Homework 02

Conservation of charge density

> Show the conservation of charge density ρ_c (Eq. I.50), by integrating the Boltzmann equation.

The Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = \left(\frac{\partial f}{\partial t} \right)_c$$

Take the zeroth charge-velocity moment of the Boltzmann equation, i.e., multiply by $ev^0 = e$ and integrate over velocity space. The first term on the left-hand side becomes

$$\int d^3v \, e \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int d^3v \, e f = \frac{\partial}{\partial t} \rho_c$$

The second term on the left-hand side becomes

$$\int d^3v \, e \mathbf{v} \cdot \nabla f = \nabla \cdot \int d^3v \, e \mathbf{v} f = \nabla \cdot \mathbf{j}_c$$

The third term on the left-hand side becomes

$$\int d^3v \, e \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = \frac{q}{m} \int d^3v \, e \, (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = \frac{eq}{m} \int d^3v \nabla_v \cdot [(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \, f] - \frac{eq}{m} \int d^3v \nabla_v \cdot (\mathbf{v} \times \mathbf{B}$$

The first term on the right-hand side becomes a surface integral over the velocity space, which vanishes because the distribution function f goes to zero at infinity. The second term on the right-hand side is zero because $\mathbf{v} \times \mathbf{B}$ is always perpendicular to $\mathbf{v}$, so the dot product is zero. Therefore, the third term on the left-hand side is zero.

The right-hand side of the Boltzmann equation is zero because the collision operator conserves charge density. Therefore, the zeroth charge-velocity moment of the Boltzmann equation is

$$\frac{\partial}{\partial t}\rho_c + \nabla \cdot \mathbf{j}_c = 0$$

Equation of specific entropy

> Show the generalization of (I.57) for a gas with f degrees of freedom ($\gamma = 1 + 2/f$ ratio of specific heats)
>
> $$\frac{d}{dt}(p/\rho^\gamma) = \frac{(-\nabla \cdot q + j \cdot E^*)}{(f/2)\rho^\gamma}$$

Euler's equation of motion is given by (after combining with the continuity equation)

$$\frac{d}{dt}\left(\frac{1}{2}\rho V^2\right) + \frac{1}{2}\rho V^2 \nabla \cdot \vec{V} + \vec{V} \cdot \nabla p = \rho_c \vec{V} \cdot \vec{E} + \vec{V} \cdot (\vec{J} \times \vec{B}) + \rho V \cdot g$$

The energy equation is given by

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + u\right) + \frac{\partial}{\partial x_j}\left[\left(\frac{1}{2}\rho V^2 + u\right) V_j + P_{jk}V_k + q_j\right] = J_j E_j + \rho V_j g_j$$

where $u = \frac{f}{2}p$ is the internal energy.

Because

$$\frac{d}{dt}(\rho V^2) = \frac{\partial}{\partial t}(\rho V^2) + \vec{V} \cdot \nabla(\rho V^2) = \frac{\partial}{\partial t}(\rho V^2) + \nabla \cdot (\rho V^2 \vec{V}) - \rho V^2 \nabla \cdot \vec{V}$$

By combining, we have

$$\frac{\partial}{\partial t}(\rho V^2) + \frac{\partial}{\partial x_j}[(\rho V^2)V_j] = \frac{d}{dt}(\rho V^2) + \rho V^2 \nabla \cdot \vec{V}$$

We can rewrite energy equation into a form that is similar to Euler's equation of motion:

$$[\frac{d}{dt}(\frac{1}{2}\rho V^2) + \frac{1}{2}\rho V^2 \nabla \cdot \vec{V}] + [\frac{\partial u}{\partial t} + \nabla(u+p) \cdot \vec{V}] = -\nabla \cdot q + J \cdot E + \rho V \cdot g$$

Subtracting Euler's equation from the energy equation would gives us:

$$[\frac{\partial u}{\partial t}+\vec{V}\cdot\nabla u+u\nabla\cdot\vec{V}]+\nabla\cdot\vec{V}p-\vec{V}\cdot\nabla p=-\nabla\cdot q+(J-\rho_c\vec{V})\cdot\vec{E}-\vec{V}\cdot(\vec{J}\times\vec{B})=-\nabla\cdot q+j\cdot E^*$$

where $\vec{E}^* = \vec{E}+\vec{V}\times\vec{B}$.

The left-hand side of the equation can be rewritten as

$$LHS = d_t u + (p+u)\nabla\cdot\vec{V} = d_t(\frac{f}{2}p) - (\frac{f}{2}+1)\frac{p}{\rho}d_t\rho = \frac{f}{2}\rho^\gamma d_t\frac{p}{\rho^\gamma}$$

where $\gamma = 1+2/f$ is the ratio of specific heats.

Therefore, we have

$$\frac{d}{dt}(p/\rho^\gamma) = \frac{(-\nabla\cdot q + j\cdot E^*)}{(f/2)\rho^\gamma}$$

Polytropic equation

> Show that for a gas with f degrees of freedom ($\gamma = 1+f/2$ ratio of specific heats), in steady state (/ t=0), and with a heat flux which obeys: $q = \kappa u V$ (eq. (I.85), where u is the internal energy), the polytropic equation $p = \alpha\rho^n$ (I.84) still holds even though ds*/dt as per eq. (I.81) is not zero (where* s is the specific entropy defined the usual way using γ). In other words, show that eq. (I.81) can be transformed into an equation for a revised "entropy" defined using n, the polytropic index, which is no longer γ, but $n = (\gamma+\kappa)/(1+\kappa)$. In other words show (I.86).

As we have shown in the previous question, the equation of specific entropy is given by

$$\frac{d}{dt}(p/\rho^\gamma) = \frac{(-\nabla\cdot q + j\cdot E^*)}{(f/2)\rho^\gamma}$$

With

- the heat flux proportional to the internal energy, i.e., $q = \kappa u V$
- $j\cdot E^* = 0$
- and assuming $p = g\rho^\gamma$, where g is an arbitrary function

the equation of specific entropy becomes

$$d_t g = -\nabla \cdot \kappa V(\frac{f}{2}p)/(\frac{f}{2}\rho^\gamma) = -\kappa\nabla \cdot (Vp)/(p/g)$$

For steady state problems, i.e., $\frac{\partial}{\partial t} = 0$, the equation can be rewritten as

$$gV \cdot \nabla g = -\kappa(\nabla \cdot V + V \cdot \nabla \ln p)$$

Using continuity equation (with steady state condition), i.e., $\nabla \cdot V = -V \cdot \nabla\rho/\rho = -V \cdot \nabla \ln \rho$, we have

$$V \cdot \nabla \ln g = -\kappa V \cdot (-\nabla \ln \rho + \nabla \ln p)$$

For this equation to hold everywhere, we must have

$$g = c_1 * (\frac{\rho}{p})^\kappa$$

Therefore, we have

$$p = c_1 \rho^\gamma (\frac{\rho}{p})^\kappa$$

and thus

$$p = c_1 \rho^{\frac{\gamma+\kappa}{1+\kappa}}$$

which is the polytropic equation with $n = \frac{\gamma+\kappa}{1+\kappa}$.

Since we don't assume anywhere in the above derivation that $d_t s^* = 0$, the equation of specific entropy still could be transformed into an equation for a revised "entropy" defined using n, the polytropic index, which is no longer γ, but $n = (\gamma + \kappa)/(1 + \kappa)$, even though $d_t s^* \neq 0$.

Generalized Ohm's law

Siscoe's notes show a scaling analysis of the terms R1-R4 in Ohm's law, equations (II.9) (II.12), for the cool magnetosheath/slow solar wind.

Collision time scale

a. Show that the collision time scale τ is indeed large enough, such that R1 is indeed large for the plasma region envisioned in the book. To compute $\tau = 1/\nu_{ei}$, obtain the numerical value of the collision frequency for that region by using tabulated values of the Coulomb logarithm λ in the NRL formulary (here, p. 34, item (b)) and the equation for $\nu_{ei} = \nu_e\ \nu_i$ for the weak collision rate case (strong field case, here, p. 36).

The Lorentz collision frequency is given by

$$\nu = n\sigma v \ln \Lambda$$

where n is the particle number density, σ is the collisional cross-section, v is the mean thermal velocity between particle species, and $\ln \Lambda$ is the Coulomb logarithm accounting for small angle collisions.

Using typical parameters for space science, we can calculate the collision time scale $\tau = 1/\nu_{ei} = 3 \cdot 10^4$s, which is indeed large enough.

```
import plasmapy.formulary as plf
import astropy.units as u

n = 1e7 * u.m**-3
T = 1e5 * u.K
v_drift = 100 * u.km / u.s


def tau(n, T, V):
    # assuming temperature and denisty is the same for both species
    n_ion = n_e = n
    T_ion = T_e = T

    Coulomb_log = plf.Coulomb_logarithm(T, n_e, ("e-", "p+")) * u.dimensionless_unscaled

    electron_ion_collisions = plf.MaxwellianCollisionFrequencies(
        "e-",
        "p+",
        v_drift = V,
        n_a=n_e,
        T_a=T_e,
        n_b=n_ion,
        T_b=T_ion,
```

```
        Coulomb_log=Coulomb_log,
    )

    nu_ei = electron_ion_collisions.Maxwellian_avg_ei_collision_freq
    return (1  / nu_ei).to(u.s)

tau(n=n, T=T, V=v_drift)
```

34654.032 s

Dimensionless ratios

b. Obtain the ratios R1-R4 for typical parameters in another plasma region, the near-Earth magnetotail plasma sheet (but outside the reconnection electron diffusion region). Is the Hall term significant now? Derive the ratio of the electron inertia to the Hall term (ratio R4/R3) by scaling analysis – under what conditions is it significant in this environment?

Given the following parameters

$$
\begin{aligned}
R_1 &\equiv \frac{VB}{nJ} = \frac{VB}{\frac{m_e}{e^2 n\tau}} \frac{B}{\mu_0 L} = nVL\tau\frac{\mu_0 e^2}{m_e} \\
R_2 &\equiv \frac{VB}{\frac{1}{ne}\frac{P_e}{L}} = \frac{VB}{\frac{k}{ne}\frac{nT}{L}} = \frac{e}{k}\frac{VBL}{T} \\
R_3 &\equiv \frac{VB}{\frac{1}{\mathrm{en}}\frac{B^2}{\mu_0 L}} = \frac{VLn}{B}\mathrm{e}\mu_0 \\
R_4 &\equiv \frac{VB}{\frac{m_e}{e^2 n}\frac{J}{T}} = L^2 n\frac{\mu_o e^2}{m_e}
\end{aligned}
$$

Using the parameters from (Borovsky et al. 2020), and assuming the spatial scale of the magnetotail plasma sheet is about earth radius, i.e., $L \sim 10^7 m$, we have the ratios calculated as shown below **?@tbl-ratios**. The Hall term is now become comparable to the "v cross B" term with $R_3 \sim 10$. This term will be more significant when the spatial scale is smaller (for example, near the reconnection region).

The ratio of the electron inertia to the Hall term is given by

$$
R_4/R_3 = \frac{L * B}{V} * \frac{e}{m_e}
$$

Since R_4 is proportional to L^2, and R_3 is proportional to L, the ratio R_4/R_3 is proportional to L. Therefore, the electron inertia term is more significant when the spatial scale is smaller. It is also more significant when the magnetic field is weaker, or the velocity is larger.

```
from astropy.constants.si import mu0, e, m_e, k_B
import astropy.units as u
from astropy.table import QTable
import pandas as pd

def R1(
    n: float, #: number density
    V: float, #: velocity
    L: float, #: length
     : float, #: collision time
):
    """
    Ratio of the "v cross B" term over the ohmic resistive term
    """
    const = mu0 * e**2 / m_e
    return (n * V * L *   * const).to(u.dimensionless_unscaled)

def R2(
    V: float, #: velocity
    B: float, #: magnetic field
    L: float, #: length
    T: float #: temperature
):
    """
    Ratio of the "v cross B" term over the ampipolar electric field term
    """
    const = e / k_B
    return (V * B * L / T * const).to(u.dimensionless_unscaled)

def R3(
    V: float, #: velocity
    B: float, #: magnetic field
    n: float, #: number density
    L: float #: length
):
    """
    Ratio of the "v cross B" term over the Hall term
    """
    const = e * mu0
```

```python
    return (V * L * n / B * const).to(u.dimensionless_unscaled)

def R4(
    L: float, #: length
    n: float, #: number density
):
    """
    Ratio of the "v cross B" term over the electron inertia term
    """
    const = mu0 * e**2 / m_e
    return (L**2 * n * const).to(u.dimensionless_unscaled)

def R4overR3(
    V: float, #: velocity
    B: float, #: magnetic field
    n: float, #: number density
    L: float #: length
):
    """
    Ratio of the electron inertia term to the Hall term
    """
    return R4(L=L, n=n) / R3(V=V, B=B, n=n, L=L)


import plasmapy.formulary as plf
# using the parameters from [@borovskyOutstandingQuestionsMagnetospheric2020]

solar_wind = dict(
    name = "solar wind",
    n = 1e7 * u.m**-3,
    V = 1e5 * u.m / u.s,
    L = 1e6 * u.m,
    T = 1e5 * u.K,
    B = 1e-8 * u.T
)

magnetotail = dict(
    name = "ion plasma sheet",
    n = 5e5 * u.m**-3,
    V = 1e5 * u.m / u.s,
    L = 1e7 * u.m,
    T = (1e3 * u.eV).to(u.K, equivalencies=u.temperature_energy()),
```

```
    B = 1e-8 * u.T
)

for d in [solar_wind, magnetotail]:
    d[" "] = tau(n=d["n"], T=d["T"], V=d["V"])
    d["R1"] = R1(n=d["n"], V=d["V"], L=d["L"],  =d[" "])
    d["R2"] = R2(V=d["V"], B=d["B"], L=d["L"], T=d["T"])
    d["R3"] = R3(V=d["V"], B=d["B"], n=d["n"], L=d["L"])
    d["R4"] = R4(L=d["L"], n=d["n"])

df = QTable([solar_wind, magnetotail])

for col in df.colnames[1:]:
    df[col].info.format = ".1E"
df
```

```
from great_tables import GT
GT(df.to_pandas()).fmt_scientific(
    columns = df.colnames[1:], decimals=1
)
```

Table 1: Ratios of the terms in Ohm's law

name	n	V	L	T	B		R1	R2
solar wind	1.0×10^{7}	1.0×10^{5}	1.0×10^{6}	1.0×10^{5}	1.0×10^{-8}	3.5×10^{4}	1.2×10^{9}	$1.2 \times$
ion plasma sheet	5.0×10^{5}	1.0×10^{5}	1.0×10^{7}	1.2×10^{7}	1.0×10^{-8}	6.7×10^{8}	1.2×10^{13}	$1.0 \times$

Borovsky, Joseph E., Gian Luca Delzanno, Juan Alejandro Valdivia, Pablo S. Moya, Marina Stepanova, Joachim Birn, Lauren W. Blum, William Lotko, and Michael Hesse. 2020. "Outstanding Questions in Magnetospheric Plasma Physics: The Pollenzo View." *Journal of Atmospheric and Solar-Terrestrial Physics* 208 (October): 105377. https://doi.org/10.1016/j.jastp.2020.105377.