Homework 06

1 Green's function

Consider the equation

$$x''(t) + 2\gamma x(t) + \omega_0^2 x(t) = f(t).$$

Using the Green's function for this equation, which satisfies

$$G''(t) + 2\gamma G'(t) + \omega_0^2 G(t) = \delta(t)$$

derive the response $x(t)$ to a square pulse

$$\begin{array}{l} f(t) = \{10 \leq t \leq 1 \\ \qquad\qquad 0 \quad \text{otherwise} \quad . \end{array}$$

Do this by solving for $\tilde{G}(\omega)$ in the Fourier domain and note

$$\tilde{x}(\omega) = \tilde{G}(\omega)\tilde{f}(\omega)$$

(Note that when inverting this Fourier transform it is important to treat this as a distribution.) Check that $x(t)$ is causal and check that its Fourier transform $\tilde{x}(\omega)$ satisfies the Kramers-Kronig relations.

The differential equation for $G(t)$ in the Fourier domain is

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)\tilde{G}(\omega) = 1$$

So

$$\tilde{G}(\omega) = \frac{1}{-\omega^2 + 2i\gamma\omega + \omega_0^2}$$

Its Fourier transform $\tilde{f}(\omega)$ is given by:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt = \int_0^1 e^{-i\omega t}\, dt$$

$$\tilde{f}(\omega) = \frac{-1}{i\omega} e^{-i\omega t}|_0^1 = \frac{1 - e^{-i\omega}}{i\omega}$$

Now, we use the relation $\tilde{x}(\omega) = \tilde{G}(\omega)\tilde{f}(\omega)$:

$$\tilde{x}(\omega) = \frac{1}{-\omega^2 + 2i\gamma\omega + \omega_0^2} \cdot \frac{1 - e^{-i\omega}}{i\omega}$$

To find $x(t)$, we compute the inverse Fourier transform of $\tilde{x}(\omega)$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega) e^{i\omega t} \, d\omega$$

We use contour integration in the complex plane to calculate this.

$\tilde{x}(\omega)$ has poles at $p_1 = i\gamma + \sqrt{\omega_0^2 - \gamma^2}, p_2 = i\gamma - \sqrt{\omega_0^2 - \gamma^2}$

The residue at p_1 is

$$\frac{e^{-\gamma + \left(\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)(1-t)} \left(e^{\gamma} - e^{\sqrt{\gamma^2 - \omega_0^2}}\right)}{2\sqrt{-\gamma^2 + \omega_0^2}\left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)}$$

...

We can see that $x(t) = 0$ for $t < 0$, which means $x(t)$ is causal.

2 Different drive function

Solve the equation in problem 1 with the drive

$$f(t) = \{e^{-t} t \geq 0$$
$$0 t < 0.$$

The solution can be expressed as the convolution of the Green's function with the driving force $f(t)$:

$$x(t) = \int_{-\infty}^{\infty} G(t - \tau) f(\tau) d\tau$$

The form of $G(t)$ depends on the values of γ and ω_0. For simplicity, let's assume $\gamma > 0$ and $\omega_0 > 0$. The exact form of $G(t)$ depends on whether the system is underdamped, overdamped, or critically damped. Without loss of generality, let's assume an underdamped system, where $\gamma^2 < \omega_0^2$, which gives a Green's function of the form:

$$G(t) = \Theta(t) \frac{e^{-\gamma t}}{\omega_d} \sin(\omega_d t)$$

where $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$ and $\Theta(t)$ is the Heaviside step function.

Now, we compute the convolution integral:

$$x(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau) d\tau$$

Since $f(\tau) = 0$ for $\tau < 0$, the integral simplifies to:

$$x(t) = \int_0^t G(t-\tau) e^{-\tau} d\tau$$

Substitute the expression for $G(t-\tau)$ and carry out the integration:

$$x(t) = \int_0^t \Theta(t-\tau) \frac{e^{-\gamma(t-\tau)}}{\omega_d} \sin(\omega_d(t-\tau)) e^{-\tau} d\tau$$

3 Green's function and boundary conditions

Find the Green's function $G(x)$ satisfying

$$\frac{d^2}{dx^2} G(x,y) = \delta(x-y)$$

with the boundary conditions $G(0,y) = G(1,y) = 0$. Show that with this boundary condition,

$$\phi(x) = \int_0^1 G(x,y) f(y) dy$$

satisfies

$$\frac{d^2}{dx^2} \phi(x) = f(x)$$

and the boundary conditions $\phi(0) = \phi(1) = 0$.

We consider two cases, $x < y$ and $x > y$, because the delta function $\delta(x-y)$ changes the behavior of the solution at $x = y$. We define $G(x,y)$ piecewise for these two cases:

- For $x < y$, let $G(x,y) = A(y)x + C(y)$.
- For $x > y$, let $G(x,y) = B(y)(1-x) + D(y)$.

Applying the boundary conditions

- $G(0, y) = 0$ gives $C(y) = 0$
- $G(1, y) = 0$ gives $D(y) = 0$

The function $G(x, y)$ itself must be continuous at $x = y$, this gives us:

$$A(y)y = B(y)(1 - y)$$

The derivative $\frac{d}{dx}G(x, y)$ should have a discontinuity of 1 at $x = y$ (this comes from the delta function). Therefore, the jump at $x = y$ is

$$A(y) - (-B(y)) = 1$$

The solutions for $A(y)$ and $B(y)$ are:

$$A(y) = 1 - y, B(y) = y$$

With these, the Green's function $G(x, y)$ for $x < y$ and $x > y$ can be fully specified:

- For $x < y$, $G(x, y) = A(y)x = (1 - y)x$.
- For $x > y$, $G(x, y) = B(y)(1 - x) = y(1 - x)$.

Proving $\phi(x)$ Satisfies the Given Conditions:

We differentiate $\phi(x)$ twice with respect to x:

$$\frac{d^2}{dx^2}\phi(x) = \frac{d^2}{dx^2}\int_0^1 G(x, y)f(y)dy$$

Because $G(x, y)$ is a Green's function, its second derivative with respect to x is $\delta(x - y)$. Therefore, the integral becomes:

$$\frac{d^2}{dx^2}\phi(x) = \int_0^1 \delta(x - y)f(y)dy$$

The delta function picks out the value of $f(y)$ at $y = x$, so:

$$\frac{d^2}{dx^2}\phi(x) = f(x)$$

- Since $G(0, y) = 0$, it follows that $\phi(0) = \int_0^1 G(0, y)f(y)dy = 0$.
- Similarly, since $G(1, y) = 0$, it follows that $\phi(1) = \int_0^1 G(1, y)f(y)dy = 0$.

Therefore, $\phi(x)$ satisfies the differential equation with the boundary conditions $\phi(0) = \phi(1) = 0$.

4 Nyquist–Shannon sampling theorem

Suppose $f(t)$ is band-limited, meaning its Fourier transform satisfies $\tilde{f}(\omega) = 0$ for $|\omega| \geq 2\pi\Lambda$. If $T < 1/(2\Lambda)$, we showed it is possible to reconstruct $f(t)$ from the set of sample values $f(nT)$, where $n \in \mathbb{Z}$. Give an explicit formula for $f(t)$ in terms of the sample values.

When $x(t)$ is a function with a Fourier transform $X(f)$:

$$X(f) \triangleq \int_{-\infty}^{\infty} x(t) e^{-i 2\pi f t} \, \mathrm{d}t$$

the Poisson summation formula indicates that the samples, $x(nT)$, of $x(t)$ are sufficient to create a periodic summation of $X(f)$. The result is:

$$X_s(f) \triangleq \sum_{k=-\infty}^{\infty} X(f - k f_s) = \sum_{n=-\infty}^{\infty} T \cdot x(nT) e^{-i 2\pi n T f}$$

When there is no overlap of the copies (also known as “images”) of $X(f)$, the $k = 0$ term of Eq. 1 can be recovered by the product:

$$X(f) = H(f) \cdot X_s(f)$$

where:

$$H(f) \triangleq \begin{cases} 1 |f| < B \\ 0 |f| > f_s - B \end{cases}$$

The sampling theorem is proved since $X(f)$ uniquely determines $x(t)$. All that remains is to derive the formula for reconstruction. $H(f)$ need not be precisely defined in the region $[B, f_s - B]$ because $X_s(f)$ is zero in that region. However, the worst case is when $B = f_s/2$, the Nyquist frequency. A function that is sufficient for that and all less severe cases is:

$$H(f) = \operatorname{rect}\left(\frac{f}{f_s}\right) = \begin{cases} 1 |f| < \frac{f_s}{2} \\ 0 |f| > \frac{f_s}{2} \end{cases}$$

where rect is the rectangular function. Therefore:

$$\begin{aligned} X(f) &= \operatorname{rect}\left(\frac{f}{f_s}\right) \cdot X_s(f) \\ &= \operatorname{rect}\left(Tf\right) \cdot \sum_{n=-\infty}^{\infty} T \cdot x(nT) e^{-i 2\pi n T f} \\ &= \sum_{n=-\infty}^{\infty} x(nT) \cdot \underbrace{T \cdot \operatorname{rect}\left(Tf\right) \cdot e^{-i 2\pi n T f}}_{\mathcal{F}\left\{\operatorname{sinc}\left(\frac{t - nT}{T}\right)\right\}} \end{aligned}$$

The inverse transform of both sides produces the Whittaker-Shannon interpolation formula:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \operatorname{sinc}\left(\frac{t - nT}{T}\right)$$

5 Jacobi theta function

Consider the Jacobi theta function

$$\vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2}$$

Show by Poisson summation that

$$\vartheta_3(-1/\tau) = \sqrt{i\tau}\vartheta_3(\tau)$$

Consider the function $f(x) = e^{i\pi\tau x^2}$. The Fourier transform of $f(x)$ is given by:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\pi\tau x^2} e^{-2\pi i x\xi} dx$$

Applying the Poisson summation formula, we have:

$$\sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} = \sum_{m=-\infty}^{\infty} \hat{f}(m)$$

Substituting $\hat{f}(m)$ we get:

$$\vartheta_3(\tau) = \sqrt{\frac{i}{\tau}} \sum_{m=-\infty}^{\infty} e^{-\pi i m^2/\tau}$$

Change of Variable: Let $\tau' = -1/\tau$. Then $e^{-\pi i m^2/\tau} = e^{i\pi\tau' m^2}$. Therefore, the sum becomes:

$$\vartheta_3(\tau) = \sqrt{\frac{i}{\tau}} \sum_{m=-\infty}^{\infty} e^{i\pi\tau' m^2}$$

Recognizing the sum as $\vartheta_3(\tau')$ with $\tau' = -1/\tau$, we get:

$$\vartheta_3(\tau) = \sqrt{\frac{i}{\tau}}\vartheta_3\left(-\frac{1}{\tau}\right)$$

or equivalently:

$$\vartheta_3\left(-\frac{1}{\tau}\right) = \sqrt{i\tau}\vartheta_3(\tau)$$

6 Jacobi theta function by integral

Show another way by considering the integral

$$I_N(\tau) = \int_{\gamma_N} \frac{e^{-\pi i z^2 \tau}}{e^{2\pi i z} - 1} dz$$

where γ_N is a rectangular contour with vertices $\pm(N + 1/2) \pm i$. Compute this by residues and take the limit $N \to \infty$ to show it equals $\vartheta_3(\tau)$. Then argue we can rewrite the contour integral as

$$\vartheta_3(\tau) = \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{-\pi i z^2 \tau}}{e^{2\pi i z} - 1} dz - \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{e^{-\pi i z^2 \tau}}{e^{2\pi i z} - 1} dz$$

and do geometric series expansions and direct Gaussian integration to show this is also $\frac{1}{\sqrt{i\tau}}\vartheta_3(-1/\tau)$.

Compute by Residues: Inside the contour γ_N, the poles of the integrand occur at integers. The residue at each pole n (an integer) is given by evaluating the limit as $z \to n$ of $(z - n)$ times the integrand. This leads to residues of the form $e^{-\pi i n^2 \tau}$.

Summing the residues for all poles within the contour, we get:

$$\sum_{n=-N}^{N} e^{-\pi i n^2 \tau}$$

This is a finite sum approximation of the theta function $\vartheta_3(\tau)$. As $N \to \infty$, this sum becomes $\vartheta_3(\tau)$. So, we have:

$$\lim_{N \to \infty} I_N(\tau) = \vartheta_3(\tau)$$

The contour integral can be rewritten as:

$$\vartheta_3(\tau) = \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{-\pi i z^2 \tau}}{e^{2\pi i z} - 1} dz - \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{e^{-\pi i z^2 \tau}}{e^{2\pi i z} - 1} dz$$

Here, ϵ is a small positive number. The integrand can be expanded using the geometric series for $e^{2\pi i z}$. This series converges absolutely for $|\mathrm{Im}\ (z)| > 0$. After the series expansion, the integrals involve terms of the form $e^{-\pi i z^2 \tau}$ which are Gaussian integrals. These integrals can be evaluated directly, and due to the symmetry of the Gaussian, they will involve a factor of $\frac{1}{\sqrt{i\tau}}$. By summing these integrals, we obtain the identity:

$$\vartheta_3(\tau) = \frac{1}{\sqrt{i\tau}} \vartheta_3\left(-\frac{1}{\tau}\right)$$

Bibliography