Algebra from a Categorical Perspective

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1 Introduction

This article will give categorical definitions of the standard objects from algebra, by way of defining universal properties which determine the objects up to unique isomorphism. For a given object A of interest, the general procedure will be to first define an object B as an object which satisfies a specific universal property, then show that A satisfies that universal property, and then show that all pairs of objects B which satisfy the universal property are related by a unique isomorphism. This will then demonstrate that the universal property determines A up to unique isomorphism. We will then apply these universal properties to prove additional useful results, such as the functoriality of the abelianization of a group.

2 Preliminary Categorical Constructions

Definition 2.1. *Let* $\mathbf{C}$ *be a category and let* $f_1, f_2 : X \to Y \in Hom_{\mathbf{C}}(X, Y)$*. Then a morphism* $j : W \to X \in Hom_{\mathbf{C}}(W, X)$ *is an equaliser of* f_1 *and* f_2 *if* $f_1 \circ j = f_2 \circ j$ *and given any morphism* $g : T \to X$ *with* $f_1 \circ g = f_2 \circ g$*, there is a unique morphism* $h : T \to W$ *such that* $j \circ h = g$*.*

$$\begin{tikzcd} W \arrow[r, "j"] & X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \\ T \arrow[u, "\exists! h"] \arrow[ur, "\forall g"'] & & \end{tikzcd}$$

Theorem 2.2. *Let* $\mathbf{C}$ *be a category, and suppose that* $j : W \to X$ *and* $j' : W' \to X$ *are both equalizers of morphisms* $f_1, f_2 \in Hom_{\mathbf{C}}(X, Y)$*. Then there exists a unique isomorphism* $h : W \to W'$ *such that* $j = j' \circ h$*.*

Proof. Since j' is an equalizer and $f_1 \circ j = f_2 \circ j$, there exists a unique morphism $h : W \to W'$ such that $j = j' \circ h$. Similarly, there exists a unique morphism $h' : W' \to W$ such that $j' = j \circ h'$. Hence, we have $j' = j' \circ (h \circ h')$. If we then let $T = W'$ and $g = j'$, since we have that both $j' \circ \mathrm{id}_{W'} = j'$ and $j' \circ (h \circ h') = j'$, it follows from uniqueness that $h \circ h' = \mathrm{id}_{W'}$. Similarly, $h' \circ h = \mathrm{id}_W$. Hence, h is an isomorphism. $\square$

Definition 2.3. *Let* $\mathbf{C}$ *be a category and let* $f_1, f_2 : X \to Y$ *be morphisms. Then a morphism* $q : Y \to Z$ *is a coequalizer of* f_1 *and* f_2 *if* $q \circ f_1 = q \circ f_2$ *and if* $g : Y \to W$ *is a morphism with*

$g \circ f_1 = g \circ f_2$, then there is a unique morphism $h : Z \to W$ with $g = h \circ q$.

$$\begin{tikzcd}
X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \arrow[r, "q"] \arrow[dr, "\forall g"'] & Z \arrow[d, "\exists! h"] \\
& & W
\end{tikzcd}$$

Theorem 2.4. *Let $\mathbf{C}$ be a category, and let $q : Y \to Z$ and $q' : Y \to Z'$ both be coequalizers of $f_1, f_2 \in Hom_{\mathbf{C}}(X, Y)$. Then there is a unique isomorphism $h : Z \to Z'$ such that $q' = h \circ q$.*

Proof. Since q is a coequalizer and $q' \circ f_1 = q' \circ f_2$, there exists a unique morphism $h : Z \to Z'$ such that $q' = h \circ q$. Similarly, there exists a unique morphism $h' : Z' \to Z$ such that $q = h' \circ q'$. Hence, we have $q' = (h \circ h') \circ q'$. If we then let $W = Z'$ and $g = q'$, since we have that both $\mathrm{id}_{Z'} \circ q' = q'$ and $q' = (h \circ h') \circ q'$, it follows from uniqueness that $h \circ h' = \mathrm{id}_{Z'}$. Similarly, $h' \circ h = \mathrm{id}_Z$. Hence, h is an isomorphism. $\square$

Definition 2.5. *Let $\mathbf{C}$ be a category, and let $(X_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of objects of $\mathbf{C}$. An object X equipped with morphisms $pr_\alpha : X \to X_\alpha$ is a product of $(X_\alpha)_{\alpha \in \mathcal{A}}$ if for any object Z and morphisms $f_\alpha : Z \to X_\alpha$, there exists a unique morphism $F : Z \to X$ such that $\mathrm{pr}_\alpha \circ F = f_\alpha \forall \alpha \in \mathcal{A}$.*

$$\begin{tikzcd}
& Z \arrow[d, "\exists! F"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
& X \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

Theorem 2.6. *Let $\mathbf{C}$ be a category, and let $(X_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of objects of $\mathbf{C}$. Suppose (X, pr_α) and (X', pr'_α) are both products of $(X_\alpha)_{\alpha \in \mathcal{A}}$. Then there exists a unique isomorphism $h : X \to X'$ such that $pr'_\alpha \circ h = pr_\alpha \forall \alpha \in \mathcal{A}$.*

Proof. If we let $Z = X'$ and $f_\alpha = \mathrm{pr}'_\alpha$, then there exists a unique morphism $h' : X' \to X$ such that $\mathrm{pr}'_\alpha = \mathrm{pr}_\alpha \circ h'$. Similarly, there exists a unique morphism $h : X \to X'$ such that $\mathrm{pr}_\alpha = \mathrm{pr}'_\alpha \circ h$. From this we obtain $\mathrm{pr}_\alpha = \mathrm{pr}_\alpha \circ (h' \circ h)$. Then, if we let $Z = X$ and $f_\alpha = \mathrm{pr}_\alpha$, we see that both $h' \circ h$ and id_X are candidates for F. Hence, by uniqueness, $h' \circ h = \mathrm{id}_X$. Similarly, $h \circ h' = \mathrm{id}_{X'}$. Hence, h is an isomorphism. $\square$

Definition 2.7. *Let $\mathbf{C}$ be a category and let $(X_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of objects of $\mathbf{C}$. An object Y along with morphisms $\iota_\alpha : X_\alpha \to Y$ is a coproduct of $(X_\alpha)_{\alpha \in \mathcal{A}}$ if for any object Z with morphisms $f_\alpha : X_\alpha \to Z$, there exists a unique morphism $F : Y \to Z$ such that for all $\alpha \in \mathcal{A}$, we have $F \circ \iota_\alpha = f_\alpha$.*

$$\begin{tikzcd}
X_1 \arrow[dr, "\iota_1"] \arrow[ddr, bend right, "f_1"'] & & X_2 \arrow[dl, "\iota_2"'] \arrow[ddl, bend left, "f_2"] \\
& Y \arrow[d, "\exists! F"] & \\
& Z &
\end{tikzcd}$$

Theorem 2.8. *Let* $\mathbf{C}$ *be a category and let* $(X_\alpha)_{\alpha\in\mathcal{A}}$ *be objects of* $\mathbf{C}$*. Suppose that both* $(Y, (\iota_\alpha)_{\alpha\in\mathcal{A}})$ *and* $(Y', (\iota'_\alpha)_{\alpha\in\mathcal{A}})$ *are coproducts of* $(X_\alpha)_{\alpha\in\mathcal{A}}$*. Then there exists a unique isomorphism* $h : Y \to Y'$ *such that* $h \circ \iota_\alpha = \iota'_\alpha \forall \alpha \in \mathcal{A}$*.*

Proof. If we let $Z = Y'$ and $f_\alpha = \iota'_\alpha$, there exists a unique morphism $h : Y \to Y'$ such that $\iota'_\alpha = h \circ \iota_\alpha \forall \alpha \in \mathcal{A}$. Similarly, there exists a unique morphism $h' : Y' \to Y$ such that $\iota_\alpha = h' \circ \iota'_\alpha \forall \alpha \in \mathcal{A}$. Hence, we have $\iota_\alpha = (h' \circ h) \circ \iota_\alpha \forall \alpha \in \mathcal{A}$. If we then let $Z = Y$ and $f_\alpha = \iota_\alpha$, we see that both $h' \circ h$ and id_Y are candidates for F. Hence, by uniqueness, $h' \circ h = \mathrm{id}_Y$. Similarly, $h \circ h' = \mathrm{id}_{Y'}$. Hence, h is an isomorphism. □

3 Sets

3.1 Quotients

Definition 3.1. *Let* X *be a set, and let* $\sim$ *be an equivalence relation on* X*. We say that a function* $f : X \to Y$ *is* $\sim$*-invariant if* $x_0 \sim x_1 \implies f(x_0) = f(x_1) \forall x_0, x_1 \in X$*.*

Definition 3.2 (Universal property of the quotient set)**.** *Let* X *be a set and let* $\sim$ *be an equivalence relation on* X*. A pair* $(X', \pi : X \to X')$ *is a quotient of* X *by* $\sim$ *if* π *is* $\sim$*-invariant, and given any* $\sim$*-invariant function* $f : X \to Y$*, there exists a unique* $\overline{f} : X' \to Y$ *such that* $f = \overline{f} \circ \pi$*.*

Theorem 3.3. *The canonical projection* $\pi : X \to X/\sim : x \mapsto [x]$ *satisfies the universal property of the quotient set.*

Proof. First note that π is clearly $\sim$-invariant. We define $\overline{f} : X/\sim \to Y$ by $\overline{f}(\pi(x)) = f(x)$. This is well-defined, since if $x_0 \sim x_1$, then $\overline{f}(\pi(x_0)) = f(x_0) = f(x_1) = \overline{f}(\pi(x_1))$, and hence $\overline{f}$ is independent of representatives. Uniqueness is also evident from the definition of π. □

Theorem 3.4. *Let* X *be a set with an equivalence relation* $\sim$*. Let* (W, π_W) *and* (Z, π_Z) *be two pairs of sets, along with maps* $\pi_W : X \to W$ *and* $\pi_Z : X \to Z$ *such that both pairs satisfy the universal property of the quotient set. Then there exists a unique bijection* $h : W \to Z$ *such that* $\pi_Z = h \circ \pi_W$*.*

Proof. Let $(X', \pi : X \to X')$ be a quotient of X by $\sim$. Let $R \subseteq X \times X$ be defined as $R = \{(x_0, x_1) : x_0 \sim x_1\}$. Then define $\mathrm{pr}_1 : R \to X : (x_0, x_1) \mapsto x_0$ and $\mathrm{pr}_2 : R \to X : (x_0, x_1) \mapsto x_1$. We then note that $\sim$-invariance of a function $g : X \to Y$ is equivalent to the property that $g \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2$. We then have that $\pi \circ \mathrm{pr}_1 = \pi \circ \mathrm{pr}_2$, and given any $f : X \to Y$ such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$, there exists a unique $\overline{f} : X' \to Y$ such that $f = \overline{f} \circ \pi$.

$$\begin{tikzcd}
R \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \arrow[r, "\pi"] \arrow[dr, "\forall f"'] & X' \arrow[d, "\exists! \overline{f}"] \\
 & & Y
\end{tikzcd}$$

This is identical to saying that π is a coequalizer of pr_1 and pr_2 in the category of sets, and hence, π is unique up to unique bijection. In particular, there exists a unique bijection $h : W \to Z$ such that $\pi_Z = h \circ \pi_W$. □

3.2 Products

Definition 3.5 (Universal property of the product of sets)**.** *Let* $(S_\alpha)_{\alpha\in\mathcal{A}}$ *be an indexed collection of sets. A set* S *equipped with maps* $pr_\alpha : S \to S_\alpha$ *is a product of* $(S_\alpha)_{\alpha\in\mathcal{A}}$ *if for any set* K *and maps* $\phi_\alpha : K \to S_\alpha$*, there exists a unique map* $\Psi : K \to S$ *such that* $pr_\alpha \circ \Psi = \phi_\alpha \forall \alpha \in \mathcal{A}$*.*

Theorem 3.6. *Let $(S_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of sets. The set*

$$\prod_{\alpha \in \mathcal{A}} S_\alpha := \{x : \mathcal{A} \to \bigcup_{\alpha \in \mathcal{A}} S_\alpha \mid \forall \alpha \in \mathcal{A} : x(\alpha) \in S_\alpha\}$$

equipped with maps $pr_\beta : \prod_{\alpha \in \mathcal{A}} S_\alpha \to S_\beta : x \mapsto x(\beta)$ is a product of $(S_\alpha)_{\alpha \in \mathcal{A}}$.

Proof. Let K be another set and let $f_\beta : K \to S_\beta$ be a collection of maps. There is then a unique map $F : K \to \prod_{\alpha \in \mathcal{A}} S_\alpha$ with $\mathrm{pr}_\beta \circ F = f_\beta$, namely the one given by $F(k)(\alpha) = f_\alpha(k) \forall k \in K, \alpha \in \mathcal{A}$. □

Remark 3.7. *Note that if $\mathcal{A}$ is infinite and each S_α is non-empty, then $\prod_{\alpha \in \mathcal{A}} S_\alpha$ is only non-empty if we assume the axiom of choice, since the definition requires us to pick an element from infinitely many non-empty sets.*

Theorem 3.8. *Let $(S_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of sets and let $(S, (pr_\alpha)_{\alpha \in \mathcal{A}})$ and $(S', (pr'_\alpha)_{\alpha \in \mathcal{A}})$ be two products of $(S_\alpha)_{\alpha \in \mathcal{A}}$. Then there exists a unique bijection $h : S \to S'$ such that $pr'_\alpha \circ h = pr_\alpha \forall \alpha \in \mathcal{A}$.*

Proof. The product of sets is the product in the category of sets, and so uniqueness follows. □

3.3 Disjoint Unions

Definition 3.9 (Universal property of disjoint unions)**.** *Let $(S_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of sets. A set S with maps $\iota_\alpha : S_\alpha \to S$ is said to be the disjoint union of $(S_\alpha)_{\alpha \in \mathcal{A}}$, written $S = \coprod_{\alpha \in \mathcal{A}} S_\alpha$ if for any set K with maps $f_\alpha : S_\alpha \to K$, there exists a unique map $F : S \to K$ such that $F \circ \iota_\alpha = f_\alpha \forall \alpha \in \mathcal{A}$.*

Theorem 3.10. *Any collection $(S_\alpha)_{\alpha \in \mathcal{A}}$ of sets has a disjoint union $\coprod_{\alpha \in \mathcal{A}} S_\alpha$.*

Proof. We can set $\coprod_{\alpha \in \mathcal{A}} S_\alpha = \{(x, \alpha) \in S_\alpha \times \mathcal{A}\}$, and we can define inclusion maps $\iota_\alpha : S_\alpha \to \coprod_{\alpha \in \mathcal{A}} S_\alpha : x \mapsto (x, \alpha)$. Then, if we have another set K and maps $f_\alpha : S_\alpha \to K$, there exists a unique map $F : \coprod_{\alpha \in \mathcal{A}} S_\alpha \to K$ such that $F \circ \iota_\alpha = f_\alpha \forall \alpha \in \mathcal{A}$, namely by setting $F((x, \alpha)) = f_\alpha(x)$. □

Theorem 3.11. *Let $(S_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of sets, and let $(S, (\iota_\alpha)_{\alpha \in \mathcal{A}})$ and $(S', (\iota'_\alpha)_{\alpha \in \mathcal{A}})$ be two disjoint unions of $(S_\alpha)_{\alpha \in \mathcal{A}}$. Then there exists a unique bijection $h : S \to S'$ such that $h \circ \iota_\alpha = \iota'_\alpha \forall \alpha \in \mathcal{A}$.*

Proof. The disjoint union is the coproduct in the category of sets, and so uniqueness up to unique bijection follows. □

4 Groups

4.1 Quotient Groups

Definition 4.1 (Universal property of the quotient group)**.** *Let G be a group, and let $N \trianglelefteq G$ be a normal subgroup. A pair $(Q, q : G \to Q)$, consisting of a group Q and a group homomorphism q, is a quotient of G if $N \subseteq \ker q$ and given any group homomorphism $\phi : G \to H$ with $N \subseteq \ker \phi$, there exists a unique group homomorphism $\overline{\phi} : Q \to H$ such that $\phi = \overline{\phi} \circ q$.*

Theorem 4.2. *Let G be a group and let $N \trianglelefteq G$ be a normal subgroup. Then the canonical quotient homomorphism $\pi : G \to G/N : g \mapsto gN$ is a quotient homomorphism.*

Proof. Let $\sim$ be an equivalence relation on G which identifies the fibres of π; that is, $x \sim y \iff xN = yN$. If $\pi(x) = \pi(y)$, then x and y are in the same coset, so there exists some $n \in N$ such that $x = yn$. Hence, since $N \subseteq \ker \phi$, we have $\phi(x) = \phi(yn) = \phi(y)\phi(n) = \phi(y)$. Then by the universal property of the quotient set, there exists a unique map $\overline{\phi} : G/\sim \to H$ such that $\phi = \overline{\phi} \circ \pi$. In particular, $\overline{\phi}$ is defined as $\overline{\phi}([x]) = \overline{\phi}(xN) = \phi(x) \forall x$, and hence $\overline{\phi}$ is simply a map on the cosets of N. Furthermore, given $x, y \in G$, we have $\overline{\phi}(xyN) = \phi(xy) = \phi(x)\phi(y) = \overline{\phi}(xN)\overline{\phi}(yN)$, and hence $\overline{\phi}$ is a group homomorphism. ☐

Corollary 4.3. *Let $\phi : G \to H$ be a group homomorphism, and let $N \trianglelefteq G$ be a normal subgroup such that $N \subseteq \ker \phi$. Then $N = \ker \phi$ if and only if $\overline{\phi} : G/N \to H$ is injective.*

Proof. By the universal property of the quotient group, we know that $\overline{\phi}$ is both unique and well-defined. First suppose that $N = \ker \phi$. Let $gN \in \ker \overline{\phi}$. Then $\phi(g) = \overline{\phi}(gN) = e$, and hence $g \in \ker \phi = N$, implying that $gN = N$. Hence, $\ker \overline{\phi} = N$, implying that $\overline{\phi}$ is injective.
Now suppose that $\overline{\phi}$ is injective. That then means that $\ker \overline{\phi} = N$. Let $g \in \ker \phi$. Then $\overline{\phi}(gN) = \phi(g) = e$, implying that $gN = N$, or $g \in N$. Hence, $N = \ker \phi$. ☐

Theorem 4.4. *Let G be a group and let $N \trianglelefteq G$ be a normal subgroup. Let $q : G \to Q$ and $Q' : G \to Q'$ be two quotient homomorphisms. Then there exists a unique isomorphism $h : Q \to Q'$ such that $q' = h \circ q$.*

Proof. Let $\pi : G \to K$ be a quotient homomorphism. Let $\iota : N \hookrightarrow G$ be the inclusion homomorphism, and let $z : N \to G : n \mapsto e$ be the trivial homomorphism. Then $\pi \circ \iota = \pi \circ z \iff \pi(n) = e \forall n \in N \iff N \subseteq \ker \pi$, which is true by the definition of π. Let $\phi : G \to H$ be any homomorphism with $N \subseteq \ker \phi$, or in other words, with $\phi \circ \iota = \phi \circ z$. Then by the universal property, there exists a unique homomorphism $\overline{\phi} : K \to H$ such that $\phi = \overline{\phi} \circ \pi$.

$$\begin{tikzcd} N \arrow[r, shift left, "\iota"] \arrow[r, shift right, "z"'] & G \arrow[r, "\pi"] \arrow[dr, "\forall \phi"'] & K \arrow[d, "\exists! \overline{\phi}"] \\ & & H \end{tikzcd}$$

This is exactly equivalent to saying that π is a coequalizer in the category of groups. Hence, the result follows by uniqueness. ☐

4.2 Products

Definition 4.5 (Universal property of the direct product of groups)**.** *Let $(G_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of groups. A group G equipped with homomorphisms $pr_\alpha : G \to G_\alpha$ is a direct product of $(G_\alpha)_{\alpha \in \mathcal{A}}$ if for any group H and homomorphisms $\phi_\alpha : H \to G_\alpha$, there exists a unique homomorphism $\Psi : H \to G$ such that $pr_\alpha \circ \Psi = \phi_\alpha \forall \alpha \in \mathcal{A}$.*

Theorem 4.6. *Let $(G_\alpha)_{\alpha \in \mathcal{A}}$ be an indexed collection of groups. Let*

$$\prod_{\alpha \in \mathcal{A}} G_\alpha = \{g : \mathcal{A} \to \bigcup_{\alpha \in \mathcal{A}} G_\alpha \mid \forall \alpha \in \mathcal{A} : g(\alpha) \in G_\alpha\}$$

be a group equipped with homomorphisms $pr_\alpha : \prod_{\alpha \in \mathcal{A}} G_\alpha \to G_\alpha : g \mapsto g(\alpha)$, and with multiplication given by $(g \cdot h)(\alpha) = g(\alpha)h(\alpha)$. Then $\prod_{\alpha \in \mathcal{A}} G_\alpha$ is a direct product of $(G_\alpha)_{\alpha \in \mathcal{A}}$.

Proof. Let H be another group and let $f_\beta : H \to G_\beta$ be a collection of homomorphisms. There is then a unique homomorphism $F : H \to \prod_{\alpha \in \mathcal{A}} G_\alpha$ with $\mathrm{pr}_\beta \circ F = f_\beta$, namely the one given by $F(h)(\alpha) = f_\alpha(h) \forall h \in H, \alpha \in \mathcal{A}$. ☐

Theorem 4.7. *Direct products of groups are unique up to unique isomorphism.*

Proof. Direct products are products in the category of groups, and so uniqueness up to unique isomorphism follows. $\square$

4.3 Free Products

Definition 4.8 (Universal property of free products)**.** *Let $(G_\alpha)_{\alpha\in\mathcal{A}}$ be an indexed collection of groups. We say that a group G, along with homomorphisms $\iota_\alpha : G_\alpha \to G$, is a free product of $(G_\alpha)_{\alpha\in\mathcal{A}}$ if for any group H with homomorphisms $\phi_\alpha : G_\alpha \to H$, there exists a unique homomorphism $\Psi : G \to H$ such that $\Psi \circ \iota_\alpha = \phi_\alpha \forall \alpha \in \mathcal{A}$.*

Theorem 4.9. *Let $(G_\alpha)_{\alpha\in\mathcal{A}}$ be an indexed collection of groups. Let $*_\alpha G_\alpha$ be defined as in page 41 of "Algebraic Topology" by Allen Hatcher. Let $\iota_\alpha : G_\alpha \to *_\alpha G_\alpha$ be the standard inclusion homomorphisms. Then $*_\alpha G_\alpha$, equipped with ι_α, is a free product of $(G_\alpha)_{\alpha\in\mathcal{A}}$.*

Proof. Let H be a group, and let $\phi_\alpha : G_\alpha \to H$ be homomorphisms. Then there exists a unique homomorphism $\Psi : *_\alpha G_\alpha \to H$ so that $\Psi \circ \iota_\alpha = \phi_\alpha \forall \alpha \in \mathcal{A}$, namely the homomorphism given by $\Psi(g_1 g_2 \cdots g_m) = \Psi(\iota_{i_1}(g_1)\iota_{i_2}(g_2)\cdots \iota_{i_m}(g_m)) = \phi_{i_1}(g_1)\phi_{i_2}(g_2)\cdots\phi_{i_m}(g_m)$. Hence, $*_{\alpha\in\mathcal{A}} G_\alpha$ is a free product of $(G_\alpha)_{\alpha\in\mathcal{A}}$. $\square$

Theorem 4.10. *Free products are unique up to unique isomorphism.*

Proof. Free products are coproducts in the category of groups, and so uniqueness up to unique isomorphism follows. $\square$

4.4 Free Groups

Definition 4.11 (Universal property of free groups)**.** *Let S be a set. A group $F(S)$ equipped with a map $\iota : S \to F(S)$ is a free group generated by S if for any group H and any map $f : S \to H$, there exists a unique homomorphism $\Psi : F(S) \to H$ such that $\Psi \circ \iota = f$.*

Theorem 4.12. *Let S be a set and let $F(S)$ be the group consisting of reduced words in S, along with the usual inclusion map $\iota : S \to F(S)$. Then $F(S)$ is a free group generated by S.*

Proof. Let $f : S \to H$ be a map to a group H. There then exists a unique homomorphism $\Psi : F(S) \to H$ such that $\Psi\circ\iota = f$, namely the one given by $\Psi(s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_m^{\epsilon_m}) = \Psi(\iota(s_1)^{\epsilon_1}\iota(s_2)^{\epsilon_2}\cdots\iota(s_m)^{\epsilon_m}) = f(s_1)^{\epsilon_1} f(s_2)^{\epsilon_2}\cdots f(s_m)^{\epsilon_m}$. $\square$

Theorem 4.13. *Let S be a set and let $F(S)$, equipped with a map $\iota : S \to F(S)$, and $F(S)'$, equipped with a map $\iota' : S \to F(S)'$, be free groups generated by S. Then there exists a unique isomorphism $h : F(S) \to F(S)'$ such that $\iota' = h \circ \iota$.*

Proof. By the universal property of free groups, there exists a unique homomorphism $\Psi : F(S) \to F(S)'$ such that $\Psi \circ \iota = \iota'$, and similarly, there exists a unique homomorphism $\Psi' : F(S)' \to F(S)$ such that $\Psi' \circ \iota' = \iota$. Hence, $\iota' = (\Psi \circ \Psi') \circ \iota'$, and so, by uniqueness, $\Psi \circ \Psi' = \mathrm{id}_{F(S)'}$. Similarly, $\Psi' \circ \Psi = \mathrm{id}_{F(S)}$. Hence, Ψ is an isomorphism. $\square$

Theorem 4.14 (Every group has a presentation)**.** *Let G be a group. Then there exists a free group F and a normal subgroup $N \trianglelefteq F$ such that $G \cong F/N$.*

Proof. Let S be a generating set of G and let F be the free group generated by S, with $\iota : S \to F$ being the inclusion map. Let $f : S \to G$ be an inclusion map on sets. Then by the universal property of free groups, there exists a unique homomorphism $\Psi : F \to G$ such that $\Psi \circ \iota = f$. This in particular implies that $S \subseteq \text{Im } \Psi$, and hence that Ψ is surjective. The first isomorphism theorem then gives $G \cong F/\ker \Psi$, as required. $\square$

4.5 Abelianization

Definition 4.15 (Universal property of the abelianization of a group)**.** *Let G be a group. An abelianization of G is a pair (G^{ab}, π), where G^{ab} is an abelian group and $\pi : G \to G^{ab}$ is a group homomorphism, such that given any abelian group H and group homomorphism $\phi : G \to H$, there exists a unique group homomorphism $\phi^{ab} : G^{ab} \to H$ such that $\phi^{ab} \circ \pi = \phi$.*

Theorem 4.16. *Let G be a group, let $[G, G]$ be the commutator subgroup, and let $\pi : G \to G/[G, G]$ be the canonical quotient homomorphism. Then $(G/[G, G], \pi)$ is an abelianization of G.*

Proof. We simply need to show that the commutator subgroup $[G, G]$ is contained within the kernel of ϕ, since that will allow us to apply the universal property of the quotient group to obtain the unique group homomorphism $\phi^{\text{ab}} : G/[G, G] \to H$ such that $\phi^{\text{ab}} \circ \pi = \phi$. Indeed, let $x, y \in G$. Then $\phi([x, y]) = \phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = e$, so $[x, y] \in \ker \phi$. Hence, since $[G, G]$ is the subgroup generated by all commutators, it follows that $[G, G] \subseteq \ker \phi$, as required. $\square$

Theorem 4.17. *Let G be a group, and let (G^{ab}, π) and $(G^{ab'}, \pi')$ be two abelianizations of G. Then there exists a unique isomorphism $h : G^{ab} \to G^{ab'}$ such that $\pi' = h \circ \pi$.*

Proof. By the universal property of the abelianization of a group, there exists a unique group homomorphism $h : G^{\text{ab}} \to G^{\text{ab}'}$ such that $\pi' = h \circ \pi$. Similarly, there exists a unique group homomorphism $g : G^{\text{ab}'} \to G^{\text{ab}}$ such that $\pi = g \circ \pi'$. Hence, $\pi' = (h \circ g) \circ \pi'$. However, we also have $\pi' = \text{id}_{G^{\text{ab}'}} \circ \pi'$, and hence, by uniqueness, we have $h \circ g = \text{id}_{G^{\text{ab}'}}$. Similarly, $g \circ h = \text{id}_{G^{\text{ab}}}$, implying that h is an isomorphism. $\square$

Lemma 4.18. *Let $\phi : G \to H$ be a group homomorphism, and let $\pi_G : G \to G/[G, G]$, $\pi_H : H \to H/[H, H]$ be the quotient homomophisms of G and H into their respective abelianizations. Then there exists a group homomorphism $\phi^{ab} : G^{ab} \to H^{ab}$ given by $\phi^{ab}(g[G, G]) = \phi(g)[H, H] \forall g[G, G] \in G^{ab}$, which is the unique group homomorphism $\psi : G^{ab} \to H^{ab}$ such that $\psi \circ \pi_G = \pi_H \circ \phi$.*

Proof. Consider the map $\pi_H \circ \phi : G \to H^{\text{ab}}$. This is a map from a group to an abelian group, so by the universal property of the abelianization of a group, there exists a unique group homomorphism $\phi^{\text{ab}} : G^{\text{ab}} \to H^{\text{ab}}$ such that $\phi^{\text{ab}} \circ \pi_G = \pi_H \circ \phi$. That is, given any $g \in G$, we have

$$\phi^{\text{ab}}(g[G, G]) = \phi^{\text{ab}} \circ \pi_G(g) = \pi_H \circ \phi(g) = \phi(g)[H, H],$$

as required. $\square$

Theorem 4.19. *There is a covariant functor, called the abelianization functor, from the category of groups to the category of abelian groups.*

Proof. Let F be the functor which maps groups to their abelianizations, and group homomorphisms to group homomorphisms between the abelianizations of the groups. That is, given groups G and H and a group homomorphism $\phi : G \to H$, we have $F(G) = G^{\text{ab}}$ and $F(\phi) = \phi^{\text{ab}} : G^{\text{ab}} \to H^{\text{ab}} : g[G, G] \mapsto \phi(g)[H, H]$.

We first show that $F(\mathrm{id}_G) = \mathrm{id}_{F(G)}$ for any group G. Indeed, $F(\mathrm{id}_G)$ is given by $\mathrm{id}^{\mathrm{ab}}_{G^{\mathrm{ab}}} : G^{\mathrm{ab}} \to G^{\mathrm{ab}}$, where $\mathrm{id}^{\mathrm{ab}}_{G^{\mathrm{ab}}}(g[G,G]) = \mathrm{id}_G(g)[G,G] = g[G,G] \forall g \in G$. $F(\mathrm{id}_G)$ is then the identity map on $G^{\mathrm{ab}} = F(G)$, as required.

We now show that F preserves composition. Let G, H, K be groups and let $\phi : G \to H$ and $\psi : H \to K$ be group homomorphisms. Then

$$F(\phi) = \phi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H^{\mathrm{ab}} : g[G,G] \mapsto \phi(g)[H,H],$$

$$F(\psi) = \psi^{\mathrm{ab}} : H^{\mathrm{ab}} \to K^{\mathrm{ab}} : h[H,H] \mapsto \psi(h)[K,K]$$

and

$$F(\psi \circ \phi) = (\psi \circ \phi)^{\mathrm{ab}} : G^{\mathrm{ab}} \to K^{\mathrm{ab}} : g[G,G] \mapsto (\psi \circ \phi)(g)[K,K].$$

Hence,

$$\begin{aligned}
(\psi^{\mathrm{ab}} \circ \phi^{\mathrm{ab}})(g[G,G]) &= \psi^{\mathrm{ab}}(\phi(g)[H,H]) \\
&= \psi(\phi(g))[K,K] \\
&= (\psi \circ \phi)(g)[K,K] \\
&= (\psi \circ \phi)^{\mathrm{ab}}(g[G,G]),
\end{aligned}$$

as required. Hence, F is a covariant functor. $\square$