

## Homework 08

### 1 Asymptotic series for $\log(x - 2)$

Find an asymptotic series for

$$f(x) = \log(x - 2)$$

as  $x \rightarrow \infty$  in terms of  $\log x$  and inverse powers of  $x$ . (Hint: split the log and use Taylor series on one part.)

$$f(x) = \log(x - 2) = \log x + \log\left(1 - \frac{2}{x}\right)$$

The Taylor series expansion of  $\log(1 - y)$  around  $y = 0$  is:

$$\log(1 - y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \dots$$

By substituting  $y = \frac{2}{x}$  into this series, the asymptotic series for  $f(x)$  as  $x \rightarrow \infty$  is:

$$f(x) = \log x - \frac{2}{x} - \frac{2^2}{2x^2} - \frac{2^3}{3x^3} - \dots$$

### 2 Watson's lemma and $\int_0^1 \frac{e^{-sx}}{1+x^2} dx$

Use Watson's lemma to find the  $s \rightarrow \infty$  asymptotic series for

$$I(s) = \int_0^1 \frac{e^{-sx}}{1+x^2} dx$$

To use Watson's Lemma, we need to express  $\frac{1}{1+x^2}$  as a power series at  $x = 0$ . The Taylor series of  $\frac{1}{1+x^2}$  around  $x = 0$  is:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

which is valid for  $|x| < 1$ . Substituting this into  $I(s)$  gives:

$$I(s) = \int_0^1 e^{-sx} (1 - x^2 + x^4 - x^6 + \dots) dx$$

Now, we can integrate term by term:

1. For the first term:

$$\int_0^1 e^{-sx} dx = \frac{1}{s}(1 - e^{-s})$$

As  $s \rightarrow \infty$ ,  $e^{-s}$  approaches 0 faster than  $1/s$ , so this term becomes  $\frac{1}{s}$ .

2. For the second term:

$$- \int_0^1 x^2 e^{-sx} dx$$

Applying integration by parts or a similar method, we find this term is  $O(\frac{1}{s^3})$  as  $s \rightarrow \infty$ .

3. Similarly, each subsequent term will contribute higher order terms in  $\frac{1}{s}$ .

Hence, the asymptotic expansion of  $I(s)$  as  $s \rightarrow \infty$  is:

$$I(s) \sim \frac{1}{s} - \frac{1}{s^3} + \dots$$

### 3 Watson's lemma and $\int_0^\infty \sin(\sqrt{x}) e^{-sx^2} dx$

Use Watson's lemma to find the  $s \rightarrow \infty$  asymptotic series for

$$I(s) = \int_0^\infty \sin(\sqrt{x}) e^{-sx^2} dx$$

The integral  $I(s)$  does not directly fit this form of Watson's lemma due to the  $e^{-sx^2}$  term. So first, we perform a change of variable to transform the integral into a more suitable form for Watson's Lemma. Let  $u = x^2$ , then  $du = 2x dx$  or  $dx = \frac{du}{2\sqrt{u}}$ . The integral becomes:

$$I(s) = \int_0^\infty \sin(u^{1/4}) e^{-su} \frac{du}{2\sqrt{u}}$$

Expand  $\sin(u^{1/4})$  in a Taylor series about  $u = 0$ :

$$\sin(u^{1/4}) = u^{1/4} - \frac{u^{3/4}}{3!} + \frac{u^{5/4}}{5!} - \dots$$

Substituting this series into the integral, we get:

$$I(s) = \int_0^\infty \left( u^{1/4} - \frac{u^{3/4}}{3!} + \dots \right) e^{-su} \frac{du}{2\sqrt{u}} = \frac{\Gamma(3/4)}{2s^{3/4}} - \frac{\Gamma(5/4)}{12s^{5/4}} + \dots$$

## 4 Asymptotic series for the error function

Use integration by parts to give the  $x \rightarrow \infty$  asymptotic series for the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

We can apply successive integration by parts to  $\operatorname{erfc}(x)$ , in which the first step gives

$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2) dt \\ &= \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{-2t \exp(-t^2)}{-2t} dt, \left[ u = \frac{1}{-2t}, dv = -2t \exp(-t^2) dt \right] \\ &= \frac{2}{\sqrt{\pi}} \left[ \frac{\exp(-t^2)}{-2t} - \int \frac{\exp(-t^2)}{2t^2} dt \right]_x^\infty \\ &= \frac{2}{\sqrt{\pi}} \left[ \frac{\exp(-x^2)}{2x} - \int_x^\infty \frac{\exp(-t^2)}{2t^2} dt \right] \\ &= \frac{2e^{-x^2}}{\sqrt{\pi}} \left( \frac{1}{2x} \right) - \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{\exp(-t^2)}{2t^2} dt \end{aligned}$$

We have obtained the first term. And applying integration by parts to the new integral gives

$$\begin{aligned} \int_x^\infty \frac{\exp(-t^2)}{2t^2} dt &= \frac{1}{2} \int_x^\infty \frac{-2t \exp(-t^2)}{-2t^3} dt \left[ u = -\frac{1}{2t^3}, dv = -2t \exp(-t^2) dt \right] \\ &= \frac{1}{2} \left[ \frac{\exp(-t^2)}{-2t^3} - \int \frac{3 \exp(-t^2)}{2t^4} dt \right]_x^\infty \\ &= \frac{1}{2} \left[ \frac{\exp(-x^2)}{2x^3} - \int_x^\infty \frac{3 \exp(-t^2)}{2t^4} dt \right] \end{aligned}$$

Thus we have

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \left( \frac{1}{2x} - \frac{1}{4x^3} - \int_x^\infty \frac{3 \exp(-t^2)}{2t^4} dt \right),$$

Continuing with successive integration by parts we will obtain the asymptotic expansion.

## 5 Stirling's formula

Use Laplace's method to derive Stirling's formula

$$n! \sim \sqrt{2\pi n} n^n e^{-n} \quad n \rightarrow \infty$$

using the Gamma function. Also find the next term in the asymptotic series.

Gamma function, which is related to the factorial by  $\Gamma(n+1) = n!$  is defined as:

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$$

For large  $n$ , we approximate this integral using Laplace's method, which is effective for integrals of the form  $\int e^{Mf(t)} dt$  where  $M$  is a large parameter. Here, we can rewrite the Gamma function as:

$$\Gamma(n+1) = \int_0^\infty e^{n \log t - t} dt$$

Now, to apply Laplace's method, we find the maximum of the function  $f(t) = \log t - \frac{t}{n}$ . Taking the derivative and setting it to zero gives:

$$\frac{d}{dt} \left( \log t - \frac{t}{n} \right) = \frac{1}{t} - \frac{1}{n} = 0$$

This yields  $t = n$  as the point where  $f(t)$  attains its maximum. We then expand  $f(t)$  around this point:

$$f(t) \approx f(n) + \frac{1}{2} f''(n) (t - n)^2$$

where  $f(n) = \log n - 1$  and  $f''(n) = -1/n^2$ . The integral becomes:

$$\Gamma(n+1) \approx e^{n \log n - n} \int_0^\infty e^{-\frac{1}{2} \frac{(t-n)^2}{n}} dt$$

Changing variables with  $u = \frac{t-n}{\sqrt{n}}$  gives:

$$\Gamma(n+1) \approx e^{n \log n - n} \sqrt{n} \int_0^\infty e^{-\frac{1}{2} u^2} du$$

Recognizing the Gaussian integral, we get:

$$\Gamma(n+1) = n! \approx e^{n \log n - n} \sqrt{2\pi n} = \sqrt{2\pi n} n^n e^{-n} \quad n \rightarrow \infty$$

The second derivative  $f''(t)$  at the point  $t = n$  gives us the next leading term. The third derivative of  $f(t)$  is:

$$f'''(t) = \frac{2}{t^3}$$

Evaluating this at  $t = n$  gives:

$$f'''(n) = \frac{2}{n^3}$$

The corresponding term in the expansion is of the order  $\frac{1}{n}$ , which leads to the next term in the series being  $\frac{1}{12n}$ . Therefore, the improved Stirling's formula, including the next term in the asymptotic series, is:

$$n! \sim \sqrt{2\pi n} n^n e^{-n} \left(1 + \frac{1}{12n}\right)$$

## 6 Leading asymptotics of $I(x) = \int_{-1}^1 e^{-x \sin^4 t} dt$

Use Laplace's method to find the leading asymptotics of

$$I(x) = \int_{-1}^1 e^{-x \sin^4 t} dt$$

as  $x \rightarrow \infty$ .

For the given integral, we observe that  $\sin^4 t$  is maximized when  $t = 0$  within the interval  $[-1, 1]$ . Near this point, the function  $\sin^4 t$  can be approximated by its Taylor expansion:

$$\sin^4 t \approx t^4$$

for small values of  $t$ . The integral becomes:

$$I(x) \approx \int_{-1}^1 e^{-x t^4} dt$$

As  $x \rightarrow \infty$ , the contribution to the integral from regions where  $t$  is not very small becomes negligible, so we can extend the limits of the integral to infinity for the purpose of asymptotic approximation:

$$I(x) \approx \int_{-\infty}^{\infty} e^{-x t^4} dt$$

Let  $u = x^{1/4} t$ , then  $dt = x^{-1/4} du$  and the integral becomes:

$$I(x) \approx \int_{-\infty}^{\infty} e^{-u^4} x^{-1/4} du$$

The integral  $\int_{-\infty}^{\infty} e^{-u^4} du$  is a constant (independent of  $x$ ), so the leading asymptotic behavior of  $I(x)$  as  $x \rightarrow \infty$  is given by:

$$I(x) \sim Cx^{-1/4}$$

where  $C$  is the value of the integral  $\int_{-\infty}^{\infty} e^{-u^4} du$ , which can be evaluated numerically.

## **Bibliography**