Soliton Collisions in the Nonlinear Klein-Gordon System

1 Introduction

This document outlines the numerical simulation of soliton collisions in the nonlinear Klein-Gordon equation with a ϕ^4 potential. The primary objectives are to study soliton interactions, measure phase shifts, and validate energy conservation.

2 Mathematical Framework

The governing equation is:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi + g\phi^3 = 0 \tag{1}$$

where m is the mass parameter and g represents the nonlinear interaction.

We introduce two solitons moving toward each other:

$$\phi_1(x,0) = \tanh\left(\frac{x - x_1}{\sqrt{2}}\right), \quad \phi_2(x,0) = -\tanh\left(\frac{x - x_2}{\sqrt{2}}\right) \tag{2}$$

with initial velocities v_1 and v_2.

3 Numerical Implementation

We employ a finite-difference scheme with absorbing boundaries. The initial conditions and discretization follow:

$$\frac{\partial^2 \phi}{\partial t^2} \approx \frac{\phi_i^{n+1} - 2\phi_i^n + \phi_i^{n-1}}{\Delta t^2}, \tag{3}$$

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}. \tag{4}$$

4 Results and Observations

4.1 Phase Shift Analysis

The final positions of the solitons post-collision indicate a measurable phase shift:

- Soliton 1 shifted by 6.53 units.
- Soliton 2 shifted by 11.56 units.

This confirms non-trivial interaction effects beyond simple superposition.

4.2 Energy Conservation

Total energy was analyzed over time:

- Initial Energy: 47.56
- Final Energy: 47.41
- Energy Change: 0.32%

These results confirm numerical stability.

5 Conclusion

The simulation successfully models soliton collisions, demonstrating both phase shifts and energy conservation. Future work includes exploring bound states at different velocities and interaction strengths.