CO 342

Introduction to Graph Theory

Zhisu Wang
Taught by Peter Nelson
University of Waterloo
Spring 2023

Contents

1 Connectivity

1.1 Terminology

Definition 1.1: Graph

A graph is a triple $G : (V, E, i)$ where V and E are finite sets and i is a function from $V \times E$ to $\{0, 1\}$ such that for each $e \in E$, there are exactly 2 $v \in V$, for which $i(v, e) = 1$.

We provide some basic terminologies of graph theory before we start:

- e and v are **incident** in G if $i(v, e) = 1$
- v_1 and v_2 are **adjcant** in G if $\exists e$ incident to both v_1 and v_2
- G is **simple** if for each pair v_1, v_2, at most one pair is incident to both (no multi egdes)

Definition 1.2: Walk

A **walk** of a graph G is an alternating sequence

$$v_0, e_1, v_1, \ldots, e_k, v_k$$

where $v_0, v_1, \ldots, v_k$ are vertices of G (not necessarily disjoint), and $e_1, \ldots, e_k$ are edges of G such that each e_i is an edge from v_{i-1} to v_i.

Definition 1.3: Path

A **path** is a walk where $v_0, \ldots, v_k$ are distinct vertices (and therefore edges are distinct). The length of a path is defined as the number of edges.

Note that vertex disjoint implies edge disjoint, but not vice versa.

Definition 1.4: Connectivity

Two vertices v_1, v_2 of G are connected if there exists a **walk**.

2 Planarity

2.1 Terminology

Let G be a graph with vertices V, edges E, an embedding of G in $\mathbb{R}^2$ is a function φ such that:

- For each vertex v of G, φ is a point in $\mathbb{R}^2$, and no 2 vertices are mapped to the same point by φ (injective).
- For each edge e with ends u, v, $\varphi(e)$ is a curve from $\varphi(u)$ to $\varphi(v)$.
- For distinct edges e, f of G, the images of $\varphi(e)$ and $\varphi(f)$ are disjoint (as subsets of $\mathbb{R}^2$) except where e and f intersect at a vertex.
- For all $u \in V, e \in E$, u is an $\varphi(e)$ iff u is an end of e.

Definition 2.1: Planar graphs

A graph is **planar** if it has an embedding in $\mathbb{R}^2$, otherwise it's non-planar.
If φ is an embedding of G in $\mathbb{R}^2$, then $\varphi(G)$ for the union of the images of vertices and edges, are subsets of $\mathbb{R}^2$.

We can simplify this definition by the following propositions:

Proposition 2.1

If u is an open set, then x, y are connected in u iff they are polygonally connected in u.

Corollary 2.2

Given $u \subseteq \mathbb{R}^2$ open, we have

$$x, y \text{connected in } u \Leftrightarrow x, y \text{ polygonally connected in } u$$

Corollary 2.3

If G has a planar embedding φ, then it has a planar embedding where all arcs are polygonal.

Now we have the Polygonal Jordan Curve Theorem

Theorem 2.4: PJCT

If C is a polygon, then $\mathbb{R}^2 \backslash C$ has exactly two regions.

2.2 Euler's Formula

We study the Euler's Formula in this section.

Theorem 2.5: Euler's Formula

If φ is an embedding of $G = (V, E)$ in the plane, and F is the set of faces of φ, then $|V| - |E| + |F| = 1 + c$, where c is the number of components of G.

Lemma 2.6

If φ is an embedding of a graph G that contains a cycle, then the bounding of every face of G contains a cycle.

Proof: Exercise

Lemma 2.7

Each edge in a planer embedding is in ≤ 2 faces boundaries.

Proposition 2.8

If G is a simple planar graph on ≥ 3 vertices, then

$$|E(G)| \geq 3|V(G)| - 6$$

Proof:

We combine Euler's formula with an inequality relating the # edges and # faces in an embedding.

Let $V = V(G), E = E(G)$, let F be the set of faces in some planar embedding of G, and $c = \#$ components of G.

If G is a forest, then $|E| \leq |V| - 1 \leq 3|V| - 6$. Otherwise, every face boundary contains a cycle, so has ≥ 3 edges. Let $A = \{(e, f) : f \in F$, and e is the boundary of F$\}$. Since each e is in the boundary of ≤ 2 faces, we know $|A| \leq 2|E|$. Since each $f \in F$ has ≥ 3 edges in its boundary, we know $|A| \geq 3|F|$. So $3|F| \leq 2|E|$, that is, $|F| \leq \frac{2}{3}|E|$. By Euler's

Formula,

$$1 + c = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E|$$

so $|E| \leq 3(|V| - 1 - c) \leq 3|V| - 6$ (since $c \geq 1$). □

By applying this proposition we can show that K_5 is non-planar. Note that

$$|E| \leq \binom{|V|}{2}$$

which is bounded by quadratic formula of vertices, while simple planar graphs are bounded by linear formula of vertices.

2.3 2-connected planar graphs

We want to investigate under what conditions will every face boundary is a face (instead of containing a cycle). It turns out that every 2-connected graph satisfies this property:

Proposition 2.9

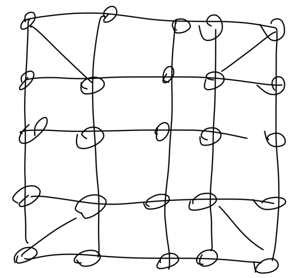
If φ is an embedding of a 2-connected graph G, then every face bounding of G is a cycle.

Proof:

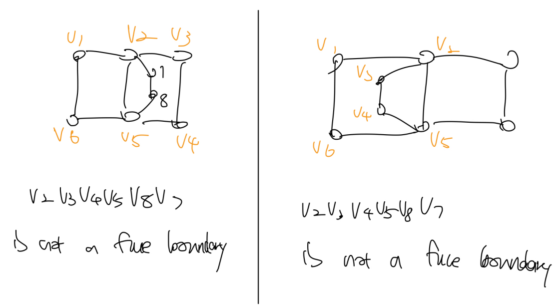
Induction with ear-decompostion: adding a path splits one face into 2 faces bounded by cycles and doesn't change any other face boundary. □

2.4 3-connected planar graphs

Given a graph G that is known to be planar, can we determine which cycles appear as face boundary in an embedding of G, without knowing the embedding?



(We can tell when embedding is given)



(But we can't tell for general graphs without embedding)

The problem is **lack of 3-connectedness**.
A cycle C of G is **non-separating** if $G - V(C)$ is connected. C is **induced** in G if there is no edge of $G \backslash E(C)$ with both ends in C (there is no chord of C).

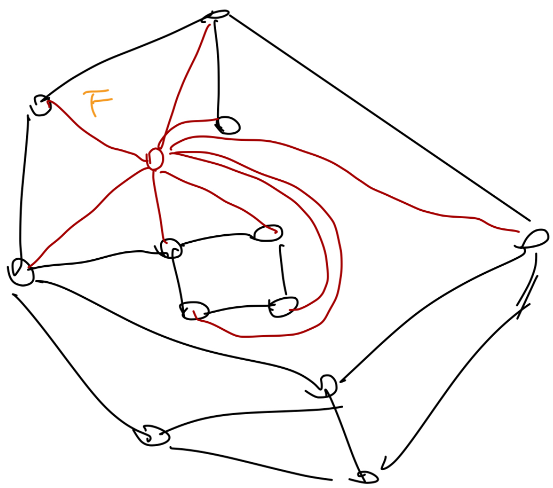
Proposition 2.10

If φ is an embedding of a 3-connected graph G, then C is a face (facial cycle) boundary iff C is non-seperating and induced by G.

We won't prove this, but give a useful lemma for the proof:

Lemma 2.11

Let G be a planar graph, and F be a face bounding in some embedding of G. Let G' be the graph obtained from G by adding a vertex v, and joining v to each vertex of F. Then $varphi$ extends to an embedding of G'.



A graphical illustration

There are more facts about 3-connected planar graphs:

1. They have a unique embedding in the plane/sphere (up to homomorphism)
2. They have a embedding in the plane with all edges are straight line segment <u>and</u> all faces are convex polygons.
3. They are exactly the stretch of polygons.

2.5 Kuratowski's Theorem

In this section we will study Kuratowski's Theorem, there are 2 versions of Kuratowski's Theorem, one in terms of **minor** and one in terms of **topological minor**. Minor is stronger than topological minor: every topological minor is a minor, but not vice versa.

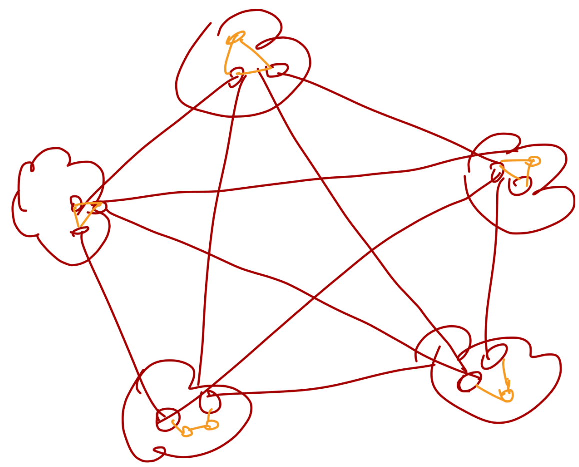
Definition 2.2: Minor

A graph H is a **minor** of a graph G if H can be botained from a subgraph G' of G by a sequence of edge-contractions.

A few notes:

- H is a minor of G iff H is obtained from G by vertex deletion, edge deletion, and edge-contractions.
- H is a minor of G iff there is a "model" of H in G
  - vertices of H: disjoint connected subgraphs of G.

– edges of H: edges of G between subgraphs

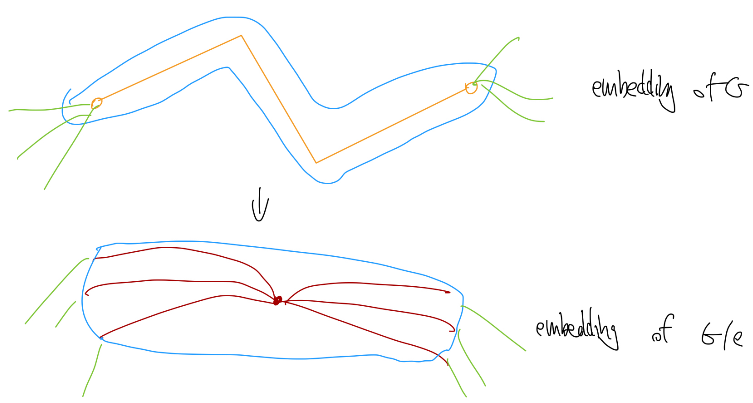


Proposition 2.12

If G is planar, and $H \leq G$, then H is planar.

Proof:

Since subgraphs of planar graphs are planar, it is enough to show that contracting a single edge in a planar graph keeps it planar.



(How to contract edges)

□

The forward direction of Kuratowski's Theorem is obvious:

Corollary 2.13

If G has $K_{3,3}$ or K_5 as a **minor**, G is non-planar.

To prove the backward direction, we need the following proposition

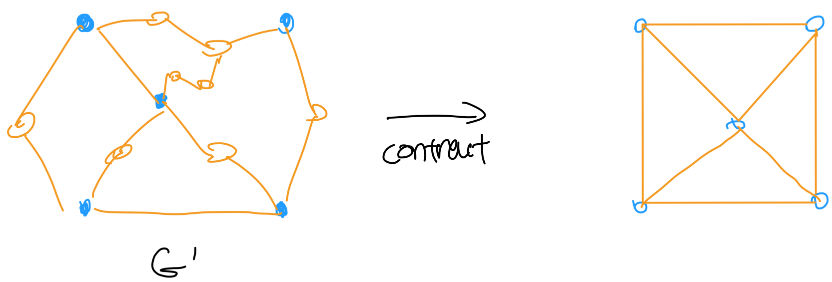
Proposition 2.14

If G has H as a topological minor, then G has H as minor.

Proof:

For each edge e of H, let P_e be the corresponding path of G. Let G' be the subgraph of G that is the union of all P_e.

Now H is obtained from G' by contracting all but one edge in each path P_e:



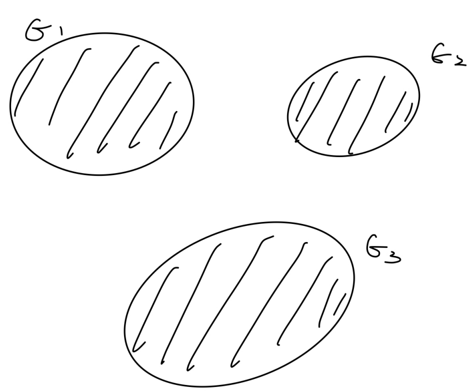
□

Now we give the proof

Theorem 2.15: Kuratowski' Thoerem

If G has no $K_{3,3}$-minor or K_5-minor, then G is planar.

The idea of proof is by contradiction to pick a minimal counter-example, with the observation that the example must have one component by minimality.

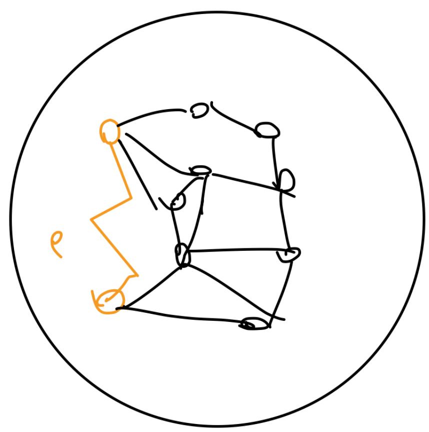


Proof:

Suppose for contradiction that G has no $K_{3,3}$ or K_5 minor, but is nonplanar. Choose G to have as few edges as possible, that is, $|V(G)| + |E(G)|$ is as small as possible.

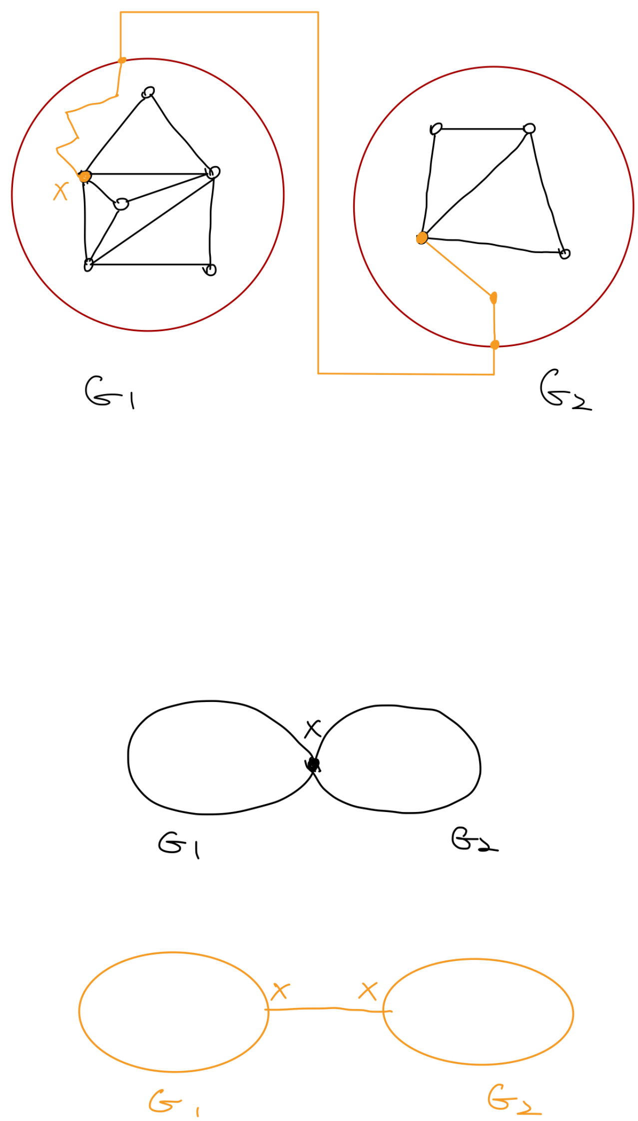
Claim 1: G is connected.
Suppose not, let $G_1, \ldots, G_k$ be its components. Since there are ≥ 2 components, we have $|V(G_i)| + |E(G_i)| < |V(G)| + |E(G)|$, so none of the G_i has $K_{3,3}$ or K_5 as a minor. Therefore, by the minimality in the choice of G, all G_i are planar. Then we can combine planar embedding of G_i to make a planar embedding of G, giving a contradiction.

To continue, we use (but not prove) the following:
Claim 2: For any embedding φ of G and any edge e (or vertex v) of G, and any disc $D \subseteq \mathbb{R}^2$, there is an embedding φ' of G such that $\varphi'(G) \subseteq D$, and e (or v) is contaiend in the boundary of the unbounded face (outside face) of φ'.



We want to show that every graph with no $K_{3,3}$-minor or K_5-minor is planar. We chose G to be a minimal counterexample. We already proved that G is connected.

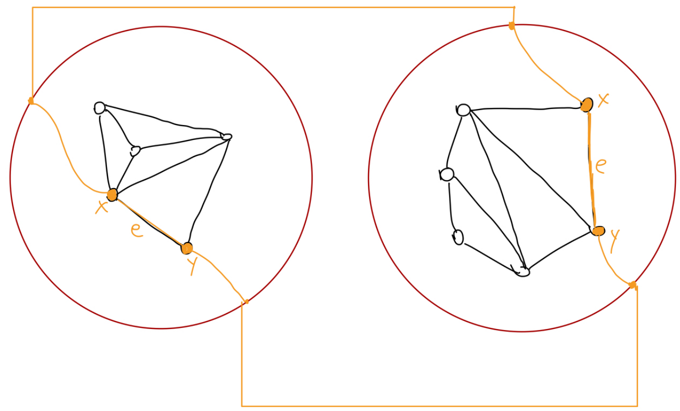
Claim 3: G is 2-connected.
If not, then G has a cut vertex x. Let G_1, G_2 be a proper subgraphs of G such that $G = G_1 \cup G_2$, and $V(G_1) \cap V(G_2) = \{x\}$. Since both G_i are smaller than G, and have no $K_{3,3}$ or K_5 minor, both are planar. Consider embeddings of G_1, G_2 in disjoint discs D_1, D_2 in the plane where x is embedded by both in the boundary of the outer face (true by Claim 2). We can find an arc e between the two copies of x in the resulting drawing to get an embedding of a graph G' such that $G'/e \cong G$ for some edge. Since G' is planar, and $G \cong G'/e$, G is also planar, a contradiction.



A graphical illustration

Claim 4: G is 3-connected.

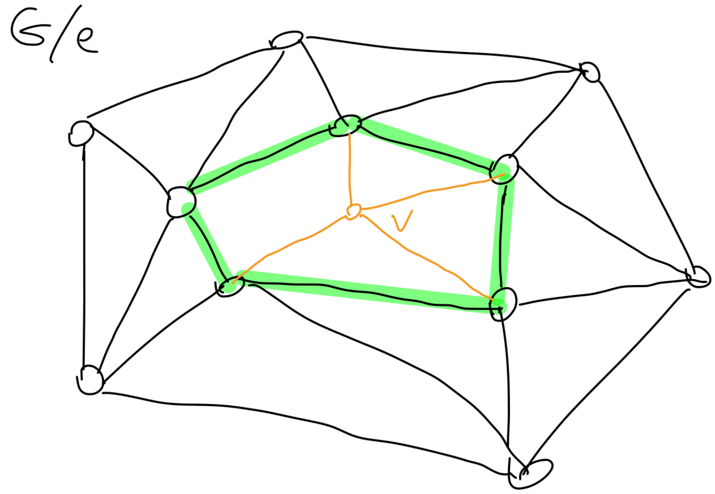
Suppose not, then there are vertices x, y of G and subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{x, y\}$. By a similar argument to the previous claim, G_1, G_2 are planar.



Let G_1' and G_2' be obtained from G_1, G_2 respectively by adding a new edge f from x to y (choose a vertex w of $G_2 - \{x, y\}$ and take a $(w, \{x, y\})$-fan to get this path, disclaimer: applying fam lemma to G, not G_2 because G_2 is not necessarily 2-connected). Since G is 2-connected, there is a path P with ≥ 2 edges in G_2 from x to y, now G' is obtained from the subgraph $G_1 \cup P$ by contracting all but one edge of P, since $K_{3,3}, K_5 \not\leq G$, and G_1' is a minor of G with fewer vertices with $K_{3,3}, K_5 \not\leq G$, so G_1' is planar. Similarly, G_2' is planar.
Now consider drawings of G_1', G_2' in disjoint discs in $\mathbb{R}^2$, where e is on the outer face. We can now combine these drawing and use connectedness of the unbounded face to obtain a planar drawing of the following graph G'. So G' is planar, so $G = G' \backslash \{e_1, e_2\}/\{f_1, f_2\}$ is planar (by contracting), a contradiction.

Now we have $K_{3,3}, K_5 \not\leq G$, G nonplanar, 3-connected. For every $e \in E(m)$, G/e and $G \backslash e$ are planar (because they are minor of G, so have no $K_{3,3}, K_5$-minor, and they are smaller than G, so are not counterexample).

Claim 5: G is simple.
If not, delete an edge e parallel to some other edge, draw $G \backslash e$, and add e back to the drawing.

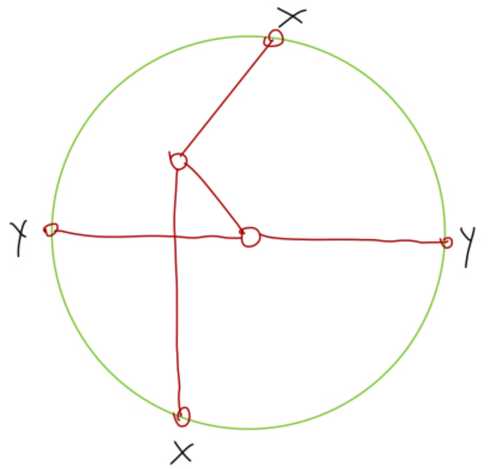


Also note that, since G nonplanar, $|V(G)| \geq 4$. By lemma from (much) earlier, G has an edge $e = xy$ such that G/e is 3-connected. We also know that G/e is planar. Let u be the vertex of G/e corresponding to e, and consider a planar embedding φ of G/e.
$(G/e) - v$ is a 2-connected planar graph, so every face bounding is a cycle, so every face bounding is a cycle, now v is embedded in some face of $(G/e) - v$ where bondary is a cycle C, and all nbrs of v in G/e lie in C.
We need a lemma

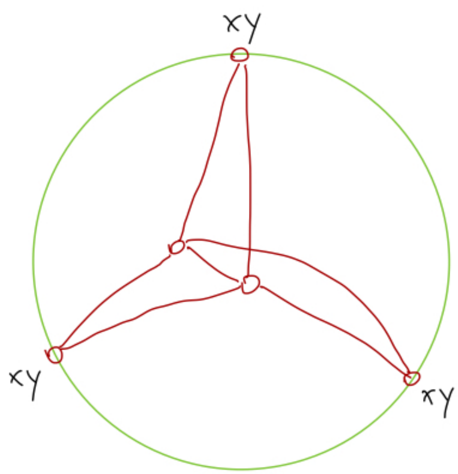
Lemma 2.16

Given sets X, Y of vertices in a cycle C, either

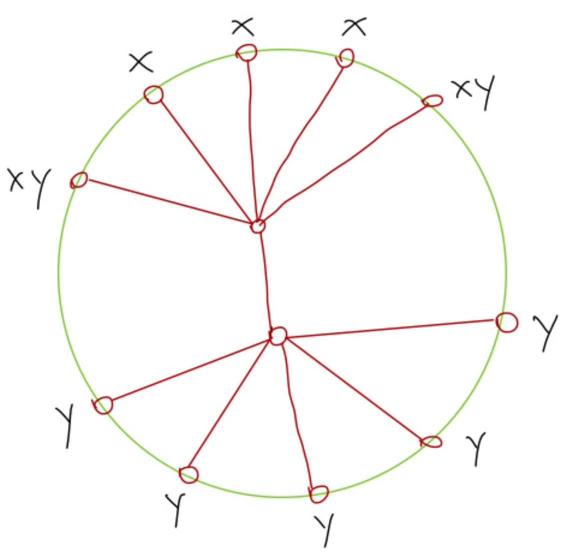
1. there exists $x, x' \in X$, $y, y' \in Y$ such that y, y' are in different components of $C - \{x, x'\}$.
2. there are paths P_x, P_y of C such that $E(P_x) \cap E(P_y) = \emptyset$, $P_x \cup P_y = C$, and $X \subseteq V(P_x), Y \subseteq V(P_y)$.
3. $|X \cap Y| \geq 3$



$\Rightarrow$ G has $K_{3,3}$-minor



$\Rightarrow$ G has K_5-minor



$\Rightarrow$ G planar

(There are only 3 possible cases)

Proof:

We may assume by symmetry that $|X| \leq |Y|$.
If $|X| \leq 1$, choose P_x to be a path with one edge containing all vertices in X, and choose P_y to be $C - p_x$, then (2) holds.
So $|X| \geq 2$. If $Y \backslash X = \emptyset$, then $X = Y$, suppose this holds, if $|X| = |Y| = 2$, then let $\{a, b\} = X = Y$, then choose P_x and P_y to be the two distinct ab-path in C. Now (2) holds.
Otherwise $|X| = |Y| \geq 3$, so $|X \cap Y| = |X| \geq 3$, so (3) holds.
So we may assume there exists $b \in Y \backslash X$. Since $b \notin X$, C is 2-connected, and $|X| \geq 2$, there is a bX-fan in C of size 2. Let P_1, P_2 be the paths in this fan. Let $P_y = P_1 \cup P_2$, since P_1, P_2 form a fan, P_y has no internal vertices in X. Let P_x be the other path in C between the ends of P_y. Since P_y has no internal vertices in X, we know that $X \subseteq V(P_x)$. If $Y \subseteq V(P_y)$, then (2) holds. Otherwise, there is some $b' \in Y$ in a different component of $C-$ {ends of P_x}, so (1) holds. □

(Back to Kuratowski) Let $X = \{\text{nbrs of } x \text{ in } C\}$, and $Y = \{\text{nbrs of } y \text{ in } C\}$. We now apply the lemma to X, Y and C.
If (3) holds, then x, y have 3 common neighbors a, b, c in C (in G). Now the vertices a, b, c, x, y are the terminals of a topological K_5-minor of G. Therefore G has a K_5-minor, a contradiction.
If (1) holds, then there exists a, b, a', b' (in that order) around C such that a, a' are neighbors of x, and b, b' are neighbors of Y. Now x, y, a, b, a', b' are the terminals of a topological $K_{3,3}$-minor of G, so G has $K_{3,3}$ as a minor ,a contradiction.
If (2) holds, we use the fact that for any polygon $C \subseteq \mathbb{R}^2$ with vertices in cyclic order $a_1, \ldots, a_t$, and any x in the interior of C, we can find arcs $A_1, A_2, \ldots, A_t$ from x to the a_i, intersecting only at x, and leaving x in the same cyclic order as the a_i occur around C (the proof is by inductivly draw the arcs one by one). Using the lemma, construct a panar embedding of G as follows:

1. take the drawing pf $G/e - u$ we were considering
2. add u back, and use the lemma to construct arcs from u to all vertices in $X \cup Y$.
3. let D be a small disc centred at u within D, split u into two vertices x, y, and use straight line segments to alter the drawing of G/e to a drawing of G. This contradicts the nonplanarity of G.

□

The quickest way to say Kuratowski thoerem is: $K_{3,3}$ and K_5 are the **excluded minors** for planarity $\Leftrightarrow$ $K_{3,3}$ and K_5 are the unique minor-minimal nonplanar graphs.

There are other interesting theorems

Theorem 2.17

G is toroidal iff G does not have S (finite set of graphs, ≥ 16000) as a minor

Theorem 2.18

G is linkless-embeddable in $\mathbb{R}^3$ iff G does not contain S (set of graphs) as a minor.

The topological Kuratowski is equivalent to the Kuratowski theorem:

Proposition 2.19

For a graph G, the following are equivalent:

1. $K_{3,3}$ or K_5 is a minor of G
2. $K_{3,3}$ or K_5 is a topological minor of G

This follows from 3 statements:

1. For all H, if H is a topological minor of G, then H is a minor of G (proved already)
2. For all H of max degree ≤ 3 (e.g. $K_3, 3$), if H is a minor of G, then H is a topological minor of G. (A3Q5)
3. If G has K_5 as a minor, it has K_5 or $K_{3,3}$ as a topological minor. (A3Q5)

Also, one can adapt one proof of Kuratowski to show that every planar graph can be drawn with all edges as straight line segments.

3 Matchings

3.1 Terminology

Recall that

Definition 3.1: Matchings

A matching in a graph G is a set M of edges of G so that no 2 edges share an end.

In planarity section, we didn't state graph has to be simple because having multigraph can make our life easier, but it is not the case for matchings. From now on, we will assume that all our graphs are simple in this chapter.
We also talked about covers in MATH 239:

Definition 3.2: Covers

A cover of G is a set $W \subseteq V(G)$ so that every edge of G has an end in W.

Observe that if M is a matching of G, and W is a cover of G, then

$$|M| \leq |W|$$

because each edge in M has an end in W, and no two have a common end. This observation leads to following corollaries:

Corollary 3.1

If M is a matching and w is a cover such that

$$|M| = |W|$$

then W contains exactly one end of each edge in M, and no other vertices

Corollary 3.2

If $\nu(G)$ is the size of maximum matching of G, and $\tau(G)$ is the size of a minimum cover of G, then $\nu(G) \leq \tau(G)$.

3.2 König's Theorem

The famous König's theorem on bibartitle graph states that $\nu(G) = \tau(G)$. The above corollary shows that one direction holds for arbitrary G. The proof we showed for König' theorem is a bit sneaky, before that, we give some useful lemmas.

Proposition 3.3

If C is an even cycle on n vertices, then $\nu(C) = \tau(C) = n$

(insert figure here) Idea of proof: {every other edge} and {every other vertices} are a matching and a cover respectively, each of size n.

Proposition 3.4

If P is a path on n vertices, then

$$\nu(P) = \tau(P) = \lfloor \frac{n}{2} \rfloor$$

Proof:

If P has vertices $v_1, \ldots, v_n$, then

- $\{v_2, v_4, \ldots, v_{2\lfloor \frac{n}{2} \rfloor}\}$ is a cover
- $\{v_1v_2, v_3v_4, \ldots\}$ is a matching

both with size $\lfloor \frac{n}{2} \rfloor$. □

However, note that statement fails for odd cycles because $\tau(C_{2n+1}) = n+1$, $\nu(C_{2n+1}) = n$.

Proposition 3.5

If $G_1, \ldots, G_k$ are the components of G, then

$$\nu(G) = \sum_{i=1}^{k} \nu(G_i)$$

and

$$\tau(G) = \sum_{i=1}^{k} \tau(G_i)$$

Proof : Easy.

Now we can prove the König's theorem:

Theorem 3.6: König's Theorem

If G is bipartite, then $\nu(G) = \tau(G)$.

That is, there is a matching M and a cover W such that $|M| = |W|$.

Proof:

We need to show that $\tau(G) \leq \nu(G)$ for bipartite G. Let G be a counterexample on as few edges as possible.

Claim: G has a vertex of degree ≥ 3.
If not, then every component is a path or a cycle, so König's thoeorem holds for each component, so it holds for G, since τ and ν are additive over components.

Let u be a vertex of degree ≥ 3, and v be a neighbor of u. We split into cases depending on whether $\nu(G-v) = \nu(G)$.
If $\nu(G-v) \leq \nu(G) - 1$, then let W_0 be a min vertex cover of $G-v$, since $G-v$ is not a counterexample, we know that

$$|W_0| = \nu(G-v) \leq \nu(G) - 1$$

Since W_0 is a cover of $G-v$, $W_0 \cup \{v\}$ is a cover of G, so

$$\tau(G) \leq |W_0 \cup \{v\}| \leq (\nu(G) - 1) + 1 = \nu(G)$$

This contradicts that G being a counterexample.
Otherwise, $\nu(G-v) = \nu(G)$. In other words, each maximum matching of $G-v$ is also a maximum matching of G. Let M be a maximum matching of both $G-v$ and G. Since $\deg(u) \geq 3$, there is an edge f incident with u but not v, such that $f \notin M$. Note that $\nu(G-f) \geq |M| = \nu(G)$. Inductively, $G-f$ has a min cover W with

$$|W| = \nu(G-f) = \nu(G) = |M|$$

so W contains an end of each edge in M, and nothing else. In particular, $u \in W$, since W is a cover of $G-f$, we have $v \in W_0$. Therefore, W covers f, so W is a cover of G of size $|M| = \nu(G)$, as required. □