Andres Fernandez – 5692442 – fr_andres@msn.com

Exercise 1

Give a real-world example of a joint distribution P (x, y) where x is discrete and y is continuous

We can model a car with 6 gears, where $x \in \{1, 2, 3, 4, 5, 6\}$ would correspond to the gear, $y \in [0, 6000]$ to the motor revolutions,and $\mathbb{P}(x, y)$ to the probability of having a breakdown after 10 years of usage.

Exercise 2

i) *What remains if I marginalize a joint distribution P (v, w, x, y, z) over five variables with respect to variables w and y?*

$$\int_{\mathbb{S}_w} \int_{\mathbb{S}_y} \mathbb{P}(v, w, x, y, z)\, dw\, dy = \int_{\mathbb{S}_w} \mathbb{P}(v, w, x, z)\, dw = \mathbb{P}(v, x, z) \tag{1}$$

ii) *What remains if I marginalize the resulting distribution with respect to v?*

$$\int_{\mathbb{S}_v} \mathbb{P}(v, x, z)\, dv = \mathbb{P}(x, z) \tag{2}$$

Exercise 3

If variables x and y are independent and variables x and z are independent, does it follow that variables y and z are independent?

No. for example, if $z := x$, it would naturally follow that

$$x \perp\!\!\!\perp y \iff z \perp\!\!\!\perp y \tag{3}$$

but $x \not\perp\!\!\!\perp y$, since the multiplication rule $\mathbb{P}(x, y) = \mathbb{P}(x)\mathbb{P}(y)$ wouldn't hold for every values of x and y. For example, given $x \sim unif\{0, 1\}$:

$$0 = \mathbb{P}(x = 1, y = 0) \neq \mathbb{P}(x = 1)\mathbb{P}(y = 1) = 0.5 \cdot 0.5 = 0.25 \tag{4}$$

$\square$

Exercise 4

Show that the following relation is true:
$\mathbb{P}(w, x, y, z) = \mathbb{P}(x, y)\mathbb{P}(z|w, x, y)\mathbb{P}(w|x, y)$

Bayes' theorem is based on the following observations:

$$\mathbb{P}(a, b) = \mathbb{P}(b, a) \tag{5}$$

$$\mathbb{P}(a|b) = \frac{\mathbb{P}(a, b)}{\mathbb{P}(b)} \tag{6}$$

$$\mathbb{P}(b|a) = \frac{\mathbb{P}(b, a)}{\mathbb{P}(a)} \tag{7}$$

Equation 5 implies the following equivalence:

$$\mathbb{P}(a|b)\mathbb{P}(b) = \mathbb{P}(b|a)\mathbb{P}(a) \tag{8}$$

this allows to define $\mathbb{P}(a|b)$ in terms of $\mathbb{P}(b|a)$ and vice versa, which is known as the **Bayes' rule**. This rule also allows to decompose any joint probability in a chain of conditional probabilities, as follows:

$$\mathbb{P}(w, x, y, z) = \mathbb{P}(z, w, x, y) = \mathbb{P}(z|w, x, y)\mathbb{P}(w, x, y) = \mathbb{P}(z|w, x, y)\mathbb{P}(w|x, y)\mathbb{P}(x, y) \tag{9}$$

Which, due to a trivial commutativity of multiplication, leads to the following equivalence:

$$\mathbb{P}(w, x, y, z) = \mathbb{P}(x, y)\mathbb{P}(z|w, x, y)\mathbb{P}(w|x, y) \tag{10}$$

□

Exercise 5

In my pocket there are two coins. Coin 1 is unbiased, so the likelihood $\mathbb{P}(h = 1|c = 1)$ of getting heads is 0.5 and the likelihood $\mathbb{P}(h = 0|c = 1)$ of getting tails is also 0.5. Coin 2 is biased, so the likelihood $\mathbb{P}(h = 1|c = 2)$ of getting heads is 0.8 and the likelihood $\mathbb{P}(h = 0|c = 2)$ of getting tails is 0.2. I reach into my pocket and draw one of the coins at random. There is an equal prior probability I might have picked either coin. I flip the coin and observe a head. Use Bayes' rule to compute the posterior probability that I chose coin 2.

Asked is $\mathbb{P}(c = 2|h = 1)$. Following Bayes' rule, we know that

$$\mathbb{P}(c = 2|h = 1) = \frac{\mathbb{P}(h = 1|c = 2)\mathbb{P}(c = 2)}{\mathbb{P}(h = 1)} \tag{11}$$

All of this terms can be calculated from the given data:

$\mathbb{P}(h=1|c=2) := 0.8$

$\mathbb{P}(c=2) := 0.5$

$\mathbb{P}(h=1) := \mathbb{P}(h=1|c=1) + \mathbb{P}(h=1|c=2) = 0.5 \cdot 0.5 + 0.5 \cdot 0.8 = 0.65$

And therefore

$$\mathbb{P}(c=2|h=1) = \frac{\mathbb{P}(h=1|c=2)\mathbb{P}(c=2)}{\mathbb{P}(h=1)} = \frac{0.8 \cdot 0.5}{0.65} \approx 0.615 \tag{12}$$

Exercise 6

Consider a biased die where the probabilities of rolling sides 1, 2, 3, 4, 5, 6 are 1/12, 1/12, 1/12, 1/12, 1/6, 1/2, respectively.

i) *What is the expected value of the die?*

The expected value for a discrete random variable can be calculated like this:

$$\mathbb{E}[X] = \sum_{x \in \mathbb{S}_X} x \cdot \rho(x) \tag{13}$$

Applying the formula to the given distribution, we obtain the following expected value: $\mathbb{E}[X] = 1 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{2} = \frac{56}{12} \approx 4.67$

ii) *If I roll the die twice, what is the expected value of the sum of the two rolls?*

From the very definition of the expected value, follows its linearity:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] \tag{14}$$

$$\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X] \qquad \lambda \in \mathbb{R} \tag{15}$$

By rolling the die twice, we could assume that Y has the same distribution as X, and both variables aren't correlated, which would be equivalent to apply equation 15 with $\lambda = 2$. In general, rolling the die n times will have an expected value of $n \cdot \mathbb{E}[X]$, which for our case means that: E[X+X] = $2 \cdot \mathbb{E}[X] = 2 \cdot \frac{56}{12} \approx 9.33$