

## Homework 05

### 1. Integral, differentiability and decay rate

Compute this generalization of an integral we did in class:

$$I_{n,a}(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(x^2 + a^2)^n} dx$$

where  $k, a \in \mathbb{R}, a > 0$  and  $n \in \mathbb{Z}, n \geq 1$ . What is the degree of differentiability of  $I_{n,a}(k)$  with respect to  $k$ ? How does this relate to the decay of the integrand?

#### Compute the Integral:

Complex function  $f(z) = \frac{e^{ikz}}{(z^2 + a^2)^n}$  has poles at  $z = \pm ia$ .

The residue at  $z = +ia$  pole

$$\text{Res}(f, ia) = \frac{1}{(n-1)!} \lim_{z \rightarrow ia} \frac{d^{n-1}}{dz^{n-1}} [(z - ia)^n f(z)]$$

Construct a semicircular contour in the upper half-plane that consists of a line segment  $C_1$  from  $-R$  to  $R$  and a semicircular arc  $C_2$  of radius  $R$  centered at the origin.

The integral over the contour  $C$  is:

$$\int_C f(z) dz = 2\pi i \times \text{Res}(f, ia)$$

As  $R \rightarrow \infty$ , the integral over  $C_2$  vanishes for  $k > 0$  due to the exponential decay of  $e^{ikz}$  in the upper half-plane.

The original integral  $I_{n,a}(k)$  is equal to the integral over  $C_1$ :

$$I_{n,a}(k) = 2\pi i \times \text{Res}(f, ia)$$

#### Degree of Differentiability

$$I_{n,a}'(k) = \int_{-\infty}^{\infty} \frac{ixe^{ikx}}{(x^2 + a^2)^n} dx$$

The corresponding complex function

$$f_k(z) = \frac{iz e^{ikz}}{(z^2 + a^2)^n}$$

has poles at  $z = \pm ia$ .

The function  $I_{n,a}(k)$  will be infinitely differentiable with respect to  $k$  because the exponential function  $e^{ikx}$  is smooth and the denominator does not depend on  $k$ .

### Relation to the Decay of the Integrand:

The decay of the integrand as  $x \rightarrow \pm\infty$  is crucial for the convergence of the integral. The factor  $(x^2 + a^2)^{-n}$  ensures that the integrand decays sufficiently fast at infinity for the integral to converge.

The faster decay of the integrand at infinity, which is ensured by a higher  $n$ , means that  $I_{n,a}(k)$  will be more regular (i.e., higher degree of differentiability).

## 2. Integral of a keyhole contour

Compute the following integral by a choice of keyhole contour, where  $a, b, c \in \mathbb{R}$  and  $b, c > 0$ :

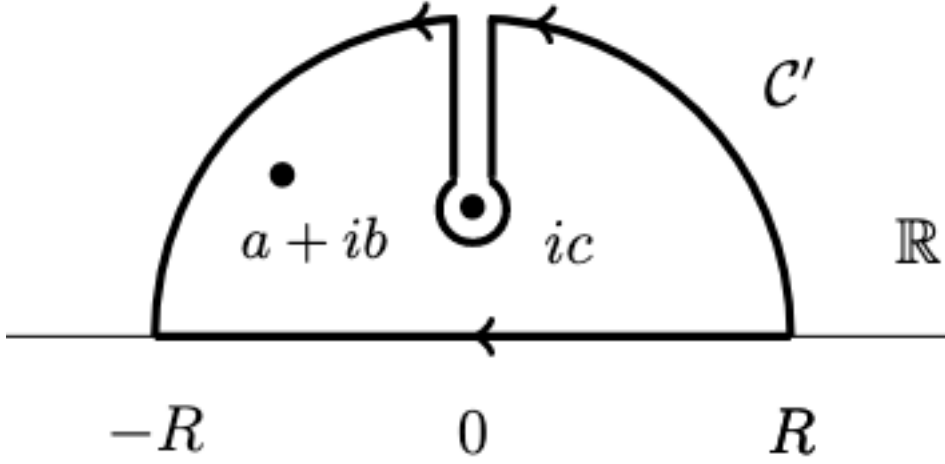
$$I(a, b, c) = \int_{-\infty}^{\infty} dx \frac{\log(x^2 + c^2)}{(x - a)^2 + b^2}$$

The complex function

$$f(z) = \frac{\log(z^2 + c^2)}{(z - a)^2 + b^2} = \frac{\log(z + ic)}{(z - a)^2 + b^2} + \frac{\log(z - ic)}{(z - a)^2 + b^2}$$

may have branch point  $z = \pm ic$  and have poles at  $a \pm ib$ .

Consider the contour integral  $\oint_C dz f(z)$  in the complex plane where  $C$  is a semicircle of radius  $R$  in the upper half-plane with a detour down and up the imaginary axis about the branch point  $z = ic$ .



The function  $\log(z + ic)$  is holomorphic in the upper half plane if we choose its branch cut to lie in the lower half plane, so the integrand is holomorphic inside the contour of  $C$  except for a simple pole at  $a + ib$ .

$$\oint_C dz \frac{\log(z + ic)}{(z - a)^2 + b^2} = 2\pi i \frac{\log(a + ib + ic)}{2ib} = \frac{\pi}{b} \log(a + ib + ic)$$

The function  $\log(z - ic)$  has a branch cut in the upper half plane. We have

$$\oint_C dz \frac{\log(z - ic)}{(z - a)^2 + b^2} = \frac{\pi}{b} \log(a + ib - ic) + \int_{ic}^{\infty} \frac{\log(iy - ic + \epsilon) - \log(iy - ic + \epsilon)}{(z - a)^2 + b^2}$$

Since the contribution from the big semi-circle tends to zero as  $R \rightarrow \infty$  and from the small circle around  $ic$  as its radius tends to zero. And

$$\int_{ic}^{\infty} \frac{\log(iy - ic + \epsilon) - \log(iy - ic + \epsilon)}{(z - a)^2 + b^2} = -2\pi \int_c^{\infty} dy \frac{1}{(iy - a)^2 + b^2} = -\frac{\pi}{b} \log \frac{a + ib - ic}{a - ib - ic}$$

So

$$\begin{aligned} I(a, b, c) &= \int_{-\infty}^{\infty} dx \frac{\log(x^2 + c^2)}{(x - a)^2 + b^2} \\ &= \frac{\pi}{b} \log(a + ib + ic) + \frac{\pi}{b} \log(a + ib - ic) - \frac{\pi}{b} \log \frac{a + ib - ic}{a - ib - ic} \\ &= \frac{\pi}{b} \log(a^2 + (b + c)^2) \end{aligned}$$

### 3. Integral and Laurent expansion

Consider the integral

$$\int_C dz \frac{z+2}{z^2-9}$$

where  $C$  is a positively-oriented circle of radius 4. Compute this integral by taking the sum of the residues inside the circle, and then again by computing the “residue at  $\infty$ ” meaning using the outer Laurent expansion

The integrand

$$f(z) = \frac{z+2}{z^2-9}$$

have simple poles at  $z = \pm 3$ .

$$\text{Res}(f, 3) = \lim_{z \rightarrow 3} (z-3) \frac{z+2}{z^2-9} = \frac{5}{6}$$

$$\text{Res}(f, -3) = \lim_{z \rightarrow -3} (z+3) \frac{z+2}{z^2-9} = \frac{1}{6}$$

By the residue theorem, the integral over the contour  $C$  is:

$$\int_C f(z), dz = 2\pi i \times (\text{Res}(f, 3) + \text{Res}(f, -3)) = 2\pi i$$

Using the outer Laurent expansion

$$\frac{z+2}{z^2-9} = \frac{1}{z} + O\left[\frac{1}{z^2}\right]$$

So using residue at  $\infty$

$$\text{Res}(f, \infty) = 1$$

$$\int_C f(z), dz = 2\pi i \times \text{Res}(f, \infty) = 2\pi i$$

## 4. Analytically continuing

Compute the integral

$$\int_0^{2\pi} \frac{d\theta}{2 - \cos \theta}$$

by analytically continuing the integrand and then using residues.

Perform a change of variables to convert the integral into a contour integral in the complex plane.

Let  $z = e^{i\theta}$ . Then,  $dz = ie^{i\theta}d\theta = izd\theta$  and  $d\theta = \frac{dz}{iz}$ . Also,  $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z})$ .

Substituting these into the integral, we get:

$$\int_0^{2\pi} \frac{d\theta}{2 - \cos \theta} = \int_{|z|=1} \frac{1}{2 - \frac{1}{2}(z + \frac{1}{z})} \times \frac{dz}{iz}$$

Simplifying this, we have:

$$\int_{|z|=1} \frac{dz}{2iz - \frac{1}{2}i(z^2 + 1)}$$

To find the poles of the integrand, we set the denominator to zero:

$$iz - \frac{1}{2}i(z^2 + 1) = 0$$

$$z_1 = 2 - \sqrt{3}, z_2 = 2 + \sqrt{3}$$

$z_1$  is within the contour, so

$$\int_{|z|=1} \frac{1}{2 - \frac{1}{2}(z + \frac{1}{z})} \times \frac{dz}{iz} = 2\pi i \frac{2}{i(z_2 - z_1)} = \frac{2\pi}{\sqrt{3}}$$

## 5. Gamma function

The Gamma function is usually defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

(This kind of integral is known as a Mellin transform, in this case of  $e^{-t}$ , and we can equivalently write it as a two-sided Laplace transform of  $e^{-e^s}$  by taking  $s = \log t$ .) Verify this integral exists for  $\Re(z) > 0$  and that

$$z\Gamma(z) = \Gamma(z+1)$$

$$\Gamma(n+1) = n!$$

The first relation can be iterated to analytically continue the  $\Gamma$  function to the whole complex plane, except for poles at the nonpositive integers. However, there is a way to analytically continue it all at once:

Split the integral into two parts  $\int_1^\infty + \int_0^1$ . Show that the first part yields an entire function in  $z$ , and that the second part equals

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n}$$

which is an entire meromorphic function with poles at the non-positive integers, thus giving an analytic continuation of  $\Gamma$  as a meromorphic function on all of  $\mathbb{C}$ .

To verify the properties of the Gamma function and its analytic continuation, we'll break down the problem into parts.

### Existence

We prove this integral converges absolutely.

$$\int_0^\infty |t^{z-1} e^{-t}| dt = \int_0^\infty t^{\operatorname{Re} z - 1} e^{-t} dt$$

For  $z \in \mathbb{R}$ ,

$$\int_0^\infty t^{z-1} e^{-t} dt = \int_0^1 t^{z-1} e^{-t} dt + \int_1^\infty t^{z-1} e^{-t} dt$$

we have for large  $N \leq t$ ,  $t^{z-1} e^{-t} \leq e^{-t/2}$

$$0 \leq t \leq N \implies t^{z-1} e^{-t} \leq t^{z-1}, \text{ and } \int_0^N t^{z-1} dt = \frac{t^z}{z} \Big|_0^N = \frac{N^z}{z}$$

$$N \leq t \implies t^{z-1} e^{-t} \leq e^{-t/2}, \text{ and } \int_N^\infty e^{-t/2} dt = -2e^{-t/2} \Big|_N^\infty = 2e^{-N/2}$$

So

$$\int_0^\infty |t^{z-1} e^{-t}| dt \leq \infty$$

Therefore, this integral exists for  $\Re(z) > 0$ .

$$z\Gamma(z) = \Gamma(z+1)$$

By definition:

$$z\Gamma(z) = z \int_0^\infty t^{z-1} e^{-t} dt$$

Using integration by parts, we get:

$$\begin{aligned} z\Gamma(z) &= t^z e^{-t} \Big|_0^\infty - \int_0^\infty t^z (-e^{-t}) dt \\ &= \int_0^\infty t^z e^{-t} dt \\ &= \Gamma(z+1) \end{aligned}$$

$$\Gamma(n+1) = n!$$

$$\begin{aligned} \Gamma(1) &= \int_0^\infty t^{1-1} e^{-t} dt \\ &= \int_0^\infty e^{-t} dt \\ &= 1. \end{aligned}$$

By induction we have  $\Gamma(n+1) = n!$

## Splitting

Now, to analytically continue the Gamma function, we split the integral into two parts:

$$\Gamma(z) = \int_1^\infty t^{z-1} e^{-t} dt + \int_0^1 t^{z-1} e^{-t} dt$$

- The first part  $\int_1^\infty t^{z-1} e^{-t} dt = \left( \int_1^N + \int_N^\infty \right) t^{z-1} e^{-t} dt$  converges for all  $z$  and is an entire function in  $z$ .
- The second part  $\int_0^1 t^{z-1} e^{-t} dt$  can be expressed as a power series using the Taylor expansion of  $e^{-t}$ :

$$\int_0^1 t^{z-1} e^{-t} dt = \int_0^1 t^{z-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^n dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{z+n-1} dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z+n}$$

This series converges for  $(z)$  not a non-positive integer and provides an analytic continuation of  $\Gamma$  as a meromorphic function on all of  $\mathbb{C}$ , with poles at the non-positive integers.

## 6. Gamma relation

Prove the relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

by combining the two Gamma integrals into

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty dt \frac{t^{z-1}}{t+1}$$

and evaluating this by residues.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

For fixed  $t$

$$\Gamma(1-z) = \int_0^\infty t^{-z} e^{-t} dt = \int_0^\infty u^{-z} e^{-u} du = t \int_0^\infty (vt)^{-z} e^{-vt} dv$$

By combining these two

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-z} dv dt = \int_0^\infty dv \frac{v^{z-1}}{v+1} = \int_0^\infty dt \frac{t^{z-1}}{t+1} = \int_{-\infty}^\infty dt \frac{e^{zx}}{1+e^x}$$

Complex function

$$\frac{e^{zx}}{1+e^x}$$

have pole at  $\pi i$ . Consider a contour  $C_R$  in the complex plane with vertices at  $R$ ,  $-R$ ,  $R + 2\pi i$ , and  $-R + 2\pi i$ , as  $R$  tends to infinity. We have

$$\int_{C_R} \frac{e^{xz}}{1+e^x} dx = -2\pi i e^{z\pi i}$$

$$\int_{C_{R3}} f(x, z) dx = -e^{2\pi i z} \int_{C_{R1}} f(x, z) dx$$

The right and left vertical sides of the rectangle tend to 0 as  $R \rightarrow \infty$  for  $z \in (0, 1)$ .

$$\int_{-\infty}^\infty dt \frac{e^{zx}}{1+e^x} = \frac{-2\pi i e^{z\pi i}}{1 - e^{2\pi i z}} = \frac{\pi}{\sin \pi z}$$

By analytic continuation, this relation is true for all  $z \in \mathbb{C}/\mathbb{Z}$

## Bibliography