Test 3 Solutions

Saturday, November 20, 2021 23:53

① 17 points

$T(5\vec{v}_1 - 3\vec{v}_2) = -4\vec{v}_1 + 4\vec{v}_2$

$5T(\vec{v}_1) - 3T(\vec{v}_2) = -4\vec{v}_1 + 4\vec{v}_2$ 2 points

$T(-3\bar{v}_1 + 2\bar{v}_2) = -2\bar{v}_1 - 5\bar{v}_2$

$-3T(\bar{v}_1) + 2T(\bar{v}_2) = -2\bar{v}_1 - 5\bar{v}_2$ 2 points

$(5T(\bar{v}_1) - 3T(\bar{v}_2) = -4\bar{v}_1 + 4\bar{v}_2) \times 2$ 2 points

$+ \ (-3T(\bar{v}_1) + 2T(\bar{v}_2) = -2\bar{v}_1 - 5\bar{v}_2) \times 3$

$T(\bar{v}_1) = -14\bar{v}_1 - 7\bar{v}_2$ 3 points

$-3T(\bar{v}_1) + 2T(\bar{v}_2) = -2\bar{v}_1 + 4\bar{v}_2$

$T(\bar{v}_2) = (+3T(\bar{v}_1) - 2\bar{v}_1 + 4\bar{v}_2)/2$ 2 points

$= (-42\bar{v}_1 - 21\bar{v}_2 - 2\bar{v}_1 - 5\bar{v}_2)/2$

$T(\bar{v}_2) = -22\bar{v}_1 - 13\bar{v}_2$ 3 points

a) $T(\bar{v}_1) = -14\bar{v}_1 - 7\bar{v}_2$

b) $T(\bar{v}_2) = -22\bar{v}_1 - 13\bar{v}_2$

c) $2T(\bar{v}_1) + 4T(\bar{v}_2) = -116\bar{v}_1 - 66\bar{v}_2$ 3 points

② 12 points

$T(u) = Au \implies u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ identity matrix

$I = u = u^{-1}$

$A = T(u)u^{-1} = T(u).I$

$A = \begin{bmatrix} 7 & 0 & 3 \\ 8 & 3 & 1 \end{bmatrix}$

③ 12 points

a) B c) D e) E

b) C d) F f) A

④ 10 points

$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \\ -2 & 2 & 0 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_2 + 2R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank(A) = 1

null(A) = 3 - 1 = 2

dim(Ker(A)) = 2

dim(Im(A)) = 1

⑤ 14 points

$A = \begin{bmatrix} -7 & -10 \\ 2 & 2 \end{bmatrix}$ $\lambda I - A = \begin{bmatrix} \lambda + 7 & 10 \\ -2 & \lambda - 2 \end{bmatrix}$ 2 points

2 points $\det(\lambda I - A) = (\lambda+7)(\lambda-2) + 20$

$= \lambda^2 + 5\lambda - 14 + 20 = \lambda^2 + 5\lambda + 6$

$= (\lambda+3)(\lambda+2) \implies \lambda = -3, \lambda = -2$

$A - 3I = \begin{bmatrix} -4 & -10 \\ 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 5/2 \\ 0 & 0 \end{bmatrix} \implies \bar{v}_1 = s\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ 2 points

s

$A - 2I = \begin{bmatrix} -5 & -10 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \implies \bar{v}_2 = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ 2 points

s

also $\bar{v}_1 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$, $\lambda_1 = -3$ 3 points

$\bar{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\lambda_2 = -2$ 3 points

⑥ 12 points

$A = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 3 & -5 \\ -1 & 1 & 0 \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} -5-\lambda & 4 & 0 \\ 0 & 3-\lambda & -5 \\ -1 & 1 & -\lambda \end{bmatrix}$

$\mathrm{Char}(A) = \det(A - \lambda I) = ((-5-\lambda)(3-\lambda)(-\lambda) + 20 + 0)$
$- (-5(-5-\lambda))$

$= (-15 + 2\lambda + \lambda^2)(-\lambda) + 20 - (25 + 5\lambda)$

$= -\lambda^3 - 2\lambda^2 + 15\lambda + 20 - 25 - 5\lambda$

$= -\lambda^3 - 2\lambda^2 + 10\lambda - 5$

also $\lambda^3 + 2\lambda^2 - 10\lambda + 5$

⑦ 8 points

a) $5^7 = 78125$

b) $1/5 = 0.2$

c) $5+3 = 8$

d) $-5 \cdot 5 = -25$

⑧ 15 points

$tr(A) = \sum_{i=1}^{K} \lambda_i$ (sum of all eigenvalues)

$\det(A) = \prod_{i=1}^{K} \lambda_i$ (product of all eigenvalues) 4 point

3×3 matrix, 3 eigenvalues $\lambda_1, \lambda_2, \lambda_3$ 2 points

$tr(A) = -2 \implies \lambda_1 + \lambda_2 + \lambda_3 = -2$

$\det(A) = -24 \implies \lambda_1 \lambda_2 \lambda_3 = -24$

2 distincts eigenvalues: WLOG assume $\lambda_1 = \lambda_2$ 2 points

$\implies \lambda_1 + \lambda_1 + \lambda_3 = -2 \implies 2\lambda_1 + \lambda_3 = -2$

$\lambda_1 \lambda_1 \lambda_3 = -24$ $\lambda_1^2 \lambda_3 = -24$

4 points

$\implies \lambda_3 = -2 - 2\lambda_1 \iff -2\lambda_1^2 - 2\lambda_1^3 = 24$

$\lambda_1^2(-2 - 2\lambda_1) = -24 \iff \lambda_1^3 + \lambda_1^2 - 12 = 0$

$\lambda_1 = 2 \implies \lambda_3 = -2 - 2\lambda_1$

$= -2 - 2(2) = -6$

Ans: $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = -6$ 3 points

$\lambda = 2$: multiplicity = 2

$\lambda = -6$: multiplicity = 1