

# Homework 03

## 1. Conformal map

Consider the half-infinite strip

$$S = \{z \mid \operatorname{Re} z > 0, 2i < \operatorname{Im} z < 5i\}$$

Find an invertible conformal map sending  $S$  to the upper half plane

$$H = \{z \mid \operatorname{Im} z > 0\},$$

we can proceed in steps using standard conformal mappings.

1. **Translate the Strip:** First, we translate the strip downwards  $T(z) = z - 2i$  so that its imaginary boundaries are on the real axis and at  $3i$ .
2. **Scale the Strip:** Next, we scale the strip so that its width becomes  $\pi$ . Define the scaling map  $D(z) = \frac{\pi}{3}z$ .
3. **Apply the Exponential Function:** The exponential function  $E(z) = e^z$  maps horizontal strips to  $\{z \mid \operatorname{Im} z > 0, |z| > 1\}$
4. **Map to the Upper Half-Plane:**  $R(z) = \frac{1}{2}(z + 1/z)$  will map to the upper half-plane.

So, the complete conformal map  $F$  from  $S$  to  $H$  is the composition of these maps:

$$F(z) = R(E(D(T(z)))) = \sqrt{e^{\frac{\pi}{3}(z-3.5i)}}.$$

This map is invertible and conformal.

Note that the inverse map of  $R(z)$

$$z = w + \sqrt{w^2 - 1}$$

has a branch cut at  $w \in (-1, 1)$ . However, for  $|z| > 1$ , we have  $\operatorname{Im} w > 0$ , so the maps are invertible.

## 2. Saddle point

Prove that if  $f = u + iv$  is holomorphic at  $z = 0$  and  $f'(z)$  has a zero of degree 1 at  $z = 0$ , that both  $u$  and  $v$  have saddle points at  $z = 0$ .

At  $z = 0$ ,  $f' = u_x + iv_x = 0$

$$\Rightarrow u_x(0) = 0 \quad \text{and} \quad v_x(0) = 0$$

$f$  is holomorphic

$$v_y = u_x = 0$$

$$u_y = -v_x = 0$$

So the Hessian determinant is

$$\begin{aligned} D_u &= u_{xx}u_{yy} - (u_{xy})^2 = -v_{yx}v_{xy} - u_{xy}^2 \\ &= -v_{xy}^2 - u_{xy}^2 \end{aligned}$$

$$D_v = v_{xx}v_{yy} - (v_{xy})^2 = -v_{xy}^2 - u_{xy}^2$$

Since  $f'$  has a zero of degree 1  $\Rightarrow$  second derivative of  $u$  and  $v$  are nonzero at  $z = 0$ , which gives  $D_u < 0$  and  $D_v < 0$ . Given first derivative is zero,  $u$  and  $v$  have saddle points at  $z = 0$

### 3. Holomorphic functions agree

Show that if two holomorphic functions agree on an interval of the real line, they agree everywhere.

Let's say two holomorphic function  $f$  and  $g$  agree on Interval  $I$ . we show  $h = f - g$  if  $h \equiv 0$  on  $I$ , then  $h$  is 0 everywhere.

$$h(z) = \sum_{n=0}^{\infty} \frac{(z-c)^n}{n!} h^{(n)}(c)$$

for  $c \in I$ . On the real line,  $h(x)$  and all its derivatives with respect to  $x$  vanish. So  $h^{(n)}(c) = 0$  for all  $n \geq 0$ . And because  $h$  is an entire function the radius of convergence should be infinite. Therefore, for any  $z \in C$  lies within the circle of convergence, we have  $h(z) = 0$ .

### 5. Mobius transformations

Show that Mobius transformations send circles and lines to circles and lines.

Mobius transformations

$$f(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{e}{z + \frac{d}{c}},$$

can be decomposed into four simple transformation of translation, dilation, and inversion.

$$f = f_4 \circ f_3 \circ f_2 \circ f_1.$$

Since translation, dilation preserve geometrical lines and circles, we only need to show that inversion  $I(z) = 1/z$  sends circles and lines to circles and lines.

$$I(z) = I(x + iy) = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - \frac{yi}{x^2 + y^2}$$

So  $I$  maps  $(x, y)$  into a  $(u, v)$  with

$$u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = \frac{-y}{x^2 + y^2}$$

For line of general form  $Ax + By = C$ , we have

$$Au - Bv = (u^2 + v^2)C$$

Thus  $I$  maps a line to a circle ( $C \neq 0$ ) or a line ( $C = 0$ ).

For circle of general form  $Dx + Ey + F(x^2 + y^2) = R$ , we have

$$Du - Ev + F = R(u^2 + v^2)$$

Thus  $I$  maps a circle to a circle ( $R \neq 0$ ) or a line ( $R = 0$ ). Note here,  $R$  is not the radius of the original circle.

## 6. cross-ratio under simultaneous Möbius transformations

Show that for any three points  $z_1, z_2, z_3$ , there is precisely one Möbius transformation sending  $z_1$  to 0,  $z_2$  to 1, and  $z_3$  to infinity. The image of a fourth point  $z_4$  under this map defines the “cross-ratio” of  $(z_1, z_2, z_3, z_4)$ . Show that the cross ratio is preserved under simultaneous Möbius transformations of these four points.

Let Möbius transformation  $f(z) = (az + b)/(cz + d)$ , satisfying

$$f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$$

Then the Möbius transformation is determined by

$$f(z_1) = 0 \Rightarrow az_1 + b = 0$$

$$f(z_2) = 1 \Rightarrow az_2 + b - cz_2 - d = 0$$

$$f(z_3) = \infty \Rightarrow cz_3 + d = 0$$

The three linear equations can be solved in the sense of their relative ratio.

And the Möbius transformation can be written as

$$f(z) = \frac{z_2 - z_3}{z_2 - z_1} \frac{z - z_1}{z - z_3}$$

So the cross ratio is

$$\frac{z_2 - z_3}{z_2 - z_1} \frac{z_4 - z_1}{z_4 - z_3}$$

Then the cross ratio of the image under the transformation of any  $f$  is

$$\frac{f(z_2) - f(z_3)}{f(z_2) - f(z_1)} \frac{f(z_4) - f(z_1)}{f(z_4) - f(z_3)}$$

Note that

$$f(x) - f(y) = \frac{ax + b}{cx + d} - \frac{ay + b}{cy + d} = \frac{(ad - bc)(x - y)}{(cx + d)(cy + d)}$$

and

$$\frac{f(x) - f(y)}{f(x) - f(z)} = \frac{(x - y)(cz + d)}{(x - z)(cy + d)}$$

So

$$\frac{f(z_2) - f(z_3)}{f(z_2) - f(z_1)} \frac{f(z_4) - f(z_1)}{f(z_4) - f(z_3)} = \frac{(z_2 - z_3)(cz_1 + d)}{(z_2 - z_1)(cz_3 + d)} \frac{(z_4 - z_1)(cz_3 + d)}{(z_4 - z_3)(cz_1 + d)} = \frac{z_2 - z_3}{z_2 - z_1} \frac{z_4 - z_1}{z_4 - z_3}$$

## Bibliography