# Algebra from a Categorical Perspective

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## 1 Introduction

This article will give categorical definitions of the standard objects from algebra, by way of defining universal properties which determine the objects up to unique isomorphism. For a given object $A$ of interest, the general procedure will be to first define an object $B$ as an object which satisfies a specific universal property, then show that $A$ satisfies that universal property, and then show that all pairs of objects $B$ which satisfy the universal property are related by a unique isomorphism. This will then demonstrate that the universal property determines $A$ up to unique isomorphism. We will then apply these universal properties to prove additional useful categorical results, such as the functoriality of the abelianization of a group.

## 2 Preliminary Categorical Constructions

**Definition 2.1.** *Let $\mathbf{C}$ be a category and let $f_1, f_2 : X \to Y \in Hom_{\mathbf{C}}(X, Y)$. Then a morphism $j : W \to X \in Hom_{\mathbf{C}}(W, X)$ is an equaliser of $f_1$ and $f_2$ if $f_1 \circ j = f_2 \circ j$ and given any morphism $g : T \to X$ with $f_1 \circ g = f_2 \circ g$, there is a unique morphism $h : T \to W$ such that $j \circ h = g$.*

$$\begin{tikzcd}
W \arrow[r, "j"] & X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \\
T \arrow[u, "\exists! h"] \arrow[ur, "\forall g"'] & &
\end{tikzcd}$$

**Theorem 2.2.** *Let $\mathbf{C}$ be a category, and suppose that $j : W \to X$ and $j' : W' \to X$ are both equalizers of morphisms $f_1, f_2 \in Hom_{\mathbf{C}}(X, Y)$. Then there exists a unique isomorphism $h : W \to W'$ such that $j = j' \circ h$.*

*Proof.* Since $j'$ is an equalizer and $f_1 \circ j = f_2 \circ j$, there exists a unique morphism $h : W \to W'$ such that $j = j' \circ h$. Similarly, there exists a unique morphism $h' : W' \to W$ such that $j' = j \circ h'$. Hence, we have $j' = j' \circ (h \circ h')$. If we then let $T = W'$ and $g = j'$, since we have that both $j' \circ \mathrm{id}_{W'} = j'$ and $j' \circ (h \circ h') = j'$, it follows from uniqueness that $h \circ h' = \mathrm{id}_{W'}$. Similarly, $h' \circ h = \mathrm{id}_W$. Hence, $h$ is an isomorphism. $\square$

**Definition 2.3.** *Let $\mathbf{C}$ be a category and let $f_1, f_2 : X \to Y$ be morphisms. Then a morphism $q : Y \to Z$ is a coequalizer of $f_1$ and $f_2$ if $q \circ f_1 = q \circ f_2$ and if $g : Y \to W$ is a morphism with*

$g \circ f_1 = g \circ f_2$, *then there is a unique morphism* $h : Z \to W$ *with* $g = h \circ q$.

$$\begin{tikzcd} X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \arrow[r, "q"] \arrow[dr, "\forall g"'] & Z \arrow[d, "\exists! h"] \\ & & W \end{tikzcd}$$

**Theorem 2.4.** *Let* $\mathbf{C}$ *be a category, and let* $q : Y \to Z$ *and* $q' : Y \to Z'$ *both be coequalizers of* $f_1, f_2 \in Hom_{\mathbf{C}}(X, Y)$. *Then there is a unique isomorphism* $h : Z \to Z'$ *such that* $q' = h \circ q$.

*Proof.* Since $q$ is a coequalizer and $q' \circ f_1 = q' \circ f_2$, there exists a unique morphism $h : Z \to Z'$ such that $q' = h \circ q$. Similarly, there exists a unique morphism $h' : Z' \to Z$ such that $q = h' \circ q'$. Hence, we have $q' = (h \circ h') \circ q'$. If we then let $W = Z'$ and $g = q'$, since we have that both $\mathrm{id}_{Z'} \circ q' = q'$ and $q' = (h \circ h') \circ q'$, it follows from uniqueness that $h \circ h' = \mathrm{id}_{Z'}$. Similarly, $h' \circ h = \mathrm{id}_Z$. Hence, $h$ is an isomorphism. $\square$

**Definition 2.5.** *Let* $\mathbf{C}$ *be a category, and let* $(X_\alpha)_{\alpha \in \mathcal{A}}$ *be an indexed collection of objects of* $\mathbf{C}$. *An object* $X$ *equipped with morphisms* $pr_\alpha : X \to X_\alpha$ *is a product of* $(X_\alpha)_{\alpha \in \mathcal{A}}$ *if for any object* $Z$ *and morphisms* $f_\alpha : Z \to X_\alpha$, *there exists a unique morphism* $F : Z \to X$ *such that* $pr_\alpha \circ F = f_\alpha \forall \alpha \in \mathcal{A}$.

$$\begin{tikzcd} & Z \arrow[dl, bend right, "f_1"'] \arrow[d, "\exists! F"] \arrow[dr, bend left, "f_2"] & \\ & X \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\ X_1 & & X_2 \end{tikzcd}$$

**Theorem 2.6.** *Let* $\mathbf{C}$ *be a category, and let* $(X_\alpha)_{\alpha \in \mathcal{A}}$ *be an indexed collection of objects of* $\mathbf{C}$. *Suppose* $(X, pr_\alpha)$ *and* $(X', pr'_\alpha)$ *are both products of* $(X_\alpha)_{\alpha \in \mathcal{A}}$. *Then there exists a unique isomorphism* $h : X \to X'$ *such that* $pr'_\alpha \circ h = pr_\alpha \forall \alpha \in \mathcal{A}$.

*Proof.* If we let $Z = X'$ and $f_\alpha = \mathrm{pr}'_\alpha$, then there exists a unique morphism $h' : X' \to X$ such that $\mathrm{pr}'_\alpha = \mathrm{pr}_\alpha \circ h'$. Similarly, there exists a unique morphism $h : X \to X'$ such that $\mathrm{pr}_\alpha = \mathrm{pr}'_\alpha \circ h$. From this we obtain $\mathrm{pr}_\alpha = \mathrm{pr}_\alpha \circ (h' \circ h)$. Then, if we let $Z = X$ and $f_\alpha = \mathrm{pr}_\alpha$, we see that both $h' \circ h$ and $\mathrm{id}_X$ are candidates for $F$. Hence, by uniqueness, $h' \circ h = \mathrm{id}_X$. Similarly, $h \circ h' = \mathrm{id}_{X'}$. Hence, $h$ is an isomorphism. $\square$

**Definition 2.7.** *Let* $\mathbf{C}$ *be a category and let* $(X_\alpha)_{\alpha \in \mathcal{A}}$ *be an indexed collection of objects of* $\mathbf{C}$. *An object* $Y$ *along with morphisms* $\iota_\alpha : X_\alpha \to Y$ *is a coproduct of* $(X_\alpha)_{\alpha \in \mathcal{A}}$ *if for any object* $Z$ *with morphisms* $f_\alpha : X_\alpha \to Z$, *there exists a unique morphism* $F : Y \to Z$ *such that for all* $\alpha \in \mathcal{A}$, *we have* $F \circ \iota_\alpha = f_\alpha$.

$$\begin{tikzcd} X_1 \arrow[dr, "\iota_1"] \arrow[ddr, bend right, "f_1"'] & & X_2 \arrow[dl, "\iota_2"'] \arrow[ddl, bend left, "f_2"] \\ & Y \arrow[d, "\exists! F"] & \\ & Z & \end{tikzcd}$$

**Theorem 2.8.** *Let* $\mathbf{C}$ *be a category and let* $(X_\alpha)_{\alpha\in\mathcal{A}}$ *be objects of* $\mathbf{C}$*. Suppose that both* $(Y, (\iota_\alpha)_{\alpha\in\mathcal{A}})$ *and* $(Y', (\iota'_\alpha)_{\alpha\in\mathcal{A}})$ *are coproducts of* $(X_\alpha)_{\alpha\in\mathcal{A}}$*. Then there exists a unique isomorphism* $h : Y \to Y'$ *such that* $h \circ \iota_\alpha = \iota'_\alpha \forall\alpha \in \mathcal{A}$*.*

*Proof.* If we let $Z = Y'$ and $f_\alpha = \iota'_\alpha$, there exists a unique morphism $h : Y \to Y'$ such that $\iota'_\alpha = h \circ \iota_\alpha \forall\alpha \in \mathcal{A}$. Similarly, there exists a unique morphism $h' : Y' \to Y$ such that $\iota_\alpha = h' \circ \iota'_\alpha \forall\alpha \in \mathcal{A}$. Hence, we have $\iota_\alpha = (h' \circ h) \circ \iota_\alpha \forall\alpha \in \mathcal{A}$. If we then let $Z = Y$ and $f_\alpha = \iota_\alpha$, we see that both $h' \circ h$ and $\mathrm{id}_Y$ are candidates for $F$. Hence, by uniqueness, $h' \circ h = \mathrm{id}_Y$. Similarly, $h \circ h' = \mathrm{id}_{Y'}$. Hence, $h$ is an isomorphism. □

# 3 Sets

**Definition 3.1.** *Let* $X$ *be a set, and let* $\sim$ *be an equivalence relation on* $X$*. We say that a function* $f : X \to Y$ *is* $\sim$*-invariant if* $x_0 \sim x_1 \implies f(x_0) = f(x_1) \forall x_0, x_1 \in X$*.*

**Definition 3.2** (Universal property of the quotient set)**.** *Let* $X$ *be a set and let* $\sim$ *be an equivalence relation on* $X$*. A pair* $(X', \pi : X \to X')$ *is a quotient of* $X$ *by* $\sim$ *if* $\pi$ *is* $\sim$*-invariant, and given any* $\sim$*-invariant function* $f : X \to Y$*, there exists a unique* $\overline{f} : X' \to Y$ *such that* $f = \overline{f} \circ \pi$*.*

**Theorem 3.3.** *The canonical projection* $\pi : X \to X/\sim : x \mapsto [x]$ *satisfies the universal property of the quotient set.*

*Proof.* First note that $\pi$ is clearly $\sim$-invariant. We define $\overline{f} : X/\sim \to Y$ by $\overline{f}(\pi(x)) = f(x)$. This is well-defined, since if $x_0 \sim x_1$, then $\overline{f}(\pi(x_0)) = f(x_0) = f(x_1) = \overline{f}(\pi(x_1))$, and hence $\overline{f}$ is independent of representatives. Uniqueness is also evident from the definition of $\pi$. □

**Theorem 3.4.** *Let* $X$ *be a set with an equivalence relation* $\sim$*. Let* $(W, \pi_W)$ *and* $(Z, \pi_Z)$ *be two pairs of sets, along with maps* $\pi_W : X \to W$ *and* $\pi_Z : X \to Z$ *such that both pairs satisfy the universal property of the quotient set. Then there exists a unique bijection* $h : W \to Z$ *such that* $\pi_Z = h \circ \pi_W$*.*

*Proof.* Let $(X', \pi : X \to X')$ be a quotient of $X$ by $\sim$. Let $R \subseteq X \times X$ be defined as $R = \{(x_0, x_1) : x_0 \sim x_1\}$. Then define $\mathrm{pr}_1 : R \to X : (x_0, x_1) \mapsto x_0$ and $\mathrm{pr}_2 : R \to X : (x_0, x_1) \mapsto x_1$. We then note that $\sim$-invariance of a function $g : X \to Y$ is equivalent to the property that $g \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2$. We then have that $\pi \circ \mathrm{pr}_1 = \pi \circ \mathrm{pr}_2$, and given any $f : X \to Y$ such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$, there exists a unique $\overline{f} : X' \to Y$ such that $f = \overline{f} \circ \pi$.

$$\begin{tikzcd} R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \arrow[r, "\pi"] \arrow[dr, "\forall f"'] & X' \arrow[d, "\exists!\overline{f}"] \\ & & Y \end{tikzcd}$$

This is identical to saying that $\pi$ is a coequalizer of $\mathrm{pr}_1$ and $\mathrm{pr}_2$ in the category of sets, and hence, $\pi$ is unique up to unique bijection. In particular, there exists a unique bijection $h : W \to Z$ such that $\pi_Z = h \circ \pi_W$. □

# 4 Groups

## 4.1 Quotient Groups

**Definition 4.1** (Universal property of the quotient group)**.** *Let* $G$ *be a group, and let* $N \trianglelefteq G$ *be a normal subgroup. A pair* $(Q, q : G \to Q)$*, consisting of a group* $Q$ *and a group homomorphism* $q$*, is* *a quotient of $G$ if $N \subseteq \ker q$ and given any group homomorphism $\phi : G \to H$ with $N \subseteq \ker \phi$, there exists a unique group homomorphism $\overline{\phi} : Q \to H$ such that $\phi = \overline{\phi} \circ q$.*

**Theorem 4.2.** *Let $G$ be a group and let $N \trianglelefteq G$ be a normal subgroup. Then the canonical quotient homomorphism $\pi : G \to G/N : g \mapsto gN$ is a quotient homomorphism.*

*Proof.* Let $\sim$ be an equivalence relation on $G$ which identifies the fibres of $\pi$; that is, $x \sim y \iff xN = yN$. If $\pi(x) = \pi(y)$, then $x$ and $y$ are in the same coset, so there exists some $n \in N$ such that $x = yn$. Hence, since $N \subseteq \ker \phi$, we have $\phi(x) = \phi(yn) = \phi(y)\phi(n) = \phi(y)$. Then by the universal property of the quotient set, there exists a unique map $\overline{\phi} : G/\sim \to H$ such that $\phi = \overline{\phi} \circ \pi$. In particular, $\overline{\phi}$ is defined as $\overline{\phi}([x]) = \overline{\phi}(xN) = \phi(x) \forall x$, and hence $\overline{\phi}$ is simply a map on the cosets of $N$. Furthermore, given $x, y \in G$, we have $\overline{\phi}(xyN) = \phi(xy) = \phi(x)\phi(y) = \overline{\phi}(xN)\overline{\phi}(yN)$, and hence $\overline{\phi}$ is a group homomorphism. $\square$

**Corollary 4.3.** *Let $\phi : G \to H$ be a group homomorphism, and let $N \trianglelefteq G$ be a normal subgroup such that $N \subseteq \ker \phi$. Then $N = \ker \phi$ if and only if $\overline{\phi} : G/N \to H$ is injective.*

*Proof.* By the universal property of the quotient group, we know that $\overline{\phi}$ is both unique and well-defined. First suppose that $N = \ker \phi$. Let $gN \in \ker \overline{\phi}$. Then $\phi(g) = \overline{\phi}(gN) = e$, and hence $g \in \ker \phi = N$, implying that $gN = N$. Hence, $\ker \overline{\phi} = N$, implying that $\overline{\phi}$ is injective.
Now suppose that $\overline{\phi}$ is injective. That then means that $\ker \overline{\phi} = N$. Let $g \in \ker \phi$. Then $\overline{\phi}(gN) = \phi(g) = e$, implying that $gN = N$, or $g \in N$. Hence, $N = \ker \phi$. $\square$

**Theorem 4.4.** *Let $G$ be a group and let $N \trianglelefteq G$ be a normal subgroup. Let $q : G \to Q$ and $Q' : G \to Q'$ be two quotient homomorphisms. Then there exists a unique isomorphism $h : Q \to Q'$ such that $q' = h \circ q$.*

*Proof.* Let $\pi : G \to K$ be a quotient homomorphism. Let $\iota : N \hookrightarrow G$ be the inclusion homomorphism, and let $z : N \to G : n \mapsto e$ be the trivial homomorphism. Then $\pi \circ \iota = \pi \circ z \iff \pi(n) = e \forall n \in N \iff N \subseteq \ker \pi$, which is true by the definition of $\pi$. Let $\phi : G \to H$ be any homomorphism with $N \subseteq \ker \phi$, or in other words, with $\phi \circ \iota = \phi \circ z$. Then by the universal property, there exists a unique homomorphism $\overline{\phi} : K \to H$ such that $\phi = \overline{\phi} \circ \pi$.

$$\begin{tikzcd} N \arrow[r, shift left, "\iota"] \arrow[r, shift right, "z"'] & G \arrow[r, "\pi"] \arrow[dr, "\forall\phi"'] & K \arrow[d, "\exists!\overline{\phi}"] \\ & & H \end{tikzcd}$$

This is exactly equivalent to saying that $\pi$ is a coequalizer in the category of groups. Hence, the result follows by uniqueness. $\square$

## 4.2 Products

## 4.3 Abelianization

**Definition 4.5** (Universal property of the abelianization of a group)**.** *Let $G$ be a group. An abelianization of $G$ is a pair $(G^{ab}, \pi)$, where $G^{ab}$ is an abelian group and $\pi : G \to G^{ab}$ is a group homomorphism, such that given any abelian group $H$ and group homomorphism $\phi : G \to H$, there exists a unique group homomorphism $\phi^{ab} : G^{ab} \to H$ such that $\phi^{ab} \circ \pi = \phi$.*

**Theorem 4.6.** *Let $G$ be a group, let $[G, G]$ be the commutator subgroup, and let $\pi : G \to G/[G, G]$ be the canonical quotient homomorphism. Then $(G/[G, G], \pi)$ is an abelianization of $G$.*

*Proof.* We simply need to show that the commutator subgroup $[G, G]$ is contained within the kernel of $\phi$, since that will allow us to apply the universal property of the quotient group to obtain the unique group homomorphism $\phi^{\mathrm{ab}} : G/[G, G] \to H$ such that $\phi^{\mathrm{ab}} \circ \pi = \phi$. Indeed, let $x, y \in G$. Then $\phi([x, y]) = \phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = e$, so $[x, y] \in \ker \phi$. Hence, since $[G, G]$ is the subgroup generated by all commutators, it follows that $[G, G] \subseteq \ker \phi$, as required. □

**Theorem 4.7.** *Let $G$ be a group, and let $(G^{ab}, \pi)$ and $(G^{ab'}, \pi')$ be two abelianizations of $G$. Then there exists a unique isomorphism $h : G^{ab} \to G^{ab'}$ such that $\pi' = h \circ \pi$.*

*Proof.* By the universal property of the abelianization of a group, there exists a unique group isomorphism $h : G^{\mathrm{ab}} \to G^{\mathrm{ab}'}$ such that $\pi' = h \circ \pi$. Similarly, there exists a unique group isomorphism $g : G^{\mathrm{ab}'} \to G^{\mathrm{ab}}$ such that $\pi = g \circ \pi'$. Hence, $\pi' = (h \circ g) \circ \pi'$. However, we also have $\pi' = \mathrm{id}_{G^{\mathrm{ab}'}} \circ \pi'$, and hence, by uniqueness, we have $h \circ g = \mathrm{id}_{G^{\mathrm{ab}'}}$. Similarly, $g \circ h = \mathrm{id}_{G^{\mathrm{ab}}}$, implying that $h$ is an isomorphism. □

**Lemma 4.8.** *Let $\phi : G \to H$ be a group homomorphism, and let $\pi_G : G \to G/[G, G]$, $\pi_H : H \to H/[H, H]$ be the quotient homomophisms of $G$ and $H$ into their respective abelianizations. Then there exists a group homomorphism $\phi^{ab} : G^{ab} \to H^{ab}$ given by $\phi^{ab}(g[G, G]) = \phi(g)[H, H] \forall g[G, G] \in G^{ab}$, which is the unique group homomorphism $\psi : G^{ab} \to H^{ab}$ such that $\psi \circ \pi_G = \pi_H \circ \phi$.*

*Proof.* Consider the map $\pi_H \circ \phi : G \to H^{\mathrm{ab}}$. This is a map from a group to an abelian group, so by the universal property of the abelianization of a group, there exists a unique group homomorphism $\phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ such that $\phi^{\mathrm{ab}} \circ \pi_G = \pi_H \circ \phi$. That is, given any $g \in G$, we have

$$\phi^{\mathrm{ab}}(g[G, G]) = \phi^{\mathrm{ab}} \circ \pi_G(g) = \pi_H \circ \phi(g) = \phi(g)[H, H],$$

as required. □

**Theorem 4.9.** *There is a covariant functor, called the abelianization functor, from the category of groups to the category of abelian groups.*

*Proof.* Let $F$ be the functor which maps groups to their abelianizations, and group homomorphisms to group homomorphisms between the abelianizations of the groups. That is, given groups $G$ and $H$ and a group homomorphism $\phi : G \to H$, we have $F(G) = G^{\mathrm{ab}}$ and $F(\phi) = \phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}} : g[G, G] \mapsto \phi(g)[H, H]$.

We first show that $F(\mathrm{id}_G) = \mathrm{id}_{F(G)}$ for any group $G$. Indeed, $F(\mathrm{id}_G)$ is given by $\mathrm{id}^{\mathrm{ab}}_{G^{\mathrm{ab}}} : G^{\mathrm{ab}} \to G^{\mathrm{ab}}$, where $\mathrm{id}^{\mathrm{ab}}_{G^{\mathrm{ab}}}(g[G, G]) = \mathrm{id}_G(g)[G, G] = g[G, G] \forall g \in G$. $F(\mathrm{id}_G)$ is then the identity map on $G^{\mathrm{ab}} = F(G)$, as required.

We now show that $F$ preserves composition. Let $G, H, K$ be groups and let $\phi : G \to H$ and $\psi : H \to K$ be group homomorphisms. Then

$$F(\phi) = \phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}} : g[G, G] \mapsto \phi(g)[H, H],$$

$$F(\psi) = \psi^{\mathrm{ab}} : H^{\mathrm{ab}} \to K^{\mathrm{ab}} : h[H, H] \mapsto \psi(h)[K, K]$$

and

$$F(\psi \circ \phi) = (\psi \circ \phi)^{\mathrm{ab}} : G^{\mathrm{ab}} \to K^{\mathrm{ab}} : g[G, G] \mapsto (\psi \circ \phi)(g)[K, K].$$

Hence,

$$\begin{aligned}
(\psi^{\mathrm{ab}} \circ \phi^{\mathrm{ab}})(g[G,G]) &= \psi^{\mathrm{ab}}(\phi(g)[H,H]) \\
&= \psi(\phi(g))[K,K] \\
&= (\psi \circ \phi)(g)[K,K] \\
&= (\psi \circ \phi)^{\mathrm{ab}}(g[G,G]),
\end{aligned}$$

as required. Hence, $F$ is a covariant functor. $\square$