Homework 03

1. Conformal map

Consider the half-infinite strip

$$S = \{z \mid \operatorname{Re} z > 0, 2i < \operatorname{Im} z < 5i\}$$

Find an invertible conformal map sending S to the upper half plane

$$H = \{z \mid \operatorname{Im} z > 0\},$$

we can proceed in steps using standard conformal mappings.

1. **Translate the Strip**: First, we translate the strip downwards $T(z) = z - 2i$ so that its imaginary boundaries are on the real axis and at $3i$.
2. **Scale the Strip**: Next, we scale the strip so that its width becomes π. Define the scaling map $D(z) = \frac{\pi}{3}z$.
3. **Apply the Exponential Function**: The exponential function $E(z) = e^z$ maps horizontal strips to $\{z \mid \operatorname{Im} z > 0, |z \mid > 1\}$
4. **Map to the Upper Half-Plane**: $R(z) = \frac{1}{2}(z + 1/z)$ will map to the upper half-plane.

So, the complete conformal map F from S to H is the composition of these maps:

$$F(z) = R(E(D(T(z)))) = \sqrt{e^{\frac{\pi}{3}(z-3.5i)}}.$$

This map is invertible and conformal.

Note that the inverse map of $R(z)$

$$z = w + \sqrt{w^2 - 1}$$

has a branch cut at $w \in (-1, 1)$. However, for $|z| > 1$, we have $\operatorname{Im} w > 0$, so the maps are inevitable.

2. Saddle point

Prove that if $f = u + iv$ is holomorphic at $z = 0$ and $f'(z)$ has a zero of degree 1 at $z = 0$, that both u and v have saddle points at $z = 0$.

At $z = 0$, $f' = u_x + iv_x = 0$

$$\Rightarrow u_x(0) = 0 \quad \text{and} \quad v_x(0) = 0$$

f is holomorphic

$$v_y = u_x = 0$$
$$u_y = -v_x = 0$$

So the Hessian determinant is

$$\begin{aligned} D_u = u_{xx}u_{yy} - \left(u_{xy}\right)^2 &= -v_{yx}v_{xy} - u_{xy}^2 \\ &= -v_{xy}^2 - u_{xy}^2 \end{aligned}$$
$$D_v = v_{xx}v_{yy} - \left(v_{xy}\right)^2 = -v_{xy}^2 - u_{xy}^2$$

Since f' has a zero of degree 1 $\Rightarrow$ second derivative of u and v are nonzero at $z = 0$, which gives $D_u < 0$ and $D_\nu < 0$. Given first derivative is zero, u and v have saddle points at $z = 0$

3. Holomorphic functions agree

> Show that if two holomorphic functions agree on an interval of the real line, they agree everywhere.

Let's say two holomorphic function f and g agree on Interval I. we show $h = f - g$ if $h \equiv 0$ on I, then h is 0 everywhere.

$$h(z) = \sum_{n=0}^{\infty} \frac{(z-c)^n}{n!} h^{(n)}(c)$$

for $c \in I$. On the real line, $h(x)$ and all its derivatives with respect to x vanish. So $h^{(n)}(c) = 0$ for all $n \geq 0$. And because h is an entire function the radius of convergence should be infinite. Therefore, for any $z \in C$ lies within the circle of convergence, we have $h(z) = 0$.

5. Mobius transformations

> Show that Mobius transformations send circles and lines to circles and lines.

Mobius transformations

$$f(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{e}{z + \frac{d}{c}},$$

can be decomposed into four simple transformation of translation, dilation, and inversion.

$$f = f_4 \circ f_3 \circ f_2 \circ f_1.$$

Since translation, dilation perserve geometrical lines and circles, we only need to show that inversion $I(z) = 1/z$ sends circles and lines to circles and lines.

$$I(z) = I(x + iy) = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - \frac{yi}{x^2 + y^2}$$

So I maps (x, y) into a (u, v) with

$$u = \frac{x}{x^2 + y^2} \quad \text{and} \quad v = \frac{-y}{x^2 + y^2}$$

For line of general form $Ax + By = C$, we have

$$Au - Bv = (u^2 + v^2)C$$

Thus I maps a line to a circle ($C \neq 0$) or a line ($C = 0$).

For cicrcle of general form $Dx + Ey + F(x^2 + y^2) = R$, we have

$$Du - Ev + F = R(u^2 + v^2)$$

Thus I maps a circle to a circle ($R \neq 0$) or a line ($R = 0$). Note here, R is not the radius of the original circle.

6. cross-ratio under simultaneous Mobius transformations

> Show that for any three points z_1, z_2, z_3, there is precisely one Mobius transformation sending z_1 to 0, z_2 to 1, and z_3 to infinity. The image of a fourth point z_4 under this map defines the "cross-ratio" of (z_1, z_2, z_3, z_4). Show that the cross ratio is preserved under simultaneous Mobius transformations of these four points.

Let Möbius transformation $f(z) = (az + b)/(cz + d)$, satisfying

$$f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$$

Then the Möbius transformation is determined by

$$\begin{aligned} f(z_1) = 0 &\Rightarrow az_1 + b = 0 \\ f(z_2) = 1 &\Rightarrow az_2 + b - cz_2 - d = 0 \\ f(z_3) = \infty &\Rightarrow cz_3 + d = 0 \end{aligned}$$

The three linear equations can be solved in the sense of their relative ratio.

And the Möbius transformation can be written as

$$f(z) = \frac{z_2 - z_3}{z_2 - z_1} \frac{z - z_1}{z - z_3}$$

So the cross ratio is

$$\frac{z_2 - z_3}{z_2 - z_1} \frac{z_4 - z_1}{z_4 - z_3}$$

Then the cross ratio of the image under the transformation of any f is

$$\frac{f(z_2) - f(z_3)}{f(z_2) - f(z_1)} \frac{f(z_4) - f(z_1)}{f(z_4) - f(z_3)}$$

Note that

$$f(x) - f(y) = \frac{ax + b}{cx + d} - \frac{ay + b}{cy + d} = \frac{(ad - bc)(x - y)}{(cx + d)(cy + d)}$$

and

$$\frac{f(x) - f(y)}{f(x) - f(z)} = \frac{(x - y)(cz + d)}{(x - z)(cy + d)}$$

So

$$\frac{f(z_2) - f(z_3)}{f(z_2) - f(z_1)} \frac{f(z_4) - f(z_1)}{f(z_4) - f(z_3)} = \frac{(z_2 - z_3)(cz_1 + d)}{(z_2 - z_1)(cz_3 + d)} \frac{(z_4 - z_1)(cz_3 + d)}{(z_4 - z_3)(cz_1 + d)} = \frac{z_2 - z_3}{z_2 - z_1} \frac{z_4 - z_1}{z_4 - z_3}$$

Bibliography