Lista 9

Allan

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Questão 1

$$f_X(k) = f_Y(k) = \begin{cases} 1, & 0 \leq k \leq 1 \\ 0, & c.c \end{cases}$$

a)

Temos que $A = \{(x,y) \in (0,1) \times (0,1) : \ |x-y| \leq \frac{1}{2}\}$

$$\begin{aligned} P\left(|X-Y| \leq \frac{1}{2}\right) =& P\left(-\frac{1}{2} \leq X - Y \leq \frac{1}{2}\right) \Leftrightarrow \\ & \Leftrightarrow P\left(Y - \frac{1}{2} \leq X \leq Y + \frac{1}{2}, 0 < Y < 1\right) \end{aligned}$$

Assim temos que:

$0 < y < 1 \ e \ \underbrace{y - 1/2}_{\leq 0} < x < \underbrace{y + 1/2}_{\leq 1}$

Assim observamos que para $1/2 < y < 1$, x varia em $y - 1/2 < x < 1$.

Para $0 < y < 1/2$, temos $0 < x < y + 1/2$.

Portanto:

$$\begin{aligned} P\left(|X-Y| \leq \frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \int_0^{y+\frac{1}{2}} 1dxdy + \int_{\frac{1}{2}}^1 \int_{y-\frac{1}{2}}^1 1dxdy \\ &= \int_0^{\frac{1}{2}} y + \frac{1}{2}dy + \int_{\frac{1}{2}}^1 \frac{3}{2} - ydy \\ &= [\frac{y^2}{2} + \frac{y}{2}]\Big|_0^{\frac{1}{2}} + [\frac{3y}{2} - \frac{y^2}{2}]\Big|_{\frac{1}{2}}^1 \\ &= \frac{1}{8} + \frac{1}{4} + \frac{3}{2} - \frac{1}{2} - \frac{3}{4} + \frac{1}{8} \\ &= \frac{1}{8} + \frac{2}{8} + \frac{12}{8} - \frac{4}{8} - \frac{6}{8} + \frac{1}{8} \\ &= \frac{6}{8} = \frac{3}{4} \end{aligned}$$

b)

$$
\begin{aligned}
P\left(|\frac{X}{Y}-1| \leq \frac{1}{2}\right) &\Leftrightarrow \\
P\left(-\frac{1}{2} \leq \frac{X}{Y}-1 \leq \frac{1}{2}\right) &\Leftrightarrow \\
P\left(1-\frac{1}{2} \leq \frac{X}{Y} \leq 1+\frac{1}{2}\right) &\Leftrightarrow \\
P\left(Y(1-\frac{1}{2}) \leq X \leq Y(1+\frac{1}{2}), 0<Y<1\right)&
\end{aligned}
$$

Assim obtemos que: $y(1-\frac{1}{2}) \leq x \leq y(1+\frac{1}{2}) \rightarrow \underbrace{\frac{y}{2}}_{\leq 0} \leq x \leq \underbrace{\frac{3y}{2}}_{\leq 1}$

Para $0<y<\frac{2}{3}$ temos $\frac{y}{2}<x<\frac{3y}{2}$

E para $\frac{2}{3}<y<1$ temos $\frac{y}{2}<x<1$

Portanto:

$$
\begin{aligned}
P\left(|\frac{X}{Y}-1| \leq \frac{1}{2}\right) &= \int_0^{\frac{2}{3}} \int_{\frac{y}{2}}^{\frac{3y}{2}} 1dxdy + \int_{\frac{2}{3}}^{1} \int_{\frac{y}{2}}^{1} 1dxdy \\
&= \int_0^{\frac{2}{3}} ydy + \int_{\frac{2}{3}}^{1} 1-\frac{y}{2}dy \\
&=[\frac{y^2}{2}]\Big|_0^{\frac{2}{3}} + [y-\frac{y^2}{4}]\Big|_{\frac{2}{3}}^{1} \\
&=\frac{4}{18}+1-\frac{1}{4}-[\frac{2}{3}-\frac{4}{36}] \\
&=\frac{8}{36}+\frac{36}{36}-\frac{9}{36}-\frac{24}{36}+\frac{4}{36} \\
&=\frac{15}{36}=\frac{5}{12}
\end{aligned}
$$

Questão 2

$$
f_X(k)=f_Y(k)=\begin{cases} e^{-\lambda k}\lambda & ,k>0 \\ 0 & ,c.c \end{cases}
$$

$$
\begin{aligned}
P(X > 2Y) =& P(X > 2Y, Y > 0) = P(X > 2Y)P(Y > 0) = \\
=& \int_0^\infty \int_{2y}^\infty \lambda^2 e^{-\lambda x}.e^{-\lambda y} dxdy \\
=& \int_0^\infty \lambda^2.e^{-\lambda y}[\frac{-1}{\lambda}e^{-\lambda x}]\Big|_{2y}^\infty dy \\
=& \int_0^\infty \lambda^2.e^{-\lambda y}\frac{1}{\lambda}e^{-\lambda 2y} dy \\
=& \int_0^\infty \lambda.e^{-\lambda 3y} dy \\
=& \frac{-1}{3\lambda}\lambda.[e^{-\lambda 3y}]\Big|_0^\infty \\
=& \frac{1}{3}
\end{aligned}
$$

Questão 3

Seja

$$f(x, y) = ce^{(x^2 - xy + 4y^2)/2}$$

Temos que:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty ce^{-(x^2-xy+4y^2)/2} dxdy = 1 = ...$$

Questão 4

Desenha ai brow

Questão 5

$$Z = X + Y$$

$$f_X(x).f_Y(y) = f_{X,Y}(x, y)$$

Temos que:

$$P(Z \leq z) = P(X + Y \leq z) = P(X \leq z - Y, Y \in \Omega) = P(X \leq z - Y)P(Y \in \Omega)$$

Logo (Convolução):

$$f_Z(z) = \int_{-\infty}^\infty f_X(z - y).f_Y(y) dy$$

a)

Tem que ser $x > 0$ e $y > 0$

Para $z > 0$

$$
\begin{aligned}
f_Z(z) = \int_0^z \overbrace{f_X(z-y)f_Y(y)}^{>0 \to 0<y<z} dy = \int_0^z \lambda_1 \lambda_2 . e^{-\lambda_1(z-y)} e^{-\lambda_2 y} dy = \\
= & \lambda_1 \lambda_2 e^{-\lambda_1 z} \int_0^z e^{\lambda_1 y} e^{-\lambda_2 y} dy = \\
= & \lambda_1 \lambda_2 e^{-\lambda_1 z} \int_0^z e^{y(\lambda_1 - \lambda_2)} dy = \\
= & \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_1 z} [e^{y(\lambda_1 - \lambda_2)}] \Big|_0^z = \\
= & \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_1 z} [e^{z(\lambda_1 - \lambda_2)} - 1] = \\
= & \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} [e^{-\lambda_2 z} - e^{-\lambda_1 z}] =
\end{aligned}
$$

b)

Tem que ser $x > 0$ e $y > 0$

Para $z > 0$

$$
\begin{aligned}
f_Z(z) = \int_0^z \overbrace{f_X(z-y)f_Y(y)}^{>0 \to 0<y<z} dy = \\
= & \int_0^z \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} e^{-\lambda(z-y)} (z-y)^{\alpha_1 - 1} \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} e^{-\lambda y} y^{\alpha_2 - 1} dy = \\
= & \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} \int_0^z (z-y)^{\alpha_1 - 1} y^{\alpha_2 - 1} dy = \\
= & \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} \int_0^z \sum_{k=0}^{\alpha_1 - 1} (-1)^k z^{\alpha_1 - 1 - k} y^k y^{\alpha_2 - 1} dy \\
= & \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} \sum_{k=0}^{\alpha_1 - 1} (-1)^k z^{\alpha_1 - 1 - k} \int_0^z y^{k + \alpha_2 - 1} dy \\
= & \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} \sum_{k=0}^{\alpha_1 - 1} (-1)^k z^{\alpha_1 - 1 - k} [\frac{y^{k + \alpha_2}}{k + \alpha_2}] \Big|_0^z = \\
= & \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} \sum_{k=0}^{\alpha_1 - 1} (-1)^k z^{\alpha_1 - 1 - k} \frac{z^{k + \alpha_2}}{k + \alpha_2} = \\
= & \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} z^{\alpha_1 + \alpha_2 - 1} \sum_{k=0}^{\alpha_1 - 1} \frac{(-1)^k}{k + \alpha_2} = ...
\end{aligned}
$$

* Fazer u=x/z

c)

Tem que ser $x \in \mathbb{R}$ e $y \in \mathbb{R}$

Para $z \in \mathbb{R}$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy =$$

$$= \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-y-\mu_1)^2}{2\sigma_1^2}}}{\sqrt{\pi}\sigma_1} \frac{e^{\frac{-(y-\mu_2)^2}{2\sigma_2^2}}}{\sqrt{\pi}\sigma_2} dy = ...$$

Sabendo que $EX = \mu_1$ e $VARX = EX^2 - (EX)^2 = \sigma_2^1$

então assumindo independência:

$$E(X+Y) = E(X) + E(Y)$$

e

$$Var(X+Y) = Var(X) + Var(Y)$$

assim $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Questão 6

...

Questão 7

$$X = Y \sim N(0, \sigma^2)$$

Temos que $P(R > 0) = 1$

onde $R = \sqrt{X^2 + Y^2}$

$R = \sqrt{X^2 + Y^2},\ X \leq \sqrt{z^2 - Y^2}, Y \in \Omega$

Temos que $A_z = \{(x, y) \in \mathbb{R}^2 :\ \sqrt{x^2 + y^2} \leq z\}$

Para $z > 0$

$$P(R \leq z) = \int_{-\infty}^{z} f_R(r) dr = \int_{-\infty}^{\infty} \int_{-\infty}^{\sqrt{z^2 - y^2}} f_{X,Y}(x, y) dx dy = \int \int_{A_z} f_{X,Y}(x, y) dx dy$$

ou por coordenadas polares (mudança de variável) faríamos:

$$P(R \leq z) = \int_0^z \int_0^{2\pi} f_{X,Y}(sen^2(\theta), cos^2(\theta)) r d\theta dr$$

Questão 8

$P(X > 0) = 1$ $P(Y > 0) = 1$

$$Z = \frac{Y}{X}$$

$P(Z > 0) = 1$

Podemos achar:

$$F_Z(z) = P(Y \leq Xz) = F_{X,Y}(\Omega, xz)$$
$$f_Z(z) = \frac{\partial}{\partial z} F_{X,Y}(\Omega, xz)$$

que no caso:

$$F_Z(z) = P(\frac{Y}{X} \leq z) = P(Y \leq Xz) = \int_0^\infty \int_0^{xz} f_{X,Y}(x,y)dydx = F_{X,Y}(\Omega, xz)$$

ou por mudança de variável:

$$\begin{cases} u = x \to x = u & , x > 0, u > 0, z > 0 \\ z = \frac{x}{y} \to y = \frac{u}{z} & , y > 0, u > 0, z > 0 \\ & , u > 0 \; e \; z > 0 \end{cases}$$

e

$J = \begin{bmatrix} 1 & 0 \\ \frac{1}{y} & \frac{-x}{y^2} \end{bmatrix} = \frac{-x}{y^2}$

assim:

$$f_{U,Z}(u,z) = \frac{y^2}{x} f_{X,Y}(u, \frac{x}{y})$$

e $f_Z(z) = \int_0^\infty f_{U,Z}(u,z)du$

Questão 9

Seja $X \sim Gamma(\alpha_1, \lambda)$ e $Y \sim Gamma(\alpha_2, \lambda)$

onde $P(X > 0) = 1 = P(Y > 0)$

Temos que $F_{X,Y}(x,y) = F_X(x)F_Y(y)$

e $Z = \frac{X}{X+Y}$

Então

$$\begin{aligned} F_Z(Z \leq z) = P(\frac{X}{X+Y} \leq z) = P(X \leq (X+Y)z) = P(X \leq Xz + Yz) = \\ = P(X - Xz \leq Yz) = P(X(1-z) \leq Yz) = \\ = P(X \leq \frac{Yz}{(1-z)}, Y \in \Omega) \end{aligned}$$

Assim

$$F_Z(Z \leq z) = \int_0^\infty \int_0^{\frac{yz}{(1-z)}} F_{X,Y}(x,y)dxdy$$

Ou por mudança de variáveis:

$Z = \frac{X}{X+Y} = \frac{1}{1+\frac{Y}{X}} = \frac{1}{1+W}$ em que $W = \frac{X}{Y}$

Assim

$$F_Z(z) = P(\frac{1}{1+W} \leq z) = P(\frac{1}{z} \leq W+1) = P(W \geq \frac{1}{z} - 1) = 1 - P(W \leq \frac{1}{z} - 1)$$

e sendo assim para $z > 0$

$$f_z(z) = f_W(\frac{1}{z} - 1).\frac{1}{z^2}$$

onde $\frac{1}{z} - 1 > 0 \to 0 < z < 1$

no qual $F_W(w) = P(W \leq w) = P(\frac{X}{Y} \leq w) = P(X \leq wY, Y \in \Omega) = F_{X,Y}(wy, \Omega)$

Fazendo uma mudança de variável de $u = \frac{x}{y}$ com $du = \frac{1}{y}dx$

assim

$$f_W(w) = \int_0^\infty y.f_{X,Y}(yu,y)dy$$

Questão 10

a)

$$Z = \rho U + \sqrt{1-\rho^2}V$$

Para $U = V \sim N(0,1)$

Seja:

$\rho U \sim N(0, \rho^2)$ $\sqrt{1-\rho^2}V \sim N(0, 1-\rho^2)$

Assim

$EZ = E(\rho U) + E(\sqrt{1-\rho^2}V) = 0 + 0 = 0$ $VarZ = Var(\rho U) + Var(\sqrt{1-\rho^2}V) = \rho^2 + 1 - \rho^2 = 1$

$Z \sim N(0,1)$

b)

para $\begin{cases} x = u & \to u = x \\ z = \rho u + \sqrt{1-\rho^2}v & \to v = \frac{-\rho}{\sqrt{1-\rho^2}}x + \frac{1}{\sqrt{1-\rho^2}}z \end{cases}$

$J = \sqrt{1-\rho^2}$

$$f_{U,Z}(u,z) = f_{X,Z}(x,z) = \frac{1}{|J|}f_{U,V}(x, \frac{-\rho}{\sqrt{1-\rho^2}}x + \frac{1}{\sqrt{1-\rho^2}})$$

c)

para $\begin{cases} x = \mu_1 + \sigma_1 u \rightarrow u = \frac{x-\mu_1}{\sigma_1} & , x \in \mathbb{R}, u \in \mathbb{R} \\ y = \mu_2 + \sigma_2 z \rightarrow z = \frac{y-\mu_2}{\sigma_2} & , y \in \mathbb{R}, z \in \mathbb{R} \end{cases}$

$J = det \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \sigma_1.\sigma_2$

$$f_{X,Y}(x,y) = \frac{1}{\sigma_1 \sigma_2} f_{U,Z}(\frac{x-\mu_1}{\sigma_1}, \frac{y-\mu_2}{\sigma_2})$$

Questão 11

$$f_R(r) = \begin{cases} \sigma^{-2} r e^{-r^2/2\sigma^2} & , r \geq 0 \\ 0 & , c.c \end{cases}$$

e

$$f_\Theta(\Theta) = \begin{cases} \frac{1}{2\pi} & , -\pi < \Theta < \pi \\ 0 & , x.x \end{cases}$$

Para

$$\begin{cases} x = rcos\theta \rightarrow r = \sqrt{x^2 + y^2} & , r > 0, x \in \mathbb{R} \\ y = rsen\theta \rightarrow \theta = arcsen(\frac{y}{x}) & , \frac{-\pi}{2} < \theta < \frac{\pi}{2}, y \in \mathbb{R} \end{cases}$$

$$J = r$$

Assim

$$f_{X,Y}(x,y) = \frac{1}{r} f_{R,\Theta}(\sqrt{x^2 + y^2}, arcsen(\frac{y}{x})) = ...$$