Double Pendulum equations of motion and numerical results

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Abstract

To be abstracted :)

Contents

1 Introduction

To be introduced :)

I'll be modelling a pendulum composed of two rigid rods as opposed to point masses hanged on ropes.

2 Equations of motion

Starting with the equations as derived on paper, will fill with explanation after!

2.1 The Lagrangian

We'll define each rod $i = \{1, 2\}$ as having length (l_i), mass (m_i) and moment of inertia about its center of mass (I_{ci}). Additionally we will have the gravitational acceleration norm g as a constant ($g \cong 9.8[\mathrm{m/s^2}]$). Normal vectors for the two "arms" of the pendulum are

$$\begin{aligned} \hat{n}_1 &= (\sin\theta, -\cos\theta)^T && (1)\\ \hat{n}_2 &= (\sin\varphi, -\cos\varphi)^T. && (2) \end{aligned}$$

Positions of the centre of mass for the two rods are

$$\begin{aligned} \vec{r}_{c1} &= \frac{l_1}{2}\hat{n}_1 && (3)\\ \vec{r}_{c2} &= l_1\hat{n}_1 + \frac{l_2}{2}\hat{n}_2. && (4) \end{aligned}$$

The Lagrangian is taken as the sum of kinetic terms minus the sum of potential terms, which gives

$$L = \frac{1}{2}m_1|\dot{\vec{r}}_{c1}|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_{c2}|^2 + \frac{1}{2}I_{c1}|\vec{\omega}_{c1}|^2 + \frac{1}{2}I_{c2}|\vec{\omega}_{c2}|^2 - m_1 g y_{c1} - m_2 g y_{c2}, \tag{5}$$

where $\dot{\vec{\omega}}_{ci}$ are the angular rates around the centres of mass for the two rods. Note that constant terms have been neglected (since a Lagrandian is unique up to linear transformation with constants). It is easy to see that $|\vec{\omega}_{c1}| = \dot{\theta}$ and $|\vec{\omega}_{c2}| = \dot{\varphi}$. Also, given a homogeneous rod of mass m and length l, the moment of inertia around its centre of mass is

$$I_c = \frac{1}{12}ml^2. \tag{6}$$

Using (1)–(4) in (5), one gets the full form

$$\begin{aligned} L = \; & \left(\tfrac{1}{6}m_1 l_1^2 + \tfrac{1}{2}m_2 l_2^2\right)\dot{\theta}^2 + \tfrac{1}{6}m_2 l_2^2\dot{\varphi}^2 + \tfrac{1}{2}m_2 l_1 l_2\cos(\theta-\varphi)\dot{\theta}\dot{\varphi} && (7)\\ & +\tfrac{1}{2}m_1 g l_1\cos\theta + m_2 g\left(l_1\cos\theta + \tfrac{l_2}{2}\cos\varphi\right). \end{aligned}$$

A lot of the clutter comes from constants, so I've chosen to simplify to

$$L = a_\theta\dot{\theta}^2 + a_\varphi\dot{\varphi}^2 + a_{\mathrm{mix}}\cos(\theta-\varphi)\dot{\theta}\dot{\varphi} + b_\theta\cos\theta + b_\varphi\cos\varphi, \tag{8}$$

where I've defined the constants

$$\begin{aligned} a_\theta &= \frac{1}{6}m_1 l_1^2 + \frac{1}{2}m_2 l_2^2 && (9)\\ a_\varphi &= \frac{1}{6}m_2 l_2^2 && (10)\\ a_{\mathrm{mix}} &= \frac{1}{2}m_2 l_1 l_2 && (11)\\ b_\theta &= l_1 g\left(\frac{m_1}{2} + m_2\right) && (12)\\ b_\varphi &= \frac{1}{2}l_2 g m_2. && (13) \end{aligned}$$

At this point the equations of motion can be readily derived.

2.2 The Equations of Motion

Lagrange's equations from a minimum action principle give dynamical equations

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}^i}\right) = \frac{dL}{dq^i}, \tag{14}$$

where q^i are the general positions (configuration space parameters) and $\dot{q}^i$ are their derivatives (general velocities). The combination of all of these $\{q^1, q^2, ..., q^N, \dot{q}^1, \dot{q}^2, ..., \dot{q}^N\}$ is the full state of the system (phase space). In our case $N = 2$ and the state is $\vec{s} = (\theta, \varphi, \dot{\theta}, \dot{\varphi})^T$. I will further denote $\dot{\theta} \equiv \omega_\theta$ and $\dot{\varphi} \equiv \omega_\varphi$.

Computing the equations (14) for the Lagrangian (8) yields the equations of motion

$$2a_\theta\dot{\omega}_\theta + a_{\text{mix}}\cos(\varphi - \theta)\dot{\omega}_\varphi \quad = -b_\theta \sin\theta + a_{\text{mix}}\sin(\varphi - \theta)\omega_\varphi^2 = f_1(\text{state}) \tag{15}$$

$$2a_\varphi\dot{\omega}_\varphi + a_{\text{mix}}\cos(\varphi - \theta)\dot{\omega}_\theta \quad = -b_\varphi \sin\varphi - a_{\text{mix}}\sin(\varphi - \theta)\omega_\theta^2 = f_2(\text{state}), \tag{16}$$

where state $\equiv (\theta, \varphi, \omega_\theta, \omega_\varphi)^T$. This can be written as a matrix equation

$$\begin{pmatrix} 2a_\theta & a_{\text{mix}}\cos(\varphi - \theta) \\ a_{\text{mix}}\cos(\varphi - \theta) & 2a_\varphi \end{pmatrix} \begin{pmatrix} \dot{\omega}_\theta \\ \dot{\omega}_\varphi \end{pmatrix} \quad = \quad \begin{pmatrix} f_1(\text{state}) \\ f_2(\text{state}). \end{pmatrix} \tag{17}$$

Complementing with the two equations $\dot{\varphi} = \omega_\varphi$ and $\dot{\theta} = \omega_\theta$ gives us a complete 1st order system of 4 non-linear differential equations. To solve it explicitly, the matrix on the left-hand-side should be invertable.

$$A_{\text{LHS}} \quad = \quad \begin{pmatrix} 2a_\theta & a_{\text{mix}}\cos(\varphi - \theta) \\ a_{\text{mix}}\cos(\varphi - \theta) & 2a_\varphi \end{pmatrix} \tag{18}$$

Looking at the determinant, the matrix will always be invertable as long as

$$\det A_{\text{LHS}} = 4a_\theta a_\varphi - a_{\text{mix}}^2\cos^2(\varphi - \theta) \neq 0. \tag{19}$$

Given the definitions of the constants a and b, this statement is equivalent to

$$\cos^2(\varphi - \theta) \neq \frac{4a_\theta a_\varphi}{a_{\text{mix}}^2} = \frac{4}{3}\left(\frac{l_2}{l_1}\right)^2\left[1 + \frac{m_1 l_1^2}{3m_2 l_2^2}\right], \tag{20}$$

which we'll show holds for all the physical cases.

To be continued :)

3 Conclussions

To be concluded :)

References

[1] H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*.