The topology of lens spaces

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April 2024

Abstract

Lens spaces are quotient spaces of the 3-sphere by cyclic groups and provide the simplest examples of manifolds (topological spaces which locally look like Euclidean space) which are homotopy-equivalent but not homeomorphic. This project will study the conditions under which lens spaces are homotopy-equivalent and homeomorphic. To this end we shall introduce the homology and cohomology groups. Being invariants of homotopy-equivalent spaces, it is expected that they would be insufficient for a classification of manifolds which can be homotopy-equivalent but not homeomorphic. However, the homology and cohomology groups are also insufficient for a homotopy-equivalence classification. The torsion linking form will be introduced for the homotopy-equivalence classification, and Reidemeister torsion for the homeomorphism classification.

Acknowledgements

Contents

1   Introduction

Algebraic topology seeks to distinguish topological spaces up to homeomorphism and homotopy equivalence by associating algebraic structures to topological spaces which are isomorphic when the spaces are homeomorphic or homotopy-equivalent. Most invariants are homotopy invariants, with homeomorphic being a stronger condition than homotopy-equivalent. The simplest example would be the number of conneted-components – a homotopy invariant. The most commonly used homotopy invariants are the fundamental group, the homology groups and the cohomology groups. This project will first study singular homology, which is the simplest and most general homology theory to study, but has the disadvantage of being difficult to perform actual computations on, due to being defined in terms of a potentially uncountably infinite number of continuous maps. The singular homology groups are defined by forming abelian groups, known as chain groups, from continuous maps of a certain dimension into the topological space which have no boundary, and quotienting by the boundaries of the maps of a dimension one higher; this makes sense because the boundary of a boundary is 0, as will be proven. We will then see a connection between the fundamental group and the first homotopy group through the Hurewicz theorem, which roughly says that the first homology group is the abelian version of the fundamental group.

We will then study the cohomology groups, which are dual to the homology groups. They have the advantage of admitting a ring structure through the cup product, and so have additional structure, giving a finer invariant. We will also see the universal coefficient theorem and Poincaré duality, which are results which connect the homology and cohomology groups. Poincaré duality however only holds for closed oriented manifolds, however, and is given by an isomorphism involving the cap product and a fundamental class; hence these concepts will also be discussed in detail. Orientation on manifolds is a generalized notion of consistent choice of unit normal for surfaces in $\mathbb{R}^3$ and can be defined in several ways. Here we shall study the homological definition, which involves a continuous choice of generator of the local homology group at each point on a manifold; local homology is a special case of relative homology: a homology given by "modding out" by a subspace of a topological space. It turns out that closed manifolds, that is manifolds without boundary, admit a simpler definition of orientability: an n–dimensional manifold is orientable if and only if it's n'th homology group is isomorphic to $\mathbb{Z}$. A generator of this group is termed a fundamental class, and represents an orientation of a manifold. Poincaré duality then relates the k'th homology group and $(n-k)$'th cohomology of an n–manifold through an explicit isomorphism given in terms of a fundamental class.

We will then use these objects to attempt to extract topological information about lens spaces, which are quotient spaces of the 3–sphere by $\mathbb{Z}_p$, where the quotient space is given by the relation that two points are equivalent if they are in the same orbit of the group action. To state it explicitly, if S^3 is given by

$$S^3 := \{(u,v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1\}$$

and q is coprime to p, then the lens space $L(p;q)$ is the quotient space given by the action generated by

$$(v,w) \mapsto (e^{2\pi i/p}v, e^{2\pi iq/p}w).$$

We will see the following result for the fundamental group, homology groups and cohomology groups

of lens spaces:

$$\pi_1(L(p;q)) \cong \mathbb{Z}_p, \ H_k(L(p;q),\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_p & \text{if } k = 1 \\ 0 & \text{if } k = 2 \\ \mathbb{Z} & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases}, \ H^k(L(p;q),\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k = 1 \\ \mathbb{Z}_p & \text{if } k = 2 \\ \mathbb{Z} & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

These groups only depend on p, which turns out to be insufficient. We will introduce two new invariants: the torsion linking form and Reidemeister torsion. The torsion linking form is a homotopy-invariant and is defined on the torsion part of the second cohomology group, where the torsion subgroup consists of all the elements of finite order. However, the torsion linking form requires explicitly defined cochains (homomorphisms from chains to an abelian group) and as such requires a simpler set of chains than the uncountably infinitely many which singular homology gives. To this end we shall introduce Delta complexes, which are simple enough to allow for lens spaces to be constructed in terms of finitely many chains, yet still allow for cup products, and hence the torsion linking form, with which it is defined, to be computed. The homeomorphism classification then requires Reidemeister torsion, which is a homeomorphism invariant – but is not a homotopy invariant – of cell complexes, where a cell complex is a topological space built combinatorially by beginning with a set of points and iteratively attaching higher-dimensional balls along their boundaries. A Delta complex is a special case of a cell complex, with additional constraints which allow for cup products to be computed, albeit at the cost of added complexity in their construction. The Reidemeister torsion for cell complexes arose out of a more general form for exact sequences of modules with coefficients in a commutative ring; a module refers to a generalisation of a vector space, where the the field is replaced by a ring, and an exact sequence refers to a sequence of modules

$$... \overset{f_n}{\to} C_n \overset{f_{n-1}}{\to} C_{n-1} \overset{f_{n-2}}{\to} C_{n-2} \overset{f_{n-3}}{\to} ...$$

where the image of f_i is the kernel of f_{i-1}. The Reidemeister torsion is then formed as the products of the determinants of change of basis matrices, and so the computation of the Reidemeister torsion of a cell complex is the easiest when the modules are 1–dimensional, since the change of basis matrices will be 1 by 1. Thankfully, it is possible to construct such a cell complex structure of a lens space if we forego the delta complex structure. The torsion linking form and Reidemeister torsion comprise the two necessary conditions needed for the homotopy and homeomorphism classification; and although the sufficiency of the condition obtained from the torsion linking form is beyond the scope of this project (it involves the higher homotopy groups and the Whitehead theorem [1], which says that a continuous map between cell complexes inducing isomorphisms on all homotopy groups is enough to make it a homotopy equivalence), it is relatively straightforward to construct the homeomorphisms which show the sufficiency of the condition given by the Reidemeister torsion. We will then obtain the following classification:

Let $L(p;q_1)$ and $L(p;q_2)$ be lens spaces.
They are homotopy equivalent if and only if $q_1 q_2 \equiv \pm n^2 \mod p$ for some $n \in \mathbb{N}$.
They are homeomorphic if and only if $q_1 \equiv \pm q_2^{\pm 1} \mod p$.

From this we can see that $L(7;1)$ and $L(7;2)$ are homotopy-equivalent but not homeomorphic, which gives the simplest example of manifolds which are homotopy-equivalent but not homeomorphic.

2 Homology

2.1 Background

We begin by recalling some basic definitions and results from algebraic topology. [2]

Definition 2.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. A map $f \colon X \to Y$ is **continuous** if $f^{-1}(U) \in \mathcal{T}$, $\forall U \in \mathcal{T}'$.*

Definition 2.2. *Let X and Y be topological spaces. A continuous map $\phi \colon X \to Y$ is a **homeomorphism** if it has a continuous inverse $\phi^{-1} \colon Y \to X$.*

Definition 2.3. *Let X be a topological space. A **path** is a continuous map γ from $[0, 1]$ to X. If $\gamma(0) = \gamma(1)$, then γ is a **loop**.*

Definition 2.4. *Let X be a topological space and $x_0 \in X$. The **fundamental group** is*

$$\pi_1(X, x_0) = \{[\gamma] : \gamma \text{ is a loop based at } x_0\}.$$

under the relation that $\gamma \sim \gamma'$ if γ is based homotopic to γ'.

This is a group with identity represented by the constant loop, multiplication given by concatenation of loops and $[\alpha]$ having inverse $[\bar{\alpha}]$, where $\bar{\alpha}(t) = \alpha(1 - t)$.

Theorem 2.5. *If X is path-connected and $x_0, y_0 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, y_0)$.*

We can then refer to the fundamental group merely as $\pi_1(X)$.

Definition 2.6. *Two continuous maps $\phi_0, \phi_1 \colon X \to Y$ of topological spaces are **homotopic** if there exists a continuous map $F \colon X \times I \to Y$ such that*

$$F(x, 0) = \phi_0(X), F(x, 1) = \phi_1(x), \forall x \in X.$$

*In this case, we write: $\phi_0 \simeq \phi_1$ and say that F is a **homotopy from ϕ_0 to ϕ_1**.*

Definition 2.7. *Let X and Y be topological spaces. If there exist continuous maps $\phi \colon X \to Y$ and $\psi \colon Y \to X$ such that $\phi \circ \psi \simeq id_Y$ and $\psi \circ \phi \simeq id_X$, then X and Y are said to be **homotopy-equivalent**. In this case, we write $X \simeq Y$.*

Theorem 2.8. *Homotopy-equivalent path-connected topological spaces have isomorphic fundamental groups.*

Definition 2.9. *Let E and X be topological spaces. A **covering map** $p : E \to X$ is a continuous surjection such that each $x \in X$ has an open neighbourhood U for which $p^{-1}(U)$ is a union of disjoint open sets S_i such that $p_{|S_i} : S_i \to U$ is a homeomorphism for each i.*

Definition 2.10. *A continuous map $f : X \to Y$ is a **local homeomorphism** if each $x \in X$ has an open neighbourhood S such that $p(S)$ is open and $p_{|S} : S \to p(S)$ is a homeomorphism.*

Proposition 2.11. *Covering maps are local homeomorphisms.*

Definition 2.12. *Let $p : E \to X$ be a covering map. A **deck transformation of** p is a homeomorphism $\phi : E \to E$ such that $p \circ \phi = p$. We denote the group of deck transformations of p by $Aut(p)$.*

Proposition 2.13. *If E is a universal cover of p then $Aut(p)$ is isomorphic to $\pi_1(X, x)$ for any $x \in X$.*

Definition 2.14. *E is a **universal cover** of X if E is simply connected and there is a covering map $p : E \to X$.*

Definition 2.15. *Let X be a topological space. We denote the group of homeomorphisms from X to X by **Homeo**(X).*

Definition 2.16. *Let $G \times E \to E, (g, e) \mapsto g \cdot e$ be a left action of a group G on a topological space E.*
*We say that the action is **continuous** if the left multiplication map $L_g : E \to E, e \mapsto g \cdot e$ is continuous for all $g \in G$.*
*We say that the action is a **covering space action** if it is continuous and every $e \in E$ has an open neighbourhood $e \in S \subset E$ such that $gS \cap S = \emptyset$ for all non-trivial elements $g \in G$.*

Theorem 2.17. *Let G be a group acting via a covering space action on a simply connected topological space E. Then $\pi_1(E/G, [e]) \cong G$ for any $[e] \in E/G$.*

This is because the quotient map is a covering map, with the group of deck transformations being isomorphic to G.

Some concepts from algebra also needs to be introduced:

Definition 2.18. *An **exact sequence** is a sequence of morphisms between objects such that the image of one morphism is the kernel of the next morphism.*

Definition 2.19. *A **short exact sequence** is an exact sequence of the form*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Definition 2.20. *A general exact sequence is also called a **long exact sequence**, in contrast to short exact sequences.*

Definition 2.21. *A **chain complex** is a sequence of abelian groups $..., A_0, A_1, A_2, A_3, ...$ connected by homomorphisms such that the composition of any two consecutive homomorphisms is the zero map.*

2.2 Singular homology

Definition 2.22. *A **k-simplex** is the convex hull*

$$[v_0, ..., v_k] := \Big\{ \sum_{i=0}^{k} \lambda_i v_i : \lambda_i \in [0, 1], \sum_{i=0}^{k} \lambda_i = 1 \Big\}$$

of $k + 1$ linearly independent vectors $v_0, ..., v_k \in \mathbb{R}^n$.
*The **standard k-simplex** is*

$$\Delta^k = [e_0, ..., e_k] = \Big\{ (t_0, ..., t_k) \in \mathbb{R}^k : t_i \geq 0, \sum_{i=0}^{k} t_i = 1 \Big\}.$$

*The vectors $v_0, ..., v_k$ define a **framing***

$$F_{v_0, ..., v_k} : \Delta^k \to [v_0, ..., v_k], (t_i) \mapsto \sum_{i=0}^{k} t_i v_i.$$

For example Δ^0 is a point, Δ^1 a line segment, Δ^2 a triangle and Δ^3 a tetrahedron.

Definition 2.23. *The **faces** of $[v_0, ..., v_k]$ are the (k-1)-simplices $[v_0, ..., \hat{v_i}, ..., v_k]$ where $\hat{v_i}$ means omit the entry v_i.*

Definition 2.24. *Let X be a topological space. A **singular k-simplex** in X is a continuous map $\sigma \colon \Delta^k \to X$. The **k-th chain group** of X is the abelian group*

$$C_k(X) := \{\sum_{i=0}^{k} n_i \sigma_i : k \in \mathbb{N}_0, n_i \in \mathbb{Z}, \sigma_i \text{ is a k-simplex}\}$$

*where the elements are called **k-chains**.*

Remark 2.25. *$C_k(X)$ is an example of a **free abelian group**; that is, an abelian group whose elements are of the form $\sum_i n_i s_i$ where each s_i belongs to some set S.*

Definition 2.26. *The **boundary map** $\partial = \partial_k \colon C_k(X) \to C_{k-1}(X)$ is given by*

$$\partial \sigma = \sum_{i=0}^{k}(-1)^i \sigma \circ F_{e_0, ..., \hat{e_i}, ..., e_k} \in C_{k-1}(X)$$

on single simplices and extended linearly to all of $C_k(X)$.
*A k-chain $c \in C_k(X)$ is a **k-cycle** if $\partial(c) = 0$.*

Intuitively, this definition means that the boundary map decomposes σ into a sum of maps to its faces–or rather the images of faces of Δ^k–taking into account orientation.

Lemma 2.27. *$\partial \circ \partial \colon C_k(X) \to C_{k-2}(X)$ is zero.*

Proof. Let $\sigma \in C_k(X)$.

$$\partial \circ \partial(\sigma) = \partial(\sum_{i=0}^{k}(-1)^i \sigma \circ F_{[e_0, ..., \hat{e_i}, ..., e_k]})$$

$$= \sum_{i=0}^{k}(-1)^i \partial \circ \sigma \circ F_{[e_0, ..., \hat{e_i}, ..., e_k]}$$

(by linearity)

$$= \sum_{i=0}^{k}(\sum_{j<i}(-1)^{i+j}\sigma \circ F_{[e_0, ..., \hat{e_j}, ..., \hat{e_i}, ..., e_k]} + \sum_{i<j}(-1)^{i+j-1}\sigma \circ F_{[e_0, ..., \hat{e_i}, ..., \hat{e_j}, ..., e_k]})$$

(since e_j gets shifted forward one spot when e_i is removed first)

$$= \sum_{i=0}^{k}(\sum_{j<i}(-1)^{i+j}\sigma \circ F_{[e_0, ..., \hat{e_j}, ..., \hat{e_i}, ..., e_k]} + \sum_{j<i}(-1)^{i+j-1}\sigma \circ F_{[e_0, ..., \hat{e_j}, ..., \hat{e_i}, ..., e_k]})$$

(interchanging j with i in the second term)

$$= \sum_{i=0}^{k}(\sum_{j<i}((-1)^{i+j} + (-1)^{1+j-1})\sigma \circ F_{[e_0, ..., \hat{e_j}, ..., \hat{e_i}, ..., e_k]}) = 0$$

$\square$

This corresponds to the fact that the boundary of a ball is a sphere, which in turn has no boundary. It also implies that $\text{im}(\partial : C_{k+1}(X) \to C_k(X)) \in \ker(\partial : C_k(X) \to C_{k-1}(X))$, allowing the following to be well-defined.

Definition 2.28. *The **degree k singular homology** of X is the abelian group*

$$H_k(X) = \frac{ker(\partial : C_k(x) \to C_{k-1}(X))}{im(\partial : C_{k+1}(x) \to C_k(X))}$$

To give a more general result we shall allow the chain groups to have coefficients in any abelian group G as opposed to merely $\mathbb{Z}$.

Definition 2.29. *The k-th singular chain group with coefficients in an abelian group G is*

$$C_k(X; G) := \{\sum_{i=1}^{k} n_i \sigma_i : k \in \mathbb{N}_0, n_i \in G, \sigma_i \text{ is a } k\text{-simplex}\}.$$

The boundary map $\partial \colon C_k(X; G) \to C_{k-1}(X; G)$ and k-th homology group with coefficients in G, $H_k(X; G)$, are then defined as above while replacing $\mathbb{Z}$ with G.

Let $f \colon X \to Y$ be a continuous map between topological spaces. Given a k–simplex $\sigma \colon \Delta^k \to X$ in X, $f \circ \sigma$ is a k–simplex in Y. This gives rise to a homomorphism

$$f_\# \colon C_k(X; G) \to C_k(Y; G) : \sum_{i=0}^{k} n_i \sigma_i \mapsto \sum_{i=0}^{k} n_i f \circ \sigma_i.$$

Definition 2.30. *This induces a **push-forward homomorphism** $f_* \colon H_k(X; G) \to H_k(Y; G)$ sending $[c]$ to $[f_\#(c)]$.*

To show that this is well-defined, note that $\partial f_\# = f_\# \partial$ and so $f_\#$ maps closed chains to closed chains. Furthermore suppose $[a - b] = [0]$. Then there exists a $\sigma \in C_{k+1}(X; G)$ such that

$$a - b = \partial \sigma \implies f_\#(a - b) = f_\# \partial \sigma = \partial f_\# \sigma \implies [f_\#(a) - f_\#(b)] = [0] = f_*([a]) - f_*([b])$$

as required.

The condition $\partial f_\# = f_\# \partial$ makes $f_\#$ a **chain map**. This argument applies more generally to show that all chain maps induce a well-defined map on homology groups.

Proposition 2.31. *The push-forward is functorial.*

Proof. Let X, Y and Z be topological spaces. Clearly $\text{id}_{X*} = \text{id}_{H_k(X;G)}$.

Now let $f \colon Y \to Z$ and $g \colon X \to X$ be continuous maps. Given $[c] = [\sum_{i=0}^{k} n_i \sigma_i] \in H_k(X; G)$,

$$(f \circ g)_*([c]) = [\sum_{i=0}^{k} n_i f \circ g(\sigma_i)] = f_*([\sum_{i=0}^{k} n_i g(\sigma_i)]) = f_* \circ g_*([c])$$

so $(f \circ g)_* = f_* \circ g_*$. $\qquad\qquad\square$

This then implies the standard results from category theory such as the push-forward sending commutative diagrams to commutative diagrams and homeomorphisms to group isomorphisms.

Theorem 2.32 (Hatcher [1, Theorem 2.10]). *If $f, g\colon X \to Y$ are homotopic continuous maps between topological spaces then $f_* = g_*\colon H_k(X) \to H_k(Y)$*

Proof. Let $F\colon X \times I \to Y$ be a homotopy from f to g. Given a singular n–simplex $\sigma\colon \Delta^n \to X$ we can define $\sigma \times \mathrm{id}\colon \Delta^n \times I \to X \times I$ and $F_\#(\sigma \times \mathrm{id})\colon \Delta^n \times I \to Y$. We now decompose $\Delta^n \times I$ as the union of $n+1$-simplices. To do this think of $\Delta^n \times I$ as the complex hull of $a_0, ..., a_n, b_0, ..., b_n$ where $a_i = (e_i, 0)$ and $b_i = (e_i, 1)$. Then $\Delta^n \times I$ is the union of the $n+1$-simplices $\alpha_i = [a_0, ..., a_i, b_i, ..., b_n]$. Now define the "prism map" $P\colon C_n(X) \to C_{n+1}(Y)$ by $P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathrm{id})|_{\alpha_i}$.

$$\partial P(\sigma) = \sum_i (-1)^i \Big[\sum_{j \leq i} (-1)^j F \circ (\sigma \times \mathrm{id})\big|_{[a_0,...,\hat{a_j},...,a_i,b_i,...,b_n]}$$
$$+ \sum_{j > i} (-1)^{(j+1)} F \circ (\sigma \times \mathrm{id})\big|_{[a_0,...,a_i,b_i,...,\hat{b_j},...,b_n]} \Big].$$

The sum of terms with $i = j$ is

$$F \circ (\sigma \times \mathrm{id})\big|_{[\hat{a_0},b_0,...,b_n]} - F \circ (\sigma \times \mathrm{id})\big|_{[a_0,\hat{b_0},b_1,...,b_n]} + F \circ (\sigma \times \mathrm{id})\big|_{[a_0,\hat{a_1},b_1,...,b_n]}$$
$$- F \circ (\sigma \times \mathrm{id})\big|_{[a_0,a_1,\hat{b_1},...,b_n]} + ... + F \circ (\sigma \times \mathrm{id})\big|_{[a_0,...,\hat{a_n},b_n]} - F \circ (\sigma \times \mathrm{id})\big|_{[a_0,...,a_n,\hat{b_n}]}$$
$$= F \circ (\sigma \times \mathrm{id})\big|_{[\hat{a_0},b_0,...,b_n]} - F \circ (\sigma \times \mathrm{id})\big|_{[a_0,...,a_n,\hat{b_n}]} = F \circ (\sigma \times \mathrm{id})|_{[b_0,...,b_n]} - F \circ (\sigma \times \mathrm{id})|_{[a_0,...,a_n]}$$

as a telescoping sum which is then $= g \circ \sigma - f \circ \sigma = (g_\# - f_\#)(\sigma)$. The terms with $i \neq j$ can be identified with $-P(\partial \sigma)$ so that $\partial P\sigma = -P(\partial \sigma) + g_\#(\sigma) - f_\#(\sigma) \implies \partial P + P\partial = g_\# - f_\#$. $\partial(\partial P + P\partial) = \partial P\partial = (\partial P + P\partial)\partial$ so $(\partial P + P\partial)$ is a chain map inducing a homomorphism on homology groups. Furthermore given a cycle σ we have $(\partial P + P\partial)(\sigma) = \partial P(\sigma) + P\partial(\sigma) = \partial P(\sigma)$ so $\partial P(\sigma)$ sends cycles to boundaries thereby inducing the zero map on homology groups. Thus $g_* - f_* = 0$ or $f_* = g_*$. $\qquad \square$

We are now ready to prove that homology is a topological invariant.

Proposition 2.33. *If X and Y are homotopy-equivalent then $H_k(X; G) \cong H_k(Y; G)$*

Proof. We have continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ giving $g_* \circ f_* = \mathrm{id}_{H_k(X;G)}$ and $f_* \circ g_* = \mathrm{id}_{H_k(Y;G)}$. Thus f_* has a left and right inverse so is a bijection so a group isomorphism. $\qquad \square$

Lemma 2.34. *If X is path-connected then $H_0(X) = \mathbb{Z}$.*

Proof. A singular 0-simplex is a map from a point to a point $x \in X$. let $c = \sum n_i \sigma_i$ be a 0-chain. $\partial c = 0$ so $\ker(\partial\colon C_0(x) \to C_{-1}(X)) = C_0(X)$ giving $H_0(X) = \frac{C_0(X)}{\partial C_1(X)}$. Let $x, y \in X$ and let σ_x and σ_y be their corresponding simplicial 0-simplices. X is path-connected so there exists a simplicial 1-simplex σ with $\partial \sigma = \sigma_x - \sigma_y$. Define $f\colon C_0(X) \to \mathbb{Z}\colon \sum n_i x_i \mapsto \sum n_i$. Clearly $\partial C_1(X) \in \ker f$. Let $z = \sum_{i=0}^{k} a_i \sigma_i \in \ker f$. Express $a_i \sigma_i$ as $\sum_{j=1}^{|a_i|} a_{ij} \sigma_i$ where a_{ij} is either all 1 or all -1. Thus $z = \sum_{j=1}^{r} b_j \tau_j$ where $\tau_j \in \{\sigma_0, ..., \sigma_k\}$ and $b_j = \pm 1$. $|\{j : b_j = 1\}| = |\{j : b_j = -1\}|$ so r is even and we can express z as a sum of terms of the form $(\sigma_\alpha - \sigma_\beta) \in \partial C_1(X)$, giving $\ker f \in \partial C_1(X)$. Thus $\ker f \in \partial C_1(X)$. f is surjective so the result follows by the first isomorphism theorem. $\qquad \square$

Lemma 2.35. $H_k(\{pt\}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$

Proof. There is only one k-simplex $\sigma_k \colon \Delta^k \to \{pt\}$, the constant map. Thus $C_k(\{pt\}) = \{n\sigma_k : n \in \mathbb{Z}\} = \mathbb{Z}$. Also $\partial_k(n\sigma_k) = \sum_{i=0}^{k}(-1)^i n\sigma_{k-1}$ which is zero if k is odd and $n\sigma_{k-1}$ if k is even. Thus the induced map on the coefficients of the boundary is 0 if k is odd and id if k is even. The homology groups will then be $\frac{\mathbb{Z}}{\mathbb{Z}} = 0$ or $\frac{\{0\}}{\{0\}} = 0$, except for H_0 which is $\mathbb{Z}$ since $\{pt\}$ is path-connected. $\qquad\square$

Lemma 2.36. *If $\{X_i\}$ is the set of path-components of a topological space X, then*

$$H_k(X; G) \cong \bigoplus_i H_k(X_i; G).$$

Proof. The image of any singular k–simplex is contained in a path-component so

$$C_k(X; G) \cong \bigoplus_i C_k(X_i; G).$$

$\qquad\square$

2.3 Hurewicz theorem

We shall now see a connection between the fundamental group and the first homology group by the Hurewicz theorem. Before stating and proving it, however, we first require some tools from group theory.

Definition 2.37. *The **commutator subgroup** of a group G, denoted $[G, G]$, is the subgroup generated by all elements, called commutators, of the form $ab(ba)^{-1}$ for any a and b in G.*

Lemma 2.38. $[G, G]$ *is a normal subgroup.*

Proof. Let $g \in G, aba^{-1}b^{-1} \in [G, G]$.

$$\begin{aligned} g^{-1}(aba^{-1}b^{-1})g &= g^{-1}agg^{-1}bgg^{-1}a^{-1}gg^{-1}b^{-1}g \\ &= (g^{-1}ag)(g^{-1}bg)(g^{-1}a^{-1}g)(g^{-1}b^{-1}g) \\ &= (g^{-1}ag)(g^{-1}bg)(g^{-1}ag)^{-1}(g^{-1}bg)^{-1} \in [G, G]. \square \end{aligned}$$

Definition 2.39. *The **abelianization** G^{ab} of a group G is $G/[G, G]$.*

As the name suggests, G^{ab} is abelian since we quotiented out by the relation that $ab = ba$ for all $a, b \in G$. If G is already abelian then $[G, G]$ is trivial so $G^{ab} = G$.

Lemma 2.40. *Given groups G and H, where H is abelian, and a group homomorphism $h \colon G \to H$, the induced homomorphism $h' \colon G^{ab} \to H : [x] \mapsto h(x)$ is well-defined.*

Proof. Let $[x] = [y] \in G^{ab}$ so that $x = y \prod_i a_i b_i a_i^{-1} b_i^{-1}$ for $a_i, b_i \in G$. Then

$$h'([x]) = h(y \prod_i a_i b_i a_i^{-1} b_i^{-1}) = h(y) \prod_i h(a_i)h(b_i)h(a_i)^{-1}h(b_i)^{-1} = h(y) = h'([y])$$

since H is abelian. $\qquad\square$

Lemma 2.41. *Let G be a group. If w is a word in G such that for any $g \in G$ the exponents of g in w sum to zero then $w \in [G, G]$*

Proof. Consider the quotient map $\pi\colon G \to G/[G,G] : g \mapsto [g]$. $G/[G,G]$ is abelian so the letters in $[w]$ can be reordered such that $[w] = \sum n_i[a_i] = 0$ implying $w \in \ker \pi = [G,G]$. $\qquad\qquad\square$

Theorem 2.42. *(Hurewicz). If X is path-connected then $H_1(X) \cong \pi_1(X)^{ab}$.*

Proof. First note that an element of $\pi_1(X, x_0)$ is of the form $[\gamma]$ for $\gamma\colon [0,1] \to X$ a loop based at x_0. $[0,1] = \Delta^1$ so γ is a 1-simplex and $\partial(\gamma) = x_0 - x_0$ making γ closed. We can then define $h\colon \pi_1(X, x_0) \to H_1(X)$ sending an element $[\gamma] \in \pi_1(X, x_0)$ to $[c_\gamma] \in H_1(X)$ where c_γ is γ as an element of $C_1(X)$. To show that h is well-defined let $[a] = [b] \in \pi_1(X, x_0)$ and let $F\colon [0,1]^2 \to X$ be a based-homotopy from a to b. Dividing $[0,1]^2$ into two triangles and parametrising each by Δ^2 allows F to be expressed as $c_F \in C_2(X)$ with $\partial c_F = c_a - c_b \implies [c_a] = [c_b]$ as required.
Next we show that h is a homomorphism.
Let $[f], [g] \in \pi_1(X, x_0)$ and define a singular 2-simplex $\sigma\colon \Delta^2 \to X$ where $\sigma_{|[e_0,e_1]} = f$, $\sigma_{|[e_1,e_2]} = g$ and $\sigma_{|[e_0,e_2]} = f \cdot g$. Then $\partial(\sigma) = g - f \cdot g + f$ so $h([f][g]) = h([f]) + h([g])$ making h a homomorphism.
We now show that h is surjective.
For the sake of lucidity we will write $[\gamma]$ for $h([\gamma])$ instead of $[c_\gamma]$

Let $[\lambda] \in H_1(X)$ where $\lambda = \sum_{i=0}^{k} n_i\sigma_i$. $\partial(\lambda) = 0 = \sum_{i=0}^{k} n_i(\sigma_i(1) - \sigma_i(0))$.

Let $\Lambda = \{\sigma_i(0)\} \cup \{\sigma_i(1)\}$ be the collection of end-points of the σ_is. For each $p \in \Lambda$ pick a path $\beta_p\colon [0,1] \to X$ from x_0 to p (which is possible because X is path-connected). Then

$$\sum_{i=0}^{k} n_i(\sigma_i(1) - \sigma_i(0)) = \sum m_p p = 0$$

where m_p is the sum of the coefficients of p in the sum. This implies that $m_p = 0 \forall p$. Now consider the loop $\eta_i = \beta_{\sigma_i(0)} \cdot \sigma_i \cdot \beta_{\sigma_i(1)}^{-1}$ based at x_0. Then

$$
\begin{aligned}
h([\eta_0^{n_0} \cdot \eta_1^{n_1} ... \cdot \eta_k^{n_k}]) &= [n_0\eta_0 + ... + n_k\eta_k] \\
&= \Big[\sum_{i=0}^{k} n_i(\beta_{\sigma_i(0)} + \sigma_i - \beta_{\sigma_i(1)})\Big] \\
&= \Big[\sum_{i=0}^{k} n_i\sigma_i + \sum_{i=0}^{k} n_i(\beta_{\sigma_i(0)} - \beta_{\sigma_i(1)})\Big] \\
&= \Big[\sum_{i=0}^{k} n_i\sigma_i - \sum m_p\beta_p\Big] \\
&= \Big[\sum_{i=0}^{k} n_i\sigma_i\Big] = [\lambda]
\end{aligned}
$$

since $m_p = 0$.
We now find the kernel.
Let $[\gamma] \in \pi_1(X, x_0)$ such that $h([\gamma]) = [0]$. Then γ considered as a 1–simplex is exact so there exists

a finite sum of singular 2-simplices $\sum_{i=0}^{k} n_i \sigma_i$ such that

$$\gamma = \partial \left(\sum_{i=0}^{k} n_i \sigma_i \right) = \sum_{i=0}^{k} n_i \partial(\sigma_i) = \sum_{i=0}^{k} n_i (\lambda_i + \mu_i + \nu_i)$$

Let $\Lambda = \{\lambda_i\} \cup \{\mu_i\} \cup \{\nu_i\}$ and for each end-point p of a singular 1-simplex in Λ let β_p be a path from x_0 to p. For each $\theta \in \Lambda$ let m_θ be the sum of coefficients of θ in $\sum_{i=0}^{k} n_i (\lambda_i + \mu_i + \nu_i)$. This sum is equal to γ thus m_θ is 1 if $\theta = \gamma$ and 0 otherwise. For each singular 2-simplex σ_i the loops $\beta_{\lambda_i(0)} \cdot \lambda_i \cdot \beta_{\lambda_i(1)}^{-1}$, $\beta_{\mu_i(0)} \cdot \mu_i \cdot \beta_{\mu_i(1)}^{-1}$ and $\beta_{\nu_i(0)} \cdot \nu_i \cdot \beta_{\nu_i(1)}^{-1}$ have null-homotopic concatenations. Let $\eta_i = \beta_{\partial \sigma_i(0)} \cdot \partial \sigma_i \cdot \beta_{\partial \sigma_i(1)}^{-1}$. Then $h([\eta_0^{n_0} \cdot ... \cdot \eta_k^{n_k} \cdot \gamma^{-1}]) = h([\gamma])h([\gamma^{-1}]) = [\gamma^{-1}]$. Now write each $[\eta_i]$ as $[\beta_{\lambda_i(0)} \cdot \lambda_i \cdot \beta_{\lambda_i(1)}^{-1}][\beta_{\mu_i(0)} \cdot \mu_i \cdot \beta_{\mu_i(1)}^{-1}][\beta_{\nu_i(0)} \cdot \nu_i \cdot \beta_{\nu_i(1)}^{-1}]$. Then for each $\theta \in \Lambda$ we have that the sum of exponents of $[\beta_{\theta(0)} \cdot \theta \cdot \beta_{\theta(1)}^{-1}]$ is m_θ The exponents of each $[\beta_{\theta(0)} \cdot \theta \cdot \beta_{\theta(1)}^{-1}]$ in $[\eta_0^{n_0} \cdot ... \cdot \eta_k^{n_k} \cdot \gamma^{-1}]$ is then 0 so by the above lemma $[\gamma^{-1}] \in [\pi_1(X, x_0), \pi_1(X, x_0)]$ meaning $[\gamma]$ is as well. Thus ker $h \subseteq [\pi_1(X, x_0), \pi_1(X, x_0)]$. We also have that $[\pi_1(X, x_0), \pi_1(X, x_0)] \subseteq$ ker h since $H_1(X)$ is abelian so ker $h = [\pi_1(X, x_0), \pi_1(X, x_0)]$. The result then follows by the first isomorphism theorem. $\qquad \square$

Corollary 2.43.

$$H_1(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_1(T^2) = \mathbb{Z}^2$$

$$H_1(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

2.4 Relative homology

Relative homology will be used to define the orientation of manifolds in terms of homology.

Definition 2.44. *[3]Let X be a topological space and let A be a subspace of X. The n'th **relative chain group** is*

$$C_n(X, A) := \frac{C_n(X)}{C_n(A)}.$$

The boundary map

$$\partial : C_n(X) \to C_{n-1}(X)$$

restricts to a boundary map

$$\partial_{|C_n(A)} : C_n(A) \to C_{n-1}(A)$$

and thus induces a boundary map ∂' on $C_n(X; A)$. This then gives a chain complex

$$... \to C_n(X, A) \xrightarrow{\partial'} C_{n-1}(X, A) \to ...$$

since $\partial'^2 = 0$.

This then allows for the following definition.

Definition 2.45. *Let X be a topological space and let A be a subspace of X. The nth **relative homology group** is*

$$H_n(X, A) := \frac{\ker \partial' : C_n(X, A) \to C_{n-1}(X, A)}{\operatorname{Im} \partial' : C_{n+1}(X, A) \to C_n(X, A)}.$$

Definition 2.46. *If $U \subseteq A \subseteq X$ are topological spaces, then U can be **excised** if the inclusion map of $(X \backslash U, A \backslash U)$ into (X, A) induces an isomorphism on the relative homologies:*

$$H_n(X \backslash U, A \backslash U) \cong H_n(X, A),$$

As before, continuous maps induce homomorphisms on relative homology groups and homotopy-equivalences induce isomorphisms.

Lemma 2.47. *Let X and A be topological spaces with A a subspace of X. Then we have a long exact sequence*

$$... \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to ...$$

Definition 2.48. *Let X be a topological space and let $x_0 \in X$. The nth **local homology group** of X at x_0 is*

$$H_n(X, X - x_0) := H_n(X, X \backslash \{x_0\}).$$

Theorem 2.49. *(Excision theorem) Let $U \subseteq A \subseteq X$ be topological spaces. If the closure of U is contained in the interior of A, then U can be excised.*

3 Cohomology

Cohomology is the dual notion to homology – instead of chains, we have cochains (homomorphisms of chains); and instead of boundaries, we have coboundaries. We shall see that the cohomology groups admit a ring structure, providing them with more content. We shall also study Poincaré duality and the universal coefficient theorem, which give relationships between the homology and cohomology groups.

3.1 Modules

[Introduction to tor [4]][Differential topology [5]] The context in which the homomorphisms hitherto mentioned arise is that of modules, which are generalisations of vector spaces, where the field is replaced by a ring. Free abelian groups, like the homology groups, are examples of modules. We shall now develop the basic theory of modules to allow for the definition of the cohomology groups.

Definition 3.1. *Let R be a commutative ring with a unit. An **R-module** M is an abelian group with a map $f\colon R \times M \to M$ such that if $f(r,m)$ is denoted by rm then $(r_1 r_2)m = r_1(r_2 m)$ and $1m = m$. We usually abbreviate R-modules to modules.*

Remark 3.2. *More specifically we defined a **left R-module**. If instead multiplication were defined to satisfy $r_1(r_2 m) = (r_2 r_1)m$ then M would be a **right R-module**.*

Definition 3.3. *Let M and N be R-modules. An **R-module homomorphism** $f\colon M \to N$ is a homomorphism of abelian groups such that $f(rm) = rf(m)$*

Example 3.4. *An abelian group G is a $\mathbb{Z}$-module given by* $ng = \begin{cases} \sum\limits_{i=1}^{n} g & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -\sum\limits_{i=1}^{-n} g & \text{if } n < 0 \end{cases}$

Definition 3.5. *Let A and B be R-modules. Their **direct sum** $A \oplus B$ is the module $A \times B$ under component-wise operations i.e. $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $r(a,b) = (ra, rb)$. This can be extended to any finite collection of R-modules.*

Definition 3.6. *A bilinear map $h\colon M \times N \to P$ is a map such that $h(m_1 + m_2, n) = h(m_1, n) + h(m_2, n)$, $h(m, n_1 + n_2) = h(m, n_1) + h(m, n_2)$ and $h(rm, n) = h(m, rn)$ for $r \in R$.*

Definition 3.7. *Given a ring R and set S, we define the module R^S to be functions from S to R, with addition and multiplication defined piecewise.*

Definition 3.8. *Given a module M, we say that $H \subseteq M$ is a **submodule** if $rh \in H \forall r \in R, h \in H$ and H is itself a module after inheriting the addition and multiplication maps of M.*

Definition 3.9. *Given a module M and submodule N, the **quotient R-module of M by N** is the quotient group M/N with multiplication given by $r(a + N) = ra + N \forall r \in R, a + N \in M/N$.*

Definition 3.10. *Let M and N be modules. Let I denote the submodule of $M \times N$ generated by the set of elements of the form $(\lambda m_1 + m_2, n) - \lambda(m_1, n) - (m_2, n)$, $(m, \lambda n_1 + n_2) - \lambda(m, n_1) - (m, n_2)$ for $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $\lambda \in R$. Then the **tensor product** of M and N is $M \otimes_R N = R^{M \times N}/I$ whose elements are represented as $m \otimes n$ for $m \in M$, $n \in N$.*

This gives rise to the relations

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \text{ and}$$

$$(rm) \otimes n = m \otimes (rn) = r(m \otimes n) \text{ for } r \in R.$$

Example 3.11. *If p and q are coprime then $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q = 0$ since there exist $m, n \in \mathbb{Z}$ such that $mp + nq = 1$ (Bezout's lemma) so $a \otimes b = (mp + nq)a \otimes b = (mpa) \otimes b + a \otimes (nqb) = 0 \forall a \otimes b \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q.$*

Example 3.12. *$C_k(X; G) = C_k(X) \otimes_{\mathbb{Z}} G.$*

Example 3.13. *$\mathbb{Z}_m \otimes \mathbb{Z}_n = \mathbb{Z}_{gcd(m,n)}.$*

Proof. We have the first isomorphism for modules that given a module homomorphism $f : M \to N$, $M/\ker f \cong \operatorname{im} f$ (the proof is analogous to the cases for groups, rings and vector spaces). So, define a module homomorphism $g \colon \mathbb{Z} \to \mathbb{Z}_m \otimes \mathbb{Z}_n : z \mapsto z(1 \otimes 1)$. g is surjective so $\operatorname{im} g = \mathbb{Z}_m \otimes \mathbb{Z}_n$. Clearly $\langle \gcd(m,n) \rangle \subseteq \ker g$. Now let $x \notin \langle \gcd(m,n) \rangle$. Then $x = s\gcd(m,n) + k$ with $0 < k < \gcd(m,n)$ by division with remainder. $g(x) = s\gcd(m,n)(1 \otimes 1) + g(k) = k(1 \otimes 1) \neq 0 \implies x \notin \ker g$ so $x \in \ker g \implies x \in \langle \gcd(m,n) \rangle$ giving $\langle \gcd(m,n) \rangle = \ker g$. $\square$

Definition 3.14. *Let M and N be modules over a ring R. The set of all module homomorphisms from M to N is denoted by $Hom_R(M, N)$. It is an abelian group under pointwise addition and a module if R is commutative.*

3.2 Cohomology

Definition 3.15. *Let X be a topological space and let G be an abelian group. The **kth cochain group with coefficients in G** is*

$$C^k(X; G) := Hom(C_k(X), G).$$

Definition 3.16. *We define $d \colon C^k(X; G) \to C^{k+1}(X; G)$ to be the dual of ∂ in the sense that if $\phi \in C^k(X; G)$ and $c \in C_{k+1}(X)$ then $(d\phi)(c) = \phi(\partial c) \in G$*

In particular, $(d \circ d\phi)(c) = (d\phi)(\partial c) = \phi(\partial \circ \partial c) = \phi(0) = 0$ so $d^2 = 0$.

Definition 3.17. *$c \in C_k(X)$ is a cycle if $\partial c = 0$ and a boundary if $c \in \operatorname{im} \partial$. $\phi \in C^k(X; G)$ is a cocycle if $d\phi = 0$ and a coboundary if $\phi \in \operatorname{im} d$.*

Definition 3.18. *The **kth cohomology group with coefficients in** G is*

$$H^k(X; G) = \frac{ker(d \colon C^k(X; G) \to C^{k+1}(X; G))}{im(d \colon C^{k-1}(X; G) \to C^k(X; G))}$$

*When G is $\mathbb{Z}$ we abbreviate it to $H^k(X)$ and call it simply the **kth cohomology group**.*

Definition 3.19. *Let $f \colon X \to Y$ be a continuous map between topological spaces and G be an abelian group. We have a homomorphism $f^{\#} \colon C^k(Y; G) \to C^k(X; G) : A \mapsto f^{\#}A$ given by $f^{\#}A(c) = A(f_{\#}(c))$. This in turn induces the **pullback homomorphism***

$$f^* \colon H^k(Y; G) \to H^k(X; G) : [A] \mapsto [f^{\#}A].$$

$df^{\#}A(c) = f^{\#}A(\partial c) = f^{\#}dA(c)$ so $df^{\#} = f^{\#}d$. Thus $f^{\#}$ is a chain map so f^* is well-defined.

Lemma 3.20. *There is a well-defined pairing $H^k(X; G) \times H_k(X) \to G$ given by $([\phi], [c]) = \phi(c)$.*

Proof. Let $[\phi] = [\psi]$ so that $\phi - \psi = d\rho$ for some $\rho \in C^{k-1}(X; G)$. Then $(\phi - \psi)(c) = d\rho(c) = \rho(\partial c) = 0$ since $c \in \ker \partial$. Thus $\phi(c) = \psi(c)$. Now let $[a] = [b]$ so that $a = b + \partial e$ for some $e \in C_{k+1}(X)$. $\phi(a) = \phi(b + \partial e) = \phi(b) + \phi(\partial e) = \phi(b) + d\phi(e) = \phi(b)$ since $\phi \in \ker d$. $\square$

This then gives that f^* is dual to f_* in the sense that $f^*[A]([c]) = [A](f_*([c]))$ for $[A] \in H^k(Y; G)$ and $[c] \in H_k(X)$.

Proposition 3.21. *Pullback is covariantly functorial.*

Proof. Let X, Y and Z be topological spaces and let G be an abelian group. Clearly $\mathrm{id}_X^* = \mathrm{id}_{H^k(X;G)}$. Now let $f \colon X \to Y$ and $g \colon Y \to Z$ be continuous maps.

$$(g \circ f)^*([A]) = [(g \circ f)^{\#}(A)] = [f^{\#} \circ g^{\#}(A)] = f^*([g^{\#}(A)]) = f^* \circ g^*([A]) \forall [A] \in H^k(Z; G).$$

Thus $(g \circ f)^* = f^* \circ g^*$. $\square$

Proposition 3.22. *Let X and Y be homotopy-equivalent topological spaces. Then*

$$H^k(X; G) \cong H^k(Y; G).$$

Proof. Dualise the proof of the homotopy-invariance of homology. $\square$

Thus the cohomology groups are also invariant under homotopy-equivalence like the homology groups.

3.3 Cup product and ring structure of cohomology

We shall now give the cohomology groups a ring structure via the cup product. This will turn out to be necessary for the definition of the torsion linking form.

Definition 3.23. *[6]Let R be a ring and X be a topological space. The **cup product** is the product on cochains*

$$C^i(X; R) \times C^j(X; R) \to C^{i+j}(X; R) : (\psi, \phi) \mapsto \psi \cup \phi$$

where

$$(\psi \cup \phi)\sigma = \psi(\sigma \circ F_{[e_0, \dots, e_i]})\phi(\sigma \circ F_{[e_i, \dots, e_{i+j}]})$$

for a singular $i + j$-simplex $\sigma \colon \Delta^{i+j} \to X$ and is extended linearly to $C_{i+j}(X)$.

Informally, this means evaluating ψ on the front face of σ, evaluating ϕ on the back face of σ, and multiplying them together.

Lemma 3.24. $d(\psi \cup \phi) = (d\psi) \cup \phi + (-1)^i \psi \cup d\phi.$

Proof. Let $\sigma \colon \Delta^{i+j+1} \to X$ be a singular simplex.

$$d(\psi \cup \phi)(\sigma) = \psi \cup \phi(\partial \sigma)$$

$$= \psi \cup \phi \big(\sum_{t=0}^{j+k+1} (-1)^t \sigma \circ F_{[e_0,...,\hat{e_t},...,e_{i+j+1}]} \big)$$

$$= \sum_{t=0}^{i+j+1} (-1)^t \psi \cup \phi(\sigma \circ F_{[e_0,...,\hat{e_t},...,e_{i+j+1}]})$$

$$= \sum_{t=0}^{i} (-1)^t \psi \cup \phi(\sigma \circ F_{[e_0,...,\hat{e_t},...,e_{i+j+1}]}) + \sum_{t=i+1}^{i+j+1} (-1)^t \psi \cup \phi(\sigma \circ F_{[e_0,...,\hat{e_t},...,e_{i+j+1}]})$$

$$= \sum_{t=0}^{i} (-1)^t \psi(\sigma \circ F_{[e_0,...,\hat{e_t},...,e_{i+1}]}) \phi(\sigma \circ F_{[e_{i+1},...,e_{i+j+1}]})$$

$$+ \sum_{t=i+1}^{i+j+1} (-1)^t \psi(\sigma \circ F_{[e_0,...,e_i]}) \phi(\sigma \circ F_{[e_i,...,\hat{e_t},...,e_{i+j+1}]})$$

$$= \sum_{t=0}^{i+1} (-1)^t \psi(\sigma \circ F_{[e_0,...,\hat{e_t},...,e_{i+1}]}) \phi(\sigma \circ F_{[e_{i+1},...,e_{i+j+1}]})$$

$$+ \sum_{t=i}^{i+j+1} (-1)^t \psi(\sigma \circ F_{[e_0,...,e_i]}) \phi(\sigma \circ F_{[e_i,...,\hat{e_t},...,e_{i+j+1}]}).$$

We have

$$(d\psi \cup \phi)(\sigma) = \sum_{t=0}^{i+1} (-1)^t \psi(\sigma \circ F_{[e_0,...,\hat{e_t},...,e_{i+1}]}) \phi(\sigma \circ F_{[e_{i+1},...,e_{i+j+1}]})$$

and

$$(-1)^i (\psi \cup d\phi)(\sigma) = \sum_{t=i}^{i+j+1} (-1)^t \psi(\sigma \circ F_{[e_0,...,e_i]}) \phi(\sigma \circ F_{[e_i,...,\hat{e_t},...,e_{i+j+1}]})$$

When these expressions are added, the last term of the first sum cancels with the first term of the second sum, leaving $d(\psi \cup \phi)(\sigma)$. $\qquad \square$

Proposition 3.25. *There is a well-defined product $H^i(X;R) \times H^j(X;R) \to H^{i+j}(X;R)$ given by $([\psi],[\phi]) \mapsto [\psi] \cup [\phi] = [\psi \cup \phi]$.*

Proof. $d(\psi \cup \phi) = 0 \cup \phi + (-1)^i \psi \cup 0 = 0$ so $\psi \cup \phi \in \ker d$. Now let $\psi = df$. Then

$$d(f \cup \phi) = (df) \cup \phi \pm f \cup d\phi = \psi \cup \phi + f \cup 0 = \psi \cup \phi.$$

Similarly if $\phi = dg$ then $d((-1)^i \psi \cup g) = \psi \cup \phi$ so adding an element in the image of d keeps $\psi \cup \phi$ in the same equivalence class. $\qquad \square$

Definition 3.26. *We call the induced product $\cup : H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$ the **cup product** on cohomology.*

We shall now see that the cup product gives cohomology a ring structure.

Definition 3.27. *A **graded ring** is a ring such that its additive group is a direct sum of abelian groups $\bigoplus_{n=0}^{\infty} R_n$ such that $R_m R_n \subseteq R_{m+n} \forall m, n \in \mathbb{N}_0$. An element of R_n is said to have **degree** n. A graded ring is **graded-commutative** if we have $xy = (-1)^{|x||y|}yx$, where $|x|$ and $|y|$ denote the degrees of x and y.*

Proposition 3.28. $H^*(X; R) = \bigoplus_{n=0}^{\infty} H^n(X; R)$ *is a graded ring with multiplication given by the cup product.*

Proof. The direct sum of abelian groups is abelian so $H^*(X; R)$ is abelian as an additive group. To show that the product is associative and distributive it suffices to show that it is for cochains. For associativity:

$$(\psi \cup \phi) \cup \theta(\sigma) = (\psi(\sigma \circ F_{[e_0,...,e_k]})\phi(\sigma \circ F_{e_k,...,e_{k+l}}))\theta(\sigma \circ F_{[e_{k+l},...,e_{k+l+m}]})$$
$$= \psi(\sigma \circ F_{[e_0,...,e_k]})(\phi(\sigma \circ F_{[e_k,...,e_{k+l}]})\theta(\sigma \circ F_{[e_{k+l},...,e_{k+l+m}]}))$$
$$= (\psi \cup \phi) \cup \theta(\sigma).$$

For distributivity:

$$\psi \cup (\phi + \theta)(\sigma) = \psi(\sigma \circ F_{[e_0,...,e_i]})(\phi(\sigma \circ F_{[e_i,...,e_{i+j}]}) + \theta(\sigma \circ F_{[e_i,...,e_{i+j}]}))$$
$$= \psi(\sigma \circ F_{[e_0,...,e_i]})\phi(\sigma \circ F_{[e_i,...,e_{i+j}]}) + \psi(\sigma \circ F_{[e_0,...,e_i]})\theta(\sigma \circ F_{[e_i,...,e_{i+j}]})$$
$$= \psi \cup \phi(\sigma) + \psi \cup \theta(\sigma).$$

If R has an identity element then there is also an identity element given by $[1] \in H^0(X; R)$ which sends each singular 0-simplex to 1. $\qquad\square$

Proposition 3.29. *Let R be a commutative ring, $\psi \in H^k(X; R)$ and $\phi \in H^l(X; R)$. Then*

$$\psi \cup \phi = (-1)^{kl}\phi \cup \psi.$$

Thus $H^*(X; R)$ is graded-commutative.

Proof. See Hatcher page 210 [1, 210]. $\qquad\square$

Proposition 3.30. *Let $f \colon X \to Y$ be a continuous map. Then $f^*(\psi \cup \phi) = f^*(\psi) \cup f^*(\phi)$.*

Proof. As before, it suffices to check this for cochains. Let $\sigma \colon \Delta^{p+q} \to X$ be a singular simplex. Then

$$(f^*\psi \cup f^*\phi)(\sigma) = f^*\psi(\sigma \circ F_{[e_0,...,e_p]})f^*\phi(\sigma \circ F_{[e_p,...,e_{p+q}]})$$
$$= \psi(f \circ \sigma \circ F_{[e_0,...,e_p]})\phi(f \circ \sigma \circ F_{[e_p,...,e_{p+q}]})$$
$$= (\psi \cup \phi)(f \circ \sigma)$$
$$= f^*(\psi \cup \phi)(\sigma).\square$$

This makes f^* a ring homomorphism.

3.4 Cap product

Poincaré duality gives explicit isomorphisms between the homology and cohomology groups. However, we first need to study the cap product, with which the Poincaré duality isomorphism is defined.

Definition 3.31. *Let X be a topological space and R a commutative ring with an identity element. The **cap product** $C_{i+j}(X;R) \times C^i(X;R) \to C_j(X;R) : (c,\phi) \mapsto c \cap \phi$ given by*

$$\sigma \cap \phi = \phi(\sigma \circ F_{[e_0,...,e_i]})\sigma \circ F_{[e_i,...,e_{i+j}]}$$

for $\sigma \colon \Delta^j \to X$ a singular simplex and $\phi \in C^{i+j}(X;R)$ and extended linearly.

Lemma 3.32. $\partial(\sigma \cap \phi) = (-1)^i(\partial \sigma \cap \phi - \sigma \cap d\phi).$

Proof. We have

$$\partial \sigma \cap \phi = \sum_{t=0}^{i}(-1)^t \phi(\sigma \circ F_{[e_0,...,\hat{e}_t,...,e_{i+1}]})\sigma \circ F_{[e_{i+1},...,e_{i+j}]} + \sum_{t=i+1}^{i+j}(-1)^t \phi(\sigma \circ F_{[e_0,..,e_i]})\sigma \circ F_{[e_i,...,\hat{e}_t,...,e_{i+j}]},$$

$$\sigma \cap d\phi = \sum_{t=0}^{i+1}(-1)^t \phi(\sigma \circ F_{[e_0,...,\hat{e}_t,..,e_{i+1}]})\sigma \circ F_{[e_{i+1},...,e_{i+j}]},$$

$$\partial(\sigma \cap \phi) = \sum_{t=i}^{i+j}(-1)^{t-i}\phi(\sigma \circ F_{[e_0,..,e_i]})\sigma \circ F_{[e_i,...,\hat{e}_t,...,e_{i+j}]}.$$

Thus

$$\text{RHS} = (-1)^i\Big((-1)^{i+1+1}\phi(\sigma \circ F_{[e_0,...,e_i]})\sigma \circ F_{[e_{i+1},...,e_{i+j}]} + \sum_{t=i+1}^{i+j}(-1)^t \phi(\sigma \circ F_{[e_0,..,e_i]})\sigma \circ F_{[e_i,...,\hat{e}_t,...,e_{i+j}]}\Big)$$

$$= (-1)^i\Big(\sum_{t=i}^{i+j}(-1)^t \phi(\sigma \circ F_{[e_0,..,e_i]})\sigma \circ F_{[e_i,...,\hat{e}_t,...,e_{i+j}]}\Big)$$

$$= \sum_{t=i}^{i+j}(-1)^{t-i}\phi(\sigma \circ F_{[e_0,..,e_i]})\sigma \circ F_{[e_i,...,\hat{e}_t,...,e_{i+j}]} = \partial(\sigma \cap \phi).$$

$\square$

Proposition 3.33. *There is an induced cap product $H_{i+j}(X;R) \times H^i(X;R) \to H_j(X;R)$ which is R-linear in each variable.*

Proof. The cap product of a cycle and a cocycle is a cycle. $\partial\sigma = 0 \implies \partial(\sigma \cap \phi) = \pm(\sigma \cap d\phi)$ so the cap product of a cycle and a coboundary is a boundary. $d\phi = 0 \implies \partial(\sigma \cap \phi) = \pm(\partial\sigma \cap \phi)$ so the cap product of a boundary and a cocycle is a boundary. $\square$

Proposition 3.34. *Let $f\colon X \to Y$ be continuous, $c \in C_{i+j}(X;R)$ and $\phi \in C^i(Y;R)$. Then*

$$f_*(c) \cap \phi = f_*(c \cap f^*(\phi)).$$

Proof. Let $\sigma\colon \Delta^{i+j} \to X$ be a singular simplex. Then $(f\circ\sigma)\cap\phi = \phi(f\circ\sigma\circ F_{[e_0,...,e_j]})f\circ\sigma\circ F_{[e_j,...,e_{i+j}]}$. The result follows by linearity. $\square$

Proposition 3.35. *Let $\psi \in C^l(X;R)$, $\alpha \in C_{k+l}(X;R)$ and $\psi \in C^k(X;R)$. Then*

$$\psi(\alpha \cap \phi) = (\phi \cup \psi)(\alpha).$$

Proof. Let $\sigma\colon \Delta^{k+l} \to X$ be a singular simplex. Then

$$\psi(\sigma \cap \phi) = \psi(\phi(\sigma \circ F_{[e_0,...,e_k]})\sigma \circ F_{[e_k,...,e_{k+l}]}) = \phi(\sigma \circ F_{[e_0,...,e_k]})\psi(\sigma \circ F_{[e_k,...,e_{k+l}]}).$$

The result then follows by linearity. $\square$

3.5 Universal coefficient theorem

[5, Differential topology] This is the final connection between homology and cohomology which we shall study. It will be used in proving that the torsion linking form is non-degenerate and in obtaining the homology and cohomology groups of lens spaces.

To state the theorem, we require an additional definition from group theory.

Definition 3.36. *Let G be an abelian group. The **torsion subgroup** $T(G)$ is the subgroup consisting of all elements of G with finite order. For a prime number p, the **p-torsion** subgroup $T_p(G)$ is the subgroup consisting of all elements with order a power of p. The **free part** of G, denoted $F(G)$, is the subgroup comprising all elements of infinite order, as well as the identity.*

We then have that $G \cong T(G) \oplus F(G)$ for any abelian group G.

Proposition 3.37. *If G is a free abelian group or $G \cong \mathbb{Q}$, then $H_k(X) \otimes_{\mathbb{Z}} G \cong H_k(X; G)$.*
If $G = \mathbb{Z}_p$, then

$$\frac{H_k(X; \mathbb{Z}_p)}{H_k(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p} \cong T_p(H_{k-1}(X)).$$

Proposition 3.38. *Let R be a commutative ring and let A be an R-module. There is a natural inclusion $H_*(X; R) \otimes_R A \hookrightarrow H_*(X; A)$*
which is an isomorphism if A is free.

Let G be an abelian group. Recall the natural pairing $H^k(X; G) \times H_k(X) \to G$ given by

$$([\psi], [c]) \mapsto \psi(c).$$

Proposition 3.39. *The induced homomorphism*

$$H^k(X; G) \to Hom(H_k(X); G)$$

is surjective.

Now consider the case $G = \mathbb{Z}$ and set

$$K = \ker(H^k(X; \mathbb{Z}) \to \mathrm{Hom}(H_k(X), \mathbb{Z}))$$

Lemma 3.40. *Let $[\psi] \in K$ and $[c] \in H_{k-1}(X)$ such that $n[c] = 0$ for some $n \in \mathbb{N}$. Pick $b \in C_k(X)$ such that $nc = \partial b$. Then $\psi(b) \bmod n$ depends only on $[\psi]$ and $[c]$. In particular, there is a well-defined pairing $K \times T(H_{k-1}(X)) \to \mathbb{Q}/\mathbb{Z}$ given by $([\psi], [c]) \mapsto \frac{1}{n}\psi(b)$.*

Proposition 3.41. *If $H_*(X)$ is finitely generated, then $K \to Hom(T(H_{k-1}(X)), \mathbb{Q}/\mathbb{Z})$ is an isomorphism.*

Remark 3.42. *An isomorphism can also be induced by $([\psi], [c]) \mapsto \frac{1}{k}\gamma(c)$, where $\gamma \in C^{k-1}(X; \mathbb{Z})$ such that $d\gamma = k\phi$ for some $k \in \mathbb{N}$. This is the isomorphism we shall make use of in proving that the torsion linking form is non-degenerate.*

Corollary 3.43. $K = T(H^k(X; \mathbb{Z}))$.

Corollary 3.44. *If $H_*(X)$ is finitely generated, then the free part of $H^k(X; \mathbb{Z})$ is naturally dual to the free part of $H_k(X; \mathbb{Z})$ and the torsion part of $H^k(X; \mathbb{Z})$ is naturally dual to the torsion part of $H_{k-1}(X)$.*

4 Topology of manifolds

Manifolds are spaces locally homeomorphic to $\mathbb{R}^n$ and allow for additional topological invariants to be defined on them. We will first see that manifolds admit a concept of orientation, as a consequence of their n'th local homology groups being isomorphic to $\mathbb{Z}$, which allows for isomorphisms between the homology and cohomology groups to be defined (Poincaré duality). We will also see that certain types of manifolds admit a topological invariant known as the torsion linking form, which will be used for the homotopy classification of lens spaces.

4.1 Orientation of manifolds

We have already seen formulations of the concept of orientation in Differential and geometric analysis [7] through orientation forms or consistent choices of unit normals on submanifolds. However, here we shall present the homological definition, and see a result that will allow us to infer the orientability of lens spaces from the orientability of S^3.

Definition 4.1. *A topological space M is an **n-dimensional manifold** if it is Hausdorff and for every $p \in M$ there is an open neighbourhood $U \subseteq M$ containing p and a homeomorphism $f : U \to V$ where $V \subseteq \mathbb{R}^n$ is open.*

Lemma 4.2. *[3]Let M be an n–manifold, let $x \in M$ and let U be a neighbourhood of x homeomorphic to $\mathbb{R}^n$. Then*
$$H_n(M, M - x) \cong \mathbb{Z}.$$

Proof. We have
$$M - U \subseteq M - x \subseteq M$$
and
$$\overline{M - U} \subset (M - x)^{\text{o}}$$
so by the excision theorem we have
$$H_n(M, M - x) \cong H_n(U, U - x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - x).$$

$\mathbb{R}^n \cong D^n$ and S^{n-1} is a deformation retract of $\mathbb{R}^n - x$ so
$$H_n(M, M - x) \cong H_n(D^n, S^{n-1}).$$

The long exact sequence stated earlier then gives the long exact sequence
$$... \to H_n(D^n) \to H_n(D^n, S^{n-1}) \to H_{n-1}(S^{n-1}) \to H_{n-1}(D^n) \to ...$$

The disk is contractible so $H_k(D^n) = 0 \forall k$. We thus have the short exact sequence
$$0 \to H_n(D^n, S^{n-1}) \to H_{n-1}(S^{n-1}) \to 0.$$

Thus
$$H_n(M, M - x) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

$\square$

Definition 4.3. *Let M be an n–manifold. A **local orientation** of M at $x \in M$ is a choice of generator $[M]_x$ of $H_n(M, M - x)$. A choice of local orientations is called **continuous** if for each $x \in M$ there is an open neighbourhood U and $\alpha \in H_n(M, M - U)$ such that the map induced by the inclusion $(M, M - U) \to (M, M - y)$ maps α to $[M]_y$ for each $y \in U$. An **orientation** of M is a continuous choice of local orientations. M is **orientable** if it has an orientation.*

Definition 4.4. *Let M and N be orientable n–manifolds with orientations $[M]_x$ and $[N]_y$. Then a homeomorphism $f : M \to N$ is **orientation-preserving** if the induced isomorphism*

$$f_* : H_n(M, M - x) \to H_n(N, N - f(x))$$

*sends $[M]_x$ to $[N]_{f(x)}$ for all $x \in M$. If instead f_* sends $[M]_x$ to $-[N]_{f(x)}$ for all $x \in M$, then f is **orientation-reversing**.*

Remark 4.5. *Note that a homeomorphism between oriented manifolds will always be either orientation-preserving or orientation-reversing at each point, since the induced map on relative homology groups will be an isomorphism, and isomorphisms map units to units. Continuity then implies that a homeomorphism will be either orientation-preserving or orientation-reversing on each connected component.*

Remark 4.6. *The homological definitions of orientation and orientation-preserving/reversing homeomorphisms are consistent with those from MA40254 [7]for submanifolds in terms of differential forms.*

Lemma 4.7 (A short survey of lens spaces [8, 1.1.6])**.** *If X is a compact Hausdorff space acted on by a finite group G by homeomorphisms, then X/G is compact Hausdorff.*

Proof. Let $p : X \to X/G \colon x \mapsto [x]$ be the quotient map. X/G is compact as the continuous image of a compact space. Now let $[x_1], [x_2] \in X/G$ with $[x_1] \neq [x_2]$. Then $g_1 \cdot x_1 \neq g_2 \cdot x_2 \forall g_1, g_2 \in G$. Thus $p^{-1}([x_1]) = G \cdot x_1$ and $p^{-1}([x_2]) = G \cdot x_2$ are disjoint. These sets are finite since G is finite. Thus, since X is Hausdorff, we can construct disjoint open sets $U_1, U_2 \in X$ such that $p^{-1}([x_1]) \subseteq U_1$ and $p^{-1}([x_2]) \subseteq U_2$.

$$p^{-1}(p(X \backslash U_i)) = \bigcup_{g \in G} g \cdot (X \backslash U_i) = \bigcup_{g \in G} \rho_g(X \backslash U_i)$$

so $p^{-1}(p(X \backslash U_i))$ is the union of finitely many closed sets so is closed in X. Thus, by the definition of the quotient topology, $p(X \backslash U_i)$ is closed in X/G and $w_i = (X/G) \backslash p(X \backslash u_i)$ is open in X/G. We have

$$p^{-1}([x_i]) \subseteq u_i \implies [x_i] \notin p(X \backslash U_i) \implies [x_i] \in w_i.$$

Also,

$$
\begin{aligned}
w_1 \cap w_2 &= ((X/G) \backslash p(X \backslash U_1)) \cap ((X/G) \backslash p(X \backslash u_2)) \\
&= (X/G) \backslash (p(X \backslash U_1) \cup p(X \backslash U_2)) \\
&= (X/G) \backslash p((X \backslash U_1) \cup (X \backslash U_2)) \\
&= (X/G) \backslash p(X \backslash (U_1 \cap U_2)) \\
&= (X/G) \backslash p(X) = \emptyset.
\end{aligned}
$$

Thus X/G is Hausdorff. $\qquad\qquad\square$

Lemma 4.8. *An open subset of an n–manifold is an n–manifold.*

Proof. Let M be an n–manifold and let $U \subseteq M$ be open. Let $x \in U$. Then there exists an open neighbourhood $S \subseteq M$ of x and a homeomorphism $f : S \to V$ where $V \subseteq \mathbb{R}^n$ is open. $S \cap U$ is then also an open neighbourhood of x and $f_{|U}U \to f(U)$ is also a homeomorphism between open sets, since the restriction of a homeomorphism is a homeomorphism and homeomorphisms map open sets to open sets. U is also Hausdorff as a subspace of a Hausdorff space so U is an n–manifold. $\qquad \square$

Proposition 4.9. *If G is a finite group acting by a covering space action on X and X is a compact n–manifold, then X/G is a compact n–manifold.*

Proof. X/G is compact Hausdorff by the above lemma. It remains to show that X/G is locally homeomorphic to $\mathbb{R}^n$. The quotient map $p : X \to X/G$ is a covering map so is a local homeomorphism. Thus given $[x] \in X/G$ there is an open neighbourhood $S \subseteq X$ of x such that $p(S)$ is an open neighbourhood of $[x]$ and $p_{|S} : S \to p(S)$ is a homeomorphism. S is also an n–manifold by the above lemma so there exists an open neighbhhourhood $U \subseteq S$ of x and a homeomorphism $f : V \to U$ where $V \subseteq \mathbb{R}^n$ is open. $p_{|U} : U \to p(U)$ is also a homeomorphism between open sets, since restrictions of homeomorphisms are homeomorphisms and homeomorphisms map open sets to open sets, so $p_{|U} \circ f : V \to p(U)$ is a homeomorphism. $\qquad \square$

Theorem 4.10. *If we have a covering space action of a group G on a connected orientable manifold M by orientation-preserving homeomorphisms, then M/G is also orientable.*

Proof. Let $p : M \to M/G$ be the quotient map and let $[M]_x$ be an orientation of M. p is a covering map so a local homeomorphism. Thus, for every $x \in M$ there exists an open neighbourhood U_x such that $p_{|U_x} \to p(U_x)$ is a homeomorphism, giving an isomorphism

$$H_n(U_x, U_x - x) \overset{(p_{|U_x})_*}{\longrightarrow} H_n(p(U_x), p(U_x) - p(x)).$$

Excision then gives isomorphisms

$$H_n(M, M - x) \cong H_n(U_x, U_x - x)$$

and

$$H_n(p(U_x), p(U_x) - p(x)) \cong H_n(M/G, M/G - p(x)),$$

implying

$$H_n(M, M - x) \cong H_n(M/G, M/G - p(x)).$$

We thus have an orientation on M/G given by $(p_{|U_x})_*([M]_x)$ which is well-defined since G acts by orientation-preserving homeomorphisms. $\qquad \square$

4.2 Poincaré duality

Having studied the cap product, we are now ready to state Poincaré duality. However, Poincaré duality doesn't hold for arbitrary topological spaces, but for closed oriented manifolds. We shall now define and study those terms.

Definition 4.11. *A manifold is **closed** if it is compact and has no boundary.*

Remark 4.12. *For a manifold to have a boundary, we would have to consider homeomorphisms to the half plane $\mathbb{H}^n := \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 \leq 0\}$. The boundary would then the union of the images of the boundary of $\mathbb{H}^n$. All manifolds defined in terms of homeomorphisms into $\mathbb{R}^n$ automatically have no boundary, so compact implies closed.*

Closed manifolds admit a simpler definition of orientation to the more general one discussed earlier. However, both formulations are equivalent.

Theorem 4.13. *Any compact oriented manifold has a fundamental class.*

Definition 4.14. *An n-dimensional closed manifold M is **orientable** if $H_n(M) \cong \mathbb{Z}$. The image of a unit is called a **fundamental class**. An **orientation** of M is a choice of fundamental class $[M] \in H_n(M)$.*

Proposition 4.15. *Let M be a closed connected n-dimensional manifold. Then $H_n(M) \cong \mathbb{Z}$ if M is orientable. Furthermore, $H_i(M) = 0$ for $i > n$.*

Definition 4.16. *Let M and N be closed connected orientable n-dimensional manifolds, let $f \colon M \to N$ be continuous and let $[M]$ and $[N]$ be fundamental classes of X and Y. Then the **degree** of f is the integer $\deg(f)$ such that $f_*([M]) = \deg(f)[N]$.*

As before, homeomorphisms between closed oriented manifolds send fundamental classes to fundamental classes.

Corollary 4.17. *The degree of a homeomorphism between closed connected orientable n-manifolds is ± 1.*

We then also have an alternate definition for orientation-preserving and orientation-reversing homeomorphisms:

Definition 4.18. *Let M be a closed orientable n–dimensional manifold, let $[M] \in H_n(X)$ be a fundamental class and let $f \colon M \to M$ be a homeomorphism. We say that f is **orientation-preserving** if $f_*([M]) = [M]$ and that f is **orientation-reversing** if $f_*([M]) = -[M]$.*

We can now state the Poincaré Duality theorem as an explicit isomorphism.

Theorem 4.19. *(Poincaré Duality) [1]. If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M; R)$, then the map $D \colon H^k(M; R) \to H_{n-k}(M; R)$ defined by $D([\alpha]) = [M] \cap [\alpha]$ is an isomorphism for all k.*

4.3   Torsion linking form

The torsion linking form is a homotopy invariant for closed oriented manifolds. We shall define it and prove basic results about it here.

Definition 4.20. *Let M^n be a closed oriented manifold with fundamental class $[M] \in H_n(M; \mathbb{Z})$. Let $[\alpha] \in H^a(M; \mathbb{Z})$ and $[\beta] \in H^{n-a+1}(M; \mathbb{Z})$ be torsion classes, i.e. $k[\alpha] = 0, m[\beta] = 0$ for some $k, m \in \mathbb{N}$. Then we can write $k\alpha = d\gamma$ for some $\gamma \in C^{a-1}(M, \mathbb{Z})$. Let A be a cycle representing the fundamental class $[M]$. We define $[\alpha] \vee [\beta] = \frac{1}{k}(\gamma \cup \beta)A \in \mathbb{Q}/\mathbb{Z}$. The **torsion linking pairing** is then*

$$T(H^a(M; \mathbb{Z})) \times T(H^{n-a+1}(M; \mathbb{Z})) \to \mathbb{Q}/\mathbb{Z} : ([\alpha], [\beta]) \mapsto [\alpha] \vee [\beta].$$

Lemma 4.21. *$[\alpha] \vee [\beta]$ is well-defined.*

Proof. Let $d\gamma = d\eta = k\alpha$ for $\eta \in C^{a-1}(M, \mathbb{Z})$. Then $d(\gamma - \eta) = 0$ so $\gamma - \eta$ is a representative of an element of $H^{a-1}(M; \mathbb{Z})$.

$$((\gamma - \eta) \cup \beta)A = ([\gamma - \eta] \cup [\beta])[M]$$

so
$$m((\gamma - \eta) \cup \beta)A = ([\gamma - \eta] \cup [m\beta])[M] = 0.$$
Thus $m(\gamma \cup \beta)A = m(\eta \cup \beta)A$ so $(\gamma \cup \beta)A = (\eta \cup \beta)A$, implying independence of γ.
Now let $[\alpha] = [\alpha'] \in H^a(M; \mathbb{Z})$. Then $\alpha' = \alpha + de$ for some $e \in C^{a-1}(M, \mathbb{Z})$, so
$$k\alpha' = d\gamma + dke = d(\gamma + ke).$$
$$\frac{1}{k}((\gamma + ke) \cup \beta)A = \frac{1}{k}(\gamma \cup \beta)A + \frac{1}{k}k(e \cup \beta)A = \frac{1}{k}(\gamma \cup \beta)A \in \mathbb{Q}/\mathbb{Z}$$
implying independence of α.
Finally, let $[\beta] = [\beta'] \in H^{n-a+1}(M; \mathbb{Z})$. Then $\beta' = \beta + df$ for some $f \in C^{n-a}(M, \mathbb{Z})$.
$$d(\gamma \cup f) = k(\alpha \cup f) + (-1)^{a-1}(\gamma \cup df)$$
so
$$\frac{1}{k}(\gamma \cup df)A = \frac{(-1)^{1-a}}{k}d(\gamma \cup f)A = \frac{(-1)^{1-a}}{k}([0], [M]) = 0.$$
Thus
$$\frac{1}{k}(\gamma \cup \beta')A = \frac{1}{k}(\gamma \cup (\beta + df))A = \frac{1}{k}(\gamma \cup \beta)A + \frac{1}{k}(\gamma \cup df)A = \frac{1}{k}(\gamma \cup \beta)A,$$
implying independence of β. Thus the pairing is well-defined. $\qquad\square$

Lemma 4.22. $[\beta] \vee [\alpha] = (-1)^{a(n-a+1)}[\alpha] \vee [\beta]$.

Proof. Let $m\beta = d\tau$ for some $\tau \in C^{n-a}(M, \mathbb{Z})$. If we consider τ and α to be representatives of elements of $H^*(M, \mathbb{Z}_m)$, we have graded commutativity. Thus
$$\begin{aligned}
[\beta] \vee [\alpha] &= \frac{1}{m}(\tau \cup \alpha)A \\
&= \frac{1}{m}([\tau] \cup [\alpha], [M]) \\
&= (-1)^{a(n-a)}\frac{1}{m}([\alpha] \cup [\tau], [M]) \\
&= (-1)^{a(n-a)}\frac{1}{m}(\alpha \cup \tau)A \\
&= \frac{(-1)^{a(n-a)}}{km}(k\alpha \cup \tau)A \\
&= \frac{(-1)^{a(n-a)}}{km}(d\gamma \cup \tau)A \in \mathbb{Q}/\mathbb{Z}.
\end{aligned}$$
Furthermore,
$$d(\gamma \cup \tau) = (d\gamma \cup \tau) + (-1)^{a-1}(\gamma \cup d\tau) = (d\gamma \cup \tau) + (-1)^{a-1}(\gamma \cup m\beta)$$
so
$$\begin{aligned}
[\beta] \vee [\alpha] = \frac{(-1)^{a(n-a)}}{km}(d\gamma \cup \tau)A &= \frac{(-1)^{a(n-a)}}{km}(d(\gamma \cup \tau)A - (-1)^{a-1}(\gamma \cup m\beta)A) \\
&= \frac{(-1)^{a(n-a)}}{km}(-1)^a(\gamma \cup m\beta)A \\
&= \frac{(-1)^{a(n-a+1)}}{k}(\gamma \cup \beta)A \\
&= (-1)^{a(n-a+1)}[\alpha] \vee [\beta]\square
\end{aligned}$$

Lemma 4.23. *The torsion linking pairing is a non-degenerate bilinear form, where non-degeneracy means that $T(H^{n-a+1}(M, \mathbb{Z})) \to \mathrm{Hom}(T(H^a(M; \mathbb{Z})), \mathbb{Q}/\mathbb{Z}) : [\beta] \mapsto ([\alpha] \mapsto [\alpha] \vee [\beta])$ is an isomorphism.*

Proof. Bilinearity follows from the bilinearity of the cup product.
By the universal coefficient theorem, we have an isomorphism

$$G : T(H_{a-1}(M; \mathbb{Z})) \to \mathrm{Hom}(T(H^a(M; \mathbb{Z})), \mathbb{Q}/\mathbb{Z}) : [x] \mapsto ([\alpha] \mapsto \frac{1}{k}\gamma_{[\alpha]}(x))$$

where $d\gamma_{[\alpha]} = k[\alpha]$. By Poincaré duality we have an isomorphism

$$D : T(H^{n-a+1}(M; \mathbb{Z})) \to T(H_{a-1}(M; \mathbb{Z})) : [\beta] \mapsto (-1)^{(n-a+1)(a-1)}[M] \cap [\beta]$$

and so

$$G \circ D : T(H^{n-a+1}(M; \mathbb{Z})) \to \mathrm{Hom}(T(H^a(M; \mathbb{Z})), \mathbb{Q}/\mathbb{Z}) : [\beta] \mapsto ([\alpha] \mapsto \frac{(-1)^{(n-a+1)(a-1)}}{k}\gamma_{[\alpha]}(A \cap \beta)$$

is also an isomorphism where

$$\frac{(-1)^{(n-a+1)(a-1)}}{k}\gamma_{[\alpha]}(A\cap\beta) = \frac{(-1)^{(n-a+1)(a-1)}}{k}(\beta\cup\gamma_{[\alpha]})A = \frac{(-1)^{2(n-a+1)(a-1)}}{k}(\gamma_{[\alpha]}\cup\beta)A = [\alpha]\vee[\beta].$$

$\square$

So in the case of 3-manifolds like lens spaces, the torsion linking pairing is a symmetric non-degenerate bilinear form on $T(H^2(M; \mathbb{Z}))$. Also, since the cup product is invariant under homotopy-equivalence, the torsion linking pairing is as well.

5 Classification of lens spaces

In this section we shall apply the theory developed erstwhile and introduce additional theory to obtain a classification of lens spaces up to homeomorphism and homotopy equivalence. We shall first compute the basic topological invariants: the fundamental group, the homology groups and the cohomology groups. These invariants are insufficient for the classification, so we shall also compute the stronger invariants of the torsion linking form and Reidemeister torsion.

The torsion linking form requires explicit descriptions of cochains, for which singular homology, having uncountably infinitely many chains on which cochains need to be evaluated, is intractable. We thus introduce delta complexes: geometric realisations of topological spaces which, in the case of lens spaces, admit chain groups with finitely many generators, making them amenable to explicitly describe cochains and compute cup products thereon.

A delta complex is a cell complex with additional requirements. This will make a mere cell complex structure more useful for the Reidemeister torsion computation, since it will be seen that the chain groups can admit a single generator, making the computation simpler. However, the extra rigidity of delta complexes allows for cup products to be computed, which will be necessary for the torsion linking form.

5.1 Lens spaces

Here we shall define lens spaces and derive the basic results about them. In particular, we shall prove that they are closed orientable 3–manifolds and compute their fundamental group, homology groups and cohomology groups.

Definition 5.1. *The **3-sphere** $S^3 := \{(u, v) \in \mathbb{C}^2 : |u|^2 + |v|^2 = 1\}$.*

Definition 5.2. *Let p and $q \in \mathbb{N}$ be coprime. Then the **lens space** $L(p; q)$ is the quotient of S^3 by $\mathbb{Z}_p$ with the action generated by $(v, w) \mapsto (e^{2\pi i/p}v, e^{2\pi iq/p}w)$.*

Example 5.3. $L(2; 1) \cong \mathbb{R}P^3$.

Proof. The group action is generated by $(v, w) \mapsto (e^{\pi i}v, e^{\pi i}w)$ and so $(v, w) \sim (-v, -w) = -(v, w)$ which is the identification which gives $\mathbb{R}P^3$. $\square$

Thus lens spaces can be considered generalisations of real projective space.

Proposition 5.4. $H_1(L(p; q)) \cong \pi_1(L(p; q)) \cong \mathbb{Z}_p$.

Proof. The multiplication map induced by the group action is clearly continuous for all elements of $\mathbb{Z}_p$. Now let $(a, b) \in S^3$. If $S = (B_{|a|\sin(\frac{\pi}{100p})}(a) \cap |a|S^1, B_{|b|\sin(\frac{\pi}{100p})}(b) \cap |b|S^1)$ then $gS \cap S = \emptyset$ for all non-trivial $g \in \mathbb{Z}_p$ so the action is a covering space action giving $\pi_1(L(p; q)) \cong \mathbb{Z}_p$. $H_1(L(p; q)) \cong \mathbb{Z}_p$ by the Hurewicz theorem. $\square$

Proposition 5.5. $L(p; q)$ *is a compact* 3*–manifold.*

Proof. S^3 is closed and bounded, and so compact by the Heine-Borel theorem. S^3 is also a manifold, which can be seen either by stereographic projection or by the regular value theorem from MA40254. The action of $\mathbb{Z}_p$ is free so $L(p; q)$ is a compact 3–manifold. $\square$

Proposition 5.6. $L(p; q)$ *is orientable.*

Proof. Since S^3 is orientable and the action yielding $L(p;q)$ is a covering space action, it suffices to show that it acts by orientation-preserving homeomorphisms. To prove this, we note that S^3 is a submanifold in $\mathbb{R}^4$ and that the group acts by diffeomorphisms, which allows us to use the theory from MA40254. We have that

$$e^{2\pi i/p} = \cos(\frac{2\pi}{p}) + i\sin(\frac{2\pi}{p})$$

so

$$e^{2\pi i/p}(x_1 + ix_2) = \cos(2\pi/p)x_1 - \sin(2\pi/p)x_2 + i\cos(2\pi/p)x_2 + i\sin(2\pi/p)x_1.$$

Similarly

$$e^{2\pi iq/p}(x_3 + ix_4) = \cos(2\pi q/p)x_3 - \sin(2\pi q/p)x_4 + i\cos(2\pi q/p)x_4 + i\sin(2\pi q/p)x_3.$$

We thus have that the action is generated by the diffeomorphism

$$\phi : S^3 \to S^3$$

which is the restriction of

$$\Phi : \mathbb{R}^4 \to \mathbb{R}^4 : (x_1, x_2, x_3, x_4) \mapsto (\cos(2\pi/p)x_1 - \sin(2\pi/p)x_2,$$
$$\cos(2\pi/p)x_2 + \sin(2\pi/p)x_1,$$
$$\cos(2\pi q/p)x_3 - \sin(2\pi q/p)x_4,$$
$$\cos(2\pi q/p)x_4 + \sin(2\pi q/p)x_3).$$

We have an orientation form $\omega \in \Omega^3(S^3)$ given by $\omega_r(v_1, v_2, v_3) = \det(r, v_1, v_2, v_3)$. $\phi^*\omega$ is then given by $(\phi^*\omega)_r = (D\phi_r)^*\omega_{\phi(r)} \in \mathrm{Alt}^3(T_r(S^3))$. Φ is a linear map so $D\Phi_r = \Phi$. We thus have that

$$(\phi^*\omega)_r(v_1, v_2, v_3) = \det(\Phi(r), \Phi(v_1), \Phi(v_2), \Phi(v_3))$$
$$= \Phi^*\det(r, v_1, v_2, v_3)$$
$$= \det(\Phi)\det(r, v_1, v_2, v_3)$$
$$= \det(\Phi)\omega_r(v_1, v_2, v_3)$$

for all $v_1, v_2, v_3 \in T_r(S^3)$, giving

$$(\phi^*\omega)_r = \det(\Phi)\omega_r.$$

Φ is represented by the following matrix with respect to the standard basis:

$$\begin{bmatrix} \cos(2\pi/p) & -\sin(2\pi/p) & 0 & 0 \\ \sin(2\pi/p) & \cos(2\pi/p) & 0 & 0 \\ 0 & 0 & \cos(2\pi q/p) & -\sin(2\pi q/p) \\ 0 & 0 & \sin(2\pi q/p) & \cos(2\pi q/p) \end{bmatrix}$$

This is a block matrix so the determinant is the product of the determinants of the two blocks, giving

$$\det(\Phi) = (\cos^2(2\pi/p) + \sin^2(2\pi/p))(\cos^2(2\pi q/p) + \sin^2(2\pi q/p)) = 1 \cdot 1 = 1.$$

Thus $(\phi^*\omega)_r = \omega_r$ and so ϕ is orientation-preserving. ϕ is the action of a generator of $\mathbb{Z}_p$ and the composition of orientation-preserving homeomorphisms is orientation preserving, hence every element of $\mathbb{Z}_p$ acts by an orientation-preserving homeomorphism. $\square$

Remark 5.7. *Alternatively, we could note that ϕ is homotopic to the identity and thus induces the identity map on homology so maps a fundamental class to itself so is orientation-preserving.*

Now that we have shown that $L(p;q)$ is a closed orientable 3–manifold, we are able to compute all the homology and cohomology groups of $L(p;q)$ just from the first homology group:

$$H^2(L(p;q);\mathbb{Z}) \cong \mathbb{Z}_p$$

by Poincaré duality.

$$H_3((L(p;q)) \cong \mathbb{Z} \cong H^0(L(p;q);\mathbb{Z})$$

since $L(p;q)$ is an orientable 3-manifold.

$$H_0(L(p;q)) \cong \mathbb{Z} \cong H^3(L(p;q);\mathbb{Z})$$

since $L(p;q)$ is path-connected.

$$T(H_2(L(p;q))) \cong T(H^3(L(p;q));\mathbb{Z}) = 0$$

and

$$F(H_2(L(p;q))) \cong F(H^2(L(p;q);\mathbb{Z})) = 0$$

by the universal coefficient theorem so

$$H_2(L(p;q)) = 0 = H^1(L(p;q);\mathbb{Z}).$$

$$H_i(L(p;q)) = H^i(L(p;q);\mathbb{Z}) = 0 \forall i > 3$$

since $L(p;q)$ is a 3-manifold.

The homology and cohomology groups depend only on p and are insufficient for a homotopy-equivalence classification.

5.2   Cell complexes

We shall now define cell complexes, and their special case of delta complexes.

Definition 5.8. *Let Y be a topological space and let $\phi\colon S^{n-1} \to Y$ be a continuous map. Define a relation on $Y \coprod B^n$ generated by $x \sim y$ if $x \in S^{n-1} \cong \partial B^n$ and $y = \phi(x)$ and let $Y \cup_\phi B^n = Y \coprod B^n/ \sim$ with the quotient topology. We call this Y with an n-cell attached via ϕ. As a set, $Y \cup_\phi B^n = Y \coprod e^n$ where the **n-cell** e^n is the image of the interior of B^n.*

Definition 5.9. *A **finite cell complex** X of dimension 0 is a discrete finite set X^0 of 0-cells. A finite cell complex X of dimension n is the result of attaching a finite number of n-cells to a finite cell complex X^{n-1} of dimension $n-1$ using attaching maps $\phi\colon S^{n-1} \to X^{n-1}$.*
*The subset $X^k \subset X$ is called the **k-skeleton**.*

So a cell complex is formed by inductively gluing n-disks along their boundaries to a cell complex of dimension $n-1$.

Definition 5.10. *Let X^n be a finite cell complex. Then $CC_k(X)$ is the free abelian group generated by the k-cells $\{e_1^k, ..., e_{r_k}^k\}$, so that $CC_k(X) \cong \mathbb{Z}^{r_k}$.*

$$CC_k(X;G) = CC_k(X) \otimes_\mathbb{Z} G$$

and

$$CC^k(X;G) = Hom(CC_k(X), G)$$

for an abelian group G.

Definition 5.11. *Let e_i^k be a k-cell with an attaching map $\phi_i \colon S^{k-1} \to X^{k-1}$. Given a $(k-1)$-cell e_j^{k-1} we have a map $\phi_i^j \colon \phi_i^{-1}(e_j^{k-1}) \to e_j^{k-1} \colon x \mapsto \phi_i(x)$. Then We define the boundary map $\partial \colon CC_k(X) \to CC_{k-1(X)}$ by $\partial e_i^k = \sum\limits_j deg(\phi_i^j)e_j^{k-1}$ and extending linearly.*

Definition 5.12. *$d \colon CC^k(X;G) \to CC^{k+1}(X;G)$ is the dual of ∂, defined the same way as for singular homology.*

Lemma 5.13. *$\partial \circ \partial = 0$ and $d \circ d = 0$.*

Definition 5.14. *The resultant homology and cohomology groups $H_k(CC_k(X;G), \partial)$ and $H^k(CC^k(X;G), d)$ are called the **cellular homology** and **cellular cohomology** groups respectively.*

Proposition 5.15. *[5]Let G be an abelian group. Then*

$$H_k(CC_k(X;G), \partial) \cong H_k(X;G)$$

and

$$H^k(CC^k(X;G), d) \cong H^k(X;G)$$

for all $k \in \mathbb{N}$.

5.3 Delta complexes

Definition 5.16. *A Δ–complex structure on a space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$, with n depending on the index α, such that:*

1. *The restriction of σ_α to the interior of Δ^n is injective, and each point of X is in the image of exactly one such restriction.*

2. *Each composition of σ_α with a framing that corresponds to deleting a vertex of Δ^n is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$.*

3. *A set $A \subset X$ is open iff σ_α^{-1} is open in Δ^n for each σ_α.*

Remark 5.17. *Condition 2 ensures that the boundary of an element of the subgroup of $C_n(X)$ generated by the elements $\Delta^n \to X$ of the Delta complex is in the subgroup of $C_{n-1}(X)$ generated by the elements $\Delta^{n-1} \to X$ of the Delta complex, making a Delta complex a chain subcomplex. In fact, the inclusion map into the singular complex induces an isomorphism on homology. Condition 2 also ensures that the cup product on the dual complex is well-defined, since the front and back faces correspond to repeated compositions of vertex-removing framings.*

Being special cases of cell complexes, the homology and cohomology groups arising from a delta-complex structure of a topological space are isomorphic to the singular homology and cohomology groups. The chain complexes of delta-complexes are usually simpler than those in the singular case. This will be seen in the computation of the torsion linking form of Lens spaces, since finite bases for the chain groups can be found, thereby making it easy to characterize cochains by their actions on said bases.

5.4 Homotopy classification

To compute the torsion linking form of a lens space, we first need to give a lens space a Δ–complex structure. To begin with, we need an easily-visualised model of S^3.

[8, Page 5] S^3 is the set of pairs of complex numbers (z_0, z_1) such that their norms sum to 1. Representing them in polar coordinates then gives $z_0 = r_0 e^{i\theta_0}$ and $z_1 = r_1 e^{i\theta_1}$, with $r_0^2 + r_1^2 = 1$. r_1 determines $r_0 = \sqrt{1 - r_1^2}$ so (z_0, z_1) can be associated with a real triple $(r_1, \theta_0, \theta_1)$, where $0 \leq r_1 \leq 1$ and $0 \leq \theta_0, \theta_1 < 2\pi$. For there to be a bijection between pairs (z_0, z_1) and triples $(r_1, \theta_0, \theta_1)$, we require the following identifications:

1. $(0, \theta_0, \theta_1) \sim (0, \theta_0, \theta_1')$ for all $0 \leq \theta_1, \theta_1' < 2\pi$, since $r_1 = 0 \implies (z_0, z_1) = (e^{i\theta_0}, 0)$ is independent of θ_1.

2. $(1, \theta_0, \theta_1) = (1, \theta_0'm, \theta_1)$ for all $0 \leq \theta_0, \theta_0' < 2\pi$, since $r_1 = 0 \implies (z_0, z_1) = (0, e^{i\theta_1})$ is independent of θ_0.
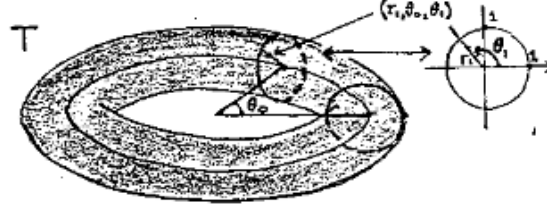
If we represent points of a solid torus T with meridianal radius 1, where θ_0 gives the longitudinal position and θ_1, r_1 give the position of the point within the cross-section determined by θ_0, as shown above, then condition 1 is satisfied, since all angles are identified on a circle with no radius.

The case where $r_1 = 1$ corresponds to the boundary of T. Varying θ_0 then traces out a latitude, so condition 2 corresponds to identifying each latitude of ∂T with a single point.

T is homeomorphic to a solid cylinder with the ends identified, and identifying each vertical line on the boundary then gives S^3 due to satisfying condition 2. Collapsing the vertical lines to points midway along the cylinder then shows that S^3 is homeomorphic to a solid ball with upper and lower hemisphered identified via orthogonal projection, where the lines collapsed to points on the equator, as shown below.
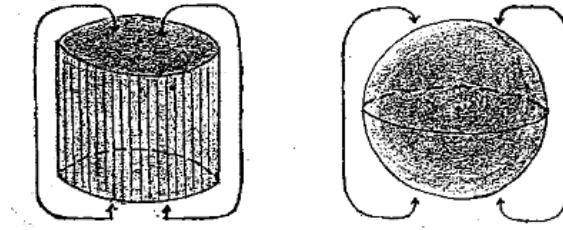
To find a geometric model of $L(p; q)$ via identifications on T, we much first determine the equivalence

classes of the action on T. We have that the action of $[m] \in \mathbb{Z}_p$ is given by

$$m \cdot (z_0, z_1) = (z_0 e^{\frac{2\pi i m}{p}}, z_1 e^{\frac{2\pi i q m}{p}}) = (r_0 e^{i(\theta_0 + \frac{2\pi m}{p})}, r_1 e^{i(\theta_1 + \frac{2\pi q m}{p})}).$$

In terms of the coordinates given to T, we have

$$m \cdot (r, \theta_0, \theta_1) = (r, \theta_0 + \frac{2\pi m}{p}, \theta_1 + \frac{2\pi q m}{p}).$$

Note that r is unaffected by the action, meaning that $\mathbb{Z}_p$ maps an individual torus to itself. We can thus express T as

$$\bigcup_{0 \leq c \leq 1} T_c,$$

where T_c is a torus of radius r, and study the action on T in terms of the actions on indivudual T_c's. For the case $0 < c < 1$, T_c can be pictured as a square with identified edges, with the horizontal axis acting as the θ_1–axis and the vertical axis as the θ_0–axis, both ranging from 0 to 2π, as shown below:
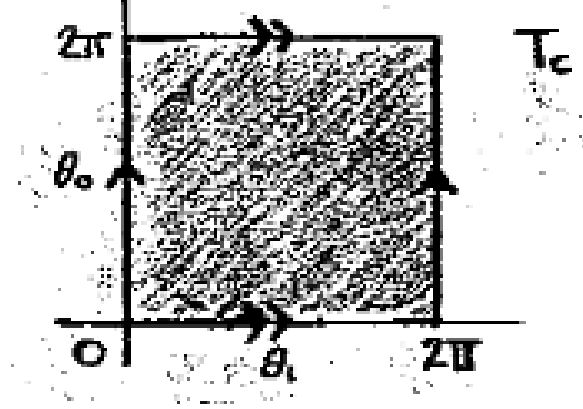
The action of $[m]$ on a point (θ_0, θ_1) results in the point $(\theta_0 + \frac{2\pi m}{p}, \theta_1 + \frac{2\pi q m}{p})$, so all actions of $\mathbb{Z}_p$ are translations along lines with gradient $\frac{2\pi m}{p} / \frac{2\pi q m}{p} = \frac{1}{q}$. If we divide the square into p horizontal bands, each of height $\frac{2\pi}{p}$, we see that an equivalence class of the action on T_c contains exactly one point in each horizontal band. We can thus define the orbit space as

$$T_c/\mathbb{Z}_p = \{(r, \theta_0, \theta_1) : r = c, 0 \leq \theta_0 < \frac{2\pi}{p}, 0 \leq \theta_1 < 2\pi\}.$$

This is a short tube with ends identified by a $\frac{2\pi q}{p}$ twist:
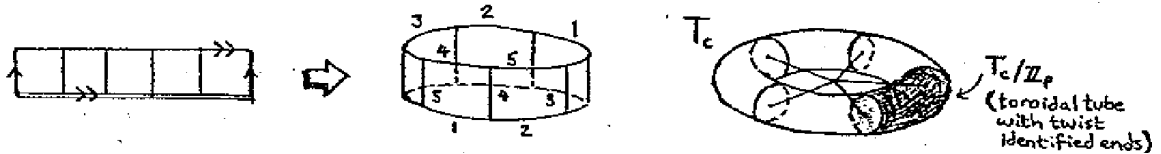
We now consider the case T_0 - the action on the central circle:
The action of $[m]$ on a point in T_0 maps θ_0 to $\theta_0 + \frac{2\pi m}{p}$, and so m acts by a $\frac{2\pi m}{p}$ rotation. Each

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equivalence class of the action on T_0 thus contains exactly one point in each $\frac{2\pi}{p}$ arc. $0 \le \theta_0 < \frac{2\pi}{p}$ thus has exactly one point from each equivalence class, so $T_0/\mathbb{Z}_p$ can be viewed as an arc with identified endpoints.

Finally we consider the case T_1 - the boundary of T with each latitude identified to a point:
Since T_1 is represented by a single meridian of ∂T, we can identify T_1 with a circle S^1. The action of $[m]$ on a point of T_1 sends θ_1 to $\theta_1 + \frac{2\pi qm}{p}$, and so acts by inducing a $\frac{2\pi qm}{p}$ rotation. Since, $\gcd(p,q) = 1$, if we partition S^1 into p equal arcs $0 \le \theta_1 < \frac{2\pi}{p}, ..., \frac{2(p-1)\pi}{p} \le \theta_1 < 2\pi$, then they all get identified.
We have
$$S^3/\mathbb{Z}_p = \bigcup_{0 \le c \le 1} T_c/\mathbb{Z}_p.$$

This is a collection of nested tubes with radii $0 \le r < 1$ whose ends are twist-identified by a $\frac{2\pi q}{p}$ radian twist, and a boundary of latitude-segments which are identified to single points, and then identified from p arcs to one.

Collapsing the latitude segments to points then gives a ball whose equator is divided into p equal arcs, all identified together. The upper and lower hemispheres then correspond to the ends of the solid cylinder and so are twist-identified by a $\frac{2\pi q}{p}$ twist.
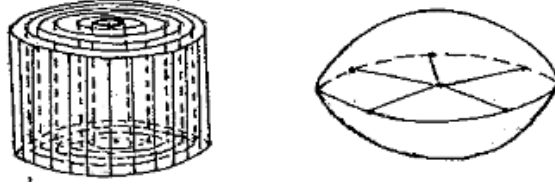
This then gives a model of $L(p;q)$, which we shall now use to obtain a delta complex.

Consider a filled-in ball with points N and S corresponding to the north and south poles and points $x_0, ..., x_{p-1}$ being evenly-spaced points along the equator. Then draw a line through the ball from N to S, lines connecting x_i to x_{i+1}, lines connecting x_i to N and lines connecting x_i to S (all geodesic). We then identify the faces on the surface by twisting the top hemisphere by $\frac{2\pi q}{p}$ radians and projecting downwards. This leaves us, after identification, with:

Points: $[N]$ and $[x_0]$ (since $\{x_0, x_q, x_{2q}, ...\}$ are identified and q is coprime to p)
Lines: $[NS]$, $[x_0 x_1]$ and $[Nx_0], ..., [Nx_{p-1}]$
Faces: $[Nx_0 x_1], ..., [Nx_{p-2} x_{p-1}], [Nx_0 x_{p-1}]$ and $[NSx_0], ..., [NSx_{p-1}]$
Slices: $[NSx_0 x_1], ..., [NSx_{p-2} x_{p-1}], [NSx_0 x_{p-1}]$.
(Here we use $[X]$ to refer to the composition of the quotient map with a singular simplex in the ball and the reason for using $[NSx_0 x_{p-1}]$ as opposed to $[NSx_{p-1} x_0]$, for example, is to comply with condition 2 of the definition of a delta complex)

Having obtained a delta-complex structure, we are now able to compute the torsion linking form of a lens space:

Let $a_{p-1} = -[NSx_0 x_{p-1}]$ and $a_i = [NSx_i x_{i+1}]$ otherwise.

Let $b_{p-1} = -[Nx_0 x_{p-1}]$ and $b_i = [Nx_i x_{i+1}]$ otherwise.

Let $c_i = [NSx_i]$.

Let $D = [x_0x_1]$.

Let $E = [NS]$.

Let $F_i = [Nx_i]$.

Lemma 5.18. $\partial b_i = D - F_{i+1} + F_i$.

Proof.

$$-\partial b_{p-1} = \partial[Nx_0x_{p-1}] = [x_0x_{p-1}] - [Nx_{p-1}] + [Nx_0] = [Nx_0] - [Nx_{p-1}] - [x_0x_1].$$

Otherwise,

$$\partial b_i = \partial[Nx_ix_{i+1}] = [x_0x_1] - [Nx_{i+1}] + [Nx_i].$$

$\square$

Lemma 5.19. $\partial c_i = E - F_i + F_{i-q}$.

Proof.

$$\partial c_i = \partial[NSx_i] = [Sx_i] - [Nx_i] + [NS] = [NS] - [Nx_i] + [Nx_{i-q}].$$

$\square$

Lemma 5.20. $\sum_{i=0}^{p-1} a_i$ *is a representative of a fundamental class.*

Proof.

$$\partial(\sum_{i=0}^{p-1} a_i) = \sum_i ([Nx_{i-q}x_{i-q+1}] - [Nx_ix_{i+1}] + [NSx_{i+1}] - [NSx_i]) = 0$$

so $\sum_{i=0}^{p-1} a_i$ is a representative of an element of $H_3(L(p;q);\mathbb{Z})$. The coefficients of a_i are all equal and $H_3(L(p;q);\mathbb{Z}) \cong \mathbb{Z}$ so $\sum_{i=0}^{p-1} a_i$ has to represent a generator, since $\partial(\sum_{i=0}^{p-1} \lambda_i a_i) = 0$ if and only if all λ_i are equal. Thus $\sum_{i=0}^{p-1} a_i$ represents a fundamental class. $\square$

Call the equivalence class it represents $[M]$.

$C_1(L(p;q),\mathbb{Z})$ has basis D, E, F_i and $C_2(L(p;q),\mathbb{Z})$ has basis b_i, c_i.

Let $\widehat{X}$ be linear map which sends X to 1 and all other elements of the basis to 0. We then have:

$$d\widehat{D} = \sum_{i=0}^{p-1} \widehat{b}_i.$$

$$d\widehat{E} = \sum_{i=0}^{p-1} \widehat{c}_i.$$

$$d\widehat{F}_i = \widehat{b}_i - \widehat{b}_{i-1} - \widehat{c}_i + \widehat{c}_{i+q}.$$

Then define

$$\gamma = \widehat{D} + (\sum_{i=0}^{p-1} i\widehat{F}_i) + q\widehat{E} \in C^1(L(p;q);\mathbb{Z}).$$

$$d\gamma = \sum_{i=0}^{p-1} \widehat{b}_i + (\sum_{i=0}^{p-1} i(\widehat{b}_i - \widehat{b}_{i-1} - \widehat{c}_i + \widehat{c}_{i+q})) + \sum_{i=0}^{p-1} q\widehat{c}_i$$

$$= \sum_{i=0}^{p-1}((i+1)\widehat{b}_i - i\widehat{b}_{i-1} - (i-q)\widehat{c}_i + i\widehat{c}_{i+q}) \equiv 0 \bmod p.$$

Let $\alpha = \frac{1}{p}d\gamma \in C^2(L(p;q);\mathbb{Z})$. Then α is a representative of an element in $H^2(L(p;q);\mathbb{Z})$ with $p[\alpha] = [0]$.

Proposition 5.21. $[\alpha] \vee [\alpha] = \frac{q}{p}$.

Proof. Note that $\gamma(E) = q$.

$$p\alpha([Sx_{q-1}x_q]) = p\alpha([Nx_{p-1}x_0]) = p\alpha(b_{p-1}) = (p-1+1)\hat{b}_{p-1}(b_{p-1}) - 0 \times \hat{b}_{-1}(b_{p-1}) = p$$

$$\implies \alpha([Sx_{q-1}x_q]) = 1.$$

Otherwise,

$$p\alpha([Sx_ix_{i+1}]) = p\alpha([Nx_{i-q}x_{i+1-q}]) = p\alpha(b_{i-q}) = (kp+i-q+1)\hat{b}_{i-q}(b_{i-q}) - (kp+i-q+1)\hat{b}_{i-q}(b_{i-q}) = 0$$

where $k \in \mathbb{Z}$ puts $kp+i-q+1$ in the compatible range for the sum of $[1, p-1]$ (The fact that it can't wrap around to 0 is why cancellation occurs). Thus,

$$[\alpha] \vee [\alpha] = \frac{1}{p}\gamma([NS])\alpha([Sx_{q-1}x_q]) = \frac{q}{p}.$$

$\square$

We then have that if $L(p;q_1)$ and $L(p;q_2)$ are homotopy-equivalent, then $\frac{q_1}{p} = \pm\frac{q_2}{p}m^2$ for some integer m, taking into account whether or not the homotopy-equivalence is orientation-reversing and choice of element of $H^2(L(p;q);\mathbb{Z})$ (since $[n\alpha] \vee [n\alpha] = n^2[\alpha] \vee [\alpha]$). This is equivalent to

$$q_1 \equiv \pm q_2 m^2 \bmod p \iff q_1 q_2^{-1} \equiv \pm m^2 \bmod p$$

$$\iff q_1 q_2 \equiv \pm(mq_2)^2 \bmod p$$

$$\iff q_2 q_2 \equiv \pm n^2 \bmod p, \text{ where } n = mq_2.$$

If we were only interested in orientation-preserving homotopy-equivalence, then the condition would be $q_2 q_2 \equiv n^2 \bmod p$; and if we were only interested in orientation-reversing homotopy-equivalence, then the condition would be $q_2 q_2 \equiv -n^2 \bmod p$. Thus $L(p; q)$ is orientation-reversing homotopy-equivalent to itself iff $q(-q)$ is a square mod $p \iff -1$ is a square mod $p \iff p \equiv 1 \bmod 4$ for p a prime. This in contrast to compact 2–manifolds, which by the classification of compact surfaces [9] always admit orientation-reversing homeomorphisms.

The proof of the sufficiency of this condition is given in exercise 29 in section 4.2 of Hatcher. [1]

5.5 Reidemeister torsion

[Notes on Reidemeister torsion [10]][Pierre ALbin [11]] We shall now define the invariant which will give us the homeomorphism classification.

Definition 5.22. Let $0 \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2} \xrightarrow{\partial} ... \xrightarrow{\partial} C_0 \to 0$ be a chain complex with coefficients in a commutative ring and with no homology and bases c_i of C_i. We can then pick auxiliary bases $b_i \cup \widetilde{b}_{i-1}$ of C_i where b_i is a basis for Im ∂_{i+1} and $\partial \widetilde{b}_{i-1} = b_{i-1}$. Then let $[b_i \cup \widetilde{b}_{i-1} | c_i]$ be the determinant of the change of basis matrix from $b_i \cup \widetilde{b}_{i-1}$ to c_i. We then define the **Reidemeister torsion**

$$\Delta(C_., c_.) = \prod_{i=0}^{n} [b_i \cup \widetilde{b}_{i-1} | c_i]^{(-1)^i}.$$

Theorem 5.23. The Reidemeister torsion is independent of choice of auxiliary bases but not of choice of c_i.

Let

$$X = \bigcup_{k=0}^{n} \bigcup_{j} e_j^k$$

be a finite cell complex and (A, β) a representation of $\pi_1(X)$ in a commutative ring A; that is, we have a homomorphism $\beta : \pi_1(X) \to A^*$. Let $\widetilde{X}$ be a universal cover of X and let $\widetilde{e}_j^k$ be a lift of e_j^k. Then every other lift of e_j^k is obtained by composition of $\widetilde{e}_j^k$ with a deck transformation (since given f' and g' lifts of f there exists a $\phi \in \text{Aut}(p)$ such that $\phi \circ f'(0) = g'(0)$ which implies $\phi \circ f = g$ by uniqueness [2]). $\text{Aut}(p) \cong \pi_1(X)$ so we can write

$$\widetilde{X} = \bigcup_{k=0}^{n} \bigcup_{g \in \pi_1(X)} \bigcup_{j} g \widetilde{e}_j^k.$$

Thus multiplication by an element of $\pi_1(X)$ permutes the cells, thereby giving $CC_.(\widetilde{X})$ a $\pi_1(X)$-action. A also has a $\pi_1(X)$ action so we can define $C_k(X; \beta) = CC_k(\widetilde{X}) \otimes_{\mathbb{Z}\pi_1(X)} A$, which is the quotient of $CC_k(\widetilde{X}) \otimes A$ by the action of $\pi_1(X)$. We have an induced boundary map $\partial : C_k(X; \beta) \to C_{k-1}(X; \beta)$ which makes $C_k(X; \beta)$ a chain complex with basis $\{\widetilde{e}_j^k\} \otimes \text{id}_A$.

Definition 5.24. *If* $C_{.}(X; \beta) \otimes_{\mathbb{Z}\pi_1(X)} A$ *has no homology, then the* **Reidemeister torsion of** X **with respect to** β *is*

$$\Delta(X; \beta) := \Delta(C_{.}(X; \beta) \otimes_{\mathbb{Z}\pi_1(X)} A).$$

This is a well-defined element of $A^*/\{\pm\beta(\pi_1(X))\}$.

Remark 5.25. *The ambiguity which required modding out by* $\{\pm\beta(\pi_1(X))\}$ *to solve arose out of the choice of lifts which form the bases* c_i *– the choice of which, unlike that of auxiliary bases,* $\Delta(C_{.}, c_{.})$ *is not independent.*

Theorem 5.26. *Let* X *and* Y *be cell complexes, let* $f : X \to Y$ *be a homeomorphism and let* (A, β) *be a representation of* $\pi_1(Y)$. *Then* $\Delta(X; \beta \circ f_*) = \Delta(Y; \beta)$

Thus the Reidemeister torsion is invariant under homeomorphism. However, it is not invariant under homotopy-equivalence.

5.6 Homeomorphism classification

[Pierre Albin [11]]
We shall now realise lens spaces as cell complexes, in preparation for computing the Reidemeister torsion to give a stronger homeomorphism invariant than the fundamental group.

We have

$$L(p; q) = \{(r_1 e^{2\pi i \theta_1}, r_2 e^{2\pi i \theta_2}) : r_1^2 + r_2^2 = 1\}/(\theta_1, \theta_2) \sim (\theta_1 + \frac{1}{p}, \theta_2 + \frac{q}{p}).$$

Let $\rho \colon S^3 \to S^3$ represent the action.
The action rotates the first coordinate by $1/p$ so let

$$e_k^3 = \{(r_1 e^{2\pi i \theta_1}, r_2 e^{2\pi i \theta_2}) \in S^3 : \frac{k}{p} < \theta_1 < \frac{k+1}{p}\}.$$

The 2-cells will then be the boundaries of the 3-cells, so let

$$e_k^2 = \{(r_1 e^{2\pi i \theta_1}, r_2 e^{2\pi i \theta_2}) \in S^3 : \theta_1 = \frac{k}{p}\}.$$

Fixing θ_1 leaves $r_2 e^{2\pi i \theta_2}$ moving in a circle, so let

$$e_k^1 = \{(r_1 e^{2\pi i \theta_1}, r_2 e^{2\pi i \theta_2}) \in S^3 : \theta_1 = 0, \frac{k}{p} < \theta_2 < \frac{k+1}{p}\}.$$

Then let $e_k^0 = \{(r_1 e^{2\pi i \theta_1}, r_2 e^{2\pi i \theta_2}) \in S^3 : \theta_1 = 0, \theta_2 = \frac{k}{p}\}$ be the boundary of e_k^1.
Then we have $\rho(e_k^3) = e_{k+1}^3$, $\rho(e_k^2) = e_{k+1}^2$, $\rho(e_k^1) = e_{k+q}^1$ and $\rho(e_k^0) = e_{k+q}^0$.
Thus ρ permutes the cells so taking the quotient gives a cell complex structure

$$L(p; q) = e^0 \cup e^1 \cup e^2 \cup e^3.$$

$$\partial e_0^3 = e_1^2 - e_0^2 = (\rho - \mathrm{id})(e_0^2),$$
$$\partial e_1^2 = e_0^1 + ... + e_{p-1}^1 = (\mathrm{id} + \rho + ... + \rho^{p-1})(e_0^1),$$
$$\partial e_0^1 = e_l^0 - e_0^0 = (\rho^l - \mathrm{id})(e_0^0),$$

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where $lq \cong 1 \mod p$.

Thus on $L(p;q)$, we have $\partial e^3 = 0$, $\partial e^2 = pe^1$, $\partial e^1 = 0$ since $p = \mathrm{id}$ modulo the group action. This then gives the same homology groups as were found earlier.

Let ω be a pth root of unity and let $\beta_\omega : \mathbb{Z}_p \to \mathbb{C}^*$ be the representation given by $1 \mapsto \omega$. In $C_\cdot(X; \beta_\omega) \otimes_{\mathbb{Z}\pi_1(X)} \mathbb{C}^*$ we have:

$$\partial(\widetilde{e}_0^3 \otimes \mathrm{id}) = (\omega - 1)(\widetilde{e}_0^2 \otimes \mathrm{id}),$$

$$\partial(\widetilde{e}_0^2 \otimes \mathrm{id}) = (1 + \omega + \omega^2 + ... + \omega^{p-1})(\widetilde{e}_0^1 \otimes \mathrm{id}),$$

$$\partial(\widetilde{e}_0^1 \otimes \mathrm{id}) = (\omega^l - 1)(\widetilde{e}_0^0 \otimes \mathrm{id}),$$

giving a complex:

$$0 \to \mathbb{C}^* \overset{\omega - 1}{\to} \mathbb{C}^* \overset{0}{\to} \mathbb{C}^* \overset{\omega^l - 1}{\to} \mathbb{C}^* \to 0 \text{ whenever } \omega \neq 1.$$

The non-zero maps are isomorphisms and so the complex has no homology. Hence for $\omega \neq 1$ we have

$$\Delta(L(p;q); \beta_\omega) = (\omega^l - 1)(\omega - 1) \in \mathbb{C}^*/\{\pm\omega^k : k \in \mathbb{Z}\}.$$

The following lemma then gives the necessary conditions for the homeomorphism classification.

Lemma 5.27. *Given q_1, q_2 and a primitive pth root of unity ω, there is another primitive root of unity ν such that $\Delta(L(p;q_1), \beta_\omega) = \Delta(L(p;q_2), \beta_\nu)$ only if $q_1 \equiv \pm q_2 \mod p$ or $q_1 q_2 \equiv \pm 1 \mod p$*

To prove the sufficiency of the above lemma for the homeomorphism classification, we require the following lemma.

Lemma 5.28. *Let X be a topological space. Let G_1 and G_2 be conjugate subgroups of $\mathrm{Homeo}(X)$ defining covering space actions of X. Then $X/G_1 \cong X/G_2$.*

Proof. We have that $G_2 = hG_1 h^{-1}$ for some $h \in \mathrm{Homeo}(X)$. The map $f : X \to X/G_2 : x \mapsto [h \cdot y]$ is continuous, since h acts by a homeomorphism and the quotient map is continuous. We also have that if $x' = g_1 \cdot y$ for some $g_1 \in G_1$, then

$$f(x') = [hg_1 \cdot x] = [(hg_1^{-1}h^{-1}) \cdot hg_1 \cdot x] = [h \cdot x] = f(x).$$

The universal property of quotient maps then implies that there exists a continuous map $\bar{f} : X/G_1 \to X/G_2$. Similarly, there exists a continuous map $\bar{g} : X/G_2 \to X/G_1 : [x] \mapsto [h^{-1} \cdot x]$.

$$\bar{g} \circ \bar{f}([x]) = [h^{-1} \cdot (h \cdot x)] = [x].$$

Similarly, $\bar{f} \circ \bar{g}([x]) = [x]$. Thus $X/G_1 \cong X/G_2$. $\square$

Lemma 5.29. $L(p;q) \cong L(p;-q)$.

Proof. $(v,w) \mapsto (e^{2\pi i/p}v, e^{2\pi iq/p}w)$ and $(v,w) \mapsto (e^{2\pi i/p}v, e^{-2\pi iq/p}w)$ are conjugate by the homeomorphism $(v,w) \mapsto (v,\bar{w})$. The result follows by the above lemma. $\square$

Lemma 5.30. *If $q_1 \equiv q_2^{-1} \mod p$ then $L(p;q_1) \cong L(p;q_2)$.*

Proof. We have that $q_1 q_2 \equiv 1 \mod p$.

$$\left(e^{2i\pi/p}\right)^{q_1} = e^{2i\pi q_1/p}, \left(e^{2i\pi/p}\right)^{q_2} = e^{2i\pi q_2/p}$$

and

$$\left(e^{2i\pi q_1/p}\right)^{q_2} = e^{2i\pi/p} = \left(e^{2i\pi q_2/p}\right)^{q_1}.$$

Thus the subgroup generated by

$$(v, w) \mapsto (e^{2i\pi/p}v, e^{2i\pi q_1/p}w)$$

is the same as the subgroup generated by

$$(v, w) \mapsto (e^{2i\pi q_2/p}v, e^{2i\pi/p}w).$$

Furthermore,

$$(v, w) \mapsto (e^{2i\pi q_2/p}v, e^{2i\pi/p}w)$$

is conjugate to

$$(v, w) \mapsto (e^{2i\pi/p}w, e^{2i\pi q_2/p}v)$$

by the homeomorphism $(v, w) \mapsto (w, v)$. Thus the subgroups generated by

$$(v, w) \mapsto (e^{2i\pi/p}v, e^{2i\pi q_1/p}w)$$

and

$$(v, w) \mapsto (e^{2i\pi/p}v, e^{2i\pi q_2/p}w)$$

are conjugate so $L(p; q_1) \cong L(p; q_2)$. $\qquad\square$

This completes the homeomorphism classification.

Theorem 5.31. *Let $L(p; q_1)$ and $L(p; q_2)$ be lens spaces.*
They are homotopy equivalent if and only if $q_1 q_2 \equiv \pm n^2 \mod p$ for some $n \in \mathbb{N}$.
They are homeomorphic if and only if $q_1 \equiv \pm q_2^{\pm 1} \mod p$.

Example 5.32. *$L(7; 1)$ and $L(7; 2)$ are homotopy-equivalent but not homeomorphic.*

This is significant, since it gives a counter-example to a once-believed conjecture that homotopy-equivalent and homeomorphic are equivalent conditions for manifolds.

References

[1] A. Hatcher, "Algebraic topology."

[2] J. Nordstrom, "Algebraic topology."

[3] Wikipedia, "Relative homology."

[4] A. Ram, "An introduction to tor: Definitions and properties."

[5] J. Nordstrom, "Differential topology."

[6] G. Quic, "Algebraic topology lecture 20."

[7] D. Calderbank, "Differential and geometric analysis."

[8] M. Watkins, "A short survey of lens spaces."

[9] G. Sankaran, "Introduction to topology."

[10] A. Ranicki, "Notes on reidemeister torsion."

[11] P. Albin, "Beyond homology: Classifying lens spaces."