Probability Theory

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$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

so

$$\mathbb{P}(A|B) = \frac{1}{\mathbb{P}(B)} \int_A \mathbf{1}_B d\mathbb{P}.$$

Hence,

$$\mathbb{E}[X|B] = \int X d\mathbb{P}(\cdot|B) = \frac{1}{\mathbb{P}(B)} \int X \mathbf{1}_B d\mathbb{P} = \frac{\mathbb{E}[X\mathbf{1}_B]}{\mathbb{P}(B)}.$$

Let X be a non-negative integer-valued random variable. Then

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}(X \geq i).$$

Exercise 0.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A_1, ..., A_n$ be measurable sets. Prove by induction that*

$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j).$$

Proof. For $n = 1$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}(A_1) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_1).$$

Now assume true for $n = k$. Then for $n = k + 1$ we have

$$\begin{aligned}\mathbb{P}\left(\bigcup_{i=1}^{k+1} A_i\right) &= \mathbb{P}\left(A_{k+1} \cup \bigcup_{i=1}^{k} A_i\right)\\ &= \mathbb{P}(A_{k+1}) + \mathbb{P}\left(\bigcup_{i=1}^{k} A_i\right) - \mathbb{P}\left(\bigcup_{i=1}^{k}(A_i \cap A_{k+1})\right)\\ &\geq \mathbb{P}(A_{k+1}) + \sum_{i=1}^{k} \mathbb{P}(A_i) - \sum_{1\leq i<j\leq k} \mathbb{P}(A_i \cap A_j) - \mathbb{P}\left(\bigcup_{i=1}^{k}(A_i \cap A_{k+1})\right)\\ &\geq \sum_{i=1}^{k+1} \mathbb{P}(A_i) - \sum_{1\leq i<j\leq k} \mathbb{P}(A_i \cap A_j) - \sum_{i=1}^{k} \mathbb{P}(A_i \cap A_{k+1})\\ &= \sum_{i=1}^{k+1} \mathbb{P}(A_i) - \sum_{1\leq i<j\leq k+1} \mathbb{P}(A_i \cap A_j).\end{aligned}$$

Hence the result holds by induction. $\square$

Exercise 0.2. *Suppose $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and $X : \Omega \to [0, \infty)$ is a non-negative random variable.*

(a) Let $A_n = \{\omega \in \Omega : X(\omega) \leq n\}$. Prove that $\mathbb{E}[X\mathbf{1}_{A_n}] \to \mathbb{E}[X]$ as $n \to \infty$.

(b) Prove that if $\mathbb{E}[X] < \infty$, then $\mathbb{E}[X\mathbf{1}_{A_n^c}] \to 0$ as $n \to \infty$.

(c) Suppose $\mathbb{E}[X] < \infty$. Prove that

$$\lim_{\delta\downarrow 0} \sup_{\{A\in\mathcal{F}:\mathbb{P}(A)<\delta\}} \mathbb{E}[X\mathbf{1}_A] = 0.$$

Proof. (a) $0 \leq X\mathbf{1}_{A_n} \uparrow X$ so apply the monotone convergence theorem.

(b) $\mathbb{E}[X\mathbf{1}_{A_n}] + \mathbb{E}[X\mathbf{1}_{A_n^c}] = \mathbb{E}[X]\forall n$ so $\lim_{n\to\infty} \mathbb{E}[X\mathbf{1}_{A_n^c}] = \mathbb{E}[X] - \lim_{n\to\infty} \mathbb{E}[X\mathbf{1}_{A_n}] = 0$.

(c) Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence such that $\delta_n \downarrow 0$. We want to show that given an $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\forall n > N$ we have

$$\sup_{\{A\in\mathcal{F}:\mathbb{P}(A)<\delta_n\}} \mathbb{E}[X\mathbf{1}_A] < \epsilon.$$

To do so, we instead show that there exists an $N \in \mathbb{N}$ such that $\forall n > N$ we have

$$\mathbb{E}[X\mathbf{1}_A] < \frac{\epsilon}{2}\forall A \in \mathcal{F} : \mathbb{P}(A) < \delta_n,$$

which would imply that

$$\sup_{\{A\in\mathcal{F}:\mathbb{P}(A)<\delta_n\}} \mathbb{E}[X\mathbf{1}_A] \leq \frac{\epsilon}{2} < \epsilon.$$

By part (b) there exists an $M \in \mathbb{N}$ such that

$$\mathbb{E}[X\mathbf{1}_{\{X>M\}}] < \frac{\epsilon}{4}.$$

Let $N \in \mathbb{N}$ be such that $\delta_n < \frac{\epsilon}{4M} \forall n > N$. Then if $A \in \mathcal{F}$ is such that $\mathbb{P}(A) < \delta_n$, we have that

$$X\mathbf{1}_A = X\mathbf{1}_{A\cap\{X>M\}} + X\mathbf{1}_{A\cap\{X\leq M\}}$$

so

$$\begin{aligned}\mathbb{E}[X\mathbf{1}_A] &\leq \mathbb{E}[X\mathbf{1}_{\{X>M\}}] + \mathbb{E}[X\mathbf{1}_{A\cap\{X\leq M\}}] \\ &< \frac{\epsilon}{4} + M\mathbb{P}(A) \\ &< \frac{\epsilon}{4} + \frac{M\epsilon}{4M} = \frac{\epsilon}{2},\end{aligned}$$

as required.

□

Exercise 0.3. *Let $\Omega = \{1,2,3,4,5,6\}$ with $\mathcal{A} = \mathcal{P}(\Omega)$ and let $\mathcal{C} = \{A, B\}$ with $A = \{1,2,3,4\}$ and $B = \{3,4,5\}$. What are the sets of $\sigma(\mathcal{C})$?*

Proof.

$$A \cap B = \{3,4\}, A^c \cap B = \{5\}, A \cap B^c = \{1,2\}, A^c \cap B^c = \{6\}.$$

Hence, if we label the sets as

$$X_1 = \{3,4\}, X_2 = \{5\}, X_3 = \{1,2\}, X_4 = \{6\}$$

then

$$\sigma(\mathcal{C}) = \{\bigcup_{i\in J} X_i : J \subseteq \{1,2,3,4\}\}.$$

□

Exercise 0.4. *Let $\Omega = \{1,2,3,4\}$ and let $\mathcal{A}$ be the σ-algebra $\sigma(\{\{1\},\{2\},\{3,4\}\})$ on Ω.*

(a) Give an example of a subset of Ω that is not in $\mathcal{A}$.

(b) Give an example of a function $f : \Omega \to \mathbb{R}$ that is not measurable.

Proof. (a) $\{1,4\}$.

(b) $\mathbf{1}_{\{1,4\}}$.

□

Exercise 0.5. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Suppose that $\mathcal{C}$ is a finite partition of Ω with $\mathcal{C} \subset \mathcal{A}$, i.e. suppose that $\mathcal{C} = \{A_1, ..., A_n\}$ with the sets A_i partitioning Ω, and with each A_i in $\mathcal{A}$. Likewise, suppose that $\mathcal{B} = \{B_1, ..., B_m\}$ is a finite partition of Ω with $\mathcal{B} \subset \mathcal{A}$.*

(a) Suppose that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all events $A \in \mathcal{C}$ and $B \in \mathcal{B}$. Prove that $\sigma(\mathcal{C})$ and $\sigma(\mathcal{B})$ are independent σ-algebras.

(b) Does the conclusion still hold if $\mathcal{C} = \{A_1, A_2, ...\}$ and $\mathcal{B} = \{B_1, B_2, ...\}$ are countably infinite partitions of Ω but otherwise things are as described above? Explain briefly.

Proof. (a) Let $A \in \sigma(\mathcal{C})$ and $B \in \sigma(\mathcal{B})$. Then $A = \bigcup_{i\in I} A_i$ and $B = \bigcup_{j\in J} B_j$ for some

$$I \subseteq \{1, ..., n\}, J \subseteq \{1, ..., m\}.$$

Hence

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}\left(\bigcup_{i\in I} A_i \cap \bigcup_{j\in J} B_j\right) \\
&= \mathbb{P}\left(\bigcup_{i\in I}\left(A_i \cap \bigcup_{j\in J} B_j\right)\right) \\
&= \sum_{i\in I} \mathbb{P}\left(A_i \cap \bigcup_{j\in J} B_j\right) \\
&= \sum_{i\in I} \mathbb{P}\left(\bigcup_{j\in J}(A_i \cap B_j)\right) \\
&= \sum_{i\in I}\sum_{j\in J} \mathbb{P}(A_i \cap B_j) \\
&= \sum_{i\in I}\sum_{j\in J} \mathbb{P}(A_i)\mathbb{P}(B_j) \\
&= \left(\sum_{i\in I} \mathbb{P}(A_i)\right)\left(\sum_{j\in J} \mathbb{P}(B_j)\right) \\
&= \mathbb{P}(A)\mathbb{P}(B).
\end{aligned}$$

(b) Yes, since the argument still holds with countably infinite indexes.

□

Exercise 0.6. *(a) Prove Markov's inequality, which says that if X is a non-negative random variable and $a > 0$, then $\mathbb{P}(X \geq a) \leq a^{-1}\mathbb{E}[X]$.*

(b) Prove that, if X is a non-negative random variable with $\mathbb{E}[X] = 0$, then $\mathbb{P}(X = 0) = 1$.

(c) Suppose X is a random variable with $\mathbb{E}[X] = \mu$ and Var$(X) = 0$. Prove that $\mathbb{P}(X = \mu) = 1$.

Proof. (a)

$$\begin{aligned}
\mathbb{P}(X \geq a) &= \mathbb{E}[\mathbf{1}_{X\geq a}] \\
&\leq \mathbb{E}\left[\frac{X}{a}\mathbf{1}_{X\geq a}\right] \\
&\leq a^{-1}\mathbb{E}[X].
\end{aligned}$$

(b) Let $\epsilon > 0$. Then $\mathbb{P}(X \geq \epsilon) \leq \epsilon^{-1} \cdot \mathbb{E}[X] = 0$. ϵ is arbitrary so $\mathbb{P}(X > 0) = \mathbb{P}(X \neq 0) = 0$. Hence $\mathbb{P}(X = 0) = 1$.

(c) $\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = 0$ so $\mathbb{P}((X-\mu)^2 = 0) = \mathbb{P}(X = \mu) = 1$.

□

Exercise 0.7. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

(a) Let $\mathcal{C}$ be the class of events of the form $\{X < t\}, t \in \mathbb{R}$. Show that $\sigma(\mathcal{C}) = \sigma(X)$.

(b) Show that $\mathcal{C}$ is a π-system.

(c) Suppose that $\mathbb{P}(\{X < a\} \cap \{Y < b\}) = \mathbb{P}(X < a)\mathbb{P}(Y < b)$ for all real numbers a and b. Let $\mathcal{C}$ be as in (a), and $\mathcal{C}' = \{\{Y < t\} : t \in \mathbb{R}\}$.

1. *Fix $A \in \mathcal{C}$ and set $\mu_A(B) = \mathbb{P}(A \cap B)$ and $\mu'_A(B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $B \in \sigma(\mathcal{C}')$. Show that $\mu_A = \mu'_A$ on $\sigma(\mathcal{C}')$.*
2. *Fix $B \in \sigma(\mathcal{C}')$, and set $\nu_B(A) = \mathbb{P}(A \cap B)$ and $\nu'_B(A) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \sigma(\mathcal{C})$. Show that $\nu_B = \nu'_B$ on $\sigma(\mathcal{C})$.*
3. *Show that X and Y are independent random variables.*

Proof. (a) $(-\infty, t)$ is a Borel set $\forall t$ so $\mathcal{C} \subseteq \sigma(X)$, and hence $\sigma(\mathcal{C}) \subseteq \sigma(X)$. $\mathcal{B} = \sigma(\mathcal{I})$, where $\mathcal{I} := \{(-\infty, t) : t \in \mathbb{R}\}$. Let $\mathcal{M} := \{A \in \mathcal{B} : X^{-1}(A) \in \sigma(\mathcal{C})\}$. Clearly $\mathcal{I} \subseteq \mathcal{M}$. Let $A_1, A_2, ... \in \mathcal{M}$. Then $X^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} X^{-1}(A_i) \in \sigma(\mathcal{C})$, since $\sigma(\mathcal{C})$ is a σ-algebra. Hence $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$. Furthermore, $X^{-1}(\emptyset) = \emptyset \in \sigma(\mathcal{C})$ so $\emptyset \in \mathcal{M}$. Finally, Let $A \in \mathcal{M}$. Then $X^{-1}(A^c) = X^{-1}(A)^c \in \sigma(\mathcal{C})$ so $A^c \in \mathcal{M}$. Hence $\mathcal{M}$ is a σ-algebra so contains $\sigma(\mathcal{I}) = \mathcal{B}$; that is, $X^{-1}(A) \in \sigma(\mathcal{C}) \forall A \in \mathcal{B}$, so $\sigma(X) \subseteq \sigma(\mathcal{C})$. Hence $\sigma(X) = \sigma(\mathcal{C})$.

(b) $\{X < t\} \cap \{X < s\} = \{X < \min(t, s)\} \in \mathcal{C}$.

(c)
1. μ_A and μ'_A agree on $\mathcal{C}'$, and $\mathcal{C}'$ is a π-system, so μ_A and μ'_A agree on $\sigma(\mathcal{C}')$ by the uniqueness lemma.
2. ν_B and ν'_B agree on $\mathcal{C}$ by the above, and $\mathcal{C}$ is a π-system, so ν_B and ν'_B agree on $\sigma(\mathcal{C})$ by the uniqueness lemma.
3. We have shown that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \forall A \in \sigma(\mathcal{C}), B \in \sigma(\mathcal{C}')$, and hence

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \forall A \in \sigma(X), B \in \sigma(Y),$$

since $\sigma(\mathcal{C}) = \sigma(X), \sigma(\mathcal{C}') = \sigma(Y)$. Thus X and Y are independent.

□

Exercise 0.8. *Let $\mathcal{F}$ be a sub-σ-algebra. Let U and V be two $\mathcal{F}$-measurable random variables.*

(a) Show that $U + V$ is $\mathcal{F}$-measurable.

(b) Show that UV is $\mathcal{F}$-measurable.

Proof. (a) $\{U + V < t\} = \bigcup_{s \in \mathbb{R}} \{U < s\} \cap \{V < t - s\}$. Furthermore, if $U(\omega) < s$ and $V(\omega) < t - s$ for some $s \in \mathbb{R}$, by the density of $\mathbb{Q}$ in $\mathbb{R}$ there will be some $q \in \mathbb{Q}$ such that $U(\omega) < q < s$ and hence $V(\omega) < t - s < t - q$, so $\omega \in \{U < q\} \cap \{V < t - q\}$. This is true for all $\omega \in \{U + V < t\}$ so $\{U + V < t\} \subseteq \bigcup_{q \in \mathbb{Q}} \{U < q\} \cap \{V < t - q\}$. The reverse inclusion clearly holds so

$$\{U + V < t\} = \bigcup_{q \in \mathbb{Q}} \{U < q\} \cap \{V < t - q\} \in \mathcal{F}.$$

Hence $U + V$ is $\mathcal{F}$-measurable.

(b) Let X be an $\mathcal{F}$-measurable random variable. Then $\{X^2 < t\} = \{-\sqrt{t} < X < \sqrt{t}\} \in \mathcal{F}$ if $t \geq 0$, and $\{X^2 < t\} = \emptyset \in \mathcal{F}$ otherwise. Hence X^2 is $\mathcal{F}$-measurable. Hence $UV = \frac{(U+V)^2-(U-V)^2}{4}$ is $\mathcal{F}$-measurable.

$\square$

Exercise 0.9. *Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables such that $\mathbb{E}[|X_i|] < \infty$ for all $i \geq 1$. For all $n \geq 1$, set $Y_n = \prod_{i=1}^n X_i$.*

(a) Prove that, for all $n \geq 1$, $Y_n = \prod_{i=1}^n X_i$ is $\sigma(X_1, \ldots, X_n)$-measurable, and deduce that $\sigma(Y_n) \subset \sigma(X_1, \ldots, X_n)$.

(b) Fix $n \geq 1$ and let

$$\mathcal{C} = \left\{ \bigcap_{i=1}^{n} \{X_i \in B_i\} : B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}) \right\}.$$

Show that $\mathcal{C}$ is a π-system.

(c) Show that, for all $A \in \sigma(X_{n+1})$ and $C \in \mathcal{C}$, $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$.

(d) Conclude that, for all $A \in \sigma(X_{n+1})$ and $C \in \sigma(X_1, \ldots, X_n)$, $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$.

(e) Deduce that, for all $n \geq 1$, Y_n and X_{n+1} are independent.

(f) By induction on n, prove that, for all $n \geq 1$,

$$\mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} \mathbb{E}[X_i]. \tag{2}$$

Proof. (a) $\sigma(X_i) \subseteq \sigma(X_1, ..., X_n) \forall i$ so each X_i is $\sigma(X_1, ..., X_n)$-measurable. The product of measurable functions is measurable so Y_n is $\sigma(X_1, ..., X_n)$-measurable. It clearly follows that $\sigma(Y_n) \subseteq \sigma(X_1, ..., X_n)$.

(b) Let $A_1, ..., A_n, B_1, ..., B_n \in \mathcal{B}$. Then

$$\bigcap_{i=1}^{n} \{X_i \in A_i\} \cap \bigcap_{i=1}^{n} \{X_i \in B_i\} = \bigcap_{i=1}^{n} \{X_i \in A_i \cap B_i\} \in \mathcal{C}.$$

(c) Let $A = \{X_{n+1} \in B_{n+1}\}$ and let $C = \bigcap_{i=1}^n \{X_i \in B_i\}$ for some $B_1, ..., B_{n+1} \in \mathcal{B}$. Then by independence we have

$$\begin{aligned}
\mathbb{P}(A \cap C) &= \mathbb{P}(\bigcap_{i=1}^{n+1} \{X_i \in B_i\}) \\
&= \prod_{i=1}^{n+1} \mathbb{P}(X_i \in B_i) \\
&= \mathbb{P}(A) \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i) \\
&= \mathbb{P}(A)\mathbb{P}(\bigcap_{i=1}^{n} \{X_i \in B_i\}) \\
&= \mathbb{P}(A)\mathbb{P}(C).
\end{aligned}$$

(d) First note that $\sigma(X_i) \subseteq \mathcal{C} \forall i \in \{1, .., n\}$ so $\sigma(X_1) \cup ... \cup \sigma(X_n) \subseteq \mathcal{C}$, and hence $\sigma(X_1, ..., X_n) \subseteq \sigma(\mathcal{C})$, and furthermore that $\mathcal{C} \subseteq \sigma(X_1, ..., X_n)$ so $\sigma(\mathcal{C}) = \sigma(X_1, ..., X_n)$. $\mathcal{C}$ is a π-system so apply the uniqueness lemma to the measures $\nu_A : \sigma(X_1, ..., X_n) \to \mathbb{R} : C \mapsto \mathbb{P}(A \cap C)$ and $\nu'_A : \sigma(X_1, ..., X_n) \to \mathbb{R} : C \mapsto \mathbb{P}(A)\mathbb{P}(C)$ for every $A \in \sigma(X_{n+1})$.

(e) Let $A \in \sigma(X_{n+1})$ and $C \in \sigma(Y_n)$. $\sigma(Y_n) \subseteq \sigma(X_1, ..., X_n)$ so $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$. Hence Y_n and X_{n+1} are independent.

(f) The base case is clear. Assume true for $n = k$:

$$\mathbb{E}\left[\prod_{i=1}^{k} X_i\right] = \prod_{i=1}^{k} \mathbb{E}[X_i].$$

For $n = k + 1$, we have that X_{k+1} and $\prod_{i=1}^{k} X_i$ are independent so

$$\mathbb{E}\left[\prod_{i=1}^{k+1} X_i\right] = \mathbb{E}\left[X_{k+1}\prod_{i=1}^{k} X_i\right] = \mathbb{E}[X_{k+1}]\mathbb{E}\left[\prod_{i=1}^{k} X_i\right] = \mathbb{E}[X_{k+1}]\prod_{i=1}^{k} \mathbb{E}[X_i] = \prod_{i=1}^{k+1} \mathbb{E}[X_i].$$

□

Exercise 0.10. *Let $p \in (0, 1)$. Let X and Y be two independent random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, such that*

$$\mathbb{P}(X = 1) = \mathbb{P}(Y = 1) = p \quad \textit{and} \quad \mathbb{P}(X = -1) = \mathbb{P}(Y = -1) = 1 - p.$$

(a) Let $A \in \mathcal{A}$ be an event. Give the formula for $\mathbb{E}[X \mid \sigma(A)]$ in terms of $\mathbb{E}[X \mid A]$ and $\mathbb{E}[X \mid A^c]$.

(b) Show that $\mathbb{E}[X \mid X + Y = 0] = 0$.

(c) Show that

$$\mathbb{E}[X \mid X + Y \neq 0] = \frac{2p - 1}{p^2 + (1 - p)^2}.$$

(d) Let $\mathcal{F}$ be the sub-σ-algebra generated by the event $\{X + Y = 0\}$. Calculate $\mathbb{E}[X \mid \mathcal{F}]$.

(e) Calculate $\mathbb{E}[Y \mid \mathcal{F}]$. Are the random variables $\mathbb{E}[X \mid \mathcal{F}]$ and $\mathbb{E}[Y \mid \mathcal{F}]$ independent? (Hint: be aware that the answer might depend on the value of p.)

Proof. (a) $\sigma(A) = \{A, A^c\}$. Hence

$$\mathbb{E}[X|\sigma(A)] = \mathbb{E}[X|A]\mathbf{1}_A + \mathbb{E}[X|A^c]\mathbf{1}_{A^c}$$

almost surely. To see this, first note that $\mathbb{E}[X|A]\mathbf{1}_A + \mathbb{E}[X|A^c]\mathbf{1}_{A^c}$ is $\sigma(A)$-measurable, and furthermore

$$\begin{aligned}\mathbb{E}[(\mathbb{E}[X|A]\mathbf{1}_A + \mathbb{E}[X|A^c]\mathbf{1}_{A^c})\mathbf{1}_A] &= \mathbb{E}[\mathbb{E}[X|A]\mathbf{1}_A] \\ &= \mathbb{E}\left[\frac{\mathbb{E}[X\mathbf{1}_A]}{\mathbb{P}(A)}\mathbf{1}_A\right] \\ &= \mathbb{E}[X\mathbf{1}_A],\end{aligned}$$

and similarly

$$\mathbb{E}[(\mathbb{E}[X|A]\mathbf{1}_A + \mathbb{E}[X|A^c]\mathbf{1}_{A^c})\mathbf{1}_{A^c}] = \mathbb{E}[X\mathbf{1}_{A^c}].$$

(b)

$$\mathbb{E}[X|X+Y=0] = \frac{\mathbb{E}[X\mathbf{1}_{X+Y=0}]}{\mathbb{P}(X+Y=0)}.$$

Since

$$\{X+Y=0\} = \{X=1, Y=-1\} \sqcup \{X=-1, Y=1\}$$

it follows that

$$\begin{aligned}\mathbb{E}[X\mathbf{1}_{X+Y=0}] &= \mathbb{P}(X=1, Y=-1) - \mathbb{P}(X=-1, Y=1)\\ &= p(1-p) - (1-p)p = 0.\end{aligned}$$

Hence

$$\mathbb{E}[X|X+Y=0] = 0.$$

(c)

$$\mathbb{E}[X|X+Y\neq 0] = \frac{\mathbb{E}[X\mathbf{1}_{X+Y\neq 0}]}{\mathbb{P}(X+Y\neq 0)}.$$

Since

$$\{X+Y\neq 0\} = \{X=1, Y=1\} \sqcup \{X=-1, Y=-1\},$$

it follows that

$$\begin{aligned}\mathbb{E}[X\mathbf{1}_{X+Y\neq 0}] &= \mathbb{P}(X=1, Y=1) - \mathbb{P}(X=-1, Y=-1)\\ &= p^2 - (1-p)^2\\ &= p^2 - 1 + 2p - p^2 = 2p-1\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(X+Y\neq 0) &= \mathbb{P}(X=1, Y=1) + \mathbb{P}(X=-1, Y=-1)\\ &= p^2 + (1-p)^2.\end{aligned}$$

Hence

$$\mathbb{E}[X|X+Y\neq 0] = \frac{2p-1}{p^2+(1-p)^2}.$$

(d)

$$\mathbb{E}[X|\mathcal{F}] = \frac{2p-1}{p^2+(1-p)^2}\mathbf{1}_{X+Y\neq 0}.$$

(e)

$$\mathbb{E}[Y|\mathcal{F}] = \mathbb{E}[X|\mathcal{F}]$$

by symmetry. First suppose that $\frac{2p-1}{p^2+(1-p)^2} \neq 0$.

$$\mathbb{P}\left(\mathbb{E}[X|\mathcal{F}] = \frac{2p-1}{p^2+(1-p)^2}, \mathbb{E}[Y|\mathcal{F}] = \frac{2p-1}{p^2+(1-p)^2}\right) = \mathbb{P}(X+Y\neq 0) = p^2+(1-p)^2.$$

and

$$\mathbb{P}\left(\mathbb{E}[X|\mathcal{F}] = \frac{2p-1}{p^2+(1-p)^2}\right)\mathbb{P}\left(\mathbb{E}[Y|\mathcal{F}] = \frac{2p-1}{p^2+(1-p)^2}\right) = \mathbb{P}(X+Y\neq 0)^2 = (p^2+(1-p)^2)^2.$$

Hence in order to have independence we need

$$\begin{aligned}&p^2 + (1-p)^2 = (p^2 + (1-p)^2)^2\\ \iff &p^2 + (1-p)^2 = 1\\ \iff &p^2 - p = 0\\ \iff &p \in \{0, 1\}.\end{aligned}$$

Now suppose $\frac{2p-1}{p^2+(1-p)^2} = 0$. Then $\mathbb{E}[Y|\mathcal{F}] = \mathbb{E}[X|\mathcal{F}] = 0$ almost surely so are independent. The only value of p which gives this is $p = \frac{1}{2}$.

$\square$

Exercise 0.11. *Let X and Y be two random variables such that $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$. Assume that*

$$\mathbb{E}[X \mid Y] = Y \text{ a.s. and } \mathbb{E}[Y \mid X] = X \text{ a.s.}$$

(a) Prove that, for all $c \in \mathbb{Q}$, $\mathbb{E}[(X-Y)\mathbf{1}_{Y \le c}] = 0$.

(b) Deduce that, for all $c \in \mathbb{Q}$, $\mathbb{E}[(X-Y)\mathbf{1}_{\{X \le c\} \cap \{Y \le c\}}] \le 0$.

(c) Using the symmetry of X and Y, deduce that, for all $c \in \mathbb{Q}$,

$$\mathbb{E}[(X-Y)\mathbf{1}_{\{X \le c\} \cap \{Y \le c\}}] = 0.$$

(d) Deduce that, for all $c \in \mathbb{Q}$, $\mathbb{P}(X > c$ and $Y \le c) = 0$.

(e) Deduce that $X = Y$ almost surely.

Proof. (a)

$$\mathbb{E}[(X-Y)\mathbf{1}_{Y \le c}] = \mathbb{E}[X\mathbf{1}_{Y \le c}] - \mathbb{E}[Y\mathbf{1}_{Y \le c}] = \mathbb{E}[X\mathbf{1}_{Y \le c}] - \mathbb{E}[\mathbb{E}[X|Y]\mathbf{1}_{Y \le c}].$$

Since

$$\{Y \le c\} \in \sigma(Y)$$

it follows that

$$\mathbb{E}[\mathbb{E}[X|Y]\mathbf{1}_{Y \le c}] = \mathbb{E}[X\mathbf{1}_{Y \le c}]$$

so

$$\mathbb{E}[(X-Y)\mathbf{1}_{Y \le c}] = 0.$$

(b)

$$\begin{aligned}\mathbb{E}[(X-Y)\mathbf{1}_{\{X \le c\} \cap \{Y \le c\}}] &= \mathbb{E}[(X-Y)\mathbf{1}_{\{Y \le c\}}] - \mathbb{E}[(X-Y)\mathbf{1}_{\{X > c\} \cap \{Y \le c\}}]\\ &= -\mathbb{E}[(X-Y)\mathbf{1}_{\{X > c\} \cap \{Y \le c\}}] \le 0.\end{aligned}$$

(c) It follows from symmetry that

$$\mathbb{E}[(Y-X)\mathbf{1}_{\{X \le c\} \cap \{Y \le c\}}] \le 0$$

so

$$\mathbb{E}[(X-Y)\mathbf{1}_{\{X \le c\} \cap \{Y \le c\}}] \ge 0$$

, and hence

$$\mathbb{E}[(X-Y)\mathbf{1}_{\{X \le c\} \cap \{Y \le c\}}] = 0.$$

(d)
$$\mathbb{E}[(X-Y)\mathbf{1}_{\{X\leq c\}\cap\{Y\leq c\}}]+\mathbb{E}[(X-Y)\mathbf{1}_{\{X>c\}\cap\{Y\leq c\}}]=\mathbb{E}[(X-Y)\mathbf{1}_{\{Y\leq c\}}]=0$$

so
$$\mathbb{E}[(X-Y)\mathbf{1}_{\{X>c\}\cap\{Y\leq c\}}]=0.$$

Since
$$(X-Y)\mathbf{1}_{\{X>c\}\cap\{Y\leq c\}}\geq 0,$$

it follows that
$$(X-Y)\mathbf{1}_{\{X>c\}\cap\{Y\leq c\}}=0$$

almost surely. That is,
$$\mathbb{P}((X-Y)\mathbf{1}_{\{X>c\}\cap\{Y\leq c\}}>0)=\mathbb{P}(X>c,Y\leq c)=0.$$

(e)
$$\{X\neq Y\}=\bigcup_{c\in\mathbb{Q}}\{X>c,Y\leq c\}\cup\bigcup_{c\in\mathbb{Q}}\{Y>c,X\leq c\}.$$

Hence,
$$\mathbb{P}(X\neq Y)=\sum_{c\in\mathbb{Q}}\mathbb{P}(X>c,Y\leq c)+\sum_{c\in\mathbb{Q}}\mathbb{P}(Y>c,X\leq c)=0$$

so
$$X=Y$$

almost surely.

□

Exercise 0.12. *Assume that we toss n fair coins and N of them come up as heads. Suppose we then roll N fair dice. Let T be the sum of the values obtained on the N dice. Express $\mathbb{E}[T|N]$ as a function of N, and then express $\mathbb{E}[T]$ and $\mathbb{E}[NT]$, both as functions of n.*

Proof.
$$\sigma(N)=\{\bigcup_{i\in I}\{N=i\}:I\subseteq\{0,...,n\}\},$$

with
$$\{N=0\},...,\{N=n\}$$

being a partition of Ω, and hence
$$\begin{aligned}\mathbb{E}[T|N]&=\sum_{i=0}^{n}\mathbb{E}[T|N=i]\mathbf{1}_{N=i}\\&=\sum_{i=0}^{n}3.5i\mathbf{1}_{N=i}\\&=3.5N.\end{aligned}$$

Next,
$$\mathbb{E}[T]=\mathbb{E}[T|N]=3.5\mathbb{E}[N]=3.5\cdot\frac{1}{2}n=1.75n.$$

Finally,

$$\mathbb{E}[NT|N] = N\mathbb{E}[T|N] = 3.5N^2.$$

Hence

$$\begin{aligned}
\mathbb{E}[NT] &= \mathbb{E}[\mathbb{E}[NT|N]] \\
&= \mathbb{E}[3.5N^2] \\
&= 3.5\sum_{i=0}^{n} i^2\mathbb{P}(N = i) \\
&= 3.5\sum_{i=0}^{n} i^2\binom{n}{i}\frac{1}{2}^i\frac{1}{2}^{n-i} \\
&= \frac{3.5}{2^n}\sum_{i=0}^{n} i^2\binom{n}{i}.
\end{aligned}$$

□

Exercise 0.13. *The aim of this question is to prove that, if X and Y are independent random variables and that both X and Y are in L^1 (i.e. $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$), then*

$$XY \in L^1 \quad \textit{and} \quad \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]. \tag{1}$$

In particular, if X and Y are independent random variables in L^2, then

$$Cov(X, Y) = 0 \quad \textit{and} \quad Var(X + Y) = Var(X) + Var(Y).$$

(Note that, without the independence assumption, in general $X, Y \in L^1$ does not *imply $XY \in L^1$.)*

(a) Prove (1) under the assumption that X and Y are non-negative and bounded, i.e. there exists some K such that $0 \le X \le K$ and $0 \le Y \le K$ almost surely.

(b) Prove (1) under the assumption that X and Y are non-negative. (Hint: set $X_n = \min(X, n)$ and $Y_n = \min(Y, n)$, for all $n \ge 1$.)

(c) Prove (1) in full generality.

Proof. (a)

$$\mathbb{E}[XY] \le \mathbb{E}[KY] = K\mathbb{E}[Y] < \infty$$

so

$$XY \in L^1.$$

Hence,

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\mathbb{E}[Y|X]] = \mathbb{E}[X\mathbb{E}[Y]] = \mathbb{E}[X]\mathbb{E}[Y].$$

(b) X_n is $\sigma(X)$-measurable $\forall n \in \mathbb{N}$, since

$$X_n = n\mathbf{1}_{\{X>n\}} + X\mathbf{1}_{\{X\le n\}}.$$

Hence

$$\sigma(X_n) \subseteq \sigma(X)\forall n$$

. Similarly,

$$\sigma(Y_n) \subseteq \sigma(Y) \forall n$$

and hence X_n and Y_n are independent $\forall n \in \mathbb{N}$. Hence, by (a),

$$\mathbb{E}[X_n Y_n] = \mathbb{E}[X_n]\mathbb{E}[Y_n] \forall n \in \mathbb{N}.$$

Furthermore,

$$0 \leq X_n \uparrow X \text{ and } 0 \leq Y_n \uparrow Y$$

so by the monotone convergence theorem

$$\mathbb{E}[XY] = \mathbb{E}[\lim_{n\to\infty} X_n Y_n] = \lim_{n\to\infty} \mathbb{E}[X_n Y_n] = \lim_{n\to\infty} \mathbb{E}[X_n]\mathbb{E}[Y_n] = \mathbb{E}[X]\mathbb{E}[Y].$$

(c) We have

$$X = X^+ - X^-, Y = Y^+ - Y^-$$

so

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[(X^+ - X^-)(Y^+ - Y^-)] \\
&= \mathbb{E}[X^+Y^+ - X^+Y^- - X^-Y^+ + X^-Y^-] \\
&= \mathbb{E}[X^+]\mathbb{E}[Y^+] - \mathbb{E}[X^+]\mathbb{E}[Y^-] - \mathbb{E}[X^-]\mathbb{E}[Y^+] + \mathbb{E}[X^-]\mathbb{E}[Y^-] \\
&= (\mathbb{E}[X^+] - \mathbb{E}[X^-])(\mathbb{E}[Y^+] - \mathbb{E}[Y^-]) \\
&= \mathbb{E}[X]\mathbb{E}[Y].
\end{aligned}$$

□

Exercise 0.14. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $\mathcal{F} \subset \mathcal{G}$ be two sub-σ-algebras of $\mathcal{A}$, and X a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$.*

(a) Prove that $\mathbb{E}[X\mathbb{E}[X \mid \mathcal{F}]] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}]^2]$.

(b) Prove that $\mathbb{E}[\mathbb{E}[X \mid \mathcal{F}]\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}]^2]$.

(c) Deduce from (i) and (ii) that

$$\mathbb{E}[(X - \mathbb{E}[X \mid \mathcal{G}])^2] + \mathbb{E}[(\mathbb{E}[X \mid \mathcal{G}] - \mathbb{E}[X \mid \mathcal{F}])^2] = \mathbb{E}[(X - \mathbb{E}[X \mid \mathcal{F}])^2].$$

(d) Recall that $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$. Similarly, we define, for all sub-σ-algebras $\mathcal{F} \subset \mathcal{A}$,

$$\mathrm{Var}(X \mid \mathcal{F}) := \mathbb{E}[(X - \mathbb{E}[X \mid \mathcal{F}])^2 \mid \mathcal{F}].$$

Taking $\mathcal{F} = \{\emptyset, \Omega\}$ in (iii), show that, for any sub-σ-algebra $\mathcal{G} \subset \mathcal{A}$,

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X \mid \mathcal{G})] + \mathrm{Var}(\mathbb{E}[X \mid \mathcal{G}]). \tag{2}$$

(Explain why $\mathcal{F} \subset \mathcal{G}$ for any sub-σ-algebra $\mathcal{G} \subset \mathcal{A}$.)

Proof. (a)

$$\begin{aligned}
\mathbb{E}[X\mathbb{E}[X|\mathcal{F}]] &= \mathbb{E}[\mathbb{E}[X\mathbb{E}[X|\mathcal{F}]|\mathcal{F}]] \\
&= \mathbb{E}[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{F}]] \\
&= \mathbb{E}[\mathbb{E}[X|\mathcal{F}]^2].
\end{aligned}$$

(b)

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X|\mathcal{F}]\mathbb{E}[X|\mathcal{G}]] &= \mathbb{E}[\mathbb{E}[\mathbb{E}[X|\mathcal{F}]X|\mathcal{G}]] \\ &= \mathbb{E}[\mathbb{E}[X|\mathcal{F}]X] \\ &= \mathbb{E}[\mathbb{E}[X|\mathcal{F}]^2].\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{E}[(X-\mathbb{E}[X \mid \mathcal{G}])^2]+\mathbb{E}[(\mathbb{E}[X \mid \mathcal{G}]-\mathbb{E}[X \mid \mathcal{F}])^2] &= \mathbb{E}[X^2-2X\mathbb{E}[X|\mathcal{G}]+2\mathbb{E}[X|\mathcal{G}]^2-2\mathbb{E}[X|\mathcal{G}]\mathbb{E}[X|\mathcal{F}]+\mathbb{E}[X|\mathcal{F}]^2] \\ &= \mathbb{E}[X^2-2\mathbb{E}[X|\mathcal{G}]^2+2\mathbb{E}[X|\mathcal{G}]^2-2\mathbb{E}[X|\mathcal{F}]^2+\mathbb{E}[X|\mathcal{F}]^2] \\ &= \mathbb{E}[X^2-\mathbb{E}[X|\mathcal{F}]^2] \\ &= \mathbb{E}[X^2-2X\mathbb{E}[X|\mathcal{F}]^2+\mathbb{E}[X|\mathcal{F}]^2] \\ &= \mathbb{E}[(X-\mathbb{E}[X|\mathcal{F}])^2].\end{aligned}$$

(d) If $\mathcal{F}=\{\emptyset,\Omega\}$ then $\mathbb{E}[X|\mathcal{F}]=\mathbb{E}[X]$ almost surely. Hence

$$\mathrm{Var}(X)=\mathbb{E}[(X-\mathbb{E}[X|\mathcal{F}])^2].$$

Furthermore,

$$\mathbb{E}[(X-\mathbb{E}[X|\mathcal{G}])^2]=\mathbb{E}[\mathbb{E}[(X-\mathbb{E}[X|\mathcal{G}])^2]|\mathcal{G}]=\mathbb{E}[\mathrm{Var}(X|\mathcal{G})]$$

and

$$\begin{aligned}\mathbb{E}[(\mathbb{E}[X \mid \mathcal{G}]-\mathbb{E}[X \mid \mathcal{F}])^2] &= \mathbb{E}[(\mathbb{E}[X \mid \mathcal{G}]-\mathbb{E}[X])^2] \\ &= \mathbb{E}[(\mathbb{E}[X \mid \mathcal{G}]-\mathbb{E}[\mathbb{E}[X|\mathcal{G}]])^2] \\ &= \mathrm{Var}(\mathbb{E}[X|\mathcal{G}]).\end{aligned}$$

Hence

$$\mathrm{Var}(X)=\mathbb{E}[\mathrm{Var}(X \mid \mathcal{G})]+\mathrm{Var}(\mathbb{E}[X \mid \mathcal{G}]).$$

□

Exercise 0.15. ***Martingales associated with the simple random walk.*** *Let $p\in(0,1)$ and $(S_n)_{n\geq 1}$ be the simple random walk with parameter p starting at 0, i.e.*

$$S_n=\sum_{i=1}^{n}X_i,$$

where $(X_i)_{i\geq 1}$ is a sequence of independent random variables satisfying

$$\mathbb{P}(X_i=1)=p \quad \textit{and} \quad \mathbb{P}(X_i=-1)=1-p \quad (\forall i\geq 1).$$

Let $(\mathcal{F}_n)_{n\geq 1}$ be the natural filtration of $(X_i)_{i\geq 1}$. For all $n\geq 0$, define

$$M_n:=S_n-n\alpha, \quad M_n':=\beta^{S_n}, \quad \textit{and} \quad M_n'':=S_n^2-n.$$

(a) Find α and β such that $(M_n)_{n\geq 1}$ and $(M_n')_{n\geq 1}$ are martingales with respect to $(\mathcal{F}_n)_{n\geq 1}$.

(b) Show that if $p = 1/2$, then $(M''_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.

Proof. Given any α, M_n is $\mathcal{F}_n$-measurable for all n. Furthermore,

$$\mathbb{E}[|M_n|] = \mathbb{E}[|S_n - n\alpha|] \leq \mathbb{E}[|S_n|] + n|\alpha| \leq n + n|\alpha| < \infty,$$

so M_n is integrable for every n and α. We now need

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n \text{ almost surely.}$$

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[S_{n+1} - (n+1)\alpha|\mathcal{F}_n] \\
&= \mathbb{E}\left[X_{n+1} + S_n - (n+1)\alpha \mid \mathcal{F}_n\right] \\
&= \mathbb{E}[X_{n+1}|\mathcal{F}_n] + S_n - (n+1)\alpha \\
&= \mathbb{E}[X_{n+1}] + S_n - (n+1)\alpha \\
&= 2p - 1 + S_n - (n+1)\alpha \\
&= M_n + 2p - 1 - \alpha \text{ almost surely.}
\end{aligned}$$

Hence M_n is a martingale when

$$2p - 1 - \alpha = 0 \iff \alpha = 2p - 1.$$

$f : \mathbb{R} \to \mathbb{R} : x \mapsto \beta^x$ is continuous $\forall \beta$ so M'_n is $\mathcal{F}_n$-measurable $\forall n, \beta$.

$$\mathbb{E}[|\beta^{S_n}|] \leq \mathbb{E}[|\beta^n|] < \infty$$

so M'_n is integrable.

$$\begin{aligned}
\mathbb{E}[M'_{n+1}|\mathcal{F}_n] &= \mathbb{E}[\beta^{S_{n+1}}|\mathcal{F}_n] \\
&= \mathbb{E}[\beta^{X_{n+1}}\beta^{S_n}|\mathcal{F}_n] \\
&= \beta^{S_n}\mathbb{E}[\beta^{X_{n+1}}|\mathcal{F}_n] \\
&= M'_n\mathbb{E}[\beta^{X_{n+1}}] \\
&= M'_n(p\beta + (1-p)\beta^{-1}) \\
&= M'_n.
\end{aligned}$$

Hence, we need

$$\begin{aligned}
&p\beta + (1-p)\beta^{-1} = 1 \\
\iff &p\beta^2 - \beta + (1-p) = 0 \\
\iff &\beta = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p} \\
\iff &\beta = \frac{1 \pm \sqrt{4p^2 - 4p + 1}}{2p} \\
\iff &\beta = \frac{1 \pm |2p - 1|}{2p}.
\end{aligned}$$

Hence either

$$\beta = \frac{1 + 2p - 1}{2p} = 1$$

or

$$\beta = \frac{1 - 2p + 1}{2p} = \frac{1-p}{p}.$$

(b) M''_n is integrable, since

$$\mathbb{E}[|M''_n|] \leq \mathbb{E}[|S_n^2|] + n \leq \mathbb{E}[|n^2|] + n < \infty.$$

M''_n is also clearly $\mathcal{F}_n$-measurable.

$$\begin{aligned}
\mathbb{E}[M''_{n+1}|\mathcal{F}_n] &= \mathbb{E}[S_{n+1}^2 - n - 1|\mathcal{F}_n] \\
&= \mathbb{E}[(S_n + X_{n+1})^2 - n - 1|\mathcal{F}_n] \\
&= \mathbb{E}[S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - n - 1|\mathcal{F}_n] \\
&= M''_n + \mathbb{E}[2S_nX_{n+1} + X_{n+1}^2 - 1|\mathcal{F}_n] \\
&= M''_n + 2S_n\mathbb{E}[X_{n+1}] + \mathbb{E}[X_{n+1}^2] - 1 \\
&= M''_n + 0 + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1)^2 - 1 \\
&= M''_n.
\end{aligned}$$

Hence M''_n is a martingale. □

Exercise 0.16. ***Martingales associated with a Galton-Watson branching process.*** *Assume that $(Z_n)_{n\geq 0}$ is the Galton-Watson branching process of offspring distribution $(p_i)_{i\geq 0}$. Assume that $\mu = \sum_{i=1}^{\infty} ip_i \in (0,\infty)$. Let $(\mathcal{F}_n)_{n\geq 1}$ be the natural filtration of $(Z_n)_{n\geq 0}$.*

(a) For all $n \geq 0$, set $M_n = \frac{Z_n}{\mu^n}$. Show that $(M_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.

(b) Assume that $\alpha > 0$ is such that $\sum_{i=0}^{\infty} \alpha^i p_i = \alpha$. For all $n \geq 1$, set $M'_n = \alpha^{Z_n}$. Show that $(M'_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.

Proof. (a) We have that

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = \mu Z_n \forall n \geq 0$$

so

$$\mathbb{E}\left[\frac{Z_{n+1}}{\mu^{n+1}}|\mathcal{F}_n\right] = \frac{\mu Z_n}{\mu^{n+1}} = \frac{Z_n}{\mu^n}$$

and hence $\frac{Z_n}{\mu^n}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.

(b) First note that

$$\mathbb{E}[\alpha^{\zeta_{n,i}}] = \sum_{i=0}^{\infty} \alpha^i p_i = \alpha.$$

M'_n is $\mathcal{F}_n$-measurable, since $f : \mathbb{R} \to \mathbb{R} : x \mapsto \alpha^x$ is continuous.

We now show that M'_n is integrable $\forall n$. Clearly $M'_1 = \alpha^{\zeta_{0,1}}$ is integrable. Now assume M'_n is integrable. We then have

$$\begin{aligned}
\mathbb{E}[|M'_{n+1}|] &= \mathbb{E}[\alpha^{Z_{n+1}}] \\
&= \mathbb{E}[\alpha^{\sum_{i=1}^{Z_n} \zeta_{n,i}}] \\
&= \mathbb{E}[\sum_{k=0}^{\infty} \mathbf{1}_{\{Z_n=k\}} \alpha^{\sum_{i=1}^{k} \zeta_{n,i}}] \\
&= \sum_{k=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{Z_n=k\}} \alpha^{\sum_{i=1}^{k} \zeta_{n,i}}].
\end{aligned}$$

For each k, we have

$$\alpha^{\sum_{i=1}^k \zeta_{n,i}} = \prod_{i=1}^k \alpha^{\zeta_{n,i}} \in L^1,$$

since $\alpha^{\zeta_{n,i}} \in L^1 \forall n, i$. Hence, by independence,

$$\begin{aligned}\sum_{k=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{Z_n=k\}} \alpha^{\sum_{i=1}^k \zeta_{n,i}}] &= \sum_{k=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{Z_n=k\}}] \mathbb{E}[\alpha^{\sum_{i=1}^k \zeta_{n,i}}] \\ &= \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k)\alpha^k \\ &= \mathbb{E}[\alpha^{Z_n}] < \infty.\end{aligned}$$

Hence M'_n is integrable $\forall n$ by induction. Finally,

$$\begin{aligned}\mathbb{E}[M'_{n+1}|\mathcal{F}_n] &= \mathbb{E}[\alpha^{Z_{n+1}}|\mathcal{F}_n] \\ &= \mathbb{E}[\alpha^{\sum_{i\geq 1} \zeta_{n,i} \mathbf{1}_{\{i \leq Z_n\}}}|\mathcal{F}_n] \\ &= \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbf{1}_{\{Z_n=k\}} \alpha^{\sum_{i=1}^k \zeta_{n,i}} |\mathcal{F}_n\right] \\ &= \sum_{k=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{Z_n=k\}} \alpha^{\sum_{i=1}^k \zeta_{n,i}}|\mathcal{F}_n] \\ &= \sum_{k=0}^{\infty} \mathbf{1}_{\{Z_n=k\}} \mathbb{E}[\alpha^{\sum_{i=1}^k \zeta_{n,i}}|\mathcal{F}_n] \\ &= \sum_{k=0}^{\infty} \mathbf{1}_{\{Z_n=k\}} \mathbb{E}[\prod_{i=1}^k \alpha^{\zeta_{n,i}}] \\ &= \sum_{k=0}^{\infty} \mathbf{1}_{\{Z_n=k\}} \alpha^k \\ &= \alpha^{Z_n} \\ &= M'_n \text{ almost surely.}\end{aligned}$$

Hence M'_n is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.

□

Exercise 0.17 (Generalised Pólya urn). *An urn initially contains $X_0 = x > 0$ grams of red powder and $Y_0 = y > 0$ grams of green powder. Sample $(V_i)_{i\geq 1}$, a sequence of non-negative i.i.d. random variables. At each time step $n + 1$ ($n \geq 0$), we pick a speck of powder uniformly at random from the urn and then return it to the urn together with V_{n+1} grams of powder of the same colour as that speck.*

More formally, let $(U_i)_{i\geq 1}$ be a sequence of i.i.d. Unif$[0, 1]$ *random variables, independent of $(V_i)_{i\geq 1}$. Define $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ by*

$$(X_{n+1}, Y_{n+1}) = \begin{cases} (\, X_n + V_{n+1},\, Y_n\,), & \text{if } U_{n+1} < \dfrac{X_n}{X_n + Y_n}, \\ (\, X_n,\, Y_n + V_{n+1}), & \text{if } U_{n+1} \geq \dfrac{X_n}{X_n + Y_n}. \end{cases}$$

For all $n \geq 0$, set

$$M_n \;=\; \frac{X_n}{X_n + Y_n}.$$

Show that $(M_n)_{n\geq 0}$ is a martingale with respect to the filtration

$$\mathcal{F}_n \;=\; \sigma\big(U_1, V_1, \ldots, U_n, V_n\big).$$

(Hint: first consider the case where the V_i are non-random.)

Proof. We show that X_n and Y_n are $\mathcal{F}_n$-measurable by induction. For $n = 0$, $X_0 = x$ and $Y_0 = y$ so $\sigma(X_0, Y_0) = \{\emptyset, \Omega\} \subseteq \mathcal{F}_0$. Now assume that X_k and Y_k are $\mathcal{F}_k$-measurable. Then

$$X_{k+1} = X_k + V_{k+1}\mathbf{1}_{\{U_{k+1} < \frac{X_k}{X_k + Y_k}\}}.$$

X_k is $\mathcal{F}_{k+1}$-measurable by the inductive hypothesis, V_{k+1} is $\mathcal{F}_{k+1}$-measurable by definition, and $\mathbf{1}_{\{U_{k+1} < \frac{X_k}{X_k + Y_k}\}}$ is $\mathcal{F}_{k+1}$ measurable since $\{U_{k+1} < \frac{X_k}{X_k+Y_k}\} \in \mathcal{F}_{k+1}$. Hence X_{k+1} is $\mathcal{F}_{k+1}$ measurable. Similarly Y_{k+1} is $\mathcal{F}_{k+1}$-measurable. Hence X_n and Y_n are both $\mathcal{F}_n$-measurable for all n by induction. hence, M_n is $\mathcal{F}_n$-measurable $\forall n$.

We now show that M_n is integrable $\forall n$. Clearly M_0 is integrable. Otherwise,

$$M_{n+1} = \frac{X_n + V_{n+1}\mathbf{1}_{\{U_{n+1} < \frac{X_n}{X_n + Y_n}\}}}{X_n + Y_n + V_{n+1}} < 1.$$

Hence M_n is integrable $\forall n$.

$$
\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\frac{X_n + V_{n+1}\mathbf{1}_{\{U_{n+1}<\frac{X_n}{X_n+Y_n}\}}}{X_n+Y_n+V_{n+1}}|\mathcal{F}_n\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{\infty}\mathbf{1}_{\{V_{n+1}=i\}}\frac{X_n + i\mathbf{1}_{\{U_{n+1}<\frac{X_n}{X_n+Y_n}\}}}{X_n+Y_n+i}|\mathcal{F}_n\right] \\
&= \sum_{i=0}^{\infty}\mathbb{E}\left[\mathbf{1}_{\{V_{n+1}=i\}}\frac{X_n + i\mathbf{1}_{\{U_{n+1}<\frac{X_n}{X_n+Y_n}\}}}{X_n+Y_n+i}|\mathcal{F}_n\right] \\
&= \sum_{i=0}^{\infty}\frac{\mathbb{E}[\mathbf{1}_{\{V_{n+1}=i\}}(X_n + i\mathbf{1}_{\{U_{n+1}<\frac{X_n}{X_n+Y_n}\}})|\mathcal{F}_n]}{X_n+Y_n+i} \\
&= \sum_{i=0}^{\infty}\frac{X_n\mathbb{E}[\mathbf{1}_{\{V_{n+1}=i\}}|\mathcal{F}_n] + \mathbb{E}[i\mathbf{1}_{\{V_{n+1}=i\}}\mathbf{1}_{\{U_{n+1}<\frac{X_n}{X_n+Y_n}\}}|\mathcal{F}_n]}{X_n+Y_n+i} \\
&= \sum_{i=0}^{\infty}\frac{X_n\mathbb{P}(V_{n+1}=i) + i\mathbb{E}[\mathbf{1}_{\{V_{n+1}=i\}}\mathbf{1}_{\{U_{n+1}<\frac{X_n}{X_n+Y_n}\}}|\mathcal{F}_n]}{X_n+Y_n+i} \\
&= \sum_{i=0}^{\infty}\frac{X_n\mathbb{P}(V_{n+1}=i) + i\mathbb{P}(V_{n+1}=i, U_{n+1}<\frac{X_n}{X_n+Y_n}|\mathcal{F}_n)}{X_n+Y_n+i} \\
&= \sum_{i=0}^{\infty}\frac{X_n\mathbb{P}(V_{n+1}=i) + i\mathbb{P}(V_{n+1}=i)M_n}{X_n+Y_n+i} \\
&= \sum_{i=0}^{\infty}\mathbb{P}(V_{n+1}=i)\frac{X_n+iM_n}{X_n+Y_n+i} \\
&= \sum_{i=0}^{\infty}\mathbb{P}(V_{n+1}=i)\frac{X_n+\frac{iX_n}{X_n+Y_n}}{X_n+Y_n+i} \\
&= \sum_{i=0}^{\infty}\mathbb{P}(V_{n+1}=i)\frac{\frac{X_n(X_n+Y_n+i)}{X_n+Y_n}}{X_n+Y_n+i} \\
&= \sum_{i=0}^{\infty}\mathbb{P}(V_{n+1}=i)M_n \\
&= M_n \text{ almost surely.}
\end{aligned}
$$

Hence M_n is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. □

Exercise 0.18. *(a) Assume that $(X_i)_{i\geq 1}$ is a sequence of random variables and $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ for all $n \geq 1$. Which of the following random variables are stopping times (use the convention $\min\emptyset = +\infty$)? Justify your answers.*

i. $T_1 = \min\{n \geq 1 : X_n \in [15, 20]\}$

ii. $T_2 = \min\{n \geq 1 : X_1 + 2X_2 + \cdots + nX_n \geq 120\}$

iii. $T_3 = 5$

iv. $T_4 = \min\{n \geq 2 : X_n = 8$ *and* $X_{n-1} = 6\}$

v. $T_5 = \min\{n \geq 1 : X_n = 8$ *and* $X_{n+1} = 6\}$

(b) Suppose S *and* T *are stopping times. Show that* $\min(S, T)$ *and* $S + T$ *are stopping times.*

Proof. (a) (i)

$$\{T_1 \leq n\} = \bigcup_{i=1}^{n} \{X_i \in [15, 20]\} \in \mathcal{F}_n$$

so T_1 is a stopping time.

(ii)

$$\{T_2 \leq n\} = \bigcup_{i=1}^{n} \left\{ \sum_{j=1}^{i} jX_j \geq 120 \right\} \in \mathcal{F}_n$$

so T_2 is a stopping time.

(iii)

$$\{T_3 = n\} \in \{\emptyset, \Omega\}$$

so T_3 is trivially a stopping time.

(iv)

$$\{T_4 = 1\} = \emptyset \in \mathcal{F}_1.$$

And for $n \geq 2$,

$$\{T_4 \leq n\} = \bigcup_{i=2}^{n} \{X_i = 8, X_{i-1} = 6\} \in \mathcal{F}_n.$$

Hence X_4 is a stopping time.

(v)

$$\{T_5 = 1\} = \{X_1 = 8, X_2 = 6\} \notin \mathcal{F}_1$$

so T_5 is not a stopping time.

(b)

$$\{\min(S, T) \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \mathcal{F}_n$$

so $\min(S, T)$ is a stopping time.

$$\{S + T = n\} = \bigcup_{i=1}^{n-1} \{S = i, T = n - i\} \in \mathcal{F}_n.$$

Hence $S + T$ is a stopping time.

□

Exercise 0.19. *An investor buys shares in a company. At the end of each six-month period, the shareholder will receive dividends of size* 0, 1, *or* 2 *per share, with probability* 1/6, 1/2, *and* 1/3, *respectively. Let* $(X_i)_{i \geq 1}$ *be the successive dividends (per share) received by the investor (assumed i.i.d.).*

The investor sells his shares at the first time when the dividend is of size 0 *(say, on the* T*-th dividend:* $X_T = 0$*). Let* $S = \sum_{i=1}^{T} X_i$ *be the total dividend income per share received by the investor before they sell. Calculate* $\mathbb{E}[S]$*.*

Proof.

$$T = \inf\{n \geq 1 : X_n = 0\},$$

which is clearly a stopping time. Furthermore,

$$\mathbb{E}[T] = \sum_{n=1}^{\infty} \mathbb{P}(T \geq n) = \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1} = 6 < \infty$$

and

$$\mathbb{E}[X_1] = \frac{1}{2} + \frac{2}{3} = \frac{7}{6} < \infty.$$

Hence Wald's equation applies to give

$$\mathbb{E}[S] = \frac{7}{6} \cdot 6 = 7.$$

□

Exercise 0.20. *Let $(S_n)_{n\geq 0}$ be the simple random walk with parameter $p \neq 1/2$. Let $a > 0$ and $b > 0$ be integers. Let*

$$T = \min\{n \geq 1 : S_n = a \text{ or } S_n = -b\}.$$

Find a formula for $\mathbb{E}[T]$.

You can use without a proof that $\mathbb{E}[T] < +\infty$. $\mathbb{P}(S_T = a) = \dfrac{\alpha^b - 1}{\alpha^{a+b} - 1}$, where $\alpha = \dfrac{1-p}{p}$.

(See Equation (8.5) in the lecture notes for the second claim. The first claim can be proved by adapting the proof of $\mathbb{E}[T] < +\infty$ in the symmetric case in Section 7.2.)

Proof. By Wald's equation,

$$\mathbb{E}[S_T] = \mathbb{E}[X_1]\mathbb{E}[T] = (2p-1)\mathbb{E}[T].$$

Furthermore,

$$\mathbb{E}[S_T] = a\mathbb{P}(S_T = a) - b\mathbb{P}(S_T = -b) = a\frac{\alpha^b - 1}{\alpha^{a+b} - 1} - b\left(1 - \frac{\alpha^b - 1}{\alpha^{a+b} - 1}\right).$$

Hence,

$$\mathbb{E}[T] = \frac{1}{2p-1}\left(a\frac{\alpha^b - 1}{\alpha^{a+b} - 1} - b\left(1 - \frac{\alpha^b - 1}{\alpha^{a+b} - 1}\right)\right).$$

□

Exercise 0.21. *The transmitter spaceship "Supermartingale" was struck by a meteorite and has started sending out random letters into space. Assume that the letters $X_1, X_2, ...$ sent after the strike are i.i.d., with each X_n uniformly distributed over the 26 letters of the alphabet. Let T denote the first time the transmitter sends the crucial sequence "SOS", i.e.*

$$T = \inf\{n \geq 3 : X_n = S, X_{n-1} = O, X_{n-2} = S\},$$

so that $\{T \leq 2\} = \emptyset$. Let $V_0 = 0$ and, for all $i \geq 1$, let $V_i = \min\{n > V_{i-1} : X_n = S\}$. Also set $W_i = V_i - V_{i-1}$ for all $i \geq 1$. Finally, for all $i \geq 1$, let Y_i denote the block of letters $(X_{V_{i-1}+1}, ..., X_{V_i})$, and set $T' = \min\{V_i : Y_i = OS\}$ be the number of letters up to the end of the first block to take the value OS.

(a) *Find* $\mathbb{E}[W_1]$ *and find* $\mathbb{P}(Y_1 = OS)$.

(b) *Let* N *be the total number of times the letter* S *is emitted up to and including time* T'. *Find* $\mathbb{E}[N]$, *justifying your answer briefly.*

(c) *Calculate* $\mathbb{E}[T']$.

(d) *Calculate* $\mathbb{E}[T]$.

Proof. (a)

$$\mathbb{E}[W_1] = \sum_{n=1}^{\infty} \mathbb{P}(W_1 \geq n) = \sum_{n=1}^{\infty} \left(\frac{25}{26}\right)^{n-1} = 26.$$

$$\mathbb{P}(Y_1 = \text{OS}) = \left(\frac{1}{26}\right)^2.$$

(b)

$$\begin{aligned}\mathbb{P}(N > n) &= \mathbb{P}(N > n | N > n-1)\mathbb{P}(N > n-1) \\ &= (1 - \mathbb{P}(Y_n = \text{OS}))\mathbb{P}(N > n-1) \\ &= (1 - \mathbb{P}(Y_1 = \text{OS}))\mathbb{P}(N > n-1) \\ &= \frac{675}{676}\mathbb{P}(N > n-1).\end{aligned}$$

Hence by induction

$$\mathbb{P}(N > n) = \left(\frac{675}{676}\right)^n$$

so

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}(N > n-1) = \sum_{n=1}^{\infty} \left(\frac{675}{676}\right)^{n-1} = 676.$$

(c)

$$T' = V_N = \sum_{i=1}^{N} W_i.$$

$W_1, W_2, ...$ are clearly i.i.d., so Wald's equation applies to give

$$\mathbb{E}[T'] = \mathbb{E}[W_1]\mathbb{E}[N] = 26 \cdot 676 = 26^3.$$

(d)

$$\begin{aligned}\mathbb{E}[T|N] &= \mathbb{E}[T\mathbf{1}_{\{N>1\}} + T\mathbf{1}_{\{N=1\}}|N] \\ &= \mathbb{E}[T'\mathbf{1}_{\{N>1\}} + T\mathbf{1}_{\{N=1\}}|N] \\ &= \mathbf{1}_{\{N>1\}}\mathbb{E}[T'|N] + \mathbf{1}_{\{N=1\}}\mathbb{E}[T|N=1] \\ &= \mathbf{1}_{\{N>1\}}\mathbb{E}[T'|N] + \mathbf{1}_{\{N=1\}}(\mathbb{E}[T'] + 2)\end{aligned}$$

so

$$
\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[\mathbf{1}_{\{N>1\}}\mathbb{E}[T'|N]] + (\mathbb{E}[T']+2)\mathbb{P}(N=1) \\
&= \mathbb{E}[T'\mathbf{1}_{\{N>1\}}] + (\mathbb{E}[T']+2)\mathbb{P}(Y_1=\mathrm{OS}) \\
&= \mathbb{E}[T'] - \mathbb{E}[T'\mathbf{1}_{\{N=1\}}] + (\mathbb{E}[T']+2)\mathbb{P}(Y_1=\mathrm{OS}) \\
&= \mathbb{E}[T'] + (\mathbb{E}[T']+2)\mathbb{P}(Y_1=\mathrm{OS}) - \mathbb{E}[T'|N=1]\mathbb{P}(Y_1=\mathrm{OS}) \\
&= \mathbb{E}[T'] + (\mathbb{E}[T']+2)\mathbb{P}(Y_1=\mathrm{OS}) - 2\mathbb{P}(Y_1=\mathrm{OS}) \\
&= 26^3 + \frac{(26^3+2)}{26^2} - \frac{2}{26^2} = 17602.
\end{aligned}
$$

□

Exercise 0.22. *Suppose you bet on successive tosses of a fair coin. You bet an amount 2^{n-1} on the n-th toss of the coin, so that, at the n-th toss you win or lose 2^{n-1} with equal probability. For all $n \geq 0$, let M_n be your total winnings after n tosses ($M_0 = 0$).*

(i) Show that $(M_n)_{n\geq 0}$ is a martingale with respect to an appropriate filtration.

(ii) Let T be the first time you win the bet; assume that you stop betting at time T. Show that $M_T = 1$ with probability 1, and deduce that the conclusion of the Optional Stopping Theorem fails. Which of the conditions of the OST fail?

Proof. (i) Let $(X_i)_{i\geq 0}$ be a sequence of i.i.d. random variables such that

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}.$$

Then

$$M_n = \sum_{i=1}^{n} 2^{i-1} X_i \forall n.$$

let $(\mathcal{F}_n)_{n\geq 0}$ be the natural filtration of $(X_i)_{i\geq 0}$. M_n is clearly $\mathcal{F}_n$-measurable $\forall n$. Furthermore,

$$\mathbb{E}[|M_n|] = \mathbb{E}\left[\left|\sum_{i=1}^{n} 2^{i-1} X_i\right|\right] \leq \left[\sum_{i=1}^{n} 2^{i-1}\right] < \infty.$$

Finally,

$$
\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[2^n X_{n+1} + M_n|\mathcal{F}_n] \\
&= 2^n \mathbb{E}[X_{n+1}|\mathcal{F}_n] + M_n \\
&= 2^n \mathbb{E}[X_{n+1}] + M_n \\
&= M_n \text{ almost surely.}
\end{aligned}
$$

Hence $(M_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.

(ii)

$$M_T = 2^{T-1} - \sum_{i=1}^{T-1} 2^{i-1} = 1.$$

This is different to $\mathbb{E}[M_0] = 0$. T is clearly a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$. Furthermore,

$$\mathbb{P}(T = \infty) = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$$

so $T < \infty$ almost surely.

$$\mathbb{E}[|M_T|] = 1 < \infty$$

so in order for the Optional stopping theorem to fail we need

$$\lim_{n\to\infty} \mathbb{E}[M_n \mathbf{1}_{\{T>n\}}] \neq 0.$$

□

Exercise 0.23. *At time 0, a bag contains one red marble and one green marble. At each time step $1, 2, \ldots,$ we draw a ball uniformly at random from the bag, and then replace it in the bag together with an additional marble of the same colour. Let R_n denote the number of red balls in the bag after n steps ($R_0 = 1$). Recall that $M_n = \frac{R_n}{n+2}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$ with $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$, for all $n \geq 1$, and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let T be the number of marbles drawn until the first draw of a red marble.*

(i) Show that T is an $(\mathcal{F}_n)_{n\geq 0}$-stopping time.

(ii) Show that $T < \infty$ almost surely.

(iii) Show that

$$\mathbb{E}\left[\frac{1}{T+2}\right] = \frac{1}{4}.$$

Proof. (i)

$$\{T > n\} = \bigcap_{i=0}^{n} \{R_i = 1\} \in \mathcal{F}_n \forall n$$

so T is an $(\mathcal{F}_n)_{n\geq 0}$-stopping time.

(ii)

$$\mathbb{P}(T = \infty) = \lim_{n\to\infty} \prod_{i=1}^{n} \frac{i}{(i+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

so $T < \infty$ almost surely.

(iii) M_n is bounded so the Optional stopping theorem applies, to give

$$\mathbb{E}[M_T] = \mathbb{E}[M_0].$$

Furthermore,

$$M_T = \frac{2}{T+2}$$

and

$$\mathbb{E}[M_0] = \frac{1}{2}$$

so

$$\mathbb{E}\left[\frac{1}{T+2}\right] = \frac{1}{4}.$$

□

Exercise 0.24. *For all $n \geq 0$, let Y_n be the assets (in GBP) of an insurance company after n years of trading. We assume that $Y_0 \in (0, \infty)$ is deterministic. Each year, the company receives a total income of P (a fixed amount) in premiums, and pays out a total of C_n, so*

$$Y_{n+1} = Y_n + P - C_{n+1} \quad \textit{for all } n \geq 0.$$

Assume that $(C_i)_{i\geq 1}$ is a sequence of i.i.d. $\mathcal{N}(\mu, \sigma^2)$ random variables with $\mu < P$, and therefore have moment generating function

$$\mathbb{E}[e^{\theta C_1}] = \exp(\mu\theta + \sigma^2\theta^2/2). \tag{1}$$

(i) Show that $M_n := e^{-\theta Y_n}$ is a martingale for a certain choice of $\theta > 0$.

(ii) We say that the company becomes bankrupt if there exists $n \geq 1$ such that $Y_n < 0$. Using the Elementary Stopping Lemma, show that

$$\mathbb{P}(Y_n < 0 \textit{ for some } n) \leq \exp(-2(P-\mu)Y_0/\sigma^2).$$

Proof. (i) Let $(\mathcal{F}_n)_{n\geq 0}$ be the natural filtration of $(Y_n)_{n\geq 0}$. Then M_n is clearly $\mathcal{F}_n$-measurable $\forall n$.

$$Y_n = Y_0 + nP - \sum_{i=1}^{n} C_i$$

so

$$M_n = e^{-\theta(Y_0 + nP - \sum_{i=1}^n C_i)} = e^{-\theta Y_0} \prod_{i=1}^{n} e^{\theta(C_i - P)}.$$

If

$$\mathbb{E}[e^{\theta(C_i - P)}] = 1 \forall i$$

then M_n is a martingale. Hence we need

$$\begin{aligned} &\exp(\mu\theta + \sigma^2\theta^2/2) = \exp(\theta P) \\ \iff &\mu\theta + \sigma^2\theta^2/2 = \theta P \\ \iff &\mu + \sigma^2\theta/2 = P \\ \iff &\theta = \frac{2}{\sigma^2}(P - \mu). \end{aligned}$$

(ii) Let $T := \inf\{n \geq 1 : Y_n < 0\}$. T is an $(\mathcal{F}_n)_{n\geq 0}$-stopping time, so by the Elementary stopping lemma

$$\mathbb{E}[M_{n\wedge T}] = \mathbb{E}[M_0] = \exp(-\theta Y_0) \forall n \geq 0.$$

Furthermore,

$$Y_n < 0 \iff M_n > 1$$

so

$$\mathbb{E}[M_{n\wedge T}] \geq \mathbb{E}[M_{n\wedge T}\mathbf{1}_{\{T\leq n\}}] = \mathbb{E}[M_T \mathbf{1}_{\{T\leq n\}}] \geq \mathbb{P}(T \leq n) \forall n.$$

Hence

$$\mathbb{P}(T < \infty) \leq \exp\left(-\frac{2}{\sigma^2}(P-\mu)Y_0\right).$$

□

Exercise 0.25. *Let $p \in (0,1) \setminus \{\frac{1}{2}\}$ and a, b be two integers such that $1 \leq a < b$. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = q = 1 - p$. Let $S_0 = a$, and $S_n = a + \sum_{i=1}^{n} X_i$. Finally, for all $n \geq 0$, set*

$$M_n = \left(\frac{q}{p}\right)^{S_n}.$$

Recall that $(M_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$, the natural filtration of $(S_n)_{n\geq 0}$. Recall that

$$T := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = b\}$$

is the stopping time when S first hits 0 or b.

(a) Show that $M_{T\wedge n}$ converges almost surely as $n \to +\infty$.

(b) Reasoning by contradiction, deduce that $\mathbb{P}(T < \infty) = 1$.

(c) Calculate $\mathbb{P}(S_T = 0)$.

(d) Show that $\mathbb{E}[S_{T\wedge n}] \to \mathbb{E}[S_T]$ as $n \uparrow \infty$.

(e) Show that, as $n \uparrow \infty$, $\mathbb{E}[T \wedge n] \to \mathbb{E}[T]$.

(f) Find an expression for $\mathbb{E}[T]$.

Proof. (a) $M_{n\wedge T}$ is a martingale, and by the elementary stopping lemma

$$\mathbb{E}[M_{n\wedge T}] = \mathbb{E}[M_0] = \left(\frac{q}{p}\right)^{a} \forall n.$$

Hence

$$\sup_{n\geq 0} \mathbb{E}[|M_{n\wedge T}|] < \infty$$

so by the martingale convergence theorem $M_{n\wedge T}$ converges almost surely.

(b) Suppose that $T = \infty$ and $M_{n\wedge T}$ converges for some $\omega \in \Omega$. If $T = \infty$ then $M_{n\wedge T} = M_n \forall n$ so

$$\lim_{n\to\infty} M_{n\wedge T} = \lim_{n\to\infty} \left(\frac{q}{p}\right)^{S_n}.$$

However, we also have

$$0 < S_n < b \forall n$$

so M_n takes finitely many values. There must be at least two values which M_n can take, since otherwise T would occur. Let ϵ be a quarter of the smallest difference between any two values which M_n takes. There then does not exist an $N \in \mathbb{N}$ such that $|M_n - M_\infty| < \epsilon \forall n > N$, which is a contradiction; hence the limit does not exist. Hence

$$\{T = \infty\} \cap \{M_{n\wedge T} \text{ converges}\} = \emptyset.$$

That is, for every $\omega \in \Omega$ such that $M_{n\wedge T}$ converges, $T < \infty$. Hence

$$\mathbb{P}(T = \infty) \leq \mathbb{P}(M_{n\wedge T} \text{ does not converge}) = 0$$

so

$$\mathbb{P}(T < \infty) = 1.$$

(c) The probability that $S_T = 0$ is the same as the probability that we hit $-a$ before $b-a$ for a random walk starting at 0. Hence

$$\begin{aligned}\mathbb{P}(S_T = 0) &= 1 - \frac{\alpha^a - 1}{\alpha^{b-a+a} - 1} \\ &= 1 - \frac{\alpha^a - 1}{\alpha^b - 1} \\ &= \frac{\alpha^b - \alpha^a}{\alpha^b - 1}\end{aligned}$$

where

$$\alpha := \frac{q}{p}.$$

(d) $|S_{n\wedge T}| \leq b \forall n$ almost surely. Furthermore, $T < \infty$ almost surely, so there exists an $N \in \mathbb{N}$ such that $S_{n\wedge T} = S_T \forall n > N$ almost surely, and hence $S_{n\wedge T} \to S_T$ as $n \uparrow \infty$ almost surely. Hence by the dominated convergence theorem

$$\mathbb{E}[S_{n\wedge T}] \to \mathbb{E}[S_T] \text{ as } n \uparrow \infty.$$

(e) T is finite almost surely so $n \wedge T \uparrow T$ almost surely. Hence by the monotone convergence theorem

$$\mathbb{E}[n \wedge T] \to \mathbb{E}[T] \text{ as } n \to \infty.$$

(f)

$$\mathbb{E}[S_T] = b\mathbb{P}(S_T = b) = b\frac{\alpha^a - 1}{\alpha^b - 1}.$$

$n \wedge T$ is a stopping time so by Wald's equation

$$\mathbb{E}[S_{n\wedge T}] - a = (p-q)\mathbb{E}[n \wedge T]$$

so

$$\lim_{n\to\infty} \mathbb{E}[S_{n\wedge T}] = a + (p-q)\mathbb{E}[T].$$

Furthermore,

$$\lim_{n\to\infty} \mathbb{E}[S_{n\wedge T}] = \mathbb{E}[S_T] = b\frac{\alpha^a - 1}{\alpha^b - 1}$$

and hence

$$\mathbb{E}[T] = \frac{1}{p-q}\left(b\frac{\alpha^a - 1}{\alpha^b - 1} - a\right).$$

□

Exercise 0.26. *Martingale formulation of Bellman's Optimality Principle. A player plays a game for N units of time. At each time $n \geq 1$, the winnings per unit stake are ε_n, where $(\varepsilon_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that, for all $n \geq 1$,*

$$\mathbb{P}(\varepsilon_n = +1) = p, \quad \mathbb{P}(\varepsilon_n = -1) = q, \quad \textit{where} \quad \frac{1}{2} < p = 1 - q < 1.$$

For all $n \geq 0$, we let Z_n denote the fortune of the player at time n. The player's stake at time n, C_n, must lie in $(0, Z_{n-1})$. The player's aim is to maximise the expected "interest rate" $\mathbb{E}[\log(Z_N/Z_0)]$. For all $n \geq 0$, we set

$$\mathcal{F}_n = \sigma(C_1, \varepsilon_1, \ldots, C_n, \varepsilon_n, C_{n+1}).$$

Let

$$\alpha := p\log p + q\log q + \log 2.$$

You can use without proof that $\log Z_n$ is integrable for all $n \geq 0$.

(a) Show that $(\log Z_n - n\alpha)_{n\geq 0}$ is a super-martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. (Hint: you may want to show that the function $x \mapsto p\log(1+x) + q\log(1-x)$ on $[0,1)$ reaches its maximum at $x = p - q$.)

(b) Deduce that $\mathbb{E}[\log(Z_N/Z_0)] \leq N\alpha$.

(c) Find a process $(C_n)_{n\geq 1}$ such that $(\log Z_n - n\alpha)_{n\geq 0}$ is an $(\mathcal{F}_n)_{n\geq 0}$-martingale. What is the best strategy for the player?

Proof. (a)

$$Z_n = Z_{n-1} + \epsilon_n C_n \forall n \geq 1.$$

$$Z_n = Z_0 + \sum_{i=1}^{n} \epsilon_i C_i \forall n$$

Z_n is clearly $\mathcal{F}_n$-measurable, and hence so is $\log Z_n - n\alpha$, as the composition of Z_n with a continuous function. Define $f : [0,1) \to \mathbb{R} : x \mapsto p\log(1+x) + q\log(1-x)$. $f'(x) = \frac{p}{1+x} - \frac{q}{1-x}$ is zero when

$$p(1-x) = q(1+x) \iff (p+q)x = p - q \iff x = \frac{p-q}{p+q} = p - q.$$

Furthermore,

$$f'(0) = p - q > 0$$

so $x = 0$ is not the maximum value of f, and

$$f'(x) < 0 \iff p(1-x) < q(1+x) \iff (p+q)x > p - q \iff x > p - q.$$

Hence $x = p - q$ is the maximum of f. Since

$$1 + (p - q) = 1 + p - (1 - p) = 2p$$

and

$$1 - (p - q) = 1 - p + 1 - p = 2q$$

It follows that

$$\begin{aligned} p\log(2p) + q\log(2q) &= p\log 2 + p\log p + q\log 2 + q\log q \\ &= p\log p + q\log q + \log 2 = \alpha \end{aligned}$$

is the maximum value of f.

$$\begin{aligned}
\mathbb{E}[\log Z_{n+1} - (n+1)\alpha | \mathcal{F}_n] &= \mathbb{E}[\log Z_{n+1} | \mathcal{F}_n] - (n+1)\alpha \\
&= \mathbb{E}[\log(Z_n + \epsilon_{n+1} C_{n+1}) | \mathcal{F}_n] - (n+1)\alpha \\
&= \mathbb{E}\left[\log\left(Z_n\left(1 + \frac{\epsilon_{n+1} C_{n+1}}{Z_n}\right)\right) | \mathcal{F}_n\right] - (n+1)\alpha \\
&= \log Z_n + \mathbb{E}\left[\log\left(1 + \frac{\epsilon_{n+1} C_{n+1}}{Z_n}\right) | \mathcal{F}_n\right] - (n+1)\alpha \\
&= \log Z_n + p \log\left(1 + \frac{C_{n+1}}{Z_n}\right) + q \log\left(1 - \frac{C_{n+1}}{Z_n}\right) - (n+1)\alpha \\
&\leq \log Z_n + \alpha - (n+1)\alpha \\
&= \log Z_n - n\alpha.
\end{aligned}$$

Hence $\log Z_n - n\alpha$ is a super-martingale.

(b)

$$\mathbb{E}[\log Z_{n+1} - (n+1)\alpha] = \mathbb{E}[\mathbb{E}[\log Z_{n+1} - (n+1)\alpha | \mathcal{F}_n]] \leq \mathbb{E}[\log Z_n - n\alpha] \forall n$$

so

$$\mathbb{E}[\log Z_{n+1}] \leq \mathbb{E}[\log Z_n] + \alpha \forall n.$$

Hence

$$\mathbb{E}[\log Z_n] \leq \mathbb{E}[\log Z_0] + N\alpha$$

so

$$\mathbb{E}[\log(Z_n/Z_0)] \leq N\alpha.$$

(c) On each turn, let

$$C_{n+1} = (p-q)Z_n.$$

□

Exercise 0.27. *Suppose that* $(M_n)_{n\geq 1}$ *is a martingale such that* $\mathbb{E}[M_n^2] < \infty$ *for all* $n \geq 1$. *We set* $M_0 = \mathbb{E}[M_1]$ *and* $\Delta_i = M_i - M_{i-1}$ *for all* $i \geq 1$. *Show that, for all* $i \neq j$, $\mathbb{E}[\Delta_i \Delta_j] = 0$, *and deduce that, for all* $n \geq 1$,

$$\mathrm{Var}(M_n) = \sum_{i=1}^{n} \mathbb{E}[\Delta_i^2]$$

(recall that, for any random variable X, *we define* $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ *whenever these expectations exist).*

Proof. Let $i > j$.

$$\mathbb{E}[\Delta_i | \mathcal{F}_{i-1}] = \mathbb{E}[M_i - M_{i-1} | \mathcal{F}_{i-1}] = M_{i-1} - M_{i-1} = 0$$

so

$$\begin{aligned}
\mathbb{E}[\Delta_i \Delta_j] &= \mathbb{E}[\mathbb{E}[\Delta_i \Delta_j | \mathcal{F}_{i-1}]] \\
&= \mathbb{E}[\Delta_j \mathbb{E}[\Delta_i | \mathcal{F}_{i-1}]] \\
&= \mathbb{E}[\Delta_j \cdot 0] = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{E}[\Delta_i^2] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} \Delta_i\right)^2\right] \\ &= \mathbb{E}[(M_n - M_0)^2] \\ &= \mathbb{E}[(M_n - \mathbb{E}[M_n])^2] \\ &= \operatorname{Var}(M_n).\end{aligned}$$

□

Exercise 0.28. *Assume that*

$$S_n = \pm 1 \pm \frac{1}{2} \pm \frac{1}{3} \pm \cdots \pm \frac{1}{n},$$

where the choice of plus or minus in each term is by independent fair coin tosses. That is, set

$$S_n = \sum_{i=1}^{n} \frac{X_i}{i}$$

where $(X_i)_{i\geq 1}$ is a sequence of i.i.d. random variables, such that

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

Show that $(S_n)_{n\geq 1}$ is bounded in L^2 and deduce that S_n converges to a limit almost surely as $n \to \infty$.

Proof. Let $(\mathcal{F}_n)_{n\geq 1}$ be the natural filtration of $(S_n)_{n\geq 1}$.

$$S_{n+1} = S_n + \frac{X_{n+1}}{n+1}$$

so

$$\begin{aligned}\mathbb{E}[S_{n+1}^2] &= \mathbb{E}\left[\mathbf{1}_{\{X_{n+1}=1\}}\left(S_n + \frac{1}{n+1}\right)^2 + \mathbf{1}_{\{X_{n+1}=-1\}}\left(S_n - \frac{1}{n+1}\right)^2\right] \\ &= \frac{1}{2}\mathbb{E}\left[\left(S_n + \frac{1}{n+1}\right)^2\right] + \frac{1}{2}\mathbb{E}\left[\left(S_n - \frac{1}{n+1}\right)^2\right] \\ &= \mathbb{E}[S_n^2] + \frac{1}{(n+1)^2}\end{aligned}$$

and hence

$$\mathbb{E}[S_n^2] = \sum_{i=1}^{n} \frac{1}{i^2} < \frac{\pi^2}{6} \forall n$$

so $(S_n)_{n\geq 1}$ is bounded in L^2. Furthermore, S_n is a sum of independent integrable random variables, each with expectation of 0, and hence is a martingale with respect to $(\mathcal{F}_n)$. Hence by the convergence of martingales bounded in L^2 there exists an almost surely finite random variable S_∞ such that $S_n \to S_\infty$ almost surely and in L^2 as $n \to \infty$. □

Exercise 0.29. *Let $(X_n)_{n\geq 1}$ be a sequence of independent (but not necessarily i.i.d.) random variables, and let $(\mathcal{F}_n)_{n\geq 1}$ be the natural filtration of $(X_n)_{n\geq 1}$. Assume that, for all $n \geq 1$, $\mathbb{E}[X_n] = 0$, and*

$$\sum_{n\geq 1} \frac{\mathrm{Var}(X_n)}{n^2} < \infty.$$

For all $n \geq 1$, we set

$$M_n = \sum_{i=1}^{n} \frac{X_i}{i} \quad \text{and} \quad S_n = \sum_{i=1}^{n} X_i.$$

(i) Show that $(M_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.

(ii) Show that $(M_n)_{n\geq 1}$ is bounded in L^2, i.e. $\sup_{n\geq 1} \mathbb{E}[|M_n|^2] < +\infty$.

(iii) Show that there exists a random variable M_∞ such that, almost surely, $\lim_{n\to+\infty} M_n = M_\infty$.

(iv) Show that, for all $n \geq 1$,

$$\frac{S_n}{n+1} = M_n - \frac{1}{n+1} \sum_{i=1}^{n} M_i.$$

(v) Prove that $S_n/n \to 0$ almost surely when $n \to +\infty$. (You can use without proof the following lemma, credited to Cesàro: if $(x_n)_{n\geq 1}$ is a sequence of real numbers such that $x_n \to x$ when $n \to +\infty$, then $\frac{1}{n}\sum_{i=1}^{n} x_i \to x$.)

Proof. (i) M_n is clearly $\mathcal{F}_n$-measurable and integrable. Finally,

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\frac{X_{n+1}}{n+1} + M_n|\mathcal{F}_n\right] \\
&= \frac{1}{n+1}\mathbb{E}[X_{n+1}] + M_n \\
&= M_n.
\end{aligned}$$

Hence $(M_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$.

(ii) We have that

$$\sum_{i=1}^{\infty} \frac{\mathbb{E}[X_i^2]}{i^2} < \infty.$$

Furthermore, by independence,

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j] = 0 \forall i \neq j.$$

Hence

$$\begin{aligned}
\mathbb{E}[|M_n|^2] &= \mathbb{E}\left[\left(\sum_{i=1}^{n}\frac{X_i}{i}\right)^2\right] \\
&= \sum_{1\le i,j\le n}\frac{\mathbb{E}[X_iX_j]}{ij} \\
&= \sum_{i=1}^{n}\frac{\mathbb{E}[X_i^2]}{i^2} \\
&\le \sum_{i=1}^{\infty}\frac{\mathbb{E}[X_i^2]}{i^2} < \infty \forall n
\end{aligned}$$

so $(M_n)_{n\ge1}$ is bounded in L^2.

(iii) Follows from the convergence theorem for martingales bounded in L^2.

(iv)

$$\begin{aligned}
M_n - \frac{1}{n+1}\sum_{i=1}^{n}M_i &= \sum_{i=1}^{n}\frac{X_i}{i} - \frac{1}{n+1}\sum_{i=1}^{n}\frac{(n-i+1)X_i}{i} \\
&= \frac{1}{n+1}\sum_{i=1}^{n}\frac{(n+1)X_i}{i} + \frac{1}{n+1}\sum_{i=1}^{n}\frac{(i-n-1)X_i}{i} \\
&= \frac{1}{n+1}\sum_{i=1}^{n}\frac{iX_i}{i} \\
&= \frac{S_n}{n+1}.
\end{aligned}$$

(v)

$$\frac{S_n}{n} = \frac{(n+1)M_n}{n} - \frac{1}{n}\sum_{i=1}^{n}M_i \to M_\infty - M_\infty = 0$$

as $n\to\infty$.

□

Exercise 0.30. *Assume that $(X_i)_{i\ge1}$ is a sequence of independent random variables such that, for all $i\ge1$, $\mathbb{E}[X_i]=0$ and $\sigma_i^2 := \mathbb{E}[X_i^2] < \infty$. For all $n\ge1$, let $M_n = \sum_{i=1}^{n}X_i$. Then $(M_n)_{n\ge0}$ is a martingale on the natural filtration of the sequence $(X_i)_{i\ge1}$. The two following questions (a) and (b) are independent.*

(a) Show that, for all $n\ge1$, $\langle M\rangle_n = \sum_{i=1}^{n-1}\sigma_i^2$ for $n\ge1$. (We recall that $\langle M\rangle$ is the previsible process in the Doob's decomposition of $(M_n^2)_{n\ge0}$.)

(b) Use Doob's martingale inequality to prove Kolmogorov's inequality which says that for any $a>0$, for any $n\ge1$,

$$\mathbb{P}\left(\max_{1\le i\le n}|M_i|\ge a\right) \le a^{-2}\sum_{i=1}^{n}\sigma_i^2.$$

Proof. (a) Let $(\mathcal{F}_n)_{n\geq 0}$ be the natural filtration of $(X_i)_{i\geq 1}$. Let $(A_n)_{n\geq 0}$ be defined by

$$A_n = \sum_{i=1}^{n-1} \sigma_i^2 \forall n \geq 0.$$

$(A_{n\geq 0})_{n\geq 0}$ is deterministic and hence $(\mathcal{F}_n)_{n\geq 0}$-predictable. Furthermore, $A_0 = 0$. It remains to show that $(M_n^2 - A_n)_{n\geq 0}$ is an $(\mathcal{F}_n)_{n\geq 0}$-martingale. $M_n^2 - A_n$ is clearly $\mathcal{F}_n$-measurable and integrable $\forall n$. Furthermore,

$$\begin{aligned}
\mathbb{E}[M_{n+1}^2 - A_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[(X_{n+1} + M_n)^2 - A_n - \sigma_n^2|\mathcal{F}_n\right] \\
&= \mathbb{E}[X_{n+1}^2] + 2M_n\mathbb{E}[X_{n+1}|\mathcal{F}_n] + M_n^2 - A_n - \sigma_n^2 \\
&= M_n^2 - A_n.
\end{aligned}$$

Hence $((M_n^2 - A_n)_{n\geq 0}, (A_n)_{n\geq 0})$ is the Doob decomposition of $(M_n^2)_{n\geq 0}$ so $\langle M\rangle_n = A_n \forall n$ almost surely.

(b) $(M_n^2)_{n\geq 0}$ is a non-negative submartingale so $\forall n \geq 1, a > 0$,

$$\mathbb{P}(\max_{1\leq i\leq n} |M_i| \geq a) = \mathbb{P}(\max_{1\leq i\leq n} M_i^2 \geq a^2) \leq \frac{\mathbb{E}[M_n^2]}{a^2} = a^{-2}\sum_{i=1}^{n} \sigma_i^2.$$

□

Exercise 0.31. *Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration. Assume that $(X_n)_{n\geq 0}$ is a non-negative super-martingale and T is a stopping time, both with respect to $(\mathcal{F}_n)_{n\geq 0}$. Recall that, by Corollary 7.7, $(X_{T\wedge n})_{n\geq 0}$ is an $(\mathcal{F}_n)_{n\geq 0}$ super-martingale.*

(a) Using Fatou's lemma, show that

$$\mathbb{E}[X_T \mathbf{1}_{T<\infty}] \leq \mathbb{E}[X_0].$$

(b) Deduce that $c\,\mathbb{P}(\sup_n X_n \geq c) \leq \mathbb{E}[X_0]$ for any $c \geq 0$.

Proof. (a) If $T < \infty$ then $\lim_{n\to\infty} X_{T\wedge n} = X_T$, and hence $\liminf_{n\to\infty} X_{T\wedge n} = X_T$. However, if $T = \infty$, then $X_{T\wedge n} = X_n \forall n$. Hence

$$\liminf_{n\to\infty} X_{T\wedge n} = X_T \mathbf{1}_{\{T<\infty\}} + \liminf_{n\to\infty} X_n \mathbf{1}_{\{T=\infty\}}.$$

Furthermore, since $(X_{T\wedge n})$ is a super-martingale,

$$\mathbb{E}[X_{T\wedge n}] \leq \mathbb{E}[X_{T\wedge 0}] = \mathbb{E}[X_0] \forall n$$

so

$$\liminf_{n\to\infty} \mathbb{E}[X_{T\wedge n}] \leq \mathbb{E}[X_0].$$

Hence, by Fatou's inequality,

$$\mathbb{E}[X_T \mathbf{1}_{\{T<\infty\}}] \leq \mathbb{E}[\liminf_{n\to\infty} X_{T\wedge n}] \leq \liminf_{n\to\infty} \mathbb{E}[X_{T\wedge n}] \leq \mathbb{E}[X_0].$$

(b) Let $T := \inf\{n \in \mathbb{N} : X_n \geq c\}$, which is clearly a stopping time. Then

$$\{T < \infty\} = \{\sup_n X_n \geq c\}.$$

Hence

$$c\mathbb{P}(\sup_n X_n \geq c) = c\mathbb{E}[\mathbf{1}_{\{T<\infty\}}] \leq \mathbb{E}[X_T \mathbf{1}_{\{T<\infty\}}] \leq \mathbb{E}[X_0].$$

□

Exercise 0.32. *The Moran model. Let $N, m \in \mathbb{N}$ with $m < N$. We consider a population of N individuals, out of which initially (in generation 0) m have the (genetic) type a, while the rest has type A. At every time-step (generation), one of the individuals (chosen uniformly at random) dies, and at the same time, a new one is born. Given all previous steps, the probability that the new individual has type a is equal to the proportion of type a in the previous generation. For all $n \geq 0$, let X_n be the proportion of individuals of type a in generation n. We also let $T := \inf\{n \in \mathbb{N} \mid X_n \in \{0, 1\}\}$.*

(a) Show that $(X_n)_{n\geq 0}$ is a martingale that converges almost surely as n tends to infinity.

(b) Show that $\mathbb{P}(T < \infty) = 1$.

(c) Calculate $\mathbb{P}(\lim_{n\uparrow\infty} X_n = 1)$.

Proof. (a) Let $(V_n)_{n\geq 1}$ be defined such that $V_n = 0$ if the person who dies in generation n has type A, and $V_n = -1$ if the person who dies in generation n has type a. Let $(W_n)_{n\geq 1}$ be defined such that $W_n = 1$ if the person born in generation n has type a, and $W_n = 0$ if the person born in generation n has type A. Then

$$X_{n+1} = \frac{V_n + W_n + NX_n}{N} = \frac{V_n + W_n}{N} + X_n.$$

Let $(\mathcal{F}_n)_{n\geq 0}$ be the natural filtration of $(X_n)_{n\geq 0}$. Then $\forall n$, X_n is $\mathcal{F}_n$-measurable by definition, and is clearly integrable. Finally,

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \frac{1}{N}\mathbb{E}[V_n + W_n|\mathcal{F}_n] + X_n \\
&= \frac{-\mathbb{P}(V_n = -1|\mathcal{F}_n) + \mathbb{P}(W_n = 1|\mathcal{F}_n)}{N} + X_n \\
&= \frac{-X_n + X_n}{N} + X_n \\
&= X_n \text{ almost surely.}
\end{aligned}$$

Hence $(X_n)_{n\geq 0}$ is an $(\mathcal{F}_n)_{n\geq 0}$-martingale. Furthermore, $(X_n)_{n\geq 0}$ is non-negative, so by the martingale convergence theorem there exists a random variable X_∞ such that $X_n \to X_\infty$ as $n \to \infty$ almost surely.

(b) Since X_n takes only values in $\{0, \frac{1}{N}, \frac{2}{N}, ..., 1\}$, X_n converges if and only if X_n is eventually constant. If $X_k = 0$ then $X_n = 0 \forall n > k$ almost surely, and if $X_k = 1$ then $X_n = 1 \forall n > k$

almost surely. However, if $X_k \notin \{0,1\}$ then $\mathbb{P}(X_n = X_k \forall n > k) = 0$ and hence

$$\begin{aligned}
\mathbb{P}(X_n \text{ converges}|T = \infty) &= \mathbb{P}\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\{X_n = X_k\}|T = \infty\right) \\
&\leq \sum_{k=1}^{\infty}\mathbb{P}\left(\bigcap_{n=k}^{\infty}\{X_n = X_k\}|T = \infty\right) \\
&= 0
\end{aligned}$$

so X_n does not converge almost surely. Suppose $\mathbb{P}(T = \infty) > 1$. Then

$$\begin{aligned}
1 = \mathbb{P}(X_n \text{ converges}) &= \mathbb{P}(X_n \text{ converges}|T < \infty)\mathbb{P}(T < \infty) + \mathbb{P}(X_n \text{ converges}|T = \infty)\mathbb{P}(T = \infty) \\
&= \mathbb{P}(X_n \text{ converges}|T < \infty)\mathbb{P}(T < \infty) \\
&= \mathbb{P}(X_n \text{ converges}|T < \infty)(1 - \mathbb{P}(T = \infty)) \\
&< 1;\ \text{a contradiction.}
\end{aligned}$$

Hence $\mathbb{P}(T < \infty) = 1$.

(c) $T < \infty$ almost surely and $(X_n)_{n\geq 0}$ is bounded, so by the optional stopping theorem

$$\mathbb{E}[X_T] = \mathbb{E}[X_0] = \frac{m}{N}.$$

Since X_n converges almost surely,

$$\mathbb{E}[X_T] = 1 \cdot \mathbb{P}(\lim_{n\to\infty} X_n = 1) + 0 \cdot \mathbb{P}(\lim_{n\to\infty} X_n = 0),$$

and hence

$$\mathbb{P}(\lim_{n\to\infty} X_n = 1) = \frac{m}{N}.$$

□