

Problem Set 3

1 Minimum and Maximum Variance Analysis

Minimize the variance σ^2 Eq. (8.4) of Ch. 8 by writing out the derivatives with respect to the vector components. Show that the solution is given by Equation (8.7) which is an eigenvalue problem with orthogonal eigenvectors.

$$\sigma^2 = \frac{1}{M} \sum_{m=1}^M |(\mathbf{B}^{(m)} - \langle \mathbf{B} \rangle) \cdot \hat{\mathbf{n}}|^2 \quad (1)$$

$$\sum_{\nu=1}^3 M_{\mu\nu}^B n_\nu = \lambda n_\mu \quad (2)$$

Expand the dot product in components:

$$(\mathbf{B}^{(m)} - \langle \mathbf{B} \rangle) \cdot \hat{\mathbf{n}} = \sum_{\mu=1}^3 (B_\mu^{(m)} - \langle B_\mu \rangle) n_\mu,$$

so that

$$\sigma^2 = \frac{1}{M} \sum_{m=1}^M \left(\sum_{\mu=1}^3 (B_\mu^{(m)} - \langle B_\mu \rangle) n_\mu \right)^2 = \sum_{\mu, \nu=1}^3 n_\mu n_\nu \left[\frac{1}{M} \sum_{m=1}^M (B_\mu^{(m)} - \langle B_\mu \rangle) (B_\nu^{(m)} - \langle B_\nu \rangle) \right]$$

Define the magnetic variance matrix

$$M_{\mu\nu}^B \equiv \langle B_\mu B_\nu \rangle - \langle B_\mu \rangle \langle B_\nu \rangle$$

so that

$$\sigma^2 = \sum_{\mu, \nu=1}^3 n_\mu M_{\mu\nu}^B n_\nu.$$

The Lagrangian is given by:

$$L(\hat{\mathbf{n}}, \lambda) = \sigma^2 - \lambda \left(\sum_{\mu=1}^3 n_\mu^2 - 1 \right) = \sum_{\mu, \nu=1}^3 n_\mu M_{\mu\nu}^B n_\nu - \lambda \left(\sum_{\mu=1}^3 n_\mu^2 - 1 \right).$$

Differentiate with Respect to the Components n_α equal to zero for each component

$$\frac{\partial}{\partial n_\alpha} \left(\sum_{\mu, \nu} n_\mu M_{\mu\nu}^B n_\nu \right) - \frac{\partial}{\partial n_\alpha} \left[\lambda \left(\sum_{\mu=1}^3 n_\mu^2 - 1 \right) \right] = 2 \sum_{\nu=1}^3 M_{\alpha\nu}^B n_\nu - 2\lambda n_\alpha.$$

we get:

$$2 \sum_{\nu=1}^3 M_{\alpha\nu}^B n_\nu - 2\lambda n_\alpha = 0.$$

This is exactly Equation (8.7): $\sum_{\nu=1}^3 M_{\mu\nu}^B n_\nu = \lambda n_\mu$. The equation is an eigenvalue problem for the symmetric 3×3 matrix $M_{\mu\nu}^B$. The eigenvalues λ are real, and the corresponding eigenvectors \hat{n} are orthogonal.

2 Electromagnetic ion cyclotron waves

Electromagnetic ion cyclotron waves are left-hand polarized waves excited by velocity space anisotropy (perpendicular gradients in velocity distribution function). They are typically below the ion cyclotron frequency for each species (H+, He+ or O+). MMS observations on 2015 Dec. 14, show that such waves were excited after the passage of a shock.

```
using Speasy
spz = speasy()

using Dates
using DimensionalData
using CairoMakie
using SpaceTools
using Unitful
using DSP
```

2.1 Spectrogram with cyclotron frequencies

[a] Obtain MMS 1 (or 2,3,4) data between 13:27 and 13:29 UT, from the FGM instrument in GSM coordinates, and plot the spectrogram of the X or Y component between 0-2Hz. Overlay the 3 cyclotron frequencies, as shown in Fig. 5 of the paper (for subset of time). [Note: you should create a low-pass filtered version of the average field at 5-10s window to compute the cyclotron frequencies with; for that use `tsmooth2` (IDL) or `pyspedas.interpolate` (PySPEDAS) after `degap/clip/deflagging` the data first].

https://cdaweb.gsfc.nasa.gov/misc/NotesM.html#MMS1_FGM_SRVY_L2

```
t0 = "2015-12-14T13:10:00"
t1 = "2015-12-14T13:50:00"
```

```
tvars = "cda/MMS1_FGM_SRVY_L2/mms1_fgm_b_gsm_srvy_l2_clean"
da = get_data(tvars, t0, t1) |> DimArray
da = SpaceTools.modify_meta(da, long_name="B")
```

Can't get MMS1_FGM_SRVY_L2/mms1_fgm_b_gsm_srvy_l2_clean without web service, switching to web service

```
r 19199x4 DimArray{Unitful.Quantity{Float32, [?] [?]-1 [?]-2,
Unitful.FreeUnits{(nT,), [?] [?]-1 [?]-2, nothing}}, 2}
mms1_fgm_b_gsm_srvy_l2_clean [
|----- dims
|
|   ↓ Ti                               Sampled{Dates.DateTime} [2015-12-14T13:10:00,
..., 2015-12-14T13:49:59.783] ForwardOrdered Irregular Points,
|   → mms1_fgm_b_gsm_srvy_l2_clean Categorical{Symbol} [Symbol("Bx GSM"),
Symbol("By GSM"), Symbol("Bz GSM"), :Bt] Unordered
|----- metadata
|
| Dict{Any, Any} with 19 entries:
| "REPRESENTATION_1" => "represent_vec_tot"
| "SCALETYP"          => "linear"
| "FILLVAL"           => Any[-1.0e31]
| "DEPEND_0"          => "Epoch"
| "FIELDNAM"          => "Magnetic field vector in GSM plus Btotal (8 or 16 S/
S...
| "VALIDMAX"           => Any[17000.0, 17000.0, 17000.0, 17000.0]
| "SI_CONVERSION"      => "1.0e-9>T"
| "LABL_PTR_1"         => Any["Bx GSM", "By GSM", "Bz GSM", "Bt   "]
| "TENSOR_ORDER"       => Any[1]
| "axes"               => VariableAxis[Speasy.VariableAxis:...
| "COORDINATE_SYSTEM" => "GSM"
| "FORMAT"             => "E13.5"
| "VAR_TYPE"           => "data"
| "CATDESC"            => "Magnetic field vector in Geocentric Solar
Magnetosphe...
| :long_name           => "B"
| "DISPLAY_TYPE"       => "time_series"
| "VALIDMIN"           => Any[-17000.0, -17000.0, -17000.0, 0.0]
| "DIM_SIZES"          => Any[4]
| "UNITS"              => "nT"
|
| : [?]
```

```
using SignalAnalysis
using PlasmaFormulary
```

```

p = SpaceTools.pspectrum(da[:, 1])
begin
    f = tplot(p; colorrange=(1e-7, 1), alpha=0.7)
    ax = current_axis()
    ylims!(ax, 0u"Hz", 2u"Hz")

    B_ts = smooth(da, 5u"s")[:, 4]

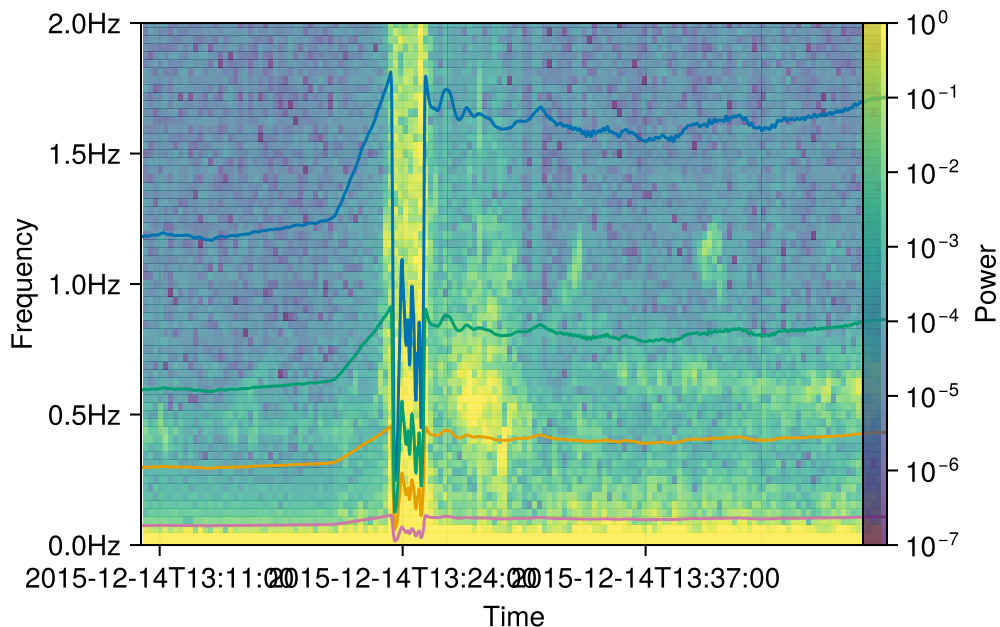
    ic_H = gyrofrequency.(B_ts, : "H+") .|> ω2f
    ic_He = gyrofrequency.(B_ts, : "He+") .|> ω2f
    ic_He2 = gyrofrequency.(B_ts, : "He2+") .|> ω2f
    ic_0 = gyrofrequency.(B_ts, : "0+") .|> ω2f
    tplot_panel!. (current_axis(), [ic_H, ic_He, ic_He2, ic_0])
    f
end

```

```

└ Warning: Time resolution is is not approximately constant (relerr ≈
0.007986138746711634)
└ @ SpaceTools ~/.julia/dev/SpaceTools/src/timeseries.jl:21
└ Warning: Time resolution is is not approximately constant (relerr ≈
0.007986138746711634)
└ @ SpaceTools ~/.julia/dev/SpaceTools/src/timeseries.jl:21

```



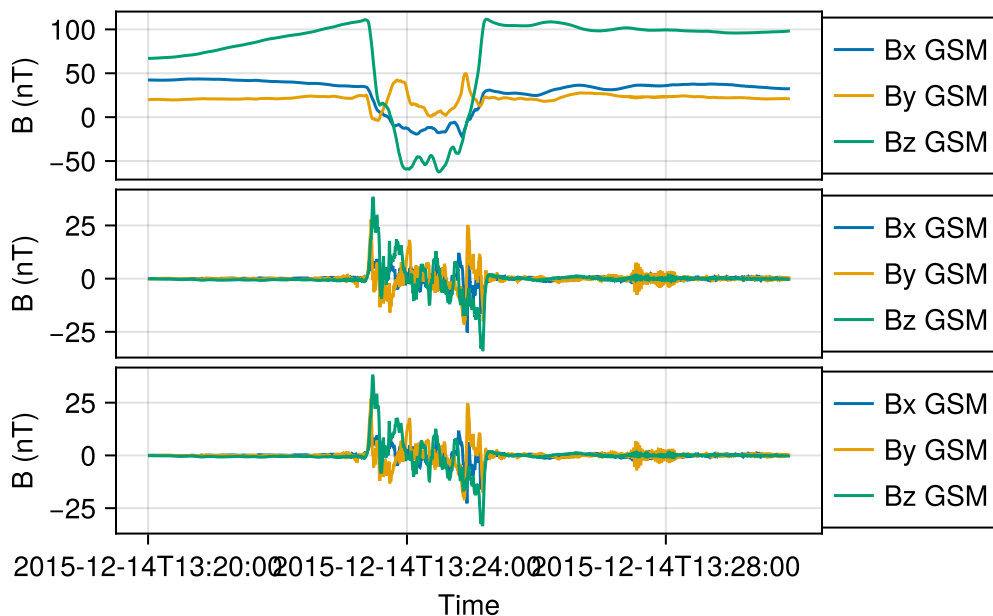
2.2 Band-pass filtered data

[b] Plot the GSM time series of the band-pass filtered data between 0.2 – 2 Hz (again using tsmooth2/pyspedas.interpol to first low-pass filter the data at 0.2Hz, subtract it from the original to create the high-pass residual >0.2Hz, then low-pass filter that with a 2Hz window).

```
time = DateTime("2015-12-14T13:20:00") .. DateTime("2015-12-14T13:30:00")
sda = da[time, 1:3]
# Band-pass filter the data between 0.2-2 Hz
# First step: Low-pass filter at 0.2 Hz
lp_02Hz = smooth(sda, 5u"s") # 0.2 Hz ≈ 5 second window
# High-pass by subtracting low-pass from original
hp_02Hz = amap(-, sda, lp_02Hz)
# Second step: Low-pass filter at 2 Hz (0.5 second window)
bp_filtered = smooth(hp_02Hz, 0.5u"s")

tplot([lp_02Hz, hp_02Hz, bp_filtered])
```

```
⚠ Warning: Time resolution is is not approximately constant (relerr ≈
0.007984877076234697)
└ @ SpaceTools ~/.julia/dev/SpaceTools/src/timeseries.jl:21
⚠ Warning: Time resolution is is not approximately constant (relerr ≈
0.007984753171800809)
└ @ SpaceTools ~/.julia/dev/SpaceTools/src/timeseries.jl:21
```

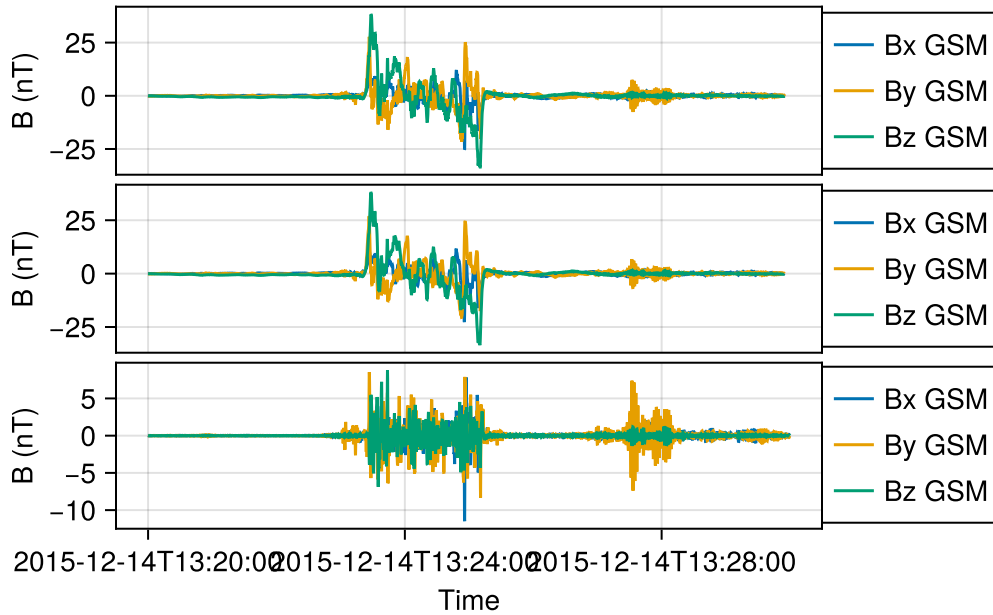


Actually, the frequency response of an M point moving average filter is not a perfect low-pass filter.

$$H[f] = \frac{\sin(\pi f M)}{M \sin(\pi f)}$$

```
bp_filtered2 = filtfilt(sda, 0.2u"Hz", 2u"Hz").data
tplot([hp_02Hz, bp_filtered, bp_filtered2])
current_figure()
```

```
⚠ Warning: Time resolution is is not approximately constant (relerr ≈
0.007984877076234697)
└ @ SpaceTools ~/.julia/dev/SpaceTools/src/timeseries.jl:21
```



2.3 Minimum Variance Analysis

[c] Use `minvar_matrix_make` (IDL or pyspedas) applied on the band-pass filtered data to obtain the minimum variance matrix, on a sliding window. You can control the width and shift of the window (see keywords). Rotate the data in i,j,k (max,int,min) coord's using `tvector_rotate` (in IDL / Py - SPEDAS; see how to use it from class examples, or from cribs / readthedocs). Plot the data in that coordinate system.

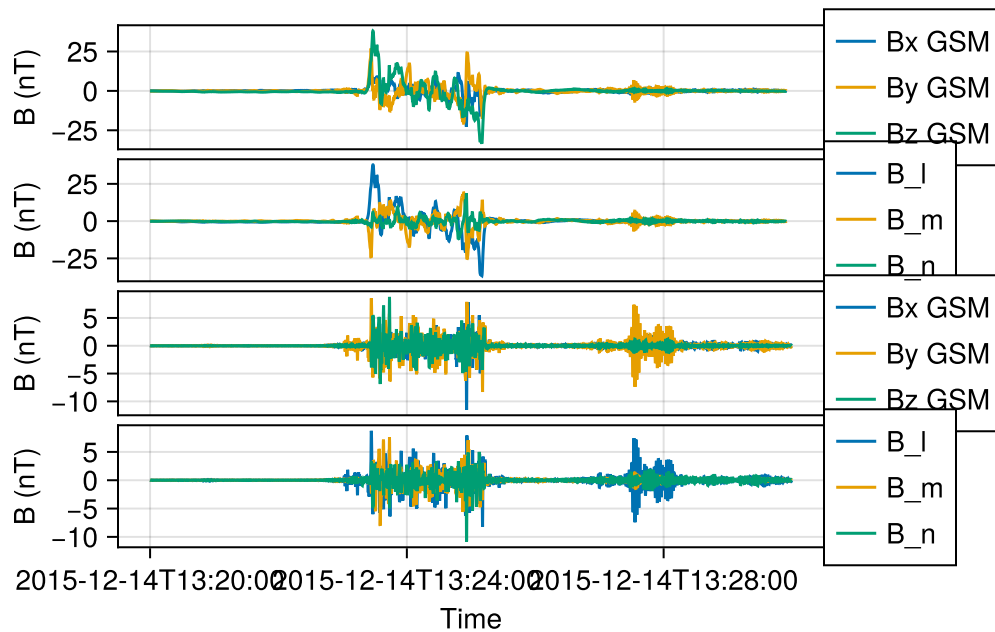
```

time = DateTime("2015-12-14T13:20:00") .. DateTime("2015-12-14T13:30:00")
B_ts = bp_filtered[time, 1:3]
B_mva_ts = mva(B_ts, B_ts)

B_ts_2 = bp_filtered2[time, 1:3]
B_mva_ts_2 = mva(B_ts_2, B_ts_2)

tplot([B_ts, B_mva_ts, B_ts_2, B_mva_ts_2])

```



2.4 Minvar direction

[d] The code `minvar_matrix_make` allows you to output the eigenvectors and eigenvalues. Plot the eigenvalues and evaluate the confidence in the minvar direction. Plot the angle between the minvar direction and the direction of the ambient magnetic field (the 5-10s average above).

```

using LinearAlgebra
using Statistics
using SpaceTools
using Dates

# Load magnetic field data around the shock crossing
t0 = DateTime("2008-09-05T15:30:00")
t1 = DateTime("2008-09-05T15:40:00")

```

```

b_data = get_data("cda/THA_L2_FGM/tha_fgs_gse", t0, t1) |> DimArray
B = b_data.data # Get the magnetic field components

# Get eigenvalues and eigenvectors
F = mva_mat(B)

# The minimum variance direction (shock normal)
n_hat = F.vectors[:, 3] # Third eigenvector corresponds to minimum variance

# Calculate  $\theta_{Bn}$  (angle between B-field and shock normal)
B_avg = mean(B, dims=1)[1, :]
B_avg_norm = B_avg / norm(B_avg)
 $\theta_{Bn}$  = acosd(abs(dot(B_avg_norm, n_hat)))

# Calculate  $B_n$  (shock-normal magnetic field component)
Bn = dot(B_avg, n_hat)

println("Results of Minimum Variance Analysis:")
println("Shock normal (GSE coordinates): ", round.(n_hat, digits=3))
println(" $\theta_{Bn}$ : ", round( $\theta_{Bn}$ , digits=1), "°")
println("Bn: ", Bn)
check_mva_mat(F; verbose=true)

```

```

[ Info: Cannot parse unit nT*GSE*(All*Qs)
Results of Minimum Variance Analysis:
Shock normal (GSE coordinates): Float32[-0.215, -0.966, 0.143]
 $\theta_{Bn}$ : 61.5°
Bn: 26.65665
Float32[0.0060435473 0.97663194 -0.21483368; 0.14459603 -0.21343324
-0.9661977; 0.9894723 0.025224842 0.14250703]
Ratio of intermediate variance to minimum variance = 1.8064607
└ Warning: Take the MVA result with a grain of salt!
└ @ SpaceTools ~/.julia/dev/SpaceTools/src/cotrans/mva.jl:82

```

2.5 Hodogram

[e] There are several bursts of EMIC wave power in the interval. Pick one clean burst with 5-10 cycles and plot the B-field hodogram in the plane of polarization (max,int). (Bandpass filtering first will help). Is it left-hand polarized as expected? [Note that the minvar code ensures a right-hand orthogonal system, but you may have to switch vectors around if the angle to B-field is not less than 90deg].

```

interval = DateTime("2015-12-14T13:40:20") .. DateTime("2015-12-14T13:40:40")
let

```



```

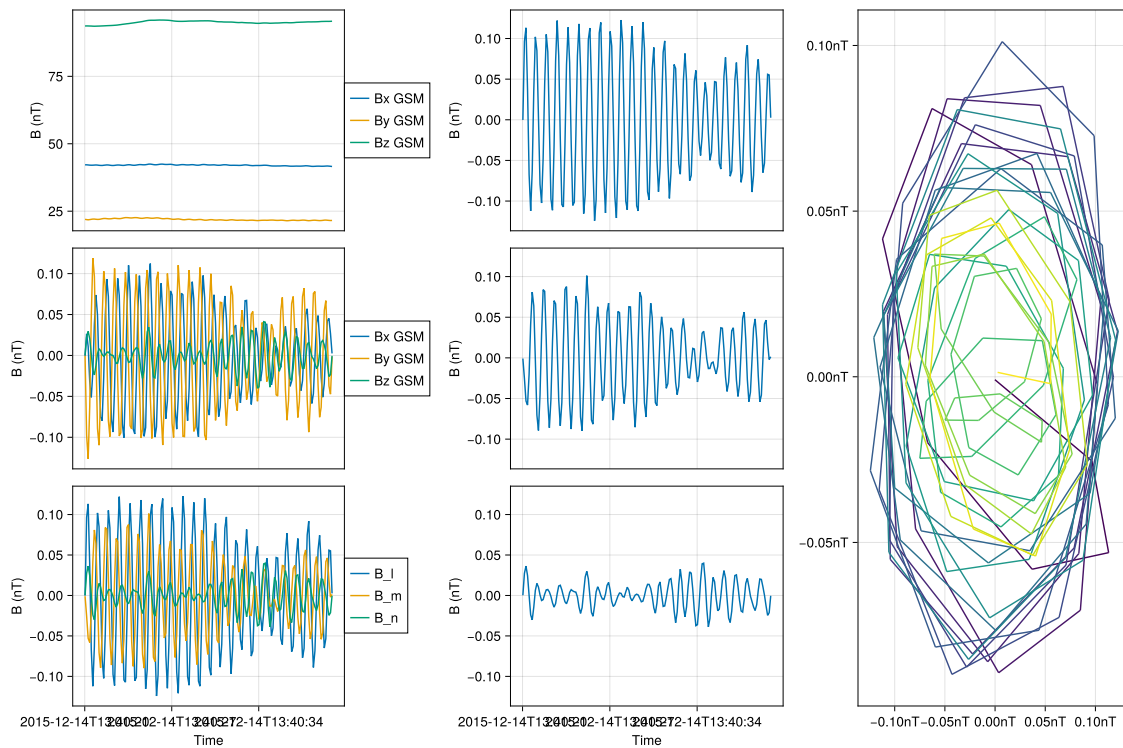
sda = da[interval, 1:3]
bp_filtered = filtfilt(sda, 0.8u"Hz", 2u"Hz").data
B_mva_ts = mva(bp_filtered, bp_filtered)
f = Figure(; size=(1200, 800))
Bx = B_mva_ts[:, 1]
By = B_mva_ts[:, 2]
Bz = B_mva_ts[:, 3]
tplot(f[1, 1], [sda, bp_filtered, B_mva_ts])
tplot(f[1, 2], [Bx, By, Bz]; link_yaxes=true)
time = dims(Bx, 1).val
color = (time .- time[1]) ./ (time[end] - time[1])
lines(f[1, 3], Bx[:, 1].data, By[:, 1].data, color=color)
f
end

```

```

Warning: Time resolution is is not approximately constant (relerr ≈
0.007949285570537276)
└ @ SpaceTools ~/.julia/dev/SpaceTools/src/timeseries.jl:21

```



Hodogram analysis indicated left handed polarization during the interval.

3 Shock crossings

Shock crossings were observed by ARTEMIS P1 (TH-B) in the solar wind on 2013-07-09 20:40UT and on 2014-06-09 16:58UT, both captured in Fast Survey and published in Fig. 3 (analysis results in Table 1 of Zhou et al., 202022). Pick one of the two to study.

```
using LinearAlgebra
using Statistics
using DataFrames, TimeSeries
```

3.1 Shock normal

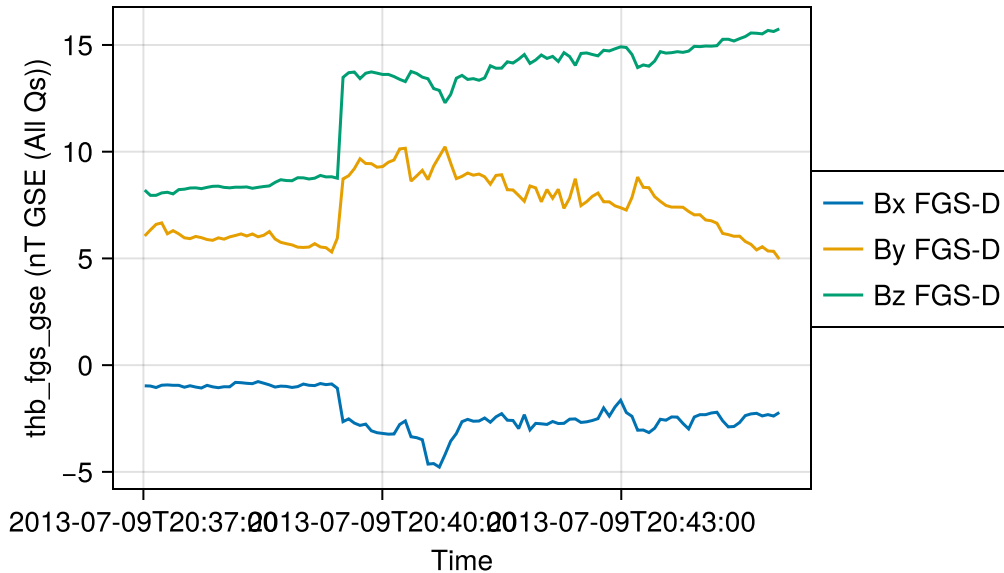
[a] Determine, using minimum variance analysis, the shock normal, the angle between the B-field and the shock normal, and the shock-normal magnetic field component, B_n .

The shock angle, θ_{Bn} , the angle between the directions of the upstream magnetic field and the shock normal;

```
# Load magnetic field data around the shock crossing
t0 = DateTime("2013-07-09T20:37:00") # Start a few minutes before the
crossing
t1 = DateTime("2013-07-09T20:45:00") # End a few minutes after
b_data = get_data("cda/THB_L2_FGM/thb_fgs_gse", t0, t1) |> DimArray
tplot(b_data; add_title=true)
```

```
[ Info: Cannot parse unit nT*GSE*(All*Qs)
```

-- FGS (spin-resolution/~3 sec) magnetic field B in XYZ GSE Coordinates



```
# Perform MVA analysis using the built-in function
F = mva_mat(b_data)

# The minimum variance direction (shock normal)
n_hat = F.vectors[:, 3] # Third eigenvector corresponds to minimum variance

# Calculate  $\theta_{Bn}$  (angle between B-field and shock normal)
B_up = vec(mean(b_data[1:10, :].data; dims=1))
 $\theta_{Bn}$  = angle(B_up, n_hat)

# Calculate Bn (shock-normal magnetic field component)
Bn = dot(B_up, n_hat)

println("Results of Minimum Variance Analysis for ARTEMIS P1 shock crossing
(2013-07-09):")
println("Shock normal (GSE coordinates): ", round.(n_hat, digits=3))
println(" $\theta_{Bn}$ : ", round( $\theta_{Bn}$ , digits=1), "°")
println("Bn: ", Bn)
check_mva_mat(F, verbose=true)
```

```
Results of Minimum Variance Analysis for ARTEMIS P1 shock crossing
(2013-07-09):
Shock normal (GSE coordinates): Float32[0.923, 0.347, 0.164]
 $\theta_{Bn}$ : 75.4°
Bn: 2.5922403
Float32[-0.27282175 0.2701564 0.9233546; 0.29161265 -0.8913902 0.3469665;
```

```
0.91680443 0.36392197 0.16440956]
Ratio of intermediate variance to minimum variance = 10.369079
[ Info: Seems to be a proper MVA attempt!
```

3.2 Minimum variance direction

[b] Note that there are significant waves downstream but not upstream, which is common for weak (subcritical) shocks. Starting from a small interval barely encompassing the shock (<1min in length), increase the interval by ~0.5 min and recalculate. Then increase again and recalculate. Do so for ~10-20 times until you reach ~10min encompassing the shock. The solution will initially fluctuate from one to the next choice of intervals, then stabilize, then start being jittery again because you are including the waves. Plot the minimum variance direction (two angles, elevation and azimuth, in GSE) and Bn, all as a function of the interval chosen. Pick and report the best solution, based on the shortest time interval choice when the solution becomes stable.

```
azimuth(x) = rad2deg(atan(x[2], x[1]))
elevation(x) = rad2deg(asin(x[3]))

# Central time of the shock crossing for ARTEMIS P1 (2013-07-09)
t_start = DateTime("2013-07-09T20:39:00")
time_shock = DateTime("2013-07-09T20:39:30")

n = 20
t_ends = [time_shock + Second(30 * (i - 1)) for i in 1:n]
b_data = get_data("cda/THB_L2_FGM/thb_fgs_gse", t_start, t_ends[end]) |>
  DimArray
b_data.metadata[:units] = "nT"

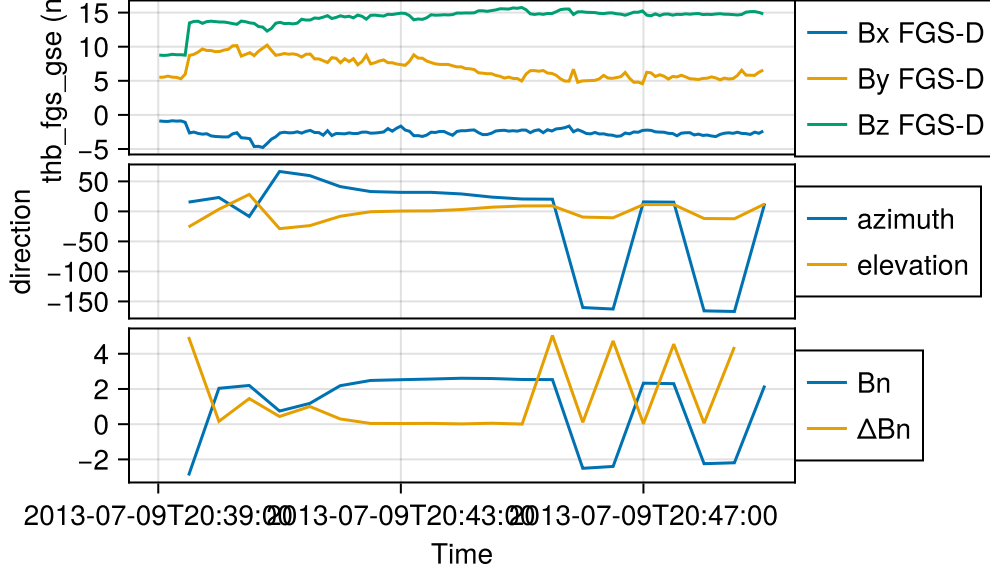
îs = map(t_ends) do t_end
  b_data_subset = b_data[t_start..t_end, :]
  F = mva_mat(b_data_subset)
  F.vectors[:, 3]
end

Bn = dot.(Ref(B_up), îs)
direction = DimArray(
  [azimuth.(îs) elevation.(îs)],
  (Ti(t_ends), Y([:azimuth, :elevation])));
  name=:direction
)
Bns = DimArray(Bn, Ti(t_ends); name=:Bn)
ΔBn = abs.(diff(Bns))
ΔBn = SpaceTools.rename(ΔBn, :ΔBn)
```

```
tvars = [b_data, direction, [Bns, ΔBn]]
f, axes = tplot(tvars; add_title=true)
```

[Info: Cannot parse unit nT*GSE*(All*Qs)

--- FGS (spin-resolution/~3 sec) magnetic field B in XYZ GSE Coordinates



The solution becomes stable around “2013-07-09T20:43:00”, about 4 mins after the shock crossing. after “2013-07-09T20:45:00” the solution becomes jittery because the waves are included.

3.3 Error estimates

[c] The noise in B_n and the minimum variance directions (angular noise) are tabulated in equations 8.24 and 8.23 of Chapter 8. Plot the ΔB_n and $\Delta\varphi$ separately. Check if they support your choice of interval.

The angular error estimates (in radians) is

$$|\Delta\varphi_{ij}| = |\Delta\varphi_{ji}| = \left\langle \left\langle (\Delta x_{ij})^2 \right\rangle \right\rangle^{1/2} = \left\langle \left\langle (\Delta x_{ji})^2 \right\rangle \right\rangle^{1/2}$$

$$= \sqrt{\frac{\lambda_3}{M-1} \frac{\lambda_i + \lambda_j - \lambda_3}{(\lambda_i - \lambda_j)^2}}, \quad i \neq j$$

The composite statistical error estimate for $\langle \mathbf{B} \cdot \mathbf{x}_3 \rangle$ is

$$|\Delta\langle \mathbf{B} \cdot \mathbf{x}_3 \rangle| = \sqrt{\frac{\lambda_3}{M-1} + (\Delta\varphi_{32}\langle \mathbf{B} \rangle \cdot \mathbf{x}_2)^2 + (\Delta\varphi_{31}\langle \mathbf{B} \rangle \cdot \mathbf{x}_1)^2}$$

```

"""
    Δφij(λi, λj, λ3, M)

Calculate the phase error between components i and j according to:
|Δφij| = |Δφji| = √(λ3/(M-1) * (λi + λj - λ3)/(λi - λj)²)

Parameters:
- λi: eigenvalue i
- λj: eigenvalue j
- λ3: smallest eigenvalue (λ3)
- M: number of samples
"""
function Δφij(λi, λj, λ3, M)
    return sqrt((λ3 / (M - 1)) * (λi + λj - λ3) / (λi - λj)²)
end

"""
Calculate the composite statistical error estimate for ⟨B·x3⟩:
|Δ(B·x3)| = √(λ3/(M-1) + (Δφ32⟨B⟩·x2)² + (Δφ31⟨B⟩·x1)²)

Parameters:
- λ1, λ2, λ3: eigenvalues in descending order
- M: number of samples
- B: mean magnetic field vector
- x1, x2, x3: eigenvectors
"""
function B_x3_error(λ1, λ2, λ3, M, B, x1, x2, x3)
    Δφ32 = Δφij(λ3, λ2, λ3, M)
    Δφ31 = Δφij(λ3, λ1, λ3, M)
    B_x2 = dot(B, x2)
    B_x1 = dot(B, x1)
    return sqrt(λ3 / (M - 1) + (Δφ32 * B_x2)² + (Δφ31 * B_x1)²)
end

B_x3_error(F::Eigen, M, B) =
    B_x3_error(F.values..., M, B, eachcol(F.vectors)...)

```

```

res = map(t_ends) do t_end
    b_data_subset = b_data[t_start..t_end, :]
    F = mva_mat(b_data_subset)
    M = size(b_data_subset, 1)
    B = mean(b_data_subset.data; dims=1)

    λ1 = F.values[1]

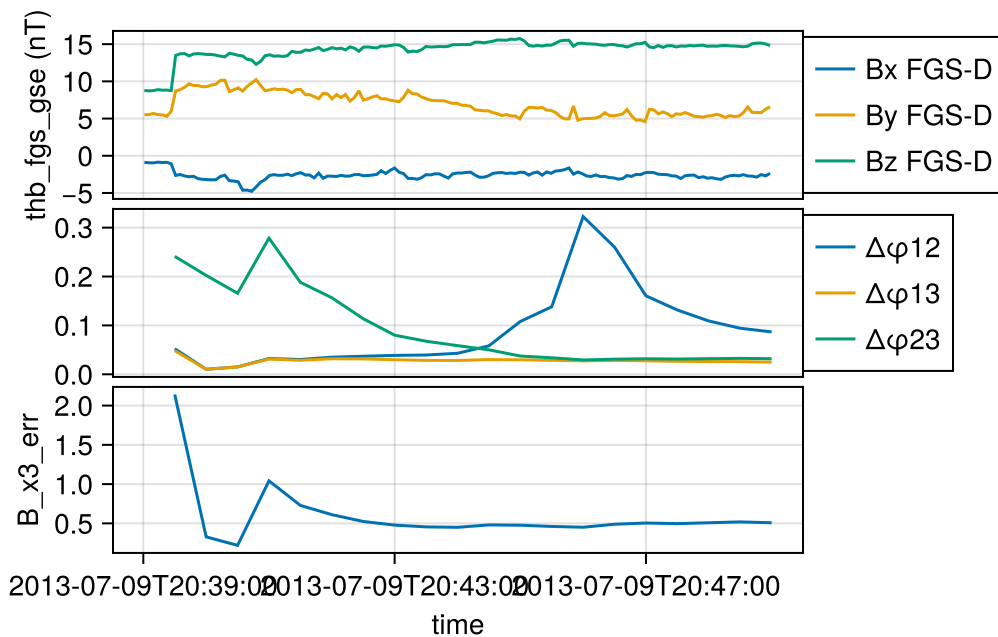
```

```

     $\lambda_2$  = F.values[2]
     $\lambda_3$  = F.values[3]
     $\Delta\phi_{12}$  = SpaceTools. $\Delta\phi_{ij}$ ( $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ , M)
     $\Delta\phi_{13}$  = SpaceTools. $\Delta\phi_{ij}$ ( $\lambda_1$ ,  $\lambda_3$ ,  $\lambda_3$ , M)
     $\Delta\phi_{23}$  = SpaceTools. $\Delta\phi_{ij}$ ( $\lambda_2$ ,  $\lambda_3$ ,  $\lambda_3$ , M)
    B_x3_err = SpaceTools.B_x3_error(F, M, B)
    (; time=t_end,  $\Delta\phi_{12}$ ,  $\Delta\phi_{13}$ ,  $\Delta\phi_{23}$ , B_x3_err)
end

errors = SpaceTools.nt2ds(res)
tvars = [
    b_data,
    [errors. $\Delta\phi_{12}$ , errors. $\Delta\phi_{13}$ , errors. $\Delta\phi_{23}$ ],
    errors.B_x3_err,
]
tplot(tvars)

```



The error estimates support our choice of interval.

Bibliography