Homework 05

1. Integral, differentiability and decay rate

Compute this generalization of an integral we did in class:

$$I_{n,a}(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(x^2 + a^2)^n} dx$$

where $k, a \in \mathbb{R}, a > 0$ and $n \in \mathbb{Z}, n \geq 1$. What is the degree of differentiability of $I_{n,a}(k)$ with respect to k? How does this relate to the decay of the integrand?

Compute the Integral:

Complex function $f(z) = \frac{e^{ikz}}{(z^2+a^2)^n}$ has poles at $z = \pm ia$.

The residue at $z = +ia$ pole

$$\text{Res } (f, ia) = \frac{1}{(n-1)!} \lim_{z \to ia} \frac{d^{n-1}}{dz^{n-1}} [(z - ia)^n f(z)]$$

Construct a semicircular contour in the upper half-plane that consists of a line segment C_1 from $-R$ to R and a semicircular arc C_2 of radius R centered at the origin.

The integral over the contour C is:

$$\int_C f(z) dz = 2\pi i \times \text{Res } (f, ia)$$

As $R \to \infty$, the integral over C_2 vanishes for $k > 0$ due to the exponential decay of e^{ikz} in the upper half-plane.

The original integral $I_{n,a}(k)$ is equal to the integral over C_1:

$$I_{n,a}(k) = 2\pi i \times \text{Res } (f, ia)$$

Degree of Differentiability

$$I_{n,a}'(k) = \int_{-\infty}^{\infty} \frac{ixe^{ikx}}{(x^2 + a^2)^n} dx$$

The corresponding complex function

$$f_k(z) = \frac{ize^{ikz}}{(z^2 + a^2)^n}$$

has poles at $z = \pm ia$.

The function $I_{n,a}(k)$ will be infinitely differentiable with respect to k because the exponential function e^{ikx} is smooth and the denominator does not depend on k.

Relation to the Decay of the Integrand:

The decay of the integrand as $x \to \pm\infty$ is crucial for the convergence of the integral. The factor $(x^2 + a^2)^{-n}$ ensures that the integrand decays sufficiently fast at infinity for the integral to converge.

The faster decay of the integrand at infinity, which is ensured by a higher n, means that $I_{n,a}(k)$ will be more regular (i.e., higher degree of differentiability).

2. Integral of a keyhole contour

Compute the following integral by a choice of keyhole contour, where $a, b, c \in \mathbb{R}$ and $b, c > 0$:

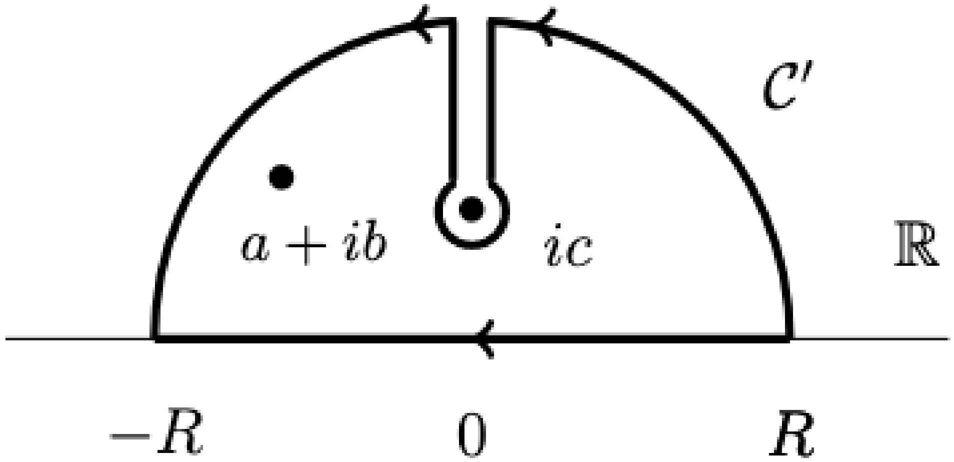
$$I(a, b, c) = \int_{-\infty}^{\infty} dx \frac{\log(x^2 + c^2)}{(x - a)^2 + b^2}$$

The complex function

$$f(z) = \frac{\log(z^2 + c^2)}{(z - a)^2 + b^2} = \frac{\log(z + ic)}{(z - a)^2 + b^2} + \frac{\log(z - ic)}{(z - a)^2 + b^2}$$

may have branch point $z = \pm ic$ and have poles at $a \pm ib$.

Consider the contour integral $\oint_C dz f(z)$in the complex plane where C is a semicircle of radius R in the upper half-plane with a detour down and up the imaginary axis about the branch point $z = ic$.



The function $\log(z+ic)$ is holomorphic in the upper half plane if we choose its branch cut to lie in the lower half plane, so the integrand is holomorphic inside the contour of C except for a simple pole at $a+ib$.

$$\oint_C dz \frac{\log(z+ic)}{(z-a)^2+b^2} = 2\pi i \frac{\log(a+ib+ic)}{2ib} = \frac{\pi}{b}\log(a+ib+ic)$$

The function $\log(z-ic)$ has a branch cut in the upper half plane. We have

$$\oint_C dz \frac{\log(z-ic)}{(z-a)^2+b^2} = \frac{\pi}{b}\log(a+ib-ic) + \int_{ic}^{\infty} \frac{\log(iy-ic+\epsilon)-\log(iy-ic+\epsilon)}{(z-a)^2+b^2}$$

Since the contribution from the big semi-circle tends to zero as $R\to\infty$ and from the small circle around ic as its radius tends to zero. And

$$\int_{ic}^{\infty} \frac{\log(iy-ic+\epsilon)-\log(iy-ic+\epsilon)}{(z-a)^2+b^2} = -2\pi\int_c^{\infty} dy \frac{1}{(iy-a)^2+b^2} = -\frac{\pi}{b}\log\frac{a+ib-ic}{a-ib-ic}$$

So

$$\begin{aligned} I(a,b,c) &= \int_{-\infty}^{\infty} dx \frac{\log(x^2+c^2)}{(x-a)^2+b^2} \\ &= \frac{\pi}{b}\log(a+ib+ic) + \frac{\pi}{b}\log(a+ib-ic) - -\frac{\pi}{b}\log\frac{a+ib-ic}{a-ib-ic} \\ &= \frac{\pi}{b}\log\left(a^2+(b+c)^2\right) \end{aligned}$$

3. Integral and Laurent expansion

Consider the integral

$$\int_C dz \frac{z+2}{z^2-9}$$

where C is a positively-oriented circle of radius 4. Compute this integral by taking the sum of the residues inside the circle, and then again by computing the "residue at ∞" meaning using the outer Laurent expansion

The integrand

$$f(z) = \frac{z+2}{z^2-9}$$

have simple poles at $z = \pm 3$.

$$\text{Res }(f, 3) = \lim_{z \to 3}(z-3)\frac{z+2}{z^2-9} = \frac{5}{6}$$

$$\text{Res }(f, -3) = \lim_{z \to -3}(z+3)\frac{z+2}{z^2-9} = \frac{1}{6}$$

By the residue theorem, the integral over the contour C is:

$$\int_C f(z), dz = 2\pi i \times (\text{Res }(f, 3) + \text{Res }(f, -3)) = 2\pi i$$

Using the outer Laurent expansion

$$\frac{z+2}{z^2-9} = \frac{1}{z} + O\left[\frac{1}{z^2}\right]$$

So using residue at ∞

$$\text{Res }(f, \infty) = 1$$

$$\int_C f(z), dz = 2\pi i \times \text{Res }(f, \infty) = 2\pi i$$

4. Analytically continuing

Compute the integral

$$\int_0^{2\pi} \frac{d\theta}{2 - \cos\theta}$$

by analytically continuing the integrand and then using residues.

Perform a change of variables to convert the integral into a contour integral in the complex plane. Let $z = e^{i\theta}$. Then, $dz = ie^{i\theta}d\theta = izd\theta$ and $d\theta = \frac{dz}{iz}$. Also, $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Substituting these into the integral, we get:

$$\int_0^{2\pi} \frac{d\theta}{2 - \cos\theta} = \int_{|z|=1} \frac{1}{2 - \frac{1}{2}\left(z + \frac{1}{z}\right)} \times \frac{dz}{iz}$$

Simplifying this, we have:

$$\int_{|z|=1} \frac{dz}{2iz - \frac{1}{2}i(z^2 + 1)}$$

To find the poles of the integrand, we set the denominator to zero:

$$iz - \frac{1}{2}i(z^2 + 1) = 0$$

$$z_1 = 2 - \sqrt{3}, z_2 = 2 + \sqrt{3}$$

z_1 is within the contour, so

$$\int_{|z|=1} \frac{1}{2 - \frac{1}{2}\left(z + \frac{1}{z}\right)} \times \frac{dz}{iz} = 2\pi i \frac{2}{i(z_2 - z_1)} = \frac{2\pi}{\sqrt{3}}$$

5. Gamma function

The Gamma function is usually defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

(This kind of integral is known as a Mellin transform, in this case of e^{-t}, and we can equivalently write it as a two-sided Laplace transform of e^{-e^s} by taking $s = \log t$.) Verify this integral exists for $\Re(z) > 0$ and that

$$z\Gamma(z) = \Gamma(z + 1)$$

$$\Gamma(n + 1) = n!$$

The first relation can be iterated to analytically continue the Γ function to the whole complex plane, except for poles at the nonpositive integers. However, there is a way to analytically continue it all at once:

Split the integral into two parts $\int_1^\infty + \int_0^1$. Show that the first part yields an entire function in z, and that the second part equals

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n}$$

which is an entire meromorphic function with poles at the non-positive integers, thus giving an analytic continuation of Γ as a meromorphic function on all of $\mathbb{C}$.

To verify the properties of the Gamma function and its analytic continuation, we'll break down the problem into parts.

Existence

We prove this integral converges absolutely.

$$\int_0^\infty \left|t^{z-1}e^{-t}\right|dt = \int_0^\infty t^{\mathrm{Re}\ z-1}e^{-t}dt$$

For $z \in \mathbb{R}$,

$$\int_0^\infty t^{z-1}e^{-t}dt = \int_0^1 t^{z-1}e^{-t}dt + \int_1^\infty t^{z-1}e^{-t}dt$$

we have for large $N \le t$, $t^{z-1}e^{-t} \le e^{-t/2}$

$$0 \le t \le N \Longrightarrow t^{z-1}e^{-t} \le t^{z-1}, \text{and} \int_0^N t^{z-1}dt = \frac{t^z}{z}|_0^N = \frac{N^z}{z}$$

$$N \le t \Longrightarrow t^{z-1}e^{-t} \le e^{-t/2}, \text{and} \int_N^\infty e^{-t/2}dt = -2e^{-t/2}|_N^\infty = 2e^{-N/2}$$

So

$$\int_0^\infty \left|t^{z-1}e^{-t}\right|dt \le \infty$$

Therefore, this integral exists for $\mathfrak{R}(z) > 0$.

$z\Gamma(z) = \Gamma(z+1)$

By definition:

$$z\Gamma(z) = z \int_0^\infty t^{z-1} e^{-t} dt$$

Using integration by parts, we get:

$$\begin{aligned} z\Gamma(z) &= t^z e^{-t}|_0^\infty - \int_0^\infty t^z (-e^{-t}) dt \\ &= \int_0^\infty t^z e^{-t} dt \\ &= \Gamma(z+1) \end{aligned}$$

$\Gamma(n+1) = n!$

$$\begin{aligned} \Gamma(1) &= \int_0^\infty t^{1-1} e^{-t}\, dt \\ &= \int_0^\infty e^{-t}\, dt \\ &= 1. \end{aligned}$$

By induction we have $\Gamma(n+1) = n!$

Splitting

Now, to analytically continue the Gamma function, we split the integral into two parts:

$$\Gamma(z) = \int_1^\infty t^{z-1} e^{-t} dt + \int_0^1 t^{z-1} e^{-t} dt$$

- The first part $\int_1^\infty t^{z-1} e^{-t} dt = \left(\int_1^N + \int_N^\infty\right) t^{z-1} e^{-t} dt$ converges for all z and is an entire function in z.
- The second part $\int_0^1 t^{z-1} e^{-t} dt$) can be expressed as a power series using the Taylor expansion of e^{-t} :

$$\int_0^1 t^{z-1} e^{-t} dt = \int_0^1 t^{z-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^n dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{z+n-1} dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z+n}$$

This series converges for (z) not a non-positive integer and provides an analytic continuation of Γ as a meromorphic function on all of $\mathbb{C}$, with poles at the non-positive integers.

6. Gamma relation

Prove the relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

by combining the two Gamma integrals into

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty dt \frac{t^{z-1}}{t+1}$$

and evaluating this by residues.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

For fixed t

$$\Gamma(1-z) = \int_0^\infty t^{-z} e^{-t} dt = \int_0^\infty u^{-z} e^{-u} du = t \int_0^\infty (vt)^{-z} e^{-vt} dv$$

By combining these two

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-z} dv dt = \int_0^\infty dv \frac{v^{z-1}}{v+1} = \int_0^\infty dt \frac{t^{z-1}}{t+1} = \int_{-\infty}^\infty dt \frac{e^{zx}}{1+e^x}$$

Complex function

$$\frac{e^{zx}}{1+e^x}$$

have pole at πi. Consider a contour C_R in the complex plane with vertices at R, $-R$, $R + 2\pi i$, and $-R + 2\pi i$, as R tends to infinity. We have

$$\int_{C_R} \frac{e^{xz}}{1+e^x} dx = -2\pi i e^{z\pi i}$$

$$\int_{C_{R3}} f(x,z) dx = -e^{2\pi i z} \int_{C_{R1}} f(x,z) dx$$

The right and left vertical sides of the rectangle tend to 0 as $R \to \infty$ for $z \in (0,1)$.

$$\int_{-\infty}^\infty dt \frac{e^{zx}}{1+e^x} = \frac{-2\pi i e^{z\pi i}}{1-e^{2\pi i z}} = \frac{\pi}{\sin \pi z}$$

By analytic continuation, this relation is true for all $z \in \mathbb{C} / \mathbb{Z}$

Bibliography