Homework 04

2. Winding number

Suppose $f(z)$ is holomorphic in the disc $|z| \leq \epsilon$ and has a zero at $z = 0$ but nowhere else in the disc $|z| \leq \epsilon$. Show by direct integration that

$$\frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f'(z)}{f(z)} dz$$

equals the winding number of the argument of f around the circle $|z| = \epsilon$. Then use the residue theorem to show that this equals the degree of the zero, in agreement with the argument principle.

The integral

$$\int_C \frac{f'(z)}{f(z)} dz = \int_C \frac{1}{f(z)} df(z) = \int_{f(C)} \frac{1}{w} dw$$

effectively measures the total change in the argument of $f(z)$ as z traverses the circle, which equals the winding number of the argument of f.

Since $f(z)$ is holomorphic in the disc $|z| \leq \epsilon$ and has a zero at $z = 0$, $f(z)$ can be locally expressed as $z^n g(z)$, where n is the degree of the zero.

Then, $f'(z) = nz^{n-1}g(z) + z^n g'(z)$, and so

$$\frac{f'(z)}{f(z)} = \frac{n}{z} + \frac{g'(z)}{g(z)}$$

The residue at $z = 0$ is the coefficient of $\frac{1}{z}$ in this expression, which is n, the degree of the zero.

4. Analytic continuation and Fourier coefficents

Give an analytic continuation of $\cos\theta$ from the unit circle $z = e^{i\theta}$ to the complex plane minus the origin.

Conclude that the Fourier coefficients c_n of $e^{-\cos\theta}$ decrease faster than any exponential, meaning $c_n = o(e^{-\alpha n})$ for all α as $n \to \pm\infty$. Compare this to the Fourier series of $1/(cos\theta - 3/2)$, what is the decay of its Fourier coefficents?

Analytic continuation

On the unit circle $z = e^{i\theta}$, $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + z^{-1}\right)$. This expression provides an analytic continuation of $\cos\theta$ to the complex plane minus the origin, as it is well-defined for all $z \neq 0$.

Fourier coefficents of $e^{-\cos\theta}$

The function $e^{-\cos\theta}$ is smooth and periodic. The Fourier coefficients c_n of a periodic function $f(\theta)$ are given by:

$$c_n = 1/2\pi \int_0^{2\pi} f(\theta) e^{in\theta} d\theta = 1/2\pi \int_0^{2\pi} e^{-\cos\theta + in\theta} d\theta$$

$$1/2\pi \int_0^{2\pi} e^{-\cos\theta + in\theta} d\theta = 1/2\pi \int_{|z|=1} e^{-z/2-1/2z} z^n \frac{dz}{iz} = Res_{z=0} e^{-z/2-1/2z} z^{n-1}$$

The coefficient a_{-n} of $g(z) = e^{-z/2-1/2z}$ at the point of $z = 0$ can de derived by expanding by separately $e^{-z/2}$ as Talyor Series and $e^{-1/(2z)}$ as a Laurent Series and then multiplying these series together

$$a_{-n} = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \frac{1}{k!} \left(-\frac{1}{2}\right)^{(n+k)} \frac{1}{(n+k)!}$$

So fourier coefficients c_n

$$c_n = a_{-n} \leq |a_{-n}| \leq \frac{1}{2^n} \frac{1}{n!}$$

decrease faster than any exponential.

Fourier coefficents of $\frac{1}{\cos\theta - 3/2}$

The Fourier coefficients c_n of a periodic function $f(\theta)$ are given by:

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{in\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\cos\theta - 3/2} e^{in\theta} d\theta \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{2z}{z^2 - 3z + 1} z^{n-1} dz \end{aligned}$$

Let $z^2 - 3z + 1 = (z - z_1)(z - z_2)$, where $z_1 = \frac{1}{2}\left(3 - \sqrt{5}\right)$, $z_2 = \frac{1}{2}\left(3 + \sqrt{5}\right)$.

$$
\begin{aligned}
c_n &= Res_{z=z_1} \frac{2z}{(z-z_1)(z-z_2)} z^{n-1} \\
&= \frac{2z_1}{z_1 - z_2} z_1^{n-1}
\end{aligned}
$$

5. Laurent series and singularity

Let's consider a function f that is holomorphic in a disc around z_0 except at z_0 itself.

1. **Removable Singularity:**

   If f has a removable singularity at z_0, it means that f can be extended to a holomorphic function at z_0. In terms of the Laurent series, this implies that all the coefficients a_n for $n < 0$ are zero because it reduces to its Taylor series.

   Conversely, if all $a_n = 0$ for $n < 0$, the Laurent series reduces to a Taylor series, implying that f is holomorphic at z_0 (since it can be expressed as a power series), and thus the singularity is removable.

2. **Pole of Order m :**

   If f has a pole of order m at z_0, it means that in the Laurent series, there is a term with $(z - z_0)^{-m}$ (where $a_{-m} \neq 0$) and no terms with higher negative powers.

   Conversely, if there is some $m < 0$ such that $a_m \neq 0$ but $a_n = 0$ for all $n < m$, then the Laurent series has a term $a_m (z - z_0)^m$ as its term with the highest negative power, indicating a pole of order m .

3. **Essential Singularity:**

   If the singularity at z_0 is neither removable nor a pole, it must be an essential singularity. This is characterized by the fact that there are infinitely many negative powers of $z - z_0$ in the Laurent series with non-zero coefficients. In other words, if the Laurent series has non-zero a_n for infinitely many $n < 0$, then z_0 is an essential singularity.

6. Euler proof of Basel problem

Using the result of the bonus problem, prove that

$$
\sin \pi z = \prod (1 - z/n) e^{z/n} = \pi z \prod (1 - z^2/n^2)
$$

Then compare the Taylor series of sin πz to the first couple terms in the expansion of the infinite product to conclude

$$\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$$

According to the Weierstrass factorization theorem, an entire function can be represented as a product over its zeros. The function $\sin \pi z$ is entire and has zeros at all integers. The product representation for $\sin \pi z$ is given by:

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right) = \sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

The Taylor series expansion of $\sin \pi z$ around $z = 0$ is:

$$\sin \pi z = \pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \frac{\pi^7 z^7}{7!} + \cdots$$

Now, let's expand the infinite product to the first couple of terms and keeping terms up to z^3, we get:

$$\begin{aligned} \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) &= \pi z \left(1 - \frac{z^2}{1^2}\right)\left(1 - \frac{z^2}{2^2}\right)\left(1 - \frac{z^2}{3^2}\right)\cdots \\ &= \pi z \left(1 - z^2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + \cdots\right) \end{aligned}$$

Comparing the coefficient of z^3 from the Taylor series and the product expansion, we have:

$$-\frac{\pi^3}{6} = -\pi\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

This is the result we want.

Bibliography