

# Homework 07

## 1 Green's function for Poisson's equation

- Use Fourier analysis to compute a Green's function for Poisson's equation on  $\mathbb{R}^3$ , satisfying

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)G(x, y, z) = \delta^{(3)}(x, y, z)$$

- Prove there is a unique such Green's function which goes to zero at infinity (hint: use Liouville's theorem for harmonic functions).
- Find the Green's function but now for functions on the unit ball around the origin, with Dirichlet boundary conditions  $\phi(x, y, z) = 0$  for  $x^2 + y^2 + z^2 = 1$ . Hint: Schwartz reflection principle/method of images.

### 1.1 Fourier Analysis

Let  $\hat{G}(k_x, k_y, k_z)$  be the Fourier transform of  $G(x, y, z)$ . After Fourier transform, the equation becomes:

$$-(k_x^2 + k_y^2 + k_z^2)\hat{G}(k_x, k_y, k_z) = 1$$

since the Fourier transform of  $\delta^{(3)}(x, y, z)$  is 1.

Thus, the solution in Fourier Space:

$$\hat{G}(k_x, k_y, k_z) = -\frac{1}{k_x^2 + k_y^2 + k_z^2}$$

To find  $G(x, y, z)$ , perform the inverse Fourier transform:

$$G(x, y, z) = \int_{\mathbb{R}^3} -\frac{e^{i(k_x x + k_y y + k_z z)}}{k_x^2 + k_y^2 + k_z^2} dk_x dk_y dk_z$$

This integral yields  $G(x, y, z) = \frac{-1}{4\pi\sqrt{x^2 + y^2 + z^2}}$ .

### 1.2 Uniqueness of the Green's Function

Liouville's theorem states that a bounded harmonic function on  $\mathbb{R}^n$  is constant. Given any two Green's functions  $G_1$  and  $G_2$  that vanish at infinity, their difference  $G_1 - G_2$  is harmonic (satisfies Laplace's equation) and bounded. By Liouville's theorem,  $G_1 - G_2$  is constant, and since both go to zero at infinity, this constant must be zero. Thus,  $G_1 = G_2$ , proving uniqueness.

### 1.3 Green's Function on the Unit Ball with Dirichlet Boundary Conditions

To find a function  $G(x, y, z; x', y', z')$  that satisfies:

1.  $(\partial_x^2 + \partial_y^2 + \partial_z^2)G(x, y, z; x', y', z') = -\delta(x - x', y - y', z - z')$  within the unit ball  $x^2 + y^2 + z^2 < 1$ .
2.  $G(x, y, z; x', y', z') = 0$  for  $x^2 + y^2 + z^2 = 1$ , enforcing the Dirichlet boundary conditions.

We place an "image" point source at  $(x'', y'', z'')$  where  $(x'', y'', z'') = \frac{(x', y', z')}{|x'|^2 + |y'|^2 + |z'|^2}$  outside the unit ball in such a way that the combined effect of the real source and the image source satisfies the boundary conditions.

The Green's function then is a combination of the solution from both the real and the image source

$$G(x, y, z; x', y', z') = \frac{1}{4\pi\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{4\pi\sqrt{(x - x'')^2 + (y - y'')^2 + (z - z'')^2}}$$

More generally, we define the point  $x^* = \frac{x}{|x|^2}$  dual to  $x$ . Therefore, a Green's function for  $B^n(0, 1)$  is given by  $G(x, y) = \Phi(y - x) - \Phi(|x|(y - x^*))$ .

## 2 Green's function for the heat equation

Find the Green's function for the heat equation

$$\partial_t u = \partial_x^2 u$$

by Fourier analysis.

We want to find a Green's function  $G(x, t; x', t')$  that satisfies the following properties:

1.  $\partial_t G = \partial_x^2 G$
2.  $G(x, t; x', t')$  behaves like a delta function as  $t \rightarrow t'^+$ , i.e.,  $G(x, t; x', t') \rightarrow \delta(x - x')$  as  $t \rightarrow t'^+$ .

By taking the Fourier transform of  $G$  with respect to  $x$ , the heat equation in Fourier space becomes

$$\partial_t \hat{G} = -k^2 \hat{G}$$

This is a first-order linear differential equation in  $t$ , the solution is

$$\hat{G}(k, t; x', t') = A(k, x', t')e^{-k^2(t-t')}$$

where  $A(k, x', t')$  is to be determined.

Now we apply the initial condition. The initial condition is that  $G$  approaches  $\delta(x - x')$  as  $t \rightarrow t'^+$ . In Fourier space, this translates to  $\hat{G}(k, t; x', t') \rightarrow e^{-ik(x-x')}$  as  $t \rightarrow t'^+$ . Thus

$$A(k, x', t') = e^{-ikx'}$$

Then take the inverse Fourier transform of  $\hat{G}(k, t; x', t')$  to get back to the spatial domain:

$$G(x, t; x', t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx'} e^{-ikx} e^{-k^2(t-t')} dk$$

This integral yields a function of the form:

$$G(x, t; x', t') = \frac{1}{\sqrt{4\pi(t-t')}} e^{-\frac{(x-x')^2}{4(t-t')}}$$

for  $t > t'$ .

So the Green's function for the heat equation is:

$$G(x, t; x', t') = \begin{cases} \frac{1}{\sqrt{4\pi(t-t')}} e^{-\frac{(x-x')^2}{4(t-t')}} & t > t' \\ 0 & \text{otherwise} \end{cases}$$

### 3 Resolvent $R(z, A) = (z\mathbb{1} - A)^{-1}$

Given a finite dimensional complex matrix  $A$ , we can define the resolvent

$$R(z, A) = (z\mathbb{1} - A)^{-1} = \frac{1}{z - A},$$

where  $\mathbb{1}$  is the identity matrix. This equation makes sense whenever  $z$  is not an eigenvalue of  $A$ , and so we can consider  $R(z, A)$  as a meromorphic, matrix-valued function of  $z$ , with poles at the eigenvalues of  $A$ . For these problems, assume  $A$  is Hermitian:

1. Show that

$$R(z, A) = \frac{P_\lambda}{z - \lambda},$$

where  $P_\lambda$  is the projector onto the eigenspace of  $\lambda$  eigenvectors.

2. Suppose  $w$  is orthogonal to the kernel of  $A$ . Prove that

$$v = \int_C \frac{dz}{2\pi i} \frac{R(z, A)}{z} w$$

solves the equation

$$Av = w,$$

where  $C$  is a contour winding once around  $z = 0$  and not enclosing any other eigenvalues.

3. Show the same as in (2) for a contour enclosing all eigenvalues except for 0.

### 3.1 Expression for $R(z, A)$

Given a Hermitian matrix  $A$ , its eigenvalues are real, and it can be diagonalized. Let  $\lambda$  be an eigenvalue of  $A$  and  $P_\lambda$  the projector onto the eigenspace of  $\lambda$ . We have spectral decomposition

$$A = \sum_{\lambda} \lambda P_{\lambda}$$

the expression for  $R(z, A)$  can be simplified:

$$R(z, A) = (z\mathbb{1} - A)^{-1} = \left( z \sum_{\lambda} P_{\lambda} - \sum_{\lambda} \lambda P_{\lambda} \right)^{-1}$$

Because the projectors  $P_{\lambda}$  are orthogonal and sum to the identity:  $R(z, A) = \sum_{\lambda} (zP_{\lambda} - \lambda P_{\lambda})^{-1}$ .

• Thus,  $R(z, A) = \sum_{\lambda} \frac{P_{\lambda}}{z - \lambda}$ .

### 3.2 Solving $Av = w$

Substitute the expression for  $R(z, A)$  into the integral:

$$v = \int_C \frac{dz}{2\pi i} \frac{\sum_{\lambda} \frac{P_{\lambda}}{z - \lambda}}{z} w$$

Evaluate it using the residue theorem. Since  $C$  winds around  $z = 0$  and does not enclose any other eigenvalues, the only contribution comes from the pole at  $z = 0$ .

The residue at  $z = 0$  gives  $P_0 w$ . So  $Av = AP_0 w = w$

### 3.3 Contour Enclosing All Eigenvalues Except 0

Use the same integral expression for  $v$ . But now the integral will pick up residues from all poles  $\lambda \neq 0$  within the contour  $C$ .

For each  $\lambda \neq 0$ , the residue at  $z = \lambda$  is  $\frac{P_{\lambda} w}{\lambda}$ .

Summing up the residues,  $v = \sum_{\lambda \neq 0} \frac{P_{\lambda} w}{\lambda}$ .

Multiplying both sides by  $A$  (and using  $AP_\lambda = \lambda P_\lambda$ ), we get  $Av = \sum_{\lambda \neq 0} P_\lambda w = w - P_0 w$ .

Since  $w$  is orthogonal to the kernel of  $A$ ,  $P_0 w = 0$ . Therefore,  $Av = w$ .

## **Bibliography**