# Homework 07

## 1 Green's function for Poisson's equation

- Use Fourier analysis to compute a Green's function for Poisson's equation on $\mathbb{R}^3$, satisfying

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) G(x, y, z) = \delta^{(3)}(x, y, z)$$

- Prove there is a unique such Green's function which goes to zero at infinity (hint: use Liouville's theorem for harmonic functions).
- Find the Green's function but now for functions on the unit ball around the origin, with Dirichlet boundary conditions $\phi(x, y, z) = 0$ for $x^2 + y^2 + z^2 = 1$. Hint: Schwartz reflection principle/method of images.

### 1.1 Fourier Analysis

Let $\hat{G}\left(k_x, k_y, k_z\right)$ be the Fourier transform of $G(x, y, z)$. After Fourier transform, the equation becomes:

$$-\left(k_x^2 + k_y^2 + k_z^2\right)\hat{G}\left(k_x, k_y, k_z\right) = 1$$

since the Fourier transform of $\delta^{(3)}(x, y, z)$ is 1.

Thus, the solution in Fourier Space:

$$\hat{G}\left(k_x, k_y, k_z\right) = -\frac{1}{k_x^2 + k_y^2 + k_z^2}$$

To find $G(x, y, z)$, perform the inverse Fourier transform:

$$G(x, y, z) = \int_{\mathbb{R}^3} -\frac{e^{i\left(k_x x + k_y y + k_z z\right)}}{k_x^2 + k_y^2 + k_z^2}\, dk_x\, dk_y\, dk_z$$

This integral yields $G(x, y, z) = \frac{-1}{4\pi\sqrt{x^2+y^2+z^2}}$.

### 1.2 Uniqueness of the Green's Function

Liouville's theorem states that a bounded harmonic function on $\mathbb{R}^n$ is constant. Given any two Green's functions $G_1$ and $G_2$ that vanish at infinity, their difference $G_1 - G_2$ is harmonic (satisfies Laplace's equation) and bounded. By Liouville's theorem, $G_1 - G_2$ is constant, and since both go to zero at infinity, this constant must be zero. Thus, $G_1 = G_2$, proving uniqueness.

### 1.3 Green's Function on the Unit Ball with Dirichlet Boundary Conditions

To find a function $G(x, y, z; x', y', z')$ that satisfies:

1. $(\partial_x^2 + \partial_y^2 + \partial_z^2) G(x, y, z; x', y', z') = -\delta(x - x', y - y', z - z')$ within the unit ball $x^2 + y^2 + z^2 < 1$.
2. $G(x, y, z; x', y', z') = 0$ for $x^2 + y^2 + z^2 = 1$, enforcing the Dirichlet boundary conditions.

We place an "image" point source at $(x'', y'', z'')$ where $(x'', y'', z'') = \frac{(x', y', z')}{|x'|^2 + |y'|^2 + |z'|^2}$ outside the unit ball in such a way that the combined effect of the real source and the image source satisfies the boundary conditions.

The Green's function then is a combination of the solution from both the real and the image source

$$G(x, y, z; x', y', z') = \frac{1}{4\pi\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{4\pi\sqrt{(x - x'')^2 + (y - y'')^2 + (z - z'')^2}}$$

More generally, we define the point $x^* = \frac{x}{|x|^2}$ dual to $x$. Therefore, a Green's function for $B^n(0, 1)$ is given by $G(x, y) = \Phi(y - x) - \Phi(|x|(y - x^*))$.

## 2 Green's function for the heat equation

Find the Green's function for the heat equation

$$\partial_t u = \partial_x^2 u$$

by Fourier analysis.

We want to find a Green's function $G(x, t; x', t')$ that satisfies the following properties:

1. $\partial_t G = \partial_x^2 G$
2. $G(x, t; x', t')$ behaves like a delta function as $t \to t'^+$, i.e., $G(x, t; x', t') \to \delta(x - x')$ as $t \to t'^+$.

By taking the Fourier transform of $G$ with respect to $x$, the heat equation in Fourier space becomes

$$\partial_t \hat{G} = -k^2 \hat{G}$$

This is a first-order linear differential equation in $t$, the solution is

$$\hat{G}(k, t; x', t') = A(k, x', t') e^{-k^2(t - t')}$$

where $A(k, x', t')$ is to be determined.

Now we apply the initial condition. The initial condition is that $G$ approaches $\delta(x - x')$ as $t \to t'^+$. In Fourier space, this translates to $\hat{G}(k, t; x', t') \to e^{-ik(x - x')}$ as $t \to t'^+$. Thus

$$A(k, x', t') = e^{-ikx'}$$

Then take the inverse Fourier transform of $\hat{G}(k,t;x',t')$ to get back to the spatial domain:

$$G(x,t;x',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx'}e^{-ikx}e^{-k^2(t-t\prime)}dk$$

This integral yields a function of the form:

$$G(x,t;x',t') = \frac{1}{\sqrt{4\pi(t-t')}}e^{-\frac{(x-x\prime)^2}{4(t-t\prime)}}$$

for $t > t'$.

So the Green's function for the heat equation is:

$$G(x,t;x',t') = \begin{cases} \frac{1}{\sqrt{4\pi(t-t\prime)}}e^{-\frac{(x-x\prime)^2}{4(t-t\prime)}}, t > t' \\ 0, & \text{otherwise} \end{cases}$$

## 3 Resolvent $R(z,A) = (z\mathbb{1} - A)^{-1}$

Given a finite dimensional complex matrix $A$, we can define the resolvent

$$R(z,A) = (z\mathbb{1} - A)^{-1} = \frac{1}{z-A},$$

where $\mathbb{1}$ is the identity matrix. This equation makes sense whenever $z$ is not an eigenvalue of $A$, and so we can consider $R(z,A)$ as a meromorphic, matrix-valued function of $z$, with poles at the eigenvalues of $A$. For these problems, assume $A$ is Hermitian:

1. Show that

$$R(z,A) = \frac{P_\lambda}{z-\lambda},$$

where $P_\lambda$ is the projector onto the eigenspace of $\lambda$ eigenvectors.

2. Suppose $w$ is orthogonal to the kernel of $A$. Prove that

$$v = \int_C \frac{dz}{2\pi i}\frac{R(z,A)}{z}w$$

solves the equation

$$Av = w,$$

where $C$ is a contour winding once around $z = 0$ and not enclosing any other eigenvalues.

3. Show the same as in (2) for a contour enclosing all eigenvalues except for 0.

### 3.1 Expression for $R(z, A)$

Given a Hermitian matrix $A$, its eigenvalues are real, and it can be diagonalized. Let $\lambda$ be an eigenvalue of $A$ and $P_\lambda$ the projector onto the eigenspace of $\lambda$. We have spectral decomposition

$$A = \sum_\lambda \lambda P_\lambda$$

the expression for $R(z, A)$ can be simplified:

$$R(z, A) = (z\mathbb{1} - A)^{-1} = \left(z \sum_\lambda P_\lambda - \sum_\lambda \lambda P_\lambda\right)^{-1}$$

Because the projectors $P_\lambda$ are orthogonal and sum to the identity: $R(z, A) = \sum_\lambda (zP_\lambda - \lambda P_\lambda)^{-1}$.

- Thus, $R(z, A) = \sum_\lambda \frac{P_\lambda}{z-\lambda}$.

### 3.2 Solving $Av = w$

Substitute the expression for $R(z, A)$ into the integral:

$$v = \int_C \frac{dz}{2\pi i} \frac{\sum_\lambda \frac{P_\lambda}{z-\lambda}}{z} w$$

Evaluate it using the residue theorem. Since $C$ winds around $z = 0$ and does not enclose any other eigenvalues, the only contribution comes from the pole at $z = 0$.

The residue at $z = 0$ gives $P_0 w$. So $Av = AP_0 w = w$

### 3.3 Contour Enclosing All Eigenvalues Except 0

Use the same integral expression for $v$. But now the integral will pick up residues from all poles $\lambda \neq 0$ within the contour $C$.

For each $\lambda \neq 0$, the residue at $z = \lambda$ is $\frac{P_\lambda w}{\lambda}$.

Summing up the residues, $v = \sum_{\lambda \neq 0} \frac{P_\lambda w}{\lambda}$.

Multiplying both sides by $A$ (and using $AP_\lambda = \lambda P_\lambda$), we get $Av = \sum_{\lambda \neq 0} P_\lambda w = w - P_0 w$.

Since $w$ is orthogonal to the kernel of $A$, $P_0 w = 0$. Therefore, $Av = w$.

## Bibliography