Lista 6

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Questão 1

$$p(x) = \begin{cases} \frac{1}{N} & , x \in \{1, ..., N\} \\ 0 & , c.c \end{cases}$$

$$EX = \sum_x xp(x) = \sum_{x=1}^{N} \frac{x}{N} = \frac{N(N+1)}{2} \frac{1}{N} = \frac{N+1}{2}$$

$$EX^2 = \sum_x x^2 p(x) = \sum_{x=1}^{N} \frac{x^2}{N} = \frac{N(N+1)(2N+1)}{6} \frac{1}{N} = \frac{(N+1)(2N+1)}{6}$$

Então

$$\begin{aligned} VarX = EX^2 - (EX)^2 = \frac{(N+1)(2N+1)}{6} - \left(\frac{N+1}{2}\right)^2 = \\ = \frac{(N+1)(2N+1)}{6} - \left(\frac{N+1}{2}\right)^2 = \frac{2(N+1)(2N+1) - 3(N+1)(N+1)}{12} = \\ = \frac{(N+1)(2(2N+1) - 3(N+1))}{12} = \frac{(N+1)(4N+2-3N-3)}{12} = \frac{(N+1)(N-1)}{12} = \\ = \frac{N^2-1}{12} \end{aligned}$$

Questão 2

a)

$$\lim_{X \to \infty} \frac{(X+1)^{-(r+2)}}{X^{-(r+2)}} = ...$$

$$\sum_{x=1}^{\infty} \frac{1}{c} x^{-(r+2)} = 1 \Rightarrow c = \sum_{x=1}^{\infty} x^{-(r+2)}$$

b)

$$EX^r = \sum_{x=1}^{\infty} x^r \frac{x^{-(r+2)}}{\sum_{x=1}^{\infty} x^{-(r+2)}} = \sum_{x=1}^{\infty} \frac{x^{-2}}{\sum_{x=1}^{\infty} x^{-(r+2)}} = \sum_{x=1}^{\infty} \frac{1}{\sum_{x=1}^{\infty} x^{-(r+2)+2}} = \sum_{x=1}^{\infty} \frac{1}{\sum_{x=1}^{\infty} x^{-r}} = ...$$

Questão 3

$$
\begin{aligned}
EX^2 = \sum_x x.xp_x(x) = \sum_{x=r}^{\infty} x.x\binom{x-1}{r-1}p^r(1-p)^{x-r} = \sum_{x=r}^{\infty} \frac{x.x!}{(x-r)!(r-1)!}p^r(1-p)^{x-r} = \\
= \sum_{x=r}^{\infty} \frac{r}{r}\frac{x.x!}{(x-r)!(r-1)!}p^r(1-p)^{x-r} = \sum_{x=r}^{\infty} r\frac{x.x!}{(x-r)!(r)!}p^{r+1-1}(1-p)^{x-r} = \\
= \frac{r}{p}\sum_{x=r}^{\infty} x\binom{x}{r}p^{r+1}(1-p)^{x-r} = \frac{r}{p}\frac{r}{p}
\end{aligned}
$$

...

Questão 4

$$Var(X^2Y) = E(X^2Y)^2 - (E(X^2Y))^2 = EX^4EY^2 - (EX^2EY)^2 = 2.1 - 0 = 2$$

Questão 5

a)

$$E\bar{X} = E\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{n}\sum_{i=1}^n E\left(X_i\right) = \frac{1}{n}\sum_{i=1}^n \mu = \mu$$

b)

$$Var\bar{X} = Var\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^n Var\left(X_i\right) = \frac{1}{n^2}\sum_{i=1}^n \sigma^2 = \frac{\sigma^2}{n}$$

c)

$$
\begin{aligned}
E\left[\sum_{i=1}^n (X_i - \bar{X})^2\right] &= \sum_{i=1}^n E\left[(X_i - \bar{X})^2\right] = \sum_{i=1}^n E\left[X_i^2 - 2X_i\bar{X} + \bar{X}^2\right] = \\
&= \sum_{i=1}^n E[X_i^2] - 2E[X_i\bar{X}] + E[\bar{X}^2] = \\
&= \sum_{i=1}^n (\sigma^2 + \mu^2) - 2E[X_i]E[\bar{X}] + (\frac{\sigma^2}{n} + \mu^2) = \\
&= \sum_{i=1}^n (\sigma^2 + \mu^2) - 2\mu^2 + (\frac{\sigma^2}{n} + \mu^2) = \\
&= \sum_{i=1}^n \sigma^2 + \frac{\sigma^2}{n} = \sum_{i=1}^n \sigma^2(1 + \frac{1}{n}) = \\
&= \sigma^2(n+1)
\end{aligned}
$$

Questão 6

a)

b)

Questão 7

$$\begin{aligned}
\rho(X_1 - X_2, X_2 + X_3) =& \frac{Cov(X_1 - X_2, X_2 + X_3)}{\sqrt{(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}} = \frac{E[(X_1 - X_2)(X_2 + X_3)] - E[(X_1 - X_2)]E[(X_2 + X_3)]}{\sqrt{(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}} = \\
=& \frac{E[(X_1X_2 + X_1X_3 - X_2X_2 - X_2X_3)] - E[(X_1 - X_2)]E[(X_2 + X_3)]}{\sqrt{(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}} = \\
=& \frac{E[X_1X_2] + E[X_1X_3] - E[X_2X_2] - E[X_2X_3] - (E[X_1] - E[X_2])(E[X_2] + E[X_3])}{\sqrt{(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}} = \\
=& \frac{E[X_1X_2] + E[X_1X_3] - E[X_2X_2] - E[X_2X_3] - E[X_1]E[X_2] - E[X_1]E[X_3] + E[X_2]E[X_2] + E[X_2]E[}{\sqrt{(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}} \\
=& \frac{-E[X_2^2] + E[X_2]^2}{\sqrt{(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}} = \frac{-(E[X_2^2] - E[X_2]^2)}{\sqrt{(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}} = \\
=& \frac{-\sigma_2^2}{\sqrt{(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}}
\end{aligned}$$

Questão 8

$$\rho(X, Y) = \frac{a}{\sqrt{1.2}} = \frac{1}{2} \Rightarrow a = \frac{\sqrt{2}}{2} = Cov(X, Y)$$

Assim

$$\begin{aligned}
Var(X - 2Y) =& Var(X) + Var(-2Y) - 2Cov(X, -2Y) = \\
=& Var(X) + 4Var(Y) + 4Cov(X, Y) = \\
=& 1 + 8 + 4\frac{\sqrt{2}}{2} = \\
=& 9 + 2\sqrt{2}
\end{aligned}$$

Questão 9

Questão 10