Representation Theory Solutions

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November 2024

Exercise 0.1. *Consider a basis of V to be a linear isomorphism $\beta : \mathbb{F}^n \to V$ (where $n = \dim V$). Associating to a matrix $A \in \mathrm{GL}(n, \mathbb{F})$ the linear operator $\theta_A : \mathbb{F}^n \to \mathbb{F}^n$, show that the map*

$$\Phi : \mathrm{GL}(n, \mathbb{F}) \to \mathrm{GL}(V) : A \mapsto \beta\theta_A\beta^{-1}$$

is an isomorphism of groups.

Proof. Φ is well-defined because the composition of isomorphisms is an isomorphism so $\beta\theta_A\beta^{-1} \in GL(V) \forall A \in GL(n, \mathbb{F})$. $\Phi(AB) = \beta\theta_{AB}\beta^{-1} = \beta\theta_A\beta^{-1}\beta\theta_B\beta^{-1} = \Phi(A)\Phi(B)$ so Φ is a homomorphism. Now let $\Phi(A) = \mathrm{Id}_V$. Then $\beta\theta_A\beta^{-1} = \mathrm{Id}_V \implies \theta_A\beta^{-1} = \beta^{-1}\mathrm{Id}_V \implies \theta_A = \beta^{-1}\beta = \mathrm{Id}_{\mathbb{F}^n} \implies A = I$. Thus Φ is injective. Given a $\phi \in GL(V)$ let A be the matrix representing ϕ with respect to β. Then $\Phi(A) = \phi$. Thus Φ is surjective so is an isomorphism of groups. $\square$

Exercise 0.2. *Let $\phi : G \to \mathrm{GL}(V)$ be a representation of G. Show that the map*

$$\alpha : G \times V \to V : (g, v) \mapsto g \cdot v = \phi(g)(v)$$

is a linear G-action.

Proof. Let $g, h \in G, v \in V$. Then $(gh) \cdot v = \phi(gh)(v) = \phi(g)(\phi(h)(v)) = \phi(g)(h \cdot v) = g \cdot (h \cdot v)$. $e \cdot v = \phi(e)(v) = \mathrm{Id}_V(v) = v$. Thus α is a G-action. Let $\lambda, \mu \in \mathbb{F}$ and $v, w \in V$. Then $g \cdot (\lambda v + \mu w) = \phi(g)(\lambda v + \mu w) = \lambda\phi(g)v + \mu\phi(g)w = \lambda(g \cdot v) + \mu(g \cdot w)$. Thus α is a linear G-action. $\square$

Exercise 0.3. *Let $\alpha : G \times X \to X : (g, x) \mapsto g \cdot x$ be a G-action and $\tilde{\alpha} : G \times \mathbb{F}X \to \mathbb{F}X : (g, f) \mapsto g \cdot f$ be its linearisation. Let $\{\delta_x : x \in X\}$ be the standard basis of $\mathbb{F}X$.*
Show that $g \cdot \delta_x = \delta_{g \cdot x}$. Deduce that the linearised action can be characterised by

$$g \cdot \sum_{x \in X} \lambda_x \delta_x = \sum_{x \in X} \lambda_x \delta_{g \cdot x}.$$

Proof. $(g \cdot \delta_x)(y) = \delta_x(g^{-1} \cdot y)$ which is 1 iff $g^{-1} \cdot y = x \iff y = g \cdot x$ and 0 otherwise. Thus $g \cdot \delta_x = \delta_{g \cdot x}$. By linearity, $g \cdot \sum_{x \in X} \lambda_x \delta_x = \sum_{x \in X} \lambda_x g \cdot \delta_x = \sum_{x \in X} \lambda_x \delta_{g \cdot x}$ $\square$

Exercise 0.4. *Find n different degree 1 representations of $\mathbb{Z}_n$ over $\mathbb{C}$ and determine which are faithful.*

Proof. $\rho_k : \mathbb{Z}_n \to \mathbb{C}^* : a \mapsto \omega^{ak}$ where $\omega = e^{\frac{2\pi i}{n}}$ and $0 \leq k \leq n - 1$ are n degree 1 representations. They're homomorphisms since $\rho_k(a + b) = \omega^{(a+b)k} = \omega^{ak}\omega^{bk} = \rho_k(a)\rho_k(b)$. ρ_k is faithful when its image is the set of all n'th roots of unity which occurs when ω^k has order n which occurs when k is coprime to n. $\square$

Exercise 0.5. *Show that the only finite subgroups of the group $\mathbb{C}^*$ are the cyclic groups of nth roots of unity, for each positive integer n.*

Proof. Let H be a subgroup of $\mathbb{C}^*$ of order n and let $h \in H$. Then $h^n = 1$ by Lagrange's theorem so h is an nth root of unity. Furthermore, since H has order n and only comprises nth roots of unity, H must then contain all nth roots of unity and so be cyclic, generated by $e^{\frac{2\pi i}{n}}$. □

Exercise 0.6. *The dihedral group D_{2n} may be characterised as a group of order $2n$ generated by two elements a and b, subject to the relations $a^n = b^2 = 1$ and $ba = a^{-1}b$.*

(i) Show that any element of D_{2n} can be written uniquely in the form $a^i b^j$ where $0 \le i < n$ and $0 \le j < 2$.

(ii) Now let H be some other group with $h, k \in H$. What are necessary and sufficient conditions for there to exist a homomorphism $\theta : D_{2n} \to H$ with $\theta(a) = h$ and $\theta(b) = k$? Show that, then, such a homomorphism is unique.

(iii) Show that D_{2n} has a representation $\rho : D_{2n} \to GL(1, \mathbb{R})$ with $\rho(a) = 1$ and $\rho(b) = -1$. What is the geometric significance of this representation?

(iv) Write down a faithful matrix representation of D_8 of degree 2.

Proof. (i) Let w be an element of D_{2n} written as $x_1^{y_1} ... x_n^{y_m}$ for $x_i \in \{a, b\}, y_i \in \mathbb{Z}$ where $y_i = 1$ if $x_i = b$. Suppose that there is some $x_i = b$ with $i \neq m$. Then $x_{i+1} = a$ so we can swap b y_{i+1} times with the rightwards a so that $w = x_1^{y_1} ... x_{i+1}^{-y_{i+1}} b x_{i+2}^{y_{i+2}} ... x_m^{y_m}$. Again simplify if possible if $x_{i+2} = b$. Repeating this process will then give $w = a^x b^j$ where $0 \le j < 2$. Then let $i = x + ln$ where $l \in \mathbb{Z}$ is such that i is in the desired range. This shows that any element of D_{2n} can be written in the desired form. To show uniqueness, note that D_{2n} has order $2n$, each of the form $a^i b^j$ with i and j in the given ranges, and there are at most $2n$ possible elements given by $a^i b^j$ so we must have uniqueness in order to cover all $2n$ elements.

(ii) We must have $h^n = b^2 = 1$ and $kh = h^{-1}k$ as a necessary condition. θ is then given as $\theta(a^i b^j) = h^i k^j$. To show that θ is a homomorphism, let $v = a^i$ and $w = a^x b^y$. Then $\theta(vw) = \theta(a^i a^x b^y) = \theta(a^{i+x} b^y) = h^{i+x} k^y = h^i h^x k^y = \theta(v)\theta(w)$. Now let $v = a^i b$. Then $\theta(vw) = \theta(a^i b a^x b^y) = \theta(a^{i-x} b^{y+1}) = h^{i-x} k^{y+1} = h^i k h^x k^y = \theta(v)\theta(w)$. Thus the necessary conditions are also sufficient conditions for there to be a homomorphism. The homomorphism is also completely determined by its values in a and b so is unique.

(iii) $a^n = (-1)^2 = 1$ and $(-1)1 = 1^{-1}(-1)$ so there exists such a representation. The representation encodes whether or not a reflection occurred on a regular n-gon.

(iv) let $\rho(a) = \begin{pmatrix} \cos(\frac{2\pi}{8}) & -\sin(\frac{2\pi}{8}) \\ \sin(\frac{2\pi}{8}) & \cos(\frac{2\pi}{8}) \end{pmatrix}$ and let $\rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\rho(a^i) = \begin{pmatrix} \cos(\frac{2\pi i}{8}) & -\sin(\frac{2\pi i}{8}) \\ \sin(\frac{2\pi i}{8}) & \cos(\frac{2\pi i}{8}) \end{pmatrix} = I \iff i \in 8\mathbb{Z} \iff a^i = 1.$$

$$\rho(a^i b) = \begin{pmatrix} \cos(\frac{2\pi i}{8}) & \sin(\frac{2\pi i}{8}) \\ \sin(\frac{2\pi i}{8}) & -\cos(\frac{2\pi i}{8}) \end{pmatrix} \text{ which is never } I.$$

This ρ has trivial kernel so is faithful.

□

Exercise 0.7. *The quaternion group Q_8 is the subgroup of $GL(2,\mathbb{C})$ generated by the matrices*

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(i) Show that $A^4 = 1$, $A^2 = B^2$ and $BAB^{-1} = A^{-1}$, and conclude that Q_8 is non-abelian.

(ii) Show that any element of Q_8 can be written uniquely in the form A^iB^j with $0 \le i < 4$ and $0 \le j < 2$. Thus confirm that Q_8 has order 8.

(iii) Is Q_8 isomorphic to D_8?

Proof. (i) $BA = A^{-1}B \neq AB$.

(ii) Let $W = X_1^{Y_1}...X_n^{Y_n}$ where X_i is A or B. Suppose that there is an i such that $X_i = B$ and $X_{i+1} = A$. Then $W = X_1^{Y_1}...X_{i+1}^{-Y_{i+1}}X_i^{Y_i}...X_n^{Y_n}$. Repeat this until $W = A^pB^q$. Then let $r = p + 4k$ such that $0 \le r < 4$ and let $s = q + 4k$ such that $0 \le s < 4$. Then $W = A^rB^s$. If $s = 0$ or $s = 1$, let $i = r$ and $j = s$. If $s = 2$, let $i = r + 2 \bmod 4$ and $j = 0$. And if $s = 3$, let $i = r + 2 \bmod 4$ and let $j = 1$. Then $W = A^iB^j$ with i and j in the desired ranges.

For uniqueness, it can be shown that A and B can generate 8 distinct values, and each can be written as one of the 8 possibilities of A^iB^j so there can be no repetition so we have uniqueness.

(iii) No. the elements of Q_8 has orders $1, 4, 2, 4, 4, 4, 4, 4$ whereas the elements of D_8 have orders $1, 4, 2, 4, 2$,etc. The groups then have different numbers of elements of order 2 so can't be isomorphic.

□

Exercise 0.8. *Show that two matrix representations of degree one, $\rho_1 : G \to GL(1,\mathbb{F})$ and $\rho_2 : G \to GL(1,\mathbb{F})$ are isomorphic if and only if $\rho_1 = \rho_2$.*

Proof. ($\Longleftarrow$) Trivial. Take $\mathrm{Id}_{\mathbb{F}}$ as the G-linear map.

($\Longrightarrow$) Let $\theta : \mathbb{F} \to \mathbb{F} : v \mapsto \lambda v$ for $\lambda \in \mathbb{F}^*$ be a G-linear isomorphism. Then $\lambda\rho_1(g) = \theta\rho_1(g) = \rho_2(g)\theta = \lambda\rho_2(g) \implies \rho_1(g) = \rho_2(g)\forall g \in G$ so $\rho_1 = \rho_2$. □

Exercise 0.9. *Consider the real matrix representation of $G = \mathbb{Z}_3$ given by*

$$\rho : \mathbb{Z}_3 \to \mathrm{GL}(2,\mathbb{R}) : k \mapsto \begin{pmatrix} \cos\frac{2\pi k}{3} & -\sin\frac{2\pi k}{3} \\ \sin\frac{2\pi k}{3} & \cos\frac{2\pi k}{3} \end{pmatrix}.$$

Show that the corresponding G-module $V = \mathbb{R}^2$ is irreducible.
On the other hand, if we use the same matrices to define a complex representation

$$\rho_{\mathbb{C}} : \mathbb{Z}_3 \to \mathrm{GL}(2,\mathbb{C}),$$

show that the corresponding G-module $V_{\mathbb{C}} = \mathbb{C}^2$ is not irreducible.

Proof. Let $W \neq \{0\}$ be a G-submodule of V. Let $0 \neq x \in W$. Then $\rho(1)(x)$ rotates x by $\frac{2\pi}{3}$ radians and so x and $\rho(1)(x)$ are linearly independent. Thus $\dim(W) = 2$ so $W = V$. Thus V is irreducible.

Let $W = \{(\lambda, \lambda i), \lambda \in \mathbb{C}\}$. Given $\lambda \in \mathbb{C}^*$, $\rho(1)(\lambda, \lambda i) = \lambda(\cos(\frac{2\pi}{3}) - i\sin(\frac{2\pi}{3}), \sin(\frac{2\pi}{3}) + i\cos(\frac{2\pi}{3})) = \lambda(\cos(\frac{2\pi}{3}) - i\sin(\frac{2\pi}{3}), (\cos(\frac{2\pi}{3}) - i\sin(\frac{2\pi}{3}))i) = (\cos(\frac{2\pi}{3}) - i\sin(\frac{2\pi}{3}))(\lambda, \lambda i) \in W$. Similarly, $\rho(2)(\lambda, \lambda i) \in W$ so W is a G-submodule. $\dim(W) = 1 < \dim(\mathbb{C}^2) = 2$ so $W \neq V_{\mathbb{C}}$. Thus $V_{\mathbb{C}}$ is not irreducible. □

Exercise 0.10. *If G acts on a non-empty set X, then the linearisation $V = \mathbb{F}X$ contains two G-submodules:*

$$W_0 = \{f \in \mathbb{F}X : \sum_{x \in X} f(x) = 0\}, \quad W_1 = \{f \in \mathbb{F}X : f \text{ is constant}\}.$$

Show that, if $\operatorname{char} \mathbb{F}$ *does not divide $|X|$, then V is a direct sum $V = W_0 \oplus W_1$. What happens if* $\operatorname{char} \mathbb{F}$ *does divide $|X|$?*

Proof. Let $f \in W_0 \cap W_1$. Then $f(x) = c \forall x \in X$ and $c|X| = 0$. $|X| \neq 0$ so $c = 0$. Thus $W_0 \cap W_1 = \{0\}$. Let $f \in V$. Let $c = \sum_{x \in X} f(x)$. If $c = 0$, then $f \in W_0$. Otherwise, define $g \in W_0$ by $g(x) = f(x) - \frac{c}{|X|}$ and $h \in W_1$ by $h(x) = \frac{c}{|X|}$. Then $f = g + h$. Thus $V = W_0 \oplus W_1$.

If char $\mathbb{F}$ divides $|X|$ then $W_1 \subseteq W_0$. □

Exercise 0.11. *Let V and W be G-modules over a field $\mathbb{F}$ and recall that $\operatorname{Hom}_{\mathbb{F}}(V, W)$ denotes the vector space of all linear maps from V to W.*

(a) Show that $\operatorname{Hom}_{\mathbb{F}}(V, W)$ becomes a G-module if we define $g \cdot T$ by

$$(g \cdot T)(v) = g \cdot \big(T(g^{-1} \cdot v)\big),$$

for $T \in \operatorname{Hom}_{\mathbb{F}}(V, W)$, $g \in G$, and $v \in V$.

(b) The dual of V is $V^* = \operatorname{Hom}_{\mathbb{F}}(V, \mathbb{F})$. Show that V^* becomes a G-module if we define $g \cdot f$ by

$$(g \cdot f)(v) = f(g^{-1} \cdot v),$$

for $f \in V^*$, $g \in G$, and $v \in V$.

Proof. (a) Let $g, h \in G$. Then $((gh) \cdot T)(v) = (gh) \cdot (T((gh)^{-1} \cdot v)) = g \cdot h \cdot (T(h^{-1} \cdot g^{-1} \cdot v)) = (g \cdot h \cdot T)(v)$.

Clearly $e \cdot T = T$.

$(g \cdot (\lambda T + \mu F))(v) = g \cdot ((\lambda T + \mu F)(g^{-1} \cdot v)) = g \cdot (\lambda T(g^{-1} \cdot v) + \mu F(g^{-1} \cdot v)) = \lambda(g \cdot T(g^{-1} \cdot v)) + \mu(g \cdot F(g^{-1} \cdot v)) = \lambda(g \cdot T)(v) + \mu(g \cdot F(v))$.

(b) Consider $\mathbb{F}$ to be a G-module equipped with the trivial action. The result then follows from part (a).

□

Exercise 0.12. *Let V be a G-module and define $V^G = \{v \in V : g \cdot v = v \ \forall g \in G\}$.*

(a) Show that V^G is a G-submodule of V.

(b) If U and V are G-modules and $\operatorname{Hom}_F(U, V)$ is made into a G-module as in Exercise 2.4(a), show that $\operatorname{Hom}_F(U, V)^G = \operatorname{Hom}_G(U, V)$, the subspace of G-linear maps.

Proof. (a) Let $v, w \in V^G$ and let $\lambda, \mu \in \mathbb{F}$. Then given any $g \in G$, $g \cdot (\lambda v + \mu w) = \lambda(g \cdot v) + \mu(g \cdot w) = \lambda v + \mu w$. Thus V^G is a linear subspace of V. Furthermore, $h \cdot (g \cdot v) = h \cdot v = v \forall h \in G$ so $g \cdot v \in V^G$. Thus V^G is closed under the group action so is a G-submodule of V.

(b) Let $\theta \in \operatorname{Hom}_{\mathbb{F}}(U, V)^G$. Let $u \in U, g \in G$. Then $g^{-1} \cdot \theta = \theta$ so $\theta(u) = g^{-1} \cdot (\theta((g^{-1})^{-1} \cdot u)) \implies g \cdot \theta(u) = \theta(g \cdot u)$. Thus $\operatorname{Hom}_{\mathbb{F}}(U, V)^G \subseteq \operatorname{Hom}_G(U, V)$. Now let $\theta \in \operatorname{Hom}_G(U, V)$. Then $\theta(g^{-1} \cdot u) = g^{-1} \cdot \theta(u) \implies (g \cdot \theta)(u) = \theta(u)$. Thus $\operatorname{Hom}_{\mathbb{F}}(U, V)^G = \operatorname{Hom}_G(U, V)$.

□

Exercise 0.13. *Let $V = U \oplus W$ be a direct sum of G-modules. Show that for any G-module X*

$$\mathrm{Hom}_G(V, X) \cong \mathrm{Hom}_G(U, X) \oplus \mathrm{Hom}_G(W, X)$$

and thus

$$\dim \mathrm{Hom}_G(V, X) = \dim \mathrm{Hom}_G(U, X) + \dim \mathrm{Hom}_G(W, X).$$

Proof. We have a G-linear projection $\pi : V \to V : u + w \mapsto w$. Now consider $\Psi : \mathrm{Hom}_G(V, X) \to \mathrm{Hom}_G(V, X) : \theta \mapsto \theta \circ \pi$. Given $\lambda, \mu \in \mathbb{F}, \theta, \phi \in \mathrm{Hom}_G(V, X)$ we have $\Psi(\lambda\theta + \mu\phi) = (\lambda\theta + \mu\phi) \circ \pi = \lambda\theta \circ \pi + \mu\phi \circ \pi = \lambda\Psi(\theta) + \mu\Psi(\phi)$ so Ψ is linear. Furthermore, $\Psi^2(\theta) = \theta \circ \pi^2 = \theta \circ \pi = \Psi(\theta)$ so Ψ is a projection. $\theta \in \mathrm{Ker}\ \Psi \iff \theta \circ \pi = 0 \iff \theta_{|W} = 0$ so $\mathrm{Ker}\ \Psi \cong \mathrm{Hom}_G(U, X)$. Also, $\theta \in \mathrm{Im}\ \Psi \iff \exists \phi \in \mathrm{Hom}_G(V, W) : \theta = \phi \circ \pi \iff \theta_{|U} = 0$ so $\mathrm{Im}\ \Psi \cong \mathrm{Hom}_G(W, X)$. $\mathrm{Hom}_G(V, X) = \mathrm{Ker}\ \Psi \oplus \mathrm{Im}\ \Psi$ so $\mathrm{Hom}_G(V, X) \cong \mathrm{Hom}_G(U, X) \oplus \mathrm{Hom}_G(W, X)$. □

Exercise 0.14. *Let U and X be two irreducible G-modules over $\mathbb{C}$.*

(a) Use Schur's Lemma to show that

$$\dim \mathrm{Hom}_G(U, X) = \begin{cases} 1 & \text{if } U \cong X, \\ 0 & \text{if } U \ncong X. \end{cases}$$

(b) Deduce that, if X is an irreducible G-module and $V = V_1 \oplus \cdots \oplus V_n$ is a direct sum of irreducible G-modules $V_1, \ldots, V_n$ with $V_i \cong X$ for $1 \leq i \leq k$ and $V_j \ncong X$ for $j > k$, then

$$\dim \mathrm{Hom}_G(V, X) = k.$$

Proof. (a) Let $U \ncong X$. Then by Schur's lemma, the only G-linear map in $\mathrm{Hom}_G(U, X)$ is 0.

Now let $U \cong X$. Then by Schur's lemma, every $\theta \in \mathrm{Hom}_G(U, X)$ is either 0 or an isomorphism. Let $\theta \in \mathrm{Hom}_G(U, X)$ be an isomorphism and let $\phi \in \mathrm{Hom}_G(U, X)$. Then $\theta^{-1} \circ \phi = \lambda \mathrm{Id}_U$ so $\phi = \lambda\theta$. Thus $\mathrm{Hom}_G(U, X) = \langle\theta\rangle$ so $\dim\ \mathrm{Hom}_G(U, X) = 1$.

(b) $\dim\ \mathrm{Hom}_G(V, X) = \sum_{i=1}^{n} \dim\ \mathrm{Hom}_G(V_i, X) = \sum_{i=1}^{k} \dim\ \mathrm{Hom}_G(V_i, X) = k$.

□

Exercise 0.15. *Let $G = S_3$ act tautologically on $X = \{1, 2, 3\}$. Consider the linearisation $V = \mathbb{C}X$ and (as in Example 1.26(b) of the notes) the submodule*

$$W_0 = \left\{ f \in \mathbb{C}X : \sum_{x \in X} f(x) = 0 \right\}.$$

Show that W_0 is equivalent to the (unique) irreducible representation of degree 2 found in Theorem 1.31 of the notes.

Proof. We have that $\mathbb{C}X = W_0 \oplus W_1$ where $W_1 = \{f \in \mathbb{C}X : f \text{ is constant}\}$. Dim $W_1 = 1$ so Dim $W_0 =$ Dim $\mathbb{C}X - 1 = 3 - 1 = 2$. Let $f \in W_0$ with $f \neq 0$ such that $\langle f \rangle$ is a G-submodule. WLOG say that $f(1) \neq 0$. Let $g = (12) \cdot f$ so that $g(1) = f(2), g(2) = f(1)$ and $g(3) = f(3)$. g is a scalar multiple of f so $g = \frac{f(2)}{f(1)} f$ and $g = f$ so $f(1) = f(2)$ which makes $f(3) = -2f(1)$. Now consider $h = (13) \cdot f$ so that $h(1) = f(3), h(2) = f(2)$ and $h(3) = f(1)$. Then $h = f$ and $h = \frac{f(3)}{f(1)} f = \frac{-2f(1)}{f(1)} f = -2f$ so $f = 0$. But we assumed that $f \neq 0$ giving a contradiction. Thus W_0 is irreducible and so is equivalent to the unique irreducible representation of S_3 of degree 2. □

Exercise 0.16. *Show that, over $\mathbb{C}$, all irreducible representations of the dihedral group D_{2n} have degree 1 or 2 and classify the irreducible representations of D_{2n} up to equivalence.*

- *(Hint: follow the method of Section 1.9 of the notes, i.e., the case of D_6.)*

Proof. D_{2n} is generated by a, b subject to the relations $a^n = b^2 = 1$ and $ba = a^{-1}b$. Let V be an irreducible D_{2n} module, with corresponding representation $\rho : D_{2n} \to GL(V)$. The action of D_{2n} is determined by the actions of a and b. The operator $\rho(a) : V \to V$ has an eigenvector v with eigenvalue ω so $a \cdot v = \rho(a)(v) = \omega v$. Furthermore, $\rho(a)^n = \rho(a^n) = \rho(1) = \mathrm{Id}_V$ so $\omega^n = 1$. Thus $\omega = e^{\frac{2\pi i k}{n}}$ for some $k \in \{0, ..., n-1\}$. Also, $a^{-1} \cdot v = \rho(a^{-1})(v) = \rho(a)^{-1}(v) = \omega^{-1}v$. Now consider $b \cdot v$. $a \cdot (b \cdot v) = (ab) \cdot v = (ba^{-1}) \cdot v = b \cdot (\omega^{-1}v) = \omega^{-1}(b \cdot v)$. Also $b \cdot (b \cdot v) = 1 \cdot v = v$. Thus $\mathrm{span}\{v, b \cdot v\}$ is a G-submodule. V is irreducible so $V = \mathrm{span}\{v, b \cdot v\}$. Thus an irreducible D_{2n}-module is at most two dimensional.

Case 1. $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ so $\omega \neq \omega^{-1}$. Then v and $b \cdot v$ are eigenvectors of $\rho(a)$ with different eigenvalues so are linearly independent. Using the basis $\{v, b \cdot v\}$ to give a linear isomorphism $\beta : \mathbb{C}^2 \to V$, V is equivalent to the matrix representation $D_{2n} \to GL(2, \mathbb{C})$ determined by

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The only non-zero proper subspaces of V which are closed under the action of a are the eigenspaces $\mathrm{span}\{v\}$ and $\mathrm{span}\{b \cdot v\}$ and these are not closed under the action of b. Thus V is irreducible. Let ρ_1, ρ_2 be representations for when $k = k_1, k_2$ respectively. The cases $k_1 = i$ and $k_2 = n - i$ are equivalent since changing k swaps ω and ω^{-1} so an equivalence can be obtained by swapping v and $b \cdot v$. Otherwise $(e^{\frac{2\pi i k_1}{n}}, e^{\frac{-2\pi i k_1}{n}}) \neq (e^{\frac{2\pi i k_2}{n}}, e^{\frac{-2\pi i k_2}{n}})$ and $(e^{\frac{2\pi i k_1}{n}}, e^{\frac{-2\pi i k_1}{n}}) \neq (e^{\frac{-2\pi i k_2}{n}}, e^{\frac{2\pi i k_2}{n}})$ so ρ_1 and ρ_2 cannot be equivalent, since equivalent linear maps have the same eigenvalues.

Case 2. $\omega = \omega^{-1}$. Then $\omega^2 = 1 \implies \omega = \pm 1$. $\mathrm{span}\{v + b \cdot v\}$ and $\mathrm{span}\{v - b \cdot v\}$ are closed under the actions of a and b so are D_{2n}-submodules. Since V is irreducible we must have that one of them is trivial with V equal to the other, since otherwise they would be subspaces of different eigenspaces and so be distinct submodules. First suppose that $v = -b \cdot v$. Then $\rho(b) = -1$.

Now suppose that $v = b \cdot v$. Then $\rho(b) = 1$. $\rho(a) = 1$ and $\rho(a) = -1$ both work, however $\rho(a)$ is only possible when n is even.

None of these representations are equivalent since $\rho(a)$ and $\rho(b)$ have different pairs of eigenvalues for each degree one representation ρ. □

Exercise 0.17. *Consider the following representation of the additive group $\mathbb{Z}$:*

$$\rho : \mathbb{Z} \to GL(2, \mathbb{F}) : n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

(a) Show that there is only one 1-dimensional submodule of $\mathbb{F}^2$, as a $\mathbb{Z}$-module (via ρ).

(b) Deduce that Maschke's Theorem can fail for infinite groups and even for finite groups when $\mathbb{F}$ has characteristic $p > 0$.

Proof. (a) Let $\langle (a, b) \rangle$ be a 1-dimensional submodule of $\mathbb{F}^2$. Then $\rho(n)(a, b) = (a + nb, b) = \lambda_n(a, b)$ for some $\lambda_n \in \mathbb{F}$. If $b \neq 0$ then $\lambda_n = 1$ so $a + nb = a \forall n \in \mathbb{Z}$ which is impossible. Thus we need $b = 0$. $\langle (a_1, 0) \rangle = \langle (a_2, 0) \rangle$ for any non-zero $a_1, a_2 \in \mathbb{F}$ so $\langle (1, 0) \rangle$ is the only 1-dimensional submodule of $\mathbb{F}^2$.

(b) There would need to be another 1-dimensional submodule of $\mathbb{F}^2$ for Maschke's theorem to hold. Suppose $\mathbb{F}$ has characteristic $p > 0$ and $\mathbb{F}^2$ has representation

$$\rho_p : \mathbb{Z}_p \to \mathrm{GL}(2, \mathbb{F}) : n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Then $\rho(x) = \rho([x]_p)$ so as before there is only one 1-dimensional submodule of $\mathbb{F}^2$.

$\square$

[Assume now that the base field $\mathbb{F}$ is $\mathbb{C}$ and G is a finite group.]

Exercise 0.18. *Let V be a nonzero G-module that is not irreducible. Show that there is a G-linear map $T : V \to V$ which is neither zero nor invertible. (Hint: use Maschke's Theorem.)*

Proof. By Maschke's theorem, $V = U \oplus W$ for non-zero G-submodules U and W. Let π be the projection map onto U. π is then a G-linear map, is non-zero (since its image is U which is non-zero) and is not invertible (since its kernel is W which is non-zero). $\square$

Exercise 0.19. *Let U be an irreducible G-module and V any G-module.*

(a) Show that V has an irreducible submodule isomorphic to U if and only if there is a non-zero G-linear map $U \to V$.

(b) Show that V has an irreducible submodule isomorphic to U if and only if there is a non-zero G-linear map $V \to U$.

Proof. (a) ($\Longrightarrow$) Let W be an irreducible G-submodule of V isomorphic to U. let $\theta : U \to W$ be the isomorphism. Then $\phi : U \to V$ given by $\phi(u) = \theta(u) \forall u \in U$ is a non-zero G-linear map.

($\Longleftarrow$) Let $T : U \to V$ be a non-zero G-linear map. By the first isomorphism theorem, $U/\text{Ker } T \cong \text{Im } T$. Ker T is a submodule of U so can be either $\{0\}$ or U since U is irreducible. However, $T \neq 0$ so Ker $T = \{0\}$. Thus Im T is a submodule of V which is isomorphic to U. Furthermore, Im T must be irreducible, since otherwise U wouldn't be irreducible.

(b) ($\Longrightarrow$) Let W be the irreducible submodule of V isomorphic to U. Let $\theta : W \to U$ be the isomorphism and let $\pi : V \to W$ be the projection map onto W which exists by Maschke's theorem. Then $\theta \circ \pi : V \to U$ is a non-zero G-linear map.

($\Longleftarrow$) Let $T : V \to U$ be a non-zero G-linear map. Im T is either $\{0\}$ or U since U is irreducible. However, if Im $T = \{0\}$ then $T = 0$; a contradiction. Thus Im $T = U$. By Maschke's theorem there exists a G-submodule W such that $V = \text{Ker } T \oplus W$. Let $\theta : W \to U$ be the restriction of T to W. Then θ is injective with image U so $U \cong W$. As before, W must also be irreducible.

$\square$

Exercise 0.20. *Suppose that all irreducible G-modules are 1-dimensional. Show that G is abelian. (Hint: take a faithful G-module and decompose it into a direct sum of irreducible G-modules.)*

Proof. Consider the regular representation of G with corresponding representation $\rho : G \to GL(\mathbb{C}G)$. By Maschke's theorem we can decompose $\mathbb{C}G$ into a direct sum of irreducible G-modules. Since all irreducible G-modules are 1-dimensional, we have $\mathbb{C}G = V_1 \oplus ... \oplus V_{|G|}$ where each V_i has degree 1. Let $g \in G$ and let $v_i \in V_i \backslash \{0\}$ so that $v_1, ..., v_{|G|}$ form a basis. Then $\rho(g)(v_i) = \lambda_i(g) v_i \forall i$ where $\lambda_i(g) \neq 0$ since V_i is 1-dimensional. The matrix representing $\rho(g)$ with respect to the basis $v_1, ..., v_{|G|}$

is then diagonal for every $g \in G$ and diagonal matrices commute under multiplication so the image of ρ is abelian. ρ is faithful so by the first isomorphism theorem $G \cong G/\{0\} = G/\text{Ker } \rho \cong \text{Im } \rho$. Thus G is abelian. $\square$

Exercise 0.21. *Let V be a 2-dimensional G-module and $\rho : G \to GL(V)$ the associated representation. Show that if Im ρ is not abelian, then V is irreducible. Deduce that the following degree 2 representation of D_{2n} $(n \geq 3)$ is irreducible:*

$$\rho : D_{2n} \to GL(2, \mathbb{C}) : \tau \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad \sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\omega \in \mathbb{C}$ is any nth root of unity such that $\omega \neq \omega^{-1}$. Here τ and σ are generators of D_{2n} subject to the relations $\sigma^2 = 1$, $\tau^n = 1$, and $\tau\sigma = \sigma\tau^{-1}$.

Proof. Suppose that V is reducible. Then by Maschke's theorem $V = U \oplus W$ where U and W have dimension 1. We have distinct $g, h \in G$ such that $\rho(g)\rho(h) \neq \rho(h)\rho(g)$. Since U is 1-dimensional, $\rho(g)_{|U} = \lambda \text{Id}_U, \rho(h)_{|U} = \mu \text{Id}_U$ for some non-zero λ, μ. Similarly, $\rho(g)_{|W} = \alpha \text{Id}_W, \rho(h)_{|W} = \beta \text{Id}_W$. Let u, w be non-zero elements of U and W respectively so that they form a basis. Then with respect to the basis, $\rho(g)$ is represented by

$$\begin{pmatrix} \lambda & 0 \\ 0 & \alpha \end{pmatrix}$$

and $\rho(h)$ is represented by

$$\begin{pmatrix} \mu & 0 \\ 0 & \beta \end{pmatrix}.$$

But then $\rho(g)\rho(h)$ and $\rho(h)\rho(g)$ are both represented by

$$\begin{pmatrix} \lambda\mu & 0 \\ 0 & \alpha\beta \end{pmatrix}.$$

so are equal; a contradiction. Thus V must be irreducible.

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix}.$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Thus Im ρ is not abelian so the representation is irreducible. $\square$

Below G is a finite group and the base field is $\mathbb{C}$. Also, C_n denotes the group of n-th roots of unity, a cyclic subgroup of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Exercise 0.22. *Prove directly that, if V is a G-module with irreducible decomposition*

$$V = V_1 \oplus \cdots \oplus V_n$$

and W is any irreducible submodule of V, then $W \cong V_j$ for some j.

Proof. Let $\pi_i : W \to V_i$ be the projection map onto V_i restricted to W. $\text{Id}_W = \sum_i \pi_i$ is non-zero so there must be a non-zero π_i. By Schur's lemma, this is then an isomorphism. $\square$

Exercise 0.23. *Recall that the* ***centre*** $Z(G)$ *of* G *is the subgroup of* G *defined by*

$$Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}.$$

Let V *be an irreducible* G*-module with the corresponding representation* $\rho : G \to \mathrm{GL}(V)$*.*

(a) Show that, for each $z \in Z(G)$*, there is* $\lambda(z) \in \mathbb{C}^*$ *such that, for all* $v \in V$*,*

$$z \cdot v = \lambda(z)v.$$

(Hint: use Schur's Lemma.)

(b) Show that $z \mapsto \lambda(z)$ *is a group homomorphism* $Z(G) \to \mathbb{C}^*$*.*

(c) Deduce that, if ρ *is faithful, then* $Z(G)$ *is cyclic. (Hint: consider* $\rho(Z(G))$ *as a subgroup of* $\mathbb{C}^*$*.)*

(d) Which of the following groups have a faithful irreducible representation: C_n*,* D_8*,* $C_2 \times D_8$*?*

Proof. (a) Given any $g \in G$, $z \cdot (g \cdot v) = (zg) \cdot v = (gz) \cdot v = g \cdot (z \cdot v)$ so $\rho(z)$ is G-linear. Thus by Schur's lemma $\rho(z) = \lambda(z)\mathrm{Id}$ for some $\lambda(z) \in \mathbb{C}^*$ (since the map is an isomorhpsm) so $z \cdot v = \lambda(z)v$.

(b) $Z(G) \to GL(V) : z \mapsto \lambda(z)\mathrm{Id}$ is a group homomorphism so $z \mapsto \lambda(z)$ clearly is as well.

(c) By the first isomorphism theorem, $Z(G) \cong \rho(Z(G))$ which we can consider to be a subgroup of $\mathbb{C}^*$. All subgroups of $\mathbb{C}^*$ are cyclic so $Z(G)$ is cyclic.

(d) C_n has a faithful irreducible representation given by $\rho : C_n \to \mathbb{C}^* : \omega \mapsto \omega$.

D_8 has a faithful irreducible representation $\rho : D_8 \to GL(n, \mathbb{C})$ given by

$$\rho(\tau) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \rho(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{4}}$.

$Z(D_8) = \{e, \tau^2\}$ so $Z(C_2 \times D_8) = \{(1, e), (-1, e), (1, \tau^2), (-1, \tau^2)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ which is not cyclic so there is no faithful irreducible representation of $C_2 \times D_8$.

□

Exercise 0.24. *Decompose the regular module* $\mathbb{C}C_3$ *as a direct sum of irreducible submodules.*

Proof. let $\omega = e^{\frac{2\pi i}{3}}$. C_3 is abelian so every irreducible submodule has dimension 1. Furthermore, every representation $\rho : C_3 \to \mathbb{C}C_3$ is equivalent to precisely one representation $\chi_i : C_3 \to \mathbb{C}^* : g \mapsto g^i$ for $i \in \{0, 1, 2\}$. Thus if $\langle f \rangle$ is an irreducible submodule of $\mathbb{C}C_3$, then $\omega \cdot f = \omega^i f$. One irreducible submodule is the one generated by $f(g) = 1 \forall g \in C_3$ which is equivalent to χ_0. Now consider $h(g) = g \forall g \in C_3$. Then $\omega^2 \cdot h = \omega h$ and $\omega \cdot h = \omega^2 h$ so $\langle h \rangle$ is a submodule which is equivalent to χ_2. Finally consider $p(g) = g^2 \forall g \in C_3$. Then $\omega \cdot p = \omega p$ and $\omega^2 \cdot p = \omega^2 p$ so $\langle p \rangle$ is an irreducible submodule equivalent to χ_i. $\mathbb{C}C_3$ has dimension 3 so $\mathbb{C}C_3 = \langle f \rangle \oplus \langle h \rangle \oplus \langle p \rangle$. □

Exercise 0.25. *Let* $\tau = (123)$ *and* $\sigma = (12)$*, which generate* S_3*. Decompose the regular module* $\mathbb{C}S_3$ *into a direct sum of its irreducible submodules by completing the following steps.*

(a) Show that the subspaces

$$U_1 = span\{\delta_e, \delta_{(123)}, \delta_{(132)}\}, \quad U_2 = span\{\delta_{(12)}, \delta_{(23)}, \delta_{(13)}\}$$

of $\mathbb{C}S_3$ are closed under the action of τ.

(b) Find the eigenvectors v_1, v_2, v_3 of τ in U_1 and compute $\sigma v_1, \sigma v_2, \sigma v_3$.

(c) Find 4 irreducible submodules V_1, V_2, V_3, V_4 of $\mathbb{C}S_3$ of degrees 1, 1, 2, 2, respectively, so that

$$\mathbb{C}S_3 = V_1 \oplus V_2 \oplus V_3 \oplus V_4.$$

(Hint: recall the classification of representations of S_3 in Section 1.7.)

Proof. (a) $(123) \cdot \delta_e = \delta_{(123)}, (123) \cdot \delta_{(123)} = \delta_{(132)}, (123) \cdot \delta_{(132)} = \delta_e$ so U_1 is closed under τ. $(123) \cdot \delta_{(12)} = \delta_{(13)}, (123) \cdot \delta_{(23)} = \delta_{(21)}, (123) \cdot \delta_{(13)} = \delta_{(23)}$ so U_2 is closed under τ.

(b) The action of τ with respect to $\delta_e, \delta_{(123)}, \delta_{(132)}$ is represented by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which has eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \omega \\ \omega^2 \\ 1 \end{pmatrix}, \begin{pmatrix} \omega^2 \\ \omega \\ 1 \end{pmatrix}$$

with corresponding eigenvalues

$$\lambda_1 = 1, \lambda_2 = \omega^2, \lambda_3 = \omega.$$

where $\omega = e^{\frac{2\pi i}{3}}$ so the eigenvectors (up to a scalar) of τ in U_1 are

$$\begin{aligned} v_1 &= \delta_e + \delta_{(123)} + \delta_{(132)}, \\ v_2 &= \omega\delta_e + \omega^2\delta_{(123)} + \delta_{(132)}, \\ v_3 &= \omega^2\delta_e + \omega\delta_{(123)} + \delta_{(132)}. \end{aligned}$$

$\sigma \cdot \delta_e = \delta_{(12)}, \sigma \cdot \delta_{(123)} = \delta_{(23)}, \sigma \cdot \delta_{(132)} = \delta_{(13)}$ so

$$\begin{aligned} \sigma \cdot v_1 &= \delta_{(12)} + \delta_{(23)} + \delta_{(13)}, \\ \sigma \cdot v_2 &= \omega\delta_{(12)} + \omega^2\delta_{(23)} + \delta_{(13)}, \\ \sigma \cdot v_3 &= \omega^2\delta_{(12)} + \omega\delta_{(23)} + \delta_{(13)}. \end{aligned}$$

(c) Note that $\sigma \cdot v_1$ is an eigenvector of τ with eigenvalue 1, $\sigma \cdot v_2$ is an eigenvector of τ with eigenvalue ω and $\sigma \cdot v_3$ is an eigenvector of τ with eigenvalue ω^2.

Let $V_1 = \langle v_1 + \sigma \cdot v_1 \rangle$. Then $\tau \cdot (v_1 + \sigma \cdot v_1) = v_1 + \sigma \cdot v_1$ and $\sigma \cdot (v_1 + \sigma \cdot v_1) = v_1 + \sigma \cdot v_1$ so V_1 has dimension 1 and both τ and σ act as the identity.

Let $V_2 = \langle v_1 - \sigma \cdot v_1 \rangle$. Then $\tau \cdot (v_1 - \sigma \cdot v_1) = v_1 - \sigma \cdot v_1$ and $\sigma \cdot (v_1 - \sigma \cdot v_1) = -(v_1 - \sigma \cdot v_1)$ so V_1 has dimension 1, τ acts as the identity and σ acts as $-\mathrm{Id}_{V_2}$.

Finally, let $V_3 = \text{span}\{v_2, \sigma \cdot v_2\}$ and let $V_4 = \text{span}\{v_3, \sigma \cdot v_3\}$.

Since $(v_1, \sigma \cdot v_1), (v_2, \sigma \cdot v_3), (v_3, \sigma \cdot v_2)$ are pairs of linearly independent elements of eigenspaces of τ for different eigenvectors and $\mathbb{C}S_3$ has dimension 6, we have that $\mathbb{C}S_3 = \text{span}\{v_1, \sigma \cdot v_1\} \oplus \text{span}\{v_2, \sigma \cdot v_3\} \oplus \text{span}\{v_3, \sigma \cdot v_2\}$ as vector spaces. Thus $\mathbb{C}S_3 = V_1 \oplus V_2 \oplus V_3 \oplus V_4$.

□

Exercise 0.26. *Let G be a group and let $d_1, \ldots, d_r$ be the degrees of a complete set of irreducible G-modules. Use the fact that $\sum_{i=1}^{r} (d_i)^2 = |G|$ to find the possible numbers of irreducible modules of G and their dimensions, when*

1. *$|G| = 6$, and*
2. *$|G| = 8$.*

(Hint: any group has at least one irreducible module of dimension 1, i.e., the trivial module.)

Proof. (a) $6 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 = 1^2 + 1^2 + 2^2$ so up to isomorphism there are either 6 irreducible G-modules (each of dimension 1) or 3 irreducible G-modules (of dimensions $1, 1, 2$).

(b) $8 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ so up to isomorphism there are either 8 irreducible G-modules (each of dimension 1) or 5 irreducible G-modules (of dimensions $1, 1, 1, 1, 2$).

□

Exercise 0.27. *Let $\tau = (123)$ and $\sigma = (23)$, which generate S_3. Let χ_1, χ_2, and χ_3 denote the three irreducible characters of S_3, where χ_1 is the character of the trivial representation, χ_2 the character of the sign representation, and χ_3 the character of the irreducible representation of degree 2. Write down the values of the χ_i to fill out the following table:*

	1	τ	τ^2	σ	$\tau\sigma$	$\tau^2\sigma$
χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1
χ_3	2	-1	-1	0	0	0

(Hint: Recall the classification of representations of S_3 from Section 1.7.)

Exercise 0.28. *Let G act on the set $X = \{1, \ldots, n\}$ and let $V = \mathbb{C}X = span\{\delta_1, \ldots, \delta_n\}$ be the associated permutation representation, i.e., $g \cdot \delta_i = \delta_{g \cdot i}$. Let χ be the character of V.*

1. *Show that $\chi(g) = |Fix(g)|$, where $Fix(g) = \{x \in X \mid g \cdot x = x\}$.*
2. *Suppose $G = S_3$, with its usual action on $X = \{1, 2, 3\}$. Compute χ and check that $\chi = \chi_1 + \chi_3$ in the notation of the previous question.*

Proof. 1. We have a basis $\alpha = \delta_1, ..., \delta_n$. Representing $\rho(g)$ as a matrix with respect to this basis, the ith column of the ith row is non-zero if and only if $g \cdot \delta_i = \delta_i \iff \delta_{g \cdot i} = \delta_i \iff g \cdot i = i \iff i \in \text{Fix}(g)$. We thus have a bijection between non-zero (in this case 1) positions on the diagonal of the matrix and elements of $\text{Fix}(g)$ so $\chi(g) = |\text{Fix}(g)|$.

2.

$$\text{Fix}(\text{Id}) = \{1, 2, 3\}.$$

$$\text{Fix}((13)) = \{2\}.$$

$$\text{Fix}((12)) = \{3\}.$$
$$\text{Fix}((23)) = \{1\}.$$
$$\text{Fix}((123)) = \{\}.$$
$$\text{Fix}((132)) = \{\}.$$

Thus

$$\begin{aligned}
\chi(1) &= 3 = \chi(1) + \chi(3), \\
\chi(\tau) &= 0 = \chi(1) + \chi(3), \\
\chi(\tau^2) &= 0 = \chi(1) + \chi(3), \\
\chi(\sigma) &= 1 = \chi(1) + \chi(3), \\
\chi(\tau\sigma) &= 1 = \chi(1) + \chi(3), \\
\chi(\tau^2\sigma) &= 1 = \chi(1) + \chi(3).
\end{aligned}$$

□

Exercise 0.29. *For $g \in G$, let $C(g) = \{h \in G \mid gh = hg\}$ be the centraliser of g, let $\langle g \rangle$ be the subgroup generated by g, and let $g^G = \{h^{-1}gh \mid h \in G\}$ be the conjugacy class of g.*

1. *Show that $\langle g \rangle \leq C(g) \leq G$.*
2. *Show that $|g^G| = \frac{|G|}{|C(g)|}$ and thus $|g^G|$ divides $|G|$.*

Proof.

1. $ge = eg$ so $e \in C(g)$. Let $h \in C(g)$. Then $gh = hg \iff g = hgh^{-1} \iff h^{-1}g = gh^{-1}$ so $h^{-1} \in C(g)$. Let $a, b \in C(g)$. Then $abg = agb = gab$ so $ab \in C(g)$. Thus $C(g) \leq G$. Let $g^i \in \langle g \rangle$. Then $gg^i = g^{i+1} = g^i g$. Thus $\langle g \rangle \leq C(g)$.
2. Define an action by $h \cdot g = h^{-1}gh$. Then $\text{Orb}(g) = g^G$ and $\text{Stab}(g) = |C(g)|$ so by the orbit stabilizer theorem $|g^G| = \frac{|G|}{|C(g)|}$ so $|g^G|$ divides $|G|$.

□

Exercise 0.30. *A function $f : G \to \mathbb{C}$ is a* class function *if f is constant on conjugacy classes in G. Show that $\mathbb{C}_{cls}(G) = \{f \in \mathbb{C}G \mid f \text{ is a class function}\}$ is a linear subspace of $\mathbb{C}G$ of dimension equal to the number of conjugacy classes in G.*

Proof. Conjugacy classes partition G so $\mathbb{C}_{\text{cls}}G$ is clearly a well-defined linear subspace of $\mathbb{C}G$. Let $g_1^G, ..., g_n^G$ be a complete list of conjugacy classes in G. Define $f_i \in \mathbb{C}G$ by $f_i(x) = 1 \forall x \in g_i^G$ and $f_i(x) = 0$ otherwise. $f_1, ..., f_n$ then forms a basis of $\mathbb{C}_{\text{cls}}G$ and n is the number of conjugacy classes in G. □

Exercise 0.31. *Any element in S_n can be written as a product of disjoint cycles, and the list of the sizes of the cycles (ordered from big to small) is called the cycle shape of the element. Two elements in S_n are conjugate if and only if they have the same cycle shape.*

(a) For each conjugacy class in the symmetric group S_4, compute the size of the class and also find $C(g)$ for one g in that class.

(b) List the shapes of the conjugacy classes in the symmetric group S_5 and compute the size of each class.

Proof. (a) $1+1+1+1 = 2+1+1 = 2+2 = 3+1 = 4$ $\{(1234), (2134), (1243), (3214), \}$

For the conjugacy class of cycles of type (4): The conjugacy class $(1234)^{S_4}$ has 6 elements and $C((1234)) = \langle(1234)\rangle$.

For the conjugacy class of cycles of type (3, 1): The conjugacy class $(123)^{S_4}$ has $\binom{4}{3} \cdot 2 = 8$ elements and $C((123)) = \langle(123)\rangle$.

For the conjugacy class of cycles of type (2, 2): The conjugacy class $((12)(34))^{S_4}$ has 3 elements and $C((12)(34)) = \{e, (12), (34), (12)(34), (13)(24), (14)(32), (1324), (1423)\}$.

For the conjugacy class of cycles of type (2, 1, 1): The conjugacy class $(12)^{S_4}$ has $\binom{4}{2} = 6$ elements and $C((12)) = \{e, (34), (12)(34)\}$.

For the conjugacy class of cycles of type (1, 1, 1, 1): The conjugacy class $()^{S_4}$ has 1 element and $C(()) = S_4$.

$6 + 8 + 3 + 6 + 1 = 24 = |S_4|$ as expected.

(b) The shapes conjugacy types are

$$\begin{aligned}(5)\\(4,1)\\(3,2)\\(3,1,1)\\(2,2,1)\\(2,1,1,1)\\(1,1,1,1,1)\end{aligned}$$

There are $4! = 24$ elements with cycle type (5).

There are $\binom{5}{4} \cdot 3! = 30$ elements with cycle type (4, 1).

There are $\binom{5}{3} \cdot 2 = 20$ elements with cycle type (3, 2).

There are 20 elements with cycle type (3, 1, 1).

There are $\binom{5}{4} \cdot = 15$ elements with cycle type (2, 2, 1).

There are $\binom{5}{2} = 10$ elements with cycle type (2, 1, 1, 1, 1).

There is 1 element with cycle type (1, 1, 1, 1, 1).

$24 + 30 + 20 + 20 + 15 + 10 + 1 = 120 = |S_5|$ as expected.

□

Exercise 0.32. *For any $g \in G$, recall that $Z(G) \leqslant C(g) \leqslant G$.*

(a) If $g \in G \setminus Z(G)$, show that both inclusions are strict. Deduce that the index of $Z(G)$ in G (i.e., $|G|/|Z(G)|$) cannot be prime.

(b) If $|G| = 12$ and G is non-abelian, show that G has at most 7 conjugacy classes.

(c) If $|G| = 12$ and G is non-abelian, show that $|Z(G)|$ is 1 or 2.

Proof. (a) $g \in C(g)$ so $Z(G) < C(g)$. $C(g) < G$ since otherwise g would commute with every element of G and so be in the centre. Suppose that the index of $Z(G)$ in G is prime. Then $|Z(G)| = \frac{|G|}{p}$ for some prime p. We also have $|Z(G)| = \frac{|C(g)|}{a}$ and $|C(g)| = \frac{|G|}{b}$ for $a, b > 1$. Thus $|Z(G)| = \frac{|G|}{ab} = \frac{|G|}{p}$ so $p = ab$. But p is prime; a contradiction.

(b) Let k be the index of $Z(G)$ in G. k cannot be prime by (a) so is either $4, 6$ or 12 meaning that $|Z(G)|$ is at most 3. To maximize the number of conjugacy classes, we must maximize the order of the centre, so let the centre have order 3. There are then at most $\frac{12-3}{2} = 4.5$ conjugacy classes of size larger than 1 so G has at most $3 + 4 = 7$ conjugacy classes.

(c) Suppose $|Z(G)| = 3$. Let $g \in G \setminus Z(G)$. Then $3 < |C(g)| < 12$ and $3||C(g)||12$ so $|C(g)| = 6$. But then the size of every conjugacy class larger than 1 is 2 implying that the order of G is odd; a contradiction. Thus $|Z(G)|$ is 1 or 2.

$\square$

Exercise 0.33.

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\psi(g)$$

makes $\mathbb{C}G$ into a unitary G-module.

Proof. Let $h \in G$ and $\phi, \psi \in \mathbb{C}G$. Then $\langle h\phi, h\psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(h^{-1}g)\psi(h^{-1}g) = \langle \phi, \psi \rangle$ since $G \to G : g \mapsto h^{-1}g$ is a bijection. $\square$

Exercise 0.34. *Let $\alpha : G \to \mathrm{GL}(V)$ be a representation of G with character χ, and let ψ be a linear character of G. Show that the map*

$$\psi\alpha : G \to \mathrm{GL}(V), \quad g \mapsto \psi(g)\alpha(g)$$

is a representation of G with character $\psi\chi$. Show further that $\psi\alpha$ is irreducible if and only if α is irreducible.

Proof. ϕ is linear so $\psi(g) \neq 0 \forall G$. Thus the codomain of $\psi\alpha$ is well-defined. Let $g, h \in G$. Then $\psi\alpha(gh) = \psi(gh)\alpha(gh) = \psi(g)\psi(h)\alpha(g)\alpha(h) = \psi_\alpha(g)\psi_\alpha(h)$ so $\psi\alpha$ is a representation. tr $\psi(g)\alpha(g) = \psi(g)$tr $\alpha(g) = \psi(g)\chi(g)\forall g \in G$ so the character of $\psi\alpha$ is $\psi\chi$. $\psi\alpha$ is irreducible $\iff$ $\langle \psi\chi, \psi\chi \rangle = 1 \iff \frac{1}{|G|} \sum_{g \in G} \overline{\psi(g)\chi(g)}\psi(g)\chi(g) = \frac{1}{|G|} \sum_{g \in G} \psi(g^{-1})\psi(g)\overline{\chi(g)}\chi(g) = \langle \chi, \chi \rangle = 1 \iff \alpha$ is irreducible. $\square$

Exercise 0.35. *Let $\chi_1, \ldots, \chi_n$ be a complete set of irreducible characters, and let χ_{reg} be the regular character. By writing $\chi_{reg} = \sum_i a_i\chi_i$ and using the orthonormality of the χ_i, show that $a_i = \chi_i(1)$ and deduce that*

$$|G| = \chi_{reg}(1) = \sum_{i=1}^{n} \chi_i(1)^2.$$

Proof. $a_i = \langle \chi_{\mathrm{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\mathrm{reg}}(g)}\chi_i(g) = \frac{|G|\chi_i(1)}{|G|} = \chi_i(1)$. $\square$

Exercise 0.36. *Show that the converse of Lemma 5.25 is true, namely that, if $f \in \mathbb{C}G$ and the map*

$$T_V(f) : V \to V : v \mapsto \sum_{g \in G} f(g)gv$$

is G-linear for every G-module V, then f is a class function. [Hint: take $V = \mathbb{C}G$.]

Proof. $\forall p \in \mathbb{C}G, a \in G$ have $aT_v(f)(p) = T_v(f)(ap)$ so $\sum_{g \in G} f(g)(ag) \cdot p = \sum_{g \in G} f(g)(ga) \cdot p$. Let $h, s \in G$. Then $\sum_{g \in G} f(g)(h^{-1}g) \cdot \delta_h = \sum_{g \in G} f(g)(gh^{-1}) \cdot \delta_h \implies \sum_{g \in G} f(g)\delta_{h^{-1}gh} = \sum_{g \in G} f(g)\delta_g \implies f(h^{-1}sh) = f(s)$. This is true for every $s, h \in G$ so f is a class function. $\square$

Exercise 0.37. *The vector space $\mathbb{C}G = \{f : G \to \mathbb{C}\}$ can be made into an algebra using the convolution product $*$, defined as follows:*

$$(f_1 * f_2)(g) = \sum_{h_1 h_2 = g} f_1(h_1) f_2(h_2).$$

[You can assume without proof that this does define an algebra, but it is not hard to check.]

1. *Show that $\delta_g * \delta_h = \delta_{gh}$ and that $*$ is the unique bilinear extension of this rule.*
2. *For the map $T_V : \mathbb{C}G \to Hom_{\mathbb{C}}(V, V)$ in Q1, show that $T_V(f_1 * f_2) = T_V(f_1) \circ T_V(f_2)$, where $\circ$ is composition of linear operators.*
3. *Show that the center of $\mathbb{C}G$, that is, $Z(\mathbb{C}G) = \{z \in \mathbb{C}G \mid z * f = f * z, \forall f \in \mathbb{C}G\}$, is precisely $\mathbb{C}_{cls}G$.*

Proof. 1. $(\delta_g * \delta_h)(a) = \sum_{h_1 h_2 = a} \delta_g(h_1)\delta_h(h_2)$. If $a = gh$ then $(\delta_g * \delta_h)(a) = 1$. Otherwise, $(\delta_g * \delta_h)(a) = 0$. Thus $\delta_g * \delta_h = \delta_{gh}$. Let $g_1, ..., g_n$ be the complete list of elements of G. let $f_1 = \sum_i \lambda_i \delta_{g_i}, f_2 = \sum_i \mu_i \delta_{g_i}$ for $\lambda_i, \mu_i \in \mathbb{C}$. Then $f_1 * f_2 = \sum_{i,j} \lambda_i \mu_j \delta_{g_i g_j}$ so $(f_1 * f_2)(g_k) = \sum_{i,j,g_i g_j = g_k} \lambda_i \mu_j = \sum_{i,j,g_i g_j = g_k} f_1(g_i) f_2(g_j) = \sum_{h_1 h_2 = g_k} f_1(h_1) f_2(h_2)$ so $*$ is the unique bilinear extension of the rule.

2. $T_V(f_1 * f_2)(v) = \sum_{g \in G} (f_1 * f_2)(g)(g \cdot v) = \sum_{g \in G} \sum_{h_1 h_2 = g} f_1(h_1) f_2(h_2)(g \cdot v) = \sum_{h_1, h_2} f_1(h_1) f_2(h_2)((h_1 h_2) \cdot v) = \sum_{h_1} f_1(h_1) \sum_{h_2} f_2(h_2)(h_1 \cdot (h_2 \cdot v)) = \sum_{h_1} f_1(h_1) h_1 \cdot (\sum_{h_2} f_2(h_2) h_2 \cdot v) = \sum_{h_1} f_1(h_1) h_1 \cdot T_V(f_2)(v) = T_V(f_1) \circ T_V(f_2)(v) \forall v \in V$.

3. Let $z \in Z(\mathbb{C}G)$. Then for every G-module V and $f \in \mathbb{C}G$ we have $T_V(z) \circ T_V(f) = T_V(f) \circ T_V(z)$. Let $h \in G$. Then $T_V(z)(hv) = \sum_{g \in G} z(g)(ghv) = \sum_{g \in G} z(g) g \sum_{a \in G} \delta_h(a) av = \sum_{g \in G} z(g) g T_V(\delta_h)(v) = T_V(z) \circ T_V(\delta_h)(v) = T_V(\delta_h) \circ T_V(z)(v) = \sum_{g \in G} \delta_h(g) g T_V(z)(v) = h T_V(z)(v) \forall v \in V$. Thus $T_V(z)$ is G-linear for every G-module V so $z \in \mathbb{C}_{\text{cls}}G$.

   Now let $z \in \mathbb{C}_{\text{cls}}G$. Then given any $h, g \in G$ we have $(z * \delta_h)(g) = \sum_{h_1 h_2 = g} z(h_1)\delta_h(h_2) = z(gh^{-1})$ and $(\delta_h * z)(g) = \sum_{h_1 h_2 = g} \delta_h(h_1) z(h_2) = z(h^{-1}g)$. $(gh^{-1})^h = h^{-1}gh^{-1}h = h^{-1}g$ so $z * \delta_h = \delta_h * z$. $\{\delta_h : h \in G\}$ is a basis of $\mathbb{C}G$ so $z * f = f * z \forall f \in \mathbb{C}G$. Thus $z \in Z(\mathbb{C}G)$. $\square$

Exercise 0.38. *Let G be a group of order 4. Using just properties of character tables, i.e., without explicitly identifying the group, complete the following partial character table of G.*

g_i	$g_1 = 1$	g_2	g_3	g_4
χ_1		1		
χ_2		i		
χ_3		-1		
χ_4		$-i$		

Proof. χ_1 is the trivial character. so the first row is all 1s.

Since there are 4 characters and the group has order 4 we have $|g_i^G| = 1 \forall i$ so $|C(g_i)| = 4 \forall i$. Thus the group is abelian so $\chi_i(1) = 1 \forall i$. i is a 4th root of 1 and not a first or second root so g_2 has order 4 so is not its own inverse. Let $g_3 = g_2^{-1}$. Then $\chi_i(g_3) = \overline{\chi_i(g_2)} \forall i$. Since g_2 and g_3 both have order

4 we must have that $g_2^2 = g_3^2 = g_4$ so $\chi_i(g_4) = \chi_i(g_2)^2 \forall i$.

g_i	$g_1 = 1$	g_2	g_3	g_4
χ_1	1	1	1	1
χ_2	1	i	$-i$	-1
χ_3	1	-1	-1	1
χ_4	1	$-i$	i	-1

□

Exercise 0.39. *Let G be a group of order 18. Complete the partial character table of G below.*

g_i	$g_1 = 1$	g_2	g_3	g_4	g_5	g_6
$\lvert C(g_i)\rvert$	18	9	9	9	9	2
χ_1	1					
χ_2	1	1	1	1	1	-1
χ_3	2	2	-1	-1	-1	0
χ_4		-1	2		-1	
χ_5		-1			-1	
χ_6					2	

Proof. We have that χ_1 is the trivial characer so the first row is all 1s. By column orthogonality we have $\sum_{k=1}^n \overline{\chi_k(g_2)}\chi_k(g_5) = 0 = 1 + 1 - 2 + 1 + 1 + 2\overline{\chi_6(g_2)} \implies \chi_6(g_2) = -1$. Since 2 is not a root of unity, we must have that χ_3, χ_4, χ_6 have dimension greater than 1. Suppose one of them has degree 3. Then by the degre-squared equation $18 = \sum_{i=1}^6 (\dim \chi_i)^2 \geq 1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 3^2 = 20$; a contradiction. Thus no irreducible character has dimension greater than 2. $1 + 1 + 1 + 4 + 4 + 4 = 15 \neq 18$ so we must have $\chi_5(1) = 2$. By row orthogonality we have $\sum_{k=1}^6 \frac{\overline{\chi_3(g_k)}\chi_4(g_k)}{|C(g_k)|} = \frac{2\cdot 2}{18} + \frac{-2}{9} + \frac{-2}{9} + \frac{-\chi_4(g_4)}{9} + \frac{1}{9} = 0 \implies \chi_4(g_4) = -1$. Again by row orthogonality we have $\frac{2}{18} - \frac{1}{9} + \frac{2}{9} - \frac{1}{9} - \frac{1}{9} + \frac{\chi_4(g_6)}{2} = 0 \implies \chi_4(g_6) = 0$. By column orthogonality we have $\sum_{k=1}^6 \overline{\chi_k(g_6)}\chi_k(g_6) = \sum_{k=1}^6 |\chi_k(g_6)|^2 = 1 + 1 + |\chi_5(g_6)|^2 + |\chi_6(g_6)|^2 = 2 \implies \chi_5(g_6) = \chi_5(g_6) = 0$. By row orthogonality we have $\frac{4}{18} + \frac{1}{9} + \frac{|\chi_5(g_3)|^2}{9} + \frac{|\chi_5(g_4)|^2}{9} + \frac{1}{9} = 1 \implies |\chi_5(g_3)|^2 + |\chi_5(g_3)|^2 = 5$. Furthermore, $\frac{4}{18} + \frac{1}{9} + \frac{|\chi_6(g_3)|^2}{9} + \frac{|\chi_6(g_4)|^2}{9} + \frac{4}{9} = 1 \implies |\chi_6(g_3)|^2 + |\chi_6(g_4)|^2 = 2$. By row orthogonality we have that $\chi_6(g_3) + \chi_6(g_4) \in \mathbb{R}$ and $2\chi_6(g_3) - \chi_6(g_4) \in \mathbb{R}$. so $\chi_6(g_3)$ and hence $\chi_6(g_4)$ are real. By row orthogonality we have $\frac{2}{18} - \frac{1}{9} + \frac{\chi_6(g_3)}{9} + \frac{\chi_6(g_3)}{9} + \frac{2}{9} = 0 \implies \chi_6(g_3) = \chi_6(g_4) = -1$. By column orthogonality we have $1 + 1 - 2 + 4 + 2\chi_5(g_3) - 2 = 0$ so $\chi_5(g_3) = -1$. Again by column orthogonality we have $1 + 1 - 2 - 2 + 2\chi_5(g_4) - 2 = 0$ so $\chi_5(g_4) = 2$.

g_i	$g_1 = 1$	g_2	g_3	g_4	g_5	g_6
$\lvert C(g_i)\rvert$	18	9	9	9	9	2
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1
χ_3	2	2	-1	-1	-1	0
χ_4	2	-1	2	-1	-1	0
χ_5	2	-1	-1	2	-1	0
χ_6	2	-1	-1	-1	2	0

□

Exercise 0.40. *Let $\mathbb{F}$ be a field which is (i) algebraically closed and (ii) of characteristic zero. Suppose that a G-module V over $\mathbb{F}$ has two irreducible decompositions*

$$U_1 \oplus \cdots \oplus U_s = V = W_1 \oplus \cdots \oplus W_t,$$

that is, where all U_i and W_j are irreducible.

(a) Explain why each U_i is isomorphic to some W_j (and vice versa).

(b) Explain why $s = t$.

Proof. (a) U_i is isomorphic to at least one component of the first irreducible decomposition so the multiplicity of U_i in V is greater than zero which implies that U_i is isomorphic to at least one W_j in the second decomposition. The same argument can be applied for W_j.

(b) Let $X_1, ..., X_n$ be a complete list of submodules in the decomposition $U_1 \oplus ... \oplus U_s$ up to isomorphism. By (a) this is also a complete list of submodules in the decomposition $W_1 \oplus ... \oplus W_t$ up to isomorphism, for every W_j is isomorphic to some U_i and thus some X_k. Furthermore, for each k the number of submodules in each of the two decompositions which are isomorphic to X_k are the same, for they are both equal to $\dim \mathrm{Hom}_G(X_k, V)$. Thus $s = \sum_{k=1}^{n} \dim \mathrm{Hom}_G(X_k, V) = t$. □

Lemma 0.41. *Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be equivalent representations. Then there exist bases of V_1 and V_2 such that the corresponding matrix representations of ρ_1 and ρ_2 are equal.*

Proof. Since ρ_1 and ρ_2 are equivalent there exists a G-linear isomorphism $\theta : V_1 \to V_2$ such that $\theta\rho_1(g)\theta^{-1} = \rho_2(g) \forall g \in G$. Let $\alpha_1, ..., \alpha_n$ be a basis of V_1. Then $\theta(\alpha_1), ..., \theta(\alpha_n)$ is a basis for V_2. Furthermore, $\rho_2(g)(\theta(\alpha_i)) = \theta(\rho_1(g)(\alpha_i)) \forall i$ so if $\rho_1(g)(\alpha_i) = \lambda_1\alpha_1 + ... + \lambda_n\alpha_n$ then $\rho_2(g)(\theta(\alpha_i)) = \lambda_1\theta(\alpha_1) + ... + \lambda_n\theta(\alpha_n)$. $(\lambda_1, ..., \lambda_n)$ is the ith column of the matrix representing $\rho_1(g)$ with respect to $\alpha_1, ..., \alpha_n$ and the matrix representing $\rho_2(g)$ with respect to $\theta(\alpha_1), ..., \theta(\alpha_n)$ so the matrices are the same. □