

Hatcher Solutions

Saxon Supple

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0 Some Underlying Geometric Notions

Exercise 0.1. *Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.*

Proof. We can model the torus as the disk $D^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ where we split the boundary into 4 equal arcs and identify opposite arcs under the quotient map $q : D^2 \rightarrow T$. Let $A := q(\partial D^2)$ be the subspace we wish to retract T onto. Without loss of generality, suppose that we remove the point $(0, 0)$. Given a point x , we want to send it to $\frac{x}{\|x\|}$ along the line segment from x to $\frac{x}{\|x\|}$. We thus define the homotopy

$$F : T \times I \rightarrow T : (q(x), t) \mapsto q\left(x(1-t) + \frac{x}{\|x\|}t\right).$$

This homotopes between the identity map and a retraction of T onto A while fixing all points of A throughout, and hence is a deformation retraction. \square

Exercise 0.2. *Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ onto S^{n-1} .*

Proof. We define a homotopy

$$F : \mathbb{R}^n \setminus \{0\} \times I \rightarrow S^{n-1} : (x, t) \mapsto x(1-t) + \frac{x}{\|x\|}t.$$

This homotopes between the identity map on $\mathbb{R}^n \setminus \{0\}$ and a retraction of $\mathbb{R}^n \setminus \{0\}$ onto S^{n-1} , and hence is a deformation retraction. \square

Exercise 0.3. (a) *Show that the composition of homotopy equivalences $X \rightarrow Y$ and $Y \rightarrow Z$ is a homotopy equivalence $X \rightarrow Z$. Deduce that homotopy equivalence is an equivalence relation.*

(b) *Show that the relation of homotopy among maps $X \rightarrow Y$ is an equivalence relation.*

(c) *Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.*

Proof. (a) Let $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ be homotopy equivalences. There then exist homotopy inverses $\phi_1 : Y \rightarrow X$ and $\psi_1 : Z \rightarrow Y$ of ϕ and ψ respectively.

$$(\phi_1 \circ \psi_1) \circ (\psi \circ \phi) = \phi_1 \circ (\psi_1 \circ \psi) \circ \phi \simeq \phi_1 \circ \phi \simeq \text{id}_X$$

and

$$(\psi \circ \phi) \circ (\phi_1 \circ \psi_1) = \psi \circ (\phi \circ \phi_1) \circ \psi_1 \simeq \psi \circ \psi_1 \simeq \text{id}_Z$$

so $\psi \circ \phi$ is a homotopy equivalence.

- (b) Let ϕ, ψ, ρ be continuous maps from X to Y . Clearly $\phi \simeq \phi$. Furthermore, if $F : X \times I \rightarrow Y$ is a homotopy from ϕ to ψ , then $G : X \times I \rightarrow Y : (x, t) \mapsto F(x, 1-t)$ is a homotopy from ψ to ϕ . Finally, let $A : X \times I \rightarrow Y$ be a homotopy from ϕ to ψ and let $B : X \times I \rightarrow Y$ be a homotopy from ψ to ρ . We define a map $C : X \times I \rightarrow Y$ by

$$C(x, t) = \begin{cases} A(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ B(x, 2t-1) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

$X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ are both closed subsets of $X \times I$ with $X \times [0, \frac{1}{2}] \cup X \times [\frac{1}{2}, 1] = X \times I$. Furthermore, $C|_{X \times [0, \frac{1}{2}]}$ and $C|_{X \times [\frac{1}{2}, 1]}$ are both continuous with respect to the induced topologies on $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ respectively. Hence, C is continuous on $X \times I$ and is thus a homotopy from ϕ to ρ .

- (c) Let $\phi : X \rightarrow Y$ be a homotopy equivalence with homotopy inverse $\psi : Y \rightarrow X$. Let $\phi_1 : X \rightarrow Y$ be homotopic to ϕ . Then

$$\phi_1 \circ \psi \simeq \phi \circ \psi \simeq \text{id}_Y$$

and

$$\psi \circ \phi_1 \simeq \psi \circ \phi \simeq \text{id}_X.$$

Hence ϕ_1 is a homotopy equivalence. □

Exercise 0.4. *A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t : X \rightarrow X$ such that $f_0 = \text{id}_X$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t . Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.*

Proof. Let $\iota : A \rightarrow X$ denote the inclusion $A \hookrightarrow X$. Define $\phi : X \rightarrow A : x \mapsto f_1(x)$, which is well-defined since $f_1(X) \subset A$. Then

$$\phi \circ \iota = f_1|_A \simeq f_0|_A = \text{id}_A.$$

This is a valid homotopy of maps $A \rightarrow A$ since $f_1(A) \subset A \forall t$. Furthermore,

$$\iota \circ \phi = f_1 \simeq \text{id}_X.$$

Hence ι is a homotopy equivalence with homotopy inverse ϕ . □

Exercise 0.5. *Show that if a space X deformation retracts to a point $x \in X$, then for each neighbourhood U of x in X there exists a neighbourhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.*

Proof. Without loss of generality, let U be an open neighbourhood of x . Define

$$F : X \times I \rightarrow X : (y, t) \mapsto f_t(y)$$

and let $A := F^{-1}(U)$. A is an open set containing the slice $\{x\} \times I$, so by the Tube Lemma there exists a tube $V \times I \subset X \times I$, where V is open and contains x . Furthermore, $V \times I \subset A$, so V consists only of points in U (since $V \times \{0\} \subset U \times \{0\}$) which stay in U throughout the homotopy f_t . Let $\iota : V \hookrightarrow U$ be the inclusion map $V \hookrightarrow U$. Then $f_t : X \rightarrow X$ restricts to a homotopy $f_t|_V : V \rightarrow U$ and hence $\iota \simeq f_1|_V$, where $f_1|_V$ is the constant map, so ι is nullhomotopic. □

Exercise 0.6. (a) Let X be the subspace of \mathbb{R}^2 consisting of the horizontal segment $[0, 1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1 - r]$ for r a rational number in $[0, 1]$. Show that X deformation retracts to any point in the segment $[0, 1] \times \{0\}$, but not to any other point.

(b) Let Y be the subspace of \mathbb{R}^2 that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.



(c) Let Z be the zigzag subspace of Y homeomorphic to \mathbb{R} indicated by the heavier line. Show there is a deformation retraction in the weak sense of Y onto Z , but no true deformation retraction.

Proof. (a) We define a homotopy

$$f_t : X \rightarrow X : (a, b) \mapsto (a, b(1 - t)).$$

f_t fixes $[0, 1] \times \{0\}$, $f_0 = \text{id}_X$ and f_1 is a retraction from X to $[0, 1] \times \{0\}$, so f_t is a deformation retraction. Hence $[0, 1] \times \{0\}$ is a deformation retract of X . Any point in $[0, 1] \times \{0\}$ is then clearly a deformation retract of $[0, 1] \times \{0\}$, so X deformation retracts to any point in $[0, 1] \times \{0\}$ by transitivity. Now suppose X deformation retracts to some other point $x \in X$. We can then find a neighbourhood U of x of the form $B_\epsilon(x) \cap X$ for some $\epsilon > 0$, which does not contain $[0, 1] \times \{0\}$. However, U is not path-connected, so there does not exist a neighbourhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic; a contradiction.

(b) Define a homotopy $f_t : Y \rightarrow Y$ where $f_t(y)$ sends y a distance of t to the right along Y . Then $f_0 = \text{id}_Y$ and $f_1(Y) = Z$, where Z is the zigzag subspace of Y . Z is contractible, being homeomorphic to \mathbb{R} , and hence there exists a homotopy $h_t : Z \rightarrow Z$ from id_Z to a constant function. We can then define a homotopy $F_t : Y \rightarrow Y$ of id_Y to a constant map by

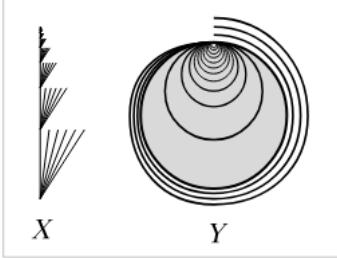
$$F_t(y) = \begin{cases} f_{2t}(y) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ h_{2t-1} \circ f_1(y) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

Now suppose Y deformation retracts onto a point y . We can then find a sufficiently small open neighbourhood U of y such that for every neighbourhood $V \subset U$ of y , U is not path-connected, and hence the inclusion map $V \hookrightarrow U$ is not nullhomotopic; a contradiction. Hence Y does not deformation retract onto a point y .

(c) Since Z deformation retracts onto a point, if there were a deformation retract of Y onto Z , then there would be a deformation retract of Y onto a point. However, (b) showed that this is not the case, and hence Y does not deformation retract onto Z . However, the homotopy f_t defined in (b) is a weak deformation retract of Y onto Z . □

Exercise 0.7. Fill in the details in the following construction from [Edwards 1999] of a compact space $Y \subset \mathbb{R}^3$ with the same properties as the space Y in Exercise 6, that is, Y is contractible but does not deformation retract to any point. To begin, let X be the union of an infinite sequence of cones on the Cantor set arranged end-to-end, as in the figure. Next, form the one-point compactification of $X \times \mathbb{R}$. This embeds in \mathbb{R}^3 as a closed disk with curved ‘fins’ attached along circular arcs, and with

the one-point compactification of X as a cross-sectional slice. The desired space Y is then obtained from this subspace of \mathbb{R}^3 by wrapping one more cone on the Cantor set around the boundary of the disk.



Exercise 0.8. For $n > 2$, construct an n -room analog of the house with two rooms.

Exercise 0.9. Show that a retract of a contractible space is contractible.

Proof. Let X be a contractible space with A a retract of X . Let $f_t : X \rightarrow X$ be a homotopy from id_X to a constant map, and let $r : X \rightarrow A$ be a retraction of X onto A . $r \circ f_0|_A = \text{id}_A$ and $r \circ f_1|_A$ is constant due to $f_1|_A$ being constant. Hence id_A is homotopic to a constant map so A is contractible. \square

Exercise 0.10. Show that a space X is contractible iff every map $f : X \rightarrow Y$, for arbitrary Y , is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \rightarrow X$ is nullhomotopic.

Proof. (\Leftarrow): Let $Y = X$ and $f = \text{id}_X$. Then id_X is nullhomotopic so X is contractible.

(\Rightarrow): Let $g_t : X \rightarrow X$ be a homotopy from id_X to a constant map. Then define a homotopy $h_t : X \rightarrow Y$ by $h_t(x) = f \circ g_t(x)$. This is a homotopy between f and a constant map, so f is nullhomotopic.

(\Leftarrow): Again, let $Y = X$ and $f = \text{id}_X$.

(\Rightarrow): $g_t \circ f$ is a homotopy from f to a constant map so f is nullhomotopic. \square

Exercise 0.11. Show that $f : X \rightarrow Y$ is a homotopy equivalence if there exist maps $g, h : Y \rightarrow X$ such that $fg \simeq \mathbb{1}$ and $hf \simeq \mathbb{1}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Proof. Let $\phi = h \circ f \circ g : Y \rightarrow X$. Then

$$f \circ \phi = f \circ (h \circ f \circ g) = f \circ (h \circ f) \circ g \simeq f \circ g \simeq \mathbb{1}$$

and

$$\phi \circ f = (h \circ f \circ g) \circ f = h \circ (f \circ g) \circ f \simeq h \circ f \simeq \mathbb{1}$$

so ϕ is a homotopy inverse of f so f is a homotopy equivalence.

Let g_1 and h_1 be the homotopy inverses of fg and hf respectively. We then have that $f(gg_1) \simeq \mathbb{1}$ and $(h_1h)f \simeq \mathbb{1}$, and hence f is a homotopy equivalence with homotopy inverse h_1hfgg_1 . \square

Exercise 0.12. Show that a homotopy equivalence $f : X \rightarrow Y$ induces a bijection between the set of path-components of X and the set of path-components of Y , and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y . Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X .

Proof. Let \sim_X be the equivalence relation on X given by $a \sim_X b$ if and only if a and b are in the same path-component. Similarly define the relation \sim_Y on Y . Let $g : Y \rightarrow X$ be a homotopy inverse of f . We let ϕ_t and ψ_t be homotopies from id_Y to $f \circ g$ and from id_X to $g \circ f$ respectively.

We first show that f induces a well-defined map on path components.

Let $x_0 \sim_X x_1$. Let $\phi : I \rightarrow X$ be a path from x_0 to x_1 . Then $f \circ \phi : I \rightarrow Y$ is a path from $f(x_0)$ to $f(x_1)$ so $f(x_0) \sim_Y f(x_1)$.

We now show that f induces a surjective map on path components.

Let $y \in Y$. Then $y = \phi_0(y) \sim_Y \phi_1(y) = f(g(y))$, and hence y is in the same path component as a point in the image of f .

We finally show that f induces an injective map on path components.

Suppose that $f(x_0) \sim_Y f(x_1)$. Then $g \circ f(x_0) \sim_X g \circ f(x_1)$ and hence $\psi_1(x_0) \sim_X \psi_1(x_1)$. We also have that $x_0 = \psi_0(x_0) \sim_X \psi_1(x_0)$ and $x_1 = \psi_0(x_1) \sim_X \psi_1(x_1)$, and so by transitivity

$$x_0 \sim_X \psi_1(x_0) \sim_X \psi_1(x_1) \sim_X x_1 \implies x_0 \sim_X x_1.$$

Hence f induces a bijection on path-components.

Let $[x]_X$ be the path component of some point $x \in X$ and let $[f(x)]_Y$ be the path component in Y of $f(x)$. f then restricts to a map

$$f|_{[x]_X} : [x]_X \rightarrow [f(x)]_Y.$$

Furthermore, $g \circ f(x) \sim_X x$, and hence g also restricts to a map

$$g|_{[f(x)]_Y} : [f(x)]_Y \rightarrow [x]_X.$$

We can then also restrict the homotopies ϕ_t and ψ_t to

$$\phi_t|_{[f(x)]_Y} : [f(x)]_Y \rightarrow [f(x)]_Y$$

and

$$\psi_t|_{[x]_X} : [x]_X \rightarrow [x]_X$$

respectively, since $\forall x_0 \in [x]_X$ we have $\psi_t(x_0) \sim_X x_0 \forall t$, and $\forall y_0 \in [f(x)]_Y$ we have $\phi_t(y_0) \sim_Y y_0 \forall t$. Hence $f|_{[x]_X}$ is a homotopy equivalence between $[x]_X$ and $[f(x)]_Y$.

Now let \sim_X and \sim_Y be the corresponding relations for connected components.

We first show that f induces a well-defined map on connected components. Let $x_0 \sim_X x_1$. Let Z be a connected subspace containing x_0 and x_1 . The image of a connected space is connected, and hence $f(x_0) \sim_Y f(x_1)$, as both are contained in $f(Z)$.

We next show that f induces a surjective map on connected components. Let $y \in Y$. Let $[y]_Y$ be the connected component of y . As shown earlier, y is in the same path component as $f \circ g(y)$, and hence is in the same connected component as $f \circ g(y)$. Hence, $[y]_Y$ is in the image of the map on connected components induced by f .

We finally show that f induces an injective map on connected components. Let $f(x_0) \sim_Y f(x_1)$. Then $g \circ f(x_0) \sim_X g \circ f(x_1)$. We then have that x_0 is in the same path-component as $g \circ f(x_0)$ and x_1 is in the same path-component as $g \circ f(x_1)$, and hence that x_0 is in the same connected component as $g \circ f(x_0)$ and x_1 is in the same connected component as $g \circ f(x_1)$. Hence by transitivity

$$x_0 \sim_X g \circ f(x_0) \sim_X g \circ f(x_1) \sim_X x_1 \implies x_0 \sim_X x_1.$$

Hence f induces a bijection on connected components.

Let $[x]_X$ be the connected component of some point $x \in X$ and let $[f(x)]_Y$ be the connected component of $f(x)$ in Y . f then restricts to a map

$$f|_{[x]_X} : [x]_X \rightarrow [f(x)]_Y.$$

Furthermore, $g \circ f(x) \sim_X x$, and hence g also restricts to a map

$$g|_{[f(x)]_Y} : [f(x)]_Y \rightarrow [x]_X.$$

We can then also restrict the homotopies ϕ_t and ψ_t to

$$\phi_t|_{[f(x)]_Y} : [f(x)]_Y \rightarrow [f(x)]_Y$$

and

$$\psi_t|_{[x]_X} : [x]_X \rightarrow [x]_X$$

respectively, since $\forall x_0 \in [x]_X$ we have $\psi_t(x_0) \sim_X x_0 \forall t$, and $\forall y_0 \in [f(x)]_Y$ we have $\phi_t(y_0) \sim_Y y_0 \forall t$. Hence $f|_{[x]_X}$ is a homotopy equivalence between $[x]_X$ and $[f(x)]_Y$.

Finally, suppose that the connected components of a space X coincide with its path-components, and let Y be homotopy equivalent to X . Let $f : X \rightarrow Y$ be a homotopy equivalence. Let

$$f_\# : \{\text{Path/connected components of } X\} \rightarrow \{\text{Path components of } Y\}$$

and

$$f_* : \{\text{Path/connected components of } X\} \rightarrow \{\text{Connected components of } Y\}$$

be the maps induced by f . We then have a bijection between path-components of Y and connected components of Y given by $f_* \circ f_\#^{-1}$. Furthermore, given any path-component $A \subseteq Y$, $f_* \circ f_\#^{-1}(A)$ is the connected component containing A . Hence there must be at most one path-component per connected component, since otherwise multiple path-components would be mapped to the same connected component by $f_* \circ f_\#^{-1}$, and it would cease to be injective. Hence the connected components of Y coincide with its path-components. \square

Exercise 0.13. Show that any two deformation retractions r_t^0 and r_t^1 of a space X onto a subspace A can be joined by a continuous family of deformation retractions r_t^s , $0 \leq s \leq 1$, of X onto A , where continuity means that the map $X \times I \times I \rightarrow X$ sending (x, s, t) to $r_t^s(x)$ is continuous.

Proof. We know what r_t^s is on $X \times \{0, 1\} \times I \cup A \times I \times I \cup X \times I \times \{0\}$. Define r_t^s by

$$r_t^s(x) = \begin{cases} r_t^0 \circ r_{2ts}^1(x) & \text{if } 0 \leq s < \frac{1}{2}, \\ r_{2t(1-s)}^0 \circ r_t^1(x) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

If $s = 0$, then $r_t^0 = r_t^0 \circ r_0^1 = r_t^0$.

If $s = 1$, then $r_t^1 = r_0^0 \circ r_t^1 = r_t^1$.

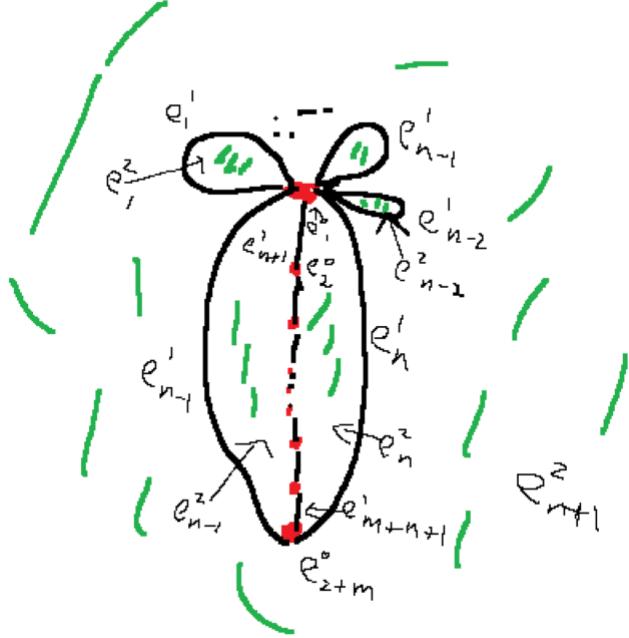
If $a \in A$, then $r_t^s(a) = a \forall s, t$.

If $t = 0$, then $r_0^s = \text{id}_X \forall s$.

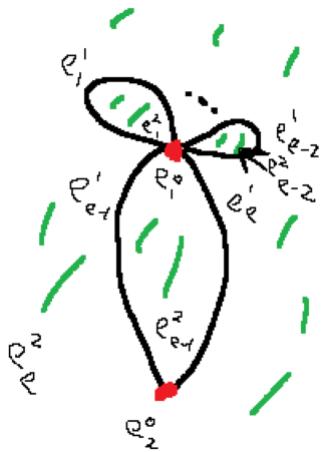
If $t = 1$, then $r_1^0(x) \in A \forall x$ and $r_1^1(x) \in A \forall x$, and hence $r_1^s(x) \in A \forall x$. \square

Exercise 0.14. Given positive integers v, e and f satisfying $v - e + f = 2$, construct a cell structure on S^2 having v 0-cells, e 1-cells, and f 2-cells.

Proof. For $(v, e, f) = (2 + m, m + n + 1, n + 1)$ with $m > 0, n > 1$, we can use the following construction: Begin with 0-cells e_1^0, \dots, e_{2+m}^0 . Then attach $n - 2$ loops e_1^1, \dots, e_{n-2}^1 to e_1^0 , and add two 1-cells e_{n-1}^1 and e_n^1 , each with one endpoint attached to e_1^0 and the other endpoint attached to e_{2+m}^0 . Then add 1-cells $e_{n+1}^1, \dots, e_{n+1+m}^1$, where e_{n+i}^1 is attached to e_i^0 and e_{i+1}^0 . Then add 2-cells e_1^2, \dots, e_{n+1}^2 , where the first $n - 2$ are attached to the $n - 2$ loops, the next two are attached to $e_1^0 \cup \dots \cup e_{2+m}^0 \cup e_{n-1}^1 \cup e_{n+1}^1 \cup \dots \cup e_{n+1+m}^1$ and $e_1^0 \cup \dots \cup e_{2+m}^0 \cup e_n^1 \cup e_{n+1}^1 \cup \dots \cup e_{n+1+m}^1$ respectively. Finally, attach the last 2-cell to $e_1^0 \cup e_2^0 \cup e_1^1 \cup \dots \cup e_n^1$.



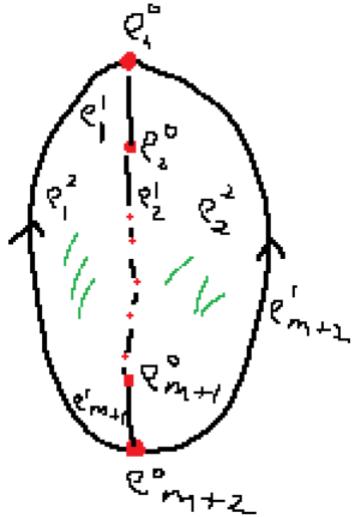
If $v = 2$, then $e = f$. First suppose $e > 1$. We can then form S^2 by starting with e_1^0 and e_2^0 , and then attaching $e - 2$ loops to e_1^0 and two 1-cells each connecting the two 0-cells, and finally attaching e 2-cells.



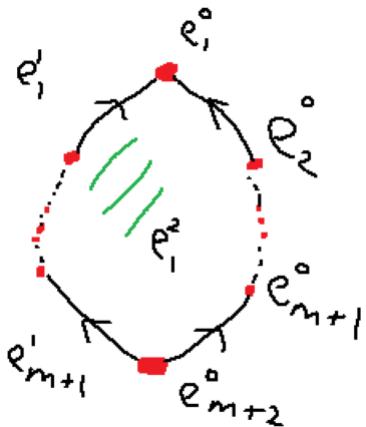
The case $(v, e, f) = (2, 1, 1)$ is then just the standard construction of starting with two points, attaching two lines joining them to form a circle, filling in the centre of the circle, and then identifying

the two lines.

Now consider the case $(v, e, f) = (2 + m, m + 2, 2)$. That is, the case where $n = 1$. The following diagram works:



For the case $(v, e, f) = (2 + m, m + 1, 1)$, which corresponds to $n = 0$, the following diagram works:



For the case $(v, e, f) = (1, x, x + 1)$, the following diagram works:



□

Exercise 0.15. Enumerate all the subcomplexes of S^∞ , with the cell structure on S^∞ that has S^n as its n -skeleton.

Proof. $X^n = S^n$ is a subcomplex of S^∞ for every n . Also, for each $X^n = S^n$, $X^n \cup e_1^{n+1}$ and $X^n \cup e_2^{n+1}$ are subcomplexes. It is not possible to have a subcomplex formed by a proper subset of X^n and some $\emptyset \neq \phi \subseteq X^{n+i} \setminus X^n$ for some $i \geq 1$, since ϕ attaches to the whole of X^n , and hence the subcomplex, being closed, would contain X^n ; a contradiction. Hence, if a subcomplex contains an i -cell for $i > 0$, it must also contain X^{i-1} . Hence the only possible subcomplexes are:

$$\emptyset, S^\infty, e_1^0, e_2^0, S^{i-1} \cup e_1^i, S^{i-1} \cup e_2^i \text{ for } i > 0, \text{ and } S^n \text{ for } n \geq 0.$$

□

Exercise 0.16. Show that S^∞ is contractible.

Proof. For $k > 0$, each k -skeleton $X^k \subseteq S^\infty$ is $S^{k-1} \cup e_1^k \cup e_2^k$, where $S^{k-1} \cup e_1^k$ and $S^{k-1} \cup e_2^k$ are both the disk D^k . Furthermore, for $k > 1$, each $S^{k-1} \cup e_i^k$ can be deformation retracted into either $X^{k-2} \cup e_1^{k-1}$ or $X^{k-2} \cup e_2^{k-1}$. We can then repeat this recursively until we reach $X^0 \cup e_i^1$, which we can then deformation retract to a single point in X^0 . For $k > 1$, let

$$F_k : S^{k-1} \cup e_1^k \times I \rightarrow S^{k-1} \cup e_1^k$$

be a deformation retraction of $S^{k-1} \cup e_1^k$ onto $X^{k-2} \cup e_1^{k-1}$, and for $k = 1$, let

$$F_1 : X^0 \cup e_1^1 \times I \rightarrow X^0 \cup e_1^1$$

be a deformation retraction of $X^0 \cup e_1^1$ onto e_1^0 . Then, given an $x \in S^\infty$, let n be the smallest positive integer such that $x \in X^{n-1} \cup e_1^n$. We then define a homotopy h_t between id_{S^∞} and a constant map by

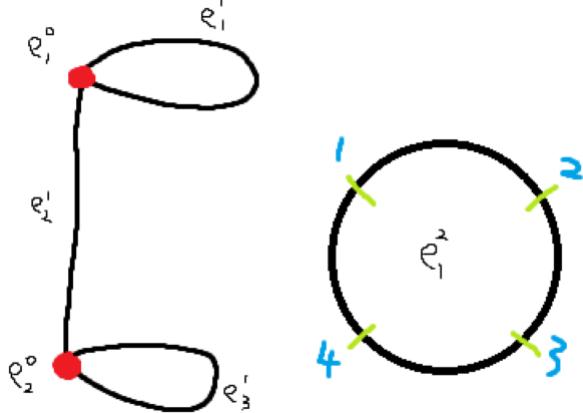
$$h_t(x) = \begin{cases} x & \text{if } 0 \leq t < \frac{1}{2^{n+1}}, \\ F_n(x, 2^n(t - \frac{1}{2^{n+1}})) & \text{if } \frac{1}{2^{n+1}} \leq t < \frac{1}{2^n}, \\ F_{n-1}(F_n(x, 1), 2^{n-1}(t - \frac{1}{2^n})) & \text{if } \frac{1}{2^n} \leq t < \frac{1}{2^{n-1}}, \\ \vdots & \vdots \\ F_1(F_2(\dots F_n(x, 1), 1), 2(t - \frac{1}{2})) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since CW complexes have the weak topology with respect to their skeleta, a map is continuous if and only if its restriction to each skeleton is continuous. This is clearly the case for h_t , and hence h_t is continuous. \square

Exercise 0.17. (a) Show that the mapping cylinder of every map $f : S^1 \rightarrow S^1$ is a CW complex.

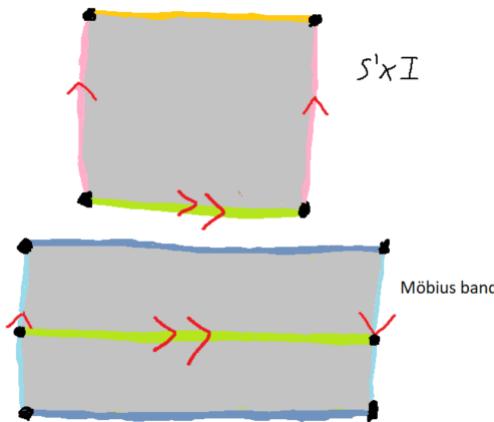
(b) Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

Proof. (a) Observe the following diagram:



We begin with a 1-skeleton consisting of $e_1^0, e_2^0, e_1^1, e_2^1, e_3^1$. We then introduce a 2-cell e_1^2 . Let $\phi : S^1 \rightarrow X^1$ be the attaching map of e_1^2 . We define ϕ as follows: Identify the arc from 1 to 2 with $e_1^0 \cup e_1^1$. Then identify the arc from 2 to 3 with $e_1^0 \cup e_2^1 \cup e_2^0$. Then identify the arc from 1 to 4 with $e_1^0 \cup e_2^1 \cup e_2^0$. Finally, let A be the arc from 3 to 4. We then have homeomorphisms $g : A/3 \sim 4 \rightarrow S^1$ and $h : S^1 \rightarrow e_3^1 \cup e_2^0$ such that $h \circ f \circ g(3) = e_2^0$, and hence we can attach A to e_3^1 under the identification $x \sim h \circ f \circ g(x) \forall x \in A$. This then gives a cell complex that is homeomorphic to M_f .

(b) We can form a CW complex out of gluing $S^1 \times I$ to the midcircle of the Möbius band, as in the diagram below:



We can then deformation retract $S^1 \times I$ onto the midcircle of the Möbius band, and we can

deformation retract the Möbius band onto $S^1 \times I$ by deformation retracting it onto its midcircle, which is identified with $S^1 \times \{0\}$.

□

Exercise 0.18. Show that $S^1 * S^1 = S^3$, and more generally $S^m * S^n = S^{m+n+1}$.

Proof. We can represent S^1 simply as $\{\theta : 0 \leq \theta < 2\pi\}$, and hence $S^1 \times S^1 \times I \cong \{(\theta_0, \theta_1, r) : 0 \leq \theta_0, \theta_1 < 2\pi, 0 \leq r \leq 1\}$. Then, applying the identifications gives

$$S^1 * S^1 \cong \{(\theta_0, \theta_1, r) : 0 \leq \theta_0, \theta_1 < 2\pi, 0 \leq r \leq 1\} / \sim,$$

where

$$(\theta_0, \theta_1, 0) \sim (\theta_0, \theta'_1, 0) \forall \theta_0, \theta_1, \theta'_1 \text{ and } (\theta_0, \theta_1, 1) \sim (\theta'_0, \theta_1, 1) \forall \theta_0, \theta'_0, \theta_1.$$

Now, S^3 is the set of pairs of complex numbers (z_0, z_1) such that the squares of their norms sum to 1. Representing them in polar coordinates then gives $z_0 = r_0 e^{i\theta_0}$ and $z_1 = r_1 e^{i\theta_1}$, where $r_0^2 + r_1^2 = 1$, $0 \leq r_0, r_1 \leq 1$ and $0 \leq \theta_0, \theta_1 < 2\pi$. r_1 determines $r_0 = \sqrt{1 - r_1^2}$, so (z_0, z_1) can be associated with a real triple $(\theta_0, \theta_1, r_1)$. However, in order to create a bijection, we need to make the following identifications:

$$(\theta_0, \theta_1, 0) \sim (\theta_0, \theta'_1, 0) \forall \theta_0, \theta_1, \theta'_1$$

and

$$(\theta_0, \theta_1, 1) \sim (\theta'_0, \theta_1, 1) \forall \theta_0, \theta'_0, \theta_1.$$

These are exactly the identifications which give $S^1 * S^1$, and hence $S^1 * S^1 = S^3$.

For the general case:

Recall that $SS^n = S^{n+1}$, and hence note that $S^n * S^0 = SS^n = S^{n+1}$, which implies that

$$S^n = S^0 * \dots * S^0 \text{ (} n+1 \text{ times)}$$

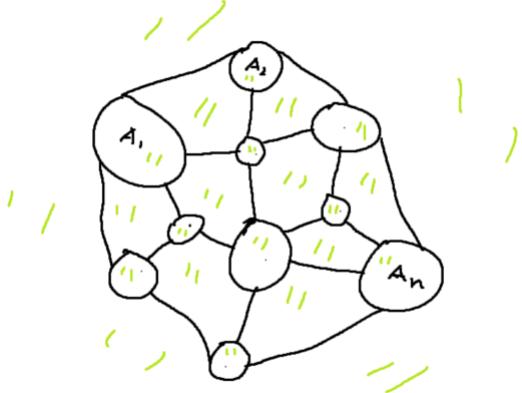
Thus

$$S^m * S^n = (S^0 * \dots * S^0) * (S^0 * \dots * S^0) = S^0 * \dots * S^0 \text{ (} n+1+m+1 \text{ times)} = S^{m+n+1}.$$

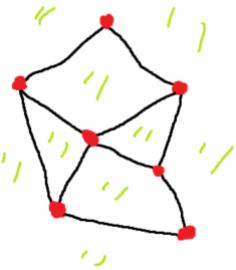
□

Exercise 0.19. Show that the space obtained from S^2 by attaching n 2-cells along any collection of n circles in S^2 is homotopy equivalent to the wedge sum of $n+1$ 2-spheres.

Proof. Let A be the space obtained from S^2 by attaching n 2-cells. We can give A a cell-complex structure as follows: We start with n discs A_1, \dots, A_n , and we then draw lines between them such that the outer circles form part of a perimeter, and every region within the perimeter includes at most one circle A_i . Then attach 2-cells to fill in the area within the perimeter, and another 2-cell to the whole perimeter. This is shown in the following diagram:

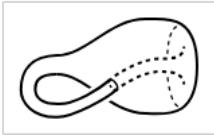


We can then attach n 2-cells to the boundaries of the regions A_1, \dots, A_n to get a space homeomorphic to A . Each A_i is a contractible subcomplex, and hence we can collapse each A_i to a point to obtain a space B homotopy-equivalent to A , which consists of a sphere S^2 with n other spheres attached to it at different points. We can then model B as a cell-complex by starting with the points where the spheres are attached to the first sphere as 0-cells, then adding 1-cells to form a connected graph, then attaching 2-cells to form a sphere, and then attaching n more 2-cells to each 0-cell, as shown in the following diagram:

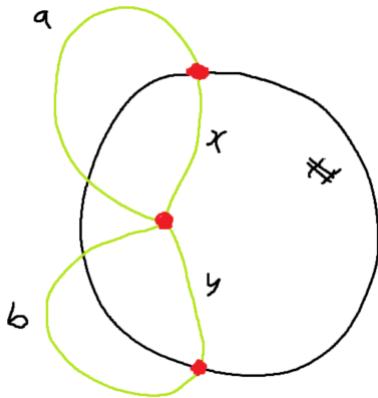


We can then repeatedly collapse each pair of vertices connected by an edge to a point, so as to end up with $\bigvee_{i=1}^{n+1} S^2$, which will be homotopy equivalent to A . \square

Exercise 0.20. Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.



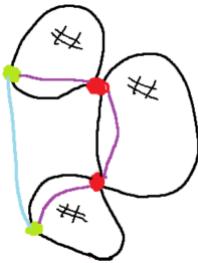
Proof. First, we can collapse the disc whose boundary is the circle of intersection to a point, to obtain the 2-sphere with 3 distinct points identified. This is then homotopy equivalent to the diagram below, since we can collapse both segments a and b to obtain a sphere with three distinct points identified.



However, we can also collapse the segments x and y instead, which then gives $S^1 \vee S^1 \vee S^2$, as required. \square

Exercise 0.21. *If X is a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy equivalent to a wedge sum of S^1 's and S^2 's.*

Proof. Firstly, we can represent the spheres as a simple connected graph, where each vertex represents a sphere, and two vertices being connected by an edge represents two spheres intersecting at a point. Let T be a spanning tree of the graph. We can construct a CW complex of X as follows: For each sphere X_i , let A_i be the set of points of intersection of X_i with other spheres which are connected to X_i by an edge in T , and let B_i be the set of points of intersection of X_i with other spheres which are connected to X_i by an edge not in T . Let C_i be a copy of B_i , so that $|\bigcup_{i=1}^n C_i| = |\bigcup_{i=1}^n B_i|$. We then form a 0-skeleton consisting of 0-cells which correspond to the points in $A_1 \cup \dots \cup A_n$ and $C_1 \cup \dots \cup C_n$. For each i , we connect all the points in $A_i \cup C_i$ by 1-cells, so as to form a simple acyclic graph, which we shall denote as G_i . We then connect each pair of points in $C_i \cup C_j$ which correspond to a single point in B_i with a 1-cell. Finally, we attach a 2-cell to each G_i , which then gives a space that is homotopy equivalent to X ; for we can collapse the 1-cells connecting the points in $C_1 \cup \dots \cup C_n$ to obtain all the intersections. This is illustrated in the following diagram, where the black circles represent 2-spheres, the red dots represent points in $A_1 \cup \dots \cup A_n$, the purple lines represent G_i 's, and the blue line represents a 1 cell connecting to points in $C_1 \cup \dots \cup C_n$.



If we instead collapse each G_i to a point, we then obtain a wedge sum of n 2-spheres, along with the loops corresponding to the 1-cells connecting the points in $C_1 \cup \dots \cup C_n$, which are now all the same point. \square

Exercise 0.22. *Let X be a finite graph lying in a half-plane $P \subseteq \mathbb{R}^3$ and intersecting the edge of P*

in a subset of the vertices of X . Describe the homotopy type of the 'surface of revolution' obtained by rotating X about the edge line of P .

Proof. First, assume that X is connected. Without loss of generality, we can assume that none of the edges of X intersect, for if they do, then we can simply add a vertex at the point of intersection. We can also assume without loss of generality that there are no edges connecting two vertices on ∂P , for if there is one, then we can add a vertex in the middle of the edge. We first consider three subgraphs of X :

1. The first, which we shall denote as T , is a spanning forest of the collection of vertices of X which do not lie in ∂P .
2. The second, which we shall denote as A , consists of all the edges of X which are not part of T , and which do not have an endpoint on ∂P , along with their endpoints.
3. The third, which we shall denote as B , consists of all vertices on ∂P , all vertices of X which are connected to a vertex on ∂P by an edge, and all edges which have an endpoint in ∂P .

Let T_1, \dots, T_n be the spanning trees which comprise T . Let V_{T_i} be the subcomplex of T_i formed by all the vertices of T_i , let v_i be a vertex in T_i that is connected to a vertex in ∂P by an edge, let V_A be all the vertices of A , and let $V_B := \{w_1, \dots, w_m\}$ be all vertices in B which do not lie on ∂P . If we define a map $f : V_A \sqcup V_B \rightarrow T$ which maps the vertices of $V_A \sqcup V_B$ to their appropriate vertices in T , we then have that $X \cong T \sqcup_f (A \sqcup B)$. We can then define a map $g : V_A \sqcup V_B \rightarrow T$ which maps every vertex of $V_A \sqcup V_B$ to some v_i . Clearly $f \simeq g$. Let

$$F : (V_A \sqcup V_B) \times I \rightarrow T$$

be a homotopy from f to g . We can then obtain a homotopy

$$G : (V_A \sqcup V_B) \times S^1 \times I \rightarrow T \times S^1 : (v, \omega, t) \mapsto (F(v, t), \omega).$$

Continuity of G follows from the universal property of the product topology, for

$$\pi_T \circ G : (V_A \sqcup V_B) \times S^1 \times I \rightarrow T : (v, \omega, t) \mapsto F(v, t)$$

and

$$\pi_{S^1} \circ G : (V_A \sqcup V_B) \times S^1 \times I \rightarrow S^1 : (v, \omega, t) \mapsto \omega$$

are both continuous. Define

$$\alpha : (V_A \sqcup V_B) \times S^1 \rightarrow T \times S^1 : (v, \omega) \mapsto G(v, \omega, 0)$$

and

$$\beta : (V_A \sqcup V_B) \times S^1 \rightarrow T \times S^1 : (v, \omega) \mapsto G(v, \omega, 1).$$

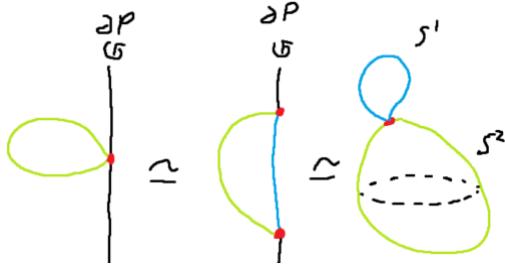
Let Y be the resulting space after rotating X about ∂P , which is homeomorphic to

$$(X \times S^1) / \sim,$$

where \sim is given by $(v, \omega) \sim (v, \omega') \forall \omega, v \in X^0 \cap \partial P$. Then if we define a relation \sim_B on $(A \sqcup B) \times S^1$ given by $(v, \omega) \sim_B (v, \omega') \forall v \in \partial P, \omega \in S^1$, we have

$$\begin{aligned} Y &\cong (T \times S^1) \sqcup_\alpha (((A \sqcup B) \times S^1) / \sim_B) \\ &\simeq (T \times S^1) \sqcup_\beta (((A \sqcup B) \times S^1) / \sim_B) \\ &\simeq (\{v_1, \dots, v_n\} \times S^1) \sqcup_{r \circ \beta} (((A \sqcup B) \times S^1) / \sim_B), \end{aligned}$$

where the last homotopy equivalence was obtained by deformation retracting $T \times S^1$ to $\{v_1, \dots, v_n\} \times S^1$, and letting $r : T \times S^1 \rightarrow \{v_1, \dots, v_n\} \times S^1$ be a retraction. Let q be the number of edges in A . Then the joining of A to $\{v_1, \dots, v_n\} \times S^1$ amounts to attaching q 2-tori onto $\{v_1, \dots, v_n\} \times S^1$ around a longitude. Finally, for each i , we choose a subspace B_i which consists of $\{v_i\} \times S^1$, a vertex in ∂P which is connected to v_i by a disk, and the connecting disk itself. Each B_i is a contractible subcomplex, and hence we can collapse it to a point. Then all the disks formed from the edges of B will become 2-spheres or horn tori, and the tori formed from the edges of A will also become horn tori. Moreover, each horn torus, being formed by a loop with a basepoint on ∂P being rotated about ∂P , is homeomorphic to a semicircle (with the line of the semicircle lying on ∂P) being rotated around ∂P and then having the line collapsed to a point. That is in turn homotopy equivalent to $S^2 \vee S^1$, for we can move one of the attaching points of the line to the other point. This is illustrated in the diagram below:



We finish by repeating the following procedure: We choose a pair of vertices on ∂P which are the endpoints of a sphere, and we choose one such sphere and collapse a contractible arc on the sphere connecting those two vertices. This will maintain the homotopy type of the sphere, while identifying two points on the other such spheres, which then gives them the homotopy types of $S^2 \vee S^1$. We repeat this until every vertex is identified with every other, which is possible, due to X being connected. The end result is a wedge sum of circles and spheres. \square

Exercise 0.23. Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

Proof. Let X be a CW complex which is a union of contractible subcomplexes A and B , where $A \cap B$ is also contractible. Then $X \simeq X/A \cong B/(A \cap B) \simeq B \simeq \{\text{pt}\}$.

We can observe the homeomorphism $X/A \cong B/(A \cap B)$ as follows:

Let $a \in A \cap B$. Let $q_X : X \rightarrow X/A$ and $q_B : B \rightarrow B/(A \cap B)$ be the quotient maps, and define $f : X/A \rightarrow B/(A \cap B)$ by

$$f([x]_X) = \begin{cases} [a]_B & \text{if } x \in A, \\ [x]_B & \text{if } x \in B. \end{cases}$$

Next, define $g : X \rightarrow B/(A \cap B)$ by

$$g(x) = \begin{cases} [a]_B & \text{if } x \in A, \\ [x]_B & \text{if } x \in B. \end{cases}$$

These maps are both well-defined, for if $x \in A \cap B$, then $[x]_B = [a]_B$. Furthermore, g is continuous by the pasting lemma, and $g = f \circ q_X$, so f is continuous by the universal property of the final topology.

Now define $f^{-1} : B/(A \cap B) \rightarrow X/A$ by

$$f^{-1}([x]_B) = \begin{cases} [a]_X & \text{if } x \in A, \\ [x]_X & \text{if } x \in B. \end{cases}$$

and define $h : B \rightarrow X/A$ by

$$h(x) = \begin{cases} [a]_X & \text{if } x \in A, \\ [x]_X & \text{if } x \in B. \end{cases}$$

Again, these maps are well-defined, for if $x \in A \cap B$, then $[x]_X = [a]_X$. Furthermore, h is continuous, due to simply being the composition of q_X with the inclusion map $B \hookrightarrow X$. Hence, f^{-1} is continuous by the universal property of the final topology, for $h = f^{-1} \circ q_B$. f and f^{-1} are two-sided inverses, and hence f is a homeomorphism. \square

Exercise 0.24. Let X and Y be CW complexes with 0-cells x_0 and y_0 . Show that the quotient spaces $X * Y / (X * \{y_0\} \cup \{x_0\} * Y)$ and $S(X \wedge Y) / S(\{x_0\} \wedge \{y_0\})$ are homeomorphic, and deduce that $X * Y \simeq S(X \wedge Y)$.

Proof.

$$\begin{aligned} \{x_0\} \wedge \{y_0\} &= (\{x_0\} \times \{y_0\}) / (\{x_0\} \times \{y_0\} \cup \{x_0\} \times \{y_0\}) \\ &= \{(x_0, y_0)\} / \{(x_0, y_0)\} \\ &= \{(x_0, y_0)\}. \end{aligned}$$

and hence

$$S(\{x_0\} \wedge \{y_0\}) = S\{(x_0, y_0)\} = \{(x_0, y_0)\} \times I.$$

Furthermore,

$$\begin{aligned} S(X \wedge Y) &= S(X \times Y / (\{x_0\} \times Y \cup X \times \{y_0\})) \\ &= X \times Y \times I / \sim \end{aligned}$$

where

$$\begin{aligned} (x, y, 0) \sim (x', y', 0) && \forall x, y, x', y', \\ (x, y, 1) \sim (x', y', 1) && \forall x, y, x', y', \\ (x_0, y, t) \sim (x_0, y_0, t) && \forall y, t, \\ (x, y_0, t) \sim (x_0, y_0, t) && \forall x, t. \end{aligned}$$

Hence

$$S(X \wedge Y) / S(\{x_0\} \wedge \{y_0\}) = X \times Y \times I / \sim$$

where

$$\begin{aligned} (x, y, 0) \sim (x', y', 0) && \forall x, y, x', y', \\ (x, y, 1) \sim (x', y', 1) && \forall x, y, x', y', \\ (x_0, y, t) \sim (x_0, y_0, 0) && \forall y, t, \\ (x, y_0, t) \sim (x_0, y_0, 0) && \forall x, t. \end{aligned}$$

which simplifies to

$$\begin{aligned}(x, y, 0) &\sim (x_0, y_0, 0) & \forall x, y, \\(x, y, 1) &\sim (x_0, y_0, 0) & \forall x, y, \\(x_0, y, t) &\sim (x_0, y_0, 0) & \forall y, t, \\(x, y_0, t) &\sim (x_0, y_0, 0) & \forall x, t.\end{aligned}$$

Next,

$$X * Y = X \times Y \times I / \sim$$

where

$$\begin{aligned}(x, y, 0) &\sim (x, y', 0) & \forall x, y, y' \\(x, y, 1) &\sim (x', y, 1) & \forall x, x', y\end{aligned}$$

Furthermore,

$$X * \{y_0\} \cup \{x_0\} * Y = (X \times \{y_0\} \times I \cup \{x_0\} \times Y \times I) / \sim$$

where

$$\begin{aligned}(x_0, y, 0) &\sim (x_0, y', 0) & \forall y, y' \\(x, y_0, 1) &\sim (x', y_0, 1) & \forall x, x'\end{aligned}$$

and hence

$$X * Y / (X * \{y_0\} \cup \{x_0\} * Y) = X \times Y \times I / \sim$$

where

$$\begin{aligned}(x, y, 0) &\sim (x, y', 0) & \forall x, y, y' \\(x, y, 1) &\sim (x', y, 1) & \forall x, x', y \\(x, y_0, t) &\sim (x_0, y_0, 0) & \forall x, t \\(x_0, y, t) &\sim (x_0, y_0, 0) & \forall y, t \\(x_0, y, 0) &\sim (x_0, y', 0) & \forall y, y' \\(x, y_0, 1) &\sim (x', y_0, 1) & \forall x, x'\end{aligned}$$

which simplifies to

$$\begin{aligned}(x, y, 0) &\sim (x, y', 0) & \forall x, y, y' \\(x, y, 1) &\sim (x', y, 1) & \forall x, x', y \\(x, y_0, t) &\sim (x_0, y_0, 0) & \forall x, t \\(x_0, y, t) &\sim (x_0, y_0, 0) & \forall y, t\end{aligned}$$

Furthermore,

$$(x, y, 0) \sim (x, y_0, 0) \sim (x_0, y_0, 0) \forall x, y,$$

and

$$(x, y, 1) \sim (x_0, y, 1) \sim (x_0, y_0, 0) \forall x, y.$$

So the final simplification is

$$\begin{aligned} (x, y, 0) &\sim (x_0, y_0, 0) & \forall x, y \\ (x, y, 1) &\sim (x_0, y_0, 1) & \forall x, y \\ (x, y_0, t) &\sim (x_0, y_0, 0) & \forall x, t \\ (x_0, y, t) &\sim (x_0, y_0, 0) & \forall y, t \end{aligned}$$

These identifications are identical to the identifications which yield $S(X \wedge Y)/S(\{x_0\} \wedge \{y_0\})$, and hence

$$X * Y / (X * \{y_0\} \cup \{x_0\} * Y) \cong S(X \wedge Y)/S(\{x_0\} \wedge \{y_0\}).$$

Finally, both $(X * \{y_0\} \cup \{x_0\} * Y)$ and $S(\{x_0\} \wedge \{y_0\})$ are contractible subcomplexes, and hence

$$X * Y \simeq X * Y / (X * \{y_0\} \cup \{x_0\} * Y) \cong S(X \wedge Y)/S(\{x_0\} \wedge \{y_0\}) \simeq S(X \wedge Y).$$

□

Exercise 0.25. If X is a CW complex with components X_α , show that the suspension SX is homotopy equivalent to $Y \bigvee_\alpha SX_\alpha$ for some graph Y . In the case that X is a finite graph, show that SX is homotopy equivalent to a wedge sum of circles and 2-spheres.

Proof. $SX \cong (\coprod_\alpha X_\alpha \times I) / \sim$, where

$$\begin{aligned} \forall \alpha : (x, 0) &\sim (x', 0) & \forall x, x' \in X_\alpha \\ \forall \alpha : (x, 1) &\sim (x', 1) & \forall x, x' \in X_\alpha \end{aligned}$$

and hence if we select an $x_\alpha \in X_\alpha \forall \alpha$, we have $SX \cong (\coprod_\alpha SX_\alpha) / \sim$ where

$$\begin{aligned} [(x_\alpha, 0)] &\sim [(x_{\alpha'}, 0)] & \forall \alpha, \alpha' \\ [(x_\alpha, 1)] &\sim [(x_{\alpha'}, 1)] & \forall \alpha, \alpha' \end{aligned}$$

This can then be expressed as the quotient of a wedge sum with chosen points $[(x_\alpha, 0)]$ as follows:

$$SX \cong \left(\bigvee_\alpha SX_\alpha \right) / \sim$$

where

$$[(x_\alpha, 1)] \sim [(x_{\alpha'}, 1)] \forall \alpha, \alpha'.$$

Now let G be a connected acyclic graph with vertices v_α . Call the set of vertices the subcomplex A . Then define $f : A \rightarrow \bigvee_\alpha SX_\alpha$ by $f(v_\alpha) = [(x_\alpha, 1)] \forall \alpha$. It then follows that $SX \cong (\bigvee_\alpha SX_\alpha) \sqcup_f G$, since we can collapse G to identify every $[(x_\alpha, 1)]$. Moreover, $f \simeq g$, where $g : A \rightarrow \bigvee_\alpha SX_\alpha$ maps every v_α to $[(x_\alpha, 0)]$, and hence

$$SX \cong \left(\bigvee_\alpha SX_\alpha \right) \sqcup_f G \cong \left(\bigvee_\alpha SX_\alpha \right) \sqcup_g G.$$

$(\bigvee_\alpha SX_\alpha) \sqcup_g G$ is in turn homeomorphic to

$$Y \bigvee_\alpha SX_\alpha,$$

where Y is a graph consisting of a single vertex and $l - 1$ loops, where l is the number of components of X .

Now let X be a finite graph. Let T_α be a spanning tree of X_α , and let n_α be the number of edges in X_α that are not part of T_α . ST_α is a contractible subcomplex, and every edge in X_α becomes a 2-sphere in SX_α . Hence, if $n_\alpha > 0$, we have

$$SX_\alpha \simeq S\left(\bigvee_{i=1}^{n_\alpha} S^1\right) \simeq \sum \left(\bigvee_{i=1}^{n_\alpha} S^1\right) = \bigvee_{i=1}^{n_\alpha} \sum S^1 = \bigvee_{i=1}^{n_\alpha} S^2.$$

Otherwise, if $n_\alpha = 0$, then SX_α is simply contractible. If we define $\bigvee_{i=1}^0 Q$ to be a point for any space Q , we then have

$$SX \simeq \bigvee_{i=1}^{l-1} S^1 \bigvee_{i=1}^{\sum_\alpha n_\alpha} S^2,$$

as required. \square

Exercise 0.26. Use Corollary 0.20 to show that if (X, A) has the homotopy extension property, then $X \times I$ deformation retracts to $X \times \{0\} \cup A \times I$. Deduce from this that Proposition 0.18 holds more generally for any pair (X_1, A) satisfying the homotopy extension property.

Proof. Firstly, $(X \times I, X \times \{0\} \cup A \times I)$ also satisfies the homotopy extension property. Let

$$\iota : X \times \{0\} \cup A \times I \rightarrow X \times I$$

be the inclusion map. Define

$$\phi : X \times I \rightarrow X \times \{0\} \cup A \times I : (x, s) \mapsto (x, 0).$$

Then

$$\iota \circ \phi(x, s) = (x, 0) \forall (x, s) \in X \times I.$$

Define

$$h_t : X \times I \rightarrow X \times I$$

by

$$h_t(x, s) = (x, ts).$$

Then $h_0 = \iota \circ \phi$ and $h_1 = \text{id}_{X \times I}$, and hence

$$\iota \circ \phi \simeq \text{id}_{X \times I}.$$

Furthermore,

$$\phi \circ \iota(x, s) = (x, 0) \forall (x, s) \in X \times \{0\} \cup A \times I.$$

Now define

$$g_t : X \times \{0\} \cup A \times I \rightarrow X \times \{0\} \cup A \times I : (x, s) \mapsto (x, ts).$$

Then $g_0 = \phi \circ \iota$ and $g_1 = \text{id}_{X \times \{0\} \cup A \times I}$, and hence

$$\phi \circ \iota \simeq \text{id}_{X \times \{0\} \cup A \times I}.$$

Corollary 0.20 then implies that $X \times I$ deformation retracts to $X \times \{0\} \cup A \times I$.

Now let (X_1, A) be a pair satisfying the homotopy extension property, let X_0 be another topological space, and let $f, g : A \rightarrow X_0$ be homotopic. Let $F : A \times I \rightarrow X_0$ be a homotopy from f to g . Consider the space $X_0 \sqcup_F (X_1 \times I)$. $X_1 \times I$ deformation retracts onto $X_1 \times \{0\} \cup A \times I$, and hence $X_0 \sqcup_F (X_1 \times I)$ deformation retracts onto $X_0 \sqcup_F (X_1 \times \{0\} \cup A \times I)$, which is in turn homeomorphic to $X_0 \sqcup_f X_1$. Similarly, $X_1 \times I$ deformation retracts onto $X_1 \times \{1\} \cup A \times I$, and hence $X_0 \sqcup_F (X_1 \times I)$ deformation retracts onto $X_0 \sqcup_g X_1$. Furthermore, these deformation retractions both fix X_0 , and hence $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0 . \square

Exercise 0.27. *Given a pair (X, A) and a homotopy equivalence $f : A \rightarrow B$, show that the natural map $X \rightarrow B \sqcup_f X$ is a homotopy equivalence if (X, A) satisfies the homotopy extension property. An interesting case is when f is a quotient map, hence the map $X \rightarrow B \sqcup_f X$ is the quotient map identifying each set $f^{-1}(b)$ to a point. When B is a point this gives another proof of Proposition 0.17.*

Proof. By Corollary 0.21, A is a deformation retract of M_f , and hence X is a deformation retract of $X \cup M_f$. By the previous exercise, $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, and hence $X \cup M_f$ is a deformation retract of $(X \times I) \sqcup B / \sim$, where \sim is given by $(a, 1) \sim f(a) \forall a \in A$. $(X \times I \sqcup B) / \sim$ in turn deformation retracts onto $(X \times \{1\} \sqcup B) / \sim$, which is homeomorphic to $B \sqcup_f X$.

$$X \xrightarrow{\iota_1} X \cup M_f \xrightarrow{\iota_2} (X \times I \sqcup B) / \sim \xrightarrow{r} B \sqcup_f X.$$

$\iota_1 : X \hookrightarrow X \cup M_f$ is a homotopy equivalence, $\iota_2 : X \cup M_f \hookrightarrow (X \times I \sqcup B) / \sim$ is a homotopy equivalence, and the retraction $r : (X \times I \sqcup B) / \sim \rightarrow B \sqcup_f X$ given by $r([(x, s)]) = [x] \forall (x, s) \in X \times I$ and $r([b]) = [b] \forall b \in B$ is a homotopy equivalence. The composition of homotopy equivalences is a homotopy equivalence, and hence $r \circ \iota_2 \circ \iota_1 : X \rightarrow B \sqcup_f X$, which is precisely the natural map $X \rightarrow B \sqcup_f X$, is a homotopy equivalence. \square

Exercise 0.28. *Show that if (X_1, A) satisfies the homotopy extension property, then so does every pair $(X_0 \sqcup_f X_1, X_0)$ obtained by attaching X_1 to a space X_0 via a map $f : A \rightarrow X_0$.*

Proof. Suppose we have a map $f_0 : X_0 \sqcup_f X_1 \rightarrow Y$ and a homotopy $f_t : X_0 \rightarrow Y$ of $f_0|_{X_0}$. This in turn induces a homotopy on A given by $f_t \circ f$. Since (X_1, A) satisfies the HEP, it follows that $f_t \circ f : A \rightarrow Y$ can be extended to a homotopy $g_t : X_1 \rightarrow Y$ with $g_0(x) = f_0([x]) \forall x \in X_1$. We can thus extend f_t to a homotopy $f_t : X_0 \sqcup_f X_1 \rightarrow Y$ given by

$$f_t([x]) = \begin{cases} g_t(x) & \text{if } x \in X_1, \\ f_t(x) & \text{if } x \in X_0. \end{cases}$$

This is well-defined, for if $x \in A$, then $g_t(x) = f_t \circ f(x) = f_t([x])$. We now show that f_t is continuous. First, define a homotopy

$$F : (X_0 \sqcup X_1) \times I \rightarrow Y : F(x, t) \mapsto f_t([x]).$$

Let ι_0, ι_1 be the inclusion maps of X_0 and X_1 respectively into $X_0 \sqcup X_1$. $\iota_0(X_0) \times I$ and $\iota_1(X_1) \times I$ are both closed in $(X_0 \sqcup X_1) \times I$ as a consequence of the coproduct topology,

$$\iota_0(X_0) \times I \cup \iota_1(X_1) \times I = (X_0 \sqcup X_1) \times I,$$

and F is continuous when restricted to both $\iota_0(X_0) \times I$ and $\iota_1(X_1) \times I$, and hence F is continuous on $(X_0 \sqcup X_1) \times I$ by the Pasting lemma. Now let

$$q : (X_0 \sqcup X_1) \times I \rightarrow (X_0 \sqcup_f X_1) \times I : (x, t) \mapsto ([x], t)$$

be the quotient map. F then descends to a map

$$G : (X_0 \sqcup_f X_1) \times I \rightarrow Y : ([x], t) \mapsto F(x, t),$$

which is continuous by the universal property of the final topology, since $F = G \circ q$. Finally,

$$G([x], t) = f_t([x]) \quad \forall [x] \in X_0 \sqcup_f X_1,$$

and hence f_t is continuous, so a homotopy. Thus $(X_0 \sqcup_f X_1, X_0)$ has the HEP.

Note - To justify that q is actually a quotient map, we invoke the following result from "General Topology" by Engelking (theorem 3.3.17):

Let X be a locally compact space. Let $q : Y \rightarrow Z$ be a quotient map. Let the map

$$f : X \times Y \rightarrow X \times Z$$

be defined by $f(x, y) = (x, q(y))$ for each $(x, y) \in X \times Y$. Then the map f is a quotient map from $X \times Y$ to $X \times Z$.

Indeed, I is a compact Hausdorff space, and hence locally compact, so the result applies. \square

Exercise 0.29. In case the CW complex X is obtained from a subcomplex A by attaching a single cell e^n , describe exactly what the extension of a homotopy $f_t : A \rightarrow Y$ to X given by the proof of Proposition 0.16 looks like. That is, for a point $x \in e^n$, describe the path $f_t(x)$ for the extended f_t .

Proof. Define $F : X \times I \rightarrow Y$ by $F(x, t) = f_t(x)$. The extension of $F : X \times \{0\} \cup A \times I \rightarrow Y$ is simply $F \circ R : X \times I \rightarrow Y$, where R is the retraction of $X \times I$ to $X \times \{0\} \cup A \times I$ given in Proposition 0.16.

Let $r : D^n \times I \rightarrow D^n \times \{0\} \cup \partial D^n \times I$ be the retraction of $D^n \times I$ described in Proposition 0.16. We consider three different cases:

1. If x is in the centre of e^n , then $r(x, t) = (x, 0) \forall t \in I$.
2. If $2 - 2\|x\| > 1$, then $r(x, t)$ simply moves x radially outwards from the centre towards $\frac{x}{\|x\|}$ as t increases, while keeping the I component at 0. In particular, if $p_x : I \rightarrow D^n : t \mapsto \frac{2x}{2-t}$ is the line x travels on outwards from the centre as t increases, then we have $r(x, t) = (p_x(t), 0)$.
3. Otherwise, $r(x, t)$ moves radially outwards from $(x, 0)$ at $t = 0$ to $\left(\frac{x}{\|x\|}, 0\right)$ at $t = 2 - 2\|x\|$. For $t \geq 2 - 2\|x\|$, we simply increase the value of the I component. Explicitly, if we let $p_x : [0, 2 - 2\|x\|] \rightarrow D^n : t \mapsto \frac{2x}{2-t}$ be the line from x to $\frac{x}{\|x\|}$, then

$$r(x, t) = \begin{cases} (p_x(t), 0) & \text{if } t < 2 - 2\|x\|, \\ \left(\frac{x}{\|x\|}, 2 - \frac{2-t}{\|x\|}\right) & \text{if } t \geq 2 - 2\|x\|. \end{cases}$$

We can now describe the path of $f_t(x)$. Let $\phi : \partial D^n \rightarrow A$ be the attaching map of e^n .

1. In the first situation, where x is in the centre of e^n , then $f_t(x) = f_0(x) \forall t$.

2. In the second situation, where $2 - 2\|x\| > 1$, then $f_t(x) = f_0\left(\frac{2x}{2-t}\right) \forall t$.

3. In the final situation, where $2 - 2\|x\| \leq 1$, then

$$f_t(x) = \begin{cases} f_0\left(\frac{2x}{2-t}\right) & \text{if } t < 2 - 2\|x\|, \\ f_{2-\frac{2-t}{\|x\|}} \circ \phi\left(\frac{x}{\|x\|}\right) & \text{if } t \geq 2 - 2\|x\|. \end{cases}$$

□

1 The Fundamental Group

1.1 Basic Constructions

Exercise 1.1. Show that the composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \sim f_1 \cdot g_1$ and $g_0 \sim g_1$ then $f_0 \sim f_1$.

Proof. Let $\overline{g_0}$ and $\overline{g_1}$ be the reversals of g_0 and g_1 respectively. Then

$$f_0 \sim f_0 \cdot g_0 \cdot \overline{g_0} \sim f_1 \cdot g_1 \cdot \overline{g_0} \sim f_1 \cdot g_1 \cdot \overline{g_1} \sim f_1.$$

□

Exercise 1.2. Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h .

Proof. Let $h \sim g$ as paths from x_0 to x_1 . Let $[f] \in \pi_1(X, x_1)$. Then

$$\beta_h([f]) = [h \cdot f \cdot \bar{h}] = [g \cdot f \cdot \bar{h}] = [g \cdot f \cdot \bar{g}] = \beta_g([f]).$$

□

Exercise 1.3. For a path-connected space X , show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms β_h depend only on the endpoints of the path h .

Proof. (\implies) : Let h and g be two paths from x_0 to x_1 . Let $[f] \in \pi_1(X, x_1)$. Then

$$\begin{aligned} \beta_h([f])\beta_g([f])^{-1} &= [h \cdot f \cdot \bar{h}][g \cdot \bar{f} \cdot \bar{g}] \\ &= [h \cdot f \cdot (\bar{h} \cdot g) \cdot \bar{f} \cdot \bar{g}] \\ &= [h \cdot (\bar{h} \cdot g) \cdot f \cdot \bar{f} \cdot \bar{g}] \\ &= e \end{aligned}$$

and hence $\beta_h([f]) = \beta_g([f])$.

(\impliedby) : Let $[f], [g] \in \pi_1(X, x_1)$. We have $\beta_{[f]} = \beta_{[g]}$, so

$$[f][g][f]^{-1} = [f \cdot g \cdot \bar{f}] = \beta_{[f]}([g]) = \beta_{[g]}([g]) = [g \cdot g \cdot \bar{g}] = [g]$$

and hence

$$[f][g] = [g][f].$$

Thus $\pi_1(X)$ is abelian. □

Exercise 1.4. A subspace $X \subset \mathbb{R}^3$ is said to be star-shaped if there is a point $x_0 \in X$ such that, for each $x \in X$, the line segment from x_0 to x lies in X . Show that if a subspace $X \subset \mathbb{R}^3$ is locally star-shaped, in the sense that every point of X has a star-shaped neighbourhood in X , then every path in X is homotopic in X to a piecewise linear path, that is, a path consisting of a finite number of straight line segments traversed at constant speed. Show this applies in particular when X is open or when X is a union of finitely many closed convex sets.

Proof. Let $p : I \rightarrow X$ be a path. For each $t \in I$, let U_t be a star-shaped open neighbourhood of $p(t)$. By compactness (since the continuous image of a compact set is compact) there exists a finite subset $J \subset I$ such that $\bigcup_{t \in J} U_t \cap p(I)$ is an open cover of $p(I)$. Suppose $J = \{t_1, \dots, t_n\}$, where $t_1 < \dots < t_n$. Let x_1, \dots, x_n be points in U_{t_1}, \dots, U_{t_n} such that the line segment between every point $x \in U_{t_i}$ and x_i is contained in U_{t_i} . Let V_1, \dots, V_n denote $p^{-1}(U_{t_1}), \dots, p^{-1}(U_{t_n})$. V_1, \dots, V_n then forms a finite open cover of I . By Lebesgue's number lemma, there exists a $\delta > 0$ such that every subset of I with diameter less than δ is contained within some V_i . Let N be some natural number such that $\frac{1}{N} < \delta$. We can then partition I as $0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1$. The diameter of $[\frac{i}{N}, \frac{i+1}{N}]$ is less than δ , and hence the interval is contained in some V_j for every i . Let k_i denote the index of a U_{t_j} containing $p([\frac{i}{N}, \frac{i+1}{N}])$. We define a map $q : I \rightarrow X$ as follows:

For each $i \in \{0, \dots, N-1\}$, we let $q(\frac{i}{N}) = p(\frac{i}{N})$, $q(\frac{i+0.5}{N}) = x_{k_i}$, $q(\frac{i+1}{N}) = p(\frac{i+1}{N})$. We then let q trace out the line segment from $q(\frac{i}{N})$ to $q(\frac{i+0.5}{N})$ on the interval $[\frac{i}{N}, \frac{i+0.5}{N}]$, and the line segment from $q(\frac{i+0.5}{N})$ to $q(\frac{i+1}{N})$ on the interval $[\frac{i+0.5}{N}, \frac{i+1}{N}]$. We can then construct a homotopy from p to q as follows:

For each i , let $h_{i,t} : [\frac{i}{N}, \frac{i+1}{N}] \rightarrow X$ be the straight-line homotopy from $p|_{[\frac{i}{N}, \frac{i+1}{N}]}$ to $\gamma_{q(\frac{i+0.5}{N})}$ (note, this homotopy is well-defined by star-shapedness). Then let $a_{i,t} : [\frac{i}{N}, \frac{i+1}{N}] \rightarrow X$ trace out the line segment from $q(\frac{i}{N})$ to $h_{i,t}(0)$, and let $b_{i,t} : [\frac{i}{N}, \frac{i+1}{N}] \rightarrow X$ trace out the line segment from $h_{i,t}(1)$ to $q(\frac{i+1}{N})$. $a_{i,t} \cdot h_{i,t} \cdot b_{i,t}$ is then a based-homotopy from $p|_{[\frac{i}{N}, \frac{i+1}{N}]}$ to $q|_{[\frac{i}{N}, \frac{i+1}{N}]}$. Combining these then gives a based homotopy from p to q . We can then reparametrize q by arc-length to obtain the desired map.

When X is open, every $x \in X$ is contained in a ball, which is star-shaped, and hence X is locally star-shaped.

Now let X be a union of finitely many closed convex sets C_1, \dots, C_n . Let $x \in X$. Let C_{i_1}, \dots, C_{i_m} be the collection of C_i 's which don't contain x . $X \setminus C_i$ is open $\forall C_i$, since C_i is closed, and hence for each C_{i_k} we can find an open ball $B_{i_k} \subseteq C_{i_k}^c$ which contains x . Consider $X \cap \bigcap_{k=1}^m B_{i_k} \subseteq X \setminus \bigcup_{k=1}^m C_{i_k}$. This set is open, as the finite intersection of open sets, and contains x . Let $y \in X \cap \bigcap_{k=1}^m B_{i_k}$. y is then contained in a C_i which also contains x . Since C_i is convex, the line segment between x and y is contained in $\bigcap_{k=1}^m B_{i_k}$, for the intersection of convex sets is convex, and hence the line segment between x and y is contained in $X \cap \bigcap_{k=1}^m B_{i_k}$. Hence $X \cap \bigcap_{k=1}^m B_{i_k}$ is a star-shaped neighbourhood of x , so X is locally star-shaped. \square

Exercise 1.5. Show that for a space X , the following three conditions are equivalent:

- (a) Every map $S^1 \rightarrow X$ is homotopic to a constant map, with image a point.
- (b) Every map $S^1 \rightarrow X$ extends to a map $D^2 \rightarrow X$.
- (c) $\pi_1(X, x_0) = 0 \forall x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \rightarrow X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

Proof. (a) \implies (b): Let $f : S^1 \rightarrow X$ be a map homotopic to a constant map γ_x . We have a homotopy $H : S^1 \times I \rightarrow X$ from f to γ_x . Define the equivalence relation \sim on $S^1 \times I$ by $(\omega, 1) \sim (\omega', 1) \forall \omega, \omega' \in S^1$. We then obtain a well-defined map $g : S^1 \times I / \sim \rightarrow X$ given by $g([\omega, t]) = H(\omega, t)$. This is well-defined because $H(\omega, 1) = x \forall \omega$. Let $\pi : S^1 \times I \rightarrow S^1 \times I / \sim$ be the quotient map. $g \circ \pi = H$ is continuous, and hence g is continuous by the universal property of the quotient topology. Furthermore, $S^1 \times I / \sim \cong D^2$. Let $\iota : S^1 \hookrightarrow S^1 \times I / \sim : \omega \mapsto [(\omega, 0)]$ be the inclusion map. We then have that $f = g \circ \iota$, and hence g is an extension of f to D^2 .

(b) \implies (c): Let $p : I \rightarrow X$ be a loop. This then induces a map $p' : S^1 \rightarrow X$, after defining S^1 as the quotient space $I/0 \sim 1$. We then have that p' extends to a map $q : D^2 \rightarrow X$, where we define D^2 as $S^1 \times I / (\omega, 1) \sim (\omega', 1)$, which is in turn equal to I^2 / \sim , where \sim is the relation given by $(0, t) \sim (1, t), (t, 1) \sim (s, 1) \forall t, s \in I$. We now define a map $F : I^2 \rightarrow X$ by $F((t, s)) = q([(t, s)])$. Let $\pi : I^2 \rightarrow I^2 / \sim$ be the quotient map. We then have that $F = q \circ \pi$, and hence F is continuous by the universal property of quotient spaces. Furthermore, $F(t, 0) = p(t)$ and $F(t, 1)$ is a constant map, and hence p is homotopic to a constant map. However, while this is a homotopy of loops (since $F(0, s) = F(1, s) \forall s$), the loops do not necessarily have the same basepoint as p . To resolve this, we define $h_t : I \rightarrow X : s \mapsto F(0, st)$, and let \bar{h}_t be the reverse path of h_t . Let $f_t : I \rightarrow X$ be the homotopy from p to the constant map given by F . We then have that $h_t \cdot f_t \cdot \bar{h}_t$ is a based-homotopy from p to $h_1 \cdot \bar{h}_1$, which is in turn based homotopic to γ_{x_0} . Hence $[p] = [\gamma_{x_0}]$. This holds for all p , and hence $\pi_1(X, x_0) = 0$. x_0 is arbitrary, so this then holds for all $x_0 \in X$.

(c) \implies (a) Suppose we have a map $f : S^1 \rightarrow X$. If we define S^1 as $I/0 \sim 1$, this gives a path $p : I \rightarrow X : t \mapsto f([t])$, which is continuous by the universal property of the quotient topology. Let $x_0 = p(0)$. Since $\pi_1(X, x_0) = 0$, it follows that p is based-homotopic to γ_{x_0} . Let $F : I^2 \rightarrow X$ be the homotopy, so that $F(t, 0) = p(t), F(t, 1) = x_0, F(0, s) = F(1, s) = x_0$. If we identify $S^1 \times I$ with $I^2 / (0, t) \sim (1, t)$, we can then obtain a homotopy $G : S^1 \times I \rightarrow X : [(t, s)] \mapsto F((t, s))$ from f to a constant map, which is continuous by the universal property of the quotient topology, and which is well-defined, since $F(0, s) = F(1, s) \forall s$. Hence f is homotopic to a constant map, with image a point.

Finally, suppose all maps $S^1 \rightarrow X$ are homotopic. That in particular implies that every map is homotopic to a constant map, and hence that $\pi_1(X, x_0) = 0 \forall x_0 \in X$ ((a) \implies (c)). Now let x_0 be some chosen point in X . We have that for every $x \in X$, the constant map $\gamma_x : S^1 \rightarrow X : \omega \mapsto x$ is homotopic to γ_{x_0} . This implies that there is a path from x to x_0 . Hence X is path-connected, so is simply-connected.

Now suppose that X is simply connected. We pick some point x_0 and will show that every map $f : S^1 \rightarrow X$ is homotopic to γ_{x_0} , since then all maps $S^1 \rightarrow X$ will be homotopic due to homotopy being an equivalence relation. Given a map $f : S^1 \rightarrow X$, we have that f is homotopic to a constant map. Let γ_x be that constant map. Since X is simply-connected, and hence path-connected, there exists a path $p : I \rightarrow X$ from x to x_0 . We can then obtain a homotopy from γ_x to γ_{x_0} given by $h_t : S^1 \rightarrow X : \omega \mapsto p(t)$. It then follows by transitivity that f is homotopic to γ_{x_0} , as required.

Hence X is simply-connected iff all maps $S^1 \rightarrow X$ are homotopic. \square

Exercise 1.6. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \rightarrow (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \rightarrow X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \rightarrow [S^1, X]$ obtained by ignoring basepoints. Show that Φ is surjective if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a bijection between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

Proof. We will assume without loss of generality that $S^1 := I/0 \sim 1$, and that $s_0 = [0]$. Suppose that X is path-connected. Let $f : S^1 \rightarrow X$ be a representative of an element of $[S^1, X]$ such that

$f(s_0) = x$, for some $x \in X$. Let $f' : I \rightarrow X$ be the corresponding loop of f , so that $f([t]) = f'(t)$. Let $p : I \rightarrow X$ be a path from x_0 to x . We can then construct a loop $g : S^1 \rightarrow X \in \pi_1(X, x_0)$ by defining its corresponding map $g' : I \rightarrow X$ by $g' = p \cdot f' \cdot \bar{p}$ (where each path is traversed at 3 times the speed), which is then homotopic to f' under the homotopy $F' : I \times I \rightarrow X$ given by

$$F'(t, s) = \begin{cases} p(s + 3t(1-s)) & \text{if } t \leq \frac{1}{3}, \\ f'(3(t - \frac{1}{3})) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ \bar{p}(3(1-s)(t - \frac{2}{3})) & \text{if } t \geq \frac{2}{3}. \end{cases}$$

Since $F'(0, s) = F'(1, s) \forall s$, it follows that F' also gives a well-defined homotopy $F : S^1 \times I \rightarrow X$ from g to f (up to reparametrization) given by $F([t], s) = F'(t, s)$. Hence $\Phi([g]) = [f]$ so Φ is surjective.

Let $f, g : (S^1, s_0) \rightarrow (X, x_0)$ be loops such that $[f]$ and $[g]$ are conjugate; that is, $[f] = [a]^{-1}[g][a]$ for some loop $a : (S^1, s_0) \rightarrow (X, x_0)$. Let f', g', a' be the corresponding maps of $I \rightarrow X$. We first observe that g' is homotopic to $\bar{a}' \cdot g' \cdot a'$ when we don't require basepoints to be fixed. To see this, observe the map $H' : I \times I \rightarrow X$ given by

$$H'(t, s) = \begin{cases} \bar{a}'(1-s+3st) & \text{if } t \leq \frac{1}{3}, \\ g'(3(t-\frac{1}{3})) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ a'(3s(t-\frac{2}{3})) & \text{if } t \geq \frac{2}{3}. \end{cases}$$

$H'(t, 0)$ is a reparametrization of g' , and $H'(t, 1) = \bar{a}' \cdot g' \cdot a'(t)$, and hence H' is a homotopy from g' to $\bar{a}' \cdot g' \cdot a'$ (up to reparametrization). It then follows from transitivity that g' is homotopic to f' , and hence that $\Phi([g]) = \Phi([f])$.

Now suppose $\Phi([f]) = \Phi([g])$. Let $'G : I \times I \rightarrow X$ be a homotopy from f' to g' , which does not necessarily preserve basepoints. Define $p : I \rightarrow X$ by $p(t) = G'(0, t)$. Consider the map $U' : I \times I \rightarrow X$ given by

$$U'(t, s) = \begin{cases} p(3st) & \text{if } t \leq \frac{1}{3}, \\ G'(3(t-\frac{1}{3}), s) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ \bar{p}(1-s+3s(t-\frac{2}{3})) & \text{if } t \geq \frac{2}{3}. \end{cases}$$

This is a based homotopy between f and $p \cdot g \cdot \bar{p}$ (up to reparametrization). Hence, $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

Let \sim be the equivalence relation of conjugation on $\pi_1(X)$. and let $\Psi : \pi_1(X)/\sim \rightarrow [S^1, X] : [a] \mapsto \Phi(a)$ be the induced map on conjugacy classes. This map is well-defined, since if $[f]$ and $[g]$ are conjugate in $\pi_1(X)$, then $\Psi([[f]]) = \Phi([f]) = \Phi([g]) = \Psi([[g]])$. Furthermore, Ψ is injective, since if $\Psi([[f]]) = \Psi([[g]])$, then $[[f]] = [[g]]$. Hence, if X is path-connected, Ψ is both injective and surjective, so is bijective. \square

Exercise 1.7. Define $f : S^1 \times I \rightarrow S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.]

Proof. Let f_t be a homotopy from $\text{id}_{S^1 \times I}$ to f with the associated map F defined as

$$F : S^1 \times I \times I \rightarrow S^1 \times I : (\theta, s, t) \mapsto (\theta + 2\pi st, s).$$

When $s = 0$, we have $f_t(\theta, 0) = (\theta, 0) \forall t$, as required.

Suppose for a contradiction that there exists a homotopy f_t from f to $\text{id}_{S^1 \times I}$ which is stationary on both boundary circles. Consider the path $p : I \rightarrow S^1 \times I : s \mapsto (\theta_0, s)$. We then have a homotopy of paths

$$f_t \circ p : I \rightarrow S^1 \times I$$

such that

$$f_0 \circ p(s) = (\theta_0 + 2\pi s, s)$$

and

$$f_1 \circ p(s) = (\theta_0, s).$$

Let $\pi_{S^1} : S^1 \times I \rightarrow S^1$ be the projection map from $S^1 \times I$ to S^1 . Then $\pi_{S^1} \circ f_t \circ p : I \rightarrow S^1$ is a homotopy from

$$\pi_{S^1} \circ f_0 \circ p : I \rightarrow S^1 : s \mapsto \theta_0 + 2\pi s$$

to

$$\pi_{S^1} \circ f_1 \circ p : I \rightarrow S^1 : s \mapsto \theta_0.$$

Since f_t is stationary on both boundary circles, it follows that

$$\pi_{S^1} \circ f_t \circ p(0) = \pi_{S^1} \circ f_t \circ p(1) = \theta_0 \forall t,$$

and hence $\pi_{S^1} \circ f_t \circ p$ is a homotopy of loops based at θ_0 . However, $\pi_{S^1} \circ f_0 \circ p$ is not homotopic to the constant loop; a contradiction. Hence no such homotopy can exist. \square

Exercise 1.8. Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \rightarrow \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

Proof.

\square