The Binomial distribution $X \sim B(n,p)$, $x = 0, 1, 2, \dots$ has parameters n and p and has probability mass function:

$$p(x) = \binom{n}{n-x} p^x (1-p)^{n-x} \tag{1.1}$$

When we start to consider the Poisson distribution, we shall start using the Binomial p.m.f. (1.1) but substitute the Poisson parameter λ $(= np)$.

<u>$P(X = 0)$</u>

Let us begin with the Binomial probability for $P(X = 0)$:

$$\begin{aligned} p(0) &= \binom{n}{n-0} p^0 (1-p)^{n-0} \\ &= 1 \times 1 \times (1-p)^n \\ &= (1-p)^n \end{aligned}$$

Expressed usuing the parameter of the Poisson distribution $\lambda = np$:

$$p(0) = \left(1 - \frac{\lambda}{n}\right)^n \tag{2.1}$$

Before we proceed any further, it is helpful to consider how we can arrive at a formula for the poisson distribution that includes a term containing $e^{-\lambda}$. We need to use techniques learned in MST124, namely the use of the binomial expansion to expand $(a+b)^n$ and then to link this to the Taylor series expansions of the function $f(x) = e^x$.

We shall return to $P(X = 0)$ later.

Binomial expansion

The binomial expansion is:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \dots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n \tag{3.1}$$

Let $a = 1$ and $b = x$:

$$
\begin{aligned}
&= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{3} x^3 + \cdots + \binom{n}{n} x^n \\
&= \frac{n!}{0!(n-0)!} x^0 + \frac{n!}{1!(n-1)!} x^1 + \ldots + \frac{n!}{(n-1)!\,1!} x^{n-1} + \frac{x^n}{n!} \\
&= \frac{n!}{(n-0)!} \frac{x^0}{0!} + \frac{n!}{(n-1)!} \frac{x^1}{1!} + \ldots + \frac{n!}{1!} \frac{x^{(n-1)}}{(n-1)!} + \frac{x^n}{n!} \\
&= 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots + \frac{x^n}{n!}
\end{aligned}
$$

Let $x = \frac{z}{n}$ with $n \to \infty$:

$$
(a+b)^n = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} \tag{3.2}
$$

<u>Taylor series</u>

The function $f(z) = e^z$ can be repeatedly differentiated and for each iteration $f^{(k)}(z) = e^z$. Therefore, we can approximate the function by Taylor expansion. The Taylor polynomial about point a is:

$$
\begin{aligned}
p(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f^{(3)}(a)}{3!}(z-a)^3 + \\
\ldots + \frac{f^{(n)}(a)}{n!}(z-a)^n
\end{aligned}
$$

When $a = 0$:

$$
\begin{aligned}
p(z) &= f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + f^{(3)}(0)\frac{z^3}{3!} + f^{(n)}(0)\frac{z^n}{n!} \\
&= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} \qquad (4.1)
\end{aligned}
$$

This is the Taylor series which approximates the function $f(z) = e^z$. This allows us to recognise that the binomial expansion of $\left(1 + \frac{z}{n}\right)^n$ tends to e^z as $n \to \infty$.

Replacing z with $-\lambda$, we get:

$$
\left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda} \qquad \text{for large } n \tag{4.2}
$$

Return to $P(X = 0)$

Now that we have seen why $\left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$ for large n we can resume our consideration of the probability that $P(X = 0)$. We can restate that:

$$\begin{aligned} p(0) &= \left(1 - \frac{\lambda}{n}\right)^n \\ &= e^{-\lambda} \qquad (5.1) \end{aligned}$$

For $P(X > 0)$

We calculate each new probability $P(X = x)$ from the previous probability $P(X = x - 1)$:

$$\begin{aligned} \frac{p(x)}{p(x-1)} &= \frac{\binom{n}{x} p^x (1-p)^{n-x}}{\binom{n}{x-1} p^{x-1} (1-p)^{n-x+1}} \qquad (6.1) \\ &= \frac{\binom{n}{x}}{\binom{n}{x-1}} \times \frac{p^x}{p^{x-1}} \times \frac{(1-p)^{n-x}}{(1-p)^{n-x+1}} \end{aligned}$$

But:

$$\begin{aligned} \frac{p^x}{p^{x-1}} &= \frac{p^x}{p^x \, p^{-1}} \\ &= p \qquad (6.2) \end{aligned}$$

Similarly:

$$\begin{aligned} \frac{(1-p)^{n-x}}{(1-p)^{n-x+1}} &= \frac{(1-p)^{n-x}}{(1-p)^{n-x} (1-p)^1} \\ &= \frac{1}{1-p} \qquad (6.3) \end{aligned}$$

So:

$$\frac{p(x)}{p(x-1)} = \frac{\binom{n}{x} p}{\binom{n}{x-1} (1-p)}$$

We can also simplify the binomial coefficients:

$$\begin{aligned} \frac{\binom{n}{x}}{\binom{n}{x-1}} &= \frac{n!}{x! \, (n-x)!} \div \frac{n!}{(x-1)! \, (n-x+1)!} \\ &= \frac{(x-1)!}{x!} \times \frac{(n-x+1)!}{(n-x)!} \\ &= \frac{1}{x} (n-x+1) \qquad (6.4) \end{aligned}$$

Replacing the terms in equation (6.1):

$$\frac{p(x)}{p(x-1)} = \frac{(n-x+1)}{x} \times \frac{p}{(1-p)} \tag{6.5}$$

Substituting $p = \frac{\lambda}{n}$ and also $1 - p = 1 - \frac{\lambda}{n}$ $(= 1 - \frac{\lambda}{n})$, we get:

$$\begin{aligned}\frac{p}{1-p} &= \frac{\lambda}{n} \div \frac{n-\lambda}{n} \\ &= \frac{\lambda}{n} \times \frac{n}{n-\lambda} \\ &= \frac{\lambda}{n-\lambda}\end{aligned}$$

Equation (6.5) becomes:

$$\frac{p(x)}{p(x-1)} = \frac{n-x+1}{x} \times \frac{\lambda}{n-\lambda}$$

Expressing this in terms of $p(x)$ gives:

$$p(x) = \frac{n-x+1}{n-\lambda} \times \frac{\lambda}{x} \times p(x-1)$$

As n becomes large, $\frac{n-x+1}{n-\lambda} \to 1$

Bringing it all together

Gathering together the expressions that we have obtained for the probabilities of $P(X = 0)$ and $P(X = x)$, we have a system of equations that can approximate the Binomial distribution with parameters n and $\frac{\lambda}{n}$, which is $X \sim B(n, \frac{\lambda}{n})$ with $x = 1, 2, 3, \ldots$ for a large sample size.

Using the system of equations to evaluate $P(X = 1)$:

$$\begin{aligned}p(1) &= \frac{\lambda}{1}\, p(0) \\ &= \frac{\lambda}{1}\, e^{-\lambda}\end{aligned}$$

Using a similar tachnique to evaluate $P(X = 2)$ and $P(X = 3)$:

$$\begin{aligned} p(2) &= \frac{\lambda}{2}\, p(1) \\ &= \frac{\lambda^2}{2 \times 1}\, e^{-\lambda} \end{aligned}$$

$$\begin{aligned} p(3) &= \frac{\lambda}{3}\, p(1) \\ &= \frac{\lambda^3}{3 \times 2 \times 1}\, e^{-\lambda} \end{aligned}$$

From this pattern, we can see that the probability mass function is:

$$p(x) = \frac{\lambda^x\, e^{-\lambda}}{x!} \qquad x = 1, 2, \ldots \tag{7.1}$$

Thus, we have shown that the Poisson distribution is derived from the Binomial distribution with parameters n and $\frac{\lambda}{n}$ for a large sample size, n.

Appendix: extracts from MST124

Example 10 *Finding a Taylor series about 0*

Find the Taylor series about 0 for the function $f(x) = e^x$.

Solution

Repeatedly differentiate f to find f', f'', $f^{(3)}$, ..., and find the values of $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, ... Then apply formula (6).

Here $f(0) = e^0 = 1$. Also, the nth derivative of the function $f(x) = e^x$ is $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$ for all positive integers n.

Hence, by the formula for a Taylor series about 0, the required Taylor series is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots .$$

Figure 1: MST124; Book D; Example 10; Page 138

When calculating the Taylor series for function $f(x) = e^x$, we use repeated differentiation of the function. For e^x:

$$f(x) = e^x \quad f'(x) = e^x \quad f''(x) = e^x \quad f^{(3)}(x) = e^x \quad f^{(4)}(x) = e^x \quad \text{etc.}$$

The derivation of the Taylor polynomial is:

Taylor polynomials

Let f be a function that is n-times differentiable at a point a. The **Taylor polynomial of degree n about a for f** is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The point a is called the **centre** of the Taylor polynomial.

When $a = 0$, the Taylor polynomial above becomes

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Figure 2: MST124; Book D; page 121

Standard Taylor series

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots \qquad \text{for } x \in \mathbb{R}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots \qquad \text{for } x \in \mathbb{R}$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \qquad \text{for } x \in \mathbb{R}$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \text{for } -1 < x < 1$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

$$(\alpha \text{ can be any real number}) \qquad \text{for } -1 < x < 1$$

Figure 3: MST124; Handbook; page 8