Measure Theory

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Exercise 0.1. *Suppose $n \in \mathbb{N}$ and X is a set with n elements. Show that the power set $\mathcal{P}(X)$ has 2^n elements.*

Proof. We proceed by induction on n. For $n = 0$ then clearly $\mathcal{P}(X) = \{\emptyset\}$ has 1 element. Assume it's true for $n = k$. Then for $n = k + 1$ we can pick an element $x \in X$ and see that there are 2^k elements in $\mathcal{P}(X \setminus \{x\})$ by the inductive hypothesis. Every set in $\mathcal{P}(X)$ either does or does not contain x so there are twice the number of elements in $\mathcal{P}(X)$ as there are in $\mathcal{P}(X \setminus \{x\})$. Thus $\mathcal{P}(X)$ has 2^{k+1} elements. □

Exercise 0.2. *Suppose X is a non-empty set and $\mathcal{A}$ is an algebra in X. Show that for any $k \in \mathbb{N}$, if $A_i \in \mathcal{A}$ for $i = 1, 2, \dots, k$ then $\bigcup_{i=1}^{k} A_i \in \mathcal{A}$.*

Proof. We proceed by induction. For $n = 1, 2$ the statement is obvious. Assume true for $n = k$. Then if $A_i \in \mathcal{A}$ for $i = 1, 2, ..., k, k + 1$ we have $\bigcup_{i=1}^{k} A_i \in \mathcal{A}$ and so $\bigcup_{i=1}^{k+1} A_i = A_{k+1} \cup \bigcup_{i=1}^{k} A_i \in \mathcal{A}$. □

Exercise 0.3. *Which of the following collections $\mathcal{M}$ of sets (in X) are σ-algebras? Which ones are algebras? Explain each answer.*

1. $X = \{1, 2, 3, 4\}$,

$$\mathcal{M} = \{\emptyset, \{1\}, \{2\}, \{3, 4\}, \{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.$$

2. $X = \{1, 2, 3, \dots\}$ *and*

$$\mathcal{M} = \{A \subset X : \textit{either } A \textit{ or } X \setminus A \textit{ is finite}\}.$$

3. *X is an uncountable set and*

$$\mathcal{M} = \{A \subset X : \textit{either } A \textit{ or } X \setminus A \textit{ is countable}\}.$$

4. *X is any set, $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$ are σ-algebras in X and $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$.*

Proof.
1. σ-algebra (and hence an algebra) since it contains the empty set and is closed under complements and countable unions.
2. Let $A_i = \{2i\} \in \mathcal{M}$. Then let $A = \bigcup_{i=1}^{\infty} A_i$. Neither A nor $X \setminus A$ is finite so $A \notin \mathcal{M}$. Thus $\mathcal{M}$ is not a σ-algebra. $\emptyset$ is finite so $\emptyset \in \mathcal{M}$. Let $P, Q \in \mathcal{M}$. If P is finite then $X \setminus (X \setminus P) = P$ is finite so $X \setminus P \in \mathcal{M}$. If $X \setminus P$ is finite then $X \setminus P \in \mathcal{M}$. Thus $\mathcal{M}$ is closed under complements. If P and Q are finite then $P \cup Q$ is finite so $P \cup Q \in \mathcal{M}$. If $X \setminus P$ and $X \setminus Q$ are finite then $X \setminus (P \cup Q) = (X \setminus P) \cap (X \setminus Q)$ is finite so $P \cup Q \in \mathcal{M}$. Without loss of generality let P be finite and let $X \setminus Q$ be finite. Then $X \setminus (P \cup Q) = (X \setminus P) \cap (X \setminus Q)$ is finite so $P \cup Q \in \mathcal{M}$. Thus $\mathcal{M}$ is an algebra.

3. $\emptyset$ is countable so $\emptyset \in \mathcal{M}$. Like in part (2) $\mathcal{M}$ is closed under complements. Let $A_1, A_2, ...$ be a collection of sets in $\mathcal{M}$. If every set is countable then $\bigcup_{i=1}^{\infty} A_i$ is countable since the union of countably many countable sets is countable. If $X \setminus A_k$ countable for some k then $X \setminus \bigcup_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} X \setminus A_i \subseteq X \setminus A_k$ is countable. Thus $\mathcal{M}$ is a σ-algebra.

4. $\emptyset \in \mathcal{M}_1$ so $\emptyset \in \mathcal{M}$. Let $A \in \mathcal{M}$. Then $A \in \mathcal{M}_i$ for some i so $A^c \in \mathcal{M}_i \subseteq \mathcal{M}$. Thus $\mathcal{M}$ is closed under complements. Let $A, B \in \mathcal{M}$. Then $A, B \in \mathcal{M}_n$ for some n so $A \cup B \in \mathcal{M}_n \subseteq \mathcal{M}$. Thus $\mathcal{M}$ is an algebra.

   Let $X = \mathbb{N}$ and let $\mathcal{M}_i = \{A \subseteq X : A \subseteq \{1, 2, ..., i\}$ or $A^c \subseteq \{1, 2, ..., i\}\}$. $\emptyset \subseteq \{1, 2, ..., i\}$ so $\emptyset \in \mathcal{M}_i$. $\mathcal{M}_i$ is clearly closed under complements. Let $A_1, A_2, ... \in \mathcal{M}_i$. Suppose every $A_k \subseteq \{1, 2, ..., i\}$. Then $\bigcup_{k=1}^{\infty} A_k \subseteq \{1, 2, ..., i\}$ so $\bigcup_{k=1}^{\infty} A_k \in \mathcal{M}_i$. Suppose there exists an A_l such that $A_l^c \in \{1, 2, ..., i\}$. Then $(\bigcup_{k=1}^{\infty} A_k)^c = \bigcap_{k=1}^{\infty} A_k^c \subseteq A_l^c \subseteq \{1, 2, ..., i\}$. Thus each $\mathcal{M}_i$ is a σ-algebra. Furthermore, $A \subseteq \{1, 2, ..., i\}$ or $A^c \subseteq \{1, 2, ..., i\} \implies A \subseteq \{1, 2, ..., i, i+1\}$ or $A^c \subseteq \{1, 2, ..., i, i+1\}$ so $\mathcal{M}_i \subseteq \mathcal{M}_{i+1}$. Let $A_i = \{2i\} \in \mathcal{M}_{2i} \subseteq \mathcal{M}$. Then $A := \bigcup_{k=1}^{\infty} A_k = 2\mathbb{N}$. Suppose $A \in \mathcal{M}$. Then there is an i such that $A \in \mathcal{M}_i$. However A is unbounded so neither A nor A^c is contained in $\{1, 2, ..., i\}$; a contradiction. Thus $\mathcal{M}$ is not a σ-algebra.

$\square$

Exercise 0.4. *Given numbers $x_{ij} \geq 0$ defined for each $i \in \mathbb{N}, j \in \mathbb{N}$, show that*

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} x_{ij} \right).$$

[Hint: first show a (weak) inequality between the two double sums.]

Proof. $\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right) = \lim_{N \to \infty} \sum_{i=1}^{N} (\sum_{j=1}^{\infty} x_{ij})$. Thus given any $a < \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right)$ there exists an $N \in \mathbb{N}$ such that $\sum_{i=1}^{N} (\sum_{j=1}^{\infty} x_{ij}) > a$. $\sum_{i=1}^{N} (\sum_{j=1}^{\infty} x_{ij}) = \sum_{i=1}^{N} (\lim_{M \to \infty} \sum_{j=1}^{M} x_{ij}) = \lim_{M \to \infty} \sum_{i=1}^{N} (\sum_{j=1}^{M} x_{ij}) > a$ so there exists an $M \in \mathbb{N}$ such that $\sum_{i=1}^{N} (\sum_{j=1}^{M} x_{ij}) > a$. $\sum_{i=1}^{N} (\sum_{j=1}^{M} x_{ij}) = \sum_{j=1}^{M} (\sum_{i=1}^{N} x_{ij}) > a$ and so $\sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} x_{ij}) > a$. This is true for every $a < \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right)$ so $\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right) \leq \sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} x_{ij})$. Similarly $\sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} x_{ij}) \leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right)$ and so $\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right) = \sum_{j=1}^{\infty} (\sum_{i=1}^{\infty} x_{ij})$. $\square$

Exercise 0.5. *Suppose X is a non-empty set and $\mathcal{X} = \{A_1, A_2, \ldots, A_k\}$, where the sets $A_1, \ldots, A_k$ are non-empty and form a partition of X, i.e., they are pairwise disjoint and $\bigcup_{i=1}^{k} A_i = X$. Show that*

$$\sigma(\mathcal{X}) = \{\bigcup_{j \in J} A_j : J \subset \{1, 2, \ldots, k\}\}.$$

Proof. We shall refer to $\{\bigcup_{j \in J} A_j : J \subset \{1, 2, \ldots, k\}\}$ as $\mathcal{B}$. $\emptyset = \bigcup_{j \in \emptyset} A_j \in \mathcal{B}$. Let $P \in \mathcal{B}$. Then $P = \bigcup_{j \in J} A_j$ for some $J \subseteq \{1, 2, ..., k\}$ so $P^c = \bigcup_{j \in J^c} A_j \in \mathcal{B}$. Let $P_1, P_2, ... \in \mathcal{B}$ so that each $P_i = \bigcup_{j \in J_i} A_j$ for some $J_i \subseteq \{1, 2, ..., k\}$. Then $\bigcup_{i=1}^{\infty} P_i = \bigcup_{i=1}^{\infty} \bigcup_{j \in J_i} A_j = \bigcup_{j \in \bigcup_{i=1}^{\infty} J_i} A_j \in \mathcal{B}$ since $\bigcup_{i=1}^{\infty} J_i \subseteq \{1, 2, ..., k\}$. Thus $\mathcal{B}$ is indeed a σ-algebra. Now suppose that $\mathcal{M}$ is another σ-algebra such that $\mathcal{X} \subseteq \mathcal{M}$. Clearly $\mathcal{B} \subseteq \mathcal{M}$ and so $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{X}$ so $\sigma(\mathcal{X}) = \mathcal{B}$. $\square$

Exercise 0.6. *Suppose X is a non-empty set and $\mathcal{X} = \{A_1, A_2, A_3, \ldots\}$, where the sets $A_i, i \geq 1$ are non-empty and form a countably infinite partition of X, i.e., they are pairwise disjoint and $\bigcup_{i=1}^{\infty} A_i = X$.*

1. *Describe the sets in the σ-algebra generated by $\mathcal{X}$.*

2. *Describe the sets in the algebra generated by $\mathcal{X}$.*

Proof. 1. Let $\mathcal{M} = \{\bigcup_{j\in J} A_j : J \subseteq \mathbb{N}\}$. $\emptyset \in \mathcal{M}$. Let $A := \bigcup_{i\in I} A_j, I \subseteq \mathbb{N}$. Then $A^c = \bigcup_{i\in I^c} A_j \in \mathcal{M}$ since $I^c \subseteq \mathbb{N}$. Let $P_i = \bigcup_{j\in J_i} A_j$ for some $J_i \subseteq \mathbb{N}$. Then $\bigcup_{i=1}^\infty P_i = \bigcup_{i=1}^\infty \bigcup_{j\in J_i} A_j = \bigcup_{j\in\bigcup_{i=1}^\infty J_i} A_j \in \mathcal{M}$ since $\bigcup_{i=1}^\infty J_i \subseteq \mathbb{N}$. Thus $\mathcal{M}$ is a σ-algebra and is clearly the smallest one containing $\mathcal{X}$ so is the σ-algebra generated by $\mathcal{X}$.

2. Let $\mathcal{M} = \{\bigcup_{j\in J} A_j : J \subseteq \mathbb{N}, J \text{ is finite or } J^c \text{ is finite}\}$. $\emptyset \in \mathcal{M}$. let $\bigcup_{j\in J} A_j \in \mathcal{M}$. If J is finite then $(J^c)^c$ is finite and if J^c is finite then J^c is finite so $(\bigcup_{j\in J} A_j)^c = \bigcup_{j\in J^c} A_j \in \mathcal{M}$. Let $P_1, P_2 \in \mathcal{M}$ where each $P_i = \bigcup_{j\in J_i} A_j$ so that $P_1 \cup P_2 = \bigcup_{j\in J_1\cup J_2} A_j$. If both J_i are finite then $J_1 \cup J_2$ is finite. If (without loss of generality) J_1^c is finite then $(J_1 \cup J_2)^c = J_1^c \cap J_2^c$ is finite. Thus $P_1 \cup P_2 \in \mathcal{M}$. $\mathcal{M}$ is then clearly the smallest algebra containing $\mathcal{X}$ so is the algebra generated by $\mathcal{X}$.

$\square$

Exercise 0.7. *Suppose $X = (0,7]$ and $C = \{(0,2],(1,5]\}$. Write down the sets in $\sigma(C)$.*

Proof. $(0,2]^c = (2,7], (1,5]^c = (0,1] \cup (5,7]$.
$(0,2] \cup (1,5] = (0,5]$.
$(0,5]^c = (5,7]$.
$(5,7] \cup (0,5] = X$.
Thus $\sigma(\mathcal{C}) \supseteq \mathcal{M} := \{\emptyset, (0,2], (1,5], (0,1]\cup(5,7], (0,5], (5,7], (0,7], (0,2]\cup(5,7], (1,7], (2,5], (0,1], (0,1]\cup (2,5], (1,2] \cup (5,7], (2,7], (1,2], (0,1] \cup (2,7]\}$. $\mathcal{P} := \{(0,1],(1,2],(2,5],(5,7]\}$ is a partition of X so $\sigma(\mathcal{P})$ has 16 elements. $\mathcal{C} \subseteq \sigma(\mathcal{P})$ so $\mathcal{M} \subseteq \sigma(\mathcal{P})$. $\mathcal{M}$ also has 16 elements so $\mathcal{M} = \sigma(\mathcal{P})$. Thus $\sigma(\mathcal{C}) = \mathcal{M}$. $\square$

Exercise 0.8. *Let*

$$\begin{aligned}
\mathcal{O} &= \{G \subset \mathbb{R} : G \text{ is open}\};\\
\mathcal{H} &= \{F \subset \mathbb{R} : F \text{ is closed}\};\\
\mathcal{K} &= \{K \subset \mathbb{R} : K \text{ is compact}\};\\
\mathcal{D} &= \{F \subset \mathbb{R} : F = (-\infty, q] \text{ for some } q \in \mathbb{Q}\}.
\end{aligned}$$

(Recall that K is compact iff K is closed and bounded.) Recall that the collection $\mathcal{B}$ of Borel sets in $\mathbb{R}$ is defined by $\mathcal{B} = \sigma(\mathcal{O})$.

1. *Construct a Borel set that is neither open nor closed, that is, it is in $\mathcal{B} \setminus (\mathcal{O} \cup \mathcal{H})$.*

2. *Prove that $\sigma(\mathcal{K}) = \mathcal{B}$.*

3. *Prove that $\sigma(\mathcal{D}) = \mathcal{B}$.*

Proof. 1. $(1,2]$.

2. $\mathcal{K} \subseteq \mathcal{H}$ so $\sigma(\mathcal{K}) \subseteq \sigma(\mathcal{H}) = \mathcal{B}$. Let $U \in \mathcal{O}$. Then for each $x \in U$ we can find $q, r \in \mathbb{Q}$ such that $x \in [q,r]$ and hence $U = \bigcup_{(q,r)\in\mathbb{Q}^2, [q,r]\subseteq U} [q,r]$ which is an element of $\sigma(\mathcal{K})$ since closed intervals are bounded and $\mathbb{Q}^2$ is countable. Thus $\mathcal{O} \subseteq \sigma(\mathcal{K})$ and so $\sigma(\mathcal{K}) = \mathcal{B}$.

3. Let $a, b \in \mathbb{Q}$ with $a < b$. Then $(a, b] = (-\infty, a]^c \cap (-\infty, b] \in \sigma(\mathcal{D})$. Let $x \in \mathbb{Q}, y \in \mathbb{R}$ such that $x < y$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists a decreasing sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = x$. Let $A := \bigcup_{n=1}^{\infty}(x_n, y] \subseteq (x, y]$. Let $\epsilon > 0$ such that $x + \epsilon \leq y$. Then $\exists N \in \mathbb{N} : x_N < x + \epsilon$ and so $x + \epsilon \in (x_N, y] \subseteq A$. Thus $(x, y] \subseteq A$ and so $(x, y] = A$. Thus $\sigma(\mathcal{D})$ contains all sets of the form $(x, y], x \in \mathbb{R}, y \in \mathbb{Q}$. Now consider $(x, y]$ where both $x, y \in \mathbb{R}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists a decreasing sequence $(y_n)_{n\in\mathbb{N}} \in \mathbb{Q}$ such that $\lim_{n\to\infty} y_n = y$. Let $A := \bigcap_{n=1}^{\infty}(x, y_n] \supseteq (x, y]$. Let $t > y$. Then $\exists N \in \mathbb{N}$ such that $y_N < t$ and hence $t \notin (x, y_N]$. Thus $(x, y] = A$ and hence $\mathcal{I} \subseteq \sigma(\mathcal{D})$, implying $\sigma(\mathcal{I}) = \mathcal{B} \subseteq \sigma(\mathcal{D})$. Let $(-\infty, q] \in \mathcal{D}$. Let $A := \bigcup_{n=1}^{\infty}(-n, q] \in \sigma(\mathcal{I})$. Then given any $x < q : \exists N \in \mathbb{N} : -N < x$ and so $x \in A$. Thus $A = (-\infty, q]$ and hence $\mathcal{D} \subseteq \sigma(\mathcal{I}) = \mathcal{B} \implies \sigma(\mathcal{D}) \subseteq \mathcal{B}$. Thus $\sigma(\mathcal{D}) = \mathcal{B}$.

□

Exercise 0.9. *Show that the examples described just after Definition 3.1 are indeed measures. [Hint: you may find Exercise 4 useful here.]*

Proof. 1. Counting measure: $\mu(\emptyset) = 0$. Let $A_1, A_2, ... \in \mathcal{M}$ be pairwise disjoint. If there exists an $N \in \mathbb{N}$ such that $A_n = \emptyset \forall n > N$ then $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{N} A_i$ so $\mu(\bigcup_{i=1}^{\infty} A_i) = \#$ of elements of $\bigcup_{i=1}^{N} A_i = \sum_{i=1}^{N} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. Otherwise, $\mu(\bigcup_{i=1}^{\infty} A_i) = \infty = \sum_{i=1}^{\infty} A_i$.

2. Dirac measure: $\delta_x(\emptyset) = 0$ since $x \notin \emptyset$. Let $A_1, A_2, ... \in \mathcal{M}$ be pairwise disjoint. Suppose $x \in \bigcup_{i=1}^{\infty} A_i$ so that $\delta_x(\bigcup_{i=1}^{\infty} A_i) = 1$. Since the A_i's are disjoint there is a single A_i for which $\delta_x(A_i) = 1$ and every other set does not include x. Thus $\sum_{i=1}^{\infty} \delta_x(A_i) = 1$. Now suppose $x \notin \bigcup_{i=1}^{\infty} A_i$ so that $\delta_x(\bigcup_{i=1}^{\infty} A_i) = 0$. Then $\delta_x(A_i) = 0 \forall i$ so $\sum_{i=1}^{\infty} \delta_x(A_i) = 0$.

3. Scalar multiples of measures: $(a\mu)(\emptyset) = a{\cdot}0 = 0$. $\sum_{i=1}^{\infty}(a\mu)(A_i) = a\sum_{i=1}^{\infty}\mu(A_i) = a\mu(\bigcup_{i=1}^{\infty} A_i) = (a\mu)(\bigcup_{i=1}^{\infty} A_i)$.

4. Countable sums of measures: $(\sum_{i=1}^{\infty}\mu_i)(\emptyset) = \sum_{i=1}^{\infty}\mu_i(\emptyset) = \sum_{i=1}^{\infty} 0 = 0$. $(\sum_{i=1}^{\infty}\mu_i)(\bigcup_{n=1}^{\infty} A_n) = \sum_{i=1}^{\infty}\mu_i(\bigcup_{n=1}^{\infty} A_n) = \sum_{i=1}^{\infty}(\sum_{n=1}^{\infty}\mu_i(A_n)) = \sum_{n=1}^{\infty}(\sum_{i=1}^{\infty}\mu_i(A_n)) = \sum_{n=1}^{\infty}(\sum_{i=1}^{\infty}\mu_i)(A_n)$.

5. Discrete measures: $\mu(\emptyset) = \sum_{i\in\emptyset} m_i = 0$. $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{i\in\bigcup_{n=1}^{\infty} A_n} m_i = \sum_{n=1}^{\infty}(\sum_{i\in A_n} m_i) = \sum_{n=1}^{\infty}\mu(A_n)$.

□

Exercise 0.10. *Let μ be a measure on $(\mathbb{R}, \mathcal{B})$ with $\mu(\mathbb{R}) < \infty$. For $x \in \mathbb{R}$, set $F(x) = \mu((-\infty, x])$. Show that F is nondecreasing and right continuous. [A function $f : \mathbb{R} \to \mathbb{R}$ is right continuous if for all $x \in \mathbb{R}$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $x < y < x + \delta$ then $|f(y) - f(x)| < \epsilon$.]*

Proof. Let $a < b$. Then $F(b) = \mu((-\infty, b]) = \mu((-\infty, a]) + \mu((a, b]) \geq F(a)$.

Let $x \in \mathbb{R}$ and let $\epsilon > 0$. Define $A_1, A_2, ... \in \mathcal{B}$ by $A_n = (-\infty, x + \frac{1}{n}]$ so that $\bigcap_{n=1}^{\infty} A_n = (-\infty, x]$. Then by downwards continuity we have $\lim_{n\to\infty}\mu(A_n) = F(x)$ so $\exists N \in \mathbb{N}$ such that $F(x + \frac{1}{N}) < F(x) + \epsilon$. Let $\delta := \frac{1}{N}$. Then $\forall y$ such that $x < y < x + \delta$ we have $|F(y) - F(x)| < F(x) + \epsilon - F(x) = \epsilon$. □

Exercise 0.11. *1. Give an example of a measure space $(X, \mathcal{M}, \mu)$ and a sequence of sets $A_1 \supset A_2 \supset A_3 \supset \ldots$ with each $A_i \in \mathcal{M}$, such that $\mu(\bigcap_{n=1}^{\infty} A_n) \neq \lim_{n\to\infty}\mu(A_n)$.*

2. Give an example of a measurable space $(X, \mathcal{M})$ and a set function $\mu : \mathcal{M} \to [0, \infty]$ such that μ is finitely additive but not countably additive. [Hint: In both cases we can take $X = \mathbb{N}$.]

Proof. 1. Let $X = \mathbb{N}$ and let $\mathcal{M} = \mathcal{P}(X)$. Let μ be the counting measure and let $A_n = 2^n\mathbb{N}$. Then $\bigcap_{n=1}^{\infty} A_n = \emptyset$ since given any $i \in \mathbb{N}$ we have $i \notin A_i$, and so $\mu(\bigcap_{n=1}^{\infty} A_n) = 0$. However, $\mu(A_n) = \infty \forall n$ so $\lim_{n\to\infty} \mu(A_n) = \infty$.

2. Let $X = \mathbb{N}$, let $\mathcal{M} = \mathcal{P}(X)$ and let $\mu(A) = \infty$ if A is infinite and 0 if A is finite. Let $A_n = \{n\}$. Then $\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\mathbb{N}) = \infty$ whereas $\sum_{n=1}^{\infty} \mu(A_n) = 0$.

□

Exercise 0.12. *Suppose X is a non-empty set and $\mathcal{X} = \{A_1, A_2, \ldots\}$ is a partition of X with $A_i \neq \emptyset$ for each $i \in \mathbb{N}$. Suppose $(a_1, a_2, a_3, \ldots)$ is a sequence of nonnegative numbers. Show that there is a unique measure μ on the measurable space $(X, \sigma(\mathcal{X}))$ with $\mu(A_i) = a_i$ for all $i \in \mathbb{N}$.*

Proof. Define a set function $\mu : \sigma(\mathcal{X}) \to [0, \infty]$ by $\mu(\bigcup_{j\in J} A_j) = \sum_{j\in J} a_j$ given $J \subseteq \mathbb{N}$. This defines μ for all elements of $\sigma(\mathcal{X})$. $\mu(\emptyset) = \sum_{j\in\emptyset} a_j = 0$. Given pairwise disjoint $\bigcup_{j\in j_1} A_j, \bigcup_{j\in j_2} A_j, \ldots$ (meaning that $J_1, J_2, \ldots$ are pairwise disjoint) we have $\mu(\bigcup_{i=1}^{\infty} \bigcup_{j\in J_i} A_j) = \mu(\bigcup_{j\in\bigcup_{i=1}^{\infty} J_i} A_j) = \sum_{j\in\bigcup_{i=1}^{\infty} J_i} a_j = \sum_{i=1}^{\infty}(\sum_{j\in J_i} a_j) = \sum_{i=1}^{\infty} \mu(\bigcup_{j\in J_i} A_j)$. Thus μ is a measure. Now suppose that ν is another measure satisfying $\nu(A_i) = a_j \forall i \in \mathbb{N}$. Then $\nu(\bigcup_{j\in J} A_j) = \sum_{j\in J} a_j = \mu(\bigcup_{j\in J} A_j) = \sum_{j\in J} a_j \forall J \subseteq \mathbb{N}$ so $\nu = \mu$. □

Exercise 0.13. *Show that if $A \subset \mathbb{R}$ is countable then $A \in \mathcal{B}$ and $\lambda_1(A) = 0$.*

Proof. Let $x \in \mathbb{R}$ ($\{x\}$ is also a Borel set since $x = ((-\infty, x) \cup (x, \infty))^c$). Then $x \in (x - \frac{1}{n}, x] \forall n \in \mathbb{N}$ so $0 \leq \lambda_1(\{x\}) \leq \lambda((x - \frac{1}{n}, x]) = \frac{1}{n} \forall n$ so $\lambda_1(\{x\}) = 0$. Now enumerate the elements of A as $x_1, x_2, x_3, \ldots \in \mathbb{R}$ (allowing for possible repetitions for if A is finite) so that $A = \bigcup_{i=1}^{\infty}\{x_i\}$. Then by countable sub-additivity $0 \leq \lambda_1(A) \leq \sum_{i=1}^{\infty} \lambda_1(\{x_i\}) = 0$. □

Exercise 0.14. *Show that for any interval I with left endpoint a and right endpoint b we have $\lambda_1(I) = b - a$ (regardless of whether $a, b \in I$ or not).*

Proof. If $I = (a, b]$ then $\lambda_1((a, b]) = \lambda((a, b]) = b - a$. If $I = (a, b)$ then $\lambda_1((a, b)) = \lambda_1((a, b]) - \lambda_1(\{b\}) = b - a - 0 = b - a$. etc. □

Exercise 0.15. *Give an example of a Borel set $A \subset \mathbb{R}$ with $\lambda_1(A) > 0$ but with no non-empty open interval contained in A.*

Proof. $\mathbb{R} \setminus \mathbb{Q}$. □

Exercise 0.16. *Given $\epsilon > 0$, give an example of an open set $U \subset \mathbb{R}$ with $\lambda_1(U) < \epsilon$ that is dense in $\mathbb{R}$, i.e., has non-empty intersection with every non-empty open interval in $\mathbb{R}$.*

Proof. Let $x_1, x_2, x_2, \ldots$ be an enumeration of $\mathbb{Q}$. Let $U = \bigcup_{i=1}^{\infty}(x_i - \frac{\epsilon}{2^{i+2}}, x_i + \frac{\epsilon}{2^{i+2}})$. Then $\lambda_i(U) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^{i+2}} = \frac{\epsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{\epsilon}{2} < \epsilon$. □

Exercise 0.17. *Suppose $A \subset \mathbb{R}$ is a bounded Borel set. Show that for all $\epsilon > 0$ there exists a set U which is a finite union of intervals, such that $\lambda_1(A\Delta U) < \epsilon$, where $A\Delta U := (A \cup U) \setminus (A \cap U)$. [Hint: use the fact that $\lambda_1(A) = \lambda^*(A)$.]*

Proof. $\lambda_1(A) = \inf\{\sum_{n=1}^{\infty} \lambda(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n; I_1, I_2, \ldots \in \overline{I}\}$ so by properties of infimums there exists $I_1, I_2, .. \in \overline{I}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\lambda_1(\bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} \lambda_1(I_n) < \frac{\epsilon}{2} + \lambda_1(A)$. Let $S := \bigcup_{n=1}^{\infty} I_n$ and let $S_N := \bigcup_{n=1}^{N} I_n$. Then by upward continuity $\lambda_1(S) = \lim_{N\to\infty} \lambda_1(S_N)$. Thus $\exists K \in \mathbb{N}$ such that $\lambda_1(S) - \lambda_1(S_K) < \frac{\epsilon}{2}$.

$$A \setminus S_K \subseteq S \setminus S_K$$

so

$$\lambda_1(A \setminus S_K) \le \lambda_1(S \setminus S_K) = \lambda_1(S) - \lambda_1(S_k) < \frac{\epsilon}{2}.$$

$S_K \setminus A \subseteq S \setminus A$ so $\lambda_1(S_K \setminus A) \le \lambda_1(S \setminus A) = \lambda_1(S) - \lambda_1(A) < \frac{\epsilon}{2}$. Thus

$$\lambda_1(S_K \Delta A) = \lambda_1(S_K \setminus A) + \lambda_1(A \setminus S_K) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Exercise 0.18. *In this question we write $\lambda^*(A)$ for the Lebesgue outer measure of A.*

1. *What is the definition of the Lebesgue outer measure of a set $A \subset \mathbb{R}$?*
2. *Show that for any (not necessarily Borel) $A \subset \mathbb{R}$ there exists a Borel set $B \subset \mathbb{R}$ with $A \subset B$ and $\lambda_1(B) = \lambda^*(A)$.*
3. *Suppose $A \subset \mathbb{R}$ is a Borel set with $\lambda_1(A) > 0$. Using the fact that $\lambda_1(A) = \lambda^*(A)$, show that for any $\epsilon > 0$ there exists a non-empty half-open interval I with $\lambda_1(A \cap I) \ge (1-\epsilon)\lambda_1(I)$.*
4. *Show that the set $A \ominus A := \{x - y : x, y \in A\}$ includes a non-empty half-open interval.*

Proof.

1. $\lambda^*(A) = \inf \{\sum_{n=1}^{\infty} \lambda(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n, I_1, I_2, ... \in \overline{I}\}$.
2. Given any $N \in \mathbb{N}$ there exists a countable union $A_N \in \mathcal{B}$ of elements in $\overline{I}$ such that $A \subseteq A_N$ and $\lambda^*(A) \le \lambda_1(A_N) < \lambda^*(A) + \frac{1}{N}$. Let $S_N := \bigcap_{n=1}^{N} A_n \in \mathcal{B}$ and let $S := \bigcap_{n=1}^{\infty} A_n \in \mathcal{B}$. By downwards continuity $\lambda_1(S) = \lim_{N\to\infty} \lambda_1(S_N)$. $A \subseteq S_N \subseteq A_N$ so

   $$\lambda^*(A) \le \lambda_1(S_N) < \lambda^*(A) + \frac{1}{N} \forall N,$$

   implying that $\lim_{N\to\infty} \lambda_1(S_N) = \lambda^*(A)$ and hence $\lambda_1(S) = \lambda^*(A)$.
3. First let A be bounded. Suppose for a contradiction that there exists an $\epsilon > 0$ such that $\lambda_1(A \cap I) < (1-\epsilon)\lambda_1(I)$ for every non-empty half-open interval I. Clearly $\epsilon < 1$. Let $\delta > 0$ be such that $(1-\epsilon)(1+\delta) < 1$. We have a countable union of half-open intervals $S = \bigcup_{n=1}^{\infty} I_n$ with $A \subseteq S$ such that $\sum_{n=1}^{\infty} \lambda_1(I_n) < \lambda_1(A) + \delta$. We also have

   $$\lambda_1(A) \le \sum_{n=1}^{\infty} \lambda_1(A \cap I_n) < (1-\epsilon) \sum_{n=1}^{\infty} \lambda_1(I_n)$$

   so $\lambda_1(A) < (1-\epsilon)(1+\delta)\lambda_1(A)$; a contradiction.

   Now let A be unbounded. Let $A_n := A \cap [-n, n]$ so that $A_n \subseteq A_{n+1} \forall n$ and $A = \bigcup_{i=1}^{\infty} A_i$. $\lambda_1(A) = \lim_{n\to\infty} \lambda_1(A_n)$ so $\exists N \in \mathbb{N}$ such that $\lambda_1(A_N) > 0$. Hence

   $$\lambda_1(A \cap I) \ge \lambda_1(A_N \cap I) \ge (1-\epsilon)\lambda_1(I)$$

   for some non-empty half-open interval I.
4. Suppose that $A \ominus A$ does not contain a non-empty half-open interval. Then $\forall \epsilon > 0$ there exists a non-empty half-open interval I with $\lambda_1(A \cap I) \ge (1-\epsilon)\lambda_1(I) \ge \lambda_1(I \cap A \ominus A)$.

   There exists a non-empty half-open interval I such that $\lambda_1(A \cap I) \ge 0.999\lambda_1(I)$. Suppose $z \notin A \ominus A$. Then $\forall x, y \in A$ we have $z \ne x - y$ so $x \ne z + y$. Hence $(z + A) \cap A = \emptyset$ so

$z + A \subseteq A^c$. Let δ be such that $I = (a, a+\delta]$. Suppose that $z \in (0, \frac{\delta}{2}]$ and $z \notin A \ominus A$. Then $(a, a+\frac{\delta}{2}] + z \subseteq (a, a+\delta]$ and so $z + (a, a+\frac{\delta}{2}] \cap A \subseteq A^c \cap (a, a+\delta]$. Hence

$$\begin{aligned}
\lambda_1(A \cap (a, a+\frac{\delta}{2}]) &\leq \lambda_1(A^c \cap (a, a+\delta]) \\
&= \lambda_1((a, a+\delta]) - \lambda_1(A \cap (a, a+\delta]) \\
&\leq \delta - 0.999\delta = 0.001\delta.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\lambda_1(A^c \cap (a, a+\delta]) &\geq \lambda_1(A \cap (a, a+\frac{\delta}{2}]) \\
&= \lambda_1(A \cap (a, a+\delta]) - \lambda_1(A \cap (a+\frac{\delta}{2}, a+\delta]) \\
&\geq 0.999\delta - 0.5\delta = 0.499\delta;
\end{aligned}$$

a contradiction. hence $(0, \frac{\delta}{2}] \subseteq A \ominus A$.

□

Exercise 0.19. *Suppose X is a non-empty set and $\mathcal{D}$ is a π-system in X. Show that for any $k \in \mathbb{N}$, if $A_i \in \mathcal{D}$ for $i = 1, 2, \ldots, k$ then $\bigcap_{i=1}^k A_i \in \mathcal{D}$.*

Proof. Induction. □

Exercise 0.20. *Let $\mathcal{I}$ denote the class of half-open intervals in $\mathbb{R}$, together with the empty set (as in the lecture notes). Define the set-function $\pi : \mathcal{I} \to [0, \infty]$ by*

$$\pi(A) := \begin{cases} 0 & \textit{if } A = \emptyset; \\ \infty & \textit{if } A \neq \emptyset. \end{cases}$$

Show that π has more than one extension to a measure on $\mathcal{B} = \sigma(\mathcal{I})$. What condition of the (Uniqueness theorem) failed here?

Proof. The counting measure and $\mu : \mathcal{B} \to [0, \infty]$ given by $\mu(A) := \begin{cases} 0 & \text{if } A = \emptyset; \\ \infty & \text{if } A \neq \emptyset. \end{cases}$ are both extensions of π to a measure on $\mathcal{B}$. The uniqueness theorem failed because π is not σ-finite. □

Exercise 0.21. *Show that λ_1 has the scaling property: for any real number $c \neq 0$ and any Borel set $B \in \mathcal{B}$, we have $\lambda_1(cB) = |c|\lambda_1(B)$. Here cB is defined to be the set $\{cx : x \in B\}$.*

Proof. Let $(a, b] \in I$ be a half-open interval so that $\lambda_1((a, b]) = b - a$. Let $c > 0$. Then $c(a, b] = (ca, cb]$ so

$$\lambda_1(c(a, b]) = cb - ca = |c|\lambda_1((a, b]).$$

Now let $c < 0$. Then $c(a, b] = [cb, ca)$ so

$$\lambda_1(c(a, b]) = cb - ca = -c(a - b) = |c|\lambda_1((a, b]).$$

The result then obviously holds for unbounded intervals. Now let B be any Borel set. Given any $\epsilon > 0$ there exists a countable collection $I_1, I_2, \ldots$ of half-open intervals such that $B \subseteq \bigcup_{n=1}^\infty I_n$ and $\sum_{n=1}^\infty \lambda_1(I_n) < \epsilon + \lambda_1(B)$. We then have $cB \subseteq c\bigcup_{n=1}^\infty I_n = \bigcup_{n=1}^\infty cI_n$ and hence

$$\lambda_1(cB) \leq \sum_{n=1}^\infty \lambda_1(cI_n) = \sum_{n=1}^\infty |c|\lambda_1(I_n) = |c| \sum_{n=1}^\infty \lambda_1(I_n) \leq |c|(\epsilon + \lambda_1(B)).$$

ϵ is arbitrary so $\lambda_1(cB) \le |c|\lambda_1(B)$. We then also have $\lambda_1(B) = \lambda_1(\frac{1}{c}cB) \le |\frac{1}{c}|\lambda_1(cB)$ and hence $|c|\lambda_1(B) \le \lambda_1(cB)$. Thus $\lambda_1(cB) = |c|\lambda_1(B)$. $\square$

Exercise 0.22. *Suppose μ is a translation-invariant measure on $(\mathbb{R}, \mathcal{B})$. Set $\gamma := \mu((0,1])$ and assume $0 < \gamma < \infty$.*

(a) Show that $\mu((0,1/n)) = \gamma/n$ for all $n \in \mathbb{N}$.

(b) Show that $\mu((0,q]) = \gamma q$ for all rational $q > 0$.

(c) Let $\mathcal{I}'$ be the class of half-open intervals in $\mathbb{R}$ with rational endpoints, i.e., the class of intervals of the form $(q,r]$ with $q \in \mathbb{Q}, r \in \mathbb{Q}$, and $q < r$. Show that $\mu(I) = \gamma\lambda_1(I)$ for all $I \in \mathcal{I}'$.

(d) Show that $\sigma(\mathcal{I}') = \mathcal{B}$. You may use without proof the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, every non-empty open interval in $\mathbb{R}$ contains at least one rational number.

(e) Use the Uniqueness Lemma to show that $\mu(B) = \gamma\lambda_1(B)$ for all $B \in \mathcal{B}$.

Proof. (a) $(0,1] = \bigcup_{i=1}^n (\frac{i-1}{n}, \frac{i}{n}]$ (pairwise disjoint) so $\gamma = \sum_{i=1}^n \mu((\frac{i-1}{n}, \frac{i}{n}]) = \sum_{i=1}^n \mu((0, \frac{1}{n}])$ by translation invariance and hence $\mu((0, \frac{1}{n}]) = \gamma/n$.

(b) Write q as a/b for $a, b \in \mathbb{N}$. Then $(0,q] = \bigcup_{i=1}^a (\frac{i-1}{b}, \frac{i}{b}]$ (pairwise disjoint) so

$$\mu((0,q]) = \sum_{i=1}^{a} \mu((\frac{i-1}{b}, \frac{i}{b}]) = \sum_{i=1}^{a} \mu((0, \frac{1}{b}]) = \frac{a}{b}\gamma = q\gamma.$$

(c) $(q,r] = (0,r] \setminus (0,q]$ so $\mu((q,r]) = \mu((0,r]) - \mu((0,q]) = \gamma r - \gamma q = \gamma\lambda_1((q,r])$.

(d) We need to show that $\mathcal{I} \subseteq \sigma(\mathcal{I}')$ since then $\mathcal{B} = \sigma(\mathcal{I}) \subseteq \sigma(\mathcal{I}')$ so $\sigma(\mathcal{I}') = \mathcal{B}$. Let $(a,b] \in \mathcal{I}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ there exist sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \mathbb{Q}$ such that x_n is increasing with $\lim_{n\to\infty} x_n = a$ and y_n is decreasing with $\lim_{n\to\infty} y_n = b$. Let $A := \bigcap_{n=1}^\infty (x_n, y_n] \in \sigma(\mathcal{I}')$. Clearly $(a,b] \subseteq A$. Let $\alpha \le a$. Then $\exists N$ such that $x_N \ge \alpha$ so $\alpha \notin A$. let $\beta > b$. Then $\exists K$ such that $y_K < \beta$ so $\beta \notin A$. Thus $A = (a,b]$ as required.

(e) $\mathcal{I}'$ is a π-system in $\mathbb{R}$ and $\gamma\lambda_1$ is a measure on $(\mathbb{R}, \mathcal{B})$ which is σ-finite on $\mathcal{I}'$. μ is also a measure on $(\mathbb{R}, \mathcal{B})$ and agrees with $\gamma\lambda_1$ on $\mathcal{I}'$ so by the uniqueness lemma $\mu(B) = \gamma\lambda_1(B) \forall B \in \mathcal{B}$. $\square$

Exercise 0.23. *Suppose X is a non-empty set and $\mathcal{S}$ is a semi-algebra in X. As in Chapter 6 of the notes, let $\mathcal{U}$ be the class of sets of the form $\bigcup_{i=1}^k A_i$ with $k \in \mathbb{N}$ and $A_1, \ldots, A_k$ pairwise disjoint sets in $\mathcal{S}$.*

(a) Show by induction on k that if $A \in \mathcal{U}$ then $A^c \in \mathcal{U}$, i.e., $\mathcal{U}$ is closed under complementation.

(b) Show also that $\mathcal{U}$ is closed under pairwise intersections and deduce that $\mathcal{U}$ is an algebra.

(c) Deduce that $\mathcal{U}$ is the algebra generated by $\mathcal{S}$. (Generated algebras are defined analogously to generated σ-algebras. Write $\mathcal{A}(\mathcal{S})$ for the algebra generated by $\mathcal{S}$.)

Proof. (a) For $k = 1$: A_1^c is a finite union of disjoint sets in $\mathcal{S}$ so $A_1^c \in \mathcal{U}$. Assume true for $k = n$. For $k = n + 1$,

$$(\bigcup_{i=1}^{n+1} A_i)^c = ((\bigcup_{i=1}^{n} A_i)^c) \cap A_{n+1}^c.$$

We have $A_{n+1}^c = \bigcup_{i=1}^b D_i$ for some pairwise disjoint $D_i \in \mathcal{S}$, and by the inductive hypothesis we have $(\bigcup_{i=1}^n A_i)^c = \bigcup_{i=1}^a C_i$ for some pairwise disjoint $C_i \in \mathcal{S}$.

$$\begin{aligned}(\bigcup_{i=1}^{a} C_i) \cap (\bigcup_{i=1}^{b} D_i) &= \bigcup_{i=1}^{a}(C_i \cap (\bigcup_{j=1}^{b} D_j)) \\ &= \bigcup_{i=1}^{a}\bigcup_{j=1}^{b}(C_i \cap D_j) \\ &= \bigcup_{(i,j),1\leq i\leq a,1\leq j\leq b} (C_i \cap D_j).\end{aligned}$$

Each $C_i \cap D_j \in \mathcal{S}$ since $\mathcal{S}$ is a π-system. Furthermore, given $C_i \cap D_j$ and $C_x \cap D_y$ where $i \neq x$ (without loss of generality), then $C_i \cap C_x = \emptyset$ so $(C_i \cap D_j) \cap (C_x \cap D_y) = \emptyset$. Thus $(\bigcup_{i=1}^{n+1} A_i)^c \in \mathcal{U}$ so by induction $\mathcal{U}$ is closed under complementation.

(b) Closure under pairwise intersections was proven in part (a). $\emptyset \in \mathcal{S}$ so $\emptyset \in \mathcal{U}$. $\mathcal{U}$ is also closed under complements. Let $A, B \in \mathcal{U}$. Then $A^c, B^c \in \mathcal{U}$ so $A \cup B = (A^c \cap B^c)^c \in \mathcal{U}$. Thus $\mathcal{U}$ is an algebra.

(c) $\mathcal{U} \subseteq \mathcal{A}(\mathcal{S})$ since algebras are closed under finite unions. Since $\mathcal{U}$ is also an algebra it follows that $\mathcal{U} = \mathcal{A}(\mathcal{S})$.

□

Exercise 0.24. *Suppose X is a non-empty set, $\mathcal{S}$ is a semi-algebra in X, and π is a pre-measure on $(X, \mathcal{S})$.*

(a) Show that if $A, A_1, \dots, A_k \in \mathcal{S}$ with $A_1, \dots, A_k$ pairwise disjoint and $\bigcup_{i=1}^k A_i \subseteq A$, then $\sum_{i=1}^k \pi(A_i) \leq \pi(A)$.

(b) Show that π is countably additive, i.e., $\pi(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \pi(A_n)$ whenever $A_1, A_2, \dots \in \mathcal{S}$ are pairwise disjoint with $\bigcup_{n=1}^\infty A_n \in \mathcal{S}$.

Hint: The result from Question 23 might be useful.

Proof. (a) We have $\sum_{i=1}^k \pi(A_i) = \pi(\bigcup_{i=1}^k A_i)$. Let $B_1 = A$ and $B_i = \emptyset \forall i > 1$. $\bigcup_{i=1}^k A_i \subseteq \bigcup_{i=1}^\infty B_i$ so by countable sub-additivity $\sum_{i=1}^k \pi(A_i) = \pi(\bigcup_{i=1}^k A_i) \leq \sum_{i=1}^\infty \pi(B_i) = \pi(A)$.

(b) By countable sub-additivity we have $\pi(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \pi(A_n)$. Also, $\forall N \in \mathbb{N}$ we have $\sum_{n=1}^N \pi(A_n) \leq \pi(\bigcup_{n=1}^\infty A_n)$ so $\sum_{n=1}^\infty \pi(A_n) = \lim_{N\to\infty} \sum_{n=1}^N \pi(A_n) \leq \pi(\bigcup_{n=1}^\infty)$. Thus $\pi(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \pi(A_n)$.

□

Exercise 0.25. *Let $F : (-\infty, \infty) \to \mathbb{R}$ be a non-decreasing, right-continuous function (right continuity is defined in Question 10).*

Let $\mathcal{I}$ denote the set of bounded half-open intervals in $\mathbb{R}$ (as in lectures). For $I \in \mathcal{I}$, put

$$\lambda_F(I) = F(b) - F(a), \quad \textit{where } I = (a, b], \quad \textit{and } \lambda_F(\emptyset) = 0.$$

(a) *Check that* $\lambda_F(I) \geq 0$ *for all* $I \in \mathcal{I}$.

(b) *Show that the set function* λ_F *is finitely sub-additive on* $\mathcal{I}$*, the class of bounded half-open intervals in* $\mathbb{R}$*. That is, show that if* $A, A_1, A_2, \ldots, A_n \in \mathcal{I}$ *with* $A \subseteq \bigcup_{i=1}^n A_i$*, then* $\lambda_F(A) \leq \sum_{i=1}^n \lambda_F(A_i)$.

(c) *Show that* λ_F *is finitely additive on* $\mathcal{I}$*. That is, show that if* $A_1, A_2, \ldots, A_n \in \mathcal{I}$ *are pairwise disjoint with* $A = \bigcup_{i=1}^n A_i \in \mathcal{I}$*, then* $\lambda_F(A) = \sum_{i=1}^n \lambda_F(A_i)$.

(d) *Show that* λ_F *is countably sub-additive on* $\mathcal{I}$*. That is, show that if* $A, A_1, A_2, \cdots \in \mathcal{I}$ *with* $A \subseteq \bigcup_{i=1}^\infty A_i$*, then* $\lambda_F(A) \leq \sum_{i=1}^\infty \lambda_F(A_i)$.

Proof. (a) Let $I = (a, b]$ for $b > a$. F is non-decreasing so $F(b) \geq F(a)$ and so $\lambda_F(I) = F(b) - F(a) \geq 0$.

(b) We induct on n. For $n = 1$, $\lambda_F(A) \leq \lambda_F(A_1)$. Assume true for $n = k$. Then for $n = k + 1$, write A as $(a, b]$ and A_i as $(a_i, b_i]$. Without loss of generality, let $b_1 \leq b_2 \leq \ldots \leq b_n$. Further, assume that $A_{k+1} \cap A \neq \emptyset$, since otherwise $A \subseteq \bigcup_{i=1}^k A_i$ so by the inductive hypothersis

$$\lambda_F(A) \leq \sum_{i=1}^k \lambda_F(A_i) \leq \sum_{i=1}^{k+1} \lambda_F(A_i).$$

$A \subseteq \bigcup_{i=1}^n A_i$ so $b_n \geq b$. Furthermore, $a_n \leq b$ since $A_n \cap A \neq \emptyset$. If $a_n \leq a$ then $\lambda_F(A) \leq \lambda_F(A_n)$ so the result holds. If instead $a_n \in (a, b)$ then $(a, a_n] \subseteq \bigcup_{i=1}^k A_i$. Then by the inductive hypothesis

$$\begin{aligned}
\lambda_F(A) &= (F(b) - F(a_n)) + (F(a_n) - F(a)) \\
&= \lambda_F((a_n, b]) + \lambda_F((a, a_n]) \\
&\leq \lambda_F(A_n) + \sum_{i=1}^k \lambda_F(A_i) = \sum_{i=1}^n \lambda_F(A_i).
\end{aligned}$$

(c) We induct on n. For $n = 1$ it's immediate. Assume true for $n = k$. For $n = k + 1$, again assume without loss of generality that $b_1 \leq b_2 \leq \ldots \leq b_n$. We have $b_n = b$ and $a_n \geq a$ so $(a, a_n] = \bigcup_{i=1}^k A_i$. Thus the inductive hypothesis gives

$$\begin{aligned}
\lambda_F(A) &= (F(b) - F(a_n)) + (F(a_n) - F(a)) \\
&= \lambda_F((a_n, b]) + \lambda_F((a, a_n]) \\
&= \lambda_F(A_n) + \sum_{i=1}^k \lambda_F(A_i) = \sum_{i=1}^n \lambda_F(A_i).
\end{aligned}$$

(d) Let $\epsilon > 0$. By right-continuity there exists $a' \in (a, b)$ such that $F(a') < F(a) + \epsilon$ and $b_i' > b_i$ such that $F(b_i') < F(b_i) + 2^{-i}\epsilon$.

$$[a', b] \subseteq (a, b] \subseteq \bigcup_{i=1}^\infty (a_i, b_i] \subseteq \bigcup_{i=1}^\infty (a_i, b_i')$$

so by compactness $\exists N \in \mathbb{N}$ such that

$$(a', b] \subseteq [a', b] \subseteq \bigcup_{i=1}^{N} (a_i, b_i') \subseteq \bigcup_{i=1}^{N} (a_i, b_i'].$$

Thus

$$\lambda_F((a', b]) \leq \sum_{i=1}^{N} \lambda_F((a_i, b_i']) \leq \sum_{i=1}^{\infty} \lambda_F((a_i, b_i']).$$

$$\lambda_F((a_i, b_i']) = F(b_i') - F(b_i) + F(b_i) - F(a_i) < \lambda_F(A_i) + 2^{-i}\epsilon$$

so

$$\lambda_F((a', b]) \leq \sum_{i=1}^{\infty} \lambda_F(A_i) + \sum_{i=1}^{\infty} 2^{-i}\epsilon = \sum_{i=1}^{\infty} \lambda_F(A_i) + \epsilon.$$

$$\lambda_F((a', b]) = F(b) - F(a) - (F(a') - F(a)) \geq \lambda_F(A) - \epsilon$$

so

$$\lambda_F(A) - \epsilon \leq \sum_{i=1}^{\infty} \lambda_F(A_i) + \epsilon$$

or

$$\lambda_F(A) \leq \sum_{i=1}^{\infty} \lambda_F(A_i) + 2\epsilon.$$

ϵ is arbitrary so

$$\lambda_F(A) \leq \sum_{i=1}^{\infty} \lambda_F(A_i).$$

□

Exercise 0.26. *(a) Show that if $U \subseteq \mathbb{R}^2$ is open and $x \in U$, then we can find a rectangle $R \subseteq \mathbb{R}^2$ with corners having rational coordinates such that $x \in R \subseteq U$. [We say that a set $A \subseteq \mathbb{R}^2$ is open if for every $x \in A$ there is a disk of positive radius centered on x that is contained in A.]*

(b) Show that $\sigma(\mathcal{O}_2) = \mathcal{B}_2$, where $\mathcal{O}_2$ is the class of all open sets in $\mathbb{R}^2$, and $\mathcal{B}_2$ is the Borel σ-algebra in $\mathbb{R}^2$ (see Definition 8.1).

Proof. (a) There exists a disc or radius r centred at $x = (x_1, x_2)$, $B_r(x)$, such that $B_r(x) \subseteq U$. Let P be the square centred at x that is oriented parallel to the x and y axes and with vertices touching $\partial B_r(x)$. Let the vertices of P be given by $(a, b), (a+t, b), (a, b+t), (a+t, b+t)$ where $t > 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists rational numbers q, r, v, w such that $q \in (a, x_1), r \in (x_1, a+t), v \in (b, x_2), w \in (x_2, b+t)$. Then let $R := (q, r] \times (v, w]$.

(b) Let $U \in \mathcal{O}_2$. Let $S \subseteq \mathcal{R}_2$ be the set of all rectangles with rational coordinates that are contained within U. Clearly $\bigcup_{R \in S} R \subseteq U$. Furthermore, $\forall x \in U : \exists R \in S : x \in R$ so $U \subseteq \bigcup_{R \in S} R$ and hence $\bigcup_{R \in S} R = U$. S is in bijection with a subset of $\mathbb{Q}^4$ since each rectangle is determined by four points. $\mathbb{Q}^4$ is countable so S is countable as well. Hence U is a countable union of sets in $\mathcal{R}_2$ so $U \in \sigma(\mathcal{R}_2)$. Thus $\mathcal{O}_2 \subseteq \sigma(\mathcal{R}_2) = \mathcal{B}_2$ so $\sigma(\mathcal{O}_2) \subseteq \mathcal{B}_2$. Now let $A := (a, b] \times (x, y] \in \mathcal{R}_2$. $A = (a, \infty) \times (x, \infty) \setminus ((b, \infty) \times (x, \infty) \cup (a, \infty) \times (y, \infty)) \in \sigma(\mathcal{O}_2)$ so $\mathcal{R}_2 \subseteq \sigma(\mathcal{O}_2)$ and hence $\mathcal{B}_2 = \sigma(\mathcal{R}_2) \subseteq \sigma(\mathcal{O}_2)$. Thus $\sigma(\mathcal{O}_2) = \mathcal{B}_2$.

□

Exercise 0.27. *Suppose ρ is a rotation on $\mathbb{R}^2$, i.e., pre-multiplication by a 2×2 matrix M with $M^\top = M^{-1}$ (viewing elements of $\mathbb{R}^2$ as column vectors). Let λ_2 denote 2-dimensional Lebesgue measure (see Definition 8.10).*

(a) Show that $|\rho(x)| = |x|$ for all $x \in \mathbb{R}^2$, where for $x = (x_1, x_2)^\top \in \mathbb{R}^2$ we put $|x| = \sqrt{x_1^2 + x_2^2}$.

(b) Show that $\rho(A) \in \mathcal{B}_2$ for all $A \in \mathcal{B}_2$.

(c) Define a measure μ on $\mathcal{B}_2$ by $\mu(A) = \lambda_2(\rho(A))$ for all $A \in \mathcal{B}_2$. Show that μ is translation invariant, i.e., $\mu(A + x) = \mu(A)$ for all $A \in \mathcal{B}_2$ and all $x \in \mathbb{R}^2$.

(d) Show that the measure λ_2 is rotation invariant, i.e., $\lambda_2(\rho(A)) = \lambda_2(A)$ for all Borel $A \subseteq \mathbb{R}^2$ (and for any rotation ρ). You may use without proof the fact that every translation-invariant measure ν on $(\mathbb{R}^2, \mathcal{B}_2)$ is of the form $\nu = c \times \lambda_2$ for some constant c.

Proof. (a) $|\rho(x)| = \sqrt{\rho(x) \cdot \rho(x)} = \sqrt{(Mx)^T(Mx)} = \sqrt{x^T M^T M x} = \sqrt{x^T x} = \sqrt{x \cdot x} = |x|$.

(b) Let $\mathcal{M}$ be the set of $A \subseteq \mathbb{R}^2$ such that $\rho(A) \in \mathcal{B}_2$. Let $U \in \mathcal{O}_2$. Then $\rho(U) \in \mathcal{O}_2 \subseteq \mathcal{B}_2$ so $\mathcal{O}_2 \subseteq \mathcal{M}$. $\emptyset \in \mathcal{M}$ since $\rho(\emptyset) = \emptyset \in \mathcal{B}_2$. If $A \in \mathcal{M}$ then $\rho(A^c) = \rho(A)^c \in \mathcal{B}_2$ so $\mathcal{M}$ is closed under complements. If $A_1, A_2, ... \in \mathcal{M}$ then $\rho(\bigcup_{i=1}^\infty A_i) = \bigcup_{i=1}^\infty \rho(A_i) \in \mathcal{B}_2$ so $\mathcal{M}$ is closed under countable unions. Thus $\mathcal{M}$ is a σ-algebra and so $\mathcal{B}_2 = \sigma(\mathcal{O}_2) \subseteq \mathcal{M}$. Thus $\rho(A) \in \mathcal{B}_2 \forall A \in \mathcal{B}_2$.

(c) Let $x \in \mathbb{R}^2$ and let $A \in \mathcal{B}_2$. Assume that $\mu(A) < \infty$. Let $\epsilon > 0$. There exist $R_1, R_2, ... \in \mathcal{R}_2$ such that $A \subseteq \bigcup_{i=1}^\infty R_i$ and $\sum_{i=1}^\infty \mu(R_i) < \mu(A) + \epsilon$. Then $\rho(A + x) \subseteq \bigcup_{i=1}^\infty \rho(R_i + x)$

Define $\nu : \mathcal{B}_2 \to [0, \infty] : A \mapsto \lambda_2(\rho(A + x))$. $\nu(\emptyset) = \lambda_2(\rho(\emptyset)) = \lambda_2(\emptyset) = 0$. Let $A_1, A_2, ... \in \mathcal{B}_2$ be disjoint. Then

$$\begin{aligned}
\nu(\bigcup_{i=1}^\infty A_i) &= \lambda_2(\rho((\bigcup_{i=1}^\infty A_i) + x)) \\
&= \lambda_2(\rho(\bigcup_{i=1}^\infty (A_i + x))) \\
&= \lambda_2(\bigcup_{i=1}^\infty \rho(A_i + x)) \\
&= \sum_{i=1}^\infty \lambda_2(\rho(A_i + x)) \\
&= \sum_{i=1}^\infty \nu(A_i).
\end{aligned}$$

Thus ν is a measure. Let $R := (a, b] \times (x, y] \in \mathcal{R}_2$. Then $\nu(R) = \lambda_2(\rho(R + x)) = \lambda_2(\rho(R) + \rho(x)) = \lambda_2(\rho(R)) = \mu(R)$ since λ_2 is translation invariant. $\mathcal{R}_2$ is a π-system, $\sigma(\mathcal{R}_2) = \mathcal{B}_2$ and μ is σ-finite on $\mathcal{R}_2$ so by the uniqueness lemma ν agrees with μ on $\mathcal{B}_2$.

(d) Define $\mu : \mathcal{B}_2 \to [0, \infty] : A \mapsto \lambda_2(\rho(A))$. Given $x \in \mathbb{R}^2, A \in \mathcal{B}_2$, $\mu(A + x) = \lambda_2(\rho(A + x)) = \lambda_2(\rho(A) + \rho(x)) = \lambda_2(\rho(A)) = \mu(A)$ so μ is translation-invariant and thus of the form $c \times \lambda_2$ for some $c \in \mathbb{R}$. Let $A := B_1((0, 0))$. Then $\rho(A) = A$ so $\mu(A) = c\lambda_2(A) = \lambda_2(A)$ and hence $c = 1$ so λ_2 is rotation-invariant.

$\square$

Exercise 0.28. *(a) Show that $\lambda_2(L) = 0$ for any line segment $L \subseteq \mathbb{R}^2$. [You may use the result from Question 27 without proof.]*

(b) Let $r > 0$ and set $D := \{x \in \mathbb{R}^2 : |x| < r\}$, the open disk of radius r in $\mathbb{R}^2$ centered on the origin (we define $|x|$ as in the previous question). By approximating D by an increasing sequence of regular polygons contained in D, show that $\lambda_2(D) = \pi r^2$. You may use without proof the 'half base times height' formula for the Lebesgue measure (area) of a triangle. You may also use without proof the fact that $(\sin x)/x \to 1$ as $x \to 0$.

Proof. (a) Suppose that $\lambda_2(L) > 0$. Since λ_2 is translation-invariant, assume without loss of generality that an end-point of L is $(0,0)$. Let R be the length of L. Let ρ_n be a rotation of $\frac{2\pi}{n}$ radians and let $A := \bigcup_{n=1}^{\infty} \rho_n(L)$. $A \subseteq \overline{B_R((0,0))}$ so $\lambda_2(A) = \sum_{n=1}^{\infty} \lambda_2(\rho_n(L)) = \sum_{n=1}^{\infty} \lambda_2(L) = \infty \leq \lambda_2(\overline{B_R((0,0))}) < \infty$; a contradiction. Thus $\lambda_2(L) = 0$.

(b) Let A_i be the interior of a regular $3 \cdot 2^i$-sided polygon centred at the origin with a vertex at $(r, 0)$. $\lambda_2(A_i) = 3 \cdot 2^i \frac{r^2 \sin(\frac{2\pi}{3 \cdot 2^i})}{2}$. Furthermore, $A_i \subseteq A_{i+1} \forall i$ and $\bigcup_{i=1}^{\infty} A_i = D$ so by upward continuity

$$\begin{aligned}
\lambda_2(D) &= \lim_{i\to\infty} \frac{3 \cdot 2^i}{2} r^2 \sin(\frac{2\pi}{3 \cdot 2^i}) \\
&= \frac{r^2}{2} \lim_{i\to\infty} i \sin(\frac{2\pi}{i}) \\
&= \frac{r^2}{2} \lim_{n\to 0^+} \frac{\sin(2\pi n)}{n} \\
&= \frac{2\pi r^2}{2} \lim_{n\to 0^+} \frac{\sin(2\pi n)}{2\pi n} \\
&= \pi r^2.
\end{aligned}$$

□

Exercise 0.29. *Suppose F is a function with the properties assumed in Exercise 25.*

(a) Prove that there is a unique measure μ_F on $(\mathbb{R}, \mathcal{B})$ with the property that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$ with $a < b$. [You may assume without proof Carathéodory's extension theorem, along with the results of Exercise 25.]

(b) Given $y \in \mathbb{R}$, show that the μ_F-measure of the one-point set $\{y\}$ is $\mu_F(\{y\}) = F(y) - F(y^-)$, where $F(y^-) = \lim_{x\to y^-} F(x)$.

(c) Show that $\mu_F([a, b]) = F(b) - F(a^-)$, and also find the formulas for $\mu_F((a, b))$ and $\mu_F([a, b))$, when $-\infty < a < b < \infty$.

Remark: The measure μ_F is called the Lebesgue-Stieltjes measure corresponding to the function F.

Proof. (a) λ_F as defined in question 25 is a σ-finite pre-measure on $\overline{\mathcal{I}}$ with the property that $\lambda_F((a, b]) = F(b) - F(a)$. Thus by the Caratheodory extension theorem there exists a unique measure μ_F on $(\mathbb{B}, \mathcal{B})$ which agrees with λ_F on $\mathcal{I}$.

(b) Let $A_n := (y - \frac{1}{n}, y]$. Then $A_{n+1} \subseteq A_n \forall n$, $\bigcap_{n=1}^{\infty} A_n = \{y\}$ and $\mu_F(A_1) < \infty$ so by downwards continuity $\mu_F(\{y\}) = \lim_{n\to\infty} \mu_F(A_n) = \lim_{n\to\infty}(F(y) - F(y - \frac{1}{n})) = F(y) - F(y^-)$.

(c) Let $A_n := (a - \frac{1}{n}, b]$. Then as before $\mu_F([a,b]) = \lim_{n\to\infty}(F(b) - F(a - \frac{1}{n})) = F(b) - F(a^-)$. Then

$$\begin{aligned}\mu_F((a,b)) &= \mu_F([a,b]) - \mu_F(\{a\}) - \mu_F(\{b\}) \\ &= F(b) - F(a^-) - F(b) + F(b^-) - F(a) + F(a^-) \\ &= F(b^-) - F(a).\end{aligned}$$

and

$$\begin{aligned}\mu_F([a,b)) &= \mu_F([a,b]) - \mu_F(\{b\}) \\ &= F(b) - F(a^-) - F(b) + F(b^-) \\ &= F(b^-) - F(a^-).\end{aligned}$$

□

Exercise 0.30. *Prove that if $W \subseteq \mathbb{R}$ is a Borel set, and $f : W \to \mathbb{R}$ is an increasing function (i.e., $f(x) \leq f(y)$ whenever $x, y \in W$ with $x < y$), then f is Borel measurable.*

Proof. Let $\alpha \in \mathbb{R}$ and let $t = \inf(f^{-1}((\alpha,\infty]))$. Then $f^{-1}((\alpha,\infty]) = W \cap (t,\infty)$ or $f^{-1}((\alpha,\infty]) = W \cap [t,\infty)$ (since the infimum of a set may or may not be contained in the set). The intersection of Borel sets is Borel and $W \cap (t,\infty), W \cap [t,\infty) \subseteq W$ so $f^{-1}((\alpha,\infty]) \in \mathcal{B}_W$. Thus f is Borel measurable. □

Exercise 0.31. *(a) Let $(X, \mathcal{M})$ be a measurable space, and let $f_n : X \to \mathbb{R}$ be measurable functions. Show that the set of points*

$$\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}$$

is in $\mathcal{M}$.

(b) Taking $(\Omega, \mathcal{F}, P)$ to be a probability space, and random variables (i.e., measurable functions) $Y_1, Y_2, \cdots : \Omega \to \mathbb{R}$ show that for any constant $\mu \in \mathbb{R}$ the set:

$$\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i(\omega) = \mu\right\}$$

is in $\mathcal{F}$. Deduce that expressions like $\mathbb{P}[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i = \mu]$ are meaningful.

Proof. (a) Call the set A. Define

$$B := \{x \in X : \limsup_{n\to\infty} f_n(x) = -\infty\},$$

$$C := \{x \in X : \liminf_{n\to\infty} f_n(x) = \infty\}$$

and

$$D := \{x \in X : \liminf_{n\to\infty} f_n(x) < \limsup_{n\to\infty} f_n(x)\}$$

so that $A = (B \cup C \cup D)^c$.

$$B = \bigcap_{k=1}^{\infty} (\limsup_{n\to\infty} f_n)^{-1}((-\infty, -k]) \in \mathcal{M}$$

and similarly $C \in \mathcal{M}$.

$$D = (\liminf_{n\to\infty} f_n - \limsup_{n\to\infty} f_n)^{-1}((-\infty, 0)) \in \mathcal{M}.$$

Thus $A \in \mathcal{M}$.

(b) Call the set A. Define the measurable function $g_n : \Omega \to \mathbb{R} : \omega \mapsto |\frac{1}{n}\sum_{i=1}^n Y_i(\omega) - \mu|$. Then

$$\begin{aligned} A &= \{\omega \in \Omega : \forall K \in \mathbb{N} : \exists N \in \mathbb{N} : \forall n > N : g_n(\omega) < \frac{1}{K}\} \\ &= \{\omega \in \Omega : \forall K \in \mathbb{N} : \exists N \in \mathbb{N} : \forall n > N : \omega \in g_n^{-1}([0, \frac{1}{K}))\} \\ &= \bigcap_{K=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n>N} g_n^{-1}([0, \frac{1}{K})]) \in \mathcal{F}. \end{aligned}$$

□

Exercise 0.32. *Let $(X, \mathcal{M})$ be a measurable space.*

(a) Show that if $E \in \mathcal{M}$, then its indicator function $\mathbf{1}_E$ defined by $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ for $x \notin E$, is a measurable function.

(b) Let $f : X \to \mathbb{R}$ be a function with finite range $f(X) = \{\alpha_1, \ldots, \alpha_n\}$ (with $\alpha_1, \ldots, \alpha_n$ distinct), so that $f = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$, where $A_i = \{x \in X : f(x) = \alpha_i\}$. Show that f is measurable if and only if $A_1, \ldots, A_n \in \mathcal{M}$.

Proof. (a) Let $\alpha \geq 1$. Then $\mathbf{1}_E^{-1}((\alpha, \infty]) = \emptyset \in \mathcal{M}$.

Now let $0 \leq \alpha < 1$. Then $\mathbf{1}_E^{-1}((\alpha, \infty]) = E \in \mathcal{M}$.

Now let $\alpha < 0$. Then $\mathbf{1}_E^{-1}((\alpha, \infty]) = X \in \mathcal{M}$. Thus $\mathbf{1}_E$ is measurable.

(b) ($\Longleftarrow$) If $A_1, ..., A_n \in \mathcal{M}$ then $\mathbf{1}_{A_i}$ is measurable $\forall i$ so f is measurable as the sum of measurable functions.

($\Longrightarrow$) $\{a_i\}$ is a Borel set so $f^{-1}(\{a_i\}) = A_i \in \mathcal{M} \forall i$.

□

Exercise 0.33. *Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f : X \to [0, \infty]$ is measurable.*

(a) Prove that if $a \in (0, \infty)$ then $\mu(f^{-1}[a, \infty]) \leq a^{-1} \int f d\mu$. [When μ is a probability measure, this is called Markov's inequality]

(b) Prove that if $\int f d\mu = 0$, then $\mu(f^{-1}((0, \infty])) = 0$.

Proof. (a)

$$\begin{aligned} \mu(f^{-1}([a, \infty])) &= (\mu \otimes \lambda_1)(f^{-1}([a, \infty]) \times (0, 1)) \\ &= a^{-1}(\mu \otimes \lambda_1)(f^{-1}([a, \infty]) \times (0, a)) \\ &\leq a^{-1} \int_{f^{-1}([a,\infty])} f d\mu \\ &\leq a^{-1} \int f d\mu. \end{aligned}$$

(b) Define $A_n := f^{-1}([\frac{1}{n},\infty])$. Then $A_n \subseteq A_{n+1} \forall n$ and $\bigcup_{n=1}^{\infty} A_n = f^{-1}((0,\infty])$ so by upwards continuity $\mu(f^{-1}((0,\infty])) = \lim_{n\to\infty} \mu(f^{-1}([\frac{1}{n},\infty]))$. $\mu(f^{-1}([\frac{1}{n},\infty])) \leq n \int f d\mu = 0 \forall n$ so $\mu(f^{-1}((0,\infty])) = 0$.

□

Exercise 0.34. *Let $(X,\mathcal{M})$ be a measurable space. Suppose $f : X \to [0,\infty)$ and $g : X \to [0,\infty)$ are measurable functions. Define the set $A \subset X \times \mathbb{R} \times \mathbb{R}$ by*

$$A := \{(x,s,t) : f(x) > s, g(x) > t\}.$$

Let $\mathcal{B}$ denote the Borel σ-algebra in $\mathbb{R}$. Show that $A \in \mathcal{M}\otimes\mathcal{B}\otimes\mathcal{B}$, where $\mathcal{M}\otimes\mathcal{B}\otimes\mathcal{B}$ is the σ-algebra generated by the collection of all sets in $X \times \mathbb{R} \times \mathbb{R}$ of the form $B \times C \times D$ with $B \in \mathcal{M}, C \in \mathcal{B}$ and $D \in \mathcal{B}$.

Proof. If $s < f(x)$ and $t < g(x)$ then there are rational numbers $q \in (s, f(x))$ and $r \in (t, g(x))$ since $\overline{\mathbb{Q}} = \mathbb{R}$. Thus

$$\begin{aligned} A &= \bigcup_{(q,r)\in\mathbb{Q}^2} \{(x,s,t) \in X \times \mathbb{R} \times \mathbb{R} : f(x) > q > s, g(x) > r > t\} \\ &= \bigcup_{(q,r)\in\mathbb{Q}^2} (\{(x,s,t) \in X \times \mathbb{R} \times \mathbb{R} : f(x) > q > s\} \cap \{(x,s,t) \in X \times \mathbb{R} \times \mathbb{R} : g(x) > r > t\}) \\ &= \bigcup_{(q,r)\in\mathbb{Q}^2} (f^{-1}((q,\infty)) \times (-\infty,q) \times \mathbb{R} \cap g^{-1}((r,\infty)) \times \mathbb{R} \times (-\infty,r)) \in \mathcal{M}\otimes\mathcal{B}\otimes\mathcal{B}. \end{aligned}$$

□

Exercise 0.35. *(a) Let $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ be measurable spaces. Show that for all $A \subset X \times Y$ with $A \in \mathcal{M}\otimes\mathcal{N}$, and all $y \in Y$, the horizontal cross-section $A_{[y]}$ of A defined by*

$$A_{[y]} := \{x \in X : (x,y) \in A\}$$

satisfies $A_{[y]} \in \mathcal{M}$.

[Hint: First show the class of $A \subset X \times Y$ with $A_{[y]} \in \mathcal{M}$ is a σ-algebra]

(b) Suppose $f : X \to [0,\infty]$ is such that hyp$(f) \in \mathcal{M}\otimes\mathcal{B}$. Show that f is a measurable function.

Proof. (a) Let $y \in Y$. Let $\mathcal{U} := \{A \subseteq X \times Y : A_{[y]} \in \mathcal{M}\}$. $\emptyset_{[y]} = \emptyset \in \mathcal{M}$ so $\emptyset \in \mathcal{U}$. Let $B \in \mathcal{U}$. Then

$$(B^c)_{[y]} = \{x \in X : (x,y) \in B^c\} = \{x \in X : (x,y) \notin B\} = (B_{[y]})^c \in \mathcal{M}$$

so $\mathcal{U}$ is closed under complements. Now let $A_1, A_2, ... \in \mathcal{U}$. Then

$$(\bigcup_{i=1}^{\infty} A_i)_{[y]} = \{x \in X : (x,y) \in \bigcup_{i=1}^{\infty} A_i\} = \bigcup_{i=1}^{\infty}\{x \in X : (x,y) \in A_i\} = \bigcup_{i=1}^{\infty} A_{i[y]} \in \mathcal{M}.$$

Thus $\mathcal{U}$ is a σ-algebra. Let $C := M \times N \in \mathcal{M} \times \mathcal{N}$. Then

$$C_{[y]} = \{x \in X : (x,y) \in M \times N\} = \begin{cases} M \text{ if } y \in N, \\ \emptyset \text{ otherwise} \end{cases}$$

so $C_{[y]} \in \mathcal{M}$. Thus $\mathcal{M} \times \mathcal{N} \subseteq \mathcal{U}$ so $\mathcal{M}\otimes\mathcal{N} = \sigma(\mathcal{M}\times\mathcal{N}) \subseteq \mathcal{U}$. Thus given any $A \in \mathcal{M}\otimes\mathcal{N}$ we have $A_{[y]} \in \mathcal{M}$.

(b) Let $\alpha > 0$. Then

$$\begin{aligned} f^{-1}((\alpha,\infty]) &= \{x \in X : \alpha < f(x)\} \\ &= \{x \in X : (x,\alpha) \in \mathrm{hyp}(f)\} \\ &= \mathrm{hyp}(f)_{[\alpha]} \in \mathcal{M}. \end{aligned}$$

If $\alpha < 0$, then $f^{-1}((\alpha,\infty]) = X \in \mathcal{M}$. Otherwise, $f^{-1}((0,\infty]) = \bigcup_{n=1}^{\infty} f^{-1}((\frac{1}{n},\infty]) \in \mathcal{M}$. Thus f is measurable.

□

Exercise 0.36. *Let $W \in \mathcal{B}$ (the Borel sets in $\mathbb{R}$) with $W \neq \emptyset$. Recall from Definition 10.3 that $\mathcal{B}_W := \{B \subset W : B \in \mathcal{B}\}$.*

(a) Show that $\mathcal{B}_W = \{A \cap W : A \in \mathcal{B}\}$.

(b) Show that $\mathcal{B}_W$ is the σ-algebra in W generated by the collection of all sets of the form $(-\infty, a] \cap W$ with $a \in \mathbb{R}$.

Proof. (a) Let $\mathcal{C} := \{A \cap W : A \in \mathcal{B}\}$. Let $A \cap W \in \mathcal{C}$. Then $A \cap W \subseteq W$ and $A \cap W \in \mathcal{B}$ so $\mathcal{C} \subseteq \mathcal{B}_W$. Now let $B \in \mathcal{B}_W$. Then $B = B \cap W$ with $B \in \mathcal{B}$ so $\mathcal{B}_W = \mathcal{C}$.

(b) Let $\mathcal{D} := \{(-\infty, a] \cap W : a \in \mathbb{R}\}$. $(-\infty, a] \in \mathcal{B} \forall a \in \mathbb{R}$ so $\mathcal{D} \subseteq \mathcal{B}_W$ and hence $\sigma(\mathcal{D}) \subseteq \mathcal{B}_W$. Let $C := (x, y] \cap W \in \mathcal{B}_W$. Then $C = ((-\infty, y] \cap W) \cap ((-\infty, x]^c \cap W) \in \sigma(\mathcal{D})$. Thus $\{A \cap W : A \in \mathcal{I}\} \subseteq \sigma(\mathcal{D})$ so $\sigma(\{A \cap W : A \in \mathcal{I}\}) = \{A \cap W : A \in \sigma(\mathcal{I}) = \mathcal{B}\} \subseteq \sigma(\mathcal{D})$. Thus $\mathcal{B}_W = \sigma(\mathcal{D})$.

□

Exercise 0.37. *Suppose $g : \mathbb{R} \to \mathbb{R}$ is integrable (with respect to Lebesgue measure), and let $t \in \mathbb{R}$.*

(a) Show that $\int_{-\infty}^{\infty} g(x-t)dx = \int_{-\infty}^{\infty} g(x)dx$.

(b) Deduce that (with g as in (a)) for any $a, b \in \mathbb{R}$ with $a < b$, $\int_{a+t}^{b+t} g(x-t)dx = \int_a^b g(x)dx$.

[Hint: For part (a), start with the case where g is nonnegative and simple. Another way to write the result in (a) is $\int h d\lambda_1 = \int g d\lambda_1$, where we set $h(x) = g(x-t)$]

Proof. (a) First let g be non-negative and simple, so that $g = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$ for $\alpha_1, ..., \alpha_n \in \mathbb{R}_+$ and $A_1, ..., A_n \in \mathcal{B}$ pairwise disjoint. Then $\int_{-\infty}^{\infty} g(x)dx = \sum_{i=1}^n \alpha_i \lambda_1(A_i)$. Let $h(x) := g(x-t)$. Then $h = \sum_{i=1}^n \alpha_i \mathbf{1}_{t+A_i}$ so

$$\int_{-\infty}^{\infty} g(x-t)dx = \sum_{i=1}^n \alpha_i \lambda_1(t + A_i) = \sum_{i=1}^n \alpha_i \lambda_1(A_i) = \int_{-\infty}^{\infty} g(x)dx.$$

Now let g be non-negative but not necessarily simple. There exist non-negative simple functions $(g_n)_{n \in \mathbb{N}}$ such that $g_n \uparrow g$ and hence also $g_n(x-t) \uparrow g(x-t)$. Then by the monotone convergence theorem,

$$\int_{-\infty}^{\infty} g(x-t) = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x-t)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dx = \int_{-\infty}^{\infty} g(x)dx.$$

Now let g be any integrable function. Then

$$\int_{-\infty}^{\infty} g(x-t)dx = \int_{-\infty}^{\infty} g(x-t)^+dx - \int_{-\infty}^{\infty} g(x-t)^-dx = \int_{-\infty}^{\infty} g(x)^+dx - \int_{-\infty}^{\infty} g(x)^-dx = \int_{-\infty}^{\infty} g(x)dx.$$

(b)

$$\int_{a+t}^{b+t} g(x-t)dx = \int_{-\infty}^{\infty} g(x-t)\mathbf{1}_{(a+t,b+t)}(x)dx = \int_{-\infty}^{\infty} g(x)\mathbf{1}_{(a,b)}(x)dx = \int_a^b g(x)dx.$$

□

Exercise 0.38. *Let μ be counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.*

(a) Let $k \in \mathbb{N}$. Show that if $f : \mathbb{N} \to [0,\infty)$ with $f(n) = 0$ for all $n > k$, then $\int_{\mathbb{N}} f d\mu = \sum_{n=1}^{k} f(n)$. [Hint: f must be simple.]

(b) Show that if $g : \mathbb{N} \to [0,\infty)$ then $\int_{\mathbb{N}} g d\mu = \sum_{n=1}^{\infty} g(n)$. [Hint: use the Monotone Convergence theorem.]

(c) Suppose $h : \mathbb{N} \to \mathbb{R}$ with $\sum_{n=1}^{\infty} |h(n)| < \infty$. Show that $\int_{\mathbb{N}} h d\mu = \sum_{n=1}^{\infty} h(n)$.

Proof. (a) f is simple the image of f is a finite set $\{\alpha_1, ..., \alpha_m\}$ so $\int_{\mathbb{N}} f d\mu = \sum_{i=1}^{m} \alpha_i \mu(f^{-1}(\{\alpha_i\}))$. $\mu(f^{-1}(\{\alpha_i\})) = \#\{n \in \mathbb{N} : f(n) = \alpha_i\}$ so

$$\alpha_i \mu(f^{-1}(\{\alpha_i\})) = \sum_{n \in f^{-1}(\{\alpha_i\})} f(n).$$

The fibres are pairwise disjoint so

$$\int_{\mathbb{N}} f d\mu = \sum_{n \in \bigcup_{i=1}^{m} f^{-1}(\{\alpha_i\})} f(n) = \sum_{n \in \mathbb{N}} f(n) = \sum_{n=1}^{k} f(n)$$

since $f(n) = 0 \forall n > k$.

(b) Define $g_k : \mathbb{N} \to [0,\infty)$ by

$$g_k(n) = \begin{cases} g(n) \text{ if } n \leq k, \\ 0 \text{ otherwise.} \end{cases}$$

Then $g_k \uparrow g$ so by the monotone convergence theorem

$$\int_{\mathbb{N}} g d\mu = \lim_{k\to\infty} \int_{\mathbb{N}} g_k d\mu = \lim_{k\to\infty} \sum_{n=1}^{k} g(n) = \sum_{n=1}^{\infty} g(n).$$

(c)

$$\begin{aligned} \int_{\mathbb{N}} h d\mu &= \int_{\mathbb{N}} h^+ - h^- d\mu \\ &= \int_{\mathbb{N}} h^+ d\mu - \int_{\mathbb{N}} h^- d\mu \\ &= \sum_{n=1}^{\infty} h^+(n) - \sum_{n=1}^{\infty} h^-(n) \\ &= \sum_{n=1}^{\infty} h^+(n) - h^-(n) \\ &= \sum_{n=1}^{\infty} h(n). \end{aligned}$$

□

Exercise 0.39. *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $F_1, \dots, F_n$ are subsets of X with $F_i \in \mathcal{M}$ and $\mu(F_i) < \infty$ for each $i \in [n]$, where we set $[n] := \{1, \dots, n\}$. For $S \in \mathcal{P}([n])$, i.e. $S \subset [n]$, let $|S|$ denote the number of elements of S. Use the linearity of integration, and the fact that $\mu(A) = \int_X 1_A$ for any $A \in \mathcal{M}$, to prove the inclusion-exclusion formula*

$$\mu\Big(\bigcup_{i=1}^{n} F_i\Big) = \sum_{J \in \mathcal{P}([n]) \setminus \{\emptyset\}} (-1)^{|J|+1} \mu\Big(\bigcap_{j \in J} F_j\Big).$$

[Hint: for any sets $G_1, \dots, G_k \in \mathcal{M}$ we have $1_{\cap_{i=1}^k G_i} = \prod_{i=1}^k 1_{G_i}$.]

Proof.

$$\mu(\bigcup_{i=1}^{n} F_i) = \int_X \mathbf{1}_{\bigcup_{i=1}^n F_i} d\mu.$$

We prove by induction that

$$\mathbf{1}_{\bigcup_{i=1}^n F_i} = \sum_{J \in \mathcal{P}([n]) \setminus \{\emptyset\}} (-1)^{|J|+1} \mathbf{1}_{\bigcap_{j \in J} F_j}.$$

For $n = 1$ the statement is trivial. Now assume for $n = k$. Then for $n = k + 1$,

$$\begin{aligned}
\mathbf{1}_{\bigcup_{i=1}^{k+1} F_i} &= \mathbf{1}_{\bigcup_{i=1}^{k} F_i} + \mathbf{1}_{F_{k+1}} - \mathbf{1}_{\bigcup_{i=1}^{k} (F_i \cap F_{k+1})} \\
&= \mathbf{1}_{F_{k+1}} + \sum_{J \in \mathcal{P}([k]) \setminus \{\emptyset\}} (-1)^{|J|+1} \mathbf{1}_{\bigcap_{j \in J} F_j} - \sum_{J \in \mathcal{P}([k]) \setminus \{\emptyset\}} (-1)^{|J|+1} \mathbf{1}_{\bigcap_{j \in J} F_j \cap F_{k+1}} \\
&= \sum_{J \in \mathcal{P}([k]) \setminus \{\emptyset\}} (-1)^{|J|+1} \mathbf{1}_{\bigcap_{j \in J} F_j} + \sum_{J \in \mathcal{P}([k])} (-1)^{|J|+2} \mathbf{1}_{\bigcap_{j \in J} F_j \cap F_{k+1}} \\
&= \sum_{J \in \mathcal{P}([k+1]) \setminus \{\emptyset\}} (-1)^{|J|+1} \mathbf{1}_{\bigcap_{j \in J} F_j}
\end{aligned}$$

as required. Thus

$$\begin{aligned}
\mu(\bigcup_{i=1}^{n} F_i) &= \int_X \mathbf{1}_{\bigcup_{i=1}^n F_i} d\mu \\
&= \int_X \sum_{J \in \mathcal{P}([n]) \setminus \{\emptyset\}} (-1)^{|J|+1} \mathbf{1}_{\bigcap_{j \in J} F_j} d\mu \\
&= \sum_{J \in \mathcal{P}([n]) \setminus \{\emptyset\}} (-1)^{|J|+1} \int_X \mathbf{1}_{\bigcap_{j \in J} F_j} d\mu \\
&= \sum_{J \in \mathcal{P}([n]) \setminus \{\emptyset\}} (-1)^{|J|+1} \mu(\bigcap_{j \in J} F_j).
\end{aligned}$$

□

Exercise 0.40. *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $f, g, h \in L^1(\mu)$.*

(a) For $F \in L^1(\mu)$ set $\|F\|_1 := \int |F| d\mu$. Show that $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

(b) Show that $f - h \in L^1(\mu)$ and $h - g \in L^1(\mu)$ and $\|f - g\|_1 \leq \|f - h\|_1 + \|h - g\|_1$.

Proof. (a)

$$\|f+g\|_1 = \int |f+g|d\mu \le \int |f|+|g|d\mu = \int |f|d\mu + \int |g|d\mu = \|f\|_1 + \|g\|_1.$$

(b)

$$\int |f-h|d\mu \le \int |f|+|h|d\mu = \|f\|_1 + \|h\|_1 < \infty$$

so $f-h \in L^1(\mu)$. Similarly, $h-g \in L^1(\mu)$.

$$\|f-g\|_1 = \|(f-h)+(h-g)\|_1 \le \|f-h\|_1 + \|h-g\|_1.$$

□

Exercise 0.41. *A function $f:\mathbb{R}\to\mathbb{R}$ is said to have bounded support if there exists $n\in\mathbb{N}$ such that $f(x)=0$ whenever $|x|>n$.*

Suppose $f:\mathbb{R}\to\mathbb{R}$ is integrable (with respect to Lebesgue measure). Let $\varepsilon>0$. Show that there exists integrable $g:\mathbb{R}\to\mathbb{R}$ such that $\int_{-\infty}^{\infty}|f(x)-g(x)|dx<\varepsilon$, and g has bounded support.

Proof. Define $f_n := |f|\mathbf{1}_{(-n,n)}$. Then $f_n \uparrow |f|$ so by the monotone convergence theorem

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty}|f(x)|dx$$

so $\exists N\in\mathbb{N}$ such that

$$|\int_{-\infty}^{\infty}|f(x)|dx - \int_{-\infty}^{\infty} f_N(x)dx| = \int_{-\infty}^{\infty}|f(x)| - f_N(x)dx = \int_{-\infty}^{-N}|f(x)|dx + \int_{N}^{\infty}|f(x)|dx < \epsilon.$$

Let $g := f\mathbf{1}_{(-N,N)}$. Then

$$\begin{aligned}\int_{-\infty}^{\infty}|f(x)-g(x)|dx &= \int_{-\infty}^{\infty}|f(x)-f(x)\mathbf{1}_{(-N,N)}(x)|dx\\ &= \int_{-\infty}^{-N}|f(x)|dx + \int_{N}^{\infty}|f(x)|dx\\ &< \epsilon.\end{aligned}$$

□

Exercise 0.42. *A function $g:\mathbb{R}\to\mathbb{R}$ is called a* ***step function*** *if we can write*

$$g = \sum_{i=1}^{k} c_i 1_{I_i}$$

for some $k\in\mathbb{N}$, $(c_1,\ldots,c_k)\in\mathbb{R}^k$ and $I_1,\ldots,I_k$ intervals in $\mathbb{R}$.

Suppose $f:\mathbb{R}\to[0,\infty)$ is simple and has bounded support (i.e., there exists $n\in\mathbb{N}$ with $f(x)=0$ whenever $|x|>n$). Let $\varepsilon>0$. Show that there exists a step function $g:\mathbb{R}\to\mathbb{R}$ such that

$$\int_{-\infty}^{\infty}|g-f|\,dx<\varepsilon.$$

Hint: Recall Questions 17 and 23.

Proof. Let $\mathrm{Im}(f)\backslash\{0\} = \{a_1, ..., a_n\}$ and let $A_i := f^{-1}(\{a_i\})$. For each i, since A_i is a bounded Borel set, by exercise 17 there exists a finite union of half-open intervals U_i such that $\lambda_1(A_i \Delta U) < \frac{\epsilon}{|a_i|n}$, meaning that

$$\int |\mathbf{1}_{U_i} - \mathbf{1}_{A_i}| d\lambda_1 = \int \mathbf{1}_{U_i \Delta A_i} d\lambda_1 < \frac{\epsilon}{|a_i|n}.$$

Then setting $g := \sum_{i=1}^n a_i \mathbf{1}_{U_i}$ we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |g - f| dx &= \int |\sum_{i=1}^n a_i \mathbf{1}_{U_i} - \sum_{i=1}^n a_i \mathbf{1}_{A_i}| d\lambda_1 \\
&= \int |\sum_{i=1}^n a_i(\mathbf{1}_{U_i} - \mathbf{1}_{A_i})| d\lambda_1 \\
&\leq \int \sum_{i=1}^n |a_i(\mathbf{1}_{U_i} - \mathbf{1}_{A_i})| d\lambda_1 \\
&= \sum_{i=1}^n \int |a_i(\mathbf{1}_{U_i} - \mathbf{1}_{A_i})| d\lambda_1 \\
&< \sum_{i=1}^n \frac{\epsilon}{n} = \epsilon.
\end{aligned}$$

Since each U_i is in the algebra generated by $\mathcal{I}$, being a finite union of half-open intervals, we have that U_i is a finite union of pairwise disjoint $I_{i,1}, ..., I_{i,k_i} \in \mathcal{I}$ and so $a_i \mathbf{1}_{U_i} = \sum_{n=1}^{k_i} a_i \mathbf{1}_{I_{i,n}} \forall i$ and hence g is a step function. ☐

Exercise 0.43. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is in L^1. Let $\varepsilon > 0$. Using Question 42, show there exists a continuous function $p : \mathbb{R} \to \mathbb{R}$ such that*

$$\|f - p\|_1 < \varepsilon,$$

i.e.,

$$\int_{-\infty}^{\infty} |f(x) - p(x)| \, dx < \varepsilon.$$

Proof. Let $\epsilon_1 := \frac{\epsilon}{12}$. First suppose that f is non-negative. There exists an integrable function $g : \mathbb{R} \to \mathbb{R}^+$ with bounded support such that

$$\int_{-\infty}^{\infty} |f(x) - g(x)| dx < \epsilon_1.$$

Let $g_n : \mathbb{R} \to \mathbb{R}^+$ be simple approximations of g such that $g_n \uparrow g$, and hence

$$\lim_{n\to\infty} \int g_n d\lambda_1 = \int g d\lambda_1.$$

For every n there exists a step function $h_n : \mathbb{R} \to \mathbb{R}$ such that

$$\int |g_n - h_n| d\lambda_1 < \epsilon_1.$$

There exists an $N \in \mathbb{N}$ such that

$$\int g - g_N d\lambda_1 = \int g d\lambda_1 - \int g_N d\lambda_1 < \epsilon_1$$

and hence

$$\int |g - h_N| d\lambda_1 \leq \int g - g_N d\lambda_1 + \int |g_N - h_N| < 2\epsilon_1.$$

Thus

$$\int |f - h_N| d\lambda_1 \leq \int |f - g| d\lambda_1 + \int |g - h_N| d\lambda_1 < 3\epsilon_1.$$

Now let f have negative values. Then $f = f^+ - f^-$ with step functions $s^+, s^- : \mathbb{R} \to \mathbb{R}$ such that

$$\int |f^+ - s^+| d\lambda_1 < 3\epsilon_1, \int |f^- - s^-| d\lambda_1 < 3\epsilon_1.$$

Then

$$\int |f - (s^+ - s^-)| d\lambda_1 = \int |f^+ - s^+ - f^- + s^-| d\lambda_1 \leq \int |f^+ - s^+| d\lambda_1 + \int |f^- - s^-| d\lambda_1 < 6\epsilon_1 = \frac{\epsilon}{2}.$$

Then let $s := s^+ - s^- = \sum_{i=0}^{n} a_i \mathbf{1}_{A_i}$ where $a_i \neq a_{i+1} \forall i$ and the A_i's are pairwise disjoint intervals such that

$$\bigcup_{i=0}^{n} A_i = \mathbb{R}.$$

Write A_i as $\langle \alpha_i, \alpha_{i+1} \rangle$ for every i and let $x_i := \frac{\epsilon}{n|a_i - a_{i-1}|}$. Then define a function $p : \mathbb{R} \to \mathbb{R}$ where p agrees with s on $[\alpha_i + x_i, \alpha_{i+1} - x_{i+1}] \forall i \in \{1, , ..., n-1\}$ and on $(-\infty, \alpha_1 - x_1] \cup [\alpha_n + x_n, \infty)$ but otherwise forms straight lines from $(\alpha_i - x_i, a_{i-1})$ to $(\alpha_i + x_i, a_i)$ for $i \in \{1, ..., n-1\}$. Then

$$\int |p - s| d\lambda_1 = \sum_{i=1}^{n} \frac{|a_i - a_{i-1}| x_i}{2} = \sum_{i=1}^{n} \frac{|a_i - a_{i-1}| \epsilon}{2n|a_i - a_{i-1}|} = \frac{\epsilon}{2}.$$

Thus

$$\|f - p\|_1 = \int |f - p| d\lambda_1 \leq \int |f - s| d\lambda_1 + \int |p - s| d\lambda_1 < \epsilon.$$

□

Exercise 0.44. *Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $F_n \subset X$ with $F_n \in \mathcal{M}$ and $\mu(F_n) < \infty$, $\forall n \in \mathbb{N}$. Suppose also that $\mathcal{D} \subset \mathcal{M}$ is a π-system in X with $F_n \in \mathcal{D}$ for all $n \in \mathbb{N}$, and ν is a measure on $(X, \mathcal{M})$ such that $\nu(A) = \mu(A)$ for all $A \in \mathcal{D}$.*

(a) For $n \in \mathbb{N}$ set $E_n := \bigcup_{j=1}^{n} F_j$. Use the inclusion-exclusion formula from Question 39 to show for all $n \in \mathbb{N}$, $A \in \mathcal{D}$ that

$$\mu(E_n) = \nu(E_n); \quad \mu(A \cap E_n) = \nu(A \cap E_n).$$

(b) Now suppose moreover that $\bigcup_{n=1}^{\infty} F_n = X$. Show that $\mu(A) = \nu(A)$ for all $A \in \sigma(\mathcal{D})$.

Proof. (a)

$$\begin{aligned}
\mu(E_n) &= \mu(\bigcup_{j=1}^{n} F_j) \\
&= \sum_{J\in\mathcal{P}([n])\setminus\{\emptyset\}} (-1)^{|J|+1}\mu(\bigcap_{j\in J} F_j) \\
&= \sum_{J\in\mathcal{P}([n])\setminus\{\emptyset\}} (-1)^{|J|+1}\nu(\bigcap_{j\in J} F_j) \\
&= \nu(E_n)
\end{aligned}$$

since any finite intersection of F_j's is contained in $\mathcal{D}$, over which μ and ν agree. Similarly,

$$\begin{aligned}
\mu(A\cap E_n) &= \mu(A\cap \bigcup_{j=1}^{n} F_j) \\
&= \mu(\bigcup_{j=1}^{n}(A\cap F_j)) \\
&= \sum_{J\in\mathcal{P}([n])\setminus\{\emptyset\}} (-1)^{|J|+1}\mu(\bigcap_{j\in J}(A\cap F_j)) \\
&= \sum_{J\in\mathcal{P}([n])\setminus\{\emptyset\}} (-1)^{|J|+1}\nu(\bigcap_{j\in J}(A\cap F_j)) \\
&= \nu(A\cap E_n).
\end{aligned}$$

(b) Define probability measures

$$\mu_n : \mathcal{M} \to [0,\infty] : A \mapsto \frac{\mu(A\cap E_n)}{\mu(E_n)}$$

and

$$\nu_n : \mathcal{M} \to [0,\infty] : A \mapsto \frac{\nu(A\cap E_n)}{\nu(E_n)}$$

We have that μ_n agrees with ν_n on $\mathcal{D}\forall n\in\mathbb{N}$ by part (a). Hence by the uniqueness lemma for probability measures it follows that μ_n and ν_n agree on $\sigma(\mathcal{D})\forall n\in\mathbb{N}$. Furthermore, by upwards continuity we have that

$$\mu(A) = \lim_{n\to\infty}\mu(A\cap E_n)\forall A\in\sigma(\mathcal{D})$$

so

$$\mu(A) = \lim_{n\to\infty}\mu_n(A)\cdot\mu(E_n)\forall A\in\sigma(\mathcal{D}).$$

Similarly,

$$\nu(A) = \lim_{n\to\infty}\nu_n(A)\cdot\nu(E_n)\forall A\in\sigma(\mathcal{D}).$$

$\mu(E_n) = \nu(E_n)\forall n\in\mathbb{N}$ by part (a) so it follows that μ and ν agree on $\sigma(\mathcal{D})$.

□

Exercise 0.45. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Let $f : \Omega \to [0, \infty]$ be measurable, i.e. f is a nonnegative random variable. For $t \geq 0$ define $L(t) := \int_\Omega e^{-tf(\omega)}\mu(d\omega)$ (the Laplace transform of f).*

(a) Show that $\lim_{t\to\infty} L(t) = \mu(\{\omega \in \Omega : f(\omega) = 0\})$. Here we make the convention that $e^{-\infty} = 0$.

(b) Show that $\lim_{t\to 0} L(t) = \mu(\{\omega \in \Omega : f(\omega) < \infty\})$.

(c) Show that $\lim_{t\to 0} t^{-1}(L(0) - L(t)) = \int f d\mu$ if the integral on the right is finite.

[Hint: use the fact that $1 - e^{-x} \leq x$ for $x \geq 0$]. What can anything if the integral is infinite?

Proof. (a) Define $g_n : \Omega \to [0, \infty] : \omega \mapsto e^{-nf(\omega)}$ and $g : \Omega \to [0, \infty] : \omega \mapsto e^{-\infty f(\omega)}$. Then $g_n \to g$ pointwise and $|g_n(\omega)| \leq 1 \forall \omega \in \Omega$. Hence we can apply the dominated convergence theorem to obtain

$$\lim_{t\to\infty} L(t) = \lim_{n\to\infty} \int_\Omega g_n(\omega)\mu(d\omega) = \int_\Omega g\mu(d\omega).$$

If $\omega \in f^{-1}(0)$, then $g(\omega) = e^{-\infty\cdot 0} = 1$. Otherwise, $g(\omega) = 0$. Thus $g = \mathbf{1}_{f^{-1}\{0\}}$ so

$$\lim_{t\to\infty} = \mu(f^{-1}(0)) = \mu(\{\omega \in \Omega : f(\omega) = 0\}).$$

(b) Let $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \downarrow 0$ as $n \to \infty$. Then $L(t_n) = \int_\omega e^{-t_n f(\omega)}\mu(d\omega)$. Define $g_n : \Omega \to [0, \infty] : \omega \mapsto e^{-t_n f(\omega)}$ and $g : \Omega \to [0, \infty] : \omega \mapsto \mathbf{1}_{f(\omega)<\infty}$. We then have

$$g_n(\omega) = e^{-t_n f(\omega)} \leq e^{-t_{n+1} f(\omega)} = g_{n+1}(\omega)$$

and

$$\lim_{n\to\infty} g_n(\omega) = \lim_{n\to\infty} e^{-t_n f(\omega)} = e^{0\cdot f(\omega)} = g(\omega) \forall \omega \in \Omega$$

so by the monotone convergence theorem we have

$$\lim_{n\to\infty} L(t_n) = \int_\Omega g\mu(d\omega) = \int_\Omega \mathbf{1}_{f(\omega)<\infty}\mu(d\omega) = \mu(\{\omega \in \Omega : f(\omega) < \infty\}).$$

Hence

$$\lim_{t\to 0} L(t) = \mu(\{\omega \in \Omega : f(\omega) < \infty\}).$$

(c) First suppose $\int f d\mu$ is finite. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \downarrow 0$ as $n \to \infty$. Then

$$\begin{aligned} t_n^{-1}(L(0) - L(t_n)) &= t_n^{-1} \int_\Omega \mathbf{1}_{f(\omega)<\infty} - e^{-t_n f(\omega)}\mu(d\omega) \\ &= t_n^{-1} \int_{f^{-1}([0,\infty))} 1 - e^{-t_n f(\omega)}\mu(d\omega) + t_n^{-1} \int_{f^{-1}(\{\infty\})} 0\mu(d\omega) \\ &= \int_{f^{-1}([0,\infty))} \frac{1 - e^{-t_n f(\omega)}}{t_n}\mu(d\omega). \end{aligned}$$

We have $1 - e^{-t_n f(\omega)} \leq t_n f(\omega)$ so $\frac{1-e^{-t_n f(\omega)}}{t_n} \leq f(\omega) \forall \omega \in f^{-1}([0, \infty))$. Furthermore,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} 1 - e^{-t_n f(\omega)} = 0$$

so by L'hôpital's rule

$$\lim_{n\to\infty} \frac{1-e^{-t_n f(\omega)}}{t_n} = \lim_{n\to\infty} \frac{f(\omega)e^{t_n f(\omega)}}{1} = f(\omega)\forall \omega \in f^{-1}([0,\infty)).$$

Also

$$\frac{1-e^{-t_n f(\omega)}}{t_n} \le f(\omega)\forall n \in \mathbb{N}, \omega \in f^{-1}([0,\infty))$$

so by the dominated convergence theorem

$$\lim_{t\downarrow 0} t^{-1}(L(0)-L(t)) = \lim_{n\to\infty} t_n^{-1}(L(0)-L(t_n)) = \int_{f^{-1}([0,\infty))} f\mu(d\omega)$$

which equals $\int f d\mu$ since $\mu(f^{-1}(\{\infty\}))=0$.

□

Exercise 0.46. *Let $(X,\mathcal{M},\mu)$ be a σ-finite measure space. Show the following:*

(a) If $f: X \to [-\infty,\infty]$ is measurable, $E \in \mathcal{M}$, $\int_E |f|d\mu = 0$, then $f=0$ a.e. on E.

(b) If $f \in L^1(\mu)$ with $\int_E f d\mu = 0$ for all $E \in \mathcal{M}$, then $f=0$ a.e. on X.

(c) If $f \in L^1(\mu)$ with $|\int_X f d\mu| = \int_X |f|d\mu$, then either $f \ge 0$ a.e. on X, or $f \le 0$ a.e. on X.

(d) If $f: X \to \mathbb{R}$ and $g: X\to\mathbb{R}$ are measurable functions, then $\{x \in X : f(x) \ne g(x)\} \in \mathcal{M}$.

Proof. (a) Suppose that $\mu(\{x \in E : f(x) \ne 0\}) > 0$. Let $A_n := \{x \in E : |f(x)| > \frac{1}{n}\}$ and let $A := \{x \in E : |f(x)| > 0\}$. Then $A = \bigcup_{n=1}^\infty A_n$. Since $\mu(A) > 0$ there exists an $N \in \mathbb{N}$ such that $\mu(A_N) > 0$. Hence

$$\int_E |f|d\mu \ge \int_{A_N} |f|d\mu \ge \int_{A_N} \frac{1}{N} = \frac{\mu(A_N)}{N} > 0.$$

Applying the contrapositive then gives the result.

(b) Let $A := \{x \in X : f(x) > 0\}$. Then

$$\int_A f d\mu = 0$$

and hence $f=0$ almost everywhere on A so $\mu(A)=0$. Similarly $\mu(\{x \in X : f(x) < 0\}) = 0$ so $\mu(\{x \in X : f(x) \ne 0\}) = 0$ and hence $f=0$ almost everywhere on X.

(c) Let $A := \{x \in X : f(x) \ge 0\}$ and let $B := \{x \in X : f(x) \le 0\}$. Then $|f| = f\mathbf{1}_A - f\mathbf{1}_B$ and so

$$\int_X |f|d\mu = \int_X f\mathbf{1}_A - f\mathbf{1}_B d\mu.$$

Furthermore,

$$\int_X f d\mu = \int_X f\mathbf{1}_A d\mu + \int_X f\mathbf{1}_B d\mu$$

If

$$\int_X f d\mu \ge 0$$

then

$$\int_X f\mathbf{1}_A d\mu + \int_X f\mathbf{1}_B d\mu = \int_X f\mathbf{1}_A d\mu - \int_X f\mathbf{1}_B d\mu$$

and hence

$$\int_X f\mathbf{1}_B d\mu = 0,$$

implying $f = 0$ almost everywhere on B and hence that $f \geq 0$ almost everywhere on X. If instead

$$\int_X f d\mu \leq 0$$

then

$$-\int_X f\mathbf{1}_A d\mu - \int_X f\mathbf{1}_B d\mu = \int_X f\mathbf{1}_A d\mu - \int_X f\mathbf{1}_B d\mu$$

so

$$\int_X f\mathbf{1}_A = 0$$

so $f \leq 0$ almost everywhere on X.

(d) $f - g$ is measurable so $\{x \in X : f(x) - g(x) = 0\} \in \mathcal{M}$ and hence

$$\{x \in X : f(x) \neq g(x)\} = \{x \in X : f(x) - g(x) = 0\}^c \in \mathcal{M}.$$

$\square$

Exercise 0.47. *Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Suppose $\{h_n\}_{n\geq 1}$ is a sequence in $\mathbb{R}$ such that $h_n \to 0$.*

(a) Show that for any $K \in (0, \infty)$ we have $\int_{-K}^{K} |f(x + h_n) - f(x)|dx \to 0$ as $n \to \infty$.

[Hint: first suppose f is continuous, recalling that any continuous real-valued function on a compact interval is bounded. For general f, use Question 43.]

(b) Show that $\int_{-\infty}^{\infty} |f(x + h_n) - f(x)|dx \to 0$ as $n \to \infty$.

Proof.

(a) First let f be continuous. There exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M \forall x \in [-K - \max\{|h_n| : n \in \mathbb{N}\}, K + \max\{|h_n| : n \in \mathbb{N}\}]$ by the Weierstrass extreme value theorem. Hence

$$|f(x + h_n)\mathbf{1}_{[-K,K]} - f(x)\mathbf{1}_{[-K,K]}| \leq |f(x + h_n)\mathbf{1}_{[-K,K]}| + |f(x)\mathbf{1}_{[-K,K]}| \leq 2M \forall x \in \mathbb{R}.$$

Furthermore,

$$|f(x + h_n)\mathbf{1}_{[-K,K]} - f(x)\mathbf{1}_{[-K,K]}| \to 0 \forall x \in [-K, K]$$

so by the dominated convergence theorem

$$\int_{-K}^{K} |f(x + h_n) - f(x)|dx = \int_{\mathbb{R}} |f(x + h_n)\mathbf{1}_{[-K,K]} - f(x)\mathbf{1}_{[-K,K]}| d\lambda_1 \to \int_{\mathbb{R}} 0 d\lambda_1 = 0.$$

Now drop the assumption that f is continuous. Let $\epsilon > 0$. Then there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(x) - g(x)|dx < \frac{\epsilon}{3}.$$

Furthermore,

$$\begin{aligned}|f(x+h_n)-f(x)| &= |(f(x+h_n)-g(x+h_n))+(g(x+h_n)-g(x))+(g(x)-f(x))| \\ &\le |f(x+h_n)-g(x+h_n)|+|g(x+h_n)-g(x)|+|g(x)-f(x)|.\end{aligned}$$

We have shown that there exists an $N \in \mathbb{N}$ such that

$$\int_{-K}^{K} |g(x+h_n)-g(x)|dx < \frac{\epsilon}{3} \forall n > N.$$

Hence for all $n > N$ we have

$$\begin{aligned}\int_{-K}^{K} |f(x+h_n)-f(x)|dx &\le \int_{-\infty}^{\infty} |f(x+h_n)-g(x+h_n)|dx + \int_{-K}^{K} |g(x+h_n)-g(x)|dx + \int_{-\infty}^{\infty} |g(x)-f(x)|dx \\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon\end{aligned}$$

as required.

(b) Let $\epsilon > 0$. Then there exists a $K \in \mathbb{N}$ such that

$$\int_{\mathbb{R}\setminus[-K,K]} |f(x)|d\lambda_1 < \frac{\epsilon}{3}$$

and an $N \in \mathbb{N}$ such that

$$\int_{-(K+1)}^{K+1} |f(x+h_n)-f(x)|dx < \frac{\epsilon}{3} \forall n > N.$$

By the triangle inequality we have

$$\begin{aligned}\int_{\mathbb{R}\setminus[-(K+1),K+1]} |f(x+h_n)-f(x)|d\lambda_1 &\le \int_{\mathbb{R}\setminus[-(K+1),K+1]} |f(x+h_n)|d\lambda_1 + \int_{\mathbb{R}\setminus[-(K+1),K+1]} |f(x)|dx \\ &\le \int_{\mathbb{R}\setminus[-(K+1),K+1]} |f(x+h_n)|d\lambda_1 + \int_{\mathbb{R}\setminus[-K,K]} |f(x)|dx \\ &< \int_{\mathbb{R}\setminus[-(K+1),K+1]} |f(x+h_n)|d\lambda_1 + \frac{\epsilon}{3} \forall n \in \mathbb{N}.\end{aligned}$$

Now let $M \in \mathbb{N}$ be such that $|h_n| < 1 \forall n > M$. Then

$$\{x+h_n : x \in \mathbb{R}\setminus[-(K+1),K+1], n > M\} \subseteq \mathbb{R}\setminus[-K,K]$$

and hence

$$\int_{\mathbb{R}\setminus[-(K+1),K+1]} |f(x+h_n)|d\lambda_1 \le \int_{\mathbb{R}\setminus[-K,K]} |f(x)|d\lambda_1 < \frac{\epsilon}{3} \forall n > M.$$

Thus $\forall n > \max\{N, M\}$ we have

$$\int_{-\infty}^{\infty} |f(x+h_n)-f(x)|dx < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon,$$

implying that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x+h_n)-f(x)|dx = 0.$$

□

Exercise 0.48. *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Suppose $f, f_1, f_2, \cdots \in L^1(X)$ such that $f_n \uparrow f$ pointwise and moreover $f_n \in L^1(\mu)$ and $\sup_n \int f_n d\mu < \infty$. Show that $f \in L^1(\mu)$ and $\int f d\mu \to \int f_n d\mu$ as $n \to \infty$.*

Proof. Since

$$\sup_n \int f_n d\mu < \infty$$

and $f_n \uparrow f$, implying that

$$\int f_n d\mu$$

is increasing, we know that

$$\lim_{n\to\infty} \int f_n d\mu$$

exists and is finite, since increasing sequences which are bounded converge to a finite real number. Furthermore, since $|f_n^-(x)| \leq f_1^-(x) \forall n \in \mathbb{N}$, the dominated convergence theorem implies that

$$\int f^- d\mu = \lim_{n\to\infty} \int f_n^- d\mu$$

which is finite since

$$\int f_n^- d\mu \leq \int f_1^- d\mu < \infty \forall n.$$

Thus

$$\lim_{n\to\infty} \int f_n^+ d\mu = \lim_{n\to\infty} \int f_n + f_n^- d\mu = \lim_{n\to\infty} \int f_n + \lim_{n\to\infty} \int f_n^- d\mu < \infty.$$

Furthermore, by the monotone convergence theorem we have

$$\int f^+ d\mu = \lim_{n\to\infty} \int f_n^+ d\mu < \infty$$

and hence

$$\int |f| d\mu = \int f^+ + f^- d\mu < \infty$$

so $f \in L^1(\mu)$. Finally,

$$\begin{aligned}
\int f d\mu &= \int f^+ - f^- d\mu \\
&= \int f^+ d\mu - \int f^- d\mu \\
&= \lim_{n\to\infty} \int f_n^+ d\mu - \lim_{n\to\infty} \int f_n^- d\mu \\
&= \lim_{n\to\infty} \int f_n^+ - f_n^- d\mu \\
&= \lim_{n\to\infty} \int f_n d\mu.
\end{aligned}$$

□

Exercise 0.49. *Let $-\infty < a < b < \infty$. Suppose $g : [a, b] \to \mathbb{R}$ is a continuously differentiable, strictly increasing function. Show that for all bounded Borel-measurable $f : (g(a), g(b)] \to \mathbb{R}$ we have the change of variables formula $\int_{g(a)}^{g(b)} f(y)dy = \int_a^b f(g(x))g'(x)dx$.*

Hint: First verify this for $f = \mathbf{1}_{(g(a),t]}$ with $g(a) < t \le g(b)$. Then use the Monotone Class theorem.

Proof. Let $f = \mathbf{1}_{(g(a),t]}$ with $g(a) < t \le g(b)$. Then

$$\int_{g(a)}^{g(b)} f(y)dy = \int_{\mathbb{R}} \mathbf{1}_{(g(a),t]}\mathbf{1}_{(g(a),g(b))}d\mu = \int_{g(a)}^{t} dy = t - g(a)$$

and

$$\int_a^b f(g(x))g'(x)dx = \int_a^b \mathbf{1}_{g(x)\in(g(a),t]}g'(x)dx = \int_a^{g^{-1}(t)} g'(x)dx = [g(x)]_a^{g^{-1}(t)} = t - g(a).$$

Thus the result holds in this case. Now let $\mathcal{H}$ be the set of bounded Borel-measurable functions $f : (g(a), g(b)] \to \mathbb{R}$ such that the change of variables formula holds and let

$$\mathcal{D} := \{(g(a), t] : g(a) < t \le g(b)\}.$$

Given $t_1, t_2 \in (g(a), g(b)]$ we have $(g(a), t_1] \cap (g(a), t_2] = (g(a), \min(t_1, t_2)] \in \mathcal{D}$ so $\mathcal{D}$ is a π-system. Furthermore, we have shown that $A \in \mathcal{D} \implies \mathbf{1}_A \in \mathcal{H}$, with $\mathbf{1}_A$ being Borel-measurable due to A being a measurable set. We also have that $(g(a), g(b)] \in \mathcal{D}$ so the first condition of the monotone class theorem is satisfied. Now let $p, q \in \mathcal{H}$. Then

$$\begin{aligned}
\int_{g(a)}^{g(b)} (p+q)(y)dy &= \int_{g(a)}^{g(b)} p(y)dy + \int_{g(a)}^{g(b)} q(y)dy \\
&= \int_a^b p(g(x))g'(x)dx + \int_a^b q(g(x))g'(x)dx \\
&= \int_a^b (p+q)(g(x))g'(x)dx
\end{aligned}$$

so $p + q \in \mathcal{H}$. Also, given $\alpha \in \mathbb{R}$ we have

$$\int_{g(a)}^{g(b)} \alpha p(y)dy = \alpha \int_{g(a)}^{g(b)} p(y)dy = \int_a^b \alpha p(g(x))g'(x)dx$$

so $\alpha p \in \mathcal{H}$ so the second condition of the monotone class theorem is satisfied. Now let $f_n \in \mathcal{H}$ for $n \in \mathbb{N}$ with $0 \le f_n \uparrow f$ pointwise where f is bounded. Then by the monotone convergence theorem

$$\int_{g(a)}^{g(b)} f(y)dy = \lim_{n\to\infty} \int_{g(a)}^{g(b)} f_n(y)dy = \lim_{n\to\infty} \int_a^b f_n(g(x))g'(x)dx.$$

Since g is strictly increasing we have that $g' \ge 0$ so we can apply the monotone convergence theorem again to obtain

$$\int_{g(a)}^{g(b)} f(y)dy = \lim_{n\to\infty} \int_a^b f_n(g(x))g'(x)dx = \int_a^b f(g(x))g'(x)dx$$

and hence $f \in \mathcal{H}$. Thus the monotone class theorem implies that $\mathcal{H}$ contains every bounded measurable function with respect to $\sigma(\mathcal{D}) = \mathcal{B}_{(g(a),g(b)]}$ as required. $\square$

Exercise 0.50. *(a) Show that* $\{(x,y) \in \mathbb{R}^2 : x < y\} \in \mathcal{B} \otimes \mathcal{B}$.

(b) Let $c \in (0, \infty)$. *Show that* $\{(x,y) \in \mathbb{R}^2 : x < y \leq x + c\} \in \mathcal{B} \otimes \mathcal{B}$.

(c) Suppose μ *is a probability measure on* $(\mathbb{R}, \mathcal{B})$. *For* $x \in \mathbb{R}$, *let* $F(x) = \mu((-\infty, x])$. *Let* $c \in \mathbb{R}$. *Use Fubini's Theorem to show that* $\int_{-\infty}^{\infty}(F(x+c) - F(x))dx = c$.

Proof. (a) Define $\pi_1 : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto x$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto y$. Given any $\alpha \in \mathbb{R}$ we have $\pi_1^{-1}((\alpha, \infty]) = (\alpha, \infty) \times \mathbb{R} \in \mathcal{B} \otimes \mathcal{B}$ so π_1 is measurable. Similarly π_2 is measurable. Hence $\pi_1 - \pi_2$ is measurable so

$$\{(x,y) \in \mathbb{R}^2 : x < y\} = (\pi_1 - \pi_2)^{-1}((-\infty, 0)) \in \mathcal{B} \otimes \mathcal{B}.$$

(b) We have

$$\{(x,y) \in \mathbb{R}^2 : y \leq x + c\} = (\pi_2 - \pi_1)((-\infty, c]) \in \mathcal{B} \otimes \mathcal{B}$$

so

$$\{(x,y) \in \mathbb{R}^2 : x < y \leq x + c\} = \{(x,y) \in \mathbb{R}^2 : x < y\} \cap \{(x,y) \in \mathbb{R}^2 : y \leq x + c\} \in \mathcal{B} \otimes \mathcal{B}.$$

(c) We have that

$$F(y) = \int_{\mathbb{R}} \mathbf{1}_{(-\infty, y]} d\mu \forall y \in \mathbb{R}$$

so

$$\begin{aligned}
\int_{-\infty}^{\infty}(F(x+c) - F(x))dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{(-\infty, x+c]} \mu(dy) - \int_{\mathbb{R}} \mathbf{1}_{(-\infty, x]} \mu(dy) \lambda_1(dx) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{(x, x+c]} \mu(dy) \lambda_1(dx) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \mu(dy) \lambda_1(dx)
\end{aligned}$$

where we let

$$f := \mathbf{1}_{\{(x,y) \in \mathbb{R}^2 : x < y \leq x + c\}}.$$

Since we have shown that f is $\mathcal{B} \otimes \mathcal{B}$ and $f \geq 0$, Fubini's theorem implies that

$$\int_{-\infty}^{\infty}(F(x+c) - F(x))dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \lambda_1(dx) \mu(dy).$$

For a fixed y,

$$f_y := \mathbf{1}_{\{x \in \mathbb{R} : x < y \leq x + c\}} = \mathbf{1}_{\{x \in \mathbb{R} : x < y\}} \mathbf{1}_{\{x \in \mathbb{R} : x \geq y - c\}} = \mathbf{1}_{[y-c, y)}$$

so

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \lambda_1(dx) \mu(dy) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{[y-c, y)} \lambda_1(dx) \mu(dy) = \int_{\mathbb{R}} c d\mu = c \int_{\mathbb{R}} d\mu.$$

Since μ is a probability measure we have

$$\int_{\mathbb{R}} d\mu = 1$$

so

$$\int_{-\infty}^{\infty}(F(x+c) - F(x))dx = c$$

as required.

$\square$

Exercise 0.51. *For $d \in \mathbb{N}$ let λ_d denote d-dimensional Lebesgue measure.*

(a) Show that λ_2 and $\lambda_1 \otimes \lambda_1$ are the same measure on $(\mathbb{R}^2, \mathcal{B}_2)$.

(b) Let $A \subset \mathbb{R}^2$ be a Borel set, and for $x \in \mathbb{R}$ let $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$. Show that

$$\lambda_2(A) = \int_{-\infty}^{\infty} \lambda_1(A_x)dx.$$

Proof. (a) λ_2 and $\lambda_1 \otimes \lambda_1$ agree on the π-system $\mathcal{B} \times \mathcal{B}$ and $\lambda_1 \otimes \lambda_1$ is σ-finite on $\mathcal{B} \times \mathcal{B}$ so λ_2 and $\lambda_1 \otimes \lambda_1$ agree on $\sigma(\mathcal{B} \times \mathcal{B}) = \mathcal{B} \otimes \mathcal{B} = \mathcal{B}_2$ by the uniqueness lemma.

(b) By Fubini's theorem and the above we have

$$\begin{aligned}
\lambda_2(A) &= \int_{\mathbb{R}^2} \mathbf{1}_A d\lambda_2 \\
&= \int_{\mathbb{R}^2} \mathbf{1}_A d(\lambda_1 \otimes \lambda_1) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{A_x} \lambda_1(dy)\lambda_1(dx) \\
&= \int_{\mathbb{R}} \lambda_1(A_x)\lambda_1(dx) \\
&= \int_{-\infty}^{\infty} \lambda_1(A_x)dx.
\end{aligned}$$

□

Exercise 0.52. *For $A \subset \mathbb{R}^d$ and $u \in \mathbb{R}^d$ let $A + u := \{a + u : a \in A\}$. Also if $d = 2$, for $x \in \mathbb{R}$ set $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$.*

(a) Let $-\infty < a < b < \infty$, and let $I = (a, b)$. Let $y \in (0, \infty)$. Compute $\lambda_1((I + y) \setminus I)$.

(b) Let $B \subset [0, 1]^2$ and suppose B is open (see Question 26) and B is convex, i.e. for all $u, v \in B$ and $\alpha \in (0, 1)$, we have $\alpha u + (1 - \alpha)v \in B$. Let e be the unit vector $(0, 1)$ and for $t > 0$ let $B(t) := B + te$. Given $x \in \mathbb{R}$, show that $B(t)_x = B_x + t$.

(c) Let B be as in Part (b). Show that $\lambda_1((B(t) \setminus B)_x) = \min(t, \lambda_1(B_x))$.

(d) Let B be as in Part (b). Show that $\lambda_2(B(t) \setminus B) \leq t$.

(e) Let B be as in Part (b). Let $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ denote projection onto the first co-ordinate, i.e. for $(x, y) \in \mathbb{R}^2$ we set $\pi_2((x, y)) = x$. Show that $\frac{\lambda_2(B(t)\setminus B)}{t} \to \lambda_1(\pi_2(B))$ as $t \downarrow 0$.
[The hint for Question 45 is also relevant here.]

Proof. (a) If $a + y \geq b$ then

$$\lambda_1((I + y) \setminus I) = \lambda_1((a + y, b + y) \setminus (a, b)) = \lambda_1((a + y, b + y)) = b - a.$$

Otherwise,

$$\lambda_1((I + y) \setminus I) = \lambda_1((a + y, b + y) \setminus (a, b)) = \lambda_1([b, b + y)) = y.$$

Hence $\lambda_1((I + y) \setminus I) = \min(y, b - a)$.

(b)

$$\begin{aligned}B(t)_x &= \{y \in \mathbb{R} : (x, y) \in B + te\} \\ &= \{y \in \mathbb{R} : (x, y) \in B + (0, t)\} \\ &= \{y \in \mathbb{R} : (x, y - t) \in B\} \\ &= B_x + t.\end{aligned}$$

(c) We have that

$$\begin{aligned}(B(t) \setminus B)_x &= ((B + te) \setminus B)_x \\ &= \{y \in \mathbb{R} : (x, y) \in (B + te) \setminus B\} \\ &= \{y \in \mathbb{R} : (x, y) \in B + te\} \setminus \{y \in \mathbb{R} : (x, y) \in B\} \\ &= (B_x + t) \setminus B_x.\end{aligned}$$

B is convex so B_x is an interval (a, b) (with a and b not included due to B being open), and $B_x + t$ is $(a + t, b + t)$. Hence by part (a) we have $\lambda_1((B_x + t) \setminus B_x) = \min(t, \lambda_1(B_x))$ as required.

(d)

$$\begin{aligned}\lambda_2(B(t) \setminus B) &= \int_{-\infty}^{\infty} \lambda_1((B(t) \setminus B)_x) dx \\ &= \int_{-\infty}^{\infty} \min(t, \lambda_1(B_x)) dx \\ &= \int_0^1 \min(t, \lambda_1(B_x)) dx \\ &\leq \int_0^1 t dx = t.\end{aligned}$$

(e) Let $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $t_n \downarrow 0$ as $n \to \infty$. We have

$$\frac{\lambda_2(B(t_n) \setminus B)}{t_n} = \frac{1}{t_n} \int_0^1 \min(t_n, \lambda_1(B_x)) dx = \int_0^1 \min(1, \frac{\lambda_1(B_x)}{t_n}) dx.$$

If we define $f_n : [0, 1] \to [0, \infty] : x \mapsto \min(1, \frac{\lambda_1(B_x)}{t_n})$ then f_n is measurable $\forall n \in \mathbb{N}$ and $f_n \uparrow \mathbf{1}_{\pi_2(B)}$, since if $x \notin \pi_2(B)$ then $f_n(x) = 0 \forall n$, whereas if $x \in \pi_2(B)$ then $\frac{\lambda_1(B_x)}{t_n} > 1$ for sufficiently large n. Furthermore, $\mathbf{1}_{\pi_2(B)}$ is measurable as the limit of measurable functions. Hence by the monotone convergence theorem we have

$$\lim_{n\to\infty} \frac{\lambda_2(B(t_n) \setminus B)}{t_n} = \int_0^1 \mathbf{1}_{\pi_2(B)} dx = \lambda_1(\pi_2(B)).$$

Hence

$$\frac{\lambda_2(B(t) \setminus B)}{t} \to \lambda_1(\pi_2(B))$$

as $t \downarrow 0$.

$\square$

Exercise 0.53. *Let $(X, \mathcal{M})$ be a measurable space and suppose $f : X \to [0, \infty]$ and $g : X \to [0, \infty]$ are Borel functions. Show that*

$$\int_0^\infty \int_0^\infty \mu(\{x \in X : f(x) > s, g(x) > t\}) dsdt = \int_X f(x)g(x)\mu(dx).$$

Proof. Repeatedly applying Fubini's theorem gives

$$\begin{aligned}
\int_0^\infty \int_0^\infty \mu(\{x \in X : f(x) > s, g(x) > t\}) dsdt &= \int_0^\infty \int_0^\infty \mu(f^{-1}((s,\infty)) \cap g^{-1}((t,\infty))) dsdt \\
&= \int_0^\infty \int_0^\infty \int_X \mathbf{1}_{f^{-1}((s,\infty))} \mathbf{1}_{g^{-1}((t,\infty))} \mu(dx) dsdt \\
&= \int_0^\infty \int_0^\infty \int_X \mathbf{1}_{f(x)>s} \mathbf{1}_{g(x)>t} \mu(dx) dsdt \\
&= \int_0^\infty \int_X \int_0^\infty \mathbf{1}_{f(x)>s} \mathbf{1}_{g(x)>t} ds \mu(dx) dt \\
&= \int_0^\infty \int_X \mathbf{1}_{g(x)>t} \int_0^\infty \mathbf{1}_{f(x)>s} ds \mu(dx) dt \\
&= \int_0^\infty \int_X \mathbf{1}_{g(x)>t} \int_0^{f(x)} 1 ds \mu(dx) dt \\
&= \int_0^\infty \int_X \mathbf{1}_{g(x)>t} f(x) \mu(dx) dt \\
&= \int_X \int_0^\infty \mathbf{1}_{g(x)>t} f(x) dt \mu(dx) \\
&= \int_X f(x) \int_0^\infty \mathbf{1}_{g(x)>t} dt \mu(dx) \\
&= \int_X f(x)g(x)\mu(dx).
\end{aligned}$$

□

Exercise 0.54. *(a) Let $\alpha \in \mathbb{R}$ be a fixed constant. Let $f(x) = x^\alpha$ for $x \in (0, 1]$. Determine the values of $p \in [1, \infty)$ (depending on α), such that $f \in L^p((0, 1])$.*

(b) Let $\alpha \in \mathbb{R}$, and let $g(x) = x^\alpha$ for $x \in [1, \infty)$. Determine for values of $p \in [1, \infty)$ (depending on α) such that $g \in L^p([1, \infty))$.

Proof. (a) If $\alpha p \neq -1$ then

$$\int_0^1 |f(x)|^p dx = \int_0^1 x^{\alpha p} dx = [\frac{x^{\alpha p+1}}{\alpha p + 1}]_0^1$$

If $\alpha p > -1$ then the integral converges, and if $\alpha p < -1$ then the integral diverges. If $\alpha p = -1$ then

$$\int_0^1 |f(x)|^p dx = \int_0^1 \frac{1}{x} dx = \lim_{t \downarrow 0} \int_t^1 \frac{1}{x} dx = \lim_{t \downarrow 0} [\ln(x)]_t^1 = -\lim_{t \downarrow 0} \ln(t) = \infty.$$

Hence $f \in L^p((0, 1])$ if and only if $\alpha p > -1$.

(b) If $\alpha p \neq -1$ then

$$\int_1^\infty x^{\alpha p} dx = \lim_{n\to\infty} [\frac{x^{\alpha p+1}}{\alpha p+1}]_1^n = \lim_{n\to\infty} \frac{n^{\alpha p+1}}{\alpha p+1} - \frac{1}{\alpha p+1}.$$

If $\alpha p + 1 > 0$ then $\lim_{n\to\infty} n^{\alpha p+1} = \infty$. If $\alpha p + 1 < 0$ then $\lim_{n\to\infty} n^{\alpha p+1} = 0$. If $\alpha p = -1$ then

$$\int_1^\infty x^{\alpha p} dx = \lim_{n\to\infty} [\ln(x)]_1^n = \lim_{n\to\infty} \ln(n) = \infty.$$

Hence $g \in L^p([1,\infty))$ if and only if $\alpha p < -1$.

□

Exercise 0.55. *Let $p \in [1,\infty)$ and let $f \in L^p(\mathbb{R})$. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be real-valued sequences such that $\sum_{n=1}^\infty |a_n| < \infty$. Show that the sequence of functions $f_n(x) := \sum_{k=1}^n a_k f(x-b_k)$ converges in $L^p(\mathbb{R})$.*

Proof. Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that $\sum_{k=n}^\infty |a_k| < \frac{\epsilon}{\|f\|_p} \forall n \geq N$. Then $\forall n > m > N$ we have

$$\|f_n - f_m\|_p = \sqrt[p]{\int_{-\infty}^\infty |\sum_{k=m+1}^n a_k f(x-b_k)|^p dx} \leq \sum_{k=m+1}^n \sqrt[p]{\int_{-\infty}^\infty |a_k f(x-b_k)|^p dx}$$

by Minkowski's inequality. Hence,

$$\begin{aligned}
\|f_n - f_m\|_p &\leq \sum_{k=m+1}^n |a_k| \sqrt[p]{\int_{-\infty}^\infty |f(x-b_k)|^p dx} \\
&= \sum_{k=m+1}^n |a_k| \|f\|_p \\
&\leq \|f\|_p \sum_{k=m+1}^\infty |a_k| \\
&< \|f\|_p \cdot \frac{\epsilon}{\|f\|_p} = \epsilon.
\end{aligned}$$

Hence f_n is a Cauchy sequence in $L^p(\mathbb{R})$ so converges in $L^p(\mathbb{R})$ by the Riesz-Fischer theorem. □

Exercise 0.56. *Suppose $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of nonnegative numbers, such that $A := \sum_{n=1}^\infty a_n^{4/3} < \infty$ and $B := \sum_{n=1}^\infty b_n^4 < \infty$. Show that $\sum_{n=1}^\infty a_n b_n \leq A^{3/4} B^{1/4}$.*

Proof. $\frac{4}{3}, 4 \in (1,\infty)$ are conjugate exponents, $(a_n)_{n\in\mathbb{N}} \in \ell^{\frac{4}{3}}$ and $(b_n)_{n\in\mathbb{N}} \in \ell^4$ so by Hölder's inequality we have

$$\sum_{n=1}^\infty a_n b_n = \sum_{n=1}^\infty |a_n b_n| = \|(a_n b_n)_{n\in\mathbb{N}}\|_1 \leq \|(a_n)_{n\in\mathbb{N}}\|_{\frac{4}{3}} \|(b_n)_{n\in\mathbb{N}}\|_4 = A^{\frac{3}{4}} B^{\frac{1}{4}}.$$

□

Exercise 0.57. *Suppose that $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, and $1 \leq p < q < \infty$.*

(a) Show that if μ is a probability measure and $f \in L^q(\mu)$, then $\|f\|_p \leq \|f\|_q$.

[Hint: note that $f = f \cdot 1$, and apply Hölder's inequality]

(b) Show that if $\mu(X) < \infty$ then $L^q(\mu) \subset L^p(\mu)$.

(c) Give an example to show that if $\mu(X) = \infty$, then we might not have $L^q(\mu) \subset L^p(\mu)$.

Proof. (a) Let $r \in (1, \infty)$ be the conjugate exponent of $\frac{q}{p}$. $f^p \in L^{\frac{q}{p}}(\mu)$ so by Hölder's inequality,

$$\|f^p\|_1 \leq \|f^p\|_{\frac{q}{p}} \cdot \|1\|_r.$$

μ is a probability measure so $\|1\|_r = 1$. Hence

$$\int |f|^p d\mu = \int |f^p| d\mu \leq \left(\int |f^p|^{\frac{q}{p}} d\mu\right)^{\frac{p}{q}} = \left(\int |f|^q d\mu\right)^{\frac{p}{q}}$$

so

$$\|f\|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^q d\mu\right)^{\frac{1}{q}} = \|f\|_q.$$

(b) Let $f \in L^q(\mu)$. Let $r \in (1, \infty)$ be the conjugate exponent of $\frac{q}{p}$. We have that

$$\int |f^p|^{\frac{q}{p}} d\mu = \int |f|^q d\mu < \infty$$

so $f^p \in L^{\frac{q}{p}}(\mu)$. Hence by Hölder's inequality we have

$$\|f^p\|_1 \leq \|f^p\|_{\frac{q}{p}} \cdot \|1\|_r.$$

Thus

$$\int |f|^p d\mu = \int |f^p| d\mu \leq \left(\int |f^p|^{\frac{q}{p}} d\mu\right)^{\frac{p}{q}} \cdot \|1\|_r = \|f\|_q^p \cdot \mu(X)^{\frac{1}{r}} < \infty$$

so $f \in L^p(\mu)$.

(c) Let $X := [1, \infty)$, let $\mathcal{M} := \mathcal{B}_X$ and let $\mu := \lambda_1|_X$. Define $f : X \to \mathbb{R} : x \mapsto x^{-\frac{1}{2}}$. Then $f \in L^3(X)$, since $-\frac{1}{2} \cdot 3 < -1$. However, $-\frac{1}{2} \cdot 2 \not< -1$ so $f \notin L^2(X)$.

□

Exercise 0.58. *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $p \in (1, \infty)$. Suppose $f \in \mathbb{R}(X)$ and (for all $n \in \mathbb{N}$) $f_n \in \mathbb{R}(X)$, with $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. For all $n \in \mathbb{N}$ and $x \in X$, set*

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)| \quad \text{and} \quad g_\infty(x) = \sum_{k=1}^{\infty} |f_k(x)|.$$

(i) Show that $\|g_n\|_p \to \|g_\infty\|_p$ as $n \to \infty$, and deduce that $\|g_\infty\|_p < \infty$.

(ii) Show that the function $h(x) := \sum_{m=1}^{\infty} f_n(x)$ is well-defined and finite μ-a.e., that is, the sum converges for μ-a.e. $x \in X$.

Proof. (a) We have that $g_n^p \uparrow g_\infty^p$ pointwise so by the monotone convergence theorem,

$$\lim_{n\to\infty} \|g_n\|_p^p = \lim_{n\to\infty} \int |g_n|^p d\mu = \int |g_\infty|^p d\mu = \|g_\infty\|_p^p.$$

Sequential continuity gives $\lim_{n\to\infty} \|g_n\|_p^p = (\lim_{n\to\infty} \|g_n\|_p)^p$ so

$$\lim_{n\to\infty} \|g_n\|_p = \|g_\infty\|_p.$$

Furthermore, by Minkoski's inequality

$$\|g_n\|_p \le \sum_{k=1}^n \|f_k\|_p \le \sum_{k=1}^\infty \|f_k\|_p < \infty \forall n \in \mathbb{N}$$

so $\|g_n\|_p$ is an increasing and bounded sequence so converges to a finite limit. Hence $\|g_\infty\|_p < \infty$.

(b) Since $\|g_\infty\|_p < \infty$, it follows that $g_\infty = |g_\infty| < \infty$ μ-a.e. Hence

$$\sum_{k=1}^\infty f_k(x)$$

converges absolutely for μ-a.e. $x \in X$ so

$$\sum_{k=1}^\infty f_k(x)$$

in particular converges for μ-a.e. $x \in X$.

□

Exercise 0.59. *Let $W \in \mathcal{B}$, and for $f, g \in L^2(W)$, write $\langle f, g\rangle = \int_W f(x)g(x)dx$. Show that if also $h \in L^2(W)$ and $a, b \in \mathbb{R}$ then $\langle f, ag + bh\rangle = a\langle f, g\rangle + b\langle f, h\rangle$.*

Proof.

$$\langle f, ag + bh\rangle = \int_W f(x)(ag(x) + bh(x))dx = \int_W af(x)g(x) + bf(x)h(x)dx = a\langle f, g\rangle + b\langle f, h\rangle.$$

□

Exercise 0.60. *For $n \in \mathbb{N}$, let $f_n(x) = \sin(nx)$.*

(a) Show that for $n, m \in \mathbb{N}$ with $n \neq m$ we have $\int_0^{2\pi} f_n(x)f_m(x)dx = 0$, while $\int_0^{2\pi} (f_n(x))^2 dx = \pi$. [Hint: recall that $\cos(a + b) = \cos a \cos b - \sin a \sin b$.]

(b) Now set $g_n(x) = \sum_{k=1}^n k^{-1} f_k(x)$. Show that in $L^2([0, 2\pi])$ we have $\|g_n\|_2^2 = \pi \sum_{k=1}^n k^{-2}$.

(c) Prove that there exists $g \in L^2([0, 2\pi])$ such that $g_n \to g$ in $L^2([0, 2\pi])$ as $n \to \infty$.

Proof. (a) Let $n \neq m$

$$\begin{aligned}
I := \int_0^{2\pi} f_n(x) f_m(x) dx &= \int_0^{2\pi} \sin(nx)\sin(mx) dx \\
&= \frac{1}{m}\left[\sin(nx)\cos(mx)\right]_{2\pi}^{0} + \frac{n}{m}\int_0^{2\pi} \cos(nx)\cos(mx) dx \\
&= \frac{n}{m}\int_0^{2\pi} \cos((n+m)x) + \sin(nx)\sin(mx) dx \\
&= \frac{n}{m(n+m)}\left[\sin((n+m)x)\right]_0^{2\pi} + \frac{n}{m} I \\
&= \frac{n}{m} I
\end{aligned}$$

Hence

$$(1 - \frac{n}{m}) I = 0.$$

$\frac{n}{m} \neq 1$ so

$$I = 0.$$

Furthermore,

$$\begin{aligned}
\int_0^{2\pi} (f_n(x))^2 dx &= \int_0^{2\pi} \sin^2(nx) dx \\
&= \int_0^{2\pi} \frac{1-\cos(2x)}{2} dx \\
&= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right]_0^{2\pi} \\
&= \pi.
\end{aligned}$$

(b)

$$\begin{aligned}
\|g_n\|_2^2 &= \int_0^{2\pi} \left(\sum_{k=1}^{n} k^{-1}\sin(kx)\right)^2 dx \\
&= \int_0^{2\pi} \sum_{(i,j)\in\{1,\dots,n\}^2} (ij)^{-1}\sin(ix)\sin(jx) dx \\
&= \sum_{(ij)\in\{1,\dots,n\}^2} (ij)^{-1}\int_0^{2\pi} \sin(ix)\sin(jx) dx \\
&= \sum_{k=1}^{n} k^{-2}\int_0^{2\pi} \sin^2(ix) dx \\
&= \pi \sum_{k=1}^{n} k^{-2}
\end{aligned}$$

(c) Let $\epsilon > 0$. Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

there exists an $N \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} \frac{\pi}{k^2} < \epsilon.$$

Then for all $n > m > N$ we have

$$\begin{aligned}
\|g_n - g_m\|_2^2 &= \int_0^{2\pi} \left(\sum_{k=m+1}^{n} k^{-1} f_k(x) \right)^2 dx \\
&= \pi \sum_{k=m+1}^{n} \frac{1}{k^2} \\
&\leq \sum_{k=m+1}^{\infty} \frac{\pi}{k^2} < \epsilon.
\end{aligned}$$

Hence g_n is a Cauchy sequence in $L^2([0,2\pi])$ so converges in $L^2([0,2\pi])$ to some $g \in L^2([0,2\pi])$ by the Riesz-Fischer theorem.

□