Algebra from a Categorical Perspective

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1 Introduction

This article will give categorical definitions of the standard objects from algebra, by way of defining universal properties which determine the objects up to unique isomorphism. For a given object A of interest, the general procedure will be to first define an object B as an object which satisfies a specific universal property, then show that A satisfies that universal property, and then show that all pairs of objects B which satisfy the universal property are related by a unique isomorphism. This will then demonstrate that the universal property determines A up to unique isomorphism. We will then apply these universal properties to prove additional useful categorical results, such as the functoriality of the abelianization of a group.

2 Sets

Definition 2.1 (Universal property of the quotient set)**.** *Let X be a set and let $\sim$ be an equivalence relation on X. A pair $(X', \pi : X \to X')$ is a quotient of X by $\sim$ if given any $f : X \to Y$ such that $x_0 \sim x_1 \implies f(x_0) = f(x_1) \forall x_0, x_1 \in X$, then there exists a unique $\overline{f} : X' \to Y$ such that $f = \overline{f} \circ \pi$.*

Theorem 2.2. *The canonical projection $\pi : X \to X/\sim : x \mapsto [x]$ satisfies the universal property of the quotient set.*

Proof. We define $\overline{f} : X/\sim \to Y$ by $\overline{f}(\pi(x)) = f(x)$. This is well-defined, since if $x_0 \sim x_1$, then $\overline{f}(\pi(x_0)) = f(x_0) = f(x_1) = \overline{f}(\pi(x_1))$, and hence $\overline{f}$ is independent of representatives. Uniqueness is also evident from the definition of π. □

Theorem 2.3. *Let X be a set with an equivalence relation $\sim$. Let (W, π_W) and (Z, π_Z) be two pairs of sets, along with maps $\pi_W : X \to W$ and $\pi_Z : X \to Z$ such that both pairs satisfy the universal property of the quotient set. Then there exists a unique bijection $h : W \to Z$ such that $\pi_Z = h \circ \pi_W$.*

Proof. Let $x_0 \sim x_1$. Then $\pi_Z(x_0) = \pi_Z(x_1)$, by the definition of a quotient map. Hence, by the universal property of the quotient set, there exists a unique $h : W \to Z$ such that $\pi_Z = h \circ \pi_W$. Similarly, there exists a $g : Z \to W$ such that $\pi_W = g \circ \pi_Z$. Hence, $\pi_Z = (h \circ g) \circ \pi_Z$. However, we also have that $\pi_Z = \mathrm{id}_Z \circ \pi_Z$, and hence, by uniqueness, $h \circ g = \mathrm{id}_Z$. Similarly, $g \circ h = \mathrm{id}_W$, implying that h is a bijection, as required. □

3 Groups

3.1 Quotient Groups

Definition 3.1 (Universal property of the quotient group)**.** *Let G be a group, and let $N \trianglelefteq G$ be a normal subgroup. A pair $(Q, q : G \to Q)$, consisting of a group Q and a group homomorphism q, is a quotient of G if $N \subseteq \ker q$ and given any group homomorphism $\phi : G \to H$ with $N \subseteq \ker \phi$, there exists a unique group homomorphism $\overline{\phi} : Q \to H$ such that $\phi = \overline{\phi} \circ q$.*

Theorem 3.2. *Let G be a group and let $N \trianglelefteq G$ be a normal subgroup. Then the canonical quotient homomorphism $\pi : G \to G/N : g \mapsto gN$ is a quotient map.*

Proof. Let $\sim$ be an equivalence relation on G which identifies the fibres of π; that is, $x \sim y \iff xN = yN$. If $\pi(x) = \pi(y)$, then x and y are in the same coset, so there exists some $n \in N$ such that $x = yn$. Hence, since $N \subseteq \ker \phi$, we have $\phi(x) = \phi(yn) = \phi(y)\phi(n) = \phi(y)$. Then by the universal property of the quotient set, there exists a unique map $\overline{\phi} : G/\sim \to H$ such that $\phi = \overline{\phi} \circ \pi$. In particular, $\overline{\phi}$ is defined as $\overline{\phi}([x]) = \overline{\phi}(xN) = \phi(x) \forall x$, and hence $\overline{\phi}$ is simply a map on the cosets of N. Furthermore, given $x, y \in G$, we have $\overline{\phi}(xyN) = \phi(xy) = \phi(x)\phi(y) = \overline{\phi}(xN)\overline{\phi}(yN)$, and hence $\overline{\phi}$ is a group homomorphism. $\square$

Corollary 3.3. *Let $\phi : G \to H$ be a group homomorphism, and let $N \trianglelefteq G$ be a normal subgroup such that $N \subseteq \ker \phi$. Then $N = \ker \phi$ if and only if $\overline{\phi} : G/N \to H$ is injective.*

Proof. By the universal property of the quotient group, we know that $\overline{\phi}$ is both unique and well-defined. First suppose that $N = \ker \phi$. Let $gN \in \ker \overline{\phi}$. Then $\phi(g) = \overline{\phi}(gN) = e$, and hence $g \in \ker \phi = N$, implying that $gN = N$. Hence, $\ker \overline{\phi} = N$, implying that $\overline{\phi}$ is injective.
Now suppose that $\overline{\phi}$ is injective. That then means that $\ker \overline{\phi} = N$. Let $g \in \ker \phi$. Then $\overline{\phi}(gN) = \phi(g) = e$, implying that $gN = N$, or $g \in N$. Hence, $N = \ker \phi$. $\square$

Theorem 3.4. *Let G be a group and let $N \trianglelefteq G$ be a normal subgroup. Let $q : G \to Q$ and $Q' : G \to Q'$ be two quotient homomorphisms. Then there exists a unique isomorphism $h : Q \to Q'$ such that $q' = h \circ q$.*

Proof. By the universal property of the quotient group, there exists a unique group homomorphism $h : Q \to Q'$ such that $q' = h \circ q$, and similarly, there exists a unique group homomorphism $g : Q' \to Q$ such that $q = g \circ q'$. Hence, $q' = (h \circ g) \circ q'$. However, we also have $q' = \mathrm{id}_{Q'} \circ q'$, and so the universal property implies that $h \circ g = \mathrm{id}_{Q'}$. Similarly, $g \circ h = \mathrm{id}_Q$, and hence h is an isomorphism. $\square$

3.2 Abelianization

Definition 3.5 (Universal property of the abelianization of a group)**.** *Let G be a group. An abelianization of G is a pair (G^{ab}, π), where G^{ab} is an abelian group and $\pi : G \to G^{ab}$ is a group homomorphism, such that given any abelian group H and group homomorphism $\phi : G \to H$, there exists a unique group homomorphism $\phi^{ab} : G^{ab} \to H$ such that $\phi^{ab} \circ \pi = \phi$.*

Theorem 3.6. *Let G be a group, let $[G, G]$ be the commutator subgroup, and let $\pi : G \to G/[G, G]$ be the canonical quotient homomorphism. Then $(G/[G, G], \pi)$ is an abelianization of G.*

Proof. We simply need to show that the commutator subgroup $[G, G]$ is contained within the kernel of ϕ, since that will allow us to apply the universal property of the quotient group to obtain the unique group homomorphism $\phi^{\mathrm{ab}} : G/[G, G] \to H$ such that $\phi^{\mathrm{ab}} \circ \pi = \phi$. Indeed, let $x, y \in G$. Then $\phi([x, y]) = \phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = e$, so $[x, y] \in \ker \phi$. Hence, since $[G, G]$ is the subgroup generated by all commutators, it follows that $[G, G] \subseteq \ker \phi$, as required. $\square$

Theorem 3.7. *Let G be a group, and let (G^{ab}, π) and $(G^{ab'}, \pi')$ be two abelianizations of G. Then there exists a unique isomorphism $h : G^{ab} \to G^{ab'}$ such that $\pi' = h \circ \pi$.*

Proof. By the universal property of the abelianization of a group, there exists a unique group isomorphism $h : G^{\mathrm{ab}} \to G^{\mathrm{ab}'}$ such that $\pi' = h \circ \pi$. Similarly, there exists a unique group isomorphism $g : G^{\mathrm{ab}'} \to G^{\mathrm{ab}}$ such that $\pi = g \circ \pi'$. Hence, $\pi' = (h \circ g) \circ \pi'$. However, we also have $\pi' = \mathrm{id}_{G^{\mathrm{ab}'}} \circ \pi'$, and hence, by uniqueness, we have $h \circ g = \mathrm{id}_{G^{\mathrm{ab}'}}$. Similarly, $g \circ h = \mathrm{id}_{G^{\mathrm{ab}}}$, implying that h is an isomorphism. $\square$

Lemma 3.8. *Let $\phi : G \to H$ be a group homomorphism, and let $\pi_G : G \to G/[G, G]$, $\pi_H : H \to H/[H, H]$ be the quotient homomophisms of G and H into their respective abelianizations. Then there exists a group homomorphism $\phi^{ab} : G^{ab} \to H^{ab}$ given by $\phi^{ab}(g[G, G]) = \phi(g)[H, H] \forall g[G, G] \in G^{ab}$, which is the unique group homomorphism $\psi : G^{ab} \to H^{ab}$ such that $\psi \circ \pi_G = \pi_H \circ \phi$.*

Proof. Consider the map $\pi_H \circ \phi : G \to H^{\mathrm{ab}}$. This is a map from a group to an abelian group, so by the universal property of the abelianization of a group, there exists a unique group homomorphism $\phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ such that $\phi^{\mathrm{ab}} \circ \pi_G = \pi_H \circ \phi$. That is, given any $g \in G$, we have

$$\phi^{\mathrm{ab}}(g[G, G]) = \phi^{\mathrm{ab}} \circ \pi_G(g) = \pi_H \circ \phi(g) = \phi(g)[H, H],$$

as required. $\square$

Theorem 3.9. *There is a covariant functor, called the abelianization functor, from the category of groups to the category of abelian groups.*

Proof. Let F be the functor which maps groups to their abelianizations, and group homomorphisms to group homomorphisms between the abelianizations of the groups. That is, given groups G and H and a group homomorphism $\phi : G \to H$, we have $F(G) = G^{\mathrm{ab}}$ and $F(\phi) = \phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}} : g[G, G] \mapsto \phi(g)[H, H]$.

We first show that $F(\mathrm{id}_G) = \mathrm{id}_{F(G)}$ for any group G. Indeed, $F(\mathrm{id}_G)$ is given by $\mathrm{id}^{\mathrm{ab}}_{G^{\mathrm{ab}}} : G^{\mathrm{ab}} \to G^{\mathrm{ab}}$, where $\mathrm{id}^{\mathrm{ab}}_{G^{\mathrm{ab}}}(g[G, G]) = \mathrm{id}_G(g)[G, G] = g[G, G] \forall g \in G$. $F(\mathrm{id}_G)$ is then the identity map on $G^{\mathrm{ab}} = F(G)$, as required.

We now show that F preserves composition. Let G, H, K be groups and let $\phi : G \to H$ and $\psi : H \to K$ be group homomorphisms. Then

$$F(\phi) = \phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}} : g[G, G] \mapsto \phi(g)[H, H],$$
$$F(\psi) = \psi^{\mathrm{ab}} : H^{\mathrm{ab}} \to K^{\mathrm{ab}} : h[H, H] \mapsto \psi(h)[K, K]$$

and

$$F(\psi \circ \phi) = (\psi \circ \phi)^{\mathrm{ab}} : G^{\mathrm{ab}} \to K^{\mathrm{ab}} : g[G, G] \mapsto (\psi \circ \phi)(g)[K, K].$$

Hence,

$$\begin{aligned}(\psi^{\mathrm{ab}} \circ \phi^{\mathrm{ab}})(g[G, G]) &= \psi^{\mathrm{ab}}(\phi(g)[H, H]) \\ &= \psi(\phi(g))[K, K] \\ &= (\psi \circ \phi)(g)[K, K] \\ &= (\psi \circ \phi)^{\mathrm{ab}}(g[G, G]),\end{aligned}$$

as required. Hence, F is a covariant functor. $\square$