)}{N+1}\right] + \frac{1}{(N+1)}\left[\frac{N+1-z}{N+1}\right] \\
&= \frac{(N+1)-(z+1)}{(N+1)^2} + \frac{(N+1)-z}{(N+1)^2} \\
&= \frac{2(N+1)-2z-1}{(N+1)^2} \\
&= \frac{2(N-z)+1}{(N+1)^2}
\end{aligned}$$

$$p_Z(z) = \begin{cases} \frac{2(N-z)+1}{(N+1)^2} & , \ z = \{0,1,..,N\} \\ 0 & , \ c.c \end{cases}$$

d)

$$W = \max(X,Y), \ X, Y = \{0, 1, 2, ..., N\}$$

$W \in \{0, 1, ..., N\}$:

$$\begin{aligned}
P(W=w) &= P(\max(X,Y)=w) \\
&= P(\max(X,Y)=w, X \leq Y) + P(\max(X,Y)=w, X>Y) \\
&= P(Y=w, X \leq Y) + P(X=w, X>Y) \\
&= P(Y=w, X \leq w) + P(X=w, w>Y) \\
ind. &= P(Y=w)P(X \leq w) + P(X=w)P(Y \leq w-1) \\
&= \frac{1}{(N+1)}[P(X \leq w)] + \frac{1}{(N+1)}[P(Y \leq w-1)] \\
&= \frac{1}{(N+1)}\left[\frac{w-0+1}{N+1}\right] + \frac{1}{(N+1)}\left[\frac{w-1-0+1}{N+1}\right] \\
&= \frac{1}{(N+1)}\left[\frac{w+1}{N+1}\right] + \frac{1}{(N+1)}\left[\frac{w}{N+1}\right] \\
&= \frac{2w+1}{(N+1)^2}
\end{aligned}$$

$$p_W(w) = \begin{cases} \frac{2w+1}{(N+1)^2} & , \ w = \{0,1,..,N\} \\ 0 & , \ c.c \end{cases}$$

e)

$$U = |Y - X|,\ X, Y = \{0, 1, 2, ..., N\}$$

$U = 0:$

$$\begin{aligned} P(U = 0) &= P(|Y - X| = 0) \\ &= P(Y - X = 0) \\ &= P(Y = X) \\ &= \frac{1}{N+1} \end{aligned}$$

$U \in \{1, ..., N\}:$

$$\begin{aligned} P(u = u) &= P(|Y - X| = u) \\ &= P(|Y - X| = u, Y - X < 0) + P(|Y - X| = u, Y - X > 0) \\ &+ P(|Y - X| = u, Y - X = 0) \\ &= P(Y - X = -u, Y - X < 0) + P(Y - X = u, Y - X > 0) + 0 \\ &= P(X = Y + u, X > Y) + P(Y = X + u, Y > X) \\ &= P(X = Y + u) + P(Y = X + u) \\ i.i.d &= 2P(X = Y + u) \\ &= 2\sum_{y=0}^{N} P(X = y + u, Y = y) \\ &= 2\sum_{y=0}^{N} P(X = \overbrace{y + u}^{0 \le y+u \le N})P(Y = y) \\ (y \le N - u) &= 2\sum_{y=0}^{N-u} \frac{1}{N+1}\frac{1}{N+1} \\ &= \frac{2(N - u + 1)}{(N+1)^2} \end{aligned}$$

$$p_u(u) = \begin{cases} \frac{1}{(N+1)} & ,\ u = 0 \\ \frac{2(N-u+1)}{(N+1)^2} & ,\ u = \{1, .., N\} \\ 0 & ,\ c.c \end{cases}$$

Questão 7

$X : p = p_1\ Y : p = p_2$

$$p_x(k) = p_y(k) = \begin{cases} p(1-p)^{k-1} & ,\ k = \{1, 2, ...\} \\ 0 & ,\ c.c \end{cases}$$

a)

$$
\begin{aligned}
P(X \geq Y) &= P(Y \leq X) \\
&= P(Y \leq X, X = \mathbb{R}) \\
&= \sum_{x=1}^{\infty} P(Y \leq X, X = x) \\
&= \sum_{x=1}^{\infty} P(Y \leq x, X = x) \\
ind. &= \sum_{x=1}^{\infty} P(Y \leq x) P(X = x) \\
&= \sum_{x=1}^{\infty} [1 - (1-p_2)^x][p_1.(1-p_1)^{x-1}] \\
&= \sum_{x=1}^{\infty} [p_1.(1-p_1)^{x-1} - p_1.(1-p_1)^{x-1}.(1-p_2)^x] \\
&= \sum_{x=1}^{\infty} \left[\frac{p_1.(1-p_1)^x}{(1-p_1)} - \frac{p_1.[(1-p_1).(1-p_2)]^x}{(1-p_1)} \right] \\
&= \frac{p_1}{(1-p_1)} \sum_{x=1}^{\infty} [(1-p_1)^x - [(1-p_1).(1-p_2)]^x] \\
&= \frac{p_1}{(1-p_1)} \left[\frac{1}{1-(1-p_1)} - \frac{1}{1-(1-p_1).(1-p_2)} \right] \\
&= ...
\end{aligned}
$$

b)

$$
\begin{aligned}
P(X = Y) &= P(X = Y, Y = \mathbb{R}) \\
&= \sum_{y=1}^{\infty} P(X = Y, Y = y) \\
&= \sum_{y=1}^{\infty} P(X = y, Y = y) \\
ind. &= \sum_{y=1}^{\infty} P(X = y) P(Y = y) \\
&= \sum_{y=1}^{\infty} [p_1(1-p_1)^{y-1}][p_2.(1-p_2)^{y-1}] \\
&= p_1.p_2 \sum_{y=1}^{\infty} (1-p_1)^{y-1}(1-p_2)^{y-1} \\
&= p_1.p_2 \sum_{y=1}^{\infty} [(1-p_1)(1-p_2)]^{y-1} \\
&= \frac{p_1.p_2}{1-(1-p_1)(1-p_2)} \\
&= ...
\end{aligned}
$$

c)

$$Z = \min(X, Y),\ X, Y = \{1, 2, ...\}$$

$Z \in \{1, 2, ...\}$:

$$\begin{aligned}
P(Z = z) &= P(\min(X, Y) = z) \\
&= P(\min(X, Y) = z, X < Y) + P(\min(X, Y) = z, X \geq Y) \\
&= P(X = z, X < Y) + P(Y = z, X \geq Y) \\
&= P(X = z, z < Y) + P(Y = z, X \geq z) \\
ind. &= P(X = z)P(Y > z) + P(Y = z)P(X \geq z) \\
&= P(X = z)[1 - P(Y \leq z)] + P(Y = z)[1 - P(X \leq z - 1)] \\
&= ...(Substitui\ ai)
\end{aligned}$$

c)

$$W = X + Y$$

$W \in \{2, 3, ...\}$:

$$\begin{aligned}
P(W = w) &= P(X + Y = w) \\
&= P(X = w - Y, Y = \mathbb{R}) \\
&= \sum_{y=1}^{\infty} P(X = w - y, Y = y) \\
ind. &= \sum_{y=1}^{\infty} P(X = \overbrace{w - y}^{1 \leq w - y \leq \infty})P(Y = y) \\
(-\infty \leq y \leq w - 1) \rightarrow (1 \leq y \leq w - 1) &= \sum_{y=1}^{w-1} P(X = w - y)P(Y = y) \\
&= \sum_{y=1}^{w-1} p_1(1 - p_1)^{w-y-1} p_2(1 - p_2)^{y-1} \\
&= p_1.p_2.(1 - p_1)^w \sum_{y=1}^{w-1} (1 - p_1)^{-y-1}(1 - p_2)^{y-1} \\
&= p_1.p_2.(1 - p_1)^w \sum_{y=1}^{w-1} \frac{(1 - p_2)^{y-1}}{(1 - p_1)^{y+1}} \\
&= \frac{p_1.p_2.(1 - p_1)^w}{(1 - p_2)(1 - p_1)} \sum_{y=1}^{w-1} \left[\frac{(1 - p_2)}{(1 - p_1)}\right]^y \\
&= \frac{p_1.p_2.(1 - p_1)^w}{(1 - p_2)(1 - p_1)} \frac{1 - \left[\frac{(1-p_2)}{(1-p_1)}\right]^{w-1+1}}{1 - \left[\frac{(1-p_2)}{(1-p_1)}\right]} \\
&= \frac{p_1.p_2.(1 - p_1)^w}{(1 - p_2)(1 - p_1)} \frac{1 - \left[\frac{(1-p_2)}{(1-p_1)}\right]^{w}}{1 - \left[\frac{(1-p_2)}{(1-p_1)}\right]} \\
&= ...(Alguma\ coisa\ ai)
\end{aligned}$$

Questão 8

$$X, Y \in (1, 2, ...)$$

$Z = Y - X \to P(Y - X = z) = P(Y = z + X)$

$$Z \in \mathbb{R}$$

$W = \min(X, Y) \to P(\min(X, Y) = w) = P(X = w, X < Y) + P(Y = w, X \geq Y)$

$$W \in (1, 2, ...)$$

a) (?)

$Z \geq 0$

$$\begin{aligned} P(W = w, Z = z) &= P(X = w, X < Y, Y = z + X) + P(Y = w, X \geq Y, Y = z + X) \\ &= P(X = w, w < Y, Y = z + w) + P(Y = w, X \geq w, w = z + X) \\ &= P(X = w, Y > w, Y = z + w) + P(Y = w, X \geq w, X = w - z) \\ &= P(X = w, Y = z + w) + P(Y = w, X = w) \\ &= P(X = w)P(Y = z + w) + P(Y = w)P(X = w) \\ &= P(X = w)[P(Y = z + w) + P(Y = w)] \end{aligned}$$

Questão 9

$$Z = X + Y$$

$$P(Z = z) = P(X + Y = z) = P(X = z - Y) = P(X = z - Y, Y = \mathbb{R})$$

a) X,Y (Poisson)

$Z \in (0, 1, ...)$

$$\begin{aligned} P(Z = z) &= \sum_{y=0}^{\infty} P(X = z - y, Y = y) \\ &= \sum_{y=0}^{\infty} P(X = \overbrace{z - y}^{0 \leq z - y \leq \infty})P(Y = y) \\ (0 \leq y \leq z) &= \sum_{y=0}^{z} \frac{e^{-\lambda_1}\lambda_1^{z-y}}{(z-y)!}\frac{e^{-\lambda_2}\lambda_2^{y}}{y!} \\ &= \sum_{y=0}^{z} \frac{e^{-(\lambda_1+\lambda_2)}\lambda_1^{z-y}}{(z-y)!}\frac{\lambda_2^{y}}{y!}\frac{z!}{z!} \\ &= \sum_{y=0}^{z} \binom{z}{y}\frac{e^{-(\lambda_1+\lambda_2)}\lambda_1^{z-y}\lambda_2^{y}}{z!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^{z}}{z!} \end{aligned}$$

$$p_Z(z) = \begin{cases} \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^{z}}{z!} & , x = 0, 1, ... \\ 0 & , \ c.c \end{cases}$$

$$Z \sim Poisson(\lambda_1 + \lambda_2)$$

b) X,Y (uniforme)

$Z = (2, 3, 4, ..., 2N-1, 2N)$

$X, Y = (1, 2, 3, ..., N-1, N)$

$$\begin{aligned} P(Z=z) &= \sum_{y=1}^{N} P(X=z-y, Y=y) \\ &= \sum_{y=1}^{N} P(X = \overbrace{z-y}^{1 \le z-y \le N})P(Y=y) \end{aligned}$$

Para $\underbrace{z-N}_{=1} \le y \le \underbrace{z-1}_{=N}$

Pra $(1 \le y \le N)$ ser verdade, temos:

$Z \in (2, 3, \dots, N)$

"$1 \le y \le z-1$"

$$\begin{aligned} P(Z=z) &= \sum_{y=1}^{z-1} P(X=z-y)P(Y=y) \\ &= \sum_{y=1}^{z-1} \frac{1}{N}\frac{1}{N} \\ &= \frac{z-1}{N^2} \end{aligned}$$

$Z \in (N+1, N+2, ..., 2N)$

"$z-N \le y \le N$"

$$\begin{aligned} P(Z=z) &= \sum_{y=z-N}^{N} P(X=z-y)P(Y=y) \\ &= \sum_{y=z-N}^{N} \frac{1}{N}\frac{1}{N} \\ &= \frac{N-(z-N)+1}{N^2} \end{aligned}$$

$$p_z(z) = \begin{cases} \frac{z-1}{N^2} & , z \in \{2, 3, \dots, N\} \\ \frac{N-(z-N)+1}{N^2} & , z \in \{N+1, N+2, \dots, 2N\} \\ 0 & , \ c.c \end{cases}$$

Questão 10

$$Z = X_1 + ... + X_l$$

$$\begin{aligned}
P(Z=z) &= P(\sum_{j=1}^{l} X_j = z) \\
&= P(X_1 = z - \sum_{j=2}^{l} X_j) \\
&= P\left(\left\{X_1 = z - \sum_{j=2}^{l} x_j\right\} \bigcap_{k=2}^{l} \{X_k = x_k\}\right) \\
i.i.d &= \sum_{x_2}\sum_{x_3}...\sum_{x_l} P\left(\left\{X_1 = z - \sum_{j=2}^{l} x_j\right\}\right) \prod_{k=2}^{l} P(X_k = x_k) \\
&= \sum_{x_2}\sum_{x_3}...\sum_{x_l} \frac{e^{-\lambda_1}\lambda_1^{z-(\sum_{j=2}^{l} x_j)}}{(z-\sum_{j=2}^{l} x_j)!} \prod_{k=2}^{l} \frac{e^{-\lambda_k}\lambda_k^{x_k}}{(x_k)!} \\
&= \sum_{x_2}\sum_{x_3}...\sum_{x_l} e^{-(\sum_{k=1}^{l}\lambda_k)} \frac{\lambda_1^{z-(\sum_{j=2}^{l} x_j)}}{(z-\sum_{j=2}^{l} x_j)!} \prod_{k=2}^{l} \frac{\lambda_k^{x_k}}{(x_k)!} \\
...^{x_1 = z-(\sum_{j=2}^{l} x_j)} &= \sum_{x_2}\sum_{x_3}...\sum_{x_l} e^{-(\sum_{k=1}^{l}\lambda_k)} \frac{\lambda_1^{x_1}}{(x_1)!} \prod_{k=2}^{l} \frac{\lambda_k^{x_k}}{(x_k)!} \frac{z!}{z!} \\
&= \sum_{x_2}\sum_{x_3}...\sum_{x_l} e^{-(\sum_{k=1}^{l}\lambda_k)} \prod_{k=1}^{l} \frac{\lambda_k^{x_k}}{(x_k)!} \frac{z!}{z!} \\
&= e^{-(\sum_{k=1}^{l}\lambda_k)} \frac{1}{z!} \sum_{x_2}\sum_{x_3}...\sum_{x_l} \binom{z}{x_1, x_2, ..., x_l} \prod_{k=1}^{l} \lambda_k^{x_k} \\
&= e^{-(\sum_{k=1}^{l}\lambda_k)} \frac{\left[\sum_{k=1}^{l}\lambda_k^{x_k}\right]^z}{z!}
\end{aligned}$$

Então:

$$Z \sim Poisson\left(\sum_{j=1}^{l} x_j\right)$$

Questão 11

Os resultados possíveis são apenas 1,2 e 3, onde os respectivos números são correlacionados com a probabilidade $p_1, p_2\ , p_3$. Assim sendo com reposição a distribuição $\vec{X}$ é uma multinomial, no qual $X_1, X_2\ , X_3$ tem distribuição binomial com o complementar de p sendo o resultado diferente daquele que ele atribui.

$X_1, X_2, X_3 \in \{0, 1, ..., n\}$

a)

$$
\begin{aligned}
P(X_1 + X_2 = z) &= P(X_1 = z - X_2, X_2 = \mathbb{R}) \\
&= \sum_{k=0}^{n} P(X_1 = z-k, X_2 = k, X_3 = 0) \\
&= \sum_{k=0}^{n} \binom{n}{z-k, k, 0} p_1^{z-k} p_2^k p_3^0 \\
&= \sum_{k=0}^{n} \frac{n!}{(z-k)!k!} p_1^{z-k} p_2^k \\
&= ...
\end{aligned}
$$

b)

$y \in \mathbb{R}$

$$
\begin{aligned}
P(X_2 = y | X_1 + X_2 = z) &= \frac{P(X_2 = y, X_1 + X_2 = z)}{P(X_1 + X_2 = z)} \\
&= \frac{P(X_2 = y, X_1 = z - X_2)}{P(X_1 + X_2 = z)} \\
&= \frac{\sum_{k=0}^{n} P(X_2 = k, X_1 = z-k, X_3 = 0)}{P(X_1 + X_2 = z)} \\
&= \frac{\sum_{k=0}^{n} \binom{n}{z-k,k,0} p_1^{z-k} p_2^k p_3^0}{\binom{n}{z}(p_1+p_2)^z (1-(p_1+p_2))^{n-z}} \\
&= \frac{\sum_{k=0}^{n} \frac{n!}{(z-k)!k!} p_1^{z-k} p_2^k}{\frac{n!}{z!(n-z)!}(p_1+p_2)^z (1-(p_1+p_2))^{n-z}} \\
&= \sum_{k=0}^{n} \binom{z}{k} \left[\frac{(1-(p_1+p_2))p_1}{p_1+p_2}\right]^z \left[\frac{p_2}{p_1}\right]^k \frac{(n-z)!}{(1-(p_1+p_2))^n} \\
&= ...
\end{aligned}
$$

Questão 12

$$X \sim Poisson(\lambda)$$

a)

$\lambda = n.p = 50.(0.05) = 2.5$

$$
\begin{aligned}
P(X \leq 2) =& F_X(2) \\
=& e^{-2.5} \left[\frac{(2.5)^0}{0!} + \frac{(2.5)^1}{1!} + \frac{(2.5)^2}{2!}\right] \\
=& e^{-2.5} \frac{2 + 2(2.5) + (2.5)^2}{2}
\end{aligned}
$$

b)

$\lambda = n.p = 100.(0.03) = 3$

$$
\begin{aligned}
P(X \leq 2) =& F_X(2) \\
=& e^{-3}\left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!}\right] \\
=& e^{-3}\frac{2 + 2(3) + 9}{2}
\end{aligned}
$$

Questão 13

$$X \sim Geom(1/6)$$

a)

$$
\begin{aligned}
P(X \leq 6) =& F_X(6) \\
=& 1 - \left(\frac{5}{6}\right)^6
\end{aligned}
$$

a)

$X \in \{1, 2, ...\}$

$$
\begin{aligned}
P(X \leq x) \geq& \frac{1}{2} \\
p(1-p)^{x-1} \geq& \frac{1}{2} \\
(1-p)^x \geq& \frac{(1-p)}{2p} \\
x\log(1-p) \geq& \log\left[\frac{(1-p)}{2p}\right] \\
x \geq& \frac{\log\left[\frac{(1-p)}{2p}\right]}{\log(1-p)} \\
x \geq& \frac{\log\left[\frac{5/6}{2/6}\right]}{\log(5/6)} \\
x \geq& \frac{\log\left[\frac{5}{2}\right]}{\log(5/6)}
\end{aligned}
$$

Questão 14

$z \in \{0, 1, ...\}$

$x \in \mathbb{R}$

$$
\begin{aligned}
P(X=x|X+Y=z) =& \frac{P(X=x, X+Y=z)}{P(X+Y=z)} \\
=& \frac{P(X=x, Y=z-x)}{P(Y=z-x)} \\
=& \frac{P(X=x)P(Y=z-x)}{P(Y=z-x)} \\
=& \frac{e^{-x}\frac{\lambda_1^x}{x!}e^{-(z-x)}\frac{\lambda_2^{z-x}}{(z-x)!}}{e^{-(z-x)}\frac{\lambda_2^{z-x}}{(z-x)!}}
\end{aligned}
$$

Questão 15

$$P(X=x, Y=y, Z=z|X+Y+Z=m)$$