Problem 2

1) $x(t) = \frac{\pi t^3}{2}$; $-1 < t < 1$, $x(t+2) = x(t)$

$T = 2$

$$\therefore a_n = \frac{1}{2}\int_{-1}^{1} \frac{\pi t^3}{2} e^{-jn\pi t}\, dt$$

$$= \frac{\pi}{4}\int_{-1}^{1} t^3 e^{-jn\pi t}\, dt$$

$$= \frac{\pi}{4}\left[-t^3 \frac{e^{-jn\pi t}}{jn\pi} + 3t^2 \frac{e^{-jn\pi t}}{(n\pi)^2} + 6t \frac{e^{-jn\pi t}}{j(n\pi)^3} - 6 \frac{e^{-jn\pi t}}{(n\pi)^4} \right]_{-1}^{1}$$

$$= \frac{\pi}{4}\left[\left(\frac{-e^{-jn\pi}}{jn\pi} + \frac{3e^{-jn\pi}}{(n\pi)^2} + \frac{6e^{-jn\pi}}{j(n\pi)^3} - \frac{6e^{-jn\pi}}{(n\pi)^4} \right) - \left(\frac{e^{jn\pi}}{jn\pi} + \frac{3e^{jn\pi}}{(n\pi)^2} - \frac{6e^{jn\pi}}{j(n\pi)^3} - \frac{6e^{jn\pi}}{(n\pi)^4} \right) \right]$$

$$= \frac{\pi}{4}\left[\frac{-e^{-jn\pi}}{jn\pi} + \frac{3e^{-jn\pi}}{(n\pi)^2} + \frac{6e^{-jn\pi}}{j(n\pi)^3} - \frac{6e^{-jn\pi}}{(n\pi)^4} - \frac{e^{jn\pi}}{jn\pi} - \frac{3e^{jn\pi}}{(n\pi)^2} + \frac{6e^{jn\pi}}{j(n\pi)^3} + \frac{6e^{jn\pi}}{(n\pi)^4} \right]$$

$$= \frac{\pi}{4}\left[\frac{-1}{jn\pi}\left(e^{-jn\pi} + e^{jn\pi}\right) + \frac{3}{n^2\pi^2}\left(e^{-jn\pi} - e^{jn\pi}\right) + \frac{6}{jn^3\pi^3}\left(e^{-jn\pi} + e^{jn\pi}\right) - \frac{6}{n^4\pi^4}\left(e^{-jn\pi} - e^{jn\pi}\right) \right]$$

$$= \frac{\pi}{2}\left[\left(\frac{-1}{jn\pi} + \frac{6}{jn^3\pi^3} \right)\left(\frac{e^{-jn\pi} + e^{jn\pi}}{2} \right) + \left(\frac{3}{n^2\pi^2} - \frac{6}{n^4\pi^4} \right)\left(\frac{e^{-jn\pi} - e^{jn\pi}}{2} \right) \right]$$

$$= \frac{\pi}{2}\left[\left(\frac{-n^2\pi^2 + 6}{jn^3\pi^3} \right)\cos(n\pi)\frac{j}{j} + \cancel{j\left(\frac{3n^2\pi^2 - 6}{n^4\pi^4} \right)\sin(n\pi)} \right] \quad 0$$

$$= \frac{j}{2n^4\pi^3}\left(n\pi(n^2\pi^2 - 6)\cos(n\pi) \right)$$

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{j}{2n^4\pi^3}\left(n\pi(n^2\pi^2 - 6)\cos(n\pi) \right) \cdot e^{jn\pi t} \right] \quad \#$$

	D	I
+	t^3	$e^{-jn\pi t}$
−	$3t^2$	$\frac{e^{-jn\pi t}}{-jn\pi}$
+	$6t$	$\frac{-e^{-jn\pi t}}{(n\pi)^2}$
−	6	$\frac{e^{-jn\pi t}}{j(n\pi)^3}$
+	0	$\frac{e^{-jn\pi t}}{(n\pi)^4}$

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2) $x(t) = \pi - t \; ; -\pi \leqslant t \leqslant \pi \; ; x(t+2\pi) = x(t)$

$T = 2\pi$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - t)e^{-jnt}\,dt$$

	D	I
+	t	e^{-jnt}
-	1	$\frac{e^{-jnt}}{-jn}$
+	0	$\frac{-e^{-jnt}}{n^2}$

$$= \frac{1}{2\cancel{\pi}}\int_{-\pi}^{\pi}\cancel{\pi}e^{-jnt}\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi}te^{-jnt}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-jnt}}{-jn}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\left[\frac{-te^{-jnt}}{jn} + \frac{e^{-jnt}}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{-2jn}\left[e^{-jn\pi} - e^{jn\pi}\right] - \frac{1}{2\pi}\left[\frac{-\pi e^{-jn\pi}}{jn} + \frac{e^{-jn\pi}}{n^2} - \frac{\pi e^{jn\pi}}{jn} - \frac{e^{jn\pi}}{n^2}\right]$$

$$= \frac{\sin(n\pi)}{n} - \frac{1}{\pi}\left[\frac{-\pi}{jn}\left(\frac{e^{-jn\pi} + e^{jn\pi}}{2}\right) + \frac{1}{2n^2}\left(e^{-jn\pi} - e^{jn\pi}\right)\right]$$

$$= \frac{\sin(n\pi)}{n} - \frac{1}{\pi}\left[\frac{-\pi\cos(n\pi)}{jn} - \underbrace{\cancel{\frac{j\sin(n\pi)}{n^2}}}_{0}\right]$$

$$= \frac{\sin(n\pi)}{n} + \frac{\cos(n\pi)}{jn}$$

$$= \frac{\cos(n\pi) + j\sin(n\pi)}{jn}$$

$$= \frac{-je^{in\pi}}{n}$$

$$\therefore\; x(t) = \sum_{n=-\infty}^{\infty}\left[\frac{-je^{in\pi}}{n}\cdot e^{jn\pi t}\right]$$ ※

3) $x(t) = t^2 + \sin^3(\pi t)\ ;\ -1 \leq t \leq 1 \qquad x(t+2) = x(t)$

$T = 2 \qquad a_n = \frac{1}{2}\int_{-1}^{1} (t^2 + \sin^3(\pi t))\, e^{-jn\pi t}\, dt$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\sin^3 x = \frac{3\sin x}{4} - \frac{\sin 3x}{4}$

$$2a_n = \underbrace{\int_{-1}^{1} t^2 e^{-jn\pi t}\, dt}_{①} + \frac{3}{4}\underbrace{\int_{-1}^{1} \sin(\pi t) e^{-jn\pi t}\, dt}_{②} - \frac{1}{4}\underbrace{\int_{-1}^{1} \sin(3\pi t)\, e^{-jn\pi t}\, dt}_{③}$$

	D	I
+	t^2	$e^{-jn\pi t}$
-	$2t$	$\frac{e^{-jn\pi t}}{-jn\pi}$
+	2	$\frac{-e^{-jn\pi t}}{(n\pi)^2}$
-	0	$\frac{e^{-jn\pi t}}{j(n\pi)^3}$

① ; $\int_{-1}^{1} t^2 e^{-jn\pi t}\, dt = \left[\frac{-t^2 e^{-jn\pi t}}{jn\pi} + \frac{2t\, e^{-jn\pi t}}{n^2\pi^2} + \frac{2e^{-jn\pi t}}{j^3 n^3\pi^3} \right]_{-1}^{1}$

$= \left[\frac{-e^{-jn\pi}}{jn\pi} + \frac{2e^{-jn\pi}}{n^2\pi^2} + \frac{2e^{-jn\pi}}{j^3n^3\pi^3} - \left(\frac{-e^{jn\pi}}{jn\pi} - \frac{2e^{jn\pi}}{n^2\pi^2} + \frac{2e^{jn\pi}}{j^3n^3\pi^3} \right) \right]$

$= \left(\frac{e^{jn\pi} - e^{-jn\pi}}{jn\pi} \right) + \frac{2}{n^2\pi^2}(e^{jn\pi} + e^{-jn\pi}) - \frac{2}{j^3n^3\pi^3}(e^{jn\pi} - e^{-jn\pi})$

$e^{ix} = \cos x + j\sin x$

$e^{-ix} = \cos x - j\sin x$

$\cos x : \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x : \frac{e^{ix} - e^{-ix}}{2j}$

$= \left(\frac{1}{jn\pi} - \frac{2}{j^3n^3\pi^3} \right)(e^{jn\pi} - e^{-jn\pi}) + \frac{2}{n^2\pi^2}(e^{jn\pi} + e^{-jn\pi})$

$= \left(\frac{1}{\cancel{j}n\pi} - \frac{2}{\cancel{j}n^3\pi^3} \right) \cdot 2\cancel{j}\, \underbrace{\sin(n\pi)}_{0} + \frac{2}{n^2\pi^2} \cdot (2\cos(n\pi))$

$= \frac{2}{n^2\pi^2}\left[2\cos(n\pi) \right]$

② ; $\int_{-1}^{1} \sin(\pi t) e^{-jn\pi t}\, dt = \int_{-1}^{1} \left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j} \right) e^{-jn\pi t}\, dt$

$= \frac{1}{2j}\int_{-1}^{1} e^{j\pi(1-n)t} - e^{-j\pi(1+n)t}\, dt$

$= \frac{1}{2j}\left[\frac{e^{j\pi(1-n)t}}{j\pi(1-n)} - \frac{e^{j\pi(1+n)t}}{j\pi(1+n)}\, dt \right]_{-1}^{1}$

$= \frac{1}{2j}\left[\frac{1}{j\pi(1-n)}\left(e^{j\pi(1-n)} - e^{-j\pi(1-n)} \right) - \frac{1}{j\pi(1+n)}\left(e^{j\pi(1+n)} - e^{-j\pi(1+n)} \right) \right]$

$= \frac{1}{j\pi(1-n)}\, \underbrace{\sin[(1-n)\pi]}_{0} - \frac{1}{j\pi(1+n)}\, \underbrace{\sin[(1+n)\pi]}_{0}$

$= 0$

$$③ ; \int_{-1}^{1} \sin(3\pi t)\, e^{-jn\pi t}\, dt = \int_{-1}^{1} \left(\frac{e^{j3\pi t} - e^{-j3\pi t}}{2j} \right) e^{-jn\pi t}\, dt$$

$$= \frac{1}{2j} \int_{-1}^{1} e^{j\pi(3-n)t} - e^{-j\pi(3+n)t}\, dt$$

$$= \frac{1}{2j} \left[\frac{e^{j\pi(3-n)t}}{j\pi(3-n)} - \frac{e^{j\pi(3+n)t}}{j\pi(3+n)}\, dt \right]_{-1}^{1}$$

$$= \frac{1}{2j} \left[\frac{1}{j\pi(3-n)} \left(e^{j\pi(3-n)} - e^{-j\pi(3-n)} \right) - \frac{1}{j\pi(3+n)} \left(e^{j\pi(3+n)} - e^{-j\pi(3+n)} \right) \right]$$

$$= \frac{1}{j\pi(3-n)} \underbrace{\cancel{\sin[(3-n)\pi]}}_{0} - \frac{1}{j\pi(3+n)} \underbrace{\cancel{\sin[(3+n)\pi]}}_{0}$$

$$= 0$$

$$\therefore \quad a_n = \frac{1}{n^2\pi^2} \left[2\cos(n\pi) \right]$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2\pi^2} \left[2\cos(n\pi) \right] e^{-jn\pi t}$$

$x(t) \longleftrightarrow X(j\omega)$

$x(t-t_0) \longleftrightarrow X(j\omega)\, e^{-j\omega t_0}$

$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} X(\frac{j\omega}{\alpha})$

$t\,x(t) \longleftrightarrow j\frac{d}{d\omega} X(j\omega)$

$\frac{d}{dt} x(t) \longleftrightarrow j\omega X(j\omega)$

$x(t)y(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$

Problem 4

Given $F\{x(t)\} = X(j\omega) = \text{rect}[\frac{(\omega-1)}{2}]$

1) $x(-2t+4) = x(-2(t-2))$

$x(-2t) \longleftrightarrow \frac{1}{2} X(\frac{-j\omega}{2})$

$x(-2(t-2)) \longleftrightarrow \frac{1}{2} X(\frac{-j\omega}{2})\, e^{-j2\omega} = \frac{1}{2}\,\text{rect}\left[\frac{-\omega-2}{4}\right] e^{-j2\omega}$

2) $(t-1)\,x(t-1)$

Let $y(t) = t\,x(t) \longleftrightarrow j\frac{d}{d\omega} X(j\omega)$

$y(t-1) \longleftrightarrow j e^{-j\omega} \frac{d}{d\omega} X(j\omega) = j e^{-j\omega}[\delta(\omega-\frac{1}{2}) - \delta(\omega-\frac{3}{2})]$

3) $t\frac{d}{dt} x(t)$

Let $\frac{d}{dt} x(t) = y(t) \longleftrightarrow X_1(j\omega) = j\omega X(j\omega)$

$t\,y(t) \longleftrightarrow j\frac{d}{d\omega} X_1(j\omega) = j\frac{d}{d\omega}(j\omega X(j\omega)) = -1(X(j\omega) + \omega[\delta(\omega-\frac{1}{2}) - \delta(\omega-\frac{3}{2})])$

4) $x(2t-1)\,e^{-2jt}$

$x(2t-1) \longleftrightarrow \frac{1}{2}\,\text{rect}\left[\frac{\omega-2}{4}\right] e^{-j\omega/2}$

Let $y(t) = e^{-2jt} \longleftrightarrow 2\pi\delta(\omega+2)$

$x(2t-1)\,y(t) \longleftrightarrow \frac{\cancel{1}}{\cancel{2}\pi} \frac{1}{2}\,\text{rect}\left[\frac{-\omega-2}{4}\right] e^{-j\omega/2} * \cancel{2}\pi\,\delta(\omega+2)$

$= \frac{1}{2}\,\text{rect}\left[\frac{\omega}{4}\right] e^{-j(\frac{\omega+2}{2})}$

5) $x(t) * x(t-1)$

$\text{rect}[\frac{\omega-1}{2}] \cdot \text{rect}[\frac{\omega-1}{2}]\, e^{-j\omega} = \text{rect}^2\left[\frac{\omega-1}{2}\right] e^{-j\omega}$