

Lista 4

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Questão 1

$$p_{X,Y}(x,y) = \begin{cases} k(2x+y) & , x=1,2; y=1,2 \\ 0 & , \text{c.c} \end{cases}$$

a)

$$\begin{aligned} \sum_x \sum_y p_{X,Y}(x,y) &= 1 \\ \sum_{x=1}^2 \sum_{y=1}^2 k(2x+y) &= 1 \\ \sum_{y=1}^2 k(2+y) + k(2.2+y) &= 1 \\ k(2+1) + k(2.2+1) + k(2+2) + k(2.2+2) &= 1 \\ k(3) + k(5) + k(4) + k(6) &= 1 \\ k(18) &= 1 \\ k &= \frac{1}{18} \end{aligned}$$

b)

$$\begin{aligned} p_X(x) &= \sum_y p_{X,Y}(x,y) \\ &= \sum_{y=1}^2 k(2x+y) \\ &= k(2x+1) + k(2x+2) \\ &= 2kx + k + 2kx + 2k \\ &= 4kx + 3k \\ &= k(4x+3) \\ &= \frac{1}{18}(4x+3) \end{aligned}$$

$$\begin{aligned}
p_Y(y) &= \sum_x p_{X,Y}(x,y) \\
&= \sum_{x=1}^2 k(2x+y) \\
&= k(2+y) + k(2.2+y) \\
&= 2k + yk + 4k + yk \\
&= 2yk + 6k \\
&= k(2y+6) \\
&= \frac{1}{18}(2y+6)
\end{aligned}$$

c)

$$\begin{aligned}
p_{X,Y}(x,y) &\overset{hip.}{=} p_X(x) \cdot p_Y(y) \\
p_{X,Y}(x,y) &= \frac{(4x+3)}{18} \frac{(2y+6)}{18} \\
\frac{(2x+y)}{18} &\neq \frac{(4x+3)}{18} \frac{(2y+6)}{18}
\end{aligned}$$

Questão 2

X = números de caras de A

Y = números de caras de B

- $P(A) = \frac{1}{2}$
- $P(B) = \frac{1}{4}$

a)

$$\{(x,y) : x,y = \{0,1,2,3\}\}$$

b)

$$\{(x,y) : x,y = \{0,1,2,3\}\}$$

Jogas repetidas sem importar a ordem

$$p_x(x) = \begin{cases} \binom{3}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x} & , x = \{0,1,2,3\} \\ 0 & , c.c \end{cases}$$

$$p_y(y) = \begin{cases} \binom{3}{y} \left(\frac{1}{4}\right)^y \left(\frac{3}{4}\right)^{3-y} & , y = \{0,1,2,3\} \\ 0 & , c.c \end{cases}$$

c)

Pensar que existe 3 lançamentos onde **todas** vão ter algum resultado entre cara e coroa da escolha aleatória de moedas distintas (A e B):

$$\bar{1} \bar{2} \bar{3}$$

(Não importa a ordem e com repetição) A jogada não é enumerada e os resultado **cara** e **coroa** pode ser escolhido varias vezes de moedas distintas pra 3 lançamentos. No qual a probabilidade de escolher uma cara depende de qual foi a moeda escolhida

$$p_{x,y}(x,y) = \begin{cases} \binom{3}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{4}\right)^y \left(\frac{3}{4}\right)^{3-y} \left(\frac{1}{2}\right)^{3-x} & , x+y \leq 3 ; x,y = \{0,1,2,3\} \\ 0 & , c.c \end{cases}$$

d)

$$\begin{aligned} P(X=Y) &= \sum_{y=0}^3 P(X=y, Y=y) \\ &= \sum_{y=0}^3 P(X=y) \cdot P(Y=y) \end{aligned}$$

$$\begin{aligned} P(X>Y) &= \sum_{y=0}^3 P(X>y, Y=y) \\ &= \sum_{y=0}^3 P(X>y) \cdot P(Y=y) \end{aligned}$$

$$\begin{aligned} P(X+Y \leq 4) &= \sum_{y=0}^3 P(X \leq 4-y, Y=y) \\ &= \sum_{y=0}^3 P(X \leq 4-y) \cdot P(Y=y) \end{aligned}$$

Questão 3

$$X \sim Geo(p)$$

$$Y = \min(X, M) = \begin{cases} X & , X < M \\ M & , X \geq M \end{cases}$$

$$\begin{aligned}
P(Y = y) &= P(\min(X, M) = y, \overbrace{(x < M) \cup (x \geq M)}^{\Omega}) \\
&= P(\overbrace{\min(X, M) = y, X < M}^{Impõe X=y}) + P(\overbrace{\min(X, M) = y, X \geq M}^{M=y}) \\
&= P(\overbrace{X = y, X < M}^{y < M}) + P(\overbrace{X \geq M}^{y=M}) \\
&= \begin{cases} P(X = y) & , y = \{1, 2, \dots, M - 1\} \\ P(X \geq M) & , y = M \\ 0 & , c.c \end{cases}
\end{aligned}$$

Queremos saber sobre y, assim procuramos seus limites.

Questão 4

$$\begin{aligned}
\sum_{i=1}^6 x_i &= 10 \\
\vec{x} &= (x_1, x_2, x_3, x_4, x_5, x_6)
\end{aligned}$$

a)

Pensar que existe 10 lançamentos onde **todas** vão ter algum resultado i:

$$\overline{1} \overline{2} \overline{3} \cdots \overline{9} \overline{10}$$

(Não importa a ordem e com repetição) Pois a jogada não é enumerada e uma resultado pode ser escolhido varias vezes pra 10 lançamentos. No qual a probabilidade de escolher um resultado é de 1/10

$$p_{\vec{x}} = \begin{cases} \binom{10}{x_1, x_2, x_3, x_4, x_5, x_6} \prod_{i=1}^6 \left(\frac{1}{6}\right)^{x_i} & , \sum_{i=1}^6 x_i = 10; 0 \leq x_i \leq 10 \\ 0 & , c.c \end{cases}$$

b)

$$p_{x_i} = \sum_{\substack{0 \leq x_j \leq 10 \\ x_i + \sum_{j \neq i} x_j = 10 \\ i = (1, 2, 3, 4, 5, 6)}} P(X_i = x_i) = \binom{10}{x_i} \left(\frac{1}{6}\right)^{x_i} \left(\frac{5}{6}\right)^{10-x_i}$$

$$X_i \sim B(10, 1/6) \quad X_i = (1, 2, 3, 4, 5, 6)$$

c)

Não, pois o resultado de quantas vezes saiu o valor i, depende de quantas vezes saiu os outros valores diferentes de i. O que diz respeito a probabilidade de i (p_i) ser o complementar dos valores diferentes de i, ($\sum_{j \neq i} p_j = 1 - p_i$).

Questão 5

$$\sum_{i=1}^r x_i = 2r$$

$$\vec{X} = (X_1, X_2, X_3, \dots, X_{r-1}, X_r)$$

a)

$X_i = \{\text{caixa } i \text{ com } x_i \text{ bolas}\}$

Pensar que existe $2r$ bolas onde **todas** vão ter alguma caixa:

$$\overline{1} \overline{2} \overline{3} \cdots \overline{2r-1} \overline{2r}$$

(Não importa a ordem e com repetição) pois a bola não é enumerada e uma caixa podem ser escolhidas varias vezes pra $2r$ bolas. No qual a probabilidade de escolher uma caixa é de $1/r$

$$p_{\vec{X}}(\vec{x}) = \begin{cases} \binom{2r}{x_1, x_2, \dots, x_r} \prod_{i=1}^r \left(\frac{1}{r}\right)^{x_i} & , \sum_{i=1}^r x_i = 2r; 0 \leq x_i \leq 2r \\ 0 & , \text{c.c} \end{cases}$$

b)

$$P_{\vec{X}}(2) = \binom{2r}{2, 2, \dots, 2} \left(\frac{1}{r}\right)^{2r}$$

Questão 6

a)

$$\begin{aligned}
P(X \geq Y) &= \sum_y P(X \geq Y, Y = y) = \sum_y P(X \geq y, Y = y) \\
&= \sum_{y=0}^N P(X \geq y)P(y = y) \\
&= \sum_{y=0}^N [1 - P(X < y)]P(y = y) \\
&= \sum_{y=0}^N [1 - P(X \leq y - 1)]P(y = y) \\
&= \sum_{y=0}^N \left[1 - \frac{y - 1 - 0 + 1}{N + 1} \right] \frac{1}{N + 1} \\
&= \sum_{y=0}^N \left[1 - \frac{y}{N + 1} \right] \frac{1}{N + 1} \\
&= \frac{1}{N + 1} \left[\sum_{y=0}^N 1 - \frac{1}{N + 1} \sum_{y=0}^N y \right] \\
&= \frac{1}{N + 1} \left[(N + 1) - \frac{(N(N + 1))}{2(N + 1)} \right] \\
&= \left[1 - \frac{N}{2(N + 1)} \right]
\end{aligned}$$

b)

$$\begin{aligned}
P(X = Y) &= \sum_y P(X = Y, Y = y) = \sum_y P(X = y, Y = y) \\
&\text{ind.} = \sum_{y=0}^N P(X = y)P(y = y) \\
&= \sum_{y=0}^N \frac{1}{N + 1} \frac{1}{N + 1} \\
&= \frac{N + 1}{(N + 1)^2} \\
&= \frac{1}{(N + 1)}
\end{aligned}$$

c)

$$Z = \min(X, Y), \quad X, Y = \{0, 1, 2, \dots, N\}$$

$Z \in \{0, 1, \dots, N\} :$

$$\begin{aligned}
P(Z = z) &= P(\min(X, Y) = z) \\
&= P(\min(X, Y) = z, X < Y) + P(\min(X, Y) = z, X \geq Y) \\
&= P(X = z, X < Y) + P(Y = z, X \geq Y) \\
&= P(X = z, z < Y) + P(Y = z, X \geq z) \\
ind. &= P(X = z)P(Y > z) + P(Y = z)P(X \geq z) \\
&= \frac{1}{(N+1)}[1 - P(Y \leq z)] + \frac{1}{(N+1)}[1 - P(X \leq z-1)] \\
&= \frac{1}{(N+1)} \left[1 - \frac{z-0+1}{N+1} \right] + \frac{1}{(N+1)} \left[1 - \frac{z-1-0+1}{N+1} \right] \\
&= \frac{1}{(N+1)} \left[1 - \frac{z+1}{N+1} \right] + \frac{1}{(N+1)} \left[1 - \frac{z}{N+1} \right] \\
&= \frac{1}{(N+1)} \left[\frac{N+1-(z+1)}{N+1} \right] + \frac{1}{(N+1)} \left[\frac{N+1-z}{N+1} \right] \\
&= \frac{(N+1)-(z+1)}{(N+1)^2} + \frac{(N+1)-z}{(N+1)^2} \\
&= \frac{2(N+1)-2z-1}{(N+1)^2} \\
&= \frac{2(N-z)+1}{(N+1)^2}
\end{aligned}$$

$$p_Z(z) = \begin{cases} \frac{2(N-z)+1}{(N+1)^2} & , z = \{0, 1, \dots, N\} \\ 0 & , c.c \end{cases}$$

d)

$$W = \max(X, Y), \quad X, Y = \{0, 1, 2, \dots, N\}$$

$W \in \{0, 1, \dots, N\} :$

$$\begin{aligned}
P(W = w) &= P(\max(X, Y) = w) \\
&= P(\max(X, Y) = w, X \leq Y) + P(\max(X, Y) = w, X > Y) \\
&= P(Y = w, X \leq Y) + P(X = w, X > Y) \\
&= P(Y = w, X \leq w) + P(X = w, w > Y) \\
ind. &= P(Y = w)P(X \leq w) + P(X = w)P(Y \leq w-1) \\
&= \frac{1}{(N+1)}[P(X \leq w)] + \frac{1}{(N+1)}[P(Y \leq w-1)] \\
&= \frac{1}{(N+1)} \left[\frac{w-0+1}{N+1} \right] + \frac{1}{(N+1)} \left[\frac{w-1-0+1}{N+1} \right] \\
&= \frac{1}{(N+1)} \left[\frac{w+1}{N+1} \right] + \frac{1}{(N+1)} \left[\frac{w}{N+1} \right] \\
&= \frac{2w+1}{(N+1)^2}
\end{aligned}$$

$$p_W(w) = \begin{cases} \frac{2w+1}{(N+1)^2} & , w = \{0, 1, \dots, N\} \\ 0 & , c.c \end{cases}$$

e)

$$U = |Y - X|, \quad X, Y = \{0, 1, 2, \dots, N\}$$

$U = 0 :$

$$\begin{aligned} P(U = 0) &= P(|Y - X| = 0) \\ &= P(Y - X = 0) \\ &= P(Y = X) \\ &= \frac{1}{N+1} \end{aligned}$$

$U \in \{1, \dots, N\} :$

$$\begin{aligned} P(u = u) &= P(|Y - X| = u) \\ &= P(|Y - X| = u, Y - X < 0) + P(|Y - X| = u, Y - X > 0) \\ &\quad + P(|Y - X| = u, Y - X = 0) \\ &= P(Y - X = -u, Y - X < 0) + P(Y - X = u, Y - X > 0) + 0 \\ &= P(X = Y + u, X > Y) + P(Y = X + u, Y > X) \\ &= P(X = Y + u) + P(Y = X + u) \\ i.i.d &= 2P(X = Y + u) \\ &= 2 \sum_{y=0}^N P(X = y + u, Y = y) \\ &= 2 \sum_{y=0}^N P(X = \overbrace{y+u}^{0 \leq y+u \leq N}) P(Y = y) \\ (y \leq N - u) &= 2 \sum_{y=0}^{N-u} \frac{1}{N+1} \frac{1}{N+1} \\ &= \frac{2(N-u+1)}{(N+1)^2} \end{aligned}$$

$$p_u(u) = \begin{cases} \frac{1}{(N+1)} & , \quad u = 0 \\ \frac{2(N-u+1)}{(N+1)^2} & , \quad u = \{1, \dots, N\} \\ 0 & , \quad c.c \end{cases}$$

Questão 7

$X : p = p_1 \quad Y : p = p_2$

$$p_x(k) = p_y(k) = \begin{cases} p(1-p)^{k-1} & , \quad k = \{1, 2, \dots\} \\ 0 & , \quad c.c \end{cases}$$

a)

$$\begin{aligned}
P(X \geq Y) &= P(Y \leq X) \\
&= P(Y \leq X, X = \mathbb{R}) \\
&= \sum_{x=1}^{\infty} P(Y \leq X, X = x) \\
&= \sum_{x=1}^{\infty} P(Y \leq x, X = x) \\
ind. &= \sum_{x=1}^{\infty} P(Y \leq x)P(X = x) \\
&= \sum_{x=1}^{\infty} [1 - (1 - p_2)^x][p_1 \cdot (1 - p_1)^{x-1}] \\
&= \sum_{x=1}^{\infty} [p_1 \cdot (1 - p_1)^{x-1} - p_1 \cdot (1 - p_1)^{x-1} \cdot (1 - p_2)^x] \\
&= \sum_{x=1}^{\infty} \left[\frac{p_1 \cdot (1 - p_1)^x}{(1 - p_1)} - \frac{p_1 \cdot [(1 - p_1) \cdot (1 - p_2)]^x}{(1 - p_1)} \right] \\
&= \frac{p_1}{(1 - p_1)} \sum_{x=1}^{\infty} [(1 - p_1)^x - [(1 - p_1) \cdot (1 - p_2)]^x] \\
&= \frac{p_1}{(1 - p_1)} \left[\frac{1}{1 - (1 - p_1)} - \frac{1}{1 - (1 - p_1) \cdot (1 - p_2)} \right] \\
&= \dots
\end{aligned}$$

b)

$$\begin{aligned}
P(X = Y) &= P(X = Y, Y = \mathbb{R}) \\
&= \sum_{y=1}^{\infty} P(X = Y, Y = y) \\
&= \sum_{y=1}^{\infty} P(X = y, Y = y) \\
ind. &= \sum_{y=1}^{\infty} P(X = y)P(Y = y) \\
&= \sum_{y=1}^{\infty} [p_1(1 - p_1)^{y-1}][p_2 \cdot (1 - p_2)^{y-1}] \\
&= p_1 \cdot p_2 \sum_{y=1}^{\infty} (1 - p_1)^{y-1} (1 - p_2)^{y-1} \\
&= p_1 \cdot p_2 \sum_{y=1}^{\infty} [(1 - p_1)(1 - p_2)]^{y-1} \\
&= \frac{p_1 \cdot p_2}{1 - (1 - p_1)(1 - p_2)} \\
&= \dots
\end{aligned}$$

c)

$$Z = \min(X, Y), \quad X, Y = \{1, 2, \dots\}$$

$$Z \in \{1, 2, \dots\} :$$

$$\begin{aligned} P(Z = z) &= P(\min(X, Y) = z) \\ &= P(\min(X, Y) = z, X < Y) + P(\min(X, Y) = z, X \geq Y) \\ &= P(X = z, X < Y) + P(Y = z, X \geq Y) \\ &= P(X = z, z < Y) + P(Y = z, X \geq z) \\ ind. &= P(X = z)P(Y > z) + P(Y = z)P(X \geq z) \\ &= P(X = z)[1 - P(Y \leq z)] + P(Y = z)[1 - P(X \leq z - 1)] \\ &= \dots (Substitui ai) \end{aligned}$$

c)

$$W = X + Y$$

$$W \in \{2, 3, \dots\} :$$

$$\begin{aligned} P(W = w) &= P(X + Y = w) \\ &= P(X = w - Y, Y = \mathbb{R}) \\ &= \sum_{y=1}^{\infty} P(X = w - y, Y = y) \\ ind. &= \sum_{y=1}^{\infty} P(X = \overbrace{w-y}^{1 \leq w-y \leq \infty}) P(Y = y) \\ (-\infty \leq y \leq w-1) \rightarrow (1 \leq y \leq w-1) &= \sum_{y=1}^{w-1} P(X = w - y) P(Y = y) \\ &= \sum_{y=1}^{w-1} p_1(1 - p_1)^{w-y-1} p_2(1 - p_2)^{y-1} \\ &= p_1 \cdot p_2 \cdot (1 - p_1)^w \sum_{y=1}^{w-1} (1 - p_1)^{-y-1} (1 - p_2)^{y-1} \\ &= p_1 \cdot p_2 \cdot (1 - p_1)^w \sum_{y=1}^{w-1} \frac{(1 - p_2)^{y-1}}{(1 - p_1)^{y+1}} \\ &= \frac{p_1 \cdot p_2 \cdot (1 - p_1)^w}{(1 - p_2)(1 - p_1)} \sum_{y=1}^{w-1} \left[\frac{(1 - p_2)}{(1 - p_1)} \right]^y \\ &= \frac{p_1 \cdot p_2 \cdot (1 - p_1)^w}{(1 - p_2)(1 - p_1)} \frac{1 - \left[\frac{(1 - p_2)}{(1 - p_1)} \right]^{w-1+1}}{1 - \left[\frac{(1 - p_2)}{(1 - p_1)} \right]} \\ &= \frac{p_1 \cdot p_2 \cdot (1 - p_1)^w}{(1 - p_2)(1 - p_1)} \frac{1 - \left[\frac{(1 - p_2)}{(1 - p_1)} \right]^w}{1 - \left[\frac{(1 - p_2)}{(1 - p_1)} \right]} \\ &= \dots (Alguma coisa ai) \end{aligned}$$

Questão 8

$$X, Y \in (1, 2, \dots)$$

$$Z = Y - X \rightarrow P(Y - X = z) = P(Y = z + X)$$

$$Z \in \mathbb{R}$$

$$W = \min(X, Y) \rightarrow P(\min(X, Y) = w) = P(X = w, X < Y) + P(Y = w, X \geq Y)$$

$$W \in (1, 2, \dots)$$

a) (?)

$$Z \geq 0$$

$$\begin{aligned} P(W = w, Z = z) &= P(X = w, X < Y, Y = z + X) + P(Y = w, X \geq Y, Y = z + X) \\ &= P(X = w, w < Y, Y = z + w) + P(Y = w, X \geq w, w = z + X) \\ &= P(X = w, Y > w, Y = z + w) + P(Y = w, X \geq w, X = w - z) \\ &= P(X = w, Y = z + w) + P(Y = w, X = w) \\ &= P(X = w)P(Y = z + w) + P(Y = w)P(X = w) \\ &= P(X = w)[P(Y = z + w) + P(Y = w)] \end{aligned}$$

Questão 9

$$Z = X + Y$$

$$P(Z = z) = P(X + Y = z) = P(X = z - Y) = P(X = z - Y, Y = \mathbb{R})$$

a) X,Y (Poisson)

$$Z \in (0, 1, \dots)$$

$$\begin{aligned} P(Z = z) &= \sum_{y=0}^{\infty} P(X = z - y, Y = y) \\ &= \sum_{y=0}^{\infty} P(X = \overbrace{z - y}^{0 \leq z - y \leq \infty}) P(Y = y) \\ (0 \leq y \leq z) &= \sum_{y=0}^z \frac{e^{-\lambda_1} \lambda_1^{z-y}}{(z-y)!} \frac{e^{-\lambda_2} \lambda_2^y}{y!} \\ &= \sum_{y=0}^z \frac{e^{-(\lambda_1 + \lambda_2)} \lambda_1^{z-y} \lambda_2^y}{(z-y)! y!} \frac{z!}{z!} \\ &= \sum_{y=0}^z \binom{z}{y} \frac{e^{-(\lambda_1 + \lambda_2)} \lambda_1^{z-y} \lambda_2^y}{z!} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)} (\lambda_1 + \lambda_2)^z}{z!} \\ p_Z(z) &= \begin{cases} \frac{e^{-(\lambda_1 + \lambda_2)} (\lambda_1 + \lambda_2)^z}{z!} & , x = 0, 1, \dots \\ 0 & , c.c \end{cases} \end{aligned}$$

$$Z \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

b) X,Y (uniforme)

$$Z = (2, 3, 4, \dots, 2N - 1, 2N)$$

$$\begin{aligned} P(Z = z) &= \sum_{y=0}^{\infty} P(X = z - y, Y = y) \\ &= \sum_{y=0}^{\infty} P(X = \overbrace{z - y}^{0 \leq z - y \leq \infty}) P(Y = y) \\ (0 \leq y \leq z) &= \sum_{y=0}^z \frac{e^{-\lambda_1} \lambda_1^{z-y}}{(z-y)!} \frac{e^{-\lambda_2} \lambda_2^y}{y!} \end{aligned}$$