Homework 08

1 Asymptotic series for $\log(x-2)$

Find an asymptotic series for

$$f(x) = \log(x-2)$$

as $x \to \infty$ in terms of $\log x$ and inverse powers of x. (Hint: split the log and use Taylor series on one part.)

$$f(x) = \log(x-2) = \log x + \log\left(1 - \frac{2}{x}\right)$$

The Taylor series expansion of $\log(1-y)$ around $y = 0$ is:

$$\log(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \cdots$$

By substituting $y = \frac{2}{x}$ into this series, the asymptotic series for $f(x)$ as $x \to \infty$ is:

$$f(x) = \log x - \frac{2}{x} - \frac{2^2}{2x^2} - \frac{2^3}{3x^3} - \cdots$$

2 Watson's lemma and $\int_0^1 \frac{e^{-sx}}{1+x^2} dx$

Use Watson's lemma to find the $s \to \infty$ asymptotic series for

$$I(s) = \int_0^1 \frac{e^{-sx}}{1+x^2} dx$$

To use Watson's Lemma, we need to express $\frac{1}{1+x^2}$ as a power series at $x = 0$. The Taylor series of $\frac{1}{1+x^2}$ around $x = 0$ is:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

which is valid for $|x| < 1$. Substituting this into $I(s)$ gives:

$$I(s) = \int_0^1 e^{-sx}(1 - x^2 + x^4 - x^6 + \cdots)\, dx$$

Now, we can integrate term by term:

1. For the first term:

$$\int_0^1 e^{-sx}\, dx = \frac{1}{s}(1 - e^{-s})$$

   As $s \to \infty$, e^{-s} approaches 0 faster than $1/s$, so this term becomes $\frac{1}{s}$.

2. For the second term:

$$-\int_0^1 x^2 e^{-sx}\, dx$$

   Applying integration by parts or a similar method, we find this term is $O\left(\frac{1}{s^3}\right)$ as $s \to \infty$.

3. Similarly, each subsequent term will contribute higher order terms in $\frac{1}{s}$.

Hence, the asymptotic expansion of $I(s)$ as $s \to \infty$ is:

$$I(s) \sim \frac{1}{s} - \frac{1}{s^3} + \cdots$$

3 Watson's lemma and $\int_0^\infty \sin(\sqrt{x}) e^{-sx^2}\, dx$

Use Watson's lemma to find the $s \to \infty$ asymptotic series for

$$I(s) = \int_0^\infty \sin(\sqrt{x}) e^{-sx^2}\, dx$$

The integral $I(s)$ does not directly fit this form of Watson's lemma due to the e^{-sx^2} term. So first, we perform a change of variable to transform the integral into a more suitable form for Watson's Lemma. Let $u = x^2$, then $du = 2x\, dx$ or $dx = \frac{du}{2\sqrt{u}}$. The integral becomes:

$$I(s) = \int_0^\infty \sin(u^{1/4}) e^{-su} \frac{du}{2\sqrt{u}}$$

Expand $\sin(u^{1/4})$ in a Taylor series about $u = 0$:

$$\sin(u^{1/4}) = u^{1/4} - \frac{u^{3/4}}{3!} + \frac{u^{5/4}}{5!} - \cdots$$

Substituting this series into the integral, we get:

$$I(s)=\int_0^\infty\left(u^{1/4}-\frac{u^{3/4}}{3!}+\cdots\right)e^{-su}\frac{du}{2\sqrt{u}}=\frac{\Gamma(3/4)}{2s^{3/4}}-\frac{\Gamma(5/4)}{12s^{5/4}}+\cdots$$

4 Asymptotic series for the error function

Use integration by parts to give the $x\to\infty$ asymptotic series for the error function

$$\text{erf }(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

We can apply successive integration by parts to erfc (λ), in which the first step gives

$$\begin{aligned}
\text{erf }(x)&=\frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)dt\\
&=\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{-2t\exp(-t^2)}{-2t}dt,\left[u=\frac{1}{-2t},dv=-2t\exp(-t^2)dt\right]\\
&=\frac{2}{\sqrt{\pi}}\left[\frac{\exp(-t^2)}{-2t}-\int\frac{\exp(-t^2)}{2t^2}dt\right]_x^\infty\\
&=\frac{2}{\sqrt{\pi}}\left[\frac{\exp(-x^2)}{2x}-\int_x^\infty\frac{\exp(-t^2)}{2t^2}dt\right]\\
&=\frac{2e^{-x^2}}{\sqrt{\pi}}\left(\frac{1}{2x}\right)-\frac{2}{\sqrt{\pi}}\int_x^\infty\frac{\exp(-t^2)}{2t^2}dt
\end{aligned}$$

We have obtained the first term. And applying integration by parts to the new integral gives

$$\begin{aligned}
\int_x^\infty\frac{\exp(-t^2)}{2t^2}dt&=\frac{1}{2}\int_x^\infty\frac{-2t\exp(-t^2)}{-2t^3}dt\left[u=-\frac{1}{2t^3},dv=-2t\exp(-t^2)dt\right]\\
&=\frac{1}{2}\left[\frac{\exp(-t^2)}{-2t^3}-\int\frac{3\exp(-t^2)}{2t^4}dt\right]_x^\infty\\
&=\frac{1}{2}\left[\frac{\exp(-x^2)}{2x^3}-\int_x^\infty\frac{3\exp(-t^2)}{2t^4}dt\right]
\end{aligned}$$

Thus we have

$$\text{erfc }(x)=\frac{e^{-x^2}}{\sqrt{\pi}}\left(\frac{1}{2x}-\frac{1}{4x^3}-\int_x^\infty\frac{3\exp(-t^2)}{2t^4}dt\right),$$

Continuing with successive integration by parts we will obtain the asymptotic expansion.

5 Stirling's formula

Use Laplace's method to derive Stirling's formula

$$n! \sim \sqrt{2\pi n} n^n e^{-n} \quad n \to \infty$$

using the Gamma function. Also find the next term in the asymptotic series.

Gamma function, which is related to the factorial by $\Gamma(n+1) = n!$ is defined as:

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$$

For large n, we approximate this integral using Laplace's method, which is effective for integrals of the form $\int e^{M f(t)} dt$ where M is a large parameter. Here, we can rewrite the Gamma function as:

$$\Gamma(n+1) = \int_0^\infty e^{n \log t - t} dt$$

Now, to apply Laplace's method, we find the maximum of the function $f(t) = \log t - \frac{t}{n}$. Taking the derivative and setting it to zero gives:

$$\frac{d}{dt}\left(\log t - \frac{t}{n}\right) = \frac{1}{t} - \frac{1}{n} = 0$$

This yields $t = n$ as the point where $f(t)$ attains its maximum. We then expand $f(t)$ around this point:

$$f(t) \approx f(n) + \frac{1}{2} f''(n)(t-n)^2$$

where $f(n) = \log n - 1$ and $f''(n) = -1/n^2$. The integral becomes:

$$\Gamma(n+1) \approx e^{n \log n - n} \int_0^\infty e^{-\frac{1}{2}\frac{(t-n)^2}{n}} dt$$

Changing variables with $u = \frac{t-n}{\sqrt{n}}$ gives:

$$\Gamma(n+1) \approx e^{n \log n - n} \sqrt{n} \int_0^\infty e^{-\frac{1}{2}u^2} du$$

Recognizing the Gaussian integral, we get:

$$\Gamma(n+1) = n! \approx e^{n \log n - n} \sqrt{2\pi n} = \sqrt{2\pi n} n^n e^{-n} \quad n \to \infty$$

The second derivative $f''(t)$ at the point $t = n$ gives us the next leading term. The third derivative of $f(t)$ is:

$$f'''(t) = \frac{2}{t^3}$$

Evaluating this at $t = n$ gives:

$$f'''(n) = \frac{2}{n^3}$$

The corresponding term in the expansion is of the order $\frac{1}{n}$, which leads to the next term in the series being $\frac{1}{12n}$. Therefore, the improved Stirling's formula, including the next term in the asymptotic series, is:

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \left(1 + \frac{1}{12n}\right)$$

6 Leading asymptotics of $I(x) = \int_{-1}^{1} e^{-x \sin^4 t} dt$

Use Laplace's method to find the leading asymptotics of

$$I(x) = \int_{-1}^{1} e^{-x \sin^4 t} dt$$

as $x \to \infty$.

For the given integral, we observe that $\sin^4 t$ is maximized when $t = 0$ within the interval $[-1, 1]$. Near this point, the function $\sin^4 t$ can be approximated by its Taylor expansion:

$$\sin^4 t \approx t^4$$

for small values of t. The integral becomes:

$$I(x) \approx \int_{-1}^{1} e^{-xt^4} dt$$

As $x \to \infty$, the contribution to the integral from regions where t is not very small becomes negligible, so we can extend the limits of the integral to infinity for the purpose of asymptotic approximation:

$$I(x) \approx \int_{-\infty}^{\infty} e^{-xt^4} dt$$

Let $u = x^{1/4} t$, then $dt = x^{-1/4} du$ and the integral becomes:

$$I(x) \approx \int_{-\infty}^{\infty} e^{-u^4} x^{-1/4} du$$

The integral $\int_{-\infty}^{\infty} e^{-u^4} du$ is a constant (independent of x), so the leading asymptotic behavior of $I(x)$ as $x \to \infty$ is given by:

$$I(x) \sim C x^{-1/4}$$

where C is the value of the integral $\int_{-\infty}^{\infty} e^{-u^4} du$, which can be evaluated numerically.

Bibliography