Concerning Universal Properties and the Abelianization Functor

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Theorem 1.1 (Universal property of the quotient set)**.** *Let X and Y be sets, and let $\sim$ be an equivalence relation on X. Let $\pi : X \to X/\sim$ be the quotient map, and let $f : X \to Y$ be such that $x_0 \sim x_1 \implies f(x_0) = f(x_1) \forall x_0, x_1 \in X$. Then there exists a unique $\overline{f} : X/\sim \to Y$ such that $f = \overline{f} \circ \pi$.*

Proof. We define $\overline{f} : X/\sim \to Y$ by $\overline{f}([x]) = f(x)$. This is well-defined, since if $x_0 \sim x_1$, then $\overline{f}([x_0]) = f(x_0) = f(x_1) = \overline{f}([x_1])$, and hence $\overline{f}$ is independent of representatives. Uniqueness is also evident from the definition. □

Theorem 1.2 (Universal property of the quotient group)**.** *Let G, H be groups, and let $N \trianglelefteq G$ be a normal subgroup. Let $\pi : G \to G/N$ be the quotient map, and let $\phi : G \to H$ be a group homomorphism, with $N \subseteq \ker \phi$. Then there exists a unique group homomorphism $\overline{\phi} : G/N \to H$ such that $\phi = \overline{\phi} \circ \pi$.*

Proof. Let $\sim$ be an equivalence relation on G which identifies the fibres of π; that is, $x \sim y \iff xN = yN$. Furthermore, since $N \subseteq \ker \phi$, we have that if $\pi(x) = \pi(y)$, then x and y are in the same coset, so there exists some $n \in N$ such that $x = yn$. Hence, $\phi(x) = \phi(yn) = \phi(y)\phi(n) = \phi(y)$. Then by the universal property of the quotient set, there exists a unique map $\overline{\phi} : G/\sim \to H$ such that $\phi = \overline{\phi} \circ \pi$. In particular, $\overline{\phi}$ is defined as $\overline{\phi}([x]) = \overline{\phi}(xN) = \phi(x) \forall x$, and hence $\overline{\phi}$ is simply a map on the cosets of N. Furthermore, given $x, y \in G$, we have $\overline{\phi}(xyN) = \phi(xy) = \phi(x)\phi(y) = \overline{\phi}(xN)\overline{\phi}(yN)$, and hence $\overline{\phi}$ is a group homomorphism. □

Corollary 1.3. *Let $\phi : G \to H$ be a group homomorphism, and let $N \trianglelefteq G$ be a normal subgroup such that $N \subseteq \ker \phi$. Then $N = \ker \phi$ if and only if $\overline{\phi} : G/N \to H$ is injective.*

Proof. By the universal property of the quotient group, we know that $\overline{\phi}$ is both unique and well-defined. First suppose that $N = \ker \phi$. Let $gN \in \ker \overline{\phi}$. Then $\phi(g) = \overline{\phi}(gN) = e$, and hence $g \in \ker \phi = N$, implying that $gN = N$. Hence, $\ker \overline{\phi} = N$, implying that $\overline{\phi}$ is injective.
Now suppose that $\overline{\phi}$ is injective. That then means that $\ker \overline{\phi} = N$. Let $g \in \ker \phi$. Then $\overline{\phi}(gN) = \phi(g) = e$, implying that $gN = N$, or $g \in N$. Hence, $N = \ker \phi$. □

Theorem 1.4 (Universal property of the abelianization of a group)**.** *Let G be a group, and let $G^{ab} := G/[G, G]$ be its abelianization. Let $\pi : G \to G^{ab}$ be the quotient group homomorphism. Let H be an abelian group. Let $\phi : G \to H$ be a group homomorphism. Then there exists a unique group homomorphism $\phi^{ab} : G^{ab} \to H$ such that $\phi^{ab} \circ \pi = \phi$.*

Proof. We simply need to show that the commutator subgroup $[G, G]$ is contained within the kernel of ϕ, since that will allow us to apply the universal property of the quotient group to obtain the unique group homomorphism $\phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H$ such that $\phi^{\mathrm{ab}} \circ \pi = \phi$. Indeed, let $x, y \in G$. Then $\phi([x, y]) = \phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = e$, so $[x, y] \in \ker \phi$. Hence, since $[G, G]$ is the subgroup generated by all commutators, it follows that $[G, G] \subseteq \ker \phi$, as required. □

Lemma 1.5. *Let $\phi : G \to H$ be a group homomorphism, and let $\pi_G : G \to G/[G, G]$, $\pi_H : H \to H/[H, H]$ be the quotient homomophisms of G and H into their respective abelianizations. Then there exists a group homomorphism $\phi^{ab} : G^{ab} \to H^{ab}$ given by $\phi^{ab}(g[G, G]) = \phi(g)[H, H] \forall g[G, G] \in G^{ab}$, which is the unique group homomorphism $\psi : G^{ab} \to H^{ab}$ such that $\psi \circ \pi_G = \pi_H \circ \phi$.*

Proof. Let $\psi := \pi_H \circ \phi : G \to H^{\mathrm{ab}}$. Then by the universal property of the abelianization of a group, there exists a unique group homomorphism $\phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ such that $\phi^{\mathrm{ab}} \circ \pi_G = \pi_H \circ \phi$. That is, given any $g \in G$, we have

$$\phi^{\mathrm{ab}}(g[G, G]) = \phi^{\mathrm{ab}} \circ \pi_G(g) = \pi_H \circ \phi(g) = \phi(g)[H, H],$$

as required. □

Theorem 1.6. *There is a covariant functor, called the abelianization functor, from the category of groups to the category of abelian groups.*

Proof. Let F be the functor which maps groups to their abelianizations, and group homomorphisms to group homomorphisms between the abelianizations of the groups. That is, given groups G and H and a group homomorphism $\phi : G \to H$, we have $F(G) = G^{\mathrm{ab}}$ and $F(\phi) = \phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}} : g[G, G] \mapsto \phi(g)[H, H]$.

We first show that $F(\mathrm{id}_G) = \mathrm{id}_{F(G)}$ for any group G. Indeed, $F(\mathrm{id}_G)$ is given by $\mathrm{id}_{G^{\mathrm{ab}}}^{\mathrm{ab}} : G^{\mathrm{ab}} \to G^{\mathrm{ab}}$, where $\mathrm{id}_{G^{\mathrm{ab}}}^{\mathrm{ab}}(g[G, G]) = \mathrm{id}_G(g)[G, G] = g[G, G] \forall g \in G$. $F(\mathrm{id}_G)$ is then the identity map on $G^{\mathrm{ab}} = F(G)$, as required.

We now show that F preserves composition. Let G, H, K be groups and let $\phi : G \to H$ and $\psi : H \to K$ be group homomorphisms. Then

$$F(\phi) = \phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}} : g[G, G] \mapsto \phi(g)[H, H],$$

$$F(\psi) = \psi^{\mathrm{ab}} : H^{\mathrm{ab}} \to K^{\mathrm{ab}} : h[H, H] \mapsto \psi(h)[K, K]$$

and

$$F(\psi \circ \phi) = (\psi \circ \phi)^{\mathrm{ab}} : G^{\mathrm{ab}} \to K^{\mathrm{ab}} : g[G, G] \mapsto (\psi \circ \phi)(g)[K, K].$$

Hence,

$$\begin{aligned}
(\psi^{\mathrm{ab}} \circ \phi^{\mathrm{ab}})(g[G, G]) &= \psi^{\mathrm{ab}}(\phi(g)[H, H]) \\
&= \psi(\phi(g))[K, K] \\
&= (\psi \circ \phi)(g)[K, K] \\
&= (\psi \circ \phi)^{\mathrm{ab}}(g[G, G]),
\end{aligned}$$

as required. Hence, F is a covariant functor. □