# MATH 237 Workbook
Version 1.2



August 18, 2021

# Contents

# Credits and Acknowledgement

The Mobius courseware for MATH 237 was developed by Burcu Karabina and Amanda Garcia, with instructional design and multimedia development support provided by the Centre for Extended Learning. The course material is based on a set of course notes written by John Wainwright, Joe West, and Dan Wolczuk. Special thanks to Milad Farsi and to Chuanzheng Wang for help writing the end of unit solutions.

# About This Workbook

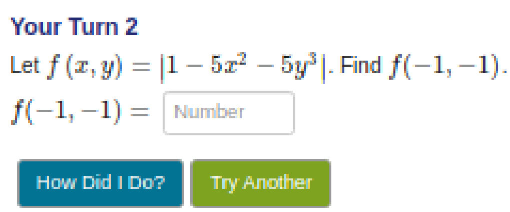
This workbook is intended as a companion to the Mobius courseware for MATH 237. This workbook is not intended to replace the online courseware. It is expected that you are working through the material online and taking notes in the workbook as you go.

The structure of this workbook, into units and lessons, follows the structure in Mobius. One exception is that the workbook does not include Unit 0. We recommend you complete Unit 0 in Mobius. You can easily navigate to units and lessons by clicking the appropriate link in the workbook's Table of Contents.

The courseware in Mobius contains several interactive elements. Here is how they will appear in your workbook:

## Your Turn Questions, Worked Examples, and Application questions

In Mobius, Your Turn questions, Worked Examples, and Application questions are opportunities for you to practice what you have just learned. These questions come in many formats including multiple choice and numerical entry.




A Your Turn question in Mobius

In this workbook, Your Turn questions will appear as rounded boxes with black borders and the heading *"A question appears in Mobius"*:


A Your Turn question in the workbook

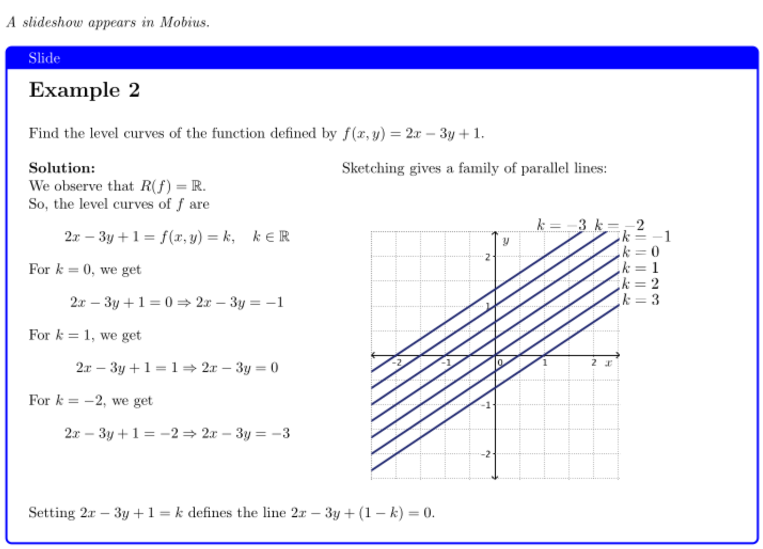
You can use the space included to write down the corresponding question from Mobius and your solution. We recommend that you practice entering your final answer into Mobius so that you familiarize yourself with Mobius syntax.

## Slideshows

In Mobius, slideshows are typically narrated videos that review examples or key concepts.


A slide in Mobius

In your workbook, each slide of a slideshow will appear inside a rounded box with blue borders. The beginning of the slideshow is indicated by the words *"A slideshow appears in Mobius"*.

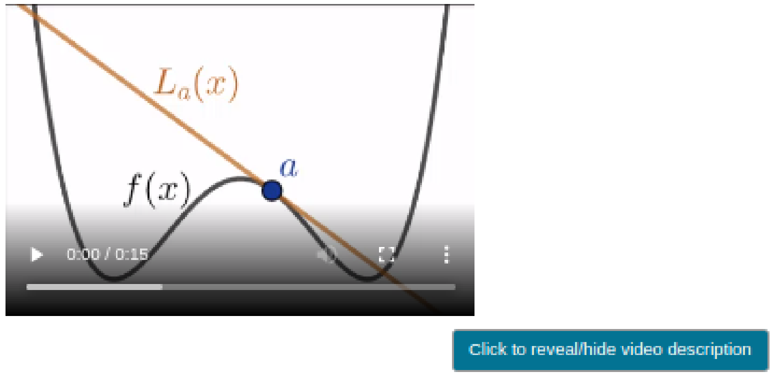



A slide in the workbook

Your workbook shows the final build of each slide. We recommend that you play the slideshow in Mobius and follow along, adding notes on the slides in your workbook.

## Videos

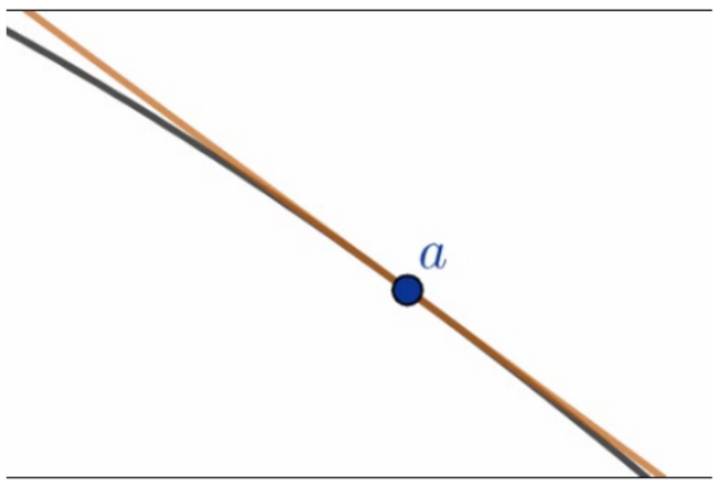
In Mobius, videos may appear within slideshows and as standalone objects (usually without audio).




A standalone video in Mobius

In this workbook, you will see the message *"A video appears here"* followed by the **last frame** of the video.

*A video appears here.*



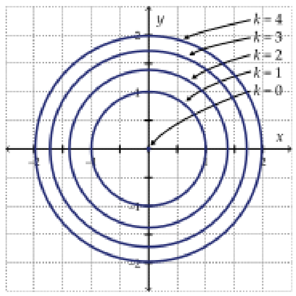

A standalone video in the workbook

In the special cases where a video is embedded in a slide, you will also see the text *"A video appears here"* along with the **last frame** of the video in that slide

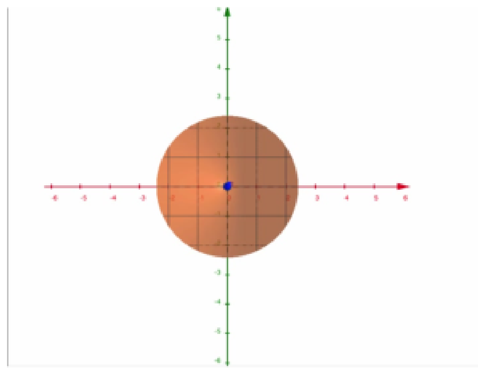
Slide

## Example 3 Continued

Sketch the level curves of $f(x, y) = x^2 + y^2$ and use them to sketch the surface $z = f(x, y)$.
**Solution:**




*A video appears here.*



We get this surface, which is called a **paraboloid**.

A video in a slide in the workbook

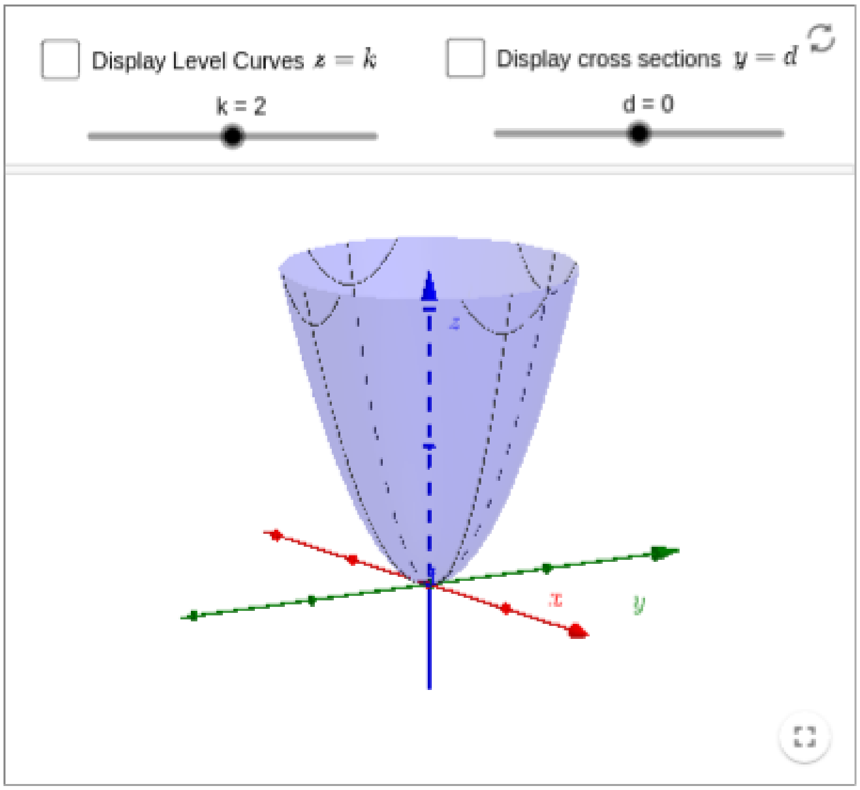
We recommend that you watch the video and write down your notes and observations in your workbook.

# GeoGebra Applets

In Mobius, there are several interactive GeoGebra applets that bring course concepts to life. Each applet is preceded by a set of instructions that guide you through the exploration.

**Instructions**

1. Check one or both of the boxes to display the level curves and/or the cross sections.
2. Drag the $k$ slider to see the level curves for different values of $k$.
3. Drag the $d$ slider to see the cross sections for different values of $d$.
4. To view the graph from different angles, click and hold on the image and then move your cursor to rotate the figure.
5. Click ↻ to reset to the original configuration.






A GeoGebra applet in Mobius

In this workbook, the instructions for the GeoGebra applet are printed along with a link to the GeoGebra resource.

**Instructions**

1. Enter a function using GeoGebra's `http://wiki.geogebra.org/en/Predefined_Functions_and_Operators`.
2. Click to start the animation. Click to pause the animation.
3. To view the graph from different angles, click and hold on the image and then move your cursor to rotate the figure.
4. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/wr6pgbx2/

Created with `https://www.geogebra.org`. 

A GeoGebra applet in the workbook

From the pdf, you can click on the link or type the link into your browser to visit the resource. We recommend that you take the time to explore each GeoGebra applet and take notes in your workbook.

# Additional Content

In Mobius, you must sometimes click a button to reveal some hints, proofs, and other content. This content **does not** appear in your workbook. In your workbook, you will see the message *"Additional content appears in Mobius"*.

# Unit 1

# Graphs of Scalar Functions

## 1.1 - Scalar Functions

### Scalar Functions

One of the most important concepts in mathematics is that of a function. Before we begin the study of scalar functions, let's review the basic vocabulary about functions in general.

- A function $f : A \to B$ associates with each element $a \in A$ a unique element $f(a) \in B$ called the **image** of $a$ under $f$.
- The set $A$ is called the **domain** of $f$ and is denoted by $D(f)$.
- The set $B$ is called the **codomain** of $f$.
- The subset of $B$ consisting of all $f(a)$ is called the **range** of $f$ and is denoted $R(f)$.

We will first extend what we did in single variable calculus to functions of several variables. We will usually look at real functions of two variables whose domain is a subset of $\mathbb{R}^2$ and whose codomain is $\mathbb{R}$. That is, we consider functions $f$ which map points $(x, y) \in \mathbb{R}^2$ to a real scalar $f(x, y) \in \mathbb{R}$. We write $z = f(x, y)$. We will also consider more general functions $f(x_1, \ldots, x_n)$ which map subsets of $\mathbb{R}^n$ to $\mathbb{R}$.

**Remark**
Although strictly speaking, $f(x, y)$ denotes the value of the function $f$ at the point $(x, y)$, it is common practice to use the phrase "the function $f(x, y)$" to stress which independent variables the function is dependent on.

**Definition: Scalar Function**
A **scalar function** $f(x_1, \ldots, x_n)$ of $n$ variables is a function whose domain is a subset of $\mathbb{R}^n$ and whose range is a subset of $\mathbb{R}$.

A question appears in Mobius

**Remark**
We will sometimes use $\vec{x}$ to represent a point in $\mathbb{R}^n$. Note that there will be several times in the course where it is convenient to view points in $\mathbb{R}^n$ as vectors in $\mathbb{R}^n$ to make use of results from linear algebra.

First, let's practise evaluating scalar functions at given points.

**Example 1**
Let $f$ be defined by $f(x, y) = 2x + 3y + 1$. Find $f(1, -4)$ and $f(1, 1)$.
**Solution:**
We have

$$f(1, -4) = 2(1) + 3(-4) + 1 = -9$$
$$f(1, 1) = 2(1) + 3(1) + 1 = 6$$

A question appears in Mobius

## Domain and Range

Next, we will study the domain and the range of scalar functions.

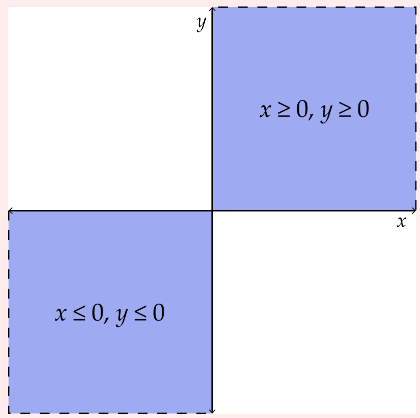
To get started, try answering the following questions.

A question appears in Mobius

A question appears in Mobius

Sometimes, it is easier to represent the domain of a scalar function with a picture.

**Example 2**
Find the domain and range of $f(x, y) = \sqrt{xy}$.
**Solution:**
We cannot take the square root of a negative number, so we require $xy \geq 0$. Thus, the domain is the set $x \geq 0$, $y \geq 0$ and $x \leq 0$, $y \leq 0$. Since this is a subset of $\mathbb{R}^2$, it is easy to represent with a picture.




For the range, we notice that $f(x, y) = \sqrt{xy} \geq 0$. To see that the range of $f$ contains all non-negative real numbers, observe that for any non-negative real number $c$ we have that $f(c^2, 1) = \sqrt{c^2} = |c| = c$.

Now, let's take a look at a more challenging example.

**Example 3**

Find the domain and range of $g(x,y) = \dfrac{x^2 - y^2}{|x| + |y|}$.

**Solution:**

Observe $g$ is undefined whenever $(x,y) = (0,0)$. So, the domain is $\mathbb{R}^2 - \{(0,0)\}$.

The range is a little more difficult to see. We need to determine all values we can get from $g$ by taking points in our domain.

First, let's try to see if $g$ can output all possible positive values. We do so by considering points $(c,0)$, $c \neq 0$. We get

$$g(c,0) = \frac{c^2 - 0^2}{|c| + |0|} = |c|$$

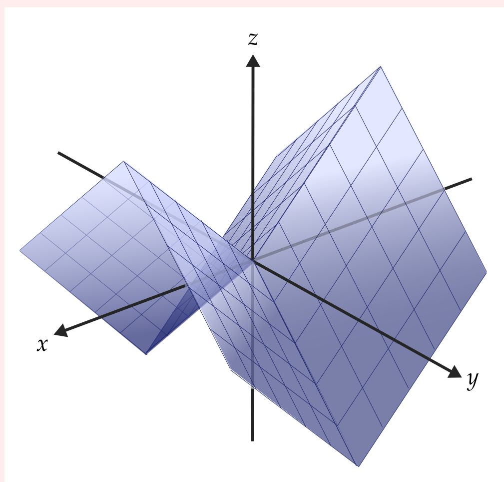
Hence, $g$ can take any positive value.

Similarly, we check whether $g$ can output all possible negative values using the points $(0,d)$, $d \neq 0$ which give

$$g(0,d) = \frac{0^2 - d^2}{|0| + |d|} = -|d|$$

Thus, $g$ can also take any negative value.

Finally, observe that $g(1,1) = 0$. Therefore, the range of $g$ is $\mathbb{R}$.




**Your Turn**

Sketch the domain and find the range of the following functions:

$f(x,y) = \ln(1 - x^2 - y^2)$

A question appears in Mobius

$g(x, y) = \sqrt{16 - x^2 + y^2}$

A question appears in Mobius

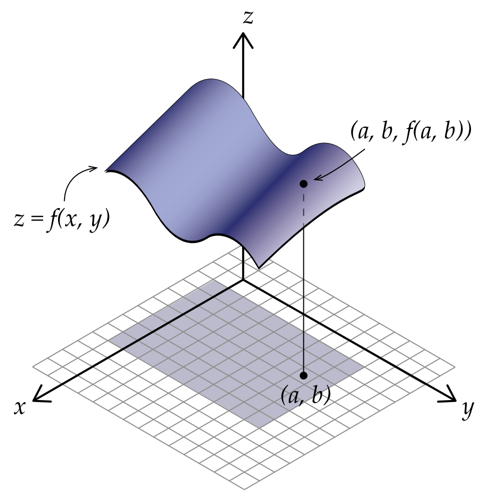
For more complicated functions, it could be extremely difficult to determine their range. When we had such situations with single variable functions we often found it helpful to sketch the graph of the function. In the next lesson, we will learn some techniques to help us sketch a graph of a function $f(x, y)$.

## 1.2 - Geometric Interpretation of $z = f(x, y)$

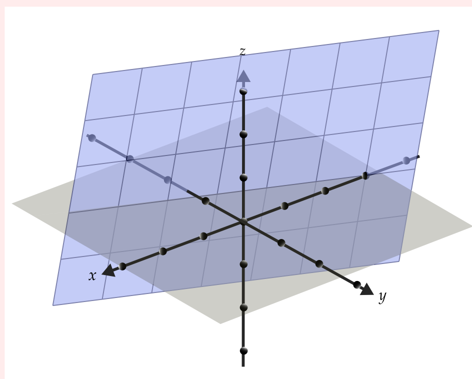
### Geometric Interpretation of $z = f(x, y)$

When we graph a function $y = f(x)$, we plot points $(a, f(a))$ in the $xy$-plane. Observe that we can think of $f(a)$ as representing the height of the graph $y = f(x)$ above (or below if negative) the $x$-axis at $x = a$.

We define the **graph** of a function $f(x, y)$ as the set of all points $(a, b, f(a, b))$ in $\mathbb{R}^3$ such that $(a, b) \in D(f)$. We think of $f(a, b)$ as representing the height of the graph $z = f(x, y)$ above (or below if negative) the $xy$-plane at the point $(x, y) = (a, b)$.




**Example 1**
Let $f$ be defined by $f(x, y) = x + 2y + 3$. We recognize this as the equation of a plane in $\mathbb{R}^3$.




In general, when $f$ is defined as $f(x, y) = c_1x + c_2y + c_3$, where $c_1, c_2, c_3$ are real constants, the graph of $z = f(x, y)$ is a plane.

Surfaces $z = f(x, y)$ can be quite complicated. To help us visualize and/or sketch these surfaces, we look at 2-dimensional slices of the surface that are called **level curves**.

## Level Curves

**Definition: Level Curves**
The **level curves** of a function $f(x, y)$ are the curves

$$f(x, y) = k$$

where the values of $k$ come from the range of $f$.

Let's see how to find the level curves of a function.

*A slideshow appears in Mobius.*

Slide

## Example 2

Find the level curves of the function defined by $f(x, y) = 2x - 3y + 1$.

**Solution:**
We observe that $R(f) = \mathbb{R}$.
So, the level curves of $f$ are

$$2x - 3y + 1 = f(x, y) = k, \quad k \in \mathbb{R}$$

For $k = 0$, we get

$$2x - 3y + 1 = 0 \Rightarrow 2x - 3y = -1$$

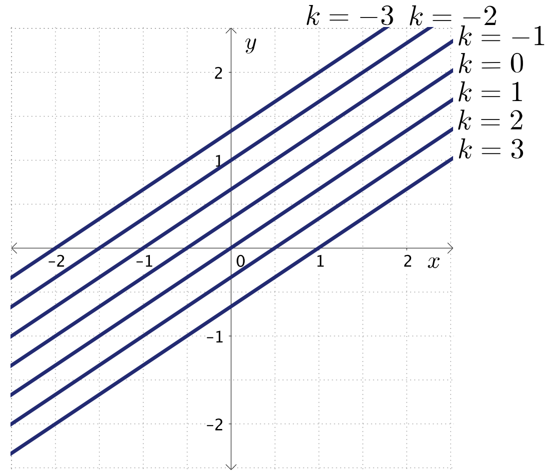
For $k = 1$, we get

$$2x - 3y + 1 = 1 \Rightarrow 2x - 3y = 0$$

For $k = -2$, we get

$$2x - 3y + 1 = -2 \Rightarrow 2x - 3y = -3$$

Sketching gives a family of parallel lines:

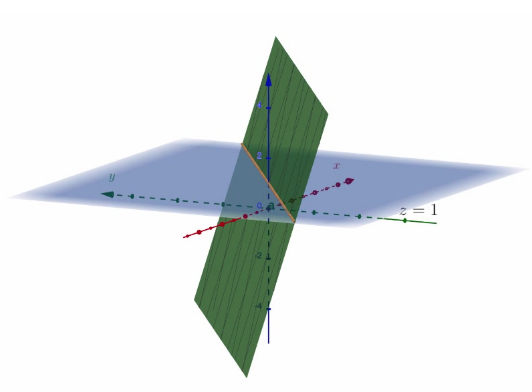


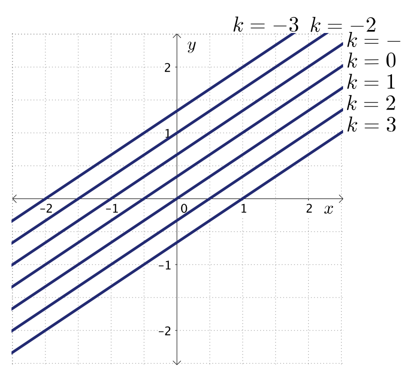

Setting $2x - 3y + 1 = k$ defines the line $2x - 3y + (1 - k) = 0$.

Slide

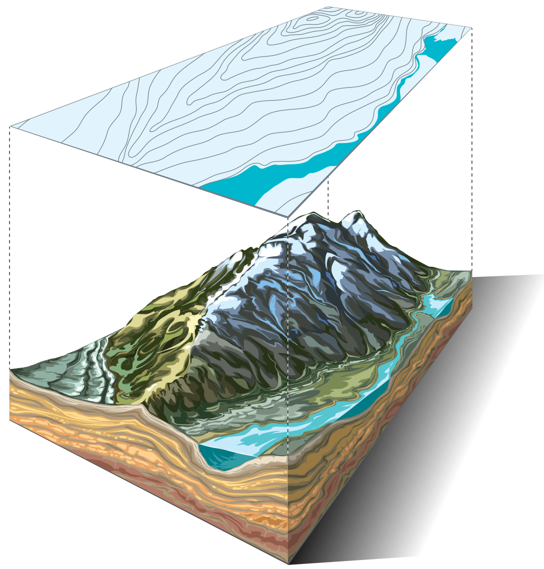
## 3D Visualization

*A video appears here.*




Observe that the level curve $f(x, y) = k$ is the intersection of $z = f(x, y)$ and the horizontal plane $z = k$. Thus, in our family of level curves, each value of $k$ represents the height of that level curve above the $xy$-plane. For this reason, the family of level curves is often called a **contour map** or a **topographic map**.



A topographic map of a landscape.
The contours, or level curves, are shown projected above the landscape.

lukaves/iStock/Getty Images

Now let's see how we can use level curves of $f(x, y)$ to sketch the surface $z = f(x, y)$.

*A slideshow appears in Mobius.*

Slide

# Example 3

Sketch the level curves of $f(x, y) = x^2 + y^2$ and use them to sketch the surface $z = f(x, y)$.

**Solution:**
We observe $R(f) = [0, \infty)$.
For $k = 0$, we get

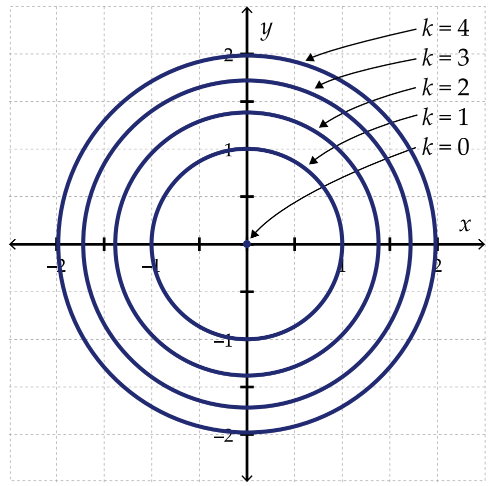
$$x^2 + y^2 = 0$$

For $k = 1$, we get

$$x^2 + y^2 = 1$$

For $k = 2$, we get

$$x^2 + y^2 = 2$$

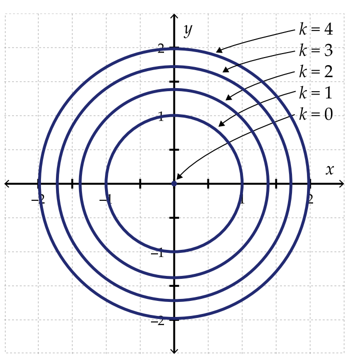



Setting $x^2 + y^2 = k$ defines a circle of radius $\sqrt{k}$ centred at the origin.

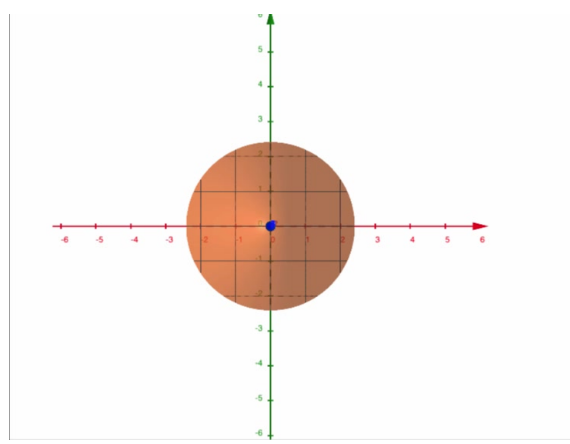
Slide

# Example 3 Continued

Sketch the level curves of $f(x, y) = x^2 + y^2$ and use them to sketch the surface $z = f(x, y)$.
**Solution:**




*A video appears here.*



We get this surface, which is called a **paraboloid**.

**Your Turn**

Use the following GeoGebra app to display the graph and the level curves of different scalar functions.
Here are some functions that you can try:

- $4x^2 + y^2 + 1$
- $\sqrt{25 - x^2 - y^2}$
- $\dfrac{-y}{x^2 + y^2}$
- $e^{x^2 - y^2}$

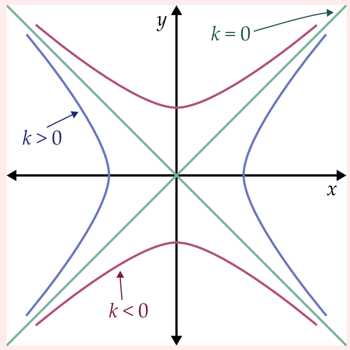
Observe that the level curves $f(x, y) = k$ are the intersection of $z = f(x, y)$ and the horizontal planes $z = k$. Viewing the graph from different angles can also help you better observe the level curves.

**Instructions**

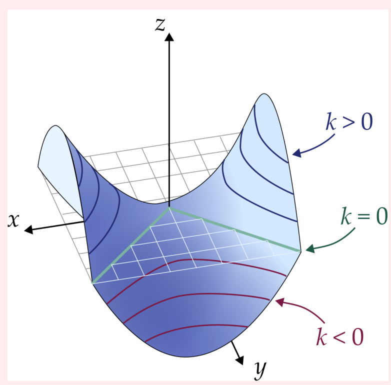
1. Enter a function using GeoGebra's `http://wiki.geogebra.org/en/Predefined_Functions_and_Operators`.
2. Click to start the animation. Click to pause the animation.
3. To view the graph from different angles, click and hold on the image and then move your cursor to rotate the figure.
4. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/wr6pgbx2/

**Example 4**
Sketch the level curves of $g(x, y) = x^2 - y^2$ and use them to sketch the surface $z = g(x, y)$.
**Solution:**
We first observe that $D(g) = \mathbb{R}^2$ and $R(g) = \mathbb{R}$. For any $k \in \mathbb{R}$ we sketch the level curves $x^2 - y^2 = k$ which we recognize as a family of hyperbolae and their asymptotes $y = \pm x$ corresponding to $x^2 - y^2 = 0$. Note that we get hyperbolae along the $x$-axis for $k > 0$ and hyperbolae along the $y$-axis for $k < 0$. For $k = 0$, we get the asymptotes $y = \pm x$.



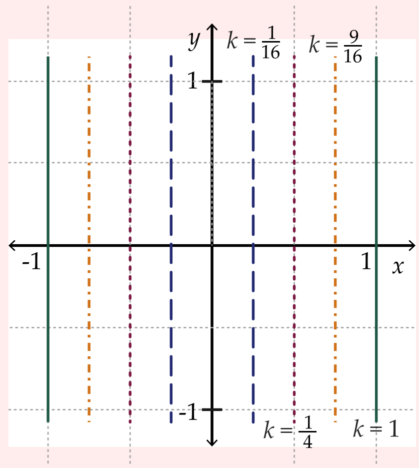

Using these to sketch the surface, we get a **saddle surface**. (A saddle surface is like the surface of a Pringles chip.)



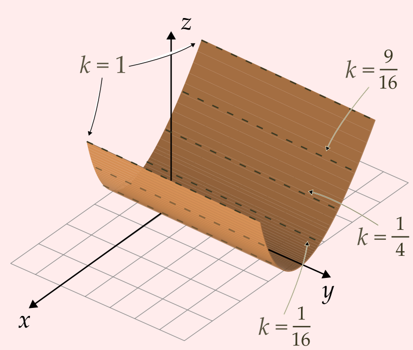
**Example 5**
Sketch the level curves of $h(x, y) = x^2$ and use them to sketch the surface $z = h(x, y)$.
**Solution:**
We have that $D(h) = \mathbb{R}^2$ and $R(h) = \{z \in \mathbb{R} \mid z \geq 0\}$. Thus, for $k \geq 0$ we have level curves

$$x^2 = k \Rightarrow x = \pm\sqrt{k}$$

Hence, the level curves are pairs of vertical straight lines.
For $k = 1$, we get $x = \pm 1$; for $k = 9/16$, we get $x = \pm 3/4$; for $k = 1/4$, we get $x = \pm 1/2$, and for $k = 1/16$, we get $x = \pm 1/4$.

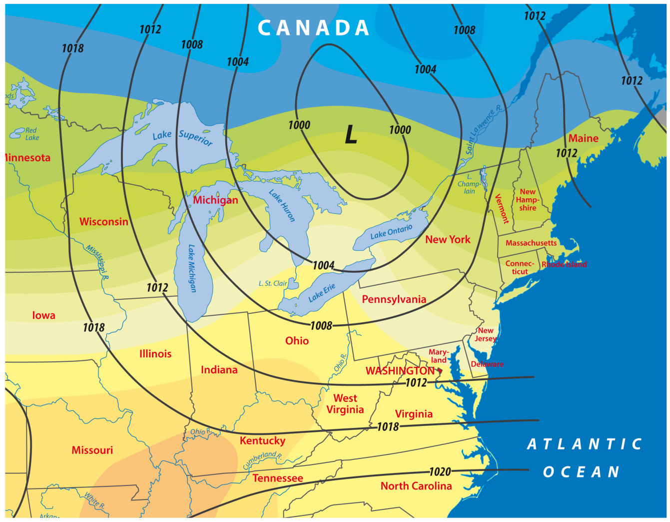



Using these to sketch the surface, we get a **parabolic cylinder**.




Level curves occur in everyday life. For example, the elevation of the earth's surface above sea level is described by an equation $z = h(x, y)$ where $x$ is the latitude and $y$ is the longitude of the position. A contour map shows the curves of constant elevation, $h(x, y) = k$, which are precisely the level curves of $h$.

Some other examples include use in weather maps to show curves of constant temperature called **isotherms**, in marine charts to indicate water depths, and in barometric pressure charts to show curves of constant pressure called **isobars**.




An isobar map of the northeast United States and southeast Canada.

Rainer Lesniewski/iStock/Getty Images

In general, it is not always possible to sketch the level curves of a given function $f(x, y)$ by inspection. Later in the course, we will develop some results which can be used to obtain information about the level curves of a function.

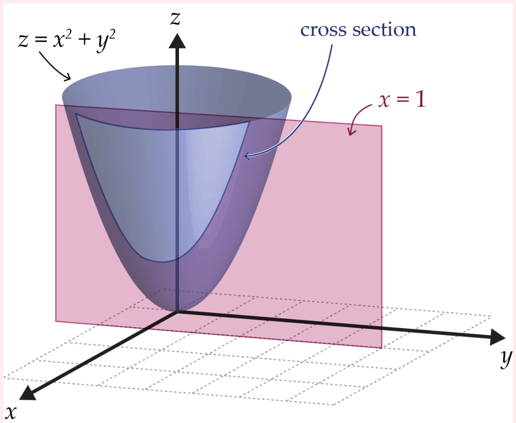
One can also obtain insight into the shape of a surface $z = f(x, y)$ by sketching the curves of intersection of the surface with vertical planes instead of horizontal planes, which we'll explore next.

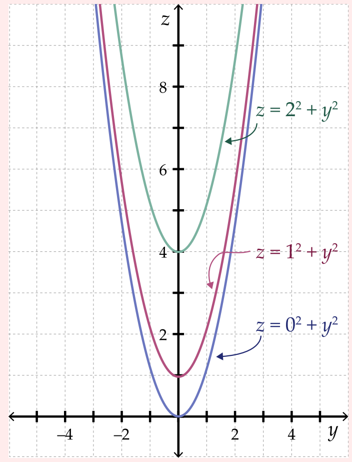
## Cross Sections

**Definition: Cross Sections**
A **cross section** of a surface $z = f(x, y)$ is the intersection of $z = f(x, y)$ with a vertical plane.

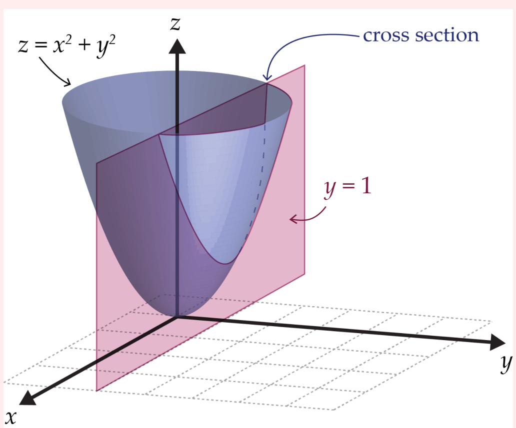
For the purpose of sketching the graph of a surface $z = f(x, y)$, it is useful to consider the cross sections formed by intersecting $z = f(x, y)$ with the vertical planes $x = c$ and $y = d$.

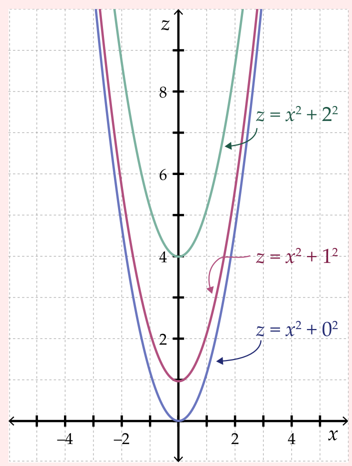
**Example 6**
Let $f(x, y) = x^2 + y^2$.
The cross sections formed by intersecting $z = f(x, y)$ with $x = c$ for $c = 0, 1, 2$, are:
$z = (0)^2 + y^2$, $z = (1)^2 + y^2$, and $z = (2)^2 + y^2$.




The cross sections formed by intersecting $z = f(x, y)$ with $y = d$ for $d = 0, 1, 2$ are:
$z = x^2 + (0)^2$, $z = x^2 + (1)^2$, and $z = x^2 + (2)^2$.




**Your Turn 1**

Use the following GeoGebra app to display both the level curves and the cross sections of the function $f(x, y) = x^2 + y^2$ from the previous example.
Observe and compare the shapes of the level curves and the cross sections.

**Instructions**

1. Check one or both of the boxes to display the level curves and/or the cross sections.
2. Drag the $k$ slider to see the level curves for different values of $k$.
3. Drag the $d$ slider to see the cross sections for different values of $d$.
4. To view the graph from different angles, click and hold on the image and then move your cursor to rotate the figure.

5. Click ↻ to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/tmrdhzed/



**Remark**
For simplicity, when you are asked to sketch the cross sections of a surface, we mean for you to sketch the family of cross sections $z = f(c, y)$ and $z = f(x, d)$ formed by the intersection of the surface with the vertical planes $x = c$ and $y = d$. As in the level curves, the values of $c$ and $d$ can take on multiple values.

### Your Turn 2

Sketch the cross sections of $g(x, y) = x^2 - y^2$.

A question appears in Mobius

Sketch the cross sections of $h(x, y) = x^2$.

A question appears in Mobius

### Your Turn 3

Sketch the level curves and cross sections of $f(x, y) = \sqrt{x^2 + y^2}$ and use them to sketch the surface $z = f(x, y)$.

A question appears in Mobius

## Generalization

When we work with functions of more than two variables, we can generalize the idea of level curves. Introducing an extra variable will also introduce a new dimension. Since we can't visualize 4D, we cannot provide any pictures in this section.

**Definition: Level Surfaces**
A **level surface** of a scalar function $f(x, y, z)$ is defined by

$$f(x, y, z) = k, \quad k \in R(f)$$

A question appears in Mobius

**Definition: Level Sets**
A **level set** of a scalar function $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$ is defined by

$$\{\vec{x} \in \mathbb{R}^n \mid f(\vec{x}) = k\}, \text{ for } k \in R(f)$$

**Example 7**
Let $f$ be defined by:

$$f(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$$

The level sets of $f(\vec{x}) = k$, in $\mathbb{R}^n (k > 0)$ are called $(n-1)$**-spheres**, denoted by $S^{n-1}$.

### Function Inventory

In this unit, we will often be working with some basic functions of two variables. When you are determining level curves or cross-sections, it may be useful to remember the behaviour of such functions. A non-exhaustive inventory is provided here.

| **Function** | **General Form** | **Level Curves** | **Cross-Sections** |
|---|---|---|---|
| Plane | $f(x, y) = ax + by + c$ | Parallel lines | Parallel lines |
| Parabolic cylinder | $f(x, y) = ax^2$ | Parallel lines | Vertical lines or parabolas |
| Elliptic paraboloid | $f(x, y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ where $a, b \in \mathbb{R}$ have the same sign | Circles or ellipses | Parabolas |
| Hyperbolic paraboloid | $f(x, y) = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ where $a, b \in \mathbb{R}$ have different signs | Hyperbolas | Parabolas |

## 1.3 - Putting It All Together

### Putting It All Together

In this lesson, we will put together everything we have learned so far to sketch the graph of a function in 3D. Examples in this section are slightly more challenging. We recommend that you study and understand the worked examples and the application problem very well before you move on to the practice problems.

When prompted, answer each question. Click 'How Did I Do' or 'Click to reveal/hide answer' to verify your answer and see any associated feedback. If you do not know how to answer the questions you should still click the button to reveal the solution. Study the solution before moving on to the next part.

Some multi-step questions have a 'Verify' button. Click this button to reveal the answer, feedback, and the next part of the question.

Let's start with a straightforward example.

### Worked Example 1

Answer the following questions for the function $f(x, y) = x^2 + y^2 - 4(x + y)$.

A question appears in Mobius

A question appears in Mobius

c. Sketch the level curves for $k = -8, -4, -1, 0, 1, 4, 8$ on the same $xy$-plane by hand and then compare your answer with the solution provided below.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

f. Explain how the level curves change as $k$ changes. What does it tell you about the shape of the surface?

A question appears in Mobius

g. Sketch the surface $z = f(x, y)$ using a computer algebra system (such as GeoGebra, Maple, MatLab, Wolfram Alpha, etc.) and compare the 3D shape with your results from previous parts.

A question appears in Mobius

## Worked Example 2

In this worked example, finding the range of the function requires a little extra work. Also, pay attention to the behaviour of the level curves and cross sections.

Answer the following questions for the function $f(x, y) = e^{1-x^2-y^2} - 1$.

A question appears in Mobius

A question appears in Mobius

c. Sketch the level curves for $k = -0.2, 0, 1, 1.2$ on the same $xy$-plane.

A question appears in Mobius

d. Sketch three cross sections for $f$ with the planes $x = 0, 1, 2$.

A question appears in Mobius

e. Sketch three cross sections for $f$ with the planes $y = 0, 1, 2$.

A question appears in Mobius

f. Explain how the level curves change as $k$ changes. What does it tell you about the shape of the surface?

A question appears in Mobius

g. Sketch the surface $z = f(x, y)$ using a computer algebra system and compare the 3D shape with your results from previous parts.

A question appears in Mobius

## Worked Example 3

In our last example, the cross sections with the planes $x = c$ and the cross sections with the planes $y = d$ are different. Let's see how this difference affects the shape of the graph in 3D.

Answer the following questions for the function $f(x, y) = 2xy - y^2$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

e. Sketch the surface $z = f(x, y)$ using a computer algebra system and compare the 3D shape with your results from previous parts.

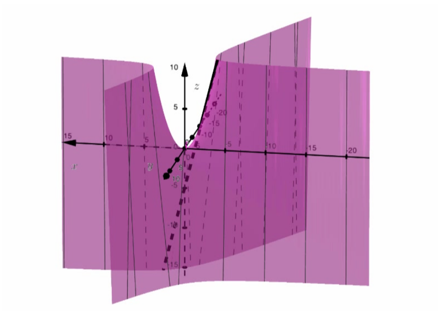
*A slideshow appears in Mobius.*

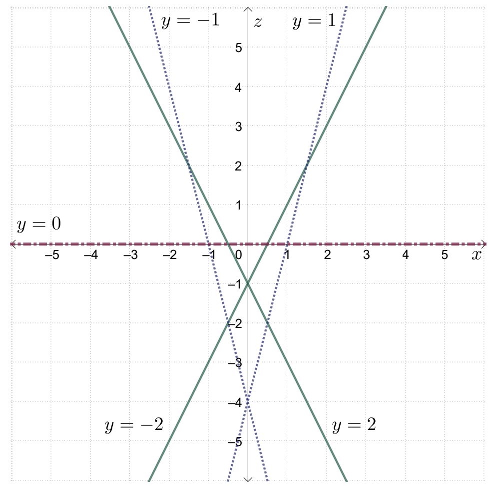
Slide

## Worked Example 3 Part e

3D view

*A video appears here.*






A question appears in Mobius

## Application 2

## Build a 3D Graph

Level curves and cross sections can be very helpful when sketching the graph of a function. In the following GeoGebra app, build the level curves and cross sections to get a sense of the basic shape of the function. Try sketching it on paper before you display the graph.

**Instructions**

1. Drag each of the sliders to build the level curves and cross sections of the function.

2. Once youve thought about the shape of the function, check the box to display and see if you were right.
3. To view the graph from different angles, click and hold on the image and then move your cursor to rotate the figure.
4. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/mg9phqmv/



## Practice Problems

1. For the function $f\left(x,y\right)=4\ x^2-y^2$, answer the following questions.
   (a) Find the domain.
   (b) Find the range.
   (c) Sketch and classify the typical level curves.
   (d) How do the level curves change as $k$ changes?
   (e) Sketch and classify typical cross sections $x=c$, and typical cross sections $y=d$.
   (f) Describe/draw/visualize the surface $z=f(x,y)$ in 3D-space.
2. For the function $f\left(x,y\right)=x^2+4\ y^2-9$, answer the following questions.
   (a) Find the domain.
   (b) Find the range.
   (c) Sketch and classify the typical level curves.
   (d) How do the level curves change as $k$ changes?
   (e) Sketch and classify typical cross sections $x=c$, and typical cross sections $y=d$.
   (f) Describe/draw/visualize the surface $z=f(x,y)$ in 3D-space.
3. For the function $f(x,y)=1-x^4-y^4$, answer the following questions.
   (a) Find the domain.
   (b) Find the range.
   (c) Sketch and classify the typical level curves.
   (d) How do the level curves change as $k$ changes?
   (e) Sketch and classify typical cross sections $x=c$, and typical cross sections $y=d$.
   (f) Describe/draw/visualize the surface $z=f(x,y)$ in 3D-space.
4. For the function $f(x,y)=1-(x^2+y^2-4)^2$, answer the following questions.
   (a) Find the domain.
   (b) Find the range.
   (c) Sketch and classify the typical level curves.
   (d) How do the level curves change as $k$ changes?
   (e) Sketch and classify typical cross sections $x=c$, and typical cross sections $y=d$.
   (f) Describe/draw/visualize the surface $z=f(x,y)$ in 3D-space.
5. For the function $f(x,y)=|1-x^2-y^2|$, answer the following questions.

(a) Find the domain.
(b) Find the range.
(c) Sketch and classify the typical level curves.
(d) How do the level curves change as $k$ changes?
(e) Sketch and classify typical cross sections $x = c$, and typical cross sections $y = d$.
(f) Describe/draw/visualize the surface $z = f(x, y)$ in 3D-space.

6. For the function $f(x, y) = \sqrt{x^2 - y^2}$, answer the following questions.
   (a) Find the domain.
   (b) Find the range.
   (c) Sketch and classify the typical level curves.
   (d) How do the level curves change as $k$ changes?
   (e) Sketch and classify typical cross sections $x = c$, and typical cross sections $y = d$.
   (f) Describe/draw/visualize the surface $z = f(x, y)$ in 3D-space.

7. *[Challenge question]* The temperature of a metal rod at position $x$, $0 \leq x \leq 1$, and at time $t$, $t \geq 0$ is given by $u(t, x) = 100e^{-t} \sin \pi x$. Sketch the level curves $u = 0, 25, 75, 100$.

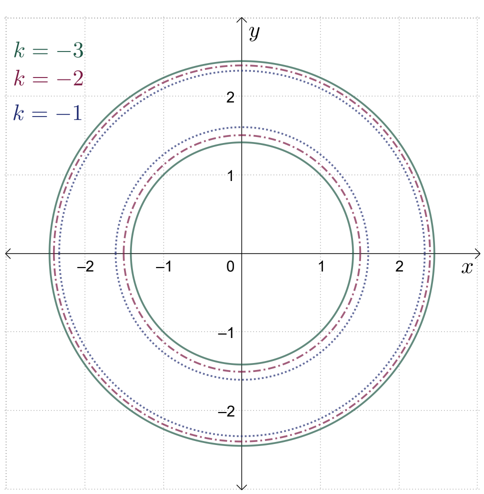
8. *[Challenge question]* A function $g$ is defined by

$$g(x, y) = \int_x^y e^{-t^2} dt$$

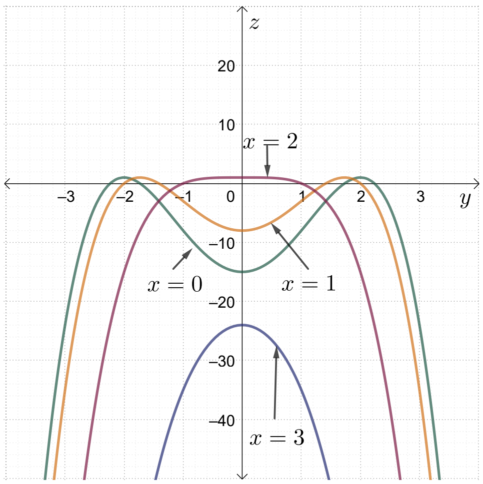
   Sketch the level curves of $g$.

## Selected Answers and Solutions

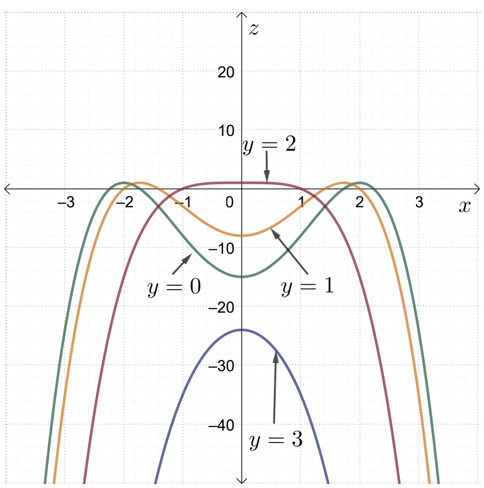
1. No answer provided.
2. No answer provided.
3. No answer provided.
4. (a) First note that the domain of $f(x, y) = 1 - (x^2 + y^2 - 4)^2$ is $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$ since any values of $x$ and $y$ can be inputted into $f(x, y)$. Since $(x^2 + y^2 - 4)^2 \geq 0$, the range is $(-\infty, 1]$.
   (b) Note that for $k = -3, -2, -1$, we get a pair of circles centred at the origin. It is a good question to ask whether this is the case for all $k$ values.



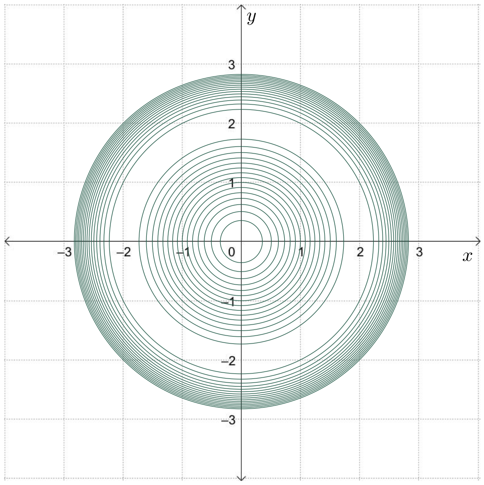
(c) Cross sections are formed by intersecting $z = f(x, y)$ with $x = c$.
For $x = 0$, we have $z = 1 - (y^2 - 4)^2$.
For $x = 1$, we have $z = 1 - (-3 + y^2)^2$.
For $x = 2$, we have $z = 1 - (y^2)^2 = 1 - y^4$.
For $x = 3$, we have $z = 1 - (5 + y^2)^2$.



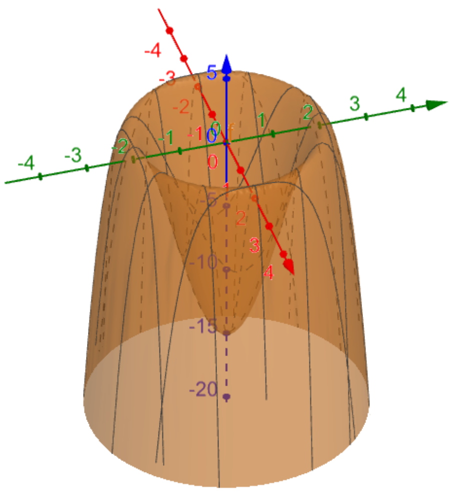

(d) Cross sections are formed by intersecting $z = f(x, y)$ with $y = c$.
For $y = 0$, we have $z = 1 - (x^2 - 4)^2$.
For $y = 1$, we have $z = 1 - (-3 + x^2)^2$.
For $y = 2$, we have $z = 1 - (x^2)^2 = 1 - x^4$.
For $y = 3$, we have $z = 1 - (5 + x^2)^2$.




(e) Note that as $k$ decreases, the level curves get closer to each other.



(f) In 3D-space we have the following shape:



5. (a) $\mathbb{R}^2$

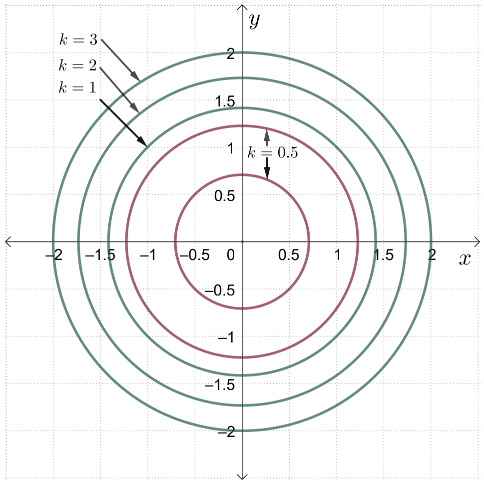
(b) Consider the range of $z = 1 - x^2 - y^2 = 1 - (x^2 + y^2)$, which is $z \leq 1$. The absolute value will take all the negative values of this function and will change their signs. We get the range of $f$ is $[0, \infty)$.

(c) Level curves are defined as $f(x, y) = k$. The level curves will have the general equation of the form $|1 - (x^2 + y^2)| = k$.

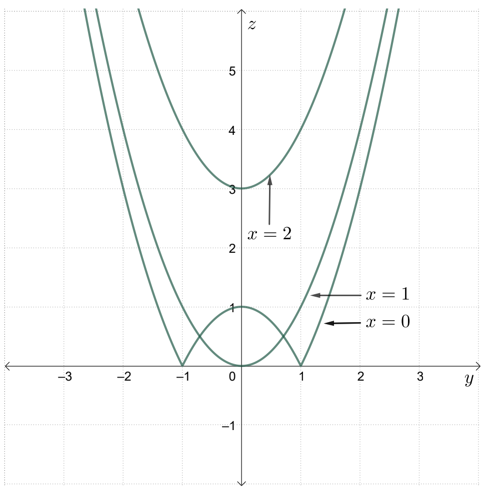
Case 1: When $0 \leq k < 1$, we get $1 - (x^2 + y^2) = k$ and $1 - (x^2 + y^2) = -k$, therefore we get a pair of circles $(x^2 + y^2) = 1 - k$ and $(x^2 + y^2) = 1 + k$.

Case 2: When $k \geq 1$, $1 - (x^2 + y^2) = -k$ which gives us a circle $(x^2 + y^2) = 1 + k$

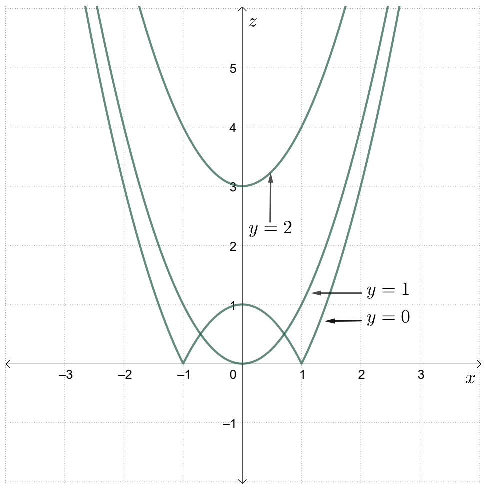
Level curves of $f$ can be classified as a family of circles.



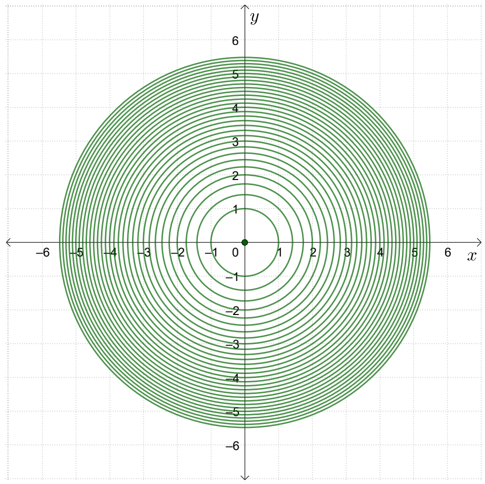

(d) Cross sections are formed by intersecting $z = f(x, y)$ with $x = c$.
For $x = 0$, we have $z = |1 - y^2|$.
For $x = 1$, we have $z = |-y^2| = y^2$.
For $x = 2$, we have $z = |1 - 4 - y^2| = |-3 - y^2| = 3 + y^2$.



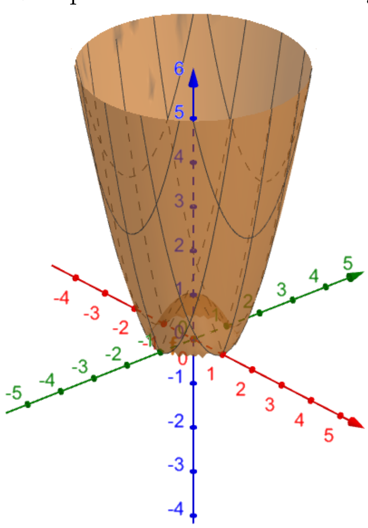

(e) Cross sections are formed by intersecting $z = f(x, y)$ with $y = d$.
For $y = 0$, we have $z = |1 - x^2|$.
For $y = 1$, we have $z = |-x^2| = x^2$.
For $y = 2$, we have $z = |1 - 4 - x^2| = |-3 - x^2| = 3 + x^2$.




(f) Note that as $k$ increases, the level curves get closer to each other.



(g) In 3D-space we have the following shape:



6. No answer provided
7. No answer provided.
8. No answer provided.

# Unit 2

# Limits

## 2.1 - Definition of a Limit

### Definition of a Limit for Functions of One Variable

Before we define the limit for functions of two variables, let's recall the limit definition in one variable.

For a real-valued function $f(x)$ we defined $\lim_{x \to a} f(x) = L$ to mean that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently close to $a$. More precisely, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

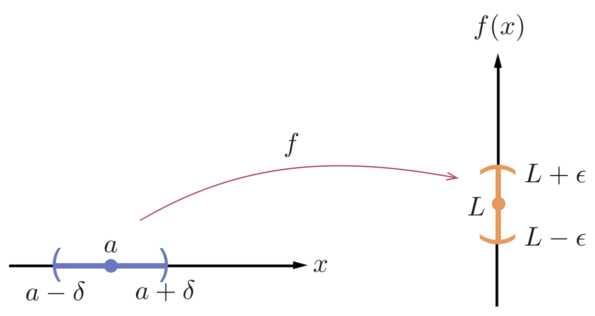
$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

and $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.

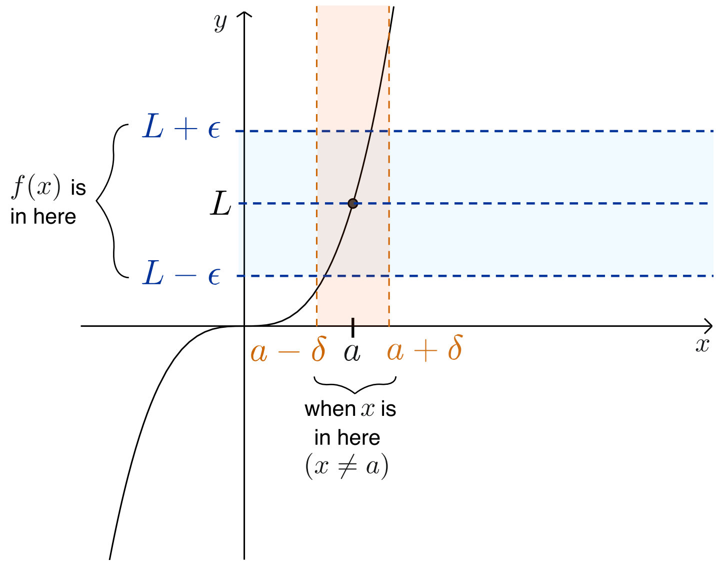
This means that no matter what $\epsilon > 0$ value we choose, we can always find a corresponding $\delta > 0$ value that would satisfy the following condition

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

A visual representation of the epsilon-delta definition in one dimension. The value of $f(x)$ is within a distance of $\varepsilon$ of the limit point $L$ when the value of $a$ is within a distance of $\delta$ of a given $x$.




A visual representation of the epsilon-delta definition in two dimensions. The value of $f(x)$ is within a distance of $\varepsilon$ of the limit point $L$ when the value of $a$ is within a distance of $\delta$ of a given $x$.




We will demonstrate the definition of limit using the following GeoGebra activity.

**Your Turn**

Beginning with $\epsilon = 2$, find a corresponding $\delta$ that satisfies the condition that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Try smaller values of $\epsilon$, and convince yourself that for every $\epsilon > 0$ there is a corresponding $\delta$. Try changing the value of $a$ to explore other limit points.

Note that this demo assumes $f(x)$ is continuous and monotone in the interval $(x - \delta, \;\; x + \delta)$.

**Instructions**

1. Use the slider to select a value of $\epsilon > 0$.
2. Adjust the slider for $\delta$ until you find a value that works, indicated by the note below the slider and on the graph.
3. Click and drag the point $a$ to explore other limit points.

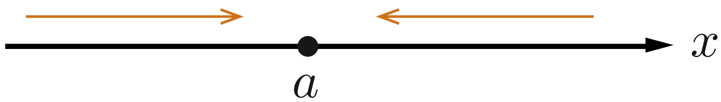
External resource: https://www.geogebra.org/material/iframe/id/r7vexkfy/



## Definition of a Limit for Functions of Two Variables

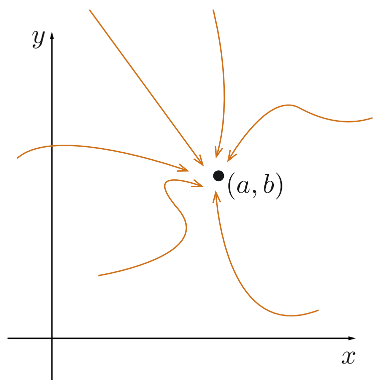
We define the limit for functions of two variables in a very similar way to the limit of functions of one variable. For a scalar function $f(x, y)$, we want $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$ to mean that the values of $f(x, y)$ can be made arbitrarily close to $L$ by taking $(x, y)$ sufficiently close to $(a, b)$.

For the one variable case, we can only approach the limit from two directions: left and right. And, $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.




Two ways to approach $a$

For multivariable scalar functions our domain is now multidimensional, so we can approach the limit from many directions: in fact, from infinitely many directions! Moreover, we are not restricted to straight lines either; we can approach $(a, b)$ along any smooth curve.




Infinitely many ways to approach $(a, b)$

Before we generalize the precise definition of a limit, we need to generalize the concept of an interval.

An **open interval** is defined as

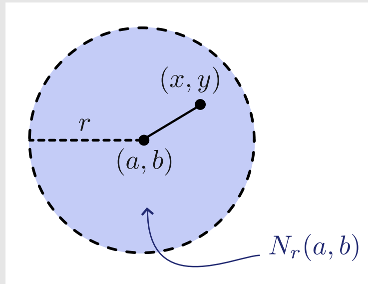
$$(-r, r) = \{x : |x| < r\}$$

where $r \in \mathbb{R}$.

**Definition: Neighbourhood**

An $r$-**neighbourhood** of a point $(a,b) \in \mathbb{R}^2$ is a set

$$N_r(a,b) = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y) - (a,b)\| < r\}, \quad r \in \mathbb{R}$$




Note that if $r < 0$, $N_r$ is the empty set.

**Remark**

Recall that $\|(x,y) - (a,b)\|$ is the Euclidean distance in $\mathbb{R}^2$. That is,

$$\|(x,y) - (a,b)\| = \sqrt{(x-a)^2 + (y-b)^2}$$

Thus, we get:

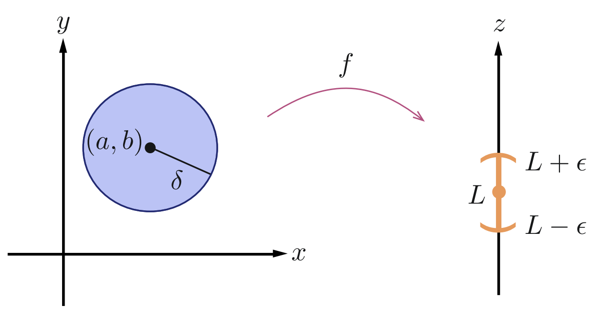
**Definition: Limit**

Assume $f(x,y)$ is defined in a neighbourhood of $(a,b)$, except possibly at $(a,b)$. If, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

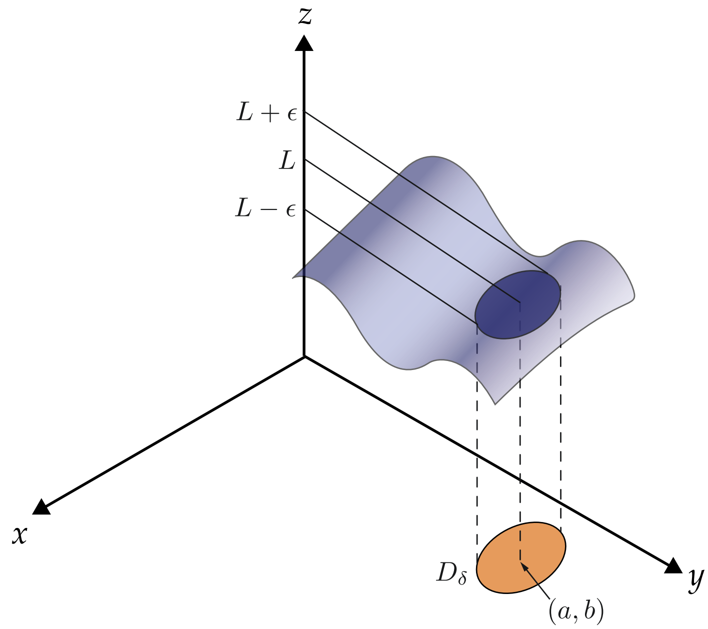
$$0 < \|(x,y) - (a,b)\| < \delta \quad \text{implies} \quad |f(x,y) - L| < \epsilon$$

then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$



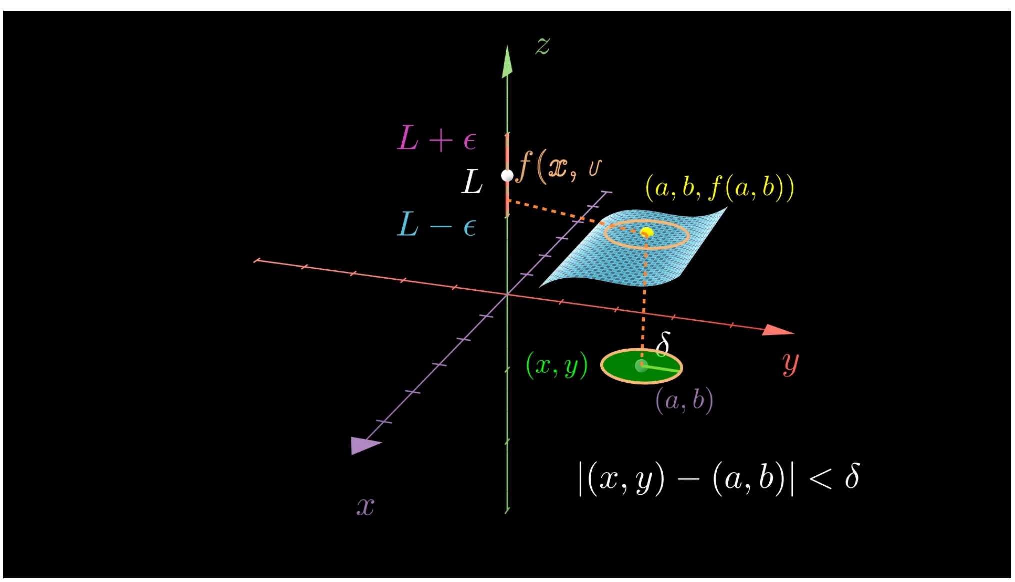

A visual representation of the epsilon-delta definition in two dimensions. The value of $f(x,y)$ is within a distance of $\varepsilon$ of the limit point $L$ when the value of $(x,y)$ is within a radius of $\delta$ of a given $(a,b)$.




A visual representation of the epsilon-delta definition in three dimensions. The value of $f(x, y)$ is within a distance of $\varepsilon$ of the limit point $L$ when the value of $(x, y)$ is within a radius of $\delta$ of a given $(a, b)$.

In the following video, we will construct the figure above one step at a time and review the definition of limit. Pause the video as needed to make connections between the figure and the definition of limit.

*A video appears here.*




Previously, we said we can approach the limits from infinitely many directions, but note that the definition of limit does not refer to any direction at all; it only refers to the distance between $(x, y)$ and $(a, b)$. In addition, using the precise definition can be quite complicated even for relatively simple limits. Thus, we will instead use the definition of a limit to prove theorems that make finding limits easier.

## 2.2 - Limit Theorems

### Limit Theorems

In extending our definition of a limit to functions of two variables, $f(x,y)$, we do indeed preserve all of the properties of limits that we had for single variable functions.

**Limit Theorem 1**
If $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y)$ both exist, then

1. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y)$.
2. $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left[\lim\limits_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim\limits_{(x,y)\to(a,b)} g(x,y)\right]$.
3. $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim\limits_{(x,y)\to(a,b)} f(x,y)}{\lim\limits_{(x,y)\to(a,b)} g(x,y)}$, provided $\lim\limits_{(x,y)\to(a,b)} g(x,y) \neq 0$.

**Proof: Part (a)**

Let $\epsilon > 0$.

Since $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L_1$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = L_2$ both exist, by definition of a limit, there exists a $\delta > 0$ such that

$$0 < \|(x,y)-(a,b)\| < \delta$$
$$\text{implies}$$
$$|f(x,y)-L_1| < \frac{1}{2}\epsilon$$
$$\text{and}$$
$$|g(x,y)-L_2| < \frac{1}{2}\epsilon$$

Thus, if $0 < \|(x,y)-(a,b)\| < \delta$, then

$$\begin{aligned}
\big|f(x,y)+g(x,y)-(L_1+L_2)\big| &= \big|[f(x,y)-L_1]+[g(x,y)-L_2]\big| \\
&\leq |f(x,y)-L_1| + |g(x,y)-L_2| \quad \text{by triangle inequality} \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon
\end{aligned}$$

as required.

□

Proofs for parts (b) and (c) are left as exercises.

We can use these algebraic properties to find a limit.

**Example 1**

Find the limit $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2 - xy + y^2}{x^2 + y^2}$.

**Solution:**

First, we need to check the denominator. Since $x^2 + y^2 = 1^2 + 1^2 = 1 + 1 = 2$ is non-zero, we can use part c) of the above theorem.

$$\begin{aligned}
\lim_{(x,y)\to(1,1)} \frac{x^2 - xy + y^2}{x^2 + y^2} &= \frac{\lim\limits_{(x,y)\to(1,1)} x^2 - xy + y^2}{\lim\limits_{(x,y)\to(1,1)} x^2 + y^2} \\
&= \frac{\lim\limits_{x\to1} x^2 - (\lim\limits_{x\to1} x)(\lim\limits_{y\to1} y) + \lim\limits_{y\to1} y^2}{2} \\
&= \frac{1 - 1 + 1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

A question appears in Mobius

**Limit Theorem 2**

If $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists, then the limit is unique.

### Your Turn 2

Provide a proof for the theorem above.

A question appears in Mobius

## 2.3 - Proving a Limit Does Not Exist

### Proving a Limit Does Not Exist

Recall for a function of one variable, we often showed a limit did not exist by showing the left-hand limit did not equal the right-hand limit and using the fact that the limit must be unique. For multivariable functions, we will essentially do the same thing, only now we have to remember that we are able to approach $(a, b)$ along any smooth curve.

**Example 1**

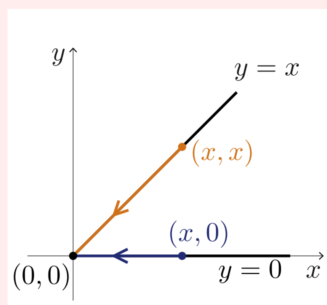
Let $f$ be defined by $f(x, y) = \dfrac{xy}{x^2 + y^2}$, for $(x, y) \neq (0, 0)$. Prove that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

**Solution:**

To prove this limit does not exist, we need to approach the limit along two paths that give different values.

We first approach the limit along the line $y = 0$. Notice that by holding $y$ constant, we are turning this limit of a function of two variables into a limit of a function of a single variable $x$. We get

$$\lim_{(x,y)\to(0,0)} f(x, 0) = \lim_{x\to 0} \frac{x(0)}{x^2 + 0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$




Now, approach the limit along the line $y = x$. This again changes the limit of a function of two variables into the limit of a function of one variable. We get

$$\lim_{(x,y)\to(0,0)} f(x, x) = \lim_{x\to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Since $f(x, y)$ approaches different values as $(x, y)$ tends to $(0, 0)$ along different paths, the limit does not exist.

We often can approach the limit along infinitely many lines or smooth curves at the same time by introducing an arbitrary coefficient $m$. If the value of the limit depends on the value of $m$, then it is not unique and hence, the limit does not exist.

**Example 2**

Prove that $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ does not exist.

**Solution:**

Approaching the limit along lines $y = mx$ we get

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\sin(x(mx))}{x^2+(mx)^2} &= \lim_{x\to 0} \frac{\sin(mx^2)}{x^2(1+m^2)} \\ &= \lim_{x\to 0} \frac{2mx\cos(mx^2)}{2x(1+m^2)} \qquad \text{by L'Hôpital's rule} \\ &= \lim_{x\to 0} \frac{m\cos(mx^2)}{1+m^2} \\ &= \frac{m}{1+m^2}\end{aligned}$$

Since the limit depends on $m$ we can get different limits along different lines $y = mx$ and hence $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ does not exist.

**Your Turn 1**

Let $f(x,y) = \frac{|x|}{|x|+y^2}$, for $(x,y) \neq (0,0)$. Show that

$$\lim_{(x,y)\to(0,0)} f(x,mx) = 1$$

for all $m \in \mathbb{R}$, but $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

**Hint:** $y = mx$ does not describe all lines through the origin

A question appears in Mobius

*A slideshow appears in Mobius.*

Slide

## Example 3

Let $f(x,y) = \dfrac{x^2y}{x^4+y^2}$ for $(x,y) \neq (0,0)$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

**Solution:**
We test the limit along the lines $y = mx$ and $x = 0$.
For $y = mx$, we get

$$\lim_{(x,y)\to(0,0)} f(x,mx) = \lim_{x\to 0} \frac{x^2(mx)}{x^4+(mx)^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0$$

For $x = 0$, we get

$$\lim_{(x,y)\to(0,0)} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$$

So far, all straight lines through $(0,0)$ give the same limit value.

Slide

## Example 3 Continued

Let $f(x,y) = \dfrac{x^2y}{x^4+y^2}$ for $(x,y) \neq (0,0)$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

**Solution:**
Next, let's test some curves passing through $(0,0)$.
For $y = x^2$, we get

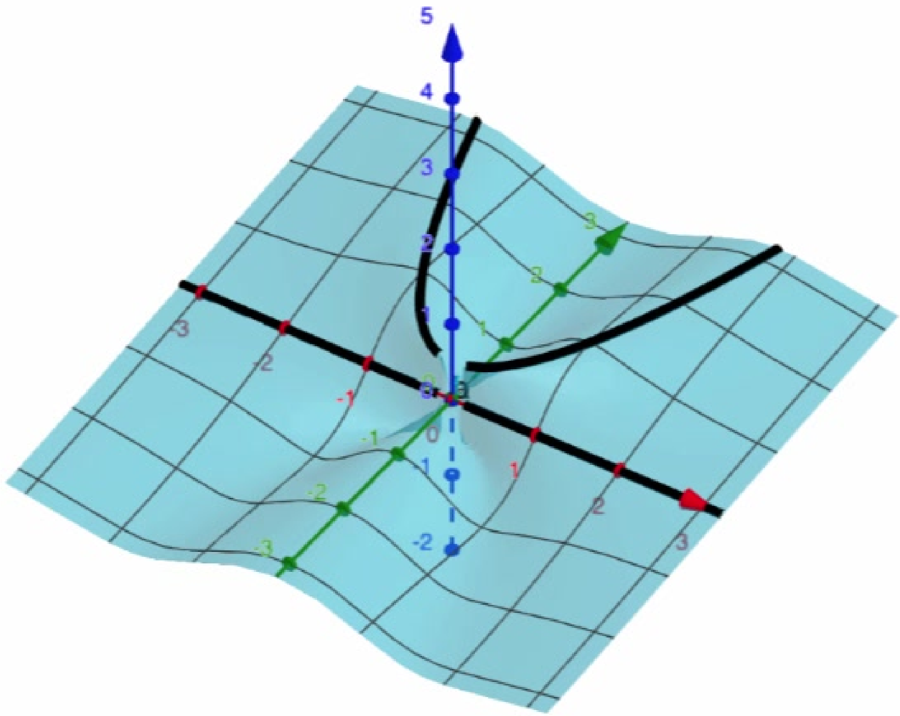
$$\lim_{(x,y)\to(0,0)} f(x,x^2) = \lim_{x\to 0} \frac{x^2x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

We have found a path with a limit that is different from 0; therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.
**How did we know to try $x^2$?**
We chose $x^2$ because we want the powers of $x$ in the numerator and denominator to match so that they cancel out.
This type of trick is common when proving that a limit does not exist.

Slide

## 3D Visualization of Example 3

*A video appears here.*




This example shows us that no matter how many lines and/or curves you test, you cannot use this method to prove a limit exists. Just because you haven't found two paths that give different values does not mean there isn't one!

**Stop and Think**

We could have done the last example more efficiently by just testing $y = mx^2$ to begin with and showing the limit depends on $m$.

**Caution**
Make sure that all lines or curves you use actually approach the limit point. A common error is to approach a limit like the one in the previous example along a line $x = 1$, which of course is meaningless as it does not pass through $(0, 0)$.

**Your Turn 2**

Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$ does not exist.

**Hint:** Try approaching $y = mx^3$.

A question appears in Mobius

Prove that $\lim_{(x,y)\to(1,0)} \dfrac{(x-1)(y+1)}{|x-1|+y}$ does not exist.

**Hint:** Does the limit exist when approaching along $x$-axis?

A question appears in Mobius

## 2.4 - Proving a Limit Exists

### Proving a Limit Exists

As we mentioned earlier, using the precise definition can be quite complicated even for relatively simple limits. Also, we cannot use the previous methods to prove a limit exists. In this section, we introduce a theorem to help us to prove a limit exists.

**Theorem 1: Squeeze Theorem**
If there exists a function $B(x, y)$ such that

$$|f(x, y) - L| \leq B(x, y), \quad \text{for all } (x, y) \neq (a, b)$$

in some neighborhood of $(a, b)$ and $\lim_{(x,y)\to(a,b)} B(x, y) = 0$, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

The Squeeze Theorem gives us a way to prove that a limit exists by bounding $|f(x, y) - L|$ above by a function $B(x, y)$ whose limit goes to zero. Note that in order to apply the Squeeze Theorem, we need a candidate limit $L$ for the function $f(x, y)$. Once we have the candidate limit, we then need to find a bounding function $B(x, y)$ whose limit is known and goes to zero.

**Proof:**

Let $\epsilon > 0$.

Since $\lim_{(x,y)\to(a,b)} B(x, y) = 0$, by the definition of limit, there exists a $\delta > 0$ such that

$$0 < \|(x, y) - (a, b)\| < \delta \quad \text{implies} \quad |B(x, y) - 0| < \epsilon$$

Hence, if $0 < \|(x, y) - (a, b)\| < \delta$, then we have

$$|f(x, y) - L| \leq B(x, y) = |B(x, y)| < \epsilon$$

as our hypothesis requires that $B(x, y) \geq 0$ for all $(x, y) \neq (a, b)$ in the neighborhood of $(a, b)$. Therefore, by definition of a limit, we have

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

□

**Stop and Think**

Our statement of the Squeeze Theorem above is not a direct generalization of the Squeeze Theorem we used in single variable calculus. What would the direct generalization of the Squeeze Theorem be? How are your generalization and the theorem above related?

## Examples

*A slideshow appears in Mobius.*

Slide

## Example 1

**Squeeze Theorem**
If there exists a function $B(x, y)$ such that

$$|f(x, y) - L| \leq B(x, y)$$

for all $(x, y) \neq (a, b)$ in some neighbourhood of $(a, b)$ and $\lim_{(x,y)\to(a,b)} B(x, y) = 0$, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

Prove that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0$.

**Solution:**

We are given $f(x, y) = \frac{x^2 y}{x^2 + y^2}$ and $L = 0$.

For $(x, y) \neq (0, 0)$ we have

$$|f(x, y) - L| = \left| \frac{x^2 y}{x^2 + y^2} - 0 \right| = \frac{x^2|y|}{x^2 + y^2}$$

Since $y^2 \geq 0$, it follows that $x^2 \leq x^2 + y^2$ , so

$$\frac{x^2|y|}{x^2 + y^2} \leq \frac{(x^2 + y^2)|y|}{x^2 + y^2} = |y|$$

So we have $0 \leq |f(x, y) - L| \leq |y|$ for all $(x, y) \neq (0, 0)$.

By inspection, $\lim_{(x,y)\to(0,0)} |y| = 0$.

Thus, by the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0$$

The next example illustrates some manipulations with inequalities. It is very common to use inequalities when taking multivariable limits; the appendix to this lesson provides some commonly used inequalities which you might find useful.

**Example 2**
Prove that

$$\frac{|2x^2 - y^2|}{|x| + |y|} \le 2|x| + |y|, \quad \text{for all } (x, y) \ne (0, 0)$$

**Solution:**
The idea is to manipulate the numerator so as to create a factor of $|x|+|y|$, which will cancel the denominator. For arbitrary $(x, y)$, consider

$$\begin{aligned} |2x^2 - y^2| &= |2x^2 + (-y^2)| \\ &\le |2x^2| + |-y^2|, \quad \text{by the Triangle Inequality} \\ &= 2|x|^2 + |y|^2 \end{aligned}$$

Since $|x| \le |x| + |y|$, and $|y| \le |x| + |y|$, we obtain

$$\begin{aligned} 2|x|^2 + |y|^2 &\le 2|x|\Big(|x| + |y|\Big) + |y|\Big(|x| + |y|\Big) \\ &= \Big(2|x| + |y|\Big)\Big(|x| + |y|\Big) \end{aligned}$$

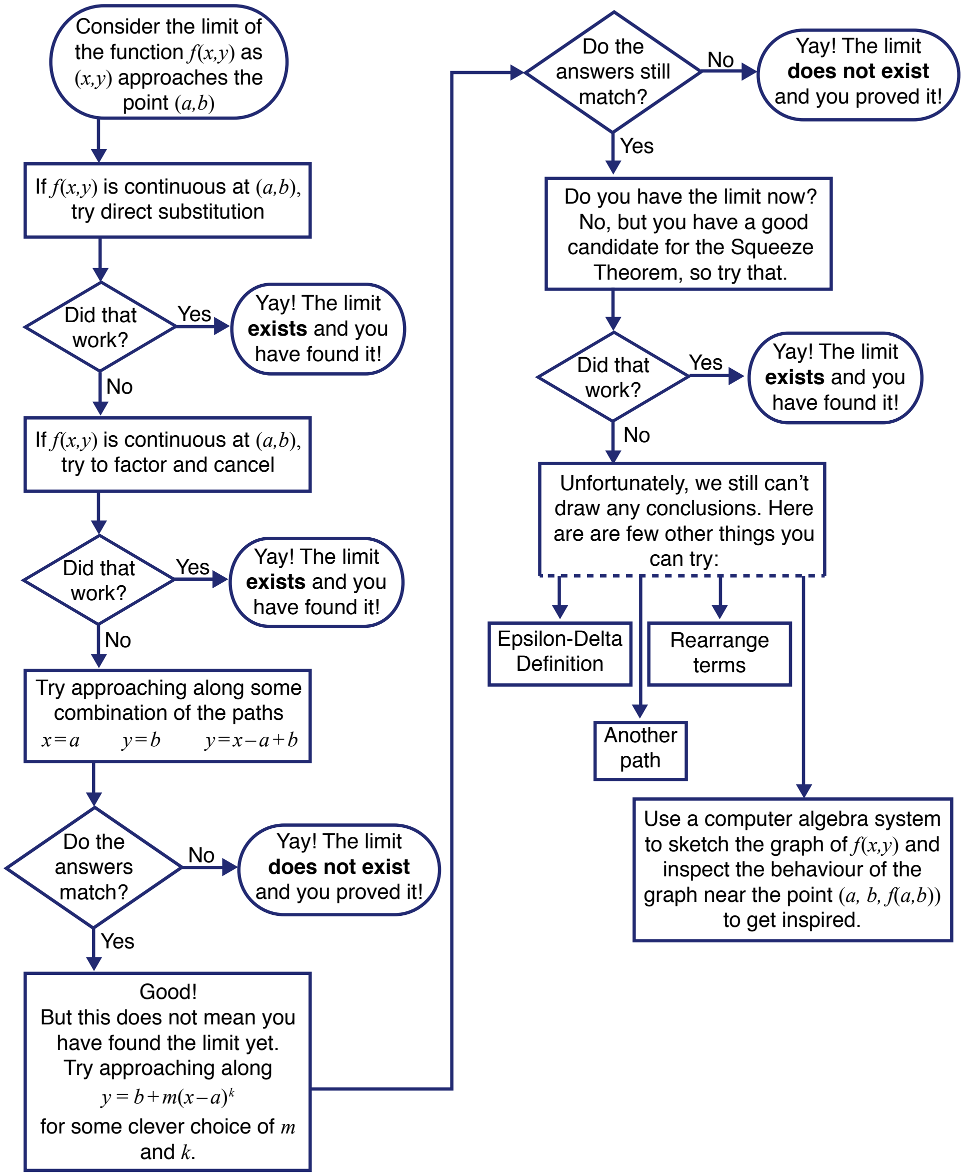
Hence,

$$\begin{aligned} \frac{|2x^2 - y^2|}{|x| + |y|} &\le \frac{(2|x| + |y|)(|x| + |y|)}{|x| + |y|} \\ &= 2|x| + |y| \end{aligned}$$

as required.

## Algorithm

The flowchart below summarizes how to tackle questions of the form "Determine whether $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists, and if so find its value"




Determining whether a limit exists and finding limits are, fundamentally, processes of trial and error. As you build experience, you will develop intuitions about what curves to try and which bounding functions to choose.

Let's see an example.

This slideshow has no audio. Clicking the arrow in the lower right will reveal the solution step by step. Try to figure out the next step for yourself before you click to reveal it.

*A slideshow appears in Mobius.*

Slide

## Example 3

Determine whether $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-|x|-|y|}{|x|+|y|}$ exists, and if so find its value.

**Solution:**

Trying lines $y = mx$ we get

$$\lim_{x\to 0} \frac{x^2-|x|-|m||x|}{|x|+|m||x|} = \lim_{x\to 0} \frac{|x|-(1+|m|)}{1+|m|} = -1$$

Since the value along each line is $L = -1$, we try to prove the limit is $-1$ with the Squeeze Theorem. Thus, we consider

$$\begin{aligned}
\left|\frac{x^2-|x|-|y|}{|x|+|y|} - (-1)\right| &= \left|\frac{x^2-|x|-|y|}{|x|+|y|} + \frac{|x|+|y|}{|x|+|y|}\right| \\
&= \frac{x^2}{|x|+|y|} \\
&= \frac{|x|\cdot|x|}{|x|+|y|} \\
&\le \frac{|x|(|x|+|y|)}{|x|+|y|} = |x| \qquad\qquad \text{since } |x| \le (|x|+|y|)
\end{aligned}$$

Since $\lim\limits_{(x,y)\to(0,0)} |x| = 0$ we get $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-|x|-|y|}{|x|+|y|} = -1$ by the Squeeze Theorem.

**Your Turn**

Consider $f$ defined by

$$f(x,y) = \frac{x^2(x-1)-y^2}{x^2+y^2}, \qquad \text{for } (x,y) \ne (0,0)$$

Determine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists, and if so find its value.

**Hint:** Try the lines $y = mx$ to find a candidate, then apply the Squeeze Theorem.

A question appears in Mobius

For another take on limits, see the `https://sequentialmath.com/comic/limits`.

## 2.5 - Appendix: Inequalities and Absolute Values

### Appendix: Inequalities and Absolute Values

When finding multivariable limits, we often make use of inequalities and absolute values. This section gathers some commonly used results for you to refer to.

The following statements can be taken as axioms (i.e., assumed properties) which define the notion of “less than” (denoted “$<$”) for real numbers. One can equivalently use the notion of “greater than”) (denoted “$>$”). The statement “$a > b$” means “$b < a$”.

- **Trichotomy Property:** For any real numbers $a$ and $b$, one and only one of the following holds:
$$a = b, \qquad a < b, \qquad b < a$$
- **Transitivity Property:** If $a < b$ and $b < c$, then $a < c$.
- **Addition Property:** If $a < b$, then for all $c$, $a + c < b + c$.
- **Multiplication Property:** If $a < b$ and $c < 0$, then $bc < ac$.
- **Multiplicative Inverse Property:** If $ab > 0$ with $a < b$, then $\dfrac{1}{b} < \dfrac{1}{a}$. Note the change in order!

Using these properties we can deduce other results.

A question appears in Mobius

The **absolute value** of a real number $a$ is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Some frequently used results, which follow from the definition of the absolute value and from the axioms on inequalities, are listed below. These are commonly used when applying the Squeeze Theorem.

1. $|a| = \sqrt{a^2}$.

2. $|a| < b$ if and only if $-b < a < b$.
3. the Triangle Inequality: $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.
4. if $c > 0$, then $a < a + c$
5. the cosine inequality $2|x||y| \leq x^2 + y^2$

One particularly common use of (4) is for things like

$$|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$$

Again, it is very important to be careful when working with inequalities.

A question appears in Mobius

**Your Turn 3**
Prove that

$$\frac{|x^3 - y^3|}{x^2 + y^2} \leq |x| + |y| \quad \text{for all} \quad (x, y) \neq (0, 0)$$

Does equality ever hold?

A question appears in Mobius

## 2.6 - Putting It All Together

### Introduction

In this lesson, we will put together everything that we have learned so far about limits and continuity. We recommend that you study and understand these examples very well before you move on to the problems assigned in the next section. Try to answer each question. If you do not know how to answer the question, click to reveal the answer to study the solution. Then, move on to the next question.

### Tips:

When you are trying to find the limit, if it exists, or show that the limit does not exist:

1. If the function is not defined piecewise, try direct substitution. If you get a finite number, congratulations, you have found the limit.
2. If 1 fails, try factoring and cancelling. If you get a finite number, congratulations, you have found the limit.
3. If 2 fails, try the following
   (a) Try $x = 0$, by inspection figure out the limit, use L'Hôpital's Rule if applicable.
   (b) Try $y = 0$, by inspection figure out the limit, use L'Hôpital's Rule if applicable.

   Did you get different results from a) and b)? Congratulations, you proved that the limit DNE.
4. Did you get the same results from a) and b)?
   Hmm, let's make sure and try $y = x$ (or another smart choice such as $y = mx$, $y = mx^2$, $y = mx^{1/2}$ to match the degrees of numerator and denominator) if it is easy to find the limit by inspection.
   Did you get a different result? Congratulations, you proved that the limit DNE.
5. Did you get the same results as a) and b)? It may be a good candidate for the limit. Try the Squeeze Theorem.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Practice Problems

Try to answer the questions. If you are having trouble, check for a hint before looking at the solutions.

1. Find the limit, if it exists, or show that the limit does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+2y^4}{x^4+y^4}$ *[Hint provided below]*

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{|x|+|y|}$ *[Hint provided below]*

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-2|x|-|y|}{2|x|+|y|}$ *[Hint provided below]*

   (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^3}{x^8+y^6}$ *[Hint provided below]*

2. Let $f(x,y)=\dfrac{|x|^a|y|^b}{|x|^c+|y|^d}$ where $a,b,c$ and $d$ are positive numbers.

   (a) Prove that if $\dfrac{a}{c}+\dfrac{b}{d}>1$ then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists and equals zero.

(b) Prove that if $\frac{a}{c} + \frac{b}{d} \leq 1$, then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

## Hints

*Additional content appears in Mobius.*

*Additional content appears in Mobius.*

*Additional content appears in Mobius.*

*Additional content appears in Mobius.*

## Select Answers and Solutions

1. (a) One possible approach is along the set of lines $y = mx$ for $m \in \mathbb{R}$.

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^4+2y^4}{x^4+y^4} &= \lim_{x\to 0} \frac{x^4+2(mx)^4}{x^4+(mx)^4} \\
&= \lim_{x\to 0} \frac{x^4+2m^4x^4}{x^4+m^4x^4} \\
&= \lim_{x\to 0} \frac{x^4(1+2m^4)}{x^4(1+m^4)} \\
&= \lim_{x\to 0} \frac{1+2m^4}{1+m^4} \\
&= \frac{1+2m^4}{1+m^4}
\end{aligned}$$

Note that the value of this limit depends on the value of $m$; hence the value of the limit depends on the direction of approach to $(0,0)$, which means that the limit does not exist.

(b) We will try along the set of lines $y = mx$ for $m \in \mathbb{R}$:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{|x|+|y|} &= \lim_{x\to 0} \frac{(x-mx)^2}{|x|+|mx|} \\
&= \lim_{x\to 0} \frac{x^2-2mx^2+m^2x^2}{|x|+|mx|} \\
&= \lim_{x\to 0} \frac{|x|(|x|-2m|x|+m|x|)}{|x|(1+|m|)} \\
&= \lim_{x\to 0} \frac{|x|-2m|x|+m|x|}{1+|m|} \\
&= 0
\end{aligned}$$

The limit does not depend on the value of $m$; this suggests $L = 0$ as a candidate limit. We must now prove this using the Squeeze Theorem.

We consider $\left|\frac{(x-y)^2}{|x|+|y|} - 0\right| = \left|\frac{(x-y)^2}{|x|+|y|}\right| = \frac{|x-y|^2}{|x|+|y|}$ and try to find an upper bound.

Note that $|x-y| \leq |x|+|y|$ and that, in general, $0 \leq a \leq b$ implies $a^2 \leq b^2$ hence

$$\frac{|x-y|^2}{|x|+|y|} \leq \frac{(|x|+|y|)^2}{|x|+|y|} = |x|+|y|$$

Since $\lim_{(x,y)\to(0,0)} |x|+|y| = 0$, we get that $\lim\limits_{(x,y)\to(0,0)} \frac{(x-y)^2}{|x|+|y|} = 0$ by the Squeeze Theorem.

(c) We will try along the set of lines $y = mx$ for $m \in \mathbb{R}$:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2 - 2|x| - |y|}{2|x| + |y|} &= \lim_{x\to 0} \frac{x^2 - 2|x| - |mx|}{2|x| + |mx|} \\ &= \lim_{x\to 0} \frac{|x|(|x| - 2 - |m|)}{|x|(2 + |m|)} \\ &= \lim_{x\to 0} \frac{|x| - 2 - |m|}{2 + |m|} \\ &= \lim_{x\to 0} \frac{-(2 + |m|)}{2 + |m|} \\ &= -1\end{aligned}$$

The limit does not depend on the value of $m$; this suggests $L = -1$ as a candidate limit. We must now prove this using the Squeeze Theorem.

We consider $\left|\dfrac{x^2 - 2|x| - |y|}{2|x| + |y|} - (-1)\right| = \left|\dfrac{x^2 - 2|x| - |y| + 2|x| + |y|}{2|x| + |y|}\right| = \left|\dfrac{x^2}{2|x| + |y|}\right|$ and try to find an upper bound.

Note that $2|x| + |y| \geq |x|$, hence $\dfrac{1}{2|x| + |y|} \leq \dfrac{1}{2|x|}$ so we have

$$\begin{aligned}\left|\frac{x^2}{2|x| + |y|}\right| &\leq \left|\frac{x^2}{2|x|}\right| \\ &= \frac{|x|}{2}\end{aligned}$$

Since $\lim\limits_{x\to 0} \dfrac{|x|}{2} = 0$, we get that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - 2|x| - |y|}{2|x| + |y|} = -1$ by the Squeeze Theorem.

(d) We start by testing the limit along the lines $y = mx^{4/3}$. We have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^4 y^3}{x^8 + y^6} &= \lim_{x\to 0} \frac{x^4 (mx^{4/3})^3}{x^8 + (mx^{4/3})^6} \\ &= \lim_{x\to 0} \frac{m^3 x^8}{x^8 + m^6 x^8} \\ &= \lim_{x\to 0} \frac{m^3}{1 + m^6}\end{aligned}$$

Note that the value of this limit depends on the value of $m$; hence the value of the limit depends on the direction of approach to $(0, 0)$, which means that the limit does not exist.

2. No solution provided.

# Unit 3

# Continuous Functions

## 3.1 - Definition of a Continuous Function

### Definition of a Continuous Function of One Variable

Take a few minutes to review the definition of a continuous function of one variable from your first year calculus textbook. You can also watch the following animation for a visual representation of continuity for a function in one variable.

*A slideshow appears in Mobius.*

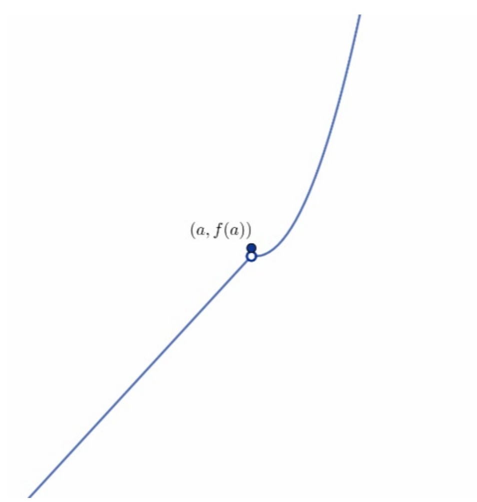
Slide

### Review of Continuity for Single-Variable Functions

A function of one variable $f(x)$ is continuous at $x = a$ if and only if

1. $f$ is defined at $x = a$
2. $\lim_{x\to a} f(x)$ exists, which means that
   (a) $\lim_{x\to a^-} f(x)$ exists; and
   (b) $\lim_{x\to a^+} f(x)$ exists; and
   (c) $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$
3. $\lim_{x\to a} f(x) = f(a)$

*A video appears here.*

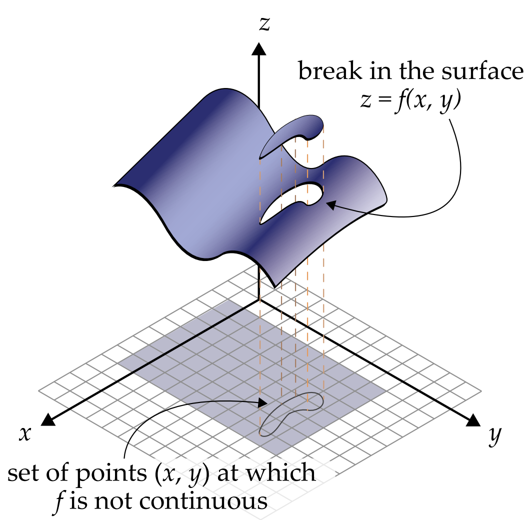



#### Your Turn

Give an example of a function $y = f(x)$ which is defined for all $x \in \mathbb{R}$, but is not continuous at $x = 0$.

A question appears in Mobius

In many situations, we require that a function $f(x, y)$ is continuous. Intuitively, this means that the graph of $f$ (the surface $z = f(x, y)$) has no "breaks" or "holes" in it.




As with functions of one variable, continuity is defined using limits.

## Definition of a Continuous Function of Two Variables

Here is the formal definition of continuity for a function of two variables.

**Definition: Continuous**
A function $f(x, y)$ is **continuous** at $(a, b)$ if and only if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Additionally, if $f$ is continuous at every point in a set $D \subset \mathbb{R}^2$, then we say that $f$ is continuous on $D$.

**Remark**
Just like in single variable calculus, there are three requirements in this definition:

1. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists,
2. $f$ is defined at $(a, b)$, and
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

When checking whether a function is continuous, it is important to remember to check each of these conditions. Let's look at a few examples to see how we apply the definition of continuity in practice.

**Example 1**
Let $f$ be defined by

$$f(x, y) = \begin{cases} \dfrac{x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Determine whether $f$ is continuous at $(0, 0)$.
**Solution:**
When we studied limits, we showed that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2 + y^2} = 0$$

Since $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2 + y^2} = 0 = f(0, 0)$, it follows that $f$ is continuous at $(0, 0)$.

To show that a function is not continuous, we need to show that **at least one** of the three requirements of the definition of continuity fails.

**Example 2**
Prove that $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ is not continuous at $(0, 0)$.
**Solution:**
To prove that $f$ is not continuous at $(0, 0)$, we need to prove that the limit

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

does not equal 0. Therefore, if we can find one path such that the limit does not equal 0, then, since the value of a limit must be unique, this will prove that the limit cannot be equal to 0.
Approaching the limit along the line $y = x$ gives

$$\lim_{x\to 0} \frac{x^2}{x^2 + x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$$

Thus, the limit cannot equal 0, so $f$ is not continuous at $(0, 0)$.

**Example 3**
Consider $f$ defined by

$$f(x,y) = \frac{\sin(xy)}{x^2+y^2}, \quad \text{if } (x,y) \neq (0,0)$$

Is it possible for $f$ to be defined at $(0,0)$ in such a way that the resulting function, whose domain is $\mathbb{R}^2$, is continuous at $(0,0)$?
**Solution:**
By definition of continuity, we must determine whether

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$$

exists. When we studied limits, we showed that this limit does not exist. Thus, no matter what value we assign to $f(0,0)$ the resulting function will not be continuous at $(0,0)$.

A question appears in Mobius

## 3.2 - The Continuity Theorems

### Basic Functions

As we saw in the previous lesson, checking whether a function is continuous can be done by verifying that the definition of continuity is satisfied. This can get a bit tedious, which is why we will now develop some results which will simplify checking for continuity. The general idea is to view a given function as being made up of simpler functions, which we know to be continuous.

We will call these simpler functions basic functions, which are known to be continuous on their domains. In this course, you can take the continuity of these functions on their domains as a given.

- the constant function $f(x,y) = k$
- the power functions $f(x,y) = x^n$, $f(x,y) = y^n$
- the logarithm function $\ln(\cdot)$
- the exponential function $e^{(\cdot)}$
- the trigonometric functions, $\sin(\cdot), \cos(\cdot)$, etc.

- the inverse trigonometric functions, $\arcsin(\cdot)$, etc.
- the absolute value function $|\cdot|$

Now that we have our list of basic functions, let's see how we can assemble them into more complicated functions using operations.

**Definition: Operations on Functions**
If $f(x, y)$ and $g(x, y)$ are scalar functions and $(x, y) \in D(f) \cap D(g)$, then:

1. the **sum** $f + g$ is defined by
$$(f + g)(x, y) = f(x, y) + g(x, y)$$
2. the **product** $fg$ is defined by
$$(fg)(x, y) = f(x, y)g(x, y)$$
3. the **quotient** $\dfrac{f}{g}$ is defined by
$$\left(\frac{f}{g}\right)(x, y) = \frac{f(x, y)}{g(x, y)}, \quad \text{if } g(x, y) \neq 0$$

A question appears in Mobius

**Definition: Composite Function**
For scalar functions $g(t)$ and $f(x, y)$ the **composite function** $g \circ f$ is defined by
$$(g \circ f)(x, y) = g(f(x, y))$$
for all $(x, y) \in D(f)$ for which $f(x, y) \in D(g)$.

**Remark**
When composing multivariable functions, it is very important to make sure that the range of the inner function is a subset of the domain of the outer function. For example, we cannot compose arbitrary scalar functions $f(x, y)$ and $h(x, y)$ since $f(x, y) \in \mathbb{R}$ which is not acceptable input into $h$.

A question appears in Mobius

## The Continuity Theorems

With basic functions and operations on functions in hand, we now state and prove a set of theorems to help us more easily prove that a function of two variables is continuous. We will refer to the following theorems collectively as the **continuity theorems**.

**Continuity Theorem 1**
If $f$ and $g$ are both continuous at $(a, b)$, then $f + g$ and $fg$ are continuous at $(a, b)$.

**Proof:**

We prove the result for $f + g$ and leave the proof for $fg$ as an exercise. By the hypothesis and the definition of continuous function we have that

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$
$$\lim_{(x,y)\to(a,b)} g(x, y) = g(a, b)$$

Hence, by definition of the sum and limit properties, we get

$$\begin{aligned}
\lim_{(x,y)\to(a,b)} (f + g)(x, y) &= \lim_{(x,y)\to(a,b)} f(x, y) + \lim_{(x,y)\to(a,b)} g(x, y) \\
&= f(a, b) + g(a, b) \\
&= (f + g)(a, b)
\end{aligned}$$

□

**Continuity Theorem 2**
If $f$ and $g$ are both continuous at $(a, b)$ and $g(a, b) \neq 0$, then the quotient $\dfrac{f}{g}$ is continuous at $(a, b)$.

**Stop and Think**

Use the limit theorems to prove Continuity Theorem 2. Where do we use the hypothesis $g(a, b) \neq 0$ explicitly?

**Continuity Theorem 3**
If $f(x, y)$ is continuous at $(a, b)$ and $g(t)$ is continuous at $f(a, b)$, then the composition $g \circ f$ is continuous at $(a, b)$.

**Proof:** Let $\epsilon > 0$. By definition of continuity we have that

$$\lim_{t \to f(a,b)} g(t) = g(f(a, b))$$

So, by definition of a limit there exists a $\delta_1 > 0$ such that

$$|t - f(a, b)| < \delta_1 \quad \text{implies} \quad |g(t) - g(f(a, b))| < \epsilon \tag{*}$$

Similarly, we have that

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

Hence, given the above $\delta_1$, there exists a $\delta > 0$ such that

$$\|(x, y) - (a, b)\| < \delta \quad \text{implies} \quad |f(x, y) - f(a, b)| < \delta_1 \tag{**}$$

Notice that the conclusion of $(**)$ is the hypothesis of $(*)$ where $t = f(x, y)$.

Hence, combining $(*)$ and $(**)$, we get

$$\begin{aligned} \|(x, y) - (a, b)\| < \delta \quad & \text{implies} \quad |f(x, y) - f(a, b)| < \delta_1 \\ & \text{implies} \quad |g(f(x, y)) - g(f(a, b))| < \epsilon \end{aligned}$$

or equivalently,

$$\|(x, y) - (a, b)\| < \delta \quad \text{implies} \quad |(g \circ f)(x, y) - (g \circ f)(a, b)| < \epsilon$$

Consequently, by definition of a limit,

$$\lim_{(x,y) \to (a,b)} (g \circ f)(x, y) = (g \circ f)(a, b)$$

which proves that $g \circ f$ is continuous at $(a, b)$.

$\square$

A question appears in Mobius

## Applying the Continuity Theorems

Let's see how we use basic functions, operations on functions, and the Continuity Theorems to prove that functions are continuous.

**Example 1**
Prove that $h(x, y) = \sin(6x^2y + 3xy^2)$ is continuous for all $(x, y) \in \mathbb{R}^2$.
**Solution:**
By applying Continuity Theorem 1 to the constant function and power functions, it follows that

$$f(x, y) = 6x^2y + 3xy^2 \tag{*}$$

is continuous for all $(x, y) \in \mathbb{R}^2$. Continuity Theorem 3, with $g(\cdot) = \sin(\cdot)$ and $f$ as in equation $(*)$, now implies that $h$ is continuous for all $(x, y) \in \mathbb{R}^2$.

A question appears in Mobius

A question appears in Mobius

The examples and Your Turn exercises in this lesson show that by using the Continuity Theorems, we can often prove the continuity of a given function "by inspection". However, we cannot always apply the Continuity Theorems. At certain points where the Continuity Theorems cannot be applied, we still have to use the definition of continuity in order to determine whether or not the function is continuous. Here is an example:

**Example 2**
Discuss the continuity of the function $f$ defined by

$$f(x,y) = \begin{cases} \dfrac{e^{xy}-1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

**Solution:**
For $(x,y) \neq (0,0)$ the Continuity Theorems immediately imply that $f$ is continuous at these points.
Observe the point $(0,0)$ is singled out in the definition of the function. Thus, the continuity theorems cannot be applied at $(0,0)$ and so we have to use the definition. That is, we have to determine whether

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$$

On the line $y = x$ we get

$$\lim_{(x,y)\to(0,0)} f(x,x) = \lim_{x\to 0} \frac{e^{x^2}-1}{2x^2} = \lim_{x\to 0} \frac{2xe^{x^2}}{4x} = \lim_{x\to 0} \frac{e^{x^2}}{2} = \frac{1}{2}$$

by L'Hôpital's rule. It follows that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not equal $f(0,0)$, and hence by definition, $f$ is not continuous at $(0,0)$.

A question appears in Mobius

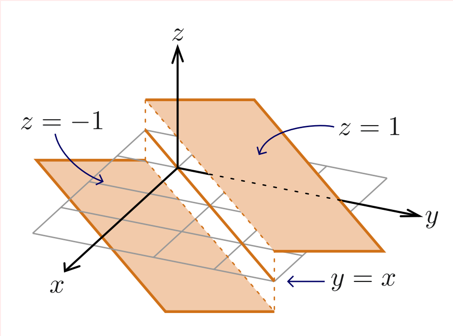
**Example 3**
Discuss the continuity of the function $f$ defined by

$$f(x,y) = \begin{cases} \dfrac{|y-x|}{y-x} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

**Solution:**
For points $(x,y)$ with $x \neq y$ the Continuity Theorems immediately imply that $f$ is continuous at these points. We can not apply the Continuity Theorems at the points $(x,y)$ with $x = y$. Consider any one of these points and denote it by $(a,a)$.
If $(x,y)$ approaches $(a,a)$ with $y - x > 0$, then $|y-x| = y - x$, and $f(x,y)$ approaches (and in fact equals) 1. On the other hand, if $(x,y)$ approaches $(a,a)$ with $y - x < 0$, then $f(x,y)$ approaches $-1$. Thus,

$$\lim_{(x,y)\to(a,a)} f(x,y)$$

does not exist. So, by definition of continuity, $f$ is not continuous at $(a,a)$.
The geometric interpretation is simple. The graph of $f$ consists of two parallel half-planes which form a "step" along the line $y = x$.



## 3.3 - Putting It All Together

### Worked Example 1

Let $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{|x| + |y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

**A question appears in Mobius**

**A question appears in Mobius**

A question appears in Mobius

A question appears in Mobius

## Application

## Limits Revisited

So far in this unit, we have shown how to prove that a function is continuous at a point "by inspection" by using the Continuity Theorems. This makes it easy to evaluate $\lim_{(x,y)\to(a,b)} f(x,y)$ if $f$ is continuous at $(a,b)$. In particular, if $f$ is continuous at $(a,b)$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ can be evaluated simply by evaluating $f(a,b)$.

**Example 1**

Define $f(x,y) = \dfrac{\cos\sqrt{x^2+y^2}}{x^2+y^2}$, for $(x,y) \neq (0,0)$. Evaluate

$$\lim_{(x,y)\to(\pi,0)} f(x,y)$$

**Solution:**
By the Continuity Theorems, $f$ is continuous for all $(x,y) \neq (0,0)$. Thus, by definition of continuity, since $(\pi,0) \neq (0,0)$

$$\lim_{(x,y)\to(\pi,0)} f(x,y) = f(\pi,0) = \frac{\cos\sqrt{\pi^2+0^2}}{\pi^2+0^2} = -\frac{1}{\pi^2}$$

A question appears in Mobius

**Remark**
When applying the Squeeze Theorem, we have to prove that $\lim\limits_{(x,y)\to(a,b)} B(x,y) = 0$. Ideally, we want to evaluate this limit by inspection, so we try to set up the inequality in the Squeeze Theorem so that $B(x,y)$ is continuous at $(a,b)$.

## Practice Problems

Try to answer the questions. If you are having trouble, check for a hint before looking at the solutions.

1. Let $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
   Determine all points where $f$ is continuous. [*Hint provided below*]
2. For each function $f$, determine (with proof) whether or not $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists. If the limit exists, define $f(0,0)$ so as to make the function continuous at $(0,0)$, if possible.
   (a) $f(x,y) = \dfrac{x^3-2y^3}{x^2+2y^2}$
   (b) $f(x,y) = \dfrac{xy^3}{x^2+y^6}$

(c) $f(x,y) = \dfrac{2|x| - |y|}{|x| + 2|y|}$

(d) $f(x,y) = \dfrac{x^2 - 6y^2}{|x| + 3|y|}$

(e) $f(x,y) = \dfrac{\sin(x^2 + 2y^2)}{x^2 + y^2}$

(f) $f(x,y) = \dfrac{y^2 - 4|y| - 2|x|}{|x| + 2|y|}$

## Hint

*Additional content appears in Mobius.*

## Select Answers and Solutions

1. By the continuity theorems, $f(x,y)$ is continuous for all $(x,y) \neq (0,0)$.

   For continuity at $(0,0)$, we need to evaluate

   $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

   .

   Approaching along the $x$-axis (where $y = 0$), we get

   $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

   Approaching along $y = x$, we get

   $$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} &= \lim_{x\to 0} \frac{x^2 - x^2}{x^2 + x^2} \\ &= \lim_{x\to 0} \frac{0}{2x^2} \\ &= 0 \end{aligned}$$

   Since we get two different limits along two different paths, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

   Hence, $f(x,y)$ is **not** continuous at $(0,0)$.

   Therefore, $f(x,y)$ is continuous for all $(x,y) \neq (0,0)$.

2. (a) Define $f(0,0) = 0$.
   (b) Since the limit as $(x,y)$ approach to $(0,0)$ does not exist, it is not possible to make $f$ continuous at $(0,0)$.
   (c) Since the limit as $(x,y)$ approach to $(0,0)$ does not exist, it is not possible to make $f$ continuous at $(0,0)$.
   (d) Define $f(0,0) = 0$.
   (e) Since the limit as $(x,y)$ approach to $(0,0)$ does not exist, it is not possible to make $f$ continuous at $(0,0)$.
   (f) Define $f(0,0) = -2$.

# Unit 4

# The Linear Approximation and Partial Derivatives

## 4.1 - Partial Derivatives

### Partial Derivatives

A scalar function $f(x, y)$ can be differentiated in two natural ways:

- By treating $y$ as a constant and differentiating with respect to $x$ to obtain $\dfrac{\partial f}{\partial x}$.
- By treating $x$ as a constant and differentiating with respect to $y$ to obtain $\dfrac{\partial f}{\partial y}$.

The derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are called the (first) **partial derivatives** of $f$; we also use the notation $f_x$ and $f_y$.

Here is the formal definition.

**Definition: Partial Derivatives**
The **partial derivatives** of $f(x, y)$ are defined by

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided that these limits exist.

Remark
It is sometimes convenient to use **operator notation** $D_1 f$ and $D_2 f$ for the partial derivatives of $f(x, y)$. The notation $D_1 f$ means: differentiate $f$ with respect to the variable in the first position, holding the other fixed. If the independent variables are $x$ and $y$, then

$$D_1 f = \frac{\partial f}{\partial x} = f_x, \qquad D_2 f = \frac{\partial f}{\partial y} = f_y$$

Sometimes, we relax the notation for the partial derivatives and use $\frac{\partial f}{\partial x}$ for $\frac{\partial f}{\partial x}(x, y)$, and similarly $\frac{\partial f}{\partial y}$ for $\frac{\partial f}{\partial y}(x, y)$.

Typically, we try to calculate the partial derivatives by using the standard rules for differentiation of functions of one variable. However, if these cannot be applied, then we must use the definition of the partial derivatives.

**Example 1**
Consider the function $f$ defined by $f(x, y) = xe^{kxy}$ where $k$ is a constant.
Determine $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
**Solution:**
By using the product rule and chain rule for differentiation, we have

$$\frac{\partial f}{\partial x} = (1)e^{kxy} + xe^{kxy}(ky) = (1 + kxy)e^{kxy}$$

and

$$\frac{\partial f}{\partial y} = xe^{kxy}(kx) = kx^2 e^{kxy}$$

A question appears in Mobius

*A slideshow appears in Mobius.*

Slide

## Example 2

Determine whether $\dfrac{\partial f}{\partial x}(0,0)$ exists for $f(x,y) = (x^3 + y^3)^{1/3}$.

**Solution:**

**Step 1:** Find $\dfrac{\partial f}{\partial x}$.

We use single-variable differentiation rules.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{3}(x^3 + y^3)^{-2/3}(3x^2) \\ &= \frac{x^2}{(x^3 + y^3)^{2/3}}\end{aligned}$$

**Step 2:** Determine whether $\dfrac{\partial f}{\partial x}$ exists at $(0,0)$.

Note that $\dfrac{\partial f}{\partial x}$ is only defined for $(x,y)$ such that $x^3 + y^3 \neq 0$.

The partial derivative is **not defined** at the point $(x,y) = (0,0)$.

Slide

## Example 2 Continued

Determine whether $\dfrac{\partial f}{\partial x}(0,0)$ exists for $f(x,y) = (x^3 + y^3)^{1/3}$.

**Solution:**

To find $\dfrac{\partial f}{\partial x}(0,0)$ we must use the definition of partial derivatives.

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

At point $(x,y) = (0,0)$, we have

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{(h^3 + 0^3)^{1/3} - 0}{h} \\ &= \lim_{h \to 0} \frac{h}{h} \\ &= 1\end{aligned}$$

Therefore, $\dfrac{\partial f}{\partial x}(0,0)$ exists and $\dfrac{\partial f}{\partial x}(0,0) = 1$.

**Example 3**

Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

Calculate $f_x(0,0)$ and $f_y(0,0)$.

**Solution:**

Since $f$ changes definition at $(0,0)$, we must use the definition of the partial derivatives. We get

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{h(0)}{h^2+0^2} - 0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{0(h)}{0^2+h^2} - 0}{h} = 0$$

**Remark**

In the unit on continuous functions, when we looked at the definition of a continuous function, we showed that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$. Here, we have just shown that its partial derivatives exist!

This demonstrates that the concept of partial derivatives **do not match** our concept of differentiability for functions of one variable, where differentiability implies continuity. We will take a closer look at this in the unit on differentiability

A question appears in Mobius

A question appears in Mobius

## Generalization

We can extend what we have done for scalar functions of two variables to scalar functions of $n$ variables $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$. To take the partial derivative of $f$ with respect to its $i$-th variable, we hold **all the other variables constant**, and differentiate with respect to the $i$-th variable

**Example 4**
Let $f(x, y, z) = xy^2z^3$. Find $f_x$, $f_y$, and $f_z$.
**Solution:**
To find $f_x$, we treat $y$ and $z$ as constants, and differentiate with respect to $x$

$$f_x(x, y, z) = y^2z^3$$

To find $f_y$, we treat $x$ and $z$ as constants, and differentiate with respect to $y$

$$f_y(x, y, z) = 2xyz^3$$

To find $f_z$, we treat $x$ and $y$ as constants, and differentiate with respect to $z$

$$f_z(x, y, z) = 3xy^2z^2$$

### Your Turn

Use the definition of partial derivatives provided earlier to write the precise definition of $f_x$, $f_y$, and $f_z$ for $f(x, y, z)$.

A question appears in Mobius

## 4.2 - Higher-Order Partial Derivatives

### Second Partial Derivatives

Observe that the partial derivatives of a scalar function of two variables are also scalar functions of two variables. Therefore, we can take the partial derivatives of the partial derivatives of any scalar function.

Since both of the partial derivatives of $f$ have two partial derivatives, there are four possible second partial derivatives of $f$. They are:

- $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$, i.e., differentiate $\dfrac{\partial f}{\partial x}$ with respect to $x$, with $y$ fixed.
- $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$, i.e., differentiate $\dfrac{\partial f}{\partial x}$ with respect to $y$, with $x$ fixed.
- $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$, i.e., differentiate $\dfrac{\partial f}{\partial y}$ with respect to $x$, with $y$ fixed.
- $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$, i.e., differentiate $\dfrac{\partial f}{\partial y}$ with respect to $y$, with $x$ fixed.

Remark
It is often convenient to use the subscript notation or the operator notation:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = D_1^2 f, \qquad \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_2 D_1 f$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_1 D_2 f, \qquad \frac{\partial^2 f}{\partial y^2} = f_{yy} = D_2^2 f$$

**Example 1**
Let $k$ be a constant. Find all the second partial derivatives of $f(x,y) = xe^{kxy}$.
**Solution:**
We first calculate the first partial derivatives. We have

$$\frac{\partial f}{\partial x}(x,y) = e^{kxy} + kxye^{kxy}$$
$$\frac{\partial f}{\partial y}(x,y) = kx^2e^{kxy}$$

Thus, we get

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left[e^{kxy} + kxye^{kxy}\right] = 2kye^{kxy} + k^2xy^2e^{kxy}$$
$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left[e^{kxy} + kxye^{kxy}\right] = 2kxe^{kxy} + k^2x^2ye^{kxy}$$
$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left[kx^2e^{kxy}\right] = 2kxe^{kxy} + k^2x^2ye^{kxy}$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left[kx^2e^{kxy}\right] = k^2x^3e^{kxy}$$

**Your Turn 1**

Find all the second partial derivatives of $f(x,y) = \cos(2x + y^2)$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

In the previous examples, you may have noticed that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

This is in fact a general property of partial derivatives, subject to a continuity requirement, as follows.

Theorem 1: Clairaut's Theorem
If $f_{xy}$ and $f_{yx}$ are defined in some neighborhood of $(a, b)$ and are both continuous at $(a, b)$, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

The proof of Clairaut's Theorem is rather technical, and is beyond the scope of this course.

A question appears in Mobius

A question appears in Mobius

## Higher-Order Partial Derivatives

We can take higher-order partial derivatives in the expected way. In particular, observe that $f(x, y)$ has eight third partial derivatives. They are:

$$f_{xxx}, \quad f_{xxy}, \quad f_{xyx}, \quad f_{xyy}, \quad f_{yxx}, \quad f_{yxy}, \quad f_{yyx}, \quad f_{yyy}$$

Clairaut's Theorem also extends to higher-order partial derivatives: if the higher-order partial derivatives are defined in a neighborhood of a point $(a, b)$ and are continuous at $(a, b)$, then $f_{i_1,\ldots,i_k} = f_{j_1,\ldots,j_k}$ whenever $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ are tuples of indices which are re-arrangements of each other.

For example, if the partial derivatives of $f$ satisfy Clairaut's Theorem, then

$$f_{xxy}(a,b) = f_{xyx}(a,b) = f_{yxx}(a,b)$$

In many situations, we will want to require that a function have continuous partial derivatives of some order. Let's introduce a bit of terminology.

If the $k$-th partial derivatives of $f(x_1, \ldots, x_n)$ are continuous, then we write

$$f \in C^k$$

and say "$f$ is in class $C^k$."

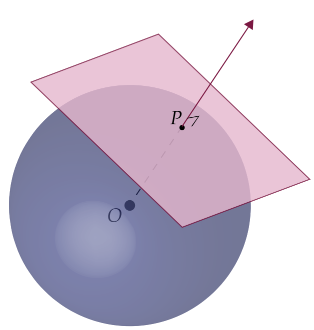
Having $f(x,y) \in C^2$, for example, means that $f$ has continuous second partial derivatives, and therefore, by Clairaut's Theorem, that $f_{xy} = f_{yx}$. More generally, $f(x,y) \in C^k$ means that $f$ has continuous $k$-th partial derivatives and that the mixed higher-order partial derivatives are equal regardless of the order in which they are taken.

A question appears in Mobius

## 4.3 - The Tangent Plane

### The Tangent Plane

The surface of a sphere has a tangent plane at each point $P$, namely, the plane through $P$ that is orthogonal to the line joining $P$ and the centre $O$. The tangent plane at $P$ can be thought of as the plane which best approximates the surface of the sphere near $P$.



This concept can be generalized to a surface defined by an equation of the form

$$z = f(x, y)$$

Let $C_1$ be the cross section $y = b$ of the surface, that is, $C_1$ is given by

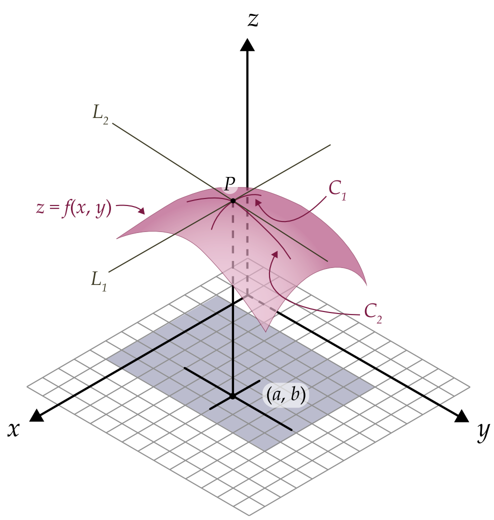
$$z = f(x, b)$$

It follows that $\dfrac{\partial f}{\partial x}(a, b)$ equals the slope of the tangent line $L_1$ of $C_1$ at the point $P(a, b, f(a, b))$.

Similarly, let $C_2$ be the cross section $x = a$ of the surface, that is, $C_2$ is given by

$$z = f(a, y)$$

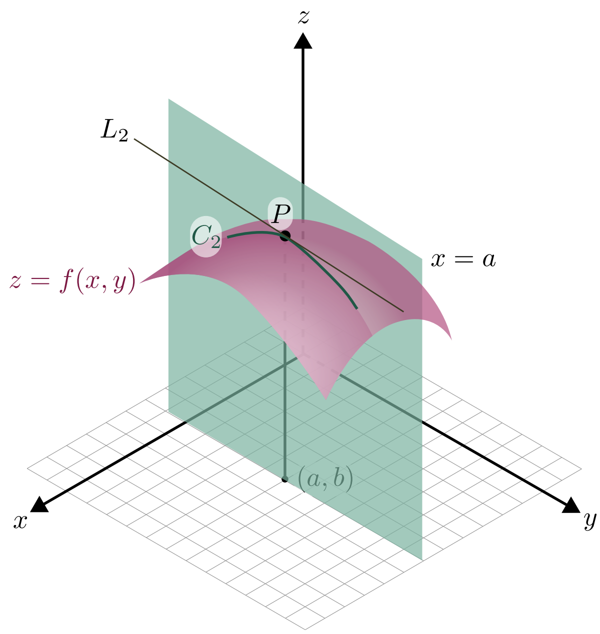
Again, it follows that $\dfrac{\partial f}{\partial y}(a, b)$ equals the slope of the tangent line $L_2$ of $C_2$ at the point $P(a, b, f(a, b))$.




*A slideshow appears in Mobius.*

Slide

## Geometric Interpretation of Partial Derivatives




$\dfrac{\partial f}{\partial x}(a, b)$ equals the slope of the tangent line $L_1$ to $C_1$ at the point $P$.
$\dfrac{\partial f}{\partial y}(a, b)$ equals the slope of the tangent line $L_2$ to $C_2$ at the point $P$.

We provisionally define the tangent plane to the surface $z = f(x, y)$ at the point $P(a, b, f(a, b))$ to be the plane which contains the tangent lines $L_1$ and $L_2$.

In order to derive the equation of the tangent plane, we note that any (non-vertical) plane through the point $P(a, b, f(a, b))$ has an equation of the form

$$z = f(a, b) + m(x - a) + n(y - b)$$

where $m$ and $n$ are constants. The intercept of this plane with the vertical plane $y = b$ is the line

$$z = f(a, b) + m(x - a) \tag{*}$$

We require this line to coincide with $L_1$. Thus the slope $m$ of the line $(*)$ must equal the slope $\dfrac{\partial f}{\partial x}(a, b)$ of the line $L_1$:

$$m = \frac{\partial f}{\partial x}(a, b)$$

A similar argument yields

$$n = \frac{\partial f}{\partial y}(a, b)$$

We make the following definition which we will formalize later.

**Definition: Tangent Plane**
The **tangent plane** to $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

For now, to make sure the tangent plane exists, we will assume that we have a differentiable function $f(x, y)$. (We will define differentiability formally in the next unit.)

**Your Turn 1**

The tangent plane at a point P is shown for the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.

Apply the definition of the tangent plane at the point $P(x, y, f(x, y))$ for different points $P$.

**Instructions:**

1. Click and drag the point $P$ to a new location.
2. Observe the changes in the partial derivatives and the equation of the tangent plane to $f$ at $P$.

External resource: https://www.geogebra.org/material/iframe/id/cbwkbgs6/



A question appears in Mobius

Remark
In the previous question, you should note that a tangent plane does not exist at the vertex $(0, 0, 0)$ of the cone, since the cone has a “cusp” there. We shall discuss the question of the existence of a tangent plane in coming lessons.

**Your Turn 3**

As you move the point A in the positive x and y direction along the following surface, observe the sign of the partial derivative with respect to x and y respectively.

**Instructions**

1. Click the button to place the point A on the $y = 0$ cross section.
2. Move the point A along the cross section $y = 0$ towards the positive $x$ axis. When the point A moves downhill on the graph towards the positive x axis, observe that the partial derivative with respect to x at point A is negative.
3. After you pass the origin, keep moving along the cross section $y = 0$ in the positive $x$ direction. The point A now moves uphill on the graph. Observe that now the partial derivative with respect to $x$ at point A becomes positive.You may wish to view the graph from a different angle, click and hold on the image and then move your cursor to rotate.
4. Next, click the button to place the point A on the $x = 0$ cross section.
5. Now move the point along the cross section $x = 0$ from the origin towards the positive $y$ axis. When the point $A$ moves downhill towards the positive $y$ axis, observe that the partial derivative with respect to $y$ at point $A$ is negative. You may need to rotate the figure so you can see the cross section $x = 0$.
6. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/u2uc58xt/



## 4.4 - Linear Approximation for $z = f(x, y)$

### Review of the 1D Case

For a function $f(x)$ the tangent line can be used to approximate the graph of the function near the point of tangency. Recall that the equation of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a)$$

The function $L_a$ defined by

$$L_a(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of $f$ at $a$ since $L_a(x)$ approximates $f(x)$ for $x$ sufficiently close to $a$.

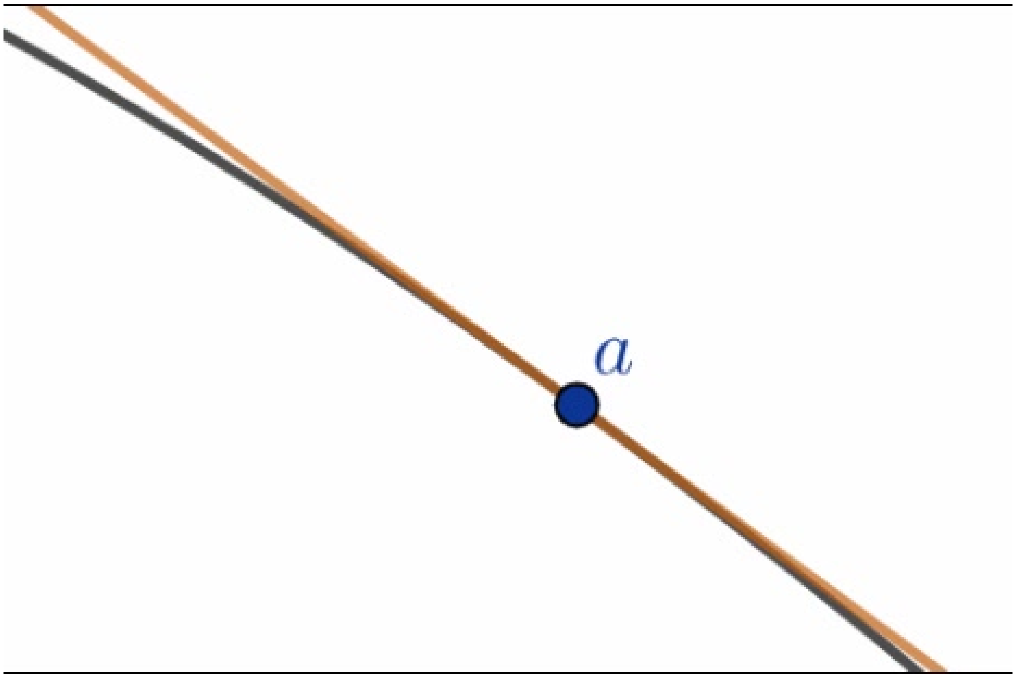
For $x$ sufficiently close to $a$, the approximation

$$f(x) \approx L_a(x)$$

is called the **linear approximation** of $f$ at $a$.

The animation below (no audio) demonstrates this concept. In the animation we zoom in on the point $a$ and observe that the linear approximation $L_a(x)$ gets very close to the graph of $f(x)$ as we approach the point of tangency $a$.

*A video appears here.*




*Additional content appears in Mobius.*

A question appears in Mobius

## The 2D Case

For a multivariable differentiable function $f(x, y)$, the tangent plane can be used to approximate the surface $z = f(x, y)$ near a point of tangency $P(a, b, f(a, b))$. Similar to the 1D-case, as we zoom in to the neighborhood of the point of tangency $P$, the surface $z = f(x, y)$ and the tangent to the plane at $P$, namely $L_{(a,b)}(x, y)$ will become very similar, almost indistinguishable. Let's formally define the linearization and linear approximation first.

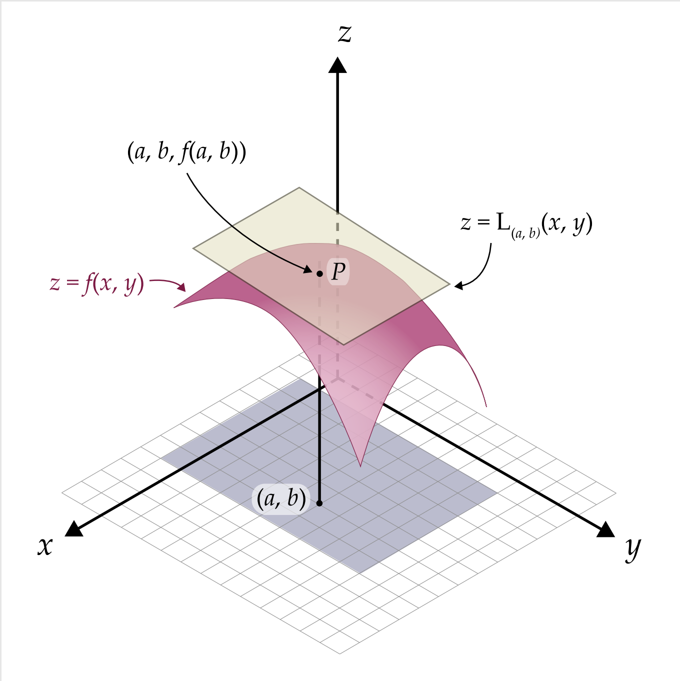
**Definition: Linearization and Linear Approximation**
For a function $f(x, y)$ we define the **linearization** $L_{(a,b)}(x, y)$ of $f$ at $(a, b)$ by

$$L_{(a,b)}(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

We call the approximation

$$f(x, y) \approx L_{(a,b)}(x, y)$$

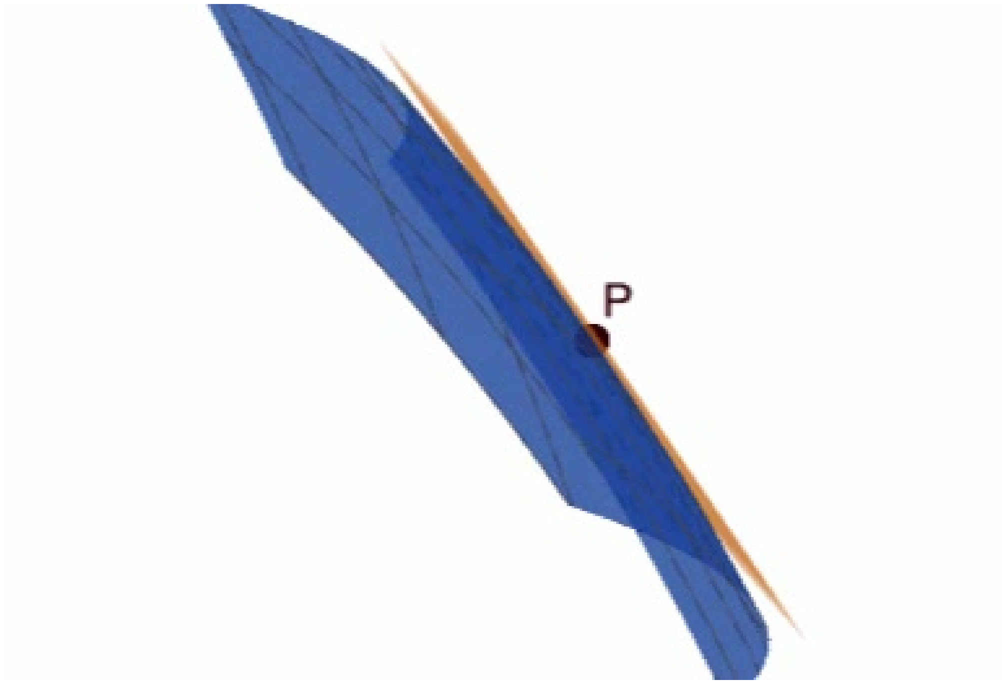
the **linear approximation** of $f(x, y)$ at $(a, b)$




In this figure, the surface $z = f(x, y)$ and the tangent plane at the point of tangency $P(a, b, f(a, b))$, namely linearization of $L_{(a,b)}(x, y)$ at of $f$ at $(a, b)$, are shown together.

The animation below (no audio) demonstrates how the linear approximation $L_{(a,b)}(x, y)$ gets very close to the graph of $f(x, y)$ as we approach the point of tangency, $P(a, b, f(a, b))$.

*A video appears here.*




*Additional content appears in Mobius.*

*A slideshow appears in Mobius.*

Slide

## Example 1

Use the linear approximation to approximate $\sqrt{(0.95)^3 + (1.98)^3}$.
**Solution:**
We choose

$$f(x, y) = \sqrt{x^3 + y^3} \quad \text{and} \quad (a, b) = (1, 2)$$

Recall the definition of linear approximation

$$L_{(a,b)}(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

So, we need to find three things:

1. $f(a, b)$,
2. $\dfrac{\partial f}{\partial x}(a, b)$, and
3. $\dfrac{\partial f}{\partial y}(a, b)$

Slide

## Example 1 Continued

Use the linear approximation to approximate $\sqrt{(0.95)^3 + (1.98)^3}$.
**Solution:**
We choose

$$f(x, y) = \sqrt{x^3 + y^3}, \quad \text{and} \quad (a, b) = (1, 2)$$

### Step 1

Find $f(a, b)$.
$f(1, 2) = \boxed{\phantom{xxxx}}$

Slide

## Example 1 Continued

Next, we find the partial derivatives of $f$.

### Step 2

Find $\dfrac{\partial f}{\partial x}(a, b)$.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\sqrt{x^3 + y^3} \\ &= \frac{\partial}{\partial x}(x^3 + y^3)^{1/2} \\ &= \frac{1}{2}(x^3 + y^3)^{-1/2}(3x^2) \\ &= \frac{3x^2}{2\sqrt{x^3 + y^3}}\end{aligned}$$

### Step 3

Find $\dfrac{\partial f}{\partial y}(a, b)$.

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\sqrt{x^3 + y^3} \\ &= \frac{\partial}{\partial y}(x^3 + y^3)^{1/2} \\ &= \frac{1}{2}(x^3 + y^3)^{-1/2}(3y^2) \\ &= \frac{3y^2}{2\sqrt{x^3 + y^3}}\end{aligned}$$

Slide

## Example 1 Continued

Therefore, we have

$$\frac{\partial f}{\partial x} = \frac{3x^2}{2\sqrt{x^3 + y^3}}, \qquad \frac{\partial f}{\partial y} = \frac{3y^2}{2\sqrt{x^3 + y^3}}$$

Finally, we evaluate the partial derivatives and $f(x, y)$ at $(1, 2)$:

$f_x(1, 2) = \boxed{\phantom{xxxx}}$ $f_y(1, 2) = \boxed{\phantom{xxxx}}$

Slide

## Example 1 Continued

Now we have

1. $f(1, 2) = 3$,
2. $f_x(1, 2) = 1/2$, and
3. $f_y(1, 2) = 2$.

Thus, the linear approximation is

$$\begin{aligned} f(x, y) &\approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \\ &\approx f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) \\ &\approx 3 + \frac{1}{2}(x - 1) + 2(y - 2) \end{aligned}$$

So, $f(x, y) \approx 3 + \frac{1}{2}(x - 1) + 2(y - 2)$ near $(1, 2)$.
Hence,

$$\begin{aligned} \sqrt{(0.95)^3 + (1.98)^3} &= f(0.95, 1.98) \\ &\approx 3 + \frac{1}{2}(0.95 - 1) + 2(1.98 - 2) \\ &\approx 3 + \frac{1}{2}(-0.05) + 2(-0.02) \\ &= 2.935 \end{aligned}$$

Note: The calculator value is 2.935943.

Remark
Resist the temptation to expand the brackets and simplify in the equation for the linearization. The bracketed terms represent small increments, and it is helpful to keep them separate.

A question appears in Mobius

A question appears in Mobius

## Increment Form of the Linear Approximation

Suppose that we know $f(a, b)$ and want to calculate $f(x, y)$ at a nearby point. Let

$$\Delta x = x - a, \quad \Delta y = y - b$$

and

$$\Delta f = f(x, y) - f(a, b)$$

The linear approximation is

$$f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

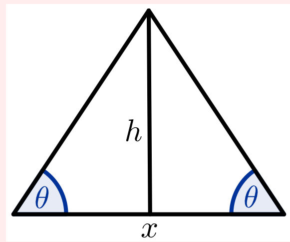
This can be rearranged to yield

$$\Delta f \approx \frac{\partial f}{\partial x}(a, b)\Delta x + \frac{\partial f}{\partial y}(a, b)\Delta y \tag{*}$$

This gives an approximation for the change $\Delta f$ in $f(x, y)$ due to a change $(\Delta x, \Delta y)$ away from the point $(a, b)$.

We shall refer to equation $(*)$ as the **increment form** of the linear approximation.

**Example 2**
An isosceles triangle has base 4 m, and equal angles of $\pi/4$ . If the base is increased by 16 cm, and the equal angles are decreased by 0.1 radians, estimate the change in the area of the triangle.
**Solution:**
Let $x$ be the length of the base of an isosceles triangle, $\theta$ be the measure of the equal angles, and $h$ be the height of the triangle.




Then the area function can be written as

$$f(x,\theta) = \frac{1}{2}xh = \frac{1}{2}x\left(\frac{x}{2}\tan\theta\right) = \frac{1}{4}x^2\tan\theta$$

Recall that

$$\Delta f \approx \frac{\partial f}{\partial x}(a,b)(\Delta x) + \frac{\partial f}{\partial y}(a,b)(\Delta y)$$

Note that the change in $x$ is $\Delta x = 16$ cm $= 0.16$ m and the change in $\theta$ is $\Delta\theta = -0.1$ radians.
Also, $f_x = \dfrac{1}{2}x\tan\theta$ and $f_\theta = \dfrac{1}{4}x^2\sec^2\theta$. So we get $f_x(4,\pi/4) = 2$ and $f_\theta(4,\pi/4) = 8$.
Using the increment form of the linear approximation, we have

$$\Delta f \approx \frac{\partial f}{\partial x}(a,b)(\Delta x) + \frac{\partial f}{\partial y}(a,b)(\Delta y) = 2(0.16) - 8(0.1) = -0.48$$

Therefore, the area decreases approximately by 0.48 m$^2$.

## 4.5 - Linear Approximation in Higher Dimensions

### Linear Approximation in $\mathbb{R}^3$

Consider a function $f(x,y,z)$. By analogy with the case of a function of two variables, we define the linearization of $f$ at $\vec{a} = (a,b,c)$ by

$$L_{\vec{a}}(x,y,z) = f(\vec{a}) + f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b) + f_z(\vec{a})(z-c)$$

The notation is becoming a bit tedious, but we can improve things by noting that the final three terms can be represented by the dot product of the vectors

$$(x-a, y-b, z-c) = (x,y,z) - (a,b,c), \quad \text{and} \quad \nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$$

since $(x-a, y-b, z-c)\cdot(f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a})) = f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b) + f_z(\vec{a})(z-c)$.

The vector $\nabla f(\vec{a})$ is called the **gradient** of $f$ at $\vec{a}$.

Using our new compact notation, we have the following definitions:

**Definition: Gradient**
Suppose that $f(x,y,z)$ has partial derivatives at $\vec{a} \in \mathbb{R}^3$. The **gradient** of $f$ at $\vec{a}$ is defined by

$$\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$$

**Definition: Linearization and Linear Approximation**
Suppose that $f(\vec{x})$, $\vec{x} \in \mathbb{R}^3$, has partial derivatives at $\vec{a} \in \mathbb{R}^3$.
The **linearization** of $f$ at $\vec{a}$ is defined by

$$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

The **linear approximation** of $f$ at $\vec{a}$ is

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

**Example 1**
Consider the function $f$ defined by

$$f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$$

Find the gradient of $f$ and the linear approximation for $f$ at $\vec{a} = (1, 2, -2)$.
**Solution:**
Differentiate to obtain

$$\nabla f(x,y,z) = \left( \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right)$$

Now, evaluate $\nabla f(x,y,z)$ at $\vec{a} = (1, 2, -2)$ to get

$$\nabla f(\vec{a}) = \left( \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right)$$

Thus,

$$\begin{aligned} L_{\vec{a}}(\vec{x}) &= f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) \\ &= 3 + \left( \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right) \cdot (x-1, y-2, z+2) \\ &= 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) - \frac{2}{3}(z+2) \end{aligned}$$

So, the linear approximation for $f$ at $(1, 2, -2)$ is

$$f(x,y,z) \approx 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) - \frac{2}{3}(z+2)$$

A question appears in Mobius

## Linear Approximation in $\mathbb{R}^n$

Recall that **linearization** of $f$ at $\vec{a}$ is defined by

$$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

and the **linear approximation** of $f$ at $\vec{a}$ is

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

The advantage of using vector notation is that the above equations hold for a function of $n$ variables $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$. For an arbitrary vector $\vec{a} \in \mathbb{R}^n$, we have

$$\Delta\vec{x} = \vec{x} - \vec{a} = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$$

and we define the gradient of $f$ at $\vec{a}$ to be

$$\nabla f(\vec{a}) = (D_1 f(\vec{a}), D_2 f(\vec{a}), \cdots, D_n f(\vec{a}))$$

Then, the increment form of the linear approximation for $f(\vec{x})$ is

$$\Delta f \approx \nabla f(\vec{a}) \cdot \Delta\vec{x}$$

Observe that this formula even works when $n = 1$. That is, for a function $g(t)$ of one variable this gives $\nabla g(a) = g'(a)$ and the increment form of the linear approximation is

$$\Delta g \approx \nabla g(a) \cdot \Delta x = g'(a)(x - a)$$

which is our familiar formula from single variable calculus.

For functions of two variables $f(x, y)$, we have $\nabla f(a, b) = (f_x(a, b), f_y(a, b))$ and the increment form of the linear approximation is

$$\Delta f \approx \nabla f(a, b) \cdot \Delta(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

which matches our previous work. We see that our generalization is indeed a true generalization.

## 4.6 - Putting It All Together

### Worked Example 1

Let $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{\ln(x^2+y^2+1)} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

A question appears in Mobius

A question appears in Mobius

### Worked Example 2

The temperature of a metal rod at position $x$, $0 \leq x \leq 1$, and at time $t$, $t \geq 0$ is given by $u(t,x) = 100e^{-t}\sin \pi x$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

e. Illustrate these rates of change by sketching the cross-sections $x = 3/4$ and $t = 1$.

A question appears in Mobius

A question appears in Mobius

## Application

In this example, we will connect some of the concepts we learned to real life.

If three resistors $R_1, R_2, R_3$ are connected in parallel, the total electrical resistance $R$ is determined by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If $R_1, R_2$ and $R_3$ initially equal 100, 200 and 300 ohms, and are increased by 1, 2, and 4 ohms, respectively, use the linear approximation to calculate the change in $R$.

**A question appears in Mobius**

b. We will use implicit differentiation to find the partial derivative of $R$ with respect to $R_1$.

Find $\dfrac{\partial R}{\partial R_1}$.

*A slideshow appears in Mobius.*

**Slide**

## Implicit Differentiation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

We will find the partial derivative of $R$ with respect to $R_1$ using implicit differentiation.

$$\begin{aligned}
\frac{\partial}{\partial R_1}\frac{1}{R} &= \frac{\partial}{\partial R_1}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \\
-\frac{1}{R^2}\frac{\partial R}{\partial R_1} &= \frac{\partial}{\partial R_1}\left(\frac{1}{R_1}\right) \\
-\frac{1}{R^2}\frac{\partial R}{\partial R_1} &= -\frac{1}{R_1^2} \\
(-\cancel{R^2})\left(-\frac{1}{\cancel{R^2}}\frac{\partial R}{\partial R_1}\right) &= (-R^2)\left(-\frac{1}{R_1^2}\right) \\
\frac{\partial R}{\partial R_1} &= \frac{R^2}{R_1^2}
\end{aligned}$$

c. By symmetry, we can also find $\dfrac{\partial R}{\partial R_2}$ and $\dfrac{\partial R}{\partial R_3}$ easily.

Now, find $\dfrac{\partial R}{\partial R_2}$ and $\dfrac{\partial R}{\partial R_3}$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Practice Problems

1. Let $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

   (a) Determine all points where $f(x,y)$ is continuous.

   (b) Find $f_x(0,0)$ and $f_y(0,0)$.

2. Find a function $g(x,y)$ such that $g(x,y)$ is continuous at $(0,0)$, but $g_x(0,0)$ and $g_y(0,0)$ both do not exist. Justify your answer.

3. Find $f_x(0,0)$ and $f_y(0,0)$ for

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} + 1 & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

4. Find the first and second partial derivatives of

   (a) $f(x,y) = \sqrt{2x^2 - y}$

   (b) $g(x,y) = xe^{x+\cos y}$

5. A function $g$ is defined by $g(x,t) = f(x-3t)$ where $f$ is a function of one variable. If $f'(2) = 3$, calculate $g_x(5,1)$, $g_t(5,1)$. Show that $g_t(x,t) = -3g_x(x,t)$ in general.

6. A function $f$ is defined by $f(x,y) = ye^{\frac{x}{y}}$, for $y \neq 0$. Verify that the second mixed partial derivatives of $f$ are equal:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

## Select Answers and Solutions

1. (a) The function is continuous everywhere

   (b) $f_x(0,0) = 1$ and $f_y(0,0) = 0$

2. The function $g(x,y) = |x| + |y|$ is continuous at $(0,0)$ but $g_x$ and $g_y$ do not exist at $(0,0)$.

3. $f_x(0,0) = 1$ and $f_y(0,0) = -1$.

4. (a) The first partial derivatives are $\dfrac{\partial f}{\partial x} = \dfrac{2x}{\sqrt{2x^2-y}}$ and $\dfrac{\partial f}{\partial y} = \dfrac{-1}{2\sqrt{2x^2-y}}$. The second partial derivatives are $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2}{\sqrt{2x^2-y}} - \dfrac{4x^2}{(2x^2-y)^{3/2}}$, $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{x}{(2x^2-y)^{3/2}}$, and $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{-1}{4(2x^2-y)^{3/2}}$

   (b) The first partial derivatives are $\dfrac{\partial f}{\partial x} = xe^{x+\cos(y)} + e^{x+\cos(y)}$ and $\dfrac{\partial f}{\partial y} = -x\sin(y)e^{x+\cos(y)}$. The second partial derivatives are $\dfrac{\partial^2 f}{\partial x^2} = xe^{x+\cos(y)} + 2e^{x+\cos(y)}$, $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = -\sin(y)e^{x+\cos(y)} - x\sin(y)e^{x+\cos(y)}$, and $\dfrac{\partial^2 f}{\partial y^2} = x\sin^2(y)e^{x+\cos(y)} - x\cos(y)e^{x+\cos(y)}$.

5. $\dfrac{\partial g}{\partial x}(x,t) = f'(x-3t)$ and $\dfrac{\partial g}{\partial t}(x,t) = -3 \cdot f'(x-3t)$

6. The first partial derivatives are $\dfrac{\partial f}{\partial x} = e^{x/y}$ and $\dfrac{\partial f}{\partial y} = e^{x/y} - \dfrac{x}{y}e^{x/y}$. The mixed partial derivatives are $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{-xe^{x/y}}{y^2}$.

# Unit 5

# Differentiable Functions

## 5.1 - Definition of Differentiability

### Definition of Differentiability

Our goal is now to extend the concept of differentiability for functions of one variable to functions of two variables. For a function of one variable, we saw that a function $g(x)$ is differentiable at $x = a$ if $g'(a)$ exists. From this, it is natural to wonder if the existence of partial derivatives is enough to define the concept of differentiability for $f(x, y)$.

Let's have a look at an example.

A question appears in Mobius

A question appears in Mobius

As seen in the example above, the concept of partial derivatives does not match with the concept of differentiability from single-variable calculus. In particular, we just saw a function whose partial derivatives exist at $(0,0)$ even though the function is not continuous at $(0,0)$.

To define the concept of differentiability for $f(x,y)$, we want to ensure that it has the same properties as the concept of differentiability for functions of one variable.

## Differentiability for Functions in One Variable

Let $g(x)$ be a single-variable real valued function. In single-variable calculus, we saw that if $g(x)$ is differentiable at $x = a$, then graph of $g(x)$ has no cusps or jumps at $x = a$, and that the linear approximation is a good approximation. In particular, we define the error in the linear approximation to be the difference between the function $g(x)$ and its approximation $L_a(x)$,

$$R_{1,a}(x) = g(x) - L_a(x)$$

The smaller the error, the better the linear approximation. This leads to the following theorem.

Note: The subscript 1 indicates the first degree approximation as we are using linear approximations.

**Theorem 1**
If $g'(a)$ exists, then $\displaystyle\lim_{x\to a} \frac{|R_{1,a}(x)|}{|x-a|} = 0$ where

$$R_{1,a}(x) = g(x) - L_a(x) = g(x) - g(a) - g'(a)(x-a)$$

**Proof:** We have

$$\frac{|R_{1,a}(x)|}{|x-a|} = \left|\frac{g(x) - g(a) - g'(a)(x-a)}{x-a}\right| = \left|\frac{g(x)-g(a)}{x-a} - g'(a)\right|$$

The result follows from taking the limit as $x \to a$ (the details are left as an exercise).

□

The theorem above says that the error $R_{1,a}(x)$ tends to zero faster than the displacement $|x-a|$. Moreover, the tangent line is the **unique** line that satisfies this property: it can be shown that if we replace the tangent line $y = L_a(x)$ by any other straight line $y = g(a) + m(x-a)$ passing through the point $(a, g(a))$, the error **will not** satisfy the conclusion of the theorem. Thus, the property

$$\lim_{x\to a} \frac{|R_{1,a}(x)|}{|x-a|} = 0$$

characterizes the tangent line at $(a, g(a))$ as **the best straight line approximation** to the graph $y = g(x)$ near $(a, g(a))$.

Therefore, to define differentiability for a function of two variables $f(x, y)$, we match this definition.

## Differentiability for Functions in Two Variables

First, we define the error in the linear approximation to be

$$R_{1,(a,b)}(x, y) = f(x, y) - L_{(a,b)}(x, y)$$

From here, we have the following definition:

**Definition: Differentiable**
A function $f(x, y)$ is **differentiable** at $(a, b)$ if

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x, y)|}{\|(x, y) - (a, b)\|} = 0$$

where

$$R_{1,(a,b)}(x, y) = f(x, y) - L_{(a,b)}(x, y)$$

As in the one dimensional case, we can prove that the only plane of the form $z = f(a, b) + c(x-a) + d(y-b)$ through the point $(a, b, f(a, b))$ that has this property is $z = L_{(a,b)}(x, y)$.

**Theorem 2**
If a function $f(x, y)$ satisfies

$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - f(a, b) - c(x-a) - d(y-b)|}{\|(x, y) - (a, b)\|} = 0$$

for some constants $c$ and $d$ then $c = f_x(a, b)$ and $d = f_y(a, b)$.

**Proof:** Since

$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - f(a, b) - c(x-a) - d(y-b)|}{\|(x, y) - (a, b)\|} = 0$$

the limit is 0 along any path. Therefore, along the path along $y = b$, we get

$$\begin{aligned}
0 &= \lim_{x\to a} \frac{|f(x, b) - f(a, b) - c(x-a) - d(b-b)|}{\|(x, b) - (a, b)\|} \\
&= \lim_{x\to a} \frac{|f(x, b) - f(a, b) - c(x-a)|}{|x-a|} \\
&= \lim_{x\to a} \left| \frac{f(x, b) - f(a, b)}{x-a} - c \right| \\
&= f_x(a, b) - c \\
\Rightarrow c &= f_x(a, b)
\end{aligned}$$

Similarly, approaching along $x = a$ we get that $d = f_y(a, b)$.

□

This implies that the tangent plane gives the best linear approximation to the graph $z = f(x, y)$ near $(a, b)$. Moreover, it tells us that the linear approximation is a "good approximation" **if and only if** $f$ is differentiable at $(a, b)$.

**Remark**
Observe that for the linear approximation to exist at $(a, b)$ both partial derivatives of $f$ must exist at $(a, b)$. However, both partial derivatives existing **does not** guarantee that $f$ will be differentiable. We say that the partial derivatives of $f$ existing at $(a, b)$ is necessary, but not sufficient.

*A slideshow appears in Mobius.*

Slide

## Differentiability and Linear Approximations

We will look at

- a function which is differentiable, and
- a function which is not differentiable.

Slide

## A Differentiable Function

Consider $f(x, y) = xy$, which is a differentiable function.
Recall:

$f(x, y)$ is differentiable at $(a, b)$ if and only if

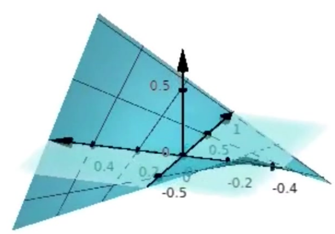
$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - L_{(a,b)}(x, y)|}{\|(x, y) - (a, b)\|} = 0$$

Let's look at how $L_{(0,0)}f(x, y)$ behaves at $(x, y) = (0, 0)$.
We have

$$\begin{aligned} L_{(0,0)}f(x, y) &= f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) \\ &= 0 \end{aligned}$$

So the linear approximation of $f(x, y)$ at $(0, 0)$ is the plane $z = 0$.
The closer we get to $(0, 0)$, the better $L_{(0,0)}$ approximates $f(x, y)$.

*A video appears here.*



Slide

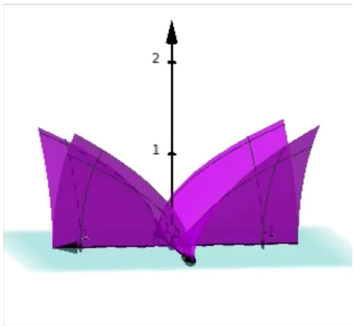
## A Function Which Is Not Differentiable at $(0,0)$

Consider $f(x,y) = \sqrt{|xy|}$, which **not** differentiable at $(0,0)$.
Let's look at how $L_{(0,0)}f(x,y)$ behaves at $(x,y) = (0,0)$.
We have

$$\begin{aligned} L_{(0,0)}f(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= 0 \end{aligned}$$

So the linear approximation of $f(x,y)$ at $(0,0)$ is the plane $z = 0$.
No matter how close we get to $(0,0)$, the linear approximation does not get any better.

*A video appears here.*




### Examples - Part 1

*A slideshow appears in Mobius.*

Slide

## Example 1

Show that $f(x,y) = x^2 + y^2$ is differentiable at $(1,0)$.
**Solution**
$f(x,y)$ is differentiable at $(a,b) = (1,0)$ if

$$\lim_{(x,y)\to(1,0)} \frac{|R_{1,(1,0)}(x,y)|}{\|(x,y)-(1,0)\|} = 0$$

Find

$$\begin{aligned} R_{1,(1,0)}(x,y) &= f(x,y) - L_{(1,0)}(x,y) \\ &= x^2 + y^2 - (1 + 2(x-1)) \\ &= (x-1)^2 + y^2 \end{aligned}$$

We are given that $f(x,y) = x^2 + y^2$.
We know that $L_{(1,0)}(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$

- $f(1,0) = 1$
- $f_x(x,y) = 2x$, so $f_x(1,0) = 2$
- $f_y(x,y) = 2y$, so $f_y(1,0) = 0$

Hence $L_{(1,0)}(x,y) = 1 + 2(x-1)$

Slide

## Example 1 Continued

Show that $f(x,y) = x^2 + y^2$ is differentiable at $(a,b) = (1,0)$
**Solution**
Recall $f(x,y)$ is differentiable at $(a,b) = (1,0)$ if

$$\lim_{(x,y)\to(1,0)} \frac{|R_{1,(1,0)}(x,y)|}{\|(x,y)-(1,0)\|} = 0$$

We found

$$R_{1,(1,0)}(x,y) = (x-1)^2 + y^2$$

Therefore,

$$\frac{|R_{1,(1,0)}(x,y)|}{\|(x,y)-(1,0)\|} = \frac{(x-1)^2+y^2}{\sqrt{(x-1)^2+y^2}} = \sqrt{(x-1)^2+y^2}$$

Hence,

$$\lim_{(x,y)\to(1,0)} \frac{|R_{1,(1,0)}(x,y)|}{\|(x,y)-(1,0)\|} = \lim_{(x,y)\to(1,0)} \sqrt{(x-1)^2+y^2} = 0 \text{ by the Continuity Theorems}$$

Therefore the function is differentiable at $(1,0)$.

Example 2

Determine whether $f(x,y) = \sqrt{|xy|}$ is differentiable at $(0,0)$.

**Solution:**

Recall that a function $f(x,y)$ is differentiable at $(a,b)$ if

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} = 0$$

where

$$\begin{aligned} R_{1,(a,b)}(x,y) &= f(x,y) - L_{(a,b)}(x,y) \\ &= f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b) \end{aligned}$$

In this example, the point is $(a,b) = (0,0)$, so we have

$$\begin{aligned} R_{1,(0,0)}(0,0) &= f(0,0) - L_{(0,0)}(x,y) \\ &= f(x,y) - f(0,0) - f_x(0,0)(x-0) - f_y(0,0)(y-0) \end{aligned}$$

We start by finding $f(0,0)$, $f_x(0,0)$, and $f_y(0,0)$:

- $f(0,0) = 0$
- $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$
- $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$

Next, we find the linear approximation,

$$\begin{aligned} L_{(0,0)}(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= 0 + 0(x-0) + 0(y-0) \\ &= 0 \end{aligned}$$

Now, the error in the linear approximation is

$$R_{1,(0,0)}(x,y) = f(x,y) - L_{(0,0)}(x,y) = \sqrt{|xy|} - 0 = \sqrt{|xy|}$$

and the magnitude of the displacement is $\|(x,y)-(0,0)\| = \sqrt{x^2+y^2}$.

Therefore,

$$\frac{|R_{1,(0,0)}(x,y)|}{\|(x,y)-(0,0)\|} = \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}, \qquad \text{for } (x,y) \neq (0,0)$$

Finally, we must determine whether

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}} = 0 \tag{*}$$

Approaching along the line $y = x$ gives

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{\sqrt{|x|^2}}{\sqrt{2x^2}} = \lim_{x\to 0} \frac{|x|}{\sqrt{2}|x|} = \frac{1}{\sqrt{2}}$$

so that

$$\lim_{(x,y)\to(0,0)} \frac{|R_{1,(0,0)}(x,y)|}{\|(x,y)-(0,0)\|} \neq 0$$

It follows that $(*)$ is false. Therefore, by the definition, $f$ is not differentiable at $(0,0)$.

Observe that in the above example we have that the partial derivatives at $(0, 0)$ both exist, but

$$\lim_{(x,y)\to(0,0)} \frac{|R_{1,(0,0)}(x, y)|}{\|(x, y) - (0, 0)\|} \neq 0$$

So, the plane $z = L_{(0,0)}(x, y) = 0$ does not give a good approximation to the surface $z = \sqrt{|xy|}$ near the origin. This can be explained geometrically. The vertical plane $y = x$ intersects the surface $z = \sqrt{|xy|}$ in the curve $z = |x|$ which has a corner at $x = 0$ and hence no tangent line. This means that the surface is not "smooth" at $(0, 0, 0)$, and hence the plane $z = L_{(0,0)}(x, y) = 0$ cannot be interpreted as a tangent plane.

**Your Turn**

The following GeoGebra app shows that graph of $z = \sqrt{|xy|}$. Observe the shape of the graph at $(0, 0, 0)$ and convince yourself that the function is not differentiable at this point.

**Instructions**

1. To view the graph from different angles, click and hold on the image and then move your cursor to rotate the figure.
2. Zoom in on the graph at $(0, 0, 0)$. To zoom, hold the Shift key while using your mouse scroll wheel. Alternatively pinch or stretch out two pingers on your mouse touchpad. You can also create a zoom window by clicking and dragging around an area of the graph.
3. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/kudqjwwz/



## Examples - Part 2

*A slideshow appears in Mobius.*

Slide

# Example 3

Determine whether $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is differentiable at $(0,0)$.

**Solution**
By the definition of partial derivatives, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{h^2(0)}{h^2+0^2} - 0}{h} = 0$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{0^2(h)}{0^2+h^2} - 0}{h} = 0$$

So, the error in the linear approximation is

$$\begin{aligned} R_{1,(0,0)}(x,y)) &= f(x,y) - f(0,0) - f_x(0,0)(x-0) - f_y(0,0)(y-0) \\ &= \frac{x^2y}{x^2+y^2} \end{aligned}$$

Slide

## Example 3 Continued

Determine whether $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is differentiable at $(0,0)$.

**Solution**

For $f$ to be differentiable at $(0,0)$, we need $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|R_{1,(0,0)}(x,y)|}{\sqrt{x^2+y^2}} = 0$.

If we approach the limit along $y = x$, we get

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{|R_{1,(0,0)}(x,y)|}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{|R_{1,(0,0)}(x,x)|}{\sqrt{x^2+x^2}} &= \lim_{x\to 0} \frac{|x^3|}{(x^2+x^2)^{3/2}} \\
&= \lim_{x\to 0} \frac{|x^3|}{(2x^2)^{3/2}} \\
&= \lim_{x\to 0} \frac{|x^3|}{2^{3/2}|x^3|} \\
&= \lim_{x\to 0} \frac{1}{2^{3/2}} \\
&= \frac{1}{2^{3/2}}
\end{aligned}$$

Therefore, the limit cannot equal 0, and hence $f$ is not differentiable at $(0,0)$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

Example 4

Determine whether $g(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is differentiable at $(0,0)$.

**Solution**
From the Your Turn exercise we have,

- $g_x(0,0) = 0$,
- $g_y(0,0) = 0$, and
- the error in the linear approximation is $R_{1,(0,0)} = \dfrac{x^2y^2}{x^2+y^2}$ .

For $g$ to be differentiable at $(0,0)$ we need $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|R_{1,(0,0)}(x,y)|}{\sqrt{x^2+y^2}} = 0$.
If we approach the limit along $y = mx$, we get

$$\begin{aligned}
\lim_{x\to 0} \frac{|R_{1,(0,0)}(x,mx)|}{\sqrt{x^2+(mx)^2}} &= \lim_{x\to 0} \frac{m^2x^4}{(x^2+m^2x^2)^{3/2}} \\
&= \lim_{x\to 0} \frac{m^2x^4}{([1+m^2]x^2)^{3/2}} \\
&= \lim_{x\to 0} \frac{m^2x^4}{(1+m^2)^{3/2}|x^3|} \\
&= \lim_{x\to 0} \frac{m^2|x|}{(1+m^2)^{3/2}} = 0
\end{aligned}$$

So, perhaps the limit exists.
We try to apply the Squeeze Theorem. We consider

$$\begin{aligned}
\left| \frac{x^2y^2}{(x^2+y^2)^{3/2}} - 0 \right| &\leq \frac{(x^2+y^2)(x^2+y^2)}{(x^2+y^2)^{3/2}} \\
&= \frac{(x^2+y^2)^2}{(x^2+y^2)^{3/2}} \\
&= (x^2+y^2)^{1/2}
\end{aligned}$$

Since

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)^{1/2} = 0$$

by the Continuity Theorems, it follows from the Squeeze Theorem that

$$\lim_{(x,y)\to(0,0)} \frac{|R_{1,(0,0)}(x,y)|}{\sqrt{x^2+y^2}} = 0$$

Hence, $g$ is differentiable at $(0,0)$.

**Remark**
In previous lessons, we showed that

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$. So, this is an example of a function that is continuous but not differentiable at a point. In the next section, we will prove that if a function is differentiable at a point, then it must be continuous at that point to match what we saw in single-variable calculus.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Tangent Plane

We can now give a formal definition of the tangent plane of $z = f(x, y)$.

**Definition: Tangent Plane**
Consider a function $f(x, y)$ which is differentiable at $(a, b)$. The **tangent plane** of the surface $z = f(x, y)$ at $(a, b, f(a, b))$ is the graph of the linearization. That is, the tangent plane is given by

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Since $f$ is assumed to be differentiable at $(a, b)$, as we have discussed earlier, the tangent plane is the plane that best approximates the surface near the point $(a, b, f(a, b))$.

**Stop and Think**

Can you invent a function $f(x, y)$ whose graph $z = f(x, y)$ is not smooth at $(1, 2, f(1, 2))$? That is, can you invent a function which is continuous but not differentiable at $(1, 2)$?

One strategy is to look for functions whose partial derivatives are not defined at $(1, 2)$.

From single-variable calculus, we know that the function $|x|$ is continuous but not differentiable at $(0, 0)$; thus, we could use this property of the absolute value function to define new function $g(x, y) = |x - 1|$. This function is continuous but not differentiable at $(1, 2)$. In fact, $g(x, y)$ is not differentiable at any point of the form $(1, y)$ for $y \in \mathbb{R}$.

If we wanted a more interesting function, we could choose $g(x, y) = |x - 1| + |y - 2|$.

Can you find another function with the same property?

# 5.2 - Differentiability and Continuity

## Differentiability and Continuity

Recall from single-variable calculus that if $g(x)$ is differentiable at $x = a$, then $g$ is continuous at $a$. We now prove that this result also holds for scalar functions $f(x, y)$.

**Theorem 1:**
If $f(x, y)$ is differentiable at $(a, b)$, then $f$ is continuous at $(a, b)$.

**Proof** The error $R_{1,(a,b)}(x, y)$ is defined by

$$R_{1,(a,b)}(x, y) = f(x, y) - L_{(a,b)}(x, y)$$

Using the definition of $L_{(a,b)}(x, y)$, this equation can be rearranged to read

$$f(x, y) = f(a, b) + \nabla f(a, b) \cdot (x - a, y - b) + R_{1,(a,b)}(x, y) \qquad (*)$$

We can write

$$R_{1,(a,b)}(x, y) = \frac{R_{1,(a,b)}(x, y)}{\|(x, y) - (a, b)\|} \|(x, y) - (a, b)\|, \quad \text{for} \quad (x, y) \neq (a, b)$$

Since $f$ is differentiable and by the Limit Theorems, we get

$$\lim_{(x,y)\to(a,b)} R_{1,(a,b)}(x, y) = 0$$

It now follows from equation $(*)$ that

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) + 0 + 0 = f(a, b)$$

and so by definition, $f$ is continuous at $(a, b)$.

□

A question appears in Mobius

A question appears in Mobius

## 5.3 - Continuous Partial Derivatives and Differentiability

### Continuous Partial Derivatives and Differentiability

We need an efficient way of proving that a given function $f$ is differentiable at a typical point. In this section, we present a theorem for this purpose. To prove this theorem, we will require an extremely important theorem from single-variable calculus: the Mean Value Theorem.

**Theorem 1: The Mean Value Theorem**
If $f(x)$ is continuous on the closed interval $[x_1, x_2]$ and $f$ is differentiable on the open interval $(x_1, x_2)$, then there exists $x_0 \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(x_0)(x_2 - x_1)$$

Now, we are ready to state the theorem.

**Theorem** 2
If the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are both continuous at $(a, b)$, then $f(x, y)$ is differentiable at $(a, b)$.

**Proof**

*Additional content appears in Mobius.*

A question appears in Mobius

**Example 1**
Determine at which points $f(x, y) = (x^2 + y^2)^{2/3}$ is differentiable.
**Solution:**
By differentiation

$$\frac{\partial f}{\partial x} = \frac{4x}{3(x^2 + y^2)^{1/3}}, \quad \text{for} \quad (x, y) \neq (0, 0)$$

By inspection, using the Continuity Theorems, $\frac{\partial f}{\partial x}$ is continuous for all $(x, y) \neq (0, 0)$. By symmetry, the same conclusion holds for $\frac{\partial f}{\partial y}$. It follows from Theorem 2 that $f$ is differentiable for all $(x, y) \neq (0, 0)$.
At the point $(0, 0)$, it is not clear whether the partial derivatives exist and so we must use the definition of partial derivative. We then have to use the definition of differentiable function. We will do this in the next Your Turn exercise.

A question appears in Mobius

**Your Turn 3**

Recall that $f(x, y) \in C^k$ means that $f$ has continuous $k$-th partial derivatives and that the mixed higher-order partial derivatives are equal regardless of the order in which they are taken.

Suppose that $f(x, y) \in C^2$ at $(a, b)$. Determine if the following statements are true or false.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

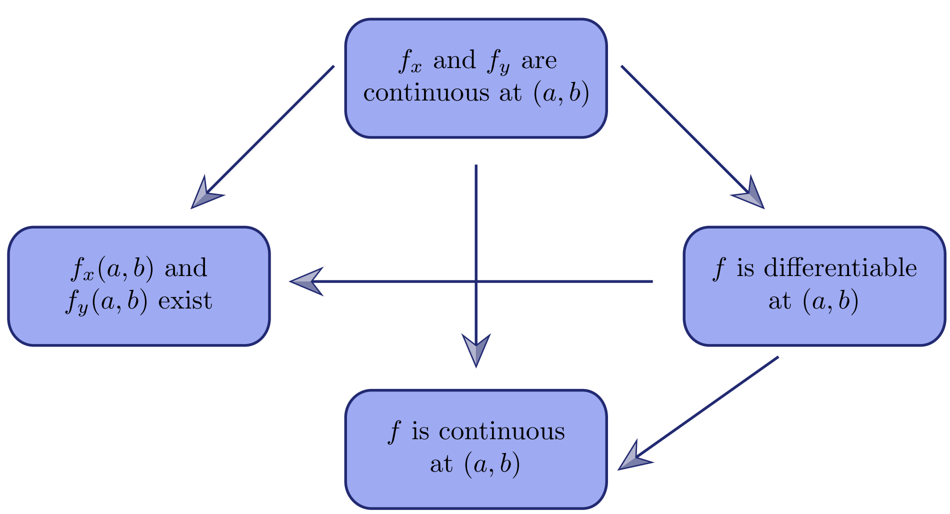
### Summary

Theorem 2 makes it easy to prove that a function $f$ is differentiable at a typical point: we differentiate $f$ to obtain the partial derivatives $f_x, f_y$, and then check that the partial derivaties are continuous functions by inspection, referring to the continuity theorems as seen previously in this course. It is only necessary to use the definition of a differentiable function at an exceptional point.

*A slideshow appears in Mobius.*

Slide

## Summary



### Generalization

The definition of a differentiable function and the main theorems in this lesson are valid for scalar functions of $n$ variables, with a few appropriate modifications.

Definition: Differentiability for $f : \mathbb{R}^n \to \mathbb{R}$
A function $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at a point $\vec{a} = (a_1, \ldots, a_n)$ if

$$\lim_{\vec{x} \to \vec{a}} \frac{|f(\vec{x}) - f(\vec{a}) - L_{\vec{a}}(\vec{x} - \vec{a})|}{\|\vec{x} - \vec{a}\|} = 0$$

where $L : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation.

Theorem 1 can be stated analogously to the two-variable case:

Theorem 1 for functions of $n$ variables
If $f(x_1, \ldots, x_n)$ is differentiable at $\vec{a} = (a_1, \ldots, a_n)$, then $f$ is continuous at $\vec{a}$.

Finally, the result from Theorem 2 also holds; the only change is that there are $n$ partial derivatives:

Theorem 2 for functions of $n$ variables
If $\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}$ are continuous at $\vec{a} = (a_1, \ldots, a_n)$, then $f(x_1, \ldots, x_n)$ is differentiable at $\vec{a}$.

Let's see an example of Theorem 2 in action.

Example
Let $f(x, y, z) = xyz$ and $\vec{a} = (1, 1, 1)$. Show that $f$ is differentiable at $\vec{a} = (1, 1, 1)$.
**Solution:**
We have

$$f_x = yz, \quad f_y = xz, \quad \text{and} \quad f_z = xy$$

By inspection, the partial derivatives of $f$ are continuous over $\mathbb{R}$; in particular, the partial derivatives are all continuous at the point $(1, 1, 1)$.
Since all of the partial derivatives of $f$ are continuous at $(1, 1, 1)$, $f$ is differentiable at $\vec{a} = (1, 1, 1)$ by Theorem 2.

## 5.4 - Linear Approximation Revisited

### Linear Approximation Revisited

The error in the linear approximation for $f(x, y)$ is defined by

$$R_{1,(a,b)}(x, y) = f(x, y) - L_{(a,b)}(x, y)$$

where

$$L_{(a,b)}(x, y) = f(a, b) + \nabla f(a, b) \cdot ((x, y) - (a, b))$$

It is convenient to rearrange the definition of $R_{1,(a,b)}(x, y)$ to read

$$f(x, y) = f(a, b) + \nabla f(a, b) \cdot (x - a, y - b) + R_{1,(a,b)}(x, y) \qquad (*)$$

The linear approximation

$$f(x, y) \approx f(a, b) + \nabla f(a, b) \cdot (x - a, y - b) \tag{**}$$

for $(x, y)$ sufficiently close to $(a, b)$, arises if we neglect the error term. In general, we have no information about $R_{1,(a,b)}(x, y)$, and so it is not clear whether the approximation is reasonable. However, we now have an important piece of information about $R_{1,(a,b)}(x, y)$, namely that if the partial derivatives of $f$ are continuous at $(a, b)$, then $f$ is differentiable and hence

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x, y)|}{\|(x, y) - (a, b)\|} = 0$$

In this case, the approximation $(**)$ is reasonable for $(x, y)$ sufficiently close to $(a, b)$, and we say that $L_{(a,b)}(x, y)$ is a **good approximation** of $f(x, y)$ near $(a, b)$.

**Example 1**
Discuss the validity of the approximation

$$(xy)^{1/3} \approx 2 + \frac{1}{3}(x - 2) + \frac{1}{6}(y - 4)$$

**Solution:**
Let $f(x, y) = (xy)^{1/3}$. By differentiation,

$$\nabla f(x, y) = \left( \frac{1}{3}x^{-2/3}y^{1/3}, \quad \frac{1}{3}x^{1/3}y^{-2/3} \right)$$

so $\nabla f(2, 4) = \left( \frac{1}{3}, \frac{1}{6} \right)$. With $(a, b) = (2, 4)$, equation $(*)$ becomes

$$(xy)^{1/3} = 2 + \frac{1}{3}(x - 2) + \frac{1}{6}(y - 4) + R_{1,(2,4)}(x, y)$$

Using the Continuity Theorems, we see that $f$ has continuous partials at the point $(2, 4)$. Thus,

$$\lim_{(x,y)\to(2,4)} \frac{|R_{1,(2,4)}(x, y)|}{\sqrt{(x - 2)^2 + (y - 4)^2}} = 0$$

It follows that for $(x, y)$ sufficiently close to $(2, 4)$, we may neglect $R_{1,(2,4)}(x, y)$. Thus,

$$(xy)^{1/3} \approx 2 + \frac{1}{3}(x - 2) + \frac{1}{6}(y - 4)$$

gives a good approximation for $(x, y)$ sufficiently close to $(2, 4)$.

A question appears in Mobius

Note that approximation is a recurring theme in calculus, and the equation

$$f(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) + R_{1,\vec{a}}(\vec{x})$$

is of fundamental importance. In the coming lessons, we shall find out more about the error term $R_{1,\vec{a}}(\vec{x})$ in terms of the second partial derivatives.

**Your Turn 2**

For each of the following statements, determine whether it is true or false. If the statement is true, briefly justify; if the statement is false, provide a counter-example.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## 5.5 - Putting It All Together

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Practice Problems

1. For each of the following functions, determine if $f$ is differentiable at $(0, 0)$.

(a) $f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2} + 1 & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

2. Let $f(x,y) = \begin{cases} \dfrac{x^3-y^4}{x^2+y^2} + 1 & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$.
   Determine all points where $f$ is differentiable.

3. Let $f(x,y) = \begin{cases} \dfrac{xy^2+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

   (a) Prove that $f$ is continuous at $(0,0)$.

   (b) Determine all points where $f$ is differentiable.

4. Invent a function $f(x,y)$ which is continuous on $\mathbb{R}^2$ but not differentiable at all points of the circle $x^2+y^2=1$. Sketch the surface $z = f(x,y)$.

## Select Answers and Solutions

1. (a) Use the Squeeze Theorem to show that $\lim_{(x,y)\to(0,0)} \dfrac{|R_{1,(0,0)}(x,y)|}{\|(x,y)-(0,0)\|} = 0$ to conclude that $f$ is differentiable at $(0,0)$.

   (b) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{|R_{1,(0,0)}(x,y)|}{\|(x,y)-(0,0)\|} \neq 0$ to conclude that $f$ is **not** differentiable at $(0,0)$.

2. (a) Note that $f(0,0) = 0$. Use the Squeeze Theorem to show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$$

   to conclude that $f$ is continuous at $(0,0)$.

   (b) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{|R_{1,(0,0)}(x,y)|}{\|(x,y)-(0,0)\|} \neq 0$ to conclude that $f$ is **not** differentiable at $(0,0)$. Use Continuity Theorems to conclude that $f_x$ and $f_y$ are both continuous for all $(x,y) \neq (0,0)$and so $f$ is differentiable for all $(x,y) \neq (0,0)$.

3. No answer provided.

4. An example is $f(x,y) = |x^2+y^2-1|$. Sketch not provided.

# Unit 6

# The Chain Rule

## 6.1 - Basic Chain Rule in Two Dimensions

### Basic Chain Rule in Two Dimensions

### Review of the Chain Rule from Single Variable Calculus

In single-variable calculus, we used the Chain Rule to take derivatives of composite functions: if $T = f(x)$ and $x(t)$ are differentiable functions, then the derivative of their composition, $T(t) = f(x(t))$ is given by

$$T'(t) = f'(x(t))x'(t)$$

If we rewrite this using the Leibniz form of the Chain Rule, we get

$$\underbrace{\frac{dT}{dt}}_{T \text{ as a composite of } t} = \underbrace{\frac{dT}{dx}}_{T \text{ as a function of } x} \frac{dx}{dt}$$

Observe that this involves the abuse of notation since $T$ is used in two different contexts. It is essential in what follows to understand these different ways of writing the Chain Rule from single variable calculus.

Before we introduce the Chain Rule for a multivariable function, first we will practice the Chain Rule from single-variable calculus.

A question appears in Mobius

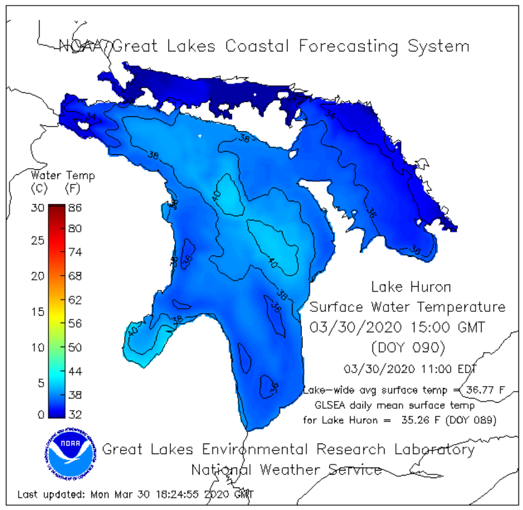
A question appears in Mobius

A question appears in Mobius

## The Chain Rule for $f(x(t), y(t))$

Now let's take a look at the Chain Rule for a simple multivariable function $f(x(t), y(t))$.

In order to provide a physical context for our discussion, suppose that the temperature of a lake is $T = f(x, y)$, as a function of position $(x, y)$. One way to represent the temperature is by using a heat map, as shown in the picture below. The darker blue represents lower temperatures and the lighter blue represents higher temperatures.




Great Lakes Environmental Research Laboratory. (n.d.). [Lake Huron Surface Water Temperature]. Retrieved March 30, 2020 from https://www.glerl.noaa.gov/res/glcfs/glcfs.php?lake=h&ext=swt&type=N&hr=21.

As the picture illustrates, the temperature on the surface of the water depends on the $(x, y)$ position. Suppose that a swimmer swims on the lake with their position given by

$$x = x(t), \quad y = y(t)$$

as a function of time $t$.

Let's find a formula for the rate of change of temperature with respect to time as experienced by the swimmer.

*A slideshow appears in Mobius.*

Slide

## The Chain Rule for $f(x(t), y(t))$

Suppose that the temperature $T$ on the surface of the water depends on the $(x, y)$ position and a swimmer is swimming in the lake with position $x = x(t), y = y(t)$ as a function of time $t$. Find a formula for the rate of change of temperature with respect to time as experienced by the swimmer.
**Solution**
The swimmer's position: $x = x(t), \quad y = y(t)$.
The temperature experienced by the swimmer: $T(t) = f(x(t), y(t))$.
As time changes by $\Delta t$,

- $x$ changes by $\Delta x = x(t + \Delta t) - x(t)$, and
- $y$ changes by $\Delta y = y(t + \Delta t) - y(t)$.

$$\begin{aligned} \Delta T &\approx \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y \\ \left(\text{Since } \lim_{\Delta t \to 0}\frac{\Delta T}{\Delta t} = \frac{dT}{dt}\right) \quad \frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} \end{aligned}$$

(Assuming $T(x, y)$ is differentiable at $(x, y)$.
As $\Delta x \to 0$ and $\Delta y \to 0$, the error in the linear approximation tends to 0.)

Slide

## The Chain Rule for $f(x(t), y(t))$ Continued

The rate of change of temperature with respect to time as experienced by the swimmer is

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

Simplest form of the Chain Rule

Recall that $T(t) = f(x(t), y(t))$.

$$T'(t) = \frac{d}{dt} f\big(x(t), y(t)\big) = f_x\big(x(t), y(t)\big)x'(t) + f_y\big(x(t), y(t)\big)y'(t)$$

Precise form of the equation which avoids abuse of notation

Remark
The preceding "derivation" is intended to make the Chain Rule plausible, but is **not** a proof. The difficulty lies in the approximation sign ($\approx$). This can be remedied by keeping track of the error in the linear approximation and leads to a proof. Note that a hypothesis on the function $f$ that is stronger than the existence of the partial derivatives is required.

Let's formalize this and look at the proof.

**Theorem 1: The Chain Rule**
Let $G(t) = f(x(t), y(t))$, and let $a = x(t_0)$ and $b = y(t_0)$. If $f$ is differentiable at $(a, b)$ and $x'(t_0)$ and $y'(t_0)$ exist, then $G'(t_0)$ exists and is given by

$$G'(t_0) = f_x(a, b)x'(t_0) + f_y(a, b)y'(t_0)$$

**Proof:**

By definition of the derivative,

$$G'(t_0) = \lim_{t \to t_0} \frac{G(t) - G(t_0)}{t - t_0} \tag{*}$$

provided that this limit exists.

By definition of $G(t)$,

$$G(t) - G(t_0) = f(x(t), y(t)) - f(x(t_0), y(t_0)) \qquad \triangle$$

Since $f$ is differentiable, we can write

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + R_{1,(a,b)}(x, y) \qquad \triangle\triangle$$

where

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x, y)|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0$$

Since $a = x(t_0)$, $b = y(t_0)$, it follows from equations ($\triangle$) and ($\triangle\triangle$) that

$$\frac{G(t) - G(t_0)}{t - t_0} = f_x(a, b) \left[\frac{x(t) - x(t_0)}{t - t_0}\right] + f_y(a, b) \left[\frac{y(t) - y(t_0)}{t - t_0}\right] + \frac{R_{1,(a,b)}(x(t), y(t))}{t - t_0} \tag{**}$$

We can now see the Chain Rule taking shape.
We have to prove that

$$\lim_{t \to t_0} \frac{|R_{1,(a,b)}(x(t), y(t))|}{|t - t_0|} = 0$$

Define $E(x, y)$ by

$$E(x, y) = \begin{cases} \dfrac{R_{1,(a,b)}(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} & \text{if } (x, y) \neq (a, b) \\ 0 & \text{if } (x, y) = (a, b) \end{cases}$$

Since $\lim_{(x,y)\to(a,b)} \dfrac{|R_{1,(a,b)}(x, y)|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0 = E(a, b)$, $E$ is continuous at $(a, b)$.

From the definition of $E$,

$$R_{1,(a,b)}(x, y) = E(x, y)\sqrt{(x - a)^2 + (y - b)^2}, \quad \text{for all } (x, y)$$

Since $a = x(t_0)$, and $b = y(t_0)$,

$$\frac{\left|R_{1,(a,b)}\Big(x(t), y(t)\Big)\right|}{|t - t_0|} = \left|E\Big(x(t), y(t)\Big)\right| \sqrt{\left[\frac{x(t) - x(t_0)}{t - t_0}\right]^2 + \left[\frac{y(t) - y(t_0)}{t - t_0}\right]^2}$$

Since $x'(t_0)$ and $y'(t_0)$ exist and $E$ is continuous at $(a, b)$ we get

$$\lim_{t \to t_0} \frac{\left|R_{1,(a,b)}\Big(x(t), y(t)\Big)\right|}{|t - t_0|} = E\Big(x(t_0), y(t_0)\Big)\sqrt{[x'(t_0)]^2 + [y'(t_0)]^2} = 0$$

since $E(a, b) = 0$.

It now follows from equation ($**$) and ($*$) that $G'(t_0)$ exists, and is given by the desired Chain Rule formula.

□

**Stop and Think**

When first studying the Chain Rule you might think that the hypothesis that $f$ is differentiable could be replaced by the weaker hypothesis that $f_x(a, b)$ and $f_y(a, b)$ exist. Is this possible?

Example 1

Use the Chain Rule to find $\dfrac{df}{dt}$ for $f(x,y) = xy^3 - x^3y$ with $x(t) = t^2+1$ and $y(t) = t^2 - 1$ at $t_0 = 1$.

**Solution:**

Let $t_0 = 1$.

First, find $a$ and $b$.

$$a = x(1) = (1)^2 + 1 = 2$$

$$b = y(1) = (1)^2 - 1 = 0$$

Next, find $x'(t) = 2t$ and $y'(t) = 2t$ to find

$$x'(1) = 2(1) = 2$$

$$y'(1) = 2(1) = 2$$

Since $f(x,y)$ is differentiable at $(a,b) = (2,0)$ and $x'(1) = 2$ and $y'(1) = 2$ exist, then $G'(1)$ exists and is given by

$$G'(t_0) = f_x(a,b)x'(t_0) + f_y(a,b)y'(t_0)$$

$$G'(1) = f_x(2,0)x'(1) + f_y(2,0)y'(1)$$

Finally, we find $f_x(x,y) = y^3 - 3x^2y$ to find $f_x(2,0) = 0$,
and $f_y(x,y) = 3xy^2 - x^3$ to find $f_y(2,0) = -8$.
Therefore,

$$G'(1) = f_x(2,0)x'(1) + f_y(2,0)y'(1) = 0(2) + (-8)(2) = -16$$

We can double check our result by writing $f$ as a function of $t$:

$$\begin{aligned} f(x(t),y(t)) &= xy^3 - x^3y \\ &= (t^2+1)(t^2-1)^3 - (t^2+1)^3(t^2-1) \end{aligned}$$

Find $f'(t) = 2t(t^2-1)^3 + 3(t^2-1)^2(2t)(t^2+1) - 3(t^2+1)^2(2t)(t^2-1) - 2t(t^2+1)^3$.
Evaluate at $t = 1$ to get $f'(1) = 0 + 0 - 0 - 2(1)(2)^3 = -16$.

A question appears in Mobius

**Remark**
In practice, it is convenient to use stronger hypotheses in the Chain Rule. In particular, we usually assume that $f$ has continuous partial derivatives at $(a, b)$ and $x'(t)$ and $y'(t)$ are both continuous at $t_0$. This also allows one to obtain the stronger conclusion that $G'(t)$ is continuous at $t_0$. These hypotheses can usually be checked quickly, either by using the Continuity Theorems, or in more theoretical situations, by using given information.

A question appears in Mobius

Now, let's consider the same lake with the same temperature function $T(x, y) = 10e^{-0.1(x^2+y^2)}$ and consider different paths for the swimmer. The path currently shown is $x(t) = 3\sin(t)$, $y(t) = 4\cos(t)$. Observe the path and how it intersects with the level curves of the temperature function. Based on the values of the level curves, try to estimate how the temperature experienced by the swimmer changes as they follow the path. You can input different paths by entering different functions for $x(t)$ and $y(t)$ in the boxes provided.

External resource: https://www.geogebra.org/material/iframe/id/mdbdfswj/



**Your Turn 1**

Let

$$T(t) = \ln(1 + x^2 + y^2), \quad \text{with} \quad x(t) = e^t \sin t, \quad y(t) = 2e^t \cos t$$

Calculate $\dfrac{dT}{dt}$ when $t = 0$ in two different ways:

**Method 1:** Substitute $x$ and $y$ in $T$

**A question appears in Mobius**

**Method 2:** Evaluate $\dfrac{dx}{dt}(0), \dfrac{dy}{dt}(0), \dfrac{\partial T}{\partial x}(0,2)$ and $\dfrac{\partial T}{\partial y}(0,2)$, and apply the Chain Rule.

**A question appears in Mobius**

We will leave it to the reader decide which method is more efficient!

**Example 3**
Define $g(t) = f(t^2 + 3, e^t)$. If $\nabla f(3, 1) = (-2, 5)$, find $g'(0)$. What condition on $f$ will guarantee the validity of your work?
**Solution:**
First, observe that $f$ is a function of two variables. Say $f = f(x, y)$. Thus, we have $g(t) = f(x(t), y(t))$ where $x(t) = t^2 + 3$ and $y(t) = e^t$.
Next, to apply the Chain Rule, we require that $f$ is differentiable at $(x(0), y(0)) = (3, 1)$.
Assuming this condition, we get

$$\begin{aligned} g'(t) &= f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) \\ &= f_x(x(t), y(t))(2t) + f_y(x(t), y(t))(e^t) \end{aligned}$$

Taking $t = 0$ and using $\nabla f(3, 1) = (-2, 5)$ gives

$$\begin{aligned} g'(0) &= f_x(x(0), y(0))2(0) + f_y(x(0), y(0))(e^0) \\ &= 0 + f_y(3, 1)(1) \\ &= 5 \end{aligned}$$

A question appears in Mobius

A question appears in Mobius

## The Vector Form of the Basic Chain Rule

We can use the dot product to rewrite the Chain Rule into a vector form. In particular, if we have

$$T(t) = f(x(t), y(t))$$

where $f(x, y)$, $x(t)$, and $y(t)$ are differentiable, then

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \\ &= \nabla f \cdot \frac{d\vec{x}}{dt}\end{aligned}$$

So, we have

$$\frac{d}{dt} f(\vec{x}(t)) = \nabla f(\vec{x}(t)) \cdot \frac{d\vec{x}}{dt}(t)$$

with $\vec{x}(t) = (x_1(t), x_2(t))$.

In this vector form, the Chain Rule holds for any differentiable function $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$, e.g., $T = f(x, y, z)$, representing temperature or some other quantity in 3-space.

**Example 4**
Let the temperature at position $(x, y, z)$ in the vicinity of the planet Mercury be given by $T = T(x, y, z)$ where $T$ is differentiable. If the path of a spaceship is $(x(t), y(t), z(t))$, then write the Chain Rule for $\frac{dT}{dt}$.
**Solution:**
We have

$$\begin{aligned}\frac{dT}{dt} &= \nabla T(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t)) \\ &= T_x(x(t), y(t), z(t))x'(t) + T_y(x(t), y(t), z(t))y'(t) + T_z(x(t), y(t), z(t))z'(t)\end{aligned}$$

### Your Turn

A differentiable function $f(x, y, z)$ is given and $g(t)$ is defined by

$$g(t) = f(x, y, z)$$

where $x(t) = t$, $y(t) = t^2$, and $z(t) = t^3$.

Write out the Chain Rule for $g'(t)$.

A question appears in Mobius

A question appears in Mobius

## 6.2 - Extensions of the Basic Chain Rule

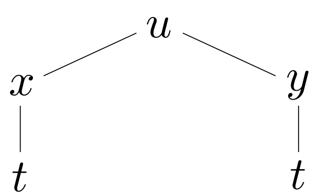
### Extensions of the Basic Chain Rule

*A slideshow appears in Mobius.*

Slide

## Functions of One Independent Variable

We have considered composite functions $u = f(x, y)$ formed from differentiable functions $x(t)$ and $y(t)$.




$u$: dependent variable

$x, y$: intermediate variables

$t$: independent variable

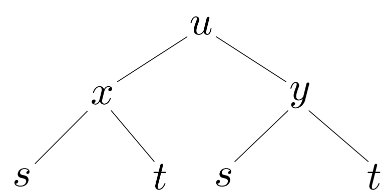
$$\frac{\partial u}{\partial t} = \text{rate of change wrt } x + \text{rate of change wrt } y$$

$$= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Slide

## Functions of Several Independent Variables

Let $u = f(x, y)$ where $x = x(s, t)$ and $y = y(s, t)$ have first order partial derivatives at $(s, t)$ and $f$ is differentiable at $(x, y) = (x(s, t), y(s, t))$.
We have the following dependence tree:




$$\begin{aligned}\frac{\partial u}{\partial s} &= \text{rate of change wrt } x + \text{rate of change wrt } y \\ &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial t} &= \text{rate of change wrt } x + \text{rate of change wrt } y \\ &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\end{aligned}$$

Slide

## Remark

Let $u = f(x, y)$ where $x = x(s, t)$ and $y = y(s, t)$ have first order partial derivatives at $(s, t)$ and $f$ is differentiable at $(x, y) = (x(s, t), y(s, t))$.
We found that

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

It's important to distinguish between the following:

| $\frac{\partial u}{\partial s}$ | $\frac{\partial u}{\partial x}$ |
|---|---|
| Think of $u$ as the composite function of $s$ and $t$ and differentiate wrt $s$, holding $t$ fixed. | Think of $u$ as the given function of $x$ and $y$ and differentiate wrt $x$, holding $y$ fixed. |

**Stop and Think**

How could the Chain Rule be motivated using the linear approximation? Where is the condition that $f(x, y)$ is differentiable used?

We can use our dependence diagrams to find the Chain Rule for more complicated situations. In particular, to obtain the Chain Rule from a dependence diagram we have the following algorithm.

**Algorithm**
To write the Chain Rule from a dependence diagram we do the following:

1. Identify all of the variables.
2. Take all possible paths from the differentiated variable to the differentiating variable.
3. For each link in a given path, differentiate the upper variable with respect to the lower variable being careful to consider if this is a derivative or a partial derivative. Multiply all such derivatives in that path.
4. Add the products from step 3 together to complete the Chain Rule.

In the next section, we will see how this algorithm works in practice.

## Examples

Example 1
Let $z = f(x, y) = (x - y)^4$ where $x = st^4$ and $y = s^4 t$. Use the Chain Rule to find the first-order partial derivatives $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.
**Solution:**

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \underbrace{4(x-y)^3}_{\frac{\partial z}{\partial x}}\underbrace{(t^4)}_{\frac{\partial x}{\partial s}} + \underbrace{4(-1)(x-y)^3}_{\frac{\partial z}{\partial y}}\underbrace{(4s^3t)}_{\frac{\partial y}{\partial s}}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \underbrace{4(x-y)^3}_{\frac{\partial z}{\partial x}}\underbrace{(4st^3)}_{\frac{\partial x}{\partial t}} + \underbrace{(4)(-1)(x-y)^3}_{\frac{\partial z}{\partial y}}\underbrace{(s^4)}_{\frac{\partial y}{\partial t}}$$

In some situations (see the example to follow) it is necessary to write a more precise form of the Chain Rule which displays the functional dependence.

Let $g$ denote the composite function of $f(u, v)$ where $u = u(x, y)$and $v = v(x, y)$:

$$g(x, y) = f(u(x, y), v(x, y))$$

Then, the first equation in $(*)$ can be written as

$$\frac{\partial g}{\partial x}(x, y) = \frac{\partial f}{\partial u}(u(x, y), v(x, y))\frac{\partial u}{\partial x}(x, y) + \frac{\partial f}{\partial v}(u(x, y), v(x, y))\frac{\partial v}{\partial x}(x, y)$$

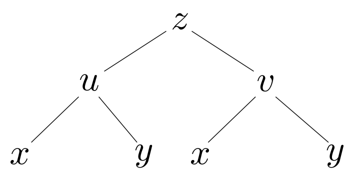
with a similar equation for $\dfrac{\partial g}{\partial y}(x, y)$.

*A slideshow appears in Mobius.*

Slide

## Example 2

Calculate $\dfrac{\partial g}{\partial x}(1,1)$ where $g(x,y)=f(2xy,x^2-y^2)$ and $f$ is a differentiable function with $\nabla f(2,0)=(2,3)$.
**Solution:**
Note that $f$ is a function of two variables: $f=f(u,v)$ where $u(x,y)=2xy$ and $v(x,y)=x^2-y^2$.




We have

$$\begin{aligned} z &= g(x,y) \\ &= f(u(x,y),v(x,y)) \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial x}(x,y) \quad &= \text{rate of change wrt } u\text{-component} + \text{rate of change wrt } v\text{-component} \\ &= \frac{\partial f}{\partial u}(u(x,y),v(x,y))\frac{\partial u}{\partial x}(x,y)+\frac{\partial f}{\partial v}(u(x,y),v(x,y))\frac{\partial v}{\partial x}(x,y) \\ &= \frac{\partial f}{\partial u}(u(x,y),v(x,y))(2y)+\frac{\partial f}{\partial v}(u(x,y),v(x,y))(2x) \end{aligned}$$

Slide

## Example 2 Continued

Calculate $\dfrac{\partial g}{\partial x}(1,1)$ where $g(x,y)=f(2xy,x^2-y^2)$ and $f$ is a differentiable function with $\nabla f(2,0)=(2,3)$.
**Solution:**
We found that

$$\frac{\partial g}{\partial x}(x,y)=\frac{\partial f}{\partial u}(u(x,y),v(x,y))(2y)+\frac{\partial f}{\partial v}(u(x,y),v(x,y))(2x)$$

Substituting $(x,y)=(1,1)$ and using $\nabla f(2,0)=(2,3)$:

$$\begin{aligned} \frac{\partial g}{\partial x}(1,1) \quad &= \frac{\partial f}{\partial u}(u(1,1),v(1,1))2(1)+\frac{\partial f}{\partial v}(u(1,1),v(1,1))2(1) \\ &= 2\frac{\partial f}{\partial u}(2,0)+2\frac{\partial f}{\partial v}(2,0) \\ &= 2(2)+2(3) \\ &= 10 \end{aligned}$$

A question appears in Mobius

**Your Turn 2**

A function $g$ is defined by

$$g(t) = f(h(t) + t, h(t) - t)$$

where $f(x, y)$ and $h(t)$ are both differentiable. Write the Chain Rule for $g'(t)$.

A question appears in Mobius

A question appears in Mobius

In the examples that we have seen so far, the dependence trees have been symmetric. As the next example shows, this is not always the case.

**Example 3**

A differentiable function $f$ is given such that $f(3,2) = 5$ and $\nabla f(3,2) = (4,-1)$. Let $g(t) = t^2 f(2t+1, 3t^3 - t)$. Calculate $g'(1)$.

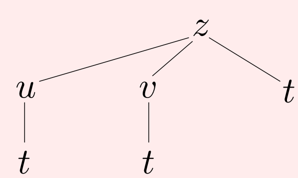
**Solution:**

We see that $f$ is a function of two variables. Say, $f = f(u, v)$. Thus, we have

$$z = g(t) = t^2 f(u(t), v(t))$$

where $u(t) = 2t + 1$ and $v(t) = 3t^3 - t$.

Observe that the dependent variable $z$ depends of the value of $t$ and the value of $f(u, v)$. Hence, $z$ is technically a function of three variables $t$, $u$, and $v$. Under composition, we get that both $u$ and $v$ are functions of $t$. Thus, we get the following dependence diagram.




The dependence diagram shows three paths to $t$. The first path goes through $u$ and gives $\dfrac{\partial z}{\partial u}\dfrac{du}{dt}$, the second path goes through $v$ and gives $\dfrac{\partial z}{\partial v}\dfrac{dv}{dt}$, and the third path goes directly to $t$, which gives $\dfrac{dz}{dt}$.

The Chain Rule is the sum of these terms. So,

$$g'(t) = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{dz}{dt}$$

To calculate $\dfrac{\partial z}{\partial u}$, we are taking the partial derivative of $z$ holding $v$ and $t$ as constants. Thus, we get

$$\frac{\partial z}{\partial u} = t^2 f_u(u, v)$$

Similarly,

$$\frac{\partial z}{\partial v} = t^2 f_v(u, v)$$

To calculate $\dfrac{dz}{dt}$ we take the partial derivative of $z$ holding $u$ and $v$ as constants. Since both $u$ and $v$ are considered constant, it means that $f(u, v)$ is constant. Hence, we get

$$\frac{dz}{dt} = 2tf(u, v)$$

Thus, we have that

$$g'(t) = t^2 f_u(u, v)(2) + t^2 f_v(u, v)(9t^2 - 1) + 2tf(u, v)$$

Hence,

$$\begin{aligned} g'(1) &= (1)^2 f_u(3,2)(2) + (1)^2 f_v(3,2)(9(1)^2 - 1) + 2(1)f(3,2) \\ &= 4(2) + (-1)(8) + 2(5) \\ &= 10 \end{aligned}$$

A question appears in Mobius

A question appears in Mobius

## Generalized Chain Rule

The Chain Rule can be generalized to functions having any number of independent variables in the following way:

Let $w = f(x_1, \ldots, x_m)$ be a differentiable function of $m$ independent variables and for $1 \leq i \leq m$ let $x_i = x_i(t_1, \ldots, t_n)$ be a differentiable function of $n$ independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for $1 \leq j \leq n$.

**Your Turn 5**

Let $u(s,t) = f(x(s,t), y(s,t), s, t)$. Write the Chain Rule for $\dfrac{\partial u}{\partial s}$, showing the functional dependence explicitly.

A question appears in Mobius

## 6.3 - The Chain Rule for Second Partial Derivatives

### The Chain Rule for Second Partial Derivatives

In some situations, it is necessary to be able to calculate second derivatives of composite functions using the Chain Rule. We encounter this problem when working with partial differential equations which involve second derivatives such as Laplace's equation

$$u_{xx} + u_{yy} = 0$$

It also arises when working with Taylor polynomials and in the proof of Taylor's theorem which we will see in a few lessons.

Let's start with an example using functions of one variable.

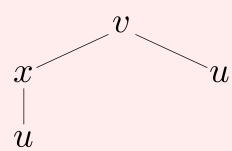
**Example 1**
If $z = f(x)$ where $f$ is twice differentiable and $x = e^u$, verify that

$$z''(u) = x^2 f''(x) + x f'(x)$$

**Solution:**
Observe that by composition we have $z = z(u)$. Since $f(x)$ and $x(u)$ are differentiable, the Chain Rule gives

$$z'(u) = f'(x)x'(u) = f'(x)e^u$$

Since $z'(u)$ is differentiable, we can apply the Chain Rule again to calculate $z''(u)$.
Let $v(x, u) = f'(x)e^u$. Note that $z'(u) = v(e^u, u)$The dependence diagram for $v(x, u)$ is as follows:




The dependence tree shows two paths from $v$ to $u$. The first path goes through $x$ and gives $\frac{\partial v}{\partial x}\frac{dx}{du}$, and the second path goes directly from $v$ to $u$, which gives $\frac{dv}{du}$.
Using our algorithm for calculating the Chain Rule we get

$$\begin{aligned} z''(u) &= \frac{\partial v}{\partial x}\frac{dx}{du} + \frac{dv}{du} \\ &= (f''(x)e^u)(e^u) + f'(x)e^u \\ &= x^2 f''(x) + x f'(x) \end{aligned}$$

**Remark**
Observe, if we had substituted in $x = e^u$ at the beginning, we would get

$$z'(u) = f'(e^u)e^u$$

Hence, taking the derivative with respect to $u$ we would get

$$\begin{aligned} z''(u) &= \frac{d}{du}\big(f'(e^u)\big)e^u + f'(e^u)\frac{d}{du}\big(e^u\big) && \text{by the product rule} \\ &= \left(f''(e^u)\frac{d}{du}\big(e^u\big)\right)e^u + f'(e^u)e^u && \text{by the Chain Rule} \\ &= \big(f''(e^u)e^u\big)e^u + f'(e^u)e^u \end{aligned}$$

which matches with the result above. Thus, we see that our dependence diagram algorithm not only calculates the necessary Chain Rules, but also includes the necessary product rules.

Example 2

Let $g(u,v) = f(u^2 - v^2, 2uv)$. Express $(g_u)^2 + (g_v)^2$ and $g_{uu} + g_{vv}$ in terms of the partial derivatives of $f$. What hypothesis must $f$ satisfy?

**Solution:**

Assume that $f \in C^2$. Let $x(u,v) = u^2 - v^2, y(u,v) = 2uv$ so that $g(u,v) = f(x(u,v), y(u,v))$. Since $f$ is differentiable, by using the Chain Rule, we get

$$g_u = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2uf_x + 2vf_y \text{ and } g_v = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2vf_x + 2uf_y$$

To find the second-order partial derivatives, we will use the product rules together with the Chain Rule:

$$\begin{aligned} g_{uu} &= 2f_x + 2u[2uf_{xx} + 2vf_{xy}] + 0 + 2v[2uf_{yx} + 2vf_{yy}] \\ &= 2f_x + 4u^2 f_{xx} + 4uvf_{xy} + 4uvf_{yx} + 4v^2 f_{yy} \end{aligned}$$

$$\begin{aligned} g_{vv} &= -2f_x - 2v[-2vf_{xx} + 2uf_{xy}] + 0 + 2u[-2vf_{yx} + 2uf_{yy}] \\ &= -2f_x + 4v^2 f_{xx} - 4uvf_{xy} - 4uvf_{yx} + 4u^2 f_{yy} \end{aligned}$$

Simplifying, we get the following interesting equalities:

$$(g_u)^2 + (g_v)^2 = 4(u^2 + v^2)(f_x^2 + f_y^2) \text{and } g_{uu} + g_{vv} = 4(u^2 + v^2)(f_{xx} + f_{yy})$$

A question appears in Mobius

A question appears in Mobius

**Your Turn 3**

Let $f(x, y) \in C^2$ and define $g$ by

$$g(s) = f(a + hs, b + ks)$$

where $(a, b)$ and $(h, k)$ are regarded as fixed. Find $g'(s)$ and $g''(s)$.

A question appears in Mobius

*A slideshow appears in Mobius.*

Slide

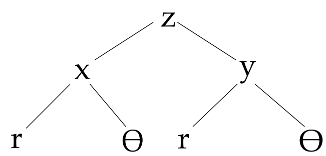
## Example 3

Let $z = f(x, y)$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$ with $f \in C^2$. Calculate $\dfrac{\partial^2 z}{\partial\theta\partial r}$.

**Solution:**

Step 1: Draw the dependence tree.

Note that $z = f(x, y)$ is a function of $x$ and $y$ where $x$ and $y$ depend on $r$ and $\theta$.




Step 2: Use Chain Rule to calculate $\dfrac{\partial z}{\partial r}$:

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)\end{aligned}$$

Slide

## Example 3 Continued

Let $z = f(x, y)$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$ with $f \in C^2$. Calculate $\dfrac{\partial^2 z}{\partial\theta\partial r}$.

**Solution:**

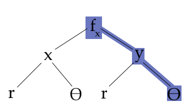
We found that $\dfrac{\partial z}{\partial r} = \cos(\theta)\dfrac{\partial f}{\partial x} + \sin(\theta)\dfrac{\partial f}{\partial y}$.

Step 3: Use the Product Rule and the Chain Rule to calculate $\dfrac{\partial^2 z}{\partial\theta\partial r}$.

Step 3a: Work on $\cos(\theta)\dfrac{\partial f}{\partial x}$:

By Product Rule, $\dfrac{\partial}{\partial\theta}\left(\cos(\theta)\dfrac{\partial f}{\partial x}\right) = -\sin(\theta)\dfrac{\partial f}{\partial x} + \cos(\theta)\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial f}{\partial x}\right)$

By Chain Rule,




$$\frac{\partial}{\partial\theta}\left(\cos(\theta)\frac{\partial f}{\partial x}\right) = -\sin(\theta)\frac{\partial f}{\partial x} + \cos(\theta)\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial\theta}\right)$$

Slide

## Example 3 Continued

Let $z = f(x, y)$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$ with $f \in C^2$. Calculate $\dfrac{\partial^2 z}{\partial\theta\partial r}$.

**Solution:**

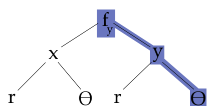
We found that $\dfrac{\partial z}{\partial r} = \cos(\theta)\dfrac{\partial f}{\partial x} + \sin(\theta)\dfrac{\partial f}{\partial y}$.

Step 3: Use the Product Rule and the Chain Rule to calculate $\dfrac{\partial^2 z}{\partial\theta\partial r}$.

Step 3b: Work on $\sin(\theta)\dfrac{\partial f}{\partial y}$:

By Product Rule, $\dfrac{\partial}{\partial\theta}\left(\sin(\theta)\dfrac{\partial f}{\partial y}\right) = \cos(\theta)\dfrac{\partial f}{\partial y} + \sin(\theta)\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial f}{\partial y}\right)$

By Chain Rule,




$$\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial y}\right) = \cos(\theta)\frac{\partial f}{\partial y} + \sin(\theta)\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial^2 y}\frac{\partial y}{\partial\theta}\right)$$

Slide

## Example 3 Continued

Let $z = f(x, y)$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$ with $f \in C^2$. Calculate $\dfrac{\partial^2 z}{\partial\theta\partial r}$.

**Solution:**

Step 4: Put everything together and simplify:

$$\begin{aligned}
\frac{\partial^2 z}{\partial\theta\partial r} &= -\sin(\theta)\frac{\partial f}{\partial x} + \cos(\theta)\left(\frac{\partial^2 f}{\partial^2 x}\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial\theta}\right)\\
&\quad + \cos(\theta)\frac{\partial f}{\partial y} + \sin(\theta)\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial^2 y}\frac{\partial y}{\partial\theta}\right)\\
&= -\sin(\theta)\frac{\partial f}{\partial x} + \cos(\theta)\frac{\partial f}{\partial y} - r\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial^2 x}\\
&\quad + r(\cos^2(\theta) - \sin^2(\theta))\frac{\partial^2 f}{\partial x\partial y} + r\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial^2 y}
\end{aligned}$$

**Stop and Think**

Where did we use the fact that $f \in C^2$?

## 6.4 - Putting It All Together

A question appears in Mobius

### Worked Examples

Let $z = f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$. Verify the following identity. State any assumptions you make in your proof.

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Let's break the problem into two steps by looking at the first degree partial derivatives, then the second degree partial derivatives. The last step of putting it all together is left to you as an exersice.

**Step 1: Find the first-order partial derivatives**

A question appears in Mobius

**Step 2: Find the second-order partial derivatives**

A question appears in Mobius

## Application

The path of a spacecraft is given by $(x, y, z) = (e^{2t}\cos t, e^{2t}\sin t, 2t + 1)$ where $t$ denotes time. The temperature at position $(x, y, z)$ is given by a function $T(x, y, z)$, and the temperature gradient at $(1, 0, 1)$ is $\nabla T(1, 0, 1) = \left(\frac{1}{5}, -\frac{1}{3}, -\frac{1}{4}\right)$.

External resource: https://www.geogebra.org/material/iframe/id/j23mhhej/



a. Find the velocity of the spacecraft at time $t$. Note: for a path $(x, y, z) = (f(t), g(t), h(t))$, the velocity is the vector $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Practice Problems

Try to answer the questions. If you are having trouble, check for a hint before looking at the solutions.

1. Write the Chain Rule for the indicated derivatives of the composite functions, assuming that the various functions are differentiable:

   (a) If $w = f(x, y, z)$, and $x = x(s, t), y = y(s, t),\ z = z(s, t)$, find $\dfrac{\partial w}{\partial t}$.

   (b) If $z = f(x, y)$, and $y = g(x)$, find $\dfrac{dz}{dx}$.

   (c) If $z = f(x, y)$, and $y = g(x), x = h(u, v)$, find $\dfrac{\partial z}{\partial u}$.

   (d) If $w = f(x, y, z)$, and $y = g(x, z), z = h(x)$, find $\dfrac{dw}{dx}$.

2. For some constant $\theta$ define
$$u(x, y) = f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$$
   Calculate $u_{xx}$ and $uxy$.

3. (a) If $F(x, y) = yf(x^2 - y^2)$, show that
$$y\frac{\partial F(x, y)}{\partial x} + x\frac{\partial F(x, y)}{\partial y} = \frac{x}{y}F(x, y)$$
   *[Hint provided below]*

   (b) If $u = x^3 f\left(\frac{y}{x}, \frac{z}{x}\right)$, show that
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$$
   *[Hint provided below]*

   (c) If $F(x, y, z) = f\left(\dfrac{y - z}{x}, \dfrac{z - x}{y}, \dfrac{x - y}{z}\right)$, show that $x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} + z\dfrac{\partial F}{\partial z} = 0$. *[Hint provided below]*

4. The position of a cellphone in your pocket is given by $(x(t), y(t)) = (\cos t, \sin t)$. At position $(x, y)$ the cellphone gets a signal strength of $F(x, y) = e^x y^2$. Using the Chain Rule, find the rate of change of the signal strength with respect to time at $t = \pi/2$.

5. A particle travels along the path $(x, y) = (t^2 - t, e^{3t})$ in a plane where the temperature at position $(x, y)$ and time $t$ is given by $T(x, y, t) = 2x^2 y\sin t$. Calculate the rate of change of temperature along the particle's path with respect to time at any time $t$.

6. Let $f$ be a function of two variables and define $g(x, y) = f(\sin y, \cos x)$. Find $g_{xx}$ and $g_{yy}$. State any assumptions you needed to make.

7. If $u = f(x + g(y))$, where $f$ and $g$ have a continuous second derivative, show that $u_x u_{xy} = u_y u_{xx}$

8. A function $g(u)$ with continuous second derivative is given, and $f$ is defined by $f(x, y) = g\left(\dfrac{x}{y}\right)$, for $y \neq 0$. Is the following statement true?
$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$$

9. Let $F(t) = f(a + th, b + tk)$, where the two-variable function $f$ has continuous second partial derivatives, and $a, b, h, k$ are constants. Show that
$$F''(t) = h^2 f_{11} + 2hkf_{12} + k^2 f_{22}$$
   where $f_{11}, f_{12}$ and $f_{22}$ are evaluated at $(a + th, b + tk)$. Can you generalize (a) to give a formula for $F'''(t)$?

10. Functions $f(x, y, z)$ which satisfy Laplace's equation $f_{xx} + f_{yy} + f_{zz} = 0$ are of interest in theoretical physics. Suppose that the single-variable function $g$ has a continuous second derivative, and $f(x, y, z) = g\left(\frac{1}{r}\right)$, where $r = \sqrt{x^2 + y^2 + z^2} > 0$. Show that

$$f_{xx} + f_{yy} + f_{zz} = \frac{1}{r^4} g''\left(\frac{1}{r}\right), \text{ for } r > 0$$

Give a function $f$, other than a linear function, which satisfies Laplace's equation.

11. Let the three-variable function $f$ be differentiable and satisfy $f(t\vec{x}) = t^p f(\vec{x})$, for all $\vec{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, where $p$ is constant. Prove that

$$\vec{x} \cdot \nabla f(\vec{x}) = p f(\vec{x}) \quad \text{for all } \vec{x} \in \mathbb{R}^3$$

## Hints

*Additional content appears in Mobius.*

*Additional content appears in Mobius.*

*Additional content appears in Mobius.*

## Select Answers and Solutions

1. (a) $\dfrac{\partial w}{\partial t} = f_x(x(s,t), y(s,t), z(s,t)) x_t(s,t) + f_y(x(s,t), y(s,t), z(s,t)) y_t(s,t) + f_z(x(s,t), y(s,t), z(s,t)) z_t(s,t)$

   (b) $\dfrac{dz}{dx} = f_x(x, g(x))(1) + f_y(x, g(x)) g'(x)$

   (c) $\dfrac{\partial z}{\partial u} = f_x(h(u,v), g(h(u,v))) h_u(u,v) + f_y(h(u,v), g(u,v)) g_x(h(u,v)) h_u(u,v)$

   (d)

$$\begin{aligned}\frac{dw}{dt} = f_x(x, g(x, h(x)), h(x)) \\ + f_y(x, g(x, h(x)), h(x))[g_x(x, h(x)) + g_z(x, h(x)) h'(x)] \\ + f_z(x, g(x, h(x)), h(x)) h'(x)\end{aligned}$$

2. Setting $u(x, y) = f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta$, $w(x, y) = x\cos\theta + y\sin\theta$, and $v(x, y) = -x\sin\theta + y\cos\theta$ we have

$$u_{xx} = f_{ww} w_x w_x + f_{wv} w_x v_x + f_{vw} w_x v_x + f_{vv} v_x v_x + f_w w_{xx} + f_v v_{xx}$$

   and

$$u_{xy} = f_{ww} w_x w_y + f_{wv} w_x v_y + f_{vw} w_y v_x + f_{vv} v_x v_y + f_w w_{xy} + f_v v_{xy}$$

3. No answer provided.

4. $-1$

5. No answer provided.

6. No answer provided.

7. $u_x = f'(x + g(y))$ and $u_{xy} = f''(x + g(y)) g'(y)$.
   $u_y = f'(x + g(y)) g'(y)$ and $u_{xx} = f''(x + g(y))$.
   Therefore, $u_x u_{xy} = u_y u_{xx}$.

8. Yes.
9. See the last Your Turn question in lesson 6.3.
10. No answer provided.
11. No answer provided.

# Unit 7

# Directional Derivatives and the Gradient Vector

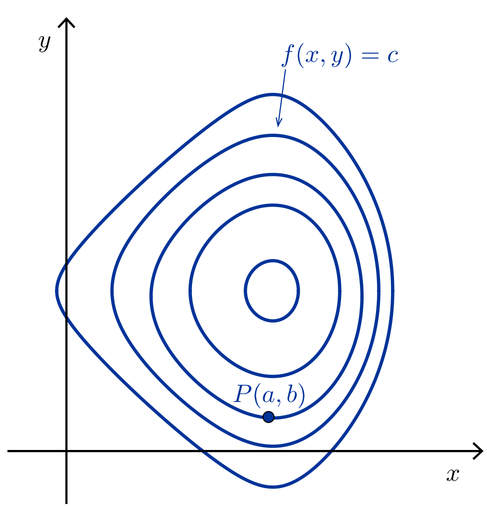
## 7.1 - Directional Derivatives

### Directional Derivatives

In this lesson, we introduce the concept of the directional derivative of a function. This leads to a geometrical interpretation of the gradient vector.

### Motivation

Let $z = f(x, y)$ represent the height of a mountain. As discussed in earlier lessons, the level curves $f(x, y) = c$ represent the contour lines or sets of points that are at the same height. Suppose that a skier is at point $P(a, b)$. In what direction should they move in order to descend as rapidly as possible?




In order to answer this question, we need to generalize the idea of the partial derivative.

We can think of the partial derivative of $f$ with repect to $x$, $f_x$, as the rate of change of $f$ in the direction of positive $x$-axis.

Similarly, we can think of the partial derivative of $f$ with repect to $y$, $f_y$, as the rate of change of $f$ in the direction of positive $y$-axis.

Our aim is to define a derivative which gives the rate of change of function $f$ in a direction specified by the unit vector $\vec{u} = (u_1, u_2)$ (i.e., $\|\vec{u}\| = 1$) from a given point $(a, b)$.

If $L$ is the line passing through $(a, b)$ in the direction $\vec{u}$, then $L$ has vector equation

$$(x, y) = (a, b) + s\vec{u} = (a + su_1, b + su_2), \text{ for } s \in \mathbb{R}$$

At points on the line $L$, $f(x, y)$ has value $f(a+su_1, b+su_2)$, and this defines a function of one variable $s$. Thus, the rate of change of $f$ at $(a, b)$ in the direction of $\vec{u}$ is the derivative of $f(a + su_1, b + su_2)$ with respect to $s$ evaluated at $s = 0$. Hence, we make the following definition.

**Definition: Directional Derivative**
The **directional derivative** of $f(x, y)$ at a point $(a, b)$ in the direction of a **unit vector** $\vec{u} = (u_1, u_2)$ where $\|\vec{u}\| = 1$ is defined by

$$D_{\vec{u}} f(a, b) = \frac{d}{ds} f(a + su_1, b + su_2)\bigg|_{s=0}$$

provided that the derivative exists.

An alternate way of stating the definition of the directional derivative of $f(x, y)$ at a point $(a, b)$ in the direction of a unit vector $\vec{u} = (u_1, u_2)$ where $\|\vec{u}\| = 1$ is

$$D_{\vec{u}} f(a, b) = \lim_{t \to 0} \frac{f((a, b) + t\vec{u}) - f(a, b)}{t}$$

**Example 1**
Find the directional derivative of $f(x, y) = x^2 - y^2$ at the point $(1, 2)$ in the direction of the vector $\vec{u} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.
**Solution:**
Note that $\|\vec{u}\| = \sqrt{(1/\sqrt{5})^2 + (2/\sqrt{5})^2} = 1$.
By definition, we get

$$\begin{aligned}
D_{\vec{u}} f(1, 2) &= \frac{d}{ds} f\left(1 + \frac{1}{\sqrt{5}}s, 2 + \frac{2}{\sqrt{5}}s\right)\bigg|_{s=0} \\
&= \frac{d}{ds}\left[\left(1 + \frac{1}{\sqrt{5}}s\right)^2 - \left(2 + \frac{2}{\sqrt{5}}s\right)^2\right]\bigg|_{s=0} \\
&= \left[\frac{2}{\sqrt{5}}\left(1 + \frac{1}{\sqrt{5}}s\right) - \frac{4}{\sqrt{5}}\left(2 + \frac{2}{\sqrt{5}}s\right)\right]\bigg|_{s=0} \\
&= -\frac{6}{\sqrt{5}}
\end{aligned}$$

Remark
When calculating the directional derivative using the definition, it's important to calculate the derivative first, then evaluate at $s = 0$.

**Your Turn 1**

Here is the graph of the function $f(x,y) = x^2 - y^2$ that we just saw in Example 1.

Instructions:

1. Drag the point A to pick a direction, then move in that direction.
2. Observe how the unit vector $u$ changes on the $xy$-plane.
3. Observe the value and the sign of the directional derivative at the point A.
4. To view the graph from different angles, click and hold on the image and then move your cursor to rotate the figure.
5. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/vubw3axv/



We now derive a simple formula for calculating the directional derivative in terms of the partial derivatives.

**Theorem 1: Directional Derivative (DD) Theorem**
If $f(x,y)$ is differentiable at $(a,b)$ and $\vec{u} = (u_1, u_2)$ where $\|\vec{u}\| = 1$ is a **unit vector**, then

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

where $\cdot$ represents the dot product.

**Proof:** Since $f$ is differentiable at $(a,b)$ we can apply the chain rule to get

$$\begin{aligned}
D_{\vec{u}} f(a,b) =& \frac{d}{ds} f(a+su_1, b+su_2)\Big|_{s=0} \\
=& \Big[ D_1 f(a+su_1, b+su_2) \frac{d}{ds}(a+su_1) \\
& + D_2 f(a+su_1, b+su_2) \frac{d}{ds}(b+su_2) \Big] \Big|_{s=0} \\
=& \left[ D_1 f(a+su_1, b+su_2) u_1 + D_2 f(a+su_1, b+su_2) u_2 \right] \Big|_{s=0} \\
=& D_1 f(a,b) u_1 + D_2 f(a,b) u_2 \\
=& \nabla f(a,b) \cdot (u_1, u_2)
\end{aligned}$$

$\square$

**Example 2**
Find the directional derivative of $f(x,y) = 2x^3 + 4xy^2 + y$ at the point $(-1,1)$ in the direction of the vector $\vec{u} = (1,1)$.
**Solution:**
Observe that the vector is not a unit vector, so we must normalize it. We get

$$\vec{u^*} = \frac{(1,1)}{\|(1,1)\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

We have

$$\nabla f(x,y) = (6x^2 + 4y^2, 8xy + 1)$$

so

$$\nabla f(-1,1) = (10, -7)$$

Since $f$ has continuous partial derivatives at $(-1,1)$, it is differentiable at $(-1,1)$. Thus, we can apply the DD Theorem to get

$$D_{\vec{u^*}} f(-1,1) = (10,-7) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$$

A question appears in Mobius

**Remarks**

1. Be careful to check the condition of the DD Theorem before applying it. If $f$ is not differentiable at $(a,b)$, then we must apply the definition of the directional derivative.

2. If we choose $\vec{u} = \hat{i} = (1,0)$ or $\vec{u} = \hat{j} = (0,1)$, then the directional derivative is equal to the partial derivatives $f_x$ or $f_y$ respectively. It is easy to follow this by the DD Theorem:

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = (f_x, f_y) \cdot (1,0) = f_x$$

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = (f_x, f_y) \cdot (0,1) = f_y$$

**Stop and Think**
The definition of the directional derivative and the DD Theorem can be extended to higher dimensions in an expected way. Can you state the definition of the directional derivative and the DD Theorem for higher dimensions?

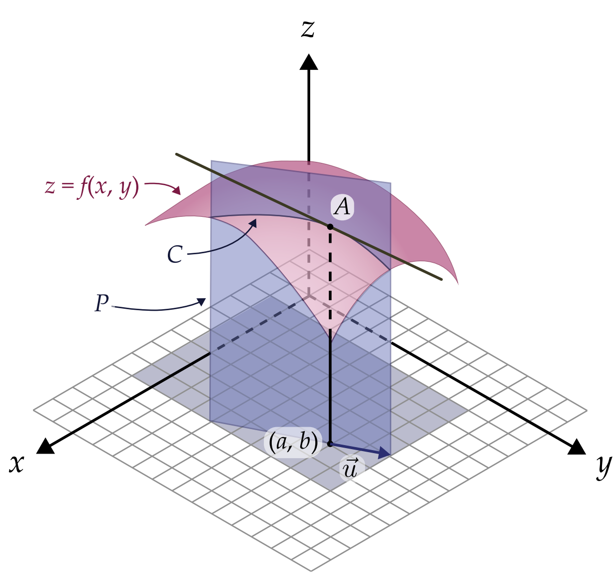
A question appears in Mobius

## Geometric Representation

When the directional derivative is applied, $(x, y)$ usually represents the position, and $f(x, y)$ represents some physical quantity such as temperature or height above sea level. Because the parameter $s$ in the definition represents the distance along the line $L$, the directional derivative represents a rate of change with respect to distance.

For example, if $f(x, y)$ gives the temperature at position $(x, y)$, then $D_{\vec{u}}f(a, b)$ is the rate of change of temperature with respect to distance at position $(a, b)$ in the direction $\vec{u}$ and has dimensions of temperature per unit length.

If $z = f(x, y)$ represents the height above sea level, then $D_{\vec{u}}f(a, b)$ equals the rate of change of height $z$ with respect to horizontal distance at position $(a, b)$ in the direction $\vec{u}$. Geometrically, it equals the slope of the tangent to the cross-section $C$ at the point $A$. (The vertical plane $P$ cuts the surface $z = f(x, y)$ along the curve $C$.)




### Your Turn

Use the following GeoGebra app to explore the geometric interpretation of the directional derivative for different curves. Then use the app to answer the subsequent Your Turn questions.

**Instructions**

1. Set the coordinates of point $P$ with the x and y sliders. (or you can drag the point $P$ around)
2. Set the direction of the unit vector $\vec{u}$ with the angle slider. Observe the vertical plane corresponding to $\vec{u}$.
3. Rotate the app so you can see the curve that results from the intersection of the surface of the function with the vertical plane corresponding to $\vec{u}$. The directional derivative of the function $f(x, y)$ at the point $P$ along the direction of the $\vec{u}$ is the slope of the tangent to this curve at $P$.
4. Change the function and repeat Steps 1-3. Choose your functions carefully to make sure it fits your GeoGebra screen. You can zoom in or out if necessary. Make sure your cursor is pointing at the origin while zooming in or out.
5. Use the app to answer the questions below.

External resource: https://www.geogebra.org/material/iframe/id/xrwye6tf/



A question appears in Mobius

A question appears in Mobius

## 7.2 - The Gradient Vector in Two Dimensions

### The Greatest Rate of Change

In general, for a function $f(x, y)$, the directional derivative $D_{\vec{u}}f(a, b)$ has infinitely many values corresponding to all possible directions $\vec{u}$ at $(a, b)$. Given this, it is a natural question to ask in which direction $D_{\vec{u}}f(a, b)$ assumes its largest value.

This is done using the next theorem and the following property of the dot product:

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$$

where $\theta$ is the angle between $\vec{x}$ and $\vec{y}$.

**Theorem 1:** The Greatest Rate of Change (GRC) Theorem
If $f(x, y)$ is differentiable at $(a, b)$ and $\nabla f(a, b) \neq (0, 0)$, then the largest value of $D_{\vec{u}}f(a, b)$ is $\|\nabla f(a, b)\|$, and occurs when $\vec{u}$ is in the direction of $\nabla f(a, b)$.

**Proof:** Since $f$ is differentiable at $(a, b)$ and $\|\vec{u}\| = 1$ we have

$$\begin{aligned} D_{\vec{u}}f(a, b) &= \nabla f(a, b) \cdot \vec{u} \\ &= \|\nabla f(a, b)\| \, \|\vec{u}\| \cos\theta \\ &= \|\nabla f(a, b)\| \cos\theta \end{aligned}$$

where $\theta$ is the angle between $\vec{u}$ and $\nabla f(a, b)$. Thus, $D_{\vec{u}}f(a, b)$ assumes its largest value when $\cos\theta = 1$, i.e., $\theta = 0$. Consequently, the largest value of $D_{\vec{u}}f(a, b)$ is $\|\nabla f(a, b)\|$ and occurs when $\vec{u}$ is in the direction of the gradient vector $\nabla f(a, b)$.

$\square$

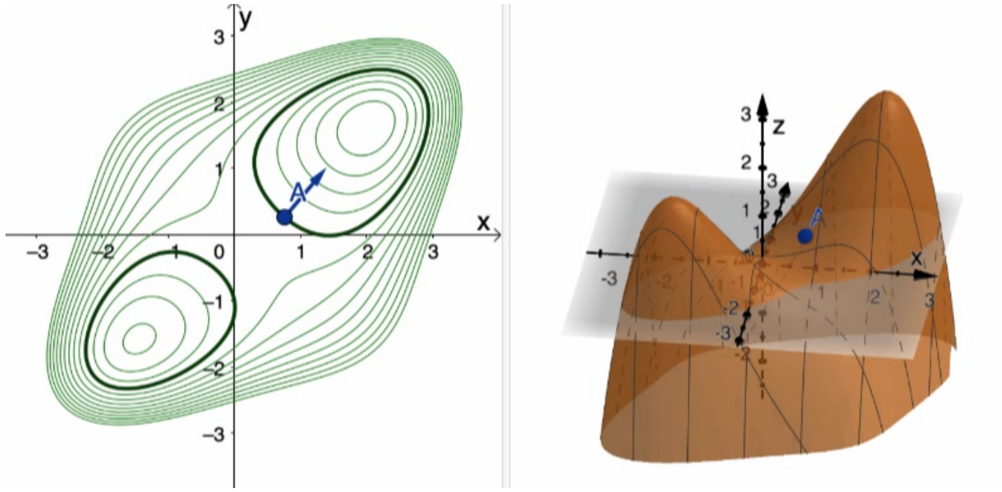
The greatest rate of change theorem tells us that the directional derivative of $f$ at $(a, b)$ assumes its largest value in the direction of the gradient vector, $\nabla f(a, b)$, and the largest value is equal to $\|\nabla f(a, b)\|$.

#### Stop and Think

Looking at the proof of GRC Theorem, can you find the minimum rate of change of $f$ at the point $(a, b)$, and the direction in which it occurs? For which direction(s) does the value of $f$ remain the same?

Watch the following video to see the geometric interpretation of the Greatest Rate of Change Theorem.

*A video appears here.*




Recording adapted from "Gradient vector field" by `https://www.geogebra.org/u/knxm`

**Example 1**
Find the largest rate of change of $f(x, y) = \sqrt{x^2 + 2y^2}$ at the point $(1, 2)$ and the direction in which it occurs.
**Solution:**
We have $\nabla f(x, y) = \left( \dfrac{x}{\sqrt{x^2 + 2y^2}}, \dfrac{2y}{\sqrt{x^2 + 2y^2}} \right)$. Thus, by the Greatest Rate of Change Theorem, the largest rate of change of $f$ at $(1, 2)$ is

$$\|\nabla f(1, 2)\| = \left\| \left( \frac{1}{3}, \frac{4}{3} \right) \right\| = \frac{\sqrt{17}}{3}$$

It occurs in the direction

$$\vec{u} = \nabla f(1, 2) = \left( \frac{1}{3}, \frac{4}{3} \right)$$

**Example 2**
Let $z = f(x, y) = 3 - x^2 + y^2$ represent the height above sea level. A hiker is at position $(1, 2, 6)$. In what direction should they move in order to follow a path of steepest ascent? What would be the slope of their path (i.e., rate of change of height with respect to horizontal distance)?
**Solution:**
The gradient of $f$ is

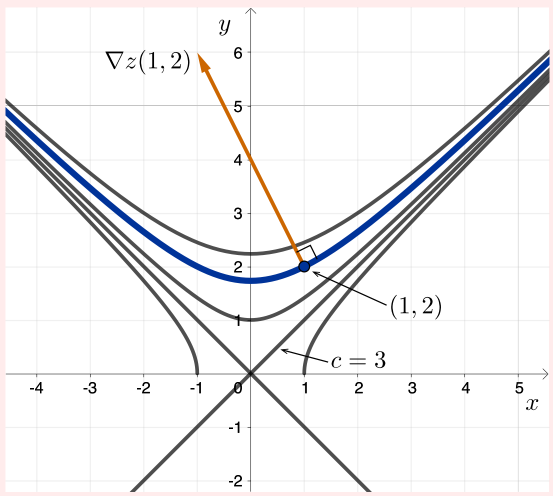
$$\nabla f(x, y) = (-2x, 2y)$$

and at the given point

$$\nabla f(1, 2) = (-2, 4)$$

By the Greatest Rate of Change Theorem, the hiker should move in the direction $\vec{u} = (-2, 4)$ in order to follow a path of steepest ascent (i.e., largest rate of change of $f$). The slope of his path would be

$$\|\nabla f(1, 2)\| = \sqrt{(-2)^2 + (4)^2} = 2\sqrt{5}$$




A question appears in Mobius

**Your Turn 2**

Give a non-constant function $f(x, y)$ and a point $(a, b)$ such that the directional derivative at $(a, b)$ is independent of the direction. What can you say about the tangent plane of the surface $z = f(x, y)$ at the point $(a, b)$?

A question appears in Mobius

The Greatest Rate of Change Theorem also applies to **any dimension**. That is, if $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$, is differentiable at $\vec{a}$ and $\vec{u} \in \mathbb{R}^n$ is a unit vector, then the largest value of $D_{\vec{u}} f(\vec{a})$ is $\|\nabla f(\vec{a})\|$ and it occurs when $\vec{u}$ is in the direction of $\nabla f(\vec{a})$.

**Example 3**
Let $f(x, y, z) = z^3 e^{x^2+y^2-2x}$. Determine the greatest rate of change of $f$ at $(1, 1, 1)$ and the direction in which it occurs.
**Solution:**
We have

$$\nabla f = \left((2x-2)z^3 e^{x^2+y^2-2x}, 2yz^3 e^{x^2+y^2-2x}, 3z^2 e^{x^2+y^2-2x}\right)$$

Thus, the greatest rate of change of $f$ at $(1, 1, 1)$ is

$$\|\nabla f(1,1,1)\| = \|(0,2,3)\| = \sqrt{0+4+9} = \sqrt{13}$$

and occurs in the direction of

$$\vec{u} = \nabla f(1,1,1) = (0,2,3)$$

A question appears in Mobius

## The Gradient and the Level Curves of $f$

People who have experience reading contour maps know that the direction of the steepest ascent is orthogonal to the contour lines. In mathematical terms, this means that the direction of the greatest rate of change of $f$, which we have shown is the direction of the gradient of $f$, is **orthogonal** to the level curves of $f$. We will now derive this result analytically.

**Theorem 2: Orthogonality Theorem**
If $f(x, y) \in C^1$ in a neighborhood of $(a, b)$ and $\nabla f(a, b) \neq (0, 0)$, then $\nabla f(a, b)$ is orthogonal to the level curve $f(x, y) = k$ through $(a, b)$.

**Proof (Optional):**

Since $\nabla f(a, b) \neq (0, 0)$, by the Implicit Function Theorem, the level curve $f(x, y) = k$ can be described by parametric equations $x = x(t)$, $y = y(t)$ for $t \in I$ for an interval $I$ where $x(t)$ and $y(t)$ are differentiable. Hence, the level curve may be written as $f(x(t), y(t)) = k$, $t \in I$. Suppose

$$a = x(t_0), \quad b = y(t_0) \quad \text{for some } t_0 \in I$$

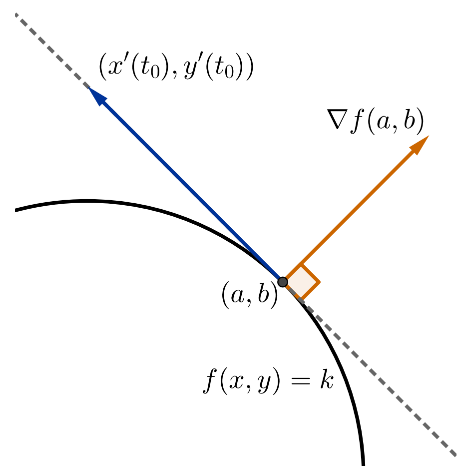
Since $f$ is differentiable, we can take the derivative of this equation with respect to $t$ using the Chain Rule to get

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0$$

Setting $t = t_0$, we get

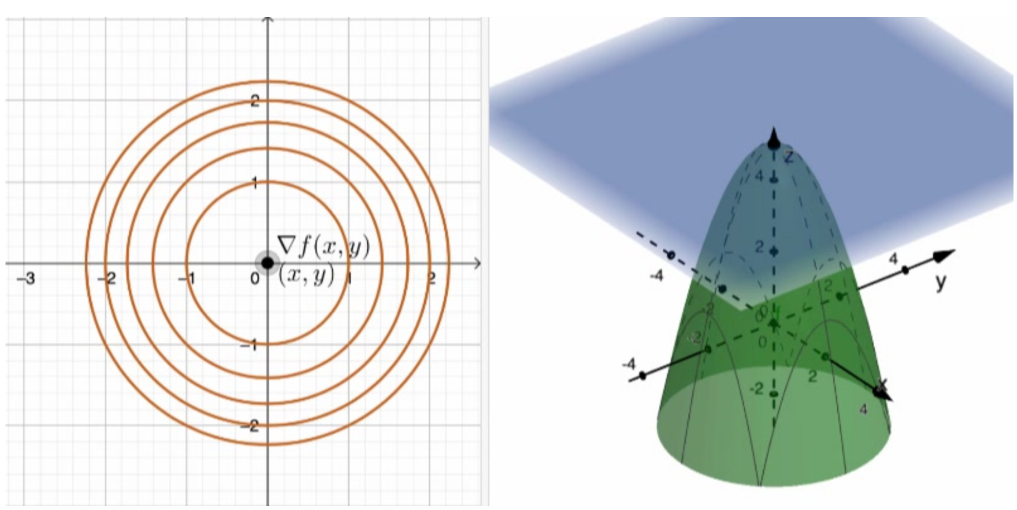
$$\nabla f(a, b) \cdot (x'(t_0), y'(t_0)) = 0$$

Thus, $\nabla f(a, b)$ is orthogonal to $(x'(t_0), y'(t_0))$ which is tangent to the level curve.




□

*A video appears here.*




**Your Turn**

Prove that the level curves of the functions $f$ and $g$ defined by

$$f(x,y) = \frac{y}{x^2}, \quad x \neq 0, \qquad g(x,y) = x^2 + 2y^2$$

intersect orthogonally. Sketch both sets of level curves to see their intersections.

A question appears in Mobius

## 7.3 - The Gradient Vector in Three Dimensions

### The Gradient Vector in Three Dimensions

We cannot visualize the graph $w = f(x,y,z)$ of a function $f(x,y,z)$, because four dimensions are required. However, we can gain insights about the function by considering the level surfaces in $\mathbb{R}^3$ defined by

$$f(x,y,z) = k, \quad \text{where } k \in R(f)$$

**Example 1**
The level surfaces of the function $f$ defined by

$$f(x,y,z) = x + 2y + 3z$$

are the parallel planes

$$x + 2y + 3z = k$$

The following GeoGebra app shows the level surfaces of $f$. Rotate the 3D image to study the behaviour of the level surfaces. Click on the restart button ( ) to go back to the original angle.

External resource: https://www.geogebra.org/material/iframe/id/gdhva3sx/

Created with `https://www.geogebra.org`. CC BY-NC-SA 3.0.

**Your Turn 1**

The level surfaces of the function

$$f(x,y,z) = x^2 + y^2 - z^2$$

are given by

$$x^2 + y^2 - z^2 = k$$

Use the GeoGebra app to explore the level surfaces of $f(x, y, z)$.

**Instructions:**

1. Use the slider to select different values of $k$ and observe the resulting level surfaces.
2. Answer the questions below the app.

External resource: https://www.geogebra.org/material/iframe/id/wtqgs94r/



A question appears in Mobius

We now discuss the interpretation of the gradient $\nabla f(a, b, c)$, for $f(x, y, z)$. As noted in the previous lesson, the Greatest Rate of Change Theorem applies in this case. That is, $\nabla f(a, b, c)$ gives the direction of the largest rate of change of $f$. We now generalize the Orthogonality Theorem to the case $f(x, y, z)$. As you might guess, we have the following theorem:

**Theorem 1: Orthogonality Theorem in Three Dimensions**
If $f(x, y, z) \in C^1$ in a neighborhood of $(a, b, c)$ and $\nabla f(a, b, c) \neq (0, 0, 0)$, then $\nabla f(a, b, c)$ is orthogonal to the level surface $f(x, y, z) = k$ through $(a, b, c)$.

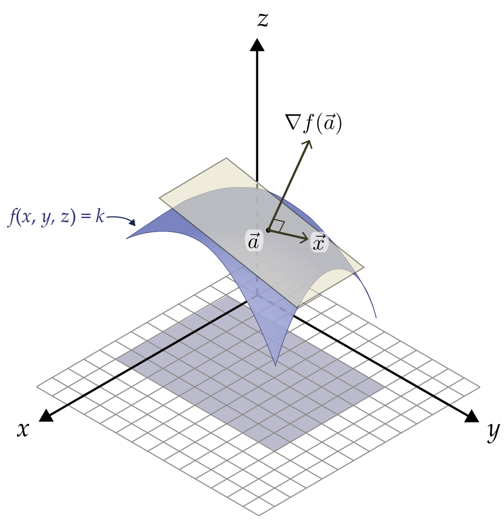
The details are similar to the proof of the Orthogonality Theorem.

Observe that the theorem above gives us a quick way to find the equation of the tangent plane of a surface in $\mathbb{R}^3$ given by

$$f(x, y, z) = k$$

If $\vec{x} \in \mathbb{R}^3$ is an arbitrary point on the tangent plane to the surface $f(x, y, z) = k$ at the point $\vec{a} \in \mathbb{R}^3$, then the vector $\vec{x} - \vec{a}$ lies on the tangent plane, and by the theorem above, is orthogonal to $\nabla f(\vec{a})$, meaning that

$$\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$$




Since this equation is satisfied for all $\vec{x}$ in the tangent plane, it is the **equation of the tangent plane**. In component form, we have

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

Example 2
Find the equation of the tangent plane to the surface $z^3 e^{x^2+y^2-2x} = 1$ at the point $(1,1,1)$.
**Solution:**
From our work above, the equation of the tangent plane is

$$\nabla f(1,1,1) \cdot (x-1, y-1, z-1) = 0$$

We have

$$\nabla f = \left((2x-2)z^3 e^{x^2+y^2-2x}, 2yz^3 e^{x^2+y^2-2x}, 3z^2 e^{x^2+y^2-2x}\right) \Rightarrow \nabla f(1,1,1) = (0,2,3)$$

Hence, we get

$$\begin{aligned}(0,2,3) \cdot (x-1, y-1, z-1) &= 0\\ 2(y-1) + 3(z-1) &= 0\end{aligned}$$

A question appears in Mobius

A question appears in Mobius

## 7.4 - Putting It All Together

### Worked Example 1

The temperature of a metal sheet as a function of position $(x, y)$ is given by $T(x, y) = 100 + 10e^{-x} \sin y$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Worked Example 2

Let $f(x, y) = \ln(x^2 + y^2)$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Practice Problems

1. Let $f(x, y, z)$ be a differentiable function such that $\nabla f(a, b, c) \neq (0, 0, 0)$. Consider the surface $f(x, y, z) = k$ and assume that $f(a, b, c) = k$. Write down the equation of the tangent plane to the surface at $(a, b, c)$, in terms of the gradient vector.

2. At a point $(a, b) \in \mathbb{R}^2$, the directional derivative of a differentiable function $f(x, y)$ in the directions $(1, 1)$ and $(1, -1)$ equals 3 and 2 respectively. Find the largest rate of change of $f(x, y)$ at $(a, b)$, and the direction in which it occurs.

3. In what directions at the point $(2, 1)$ does the directional derivative of the function $f(x, y) = xy$ equal 0? Equal $\sqrt{\frac{5}{2}}$? Express your answer by giving the angle between the required directions and the gradient of $f$ at $(2, 1)$.

4. A space-ship cruising on the sunny side of the planet Mercury starts to overheat. The space-ship is at location $(1, 1, 1)$ and the temperature of the ship's hull when at location $(x, y, z)$ will be

$$T(x, y, z) = 200 + e^{-x^2-2y^2-3z^2}$$

where $x, y, z$ are in metres.

(a) In what direction should the ship proceed in order to decrease temperature most rapidly?

(b) If the ship travels at $e^8$ m/sec, how fast will the temperature decrease (in degrees/sec) if it proceeds in that direction?

(c) The metal of the hull will crack if cooled at a rate greater than $\sqrt{14}e^2$ degrees/sec. Describe the set of possible directions in which the ship may proceed to bring the temperature down at that rate. Give a sketch.

5. Find all points on the paraboloid $z = x^2 + y^2 - 1$ at which the normal line to the surface coincides with the line joining the origin to the point. Illustrate your results with a sketch.

6. A cone, with vertex $(0, 0, -2)$ and axis the $z$-axis, intersects the plane $z = 3$ in a circle of radius $\sqrt{5}$.

   (a) Show that the tangent plane to the cone at the point $(1, -2, 3)$ cuts the $x$-axis at the point $(2, 0, 0)$. Give a sketch.

   (b) Write down a vector equation for the normal line to the cone at $(1, -2, 3)$. Hence show that this line intersects the $xy$-plane at the point $(4, -8, 0)$.

7. Chemotaxis is the chemically directed movement of organisms up a concentration gradient. The slime mold *Dictyostelium discoideum* exhibits this phenomenon. In this case, single-celled amoeba of this species move up the concentration gradient of a chemical called cyclic AMP. Suppose the concentration of cyclic AMP at the point $(x, y)$ is given by $f(x, y) = \dfrac{4}{xy + 1}$.

   (a) If you place an amoeba at the point $(3, 1)$ in the $xy$-plane, determine in which direction the amoeba will move if its movement is directed by chemotaxis.

   (b) It can be shown in general that a particle moving in the manner described above has path $y = y(x)$ satisfying the differential equation (DE) $\dfrac{dy}{dx} = f_y/f_x$ Find the path of the amoeba from part a, which has initial condition $y(3) = 1$.

8. Consider the sphere of radius 4 centered at the origin, and the sphere of radius 3 centered at the point $(0, 5, 0)$. Prove that the normal directions to these spheres at their points of intersection are orthogonal. Give a sketch.

9. An engineer wishes to build a railroad up a mountain that has the shape of an elliptic paraboloid $z = c - ax^2 - by^2$, where $a, b, c$ are positive constants. At the point $(1, 1)$, in what directions may the track be laid so that it will be climbing with a slope of 0.03 (i.e. a vertical rise of 0.03 m for each horizontal metre)? Make a sketch showing a few level curves, the gradient $\nabla z$ at $(1, 1)$, and the two possible directions for the track. Work out the details using $a = \sqrt{3}b$, $b = 0.015$.

## Select Answers and Solutions

1. $\nabla f(a, b, c) \cdot (x - a, y - b, z - c) = 0$

2. The largest rate of change is $\sqrt{13}$ in direction $(5, 1)$ (or any scalar multiple thereof).

3. No answer provided.

4. (a) $(1, 2, 3)$

   (b) $2\sqrt{14}e^2$

   (c) Cone with inside angle $\dfrac{2\pi}{3}$.

5. $(0, 0, -1)$ and all points $(x, y, -1/2)$ with $x^2 + y^2 = \dfrac{3}{2}$.

6. (a) Use the information provided to find the equation of the cone. Find the equation of the tangent plane and verify that the point $(0, 0, 2)$ is on the tangent plane.

   (b) No answer provided.

7. (a) $(-1, -3)$

(b) $y(x) = \sqrt{y^2 - 8}$

8. Use the information provided to find the equation of each sphere. Find the points on intersection of the sphere and verify that the gradients are orthogonal at those points.

9. $\pi/3$ radians from the vector $\nabla z(1,1) \approx (-0.052, -0.030)$

# Unit 8

# Taylor Polynomials and Taylor's Theorem

## 8.1 - The Taylor Polynomial of Degree 2

### The Taylor Polynomial of Degree 2

For a function of one variable $f$, the second derivative $f''$ plays an important role in approximating $f(x)$. Geometrically, $f''$ determines whether the graph of $f$ is concave up or concave down.

The second derivative can, in fact, be used to estimate the error through Taylor's formula. In addition, $f''$ can be used to increase the accuracy of the linear approximation by defining a quadratic approximation, the second degree Taylor polynomial.

In this unit, we extend these ideas to functions of two variables.

### Review of the Single Variable Case

For a function of one variable, $f(x)$, the Taylor polynomial of degree 2 at point $a$ is denoted by $P_{2,a}(x)$, and is defined as

$$P_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Observe that $P_{2,a}(x)$ is the sum of the linear approximation $L_a(x) = f(a) + f'(a)(x-a)$ and the term $\frac{1}{2}f''(a)(x-a)^2$ which is of second degree in $(x-a)$.

The coefficient of this term $\frac{1}{2}f''(a)$ is determined by requiring that the second derivative of $P_{2,a}(x)$ equals the second derivative of $f$ at $a$:

$$P''_{2,a}(a) = f''(a)$$

You should verify this by differentiating $P_{2,a}(x)$ twice.

**Your Turn**

Use the following app to convince yourself that as the degree of the Taylor Polynomial increases, we get a better accuracy.

**Instructions**

1. In the $f(x)$ input box, enter a polynomial of degree $n$ where $n \in \{1, 2, 3, 4, 5\}$
2. Use the slider to match the degree of the Taylor polynomial with the degree of $f(x)$.
3. Observe the Taylor polynomial approximation at the given point. You can change the point by dragging it on the plane.
4. Now increase the degree of the Taylor polynomial. Notice that it better approximates the function $f(x)$ at the point.
5. Repeat the steps above with other functions.

External resource: https://www.geogebra.org/material/iframe/id/vnfrhetd/



## The Two Variable Case

Suppose that $f(x, y)$ has continuous second partial derivatives at $(a, b)$. The Taylor polynomial of $f$ of degree 2 at $(a, b)$ is denoted $P_{2,(a,b)}(x, y)$ and is obtained by adding appropriate $2^{\text{nd}}$-degree terms in $(x - a)$ and $(y - b)$ to the linear approximation $L_{(a,b)}(x, y)$.

The Taylor polynomial will have the following general form:

$$P_{2,(a,b)}(x, y) = L_{(a,b)}(x, y) + A(x - a)^2 + B(x - a)(y - b) + C(y - b)^2 \tag{*}$$

where $A, B, C$ are constants. Let's find $A, B, C$.

Using $(*)$ we have that

$$\frac{\partial^2 P_{2,(a,b)}}{\partial x^2} = 2A$$

since $L_{(a,b)}(x, y)$ does not contribute to the second derivatives as it is of first degree in $x$ and $y$.

Similarly, finding the other second partial derivatives of $P_{2,(a,b)}(x, y)$ gives

$$\frac{\partial^2 P_{2,(a,b)}}{\partial x \partial y} = B$$
$$\frac{\partial^2 P_{2,(a,b)}}{\partial y^2} = 2C$$

Requiring that the second partial derivatives of $P_{2,(a,b)}$ equal the second partial derivatives of $f$ at $(a, b)$ leads to

$$2A = \frac{\partial^2 f}{\partial x^2}(a, b), \quad B = \frac{\partial^2 f}{\partial x \partial y}(a, b), \quad 2C = \frac{\partial^2 f}{\partial y^2}(a, b)$$

We then substitute these into equation $(*)$ and write out the expression for $L_{(a,b)}(x, y)$, to obtain the required formula.

**Definition: 2nd degree Taylor polynomial**
Let $f$ be a function of two variables. The **second degree Taylor polynomial** $P_{2,(a,b)}$ of $f(x, y)$ at $(a, b)$ is given by

$$\begin{aligned} P_{2,(a,b)}(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &\quad + \frac{1}{2}\Big[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\Big] \end{aligned}$$

In general, the second degree Taylor polynomial approximates $f(x, y)$ for $(x, y)$ sufficiently close to $(a, b)$:

$$f(x, y) \approx P_{2,(a,b)}(x, y)$$

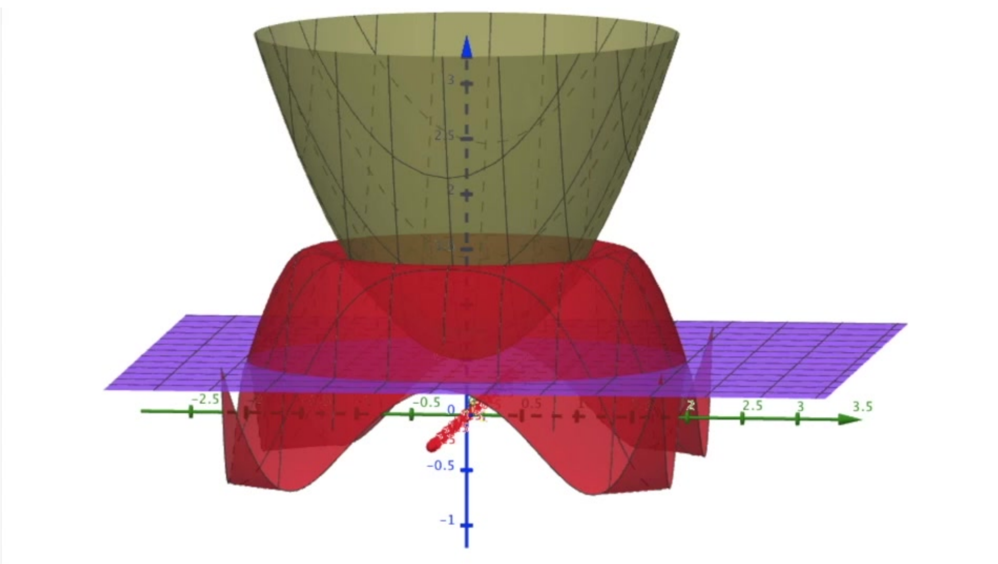
with better accuracy than the linear approximation.

The following video shows the function $\sin\left(\frac{4}{5}x^2 + \frac{4}{5}y^2\right) + \frac{1}{2}$.

- First, we see the function.
- Next, we see its linear approximation, which is a plane.
- Finally, we see its second degree Taylor approximation, which is a paraboloid. Notice that the second degree Taylor approximation is more accurate than the linear approximation.

**Note:** This video has no audio

*A video appears here.*




### Your Turn 1

Use the following app to observe the behaviour of the first and second degree approximation near a point.

**Instructions**

1. Click and hold on the image and then move your cursor to rotate the 3D-image to find the point A.
2. Move the point A around and observe the shape of the graph of $f$ near this point.
3. Click "Display the 1st degree approximation near A". Move the point around and rotate the 3D-image to observe the first approximation near point A.
4. Click "Display the 2nd degree approximation near A". Move the point around and rotate the 3D-image. Observe that the second degree approximation is more accurate than the first degree approximation near A.
5. Turn off both approximations, then repeat these steps for point B and point C.

External resource: https://www.geogebra.org/material/iframe/id/gsvypjpe/

Note that the behaviour of the first approximation is the same for all three points, but the second degree approximation differs near each point since the behaviour of the graph of $f$ is very different near these points.

Before seeing examples of second degree Taylor polynomial calculations, we introduce a useful definition:

Definition: Hessian Matrix
The **Hessian matrix** of $f(x, y)$, denoted by $Hf(x, y)$, is defined as

$$Hf(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y \end{bmatrix}$$

We can use the entries of the Hessian matrix to determine the coefficients in the second half of the second degree Taylor polynomial.

A question appears in Mobius

Previously, we used the linear approximation to estimate the value of $\sqrt{(0.95)^3 + (1.98)^3}$. The linear approximation was

$$L_{(1,2)} = 3 + \frac{1}{2}(x-1) + 2(y-2)$$

and our calculations indicated $\sqrt{(0.95)^3 + (1.98)^3} \approx 2.935$. We also claimed that higher-order approximations would give a better estimation.

Now, let's see if we do in fact get a better approximation using the Taylor polynomial.

**Example 1**
Use the Taylor polynomial of degree 2 to approximate $\sqrt{(0.95)^3 + (1.98)^3}$.
**Solution:**
Let $f(x, y) = \sqrt{x^3 + y^3}$ and $(a, b) = (1, 2)$.
By differentiating, we find

$$\nabla f(1,2) = \left(\frac{1}{2}, 2\right), \qquad Hf(1,2) = \begin{bmatrix} \frac{11}{12} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

We can use the $\nabla f(1,2)$ and $Hf(1,2)$ to determine the coefficients and set up the equation for the Taylor polynomial of degree 2 as follows:

$$P_{2,(1,2)}(x, y) = 3 + \frac{1}{2}(x-1) + 2(y-2) + \frac{1}{2}\left[\frac{11}{12}(x-1)^2 - \frac{2}{3}(x-1)(y-2) + \frac{2}{3}(y-2)^2\right]$$

This polynomial approximates $\sqrt{x^3 + y^3}$ near the point $(1, 2)$:

$$\begin{aligned} \sqrt{(0.95)^3 + (1.98)^3} &\approx P_{2,(1,2)}(0.95, 1.98) \\ &= 3 + (-0.065) + \frac{1}{2}\left(\frac{0.0227}{12}\right) \\ &= 2.935946 \end{aligned}$$

The calculator value was 2.935944. Hence, the error is 0.000002 compared with 0.000943 for the linear approximation.

A question appears in Mobius

We now ask how large the error is when we use the approximation

$$f(x, y) \approx P_{2,(a,b)}(x, y)$$

To answer this, we need to extend Taylor's Theorem to the functions of two variables $f(x, y)$. This will be the topic of the next lesson.

# 8.2 - Taylors Formula with Second Degree Remainder

## Taylor's Formula with Second Degree Remainder

## Review of the Single Variable Case

Theorem 1: Taylor Remainder for Single Variable Functions
If $f''(x)$ exists on $[a, x]$, then there exists a number $c$ between $a$ and $x$ such that

$$f(x) = f(a) + f'(a)(x-a) + R_{1,a}(x) \tag{*}$$

where

$$R_{1,a}(x) = \frac{1}{2} f''(c)(x-a)^2 \tag{**}$$

On recalling that

$$L_a(x) = f(a) + f'(a)(x-a) \tag{***}$$

we see that the term $R_{1,a}(x)$ represents the **error in the linear approximation.**

Keep in mind that we can't evaluate this expression because we don't know the value of $c$; we only know that $c$ lies somewhere between $a$ and $x$. However, this formula is useful because it gives a way of finding an upper bound for the error.

If $f$ has a continuous second derivative on an interval $[a-\delta, a+\delta]$ centered on $a$, then $f''$ is bounded on this interval. That is, there exists a number $B$ such that

$$|f''(x)| \le B, \quad \text{for all } x \in [a-\delta, a+\delta]$$

By equations $(*)$,$(**)$, and $(***)$,

$$\begin{aligned} |f(x) - L_a(x)| &= |R_{1,a}(x)| \\ &= \left| \frac{1}{2} f''(c)(x-a)^2 \right| \\ &= \frac{1}{2} |f''(c)|(x-a)^2 \\ &\le \frac{1}{2} B(x-a)^2 \end{aligned}$$

for all $x \in [a-\delta, a+\delta]$. Knowing $f''(x)$, we can find a value for the upper bound $B$.

## The Two Variable Case

In order to generalize Taylor's formula to the case of $f(x, y)$, observe that $R_{1,a}(x)$ in equation $(**)$ has the same form as the second derivative term in $P_{2,a}(x)$, except that $f''$ is evaluated at $c$ instead of at $a$. Knowing the form of $P_{2,(a,b)}(x, y)$ leads us to Taylor's Theorem for a function of two variables.

**Theorem 2: Taylor's Theorem** for Functions of Two Variables
If $f(x,y) \in C^2$ in some neighborhood $N(a,b)$ of $(a,b)$, then for all $(x,y) \in N(a,b)$ there exists a point $(c,d)$ on the line segment joining $(a,b)$ and $(x,y)$ such that

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_{1,(a,b)}(x,y)$$

where

$$R_{1,(a,b)}(x,y) = \frac{1}{2}\Big[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2\Big]$$

**Proof:**

The idea is to reduce the given function $f(x,y)$ of two variables to a function $g(t)$ of one variable by considering only points on the line segment joining $(a,b)$ and $(x,y)$.

*Additional content appears in Mobius.*

**Remark**
Like the one variable case, Taylor's Theorem for $f(x,y)$ is an **existence theorem**: it only tells us that the point $(c,d)$ exists, but it does not tell us how to find it.

Here is an example to show how Taylor's formula can be used to estimate the error when using the linear approximation formula.

**Example 1**
Let $f(x,y) = \sqrt{1+x+2y}$. Find the linear approximation near $(0,0)$ and show that if $x \geq 0$ and $y \geq 0$, we have

$$|R_{1,(0,0)}(x,y)| \leq \frac{3}{4}(x^2+y^2)$$

**Solution:**
We will solve this example as a Your Turn exercise.

A question appears in Mobius

A question appears in Mobius

**Step 3: Find the error function.**

A question appears in Mobius

**Step 4: Find an upper bound for the error.**

A question appears in Mobius

A question appears in Mobius

The most important thing about the error term $R_{1,(a,b)}(x,y)$ is not its explicit form, but rather its dependence on the magnitude of the displacement $\|(x,y)-(a,b)\|$. We state this result as a Corollary:

**Corollary**
If $f(x,y) \in C^2$ in some closed neighborhood $N(a,b)$ of $(a,b)$, then there exists a positive constant $M$ such that

$$|R_{1,(a,b)}(x,y)| \leq M\|(x,y)-(a,b)\|^2, \quad \text{for all } (x,y) \in N(a,b)$$

This result gives us an upper bound for the error in the linear approximation, which is bounded above by a scalar multiple of the magnitude of the distance between the points $(x,y)$ and $(a,b)$. Notice that this result states the existence of $M$, but does not tell us how to find this positive constant.

## 8.3 - Generalizations of the Taylor Polynomial

### Generalizations of the Taylor Polynomial

### Multi-Index Notation

In order to define the $k$-th degree Taylor polynomial for functions of two variables, we will need to introduce some notation.

If $f \in C^k$ is a function of $n$ variables, we can write a $k$-th order partial derivative of $f(x_1,\ldots,x_n)$ as

$$\partial^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f$$

where $\alpha$ is a **multi-index**; that is, $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ with $\alpha_i \in \mathbb{N}$. The sum $\alpha_1+\alpha_2+\cdots+\alpha_n = k$ is called the **order** of $\alpha$ and is sometimes denoted by $|\alpha|$. We also define $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$.

Given a multi-index of order $k$, $\partial^\alpha f$ is a partial derivative of order $k$ of $f$.

Example 1
Let $f(x,y,z) = x^3y^4z^5$ and let $\alpha = (2,1,3)$.
Then,

$$\partial^\alpha f = \left(\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial}{\partial y}\right)^1 \left(\frac{\partial}{\partial z}\right)^3 f$$

We have

$$\frac{\partial^3 f}{\partial z^3} = 60x^3y^4z^2$$

then,

$$\frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial z^3}\right) = \frac{\partial}{\partial y} 60x^3y^4z^2 = 240x^3y^3z^2$$

and finally

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial y}\frac{\partial^3 f}{\partial z^3}\right) = \frac{\partial^2}{\partial x^2} 240x^3y^3z^2 = 1440xy^3z^2$$

hence,

$$\partial^\alpha f = \left(\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial}{\partial y}\right)^1 \left(\frac{\partial}{\partial z}\right)^3 f = 1440xy^3z^2$$

A question appears in Mobius

In addition to using the multi-index notation for $k$-th order partial derivatives, we can also use it as follows.

Let $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{a} = (a_1, a_2, \ldots, a_n)$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then

$$(\vec{x} - \vec{a})^\alpha = (x_1 - a_1)^{\alpha_1}(x_2 - a_2)^{\alpha_2} \cdots (x_n - a_n)^{\alpha_n}$$

Example 2
Let $\vec{x} = (x_1, x_2, x_3)$, $\vec{a} = (2, -1, 0)$, and $\alpha = (2, 4, 1)$.
Then,

$$(\vec{x} - \vec{a})^\alpha = (x_1 - 2)^2(x_2 + 1)^4(x_3)^1$$

A question appears in Mobius

## Taylor Polynomial of Degree $k$ for Functions of Two Variables

Using the multi-index notation, we can now define the $k$-th degree Taylor polynomial for a function of two variables.

Definition: $k$-th degree Taylor polynomial

The **$k$-th degree Taylor polynomial** of a function $f(x, y)$ is

$$P_{k,(a,b)}(x, y) = \sum_{|\alpha| \leq k} \partial^\alpha f(a, b) \frac{[(x, y) - (a, b)]^\alpha}{\alpha!}$$

Note that summing over $|\alpha| \leq k$ means that we are considering all of the mixed partial derivatives whose orders sum to $k$ or less.

Example 3

Write out $P_{2,(a,b)}(x,y)$ using the subscript notation for partial derivatives.

**Solution:**

Using the definition of the $k$-th order Taylor polynomial with $k=2$, we have

$$P_{2,(a,b)}(x,y) = \sum_{|\alpha|\leq 2} \partial^\alpha f(a,b) \frac{[(x,y)-(a,b)]^\alpha}{\alpha!}$$

We need to find all the $\alpha = (\alpha_1, \alpha_2)$ such that $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 \leq 2$. There are six possibilities:

- $\alpha = (0,0)$: this gives us the term $f(a,b)$ and $\alpha! = 1$
- $\alpha = (1,0)$: this gives us the term $f_x(a,b)(x-a)^1(y-b)^0 = f_x(a,b)(x-a)$ and $\alpha! = 1$
- $\alpha = (0,1)$: this gives us the term $f_y(a,b)(x-a)^0(y-b)^1 = f_y(a,b)(y-b)$ and $\alpha! = 1$
- $\alpha = (2,0)$: this gives us the term $f_{xx}(a,b)(x-a)^2(y-b)^0 = f_{xx}(a,b)(x-a)^2$ and $\alpha! = 2$
- $\alpha = (1,1)$: this gives us the term $f_{xy}(a,b)(x-a)^1(y-b)^1 = f_{xy}(a,b)(x-a)(y-b)$ and $\alpha! = 1$
- $\alpha = (0,2)$: this gives us the term $f_{yy}(a,b)(x-a)^0(y-b)^2 = f_{yy}(a,b)(y-b)^2$ and $\alpha! = 2$

Putting everything together, we find

$$\begin{aligned} P_{2,(a,b)}(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\qquad + \frac{1}{2} f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2} f_{yy}(a,b)(y-b)^2 \end{aligned}$$

**Your Turn 1**

Write out $P_{3,(a,b)}(x,y)$ using the subscript notation for partial derivatives.

A question appears in Mobius

As we will see, all of our previous results on Taylor polynomials generalize in an expected way for all values of $k$.

**Theorem 1: Taylor's Theorem of order $k$**
If $f(x,y) \in C^{k+1}$ in some neighbourhood $N(a,b)$ of $(a,b)$, then for all $(x,y) \in N(a,b)$ there exists a point $(c,d)$ on the line segment between $(a,b)$ and $(x,y)$ such that

$$f(x,y) = P_{k,(a,b)}(x,y) + R_{k,(a,b)}(x,y)$$

where

$$R_{k,(a,b)}(x,y) = \sum_{|\alpha|=k+1} \partial^\alpha f(c,d) \frac{[(x,y)-(a,b)]^\alpha}{\alpha!}$$

From Taylor's Theorem of order $k$, it follows that when a function $f(x,y) \in C^k$ is close to the point $(a,b)$, then the Taylor polynomial is a good approximation for $f$ at $(a,b)$:

**Corollary**
If $f(x,y) \in C^k$ in some neighborhood of $(a,b)$, then

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - P_{k,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|^k} = 0$$

Notice that the ratio in the limit is the difference between $f$ and its $k$-th order Taylor polynomial over the displacement. The fact that the limit goes to zero tells us that $P_{k,(a,b)}(x,y)$ is a good approximation of $f$ at $(a,b)$.

### Your Turn 2

Use the following GeoGebra app to observe how the local approximation for a function improves as the degree of the Taylor polynomial increases.

#### Instructions

1. Observe the Taylor polynomial approximation of the function f(x,y) at the given point.
2. Notice that as the degree of the Taylor polynomial increases, the Taylor polynomial approximation improves at the point.
3. You can choose a new point to approximate locally by dragging the point on the 2D-plane.
4. Repeat the steps above with other functions by entering a new function in the f(x,y) input box.

External resource: https://www.geogebra.org/material/iframe/id/sjnsvjbf/



Finally, we have an upper bound on the error in the approximation:

**Corollary**
If $f(x,y) \in C^{k+1}$ in some closed neighborhood $N(a,b)$ of $(a,b)$, then there exists a constant $M > 0$ such that

$$|f(x,y) - P_{k,(a,b)}(x,y)| \leq M \,\|(x,y)-(a,b)\|^{k+1}$$

for all $(x,y) \in N(a,b)$.

As before, the Corollary is an existence result; it doesn't tell us how to find $M$, only that such a number exists.

### Taylor Polynomial of Degree $k$ for Functions of $n$ Variables

The final stage in the process of generalization is to consider functions of $n$ variables $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$.

Definition: Taylor polynomial of degree $k$ for functions of $n$ variables
The Taylor polynomial of degree $k$ for functions of $n$ variables is

$$P_{k,\vec{a}}(\vec{x}) = \sum_{|\alpha| \leq k} \partial^\alpha f(\vec{a}) \frac{(\vec{x} - \vec{a})^\alpha}{\alpha!}$$

## 8.4 - Putting It All Together

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

h. Show that the gradient vector of $f$ has the same direction at each point.

A question appears in Mobius

i. What conclusion can you draw about the level curves of $f$?

A question appears in Mobius

A question appears in Mobius

b. By Taylor's Theorem, the error in the linear approximation is

A question appears in Mobius

## Practice Problems

1. Find the first- and second-degree Taylor polynomials for the following functions at the given points:
   (a) $f(x,y) = e^{x^2+y^2}$ at $(0,0)$
   (b) $f(x,y) = (x-y)e^{x-y}$ at $(0,0)$
   (c) $f(x,y) = (x+y)sin(x-y)$ at $(\pi,\pi)$
2. Let $f(x,y) = e^{x-4y}$. Use Taylor's Theorem to show that the error in the linear approximation $L_{(1,1)}(x,y)$ is at most $\frac{e}{2}[5(x-1)^2 + 20(y-1)^2]$ if $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
3. Let $f(x,y) = \dfrac{1}{xy}$ for $x > 0$ and $y > 0$. Use Taylor's Theorem to show that if $x > 1$ and $y > 1$, then

$$|f(x,y) - L_{(1,1)}(x,y)| \leq \frac{3}{2}[(x-1)^2 + (y-1)^2]$$

## Select Answers and Solutions

1. Answers for problem 1
   (a) $P_{1,(0,0)}(x,y) = 1$ and $P_{2,(0,0)} = x^2 + y^2 + 1$
   (b) $P_{1,(0,0)}(x,y) = x - y$ and $P_{2,(0,0)} = x^2 - 2xy + y^2 + x - y$
   (c) $P_{1,(\pi,\pi)}(x,y) = 2\pi(x-\pi) - 2\pi(y-\pi)$ and $P_{2,(\pi,\pi)} = 2\pi(x-\pi) - 2\pi(y-\pi) + (x-\pi)^2 + (y-\pi)^2$
2. No answer provided.
3. No answer provided.

# Unit 9

# Critical Points

## 9.1 - Local Extrema and Critical Points

### Critical Points in One Variable

Let $f(x)$ be a real-valued function. A value $c \in D(f)$ is called a **critical value** of $f$ if either $f'(c) = 0$ or $f'(c)$ is undefined. Critical values are often used to find the local extrema (maxima and minima) of $f(x)$.

In this unit, we extend the definition of critical values to functions of two variables $f(x, y)$. We will use the second degree Taylor polynomial to generalize the second derivative test for local extrema. These ideas will be applied to optimization problems in the next unit. But first, let's remember the relation between critical points and local extrema with the following exercise.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Local Extrema and Critical Points

We begin with the definitions of local extrema for functions of two variables

**Definition: Local Maximum and Minimum**
A point $(a, b)$ is a **local maximum point of** $f$ if $f(x, y) \leq f(a, b)$ for all $(x, y)$ in some neighborhood of $(a, b)$.
A point $(a, b)$ is a **local minimum point of** $f$ if $f(x, y) \geq f(a, b)$ for all $(x, y)$ in some neighborhood of $(a, b)$.
Points which are either local maxima or local minima are sometimes referred to as local extrema.

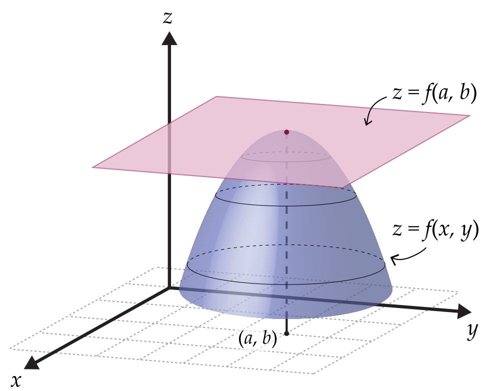
Thinking geometrically:

Let $(a, b) \in D(f)$be a local maximum/minimum point of $f$ . Then, $(a, b)$ is a local maximum/minimum point of all of the cross-sections that pass through $(a, b)$. In particular, it is a local maximum/minimum for the cross-section $f(x, b)$ and of the cross-section$f(a, y)$.

Thus, $(a, b)$ is a critical point of the cross-sections $f(x, b)$ and $f(a, y)$.

Assuming that $f$ has partial derivatives, $\dfrac{\partial f}{\partial x}(a, b) = 0$ and $\dfrac{\partial f}{\partial y}(a, b) = 0$.

This allows us to conclude that the equation of the tangent plane at $(a, b)$ is $z = f(a, b)$ which is horizontal as in the figure below.




We generalize this result in the following theorem where we also consider the possibility that one or more of the partial derivatives may not exist.

**Theorem 1:**
If $(a, b)$ is a local maximum or minimum point of $f$, then each partial derivative is either equal to zero or does not exist.

**Proof:** Consider the function $g$ defined by $g(x) = f(x, b)$. If $(a, b)$ is a local maximum/minimum point of $f$, then $x = a$ is a local maximum/minimum point of $g$, and hence either $g'(a) = 0$ or $g'(a)$ does not exist. Thus it follows that either $f_x(a, b) = 0$ or $f_x(a, b)$ does not exist. A similar argument gives $f_y(a, b) = 0$ or $f_y(a, b)$ does not exist.

Now, we are ready for the definition of critical points of functions of two variables.

**Definition: Critical Point**
A point $(a, b)$ in the domain of $f(x, y)$ is called a **critical point** of $f$ if

$$\frac{\partial f}{\partial x}(a, b) = 0 \text{ or } \frac{\partial f}{\partial x}(a, b) \text{ does not exist,}$$

and

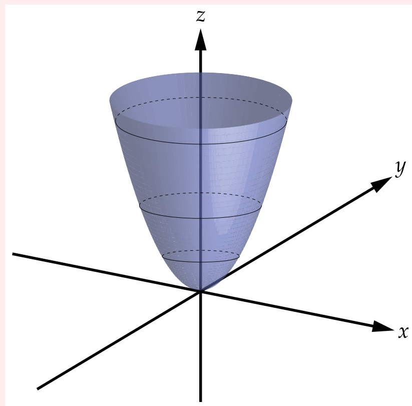
$$\frac{\partial f}{\partial y}(a, b) = 0 \text{ or } \frac{\partial f}{\partial y}(a, b) \text{ does not exist.}$$

Stop and Think
All local maxima/minima are critical points, but not all critical points are local maxima/minima. Try to come up with a function of two variables and a point $(a, b)$ with a critical point but not a local maximum or minimum.

Example 1
Find the critical points of $f(x, y) = x^2 + y^2$ and determine whether they are local maxima or local minima.
**Solution:**
We have $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$. The partial derivatives exist everywhere. However, they are equal to zero when $x = y = 0$, therefore $(0, 0)$ is the only critical point of $f$.
Observe that

$$f(x, y) = x^2 + y^2 > 0 = f(0, 0) \quad \text{for all } (x, y) \neq (0, 0)$$

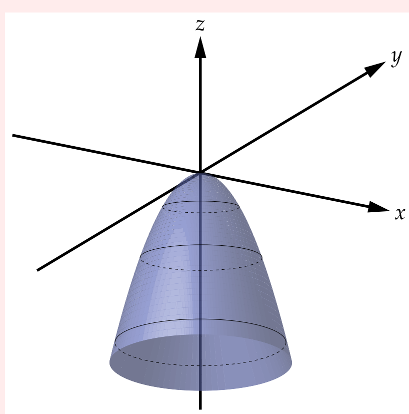
which makes $(0, 0)$ a local minimum point of $f$.
Geometrically, we see that our solution makes sense when we realize that $z = f(x, y)$ is an upward facing paraboloid.



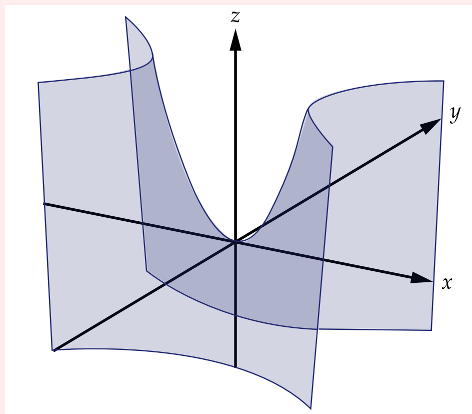

Example 2
Find the critical points of $g(x, y) = -x^2 - y^2$ and determine whether they are local maxima or local minima.
**Solution:**
We have $g_x(x, y) = -2x$ and $g_y(x, y) = -2y$. The partial derivatives exist everywhere. However, they are equal to zero when $x = y = 0$, therefore $(0, 0)$ is the only critical point of $g$.
Observe that

$$g(x, y) = -x^2 - y^2 < 0 = g(0, 0), \quad \text{for all } (x, y) \neq (0, 0)$$

which makes $(0, 0)$ a local maximum point of $g$.
Geometrically, we see that our solution makes sense when we realize that $z = g(x, y)$ is a downward-facing paraboloid.



Example 3
Find the critical points of $h(x, y) = x^2 - y^2$ and determine whether they are local maxima or local minima.
**Solution:**
We have $h_x(x, y) = 2x$ and $h_y(x, y) = -2y$. The partial derivatives exist everywhere. However, they are equal to zero when $x = y = 0$, therefore $(0, 0)$ is the only critical point of $h$.
This time, we have $h(x, 0) > h(0, 0)$ for any value of $x$ and $h(0, y) < h(0, 0)$ for any value of $y$, so $(0, 0)$ is neither a local maximum point nor a local minimum point.
Geometrically, we see that $(0, 0)$ is the point at the center of the saddle for the saddle surface $z = h(x, y)$ hence it should not be a local minimum nor a local maximum.




The last observation from Example 3 motivates the following definition.

**Definition: Saddle Point**
A critical point $(a, b)$ of $f(x, y)$ is called a **saddle point** of $f$ if in every neighborhood of $(a, b)$ there exist points $(x_1, y_1)$ and $(x_2, y_2)$ such that

$$f(x_1, y_1) > f(a, b) \text{ and } f(x_2, y_2) < f(a, b)$$

The problem of identifying local extrema for a function $f(x, y)$ is tackled in two steps:

1. Find all of the critical points of $f$.
2. Determine whether the critical points are local maxima, minima or saddle points.

Let's practice the first step, finding the critical points, with more complicated functions.

Example 4
Find all critical points of $f(x, y) = x^2y + 3xy^2 + xy$.
**Solution:**
Finding the partial derivatives, we get

$$\frac{\partial f}{\partial x}(x, y) = 2xy + 3y^2 + y, \qquad \frac{\partial f}{\partial y}(x, y) = x^2 + 6xy + x$$

In this type of problem, it is helpful to take out common factors in the expressions. To find the critical points of $f$ we will have to solve the following system of two equations

$$2xy + 3y^2 + y = 0 \Rightarrow y(2x + 3y + 1) = 0 \qquad (*)$$
$$x^2 + 6xy + x = 0 \Rightarrow x(x + 6y + 1) = 0 \qquad (**)$$

Observe that $(*)$ implies that either $y = 0$ or $2x + 3y + 1 = 0$.
We consider these two cases separately:
**Case 1:** $y = 0$.
Putting $y = 0$ into $(**)$ we get

$$\begin{aligned} x(x + 6y + 1) = 0 &\Rightarrow x(x + 6(0) + 1) = 0 \\ &\Rightarrow x(x + 1) = 0 \\ &\Rightarrow x = 0, \quad x = -1 \end{aligned}$$

The resulting two $x$ values together with the case $y = 0$, gives us two critical points $(0, 0)$ and $(-1, 0)$.
**Case 2:** $2x + 3y + 1 = 0$.
Rearranging, we have $y = \dfrac{-2x - 1}{3}$.
Putting $y = \dfrac{-2x - 1}{3}$ into $(**)$ we get

$$\begin{aligned} x(x + 6y + 1) = 0 &\Rightarrow x\left(x + 6\left(\frac{-2x - 1}{3}\right) + 1\right) = 0 \\ &\Rightarrow x(-3x - 1) = 0 \\ &\Rightarrow x = 0, \quad x = -1/3 \end{aligned}$$

giving two values $x = 0$ and $x = -1/3$.
To find the corresponding $y$ values we put these into $3y = -2x - 1$ to find $y = -1/3$ and $y = -1/9$. Thus, we get two more critical points: $(0, -1/3)$ and $(-1/3, -1/9)$.
The critical points of $f(x, y)$ are therefore $(0, 0)$, $(0, -1/3)$, $(-1, 0)$, and $(-1/3, -1/9)$.
In the following GeoGebra app, you can see the graph of $f(x, y) = x^2y + 3xy^2 + xy$ with the critical points plotted. Note the following:

- For each critical point $(a, b)$, the corresponding point on the graph has coordinates $(a, b, f(a, b))$.
- Pay close attention to the orientation of the axes.

To view the graph from different angles, click and hold on the image and then move your cursor to rotate the figure. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/hbuknaw7/

**Remark**
In the example, it was essential to solve equations $(*)$ and $(**)$ systematically, by considering all possible cases, in order to find all critical points. This systematic approach is also true in general: be sure to consider all possible cases to ensure that you find all of the critical points of a function.

In this course, we will only explicitly find the critical points for simple functions $f$. The equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$ form a system of equations which is generally non-linear, and there are no general algorithms for solving such systems exactly. There are, however, numerical methods for finding approximate solutions, one of which is a generalization of Newton's method. If you review the one variable case, you might see how to generalize it using the tangent plane. It's a challenge!

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

# 9.2 - The Second Derivative Test

## The Second Derivative Test for Functions of One Variable

For a function $f(x)$ of one variable, the second degree Taylor polynomial approximation is

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

for $x$ sufficiently close to $a$.
If $x = a$ is a critical point of $f$, then $f'(a) = 0$, and the approximation can be rearranged to give

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ f(x) &\approx f(a) + \frac{1}{2}f''(a)(x-a)^2 \\ f(x) - f(a) &\approx \frac{1}{2}f''(a)(x-a)^2 \end{aligned}$$

Note that, for $x$ sufficiently close to $a$, $f(x) - f(a)$ has the same sign as $f''(a)$.

Therefore, we make the following conclusions:

- If $f''(a) > 0$, then $f(x) - f(a) > 0$ for $x$ sufficiently close to $a$ and $x = a$ is a local minimum point.
- If $f''(a) < 0$, then $f(x) - f(a) < 0$ for $x$ sufficiently close to $a$ and $x = a$ is a local maximum point.
- If $f''(a) = 0$, we can conclude nothing.

Our next goal will be to generalize these results to functions of two variables. In order to do this, we will need to learn about objects called **quadratic forms**.

## Quadratic Forms

Let's start with a definition.

**Definition: Quadratic Form**
A function $Q$ of the form
$$Q(u,v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$$
where $a_{11}, a_{12}$ and $a_{22}$ are constants, is called a **quadratic form** on $\mathbb{R}^2$.

It is important to observe that we can use matrix notation to write
$$Q(u,v) = [u \quad v] \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

so that a quadratic form on $\mathbb{R}^2$ is determined by a $2 \times 2$ matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ be defined as the associated $2 \times 2$ matrix. Notice that the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ is symmetric.

Quadratic forms on $\mathbb{R}^2$ fall into four main classes:

1. If $Q(u,v) > 0$ for all $(u,v) \neq (0,0)$, then $Q(u,v)$ is **positive definite**.
2. If $Q(u,v) < 0$, for all $(u,v) \neq (0,0)$, then $Q(u,v)$ is **negative definite**.
3. If $Q(u,v) < 0$ for some $(u,v)$ and $Q(w,z) > 0$ for some $(w,z)$, then $Q(u,v)$ is **indefinite**.
4. If $Q(u,v)$ does not belong to classes 1) to 3), then $Q(u,v)$ is **semidefinite**. Semidefinite quadratic forms may be split into two classes:
   (a) If $Q(u,v) \geq 0$ for all $(u,v)$, then $Q(u,v)$ is **positive semidefinite.**
   (b) If $Q(u,v) \leq 0$ for all $(u,v)$, then $Q(u,v)$ is **negative semidefinite.**

These terms are also used to describe the associated symmetric matrices.

**Example 1**
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is positive definite, since the associated quadratic form $Q(u,v) = 2u^2 + 3v^2 > 0$, for all $(u,v) \neq (0,0)$.
$B = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ is indefinite, since the associated quadratic form $Q(u,v) = 2u^2 - 3v^2$, and $Q(u,0) = 2u^2 > 0$ for $u \neq 0$, and $Q(0,v) = -3v^2 < 0$ for $v \neq 0$.
$C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is semidefinite, since the associated quadratic form$Q(u,v) = 2u^2 \geq 0$ for all $(u,v)$, and $Q(0,v) = 0$ for all $v$.
More specifically, $C$ is positive semidefinite.

**Remark**
If $A$ is not a diagonal matrix, the nature of $A$ (or of $Q(u,v)$) is not immediately obvious. For example, even if all entries of $A$ are positive, it does not follow that $A$ is a positive definite matrix.

**Example 2**

Classify the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$.

**Solution:**

The associated quadratic form is

$$Q(u, v) = u^2 + 6uv + 2v^2$$

Complete the square, obtaining

$$Q(u, v) = (u + 3v)^2 - 7v^2$$

It is now clear by inspection that $A$ is indefinite, since

$$Q(u, 0) = u^2 > 0, \quad \text{for } u \neq 0$$

and

$$Q(-3v, v) = -7v^2 < 0, \quad \text{for } v \neq 0$$

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

We can also rely on the determinant of the associated matrix $A$ to classify its corresponding quadratic form using the following result:

Proposition: Determinant and Quadratic Forms
A quadratic form $Q(u,v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$ on $\mathbb{R}^2$ is

1. Positive definite if $\det(A) > 0$ and $a_{11} > 0$
2. Negative definite if $\det(A) > 0$ and $a_{11} < 0$
3. Indefinite if $\det(A) < 0$
4. Semidefinite if $\det(A) = 0$

Let's apply this result to a matrix that we've previously classified.

**Example** 3
Classify the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$.
**Solution:**
We have $\det(A) = (1)(2) - (3)(3) = -7 < 0$, therefore, by the proposition, $A$ is indefinite.

A question appears in Mobius

## The Second Derivative Test for Functions of Two Variables

Before we introduce the second derivative test for functions of two variables $f(x,y)$, we need to make some observations.

For $f(x,y) \in C^2$, recall that the second degree Taylor polynomial approximation is

$$\begin{aligned} f(x,y) &\approx P_{2,(a,b)}(x,y) \\ f(x,y) &\approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2}\Big[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\Big] \end{aligned}$$

for $(x,y)$ sufficiently close to $(a,b)$.

If $(a,b)$ is a critical point of $f$ such that

$$f_x(a,b) = 0 = f_y(a,b)$$

then the approximation can be rearranged to yield

$$\begin{aligned} f(x,y) &\approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2}\Big[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\Big] \\ f(x,y) &\approx f(a,b) + \frac{1}{2}\Big[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\Big] \\ f(x,y) - f(a,b) &\approx \frac{1}{2}\Big[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\Big] \qquad (*) \end{aligned}$$

for $(x,y)$ sufficiently close to $(a,b)$.

Note that the sign of the expression on the right of $(*)$ will determine the sign of $f(x,y) - f(a,b)$, and hence whether $(a,b)$ is a local maximum, local minimum or saddle point. We also recognize the expression on the right as a quadratic form.

Let

$$u = x - a, \qquad v = y - b$$

so that

$$f(x,y) - f(a,b) \approx \frac{1}{2}\Big[f_{xx}(a,b)u^2 + 2f_{xy}(a,b)uv + f_{yy}(a,b)v^2\Big]$$

The matrix of the quadratic form on the right is the Hessian matrix of $f$ at $(a, b)$:

$$Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

It is thus plausible that if $Hf(a,b)$ is positive definite, then

$$f(x,y) - f(a,b) > 0$$

for all $(u,v) \neq (0,0)$ i.e. for all $(x,y) \neq (a,b)$ (assuming, of course, that $(x,y)$ is sufficiently close to $(a,b)$ so that the approximation is sufficiently accurate). In other words, if $Hf(a,b)$ is positive definite, it is plausible that $(a,b)$ is a local minimum point of $f$. We can give similar arguments in the cases where $Hf(a,b)$ is negative definite or indefinite, leading to the following theorem.

**Theorem 1: Second Partial Derivatives Test**
Suppose that $f(x,y) \in C^2$ in some neighborhood of $(a,b)$ and that

$$f_x(a,b) = 0 = f_y(a,b)$$

1. If $Hf(a,b)$ is positive definite, then $(a,b)$ is a local minimum point of $f$.
2. If $Hf(a,b)$ is negative definite, then $(a,b)$ is a local maximum point of $f$.
3. If $Hf(a,b)$ is indefinite, then $(a,b)$ is a saddle point of $f$.
4. If $Hf(a,b)$ is semidefinite, then the test is inconclusive.

We will look at case 4 more carefully in the next lesson. Note that the argument preceding the theorem is not a proof since the argument involves an approximation. A rigorous proof of this result uses Taylor's formula and a continuity argument.

**Remark**
Note the analogies with the second derivative test for functions of one variable. For example, the requirement $g''(a) > 0$, which implies a local minimum, is replaced by the requirement that the matrix of second partial derivatives $Hf(a,b)$ be positive definite.

**Example 4**
Find and classify all critical points of the function $f(x,y) = x^3 - 4x^2 + 4x - 4xy^2$.
**Solution:**
To find the critical points we solve the system

$$0 = f_x(x,y) = 3x^2 - 8x + 4 - 4y^2 \qquad (*)$$
$$0 = f_y(x,y) = -8xy \qquad (**)$$

From $(**)$ we get that $x = 0$ or $y = 0$.
If $x = 0$, then $(*)$ gives $0 = 4 - 4y^2$ which then implies $y = \pm 1$.
If $y = 0$, then $(*)$ gives $0 = 3x^2 - 8x + 4 = (3x-2)(x-2)$ which then implies $x = 2$ and $x = 2/3$.
Hence, we have critical points $(0,1)$, $(0,-1)$, $(2,0)$, and $(2/3,0)$.
The second partial derivatives are

$$f_{xx}(x,y) = 6x - 8, \quad f_{xy}(x,y) = -8y, \quad f_{yy}(x,y) = -8x$$

At $(2/3,0)$, the Hessian matrix is $Hf(2/3,0) = \begin{bmatrix} -4 & 0 \\ 0 & -16/3 \end{bmatrix}$, which is negative definite since the corresponding quadratic form is $Q(u,v) = -4u^2 - \dfrac{16}{3}v^2$.
Thus, by the second partial derivative test, $(2/3,0)$ is a local maximum point.
At $(0,1)$, we get $Hf(0,1) = \begin{bmatrix} -8 & -8 \\ -8 & 0 \end{bmatrix}$. So $\det Hf(0,1) = -64 < 0$. Thus $Hf(0,1)$ is indefinite, and by the second partial derivative test, $(0,1)$ is a saddle point.
It is left as an exercise to show that $(0,-1)$ and $(2,0)$ are also saddle points.

A question appears in Mobius

**Remark**
Another way of classifying the Hessian matrix is by finding its eigenvalues. In particular, a symmetric matrix is positive definite if all of its eigenvalues are positive, negative definite if all of its eigenvalues are negative, and indefinite if has both positive and negative eigenvalues.

## Degenerate Critical Points

We have seen that quadratic forms (i.e. symmetric matrices) can be classified into four types: positive definite, negative definite, indefinite and semidefinite. Note that the second partial derivative test gives a conclusion in the first three cases but not in the semidefinite case. In fact, if $Hf(a,b)$ is semidefinite, the critical point $(a,b)$ may be a local maximum point, a local minimum point or a saddle point. We justify this statement by considering the functions

$$f(x,y) = x^4 + y^4, \quad g(x,y) = x^4 - y^4, \quad \text{and} \quad h(x,y) = -x^4 - y^4$$

For each function $(0,0)$ is the only critical point, and the Hessian matrix at $(0,0)$ is the zero matrix, which is semidefinite. However, since

$$\begin{aligned} f(x,y) - f(0,0) &\geq 0 \quad \text{for all } (x,y) \\ g(x,0) - g(0,0) &\geq 0 \quad \text{for all } x \\ g(0,y) - g(0,0) &\leq 0 \quad \text{for all } y \\ h(x,y) - h(0,0) &\leq 0 \quad \text{for all } (x,y) \end{aligned}$$

it follows that $(0,0)$ is a local minimum point for $f$, a saddle point for $g$ and a local maximum point for $h$, as seen in the following GeoGebra app. Check the boxes to display $f(x,y)$, $g(x,y)$, and $h(x,y)$.

External resource: https://www.geogebra.org/material/iframe/id/rzge8tn2/



If $Hf(a,b)$ is semidefinite, so that the second partial derivative test gives no conclusion, we say that the critical point $(a,b)$ is **degenerate**. In order to classify the critical point, we have to investigate the sign of $f(x,y) - f(a,b)$ in a small neighborhood of $(a,b)$.

**Example 5**
Show that $(0,0)$ is a degenerate critical point of $f(x,y) = 2(x-y)^2 - x^4 - y^4 + 3$ and classify it.
**Solution:**
We have

$$\nabla f(0,0) = (0,0) \quad \text{and} \qquad Hf(0,0) = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

The quadratic form associated with the Hessian is

$$Q(u,v) = 4u^2 - 8uv + 4v^2 = 4(u-v)^2 \geq 0$$

with $Q(u,u) = 0$ for all $u$, hence $Hf(0,0)$ is semidefinite. Thus, $(0,0)$ is a degenerate critical point. In order to classify it, consider

$$f(x,y) - f(0,0) = 2(x-y)^2 - x^4 - y^4$$

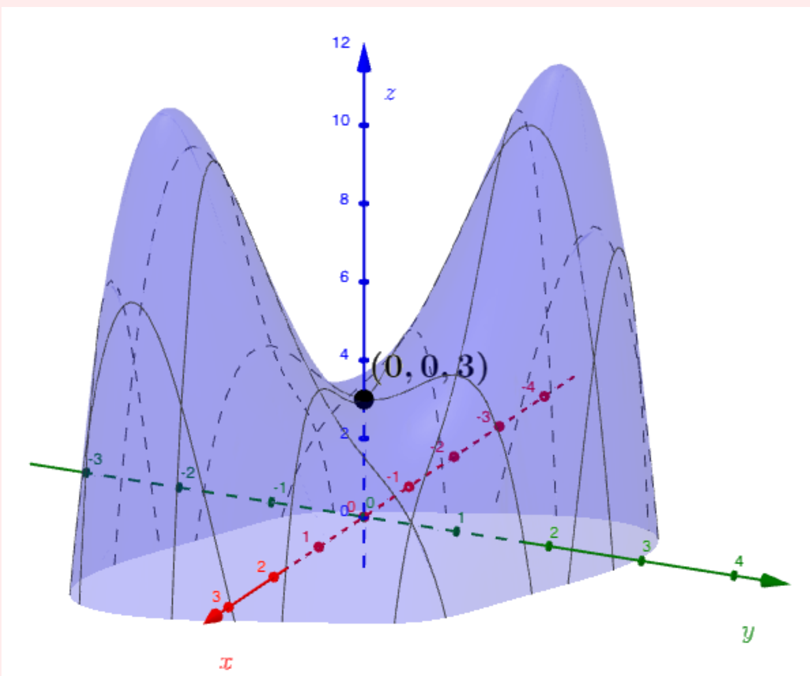
Observe that

$$f(x,x) - f(0,0) = -2x^4 < 0 \quad \text{for all } x \neq 0$$

and

$$f(x,0) - f(0,0) = 2x^2 - x^4 = x^2(2-x^2) > 0$$

for all $x$ which satisfy $0 < x^2 < 2$. So, in any sufficiently small neighborhood of $(0,0)$, $f(x,y) - f(0,0)$ assumes positive and negative values. Hence, $(0,0)$ is a saddle point.




## Generalizations

The definitions of local maximum point, local minimum point, and critical point can be generalized in a natural way to functions $f$ of $n$ variables. The Hessian matrix of $f$ at $\vec{a}$ is the $n \times n$ symmetric matrix given by

$$Hf(\vec{a}) = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \right]$$

where $i, j = 1, 2, \ldots, n$. The Hessian matrix can be classified as positive definite, negative definite, indefinite or semidefinite by considering the associated quadratic form in $\mathbb{R}^n$:

$$Q(\vec{u}) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) u_i u_j$$

The second derivative test as stated in $\mathbb{R}^2$ now holds in $\mathbb{R}^n$. It can be justified heuristically by using the second degree Taylor polynomial approximation,

$$f(\vec{x}) \approx P_{2,\vec{a}}(\vec{x})$$

which leads to

$$f(\vec{x}) - f(\vec{a}) \approx \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(x_i - a_i)(x_j - a_j)$$

generalizing equation we obtained from the second degree Taylor polynomial.

### Summary

Let's summarize what we learned about critical points and the second derivative test for functions of two variables.

To find and classify the critical points of a function of two variables $f(x, y)$:

1. Find the critical points of $f$. These are the points $(a, b)$ such that

$$\frac{\partial f}{\partial x}(a,b) = 0 \quad \text{or} \quad \frac{\partial f}{\partial x}(a,b) \quad \text{DNE} \quad \text{and} \quad \frac{\partial f}{\partial y}(a,b) = 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(a,b) \quad \text{DNE}$$

2. Classify the critical points of $f(x, y)$ by calculating $Hf(a, b)$:
   - If $Hf(a, b)$ is negative definite, then $(a, b)$ is a local maximum
   - If $Hf(a, b)$ is indefinite, then $(a, b)$ is a saddle point
   - If $Hf(a, b)$ is positive definite, then $(a, b)$ is a local minimum
   - If $Hf(a, b)$ is semidefinite, then $(a, b)$ is a degenerate critical point. Check the sign of $f(x, y) - f(a, b)$ in a small neighbourhood of $(a, b)$:
     - If the sign of $f(x, y) - f(a, b)$ is always positive, then $(a, b)$ is a local minimum
     - If the sign of $f(x, y) - f(a, b)$ is always negative, then $(a, b)$ is a local maximum
     - If $f(x, y) - f(a, b)$ assumes both positive and negative values, then $(a, b)$ is a saddle point

## 9.3 - Convex Functions

### Convex Functions

Convex functions of one variable

As we will see, the concept of convex functions can also help us classify critical points. Let's start with the definition of convex functions of one variable.

Definition: Convex and strictly convex functions of one variable
A twice differentiable function $f(x)$ is **convex** if $f''(x) \geq 0$ for all $x$ and $f$ is **strictly convex** if $f''(x) > 0$ for all $x$. Thus the term convex means "concave up".

A question appears in Mobius

Convex functions have two interesting properties.

**Theorem 1:** Properties of convex functions of one variable
If $f(x) \in C^2$ and is strictly convex, then

1. $f(x) > L_a(x) = f(a) + f'(x)(x-a)$ for all $x \neq a$, for any $a \in \mathbb{R}$.
2. For $a < b$, $f(x) < f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$ for $x \in (a,b)$.

**Proof:**

1. It follows from Taylor's Theorem that $f(x) = L_a(x) + \dfrac{f''(c)}{2}(x-a)^2$ where $c$ is between $a$ and $x$.
   Thus $R_{1,a}(x) > 0$ for $x \neq a$, giving $f(x) > L_a(x)$ for all $x \neq a$.
2. Let $g(x) = f(x) - \left[f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)\right]$.
   Then $g(a) = g(b) = 0$ and $g''(x) = f''(x) > 0$. We must show that $g(x) < 0$ for $x \in (a,b)$.
   By the Mean Value Theorem, $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in (a,b)$. Note that $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a} = f'(x) - f'(c)$. Thus $g'(c) = 0$.
   Since $g''(x) > 0$, then $g'(x)$ is strictly increasing. Since $g'(c) = 0$ then $g'(x) < 0$ on $[a,c)$ and $g'(x) > 0$ on $(c,b]$.
   This implies that $g(x)$ is strictly decreasing on $[a,c]$ and strictly increasing on $[c,b]$. Since $g(a) = 0$ and $g(b) = 0$ we get that $g(x) < 0$ on $(a,c]$ and on $[c,b)$. Therefore, $g(x) < 0$ on $(a,b)$, as required.

Part 1 of the theorem tells us that the graph of $f$ lies above any tangent line; part 2 of the theorem tells us that any secant line lies above the graph of $f$.

## Convex Functions of Two Variables

The definition of convex functions of two variables is somewhat analogous to the one-variable definition. Recall that instead of a single second derivative, the second partial derivatives of $f(x,y)$ are contained in the Hessian matrix $Hf(x,y)$.

Definition: Convex and strictly convex functions of two variables
Let $f(x, y)$ have continuous second partial derivatives. We say that $f$ is **convex** if $Hf(x, y)$ is positive semi-definite for all $(x, y)$ and that $f$ is **strictly convex** if $Hf(x, y)$ is positive definite for all $(x, y)$.

By the proposition on the determinant and quadratic forms, $f$ being strictly convex means $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ for all $(x, y)$.

**Example 1**
Determine whether $f(x, y) = x^2 + y^2$ is convex or strictly convex.
**Solution:**
We have $f_{xx} = 2$, $f_{xy} = 0$, and $f_{yy} = 2$, so $f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$, which means that $f$ is strictly convex.

A question appears in Mobius

We get a result which is analogous to Theorem 1.

**Theorem 2:** Properties of convex functions of two variables
If $f(x, y)$ has continuous second partial derivatives and is strictly convex, then

1. $f(x, y) > L_{(a,b)}(x, y)$ for all $(x, y) \neq (a, b)$, and
2. $f(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2)) < f(a_1, a_2) + t[f(b_1, b_2) - f(a_1, a_2)]$ for $0 < t < 1$, $(a_1, a_2) \neq (b_1, b_2)$.

**Proof:**

1. By Taylor's Theorem,
$$f = L_{(a,b)}(x, y) + \frac{1}{2}\left[f_{xx}(c, d)(x - a)^2 + 2f_{xy}(c, d)(x - a)(y - b) + f_{yy}(y - b)^2\right]$$
where $(c, d)$ is on the line segment from $(a, b)$ to $(x, y)$.
Since $f_{xx}(c, d) > 0$, $f_{xx}(c, d)f_{yy}(c, d) - f_{xy}(c, d)^2 > 0$, we have that $R_{1,a,b}(x, y) > 0$ for $(x, y) \neq (a, b)$. Therefore, $f(x, y) > L_{(a,b)}(x, y)$ for $(x, y) \neq (a, b)$.
2. We parameterize the line segment $L$ from $(a_1, a_2)$ to $(b_1, b_2)$ by
$$L(t) = (a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2)), \qquad 0 \leq t \leq 1$$
For simplicity, let $h = b_1 - a_1$ and $k = b_2 - a_2$. Define $g(t)$ by
$$g(t) = f(L(t)), \qquad 0 \leq t \leq 1 \tag{*}$$

Since, by hypothesis, $f$ has continuous second partial derivatives, we can apply the Chain Rule to conclude that $g'$ and $g''$ are continuous and are given by

$$g'(t) = f_x(L(t))h + f_y(L(t))k \tag{**}$$
$$g''(t) = f_{xx}(L(t))h^2 + 2f_{xy}(L(t))hk + f_{yy}(L(t))k^2 \tag{***}$$

for $0 \le t \le 1$. Since $f_{xx}(L(t)) > 0$ and $f_{xx}(L(t))f_{yy}(L(t)) - f_{xy}(L(t))^2 > 0$ for all $t$, $g''(t) > 0$ by Theorem 2. Thus, by Theorem 1, part (2):

$$g(t) < g(0) + \frac{g(1) - g(0)}{1 - 0}(t - 0), \qquad \text{for } 0 < t < 1$$

Therefore, $f(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2)) < f(a_1, a_2) + t[f(b_1, b_2) - f(a_1, a_2)]$ for $0 < t < 1$ as required.

Similarly to the one variable case, part 1 of Theorem 2 tells us that the graph of $f$ lies above the tangent plane and part 2 tells us that the cross-section of the graph of $f$ above the line segment from $(a_1, a_2)$ to $(b_1, b_2)$ lies below the secant line.

Now, let's see how convexity can help us classify some critical points.

Theorem 3: Critical points of convex and strictly convex functions
If $f(x, y) \in C^2$ is convex, then every critical point $(c, d)$ satisfies $f(x, y) \ge f(c, d)$ for all $(x, y) \ne (c, d)$.
If $f(x, y) \in C^2$ is strictly convex and has a critical point $(c, d)$, then $f(x, y) > f(c, d)$ for all $(x, y) \ne (c, d)$ and $f$ has no other critical point.

We will prove the second part of this theorem, as the first part proceeds by a similar argument.

**Proof of second statement:**

Since $f(x, y) \in C^2$, we know that $f$ has continuous second derivatives.

Also, $(c, d)$ being a critical point for $f$ implies $f_x(c, d) = 0 = f_y(c, d)$.

Therefore, $L_{(c,d)}(x, y) = f(c, d)$.

By Theorem 2, part 1, we conclude that $f(x, y) > f(c, d)$ for all $(x, y) \ne (c, d)$ as desired.

For the uniqueness part, let's assume that $f$ has a second critical point $(c_1, d_1)$such that $(c_1, d_1) \ne (c, d)$.

Then, by similar reasoning to that above, $f(x, y) > f(c_1, d_1)$ for all $(x, y) \ne (c_1, d_1)$.

Since $(c_1, d_1) \ne (c, d)$, we get $f(c, d) > f(c_1, d_1)$.

But $f(x, y) > f(c, d)$ for all $(x, y) \ne (c, d)$ implies $f(c_1, d_1) > f(c, d)$again since $(c_1, d_1) \ne (c, d)$.

And hence we get $f(c_1, d_1) > f(c, d)$ and $f(c, d) > f(c_1, d_1)$ which is a contradiction.

The first part of the theorem says that if $f(x, y) \in C^2$ is convex, then all of its critical points minimize $f$. The second part of the theorem says that if $f(x, y) \in C^2$ is strictly convex and has a critcal point, then the critical point is unique and minimizes $f$. Thus, a strictly convex function $f(x, y) \in C^2$has at most one critical point which, if it exists, must be a minimum.

A question appears in Mobius

# 9.4 - Proof of the Second Partial Derivative Test

## Appendix: Proof of the Second Partial Derivative Test

We now want to prove part (1) of the Second Partial Derivative Test. The proof depends significantly on the hypothesis that the second partial derivatives of $f$ are continuous, and on a plausible property of positive definite matrices: making a small change to the entries of a positive definite matrix leaves the new matrix positive definite. This is proved separately as Lemma 1.

**Lemma 1:**
Let $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a positive definite matrix. If $|\tilde{a}-a|$, $|\tilde{b}-b|$ and $|\tilde{c}-c|$ are sufficiently small, then $\begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{c} \end{bmatrix}$ is positive definite.

**Proof:**

Let $Q$ and $\tilde{Q}$ be the quadratic forms determined by the given matrices i.e.

$$Q(u,v) = au^2 + 2buv + cv^2 \tag{*}$$

and similarly for $\tilde{Q}(u,v)$. We perform the change of variables

$$u = r\cos\theta, \qquad v = r\sin\theta$$

to obtain

$$Q(u,v) = r^2 p(\theta) \tag{**}$$

where

$$p(\theta) = a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta$$

Since for $r=1$, $Q(u,v) = p(\theta)$, and $Q$ is positive definite, we must have $p(\theta) > 0$ for all $\theta$, $0 \le \theta \le 2\pi$.

Let

$$k = \min_{0\le\theta\le 2\pi} p(\theta)$$

Then $k > 0$

and by equation $(**)$

$$Q(u,v) \geq kr^2 \quad \text{for all } (u,v) \neq (0,0) \tag{***}$$

We are given that $|\tilde{a} - a|$, $|\tilde{b} - b|$ and $|\tilde{c} - c|$ are sufficiently small. Let

$$\delta = \max\{|\tilde{a} - a|, |\tilde{b} - b|, |\tilde{c} - c|\}$$

By equation $(*)$ and the triangle inequality,

$$\begin{aligned}
|Q(u,v) - \tilde{Q}(u,v)| &\leq |\tilde{a} - a|u^2 + 2|\tilde{b} - b||u||v| + |\tilde{c} - c|v^2 \\
&\leq \delta(u^2 + 2|u||v| + v^2) \\
&= \delta(|u| + |v|)^2 \\
&= \delta r^2(|\cos\theta| + |\sin\theta|)^2 \\
&< 4\delta r^2
\end{aligned}$$

We now choose $\delta = \dfrac{1}{8}k$. Then

$$|Q(u,v) - \tilde{Q}(u,v)| < \frac{1}{2}kr^2$$

which implies

$$\begin{aligned}
\tilde{Q}(u,v) &\geq Q(u,v) - \frac{1}{2}kr^2 \\
&\geq kr^2 - \frac{1}{2}kr^2, \qquad \text{by } (***) \\
&= \frac{1}{2}kr^2
\end{aligned}$$

This shows that $\tilde{Q}(u,v) > 0$ for all $(u,v) \neq (0,0)$. Therefore, $\tilde{Q}(u,v)$ is positive definite. $\square$

Proof taken from *Calculus 3 Course Notes for Math 237 (Edition 6.1)* as provided by D. Siegel

**Remark**
The lemma is also true if "positive definite" is replaced by "negative definite" or "indefinite".

We now prove the Second Partial Derivative Test. For convenience, we restate the theorem.

**Theorem 2: The Second Partial Derivative Test**
Suppose that $f(x,y) \in C^2$ in some neighborhood of $(a,b)$ and that

$$f_x(a,b) = 0 = f_y(a,b)$$

1. If $Hf(a,b)$ is positive definite, then $(a,b)$ is a local minimum point of $f$.
2. If $Hf(a,b)$ is negative definite, then $(a,b)$ is a local maximum point of $f$.
3. If $Hf(a,b)$ is indefinite, then $(a,b)$ is a saddle point of $f$.

**Proof:**

We will prove (1).

We apply Taylor's formula with second order remainder. Since

$$f_x(a,b) = 0 = f_y(a,b)$$

Taylor's formula can be written as

$$f(x,y) - f(a,b) = \frac{1}{2}\Big[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2\Big] \qquad (\dagger)$$

where $(c,d)$ lies on the line segment joining $(a,b)$ and $(x,y)$. The coefficient matrix in the quadratic expression on the right side of $(\dagger)$ is the Hessian matrix $Hf(c,d)$.

We are given that $Hf(a,b)$ is positive definite. By the lemma, there exists $\epsilon > 0$ such that if

$$|f_{xx}(x,y) - f_{xx}(a,b)| < \epsilon, |f_{xy}(x,y) - f_{xy}(a,b)| < \epsilon, |f_{yy}(x,y) - f_{yy}(a,b)| < \epsilon \qquad (\dagger\dagger)$$

then $Hf(x,y)$ is positive definite. Since the second partials of $f$ are continuous at $(a,b)$, the definition of continuity implies that there exists a $\delta > 0$ such that

$$\|(x,y) - (a,b)\| < \delta$$

implies $(\dagger\dagger)$ and hence that $Hf(x,y)$ is positive definite. Since

$$\|(c,d) - (a,b)\| < \|(x,y) - (a,b)\|$$

it follows that $Hf(c,d)$ is also positive definite. It now follows from equation $(\dagger)$ and the definition of positive definite matrix, that if $0 < \|(x,y)-(a,b)\| < \delta$, then $f(x,y) - f(a,b) > 0$. Thus, by definition $(a,b)$ is a local minimum point of $f$. $\square$

Parts (2) and (3) of the Second Partial Derivative Test can be proved in a similar way using the modified lemma.

## 9.5 - Putting It All Together

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Practice Problems

Find and classify the critical points of the function $f$. Use a mathematical software to sketch the graph of the function to verify your answers.

1. $f(x, y) = xy^2 - x^2y - xy + x^2$
2. $f(x, y) = xye^{x+2y}$
3. $f(x, y) = (x^2 + y^2 - 1)y$
4. $f(x, y) = x\sin(x + y)$
5. $f(x, y) = x^2 - 2x + y^3 - xy^2$
6. $f(x, y) = (x + y)(xy + 1)$
7. $f(x, y) = x^2 + y^2 + x^2y + 4$

## Select Answers and Solutions

Find and classify the critical points of the function $f$.

1. $f(x, y) = xy^2 - x^2y - xy + x^2$
   $(0, 0)$: Saddle point
   $(0, 1)$: Saddle point
   $(1, 1)$: Saddle point
   $(1/3, 2/3)$: Local minimum
2. $f(x, y) = xye^{x+2y}$
   $(0, 0)$: Saddle point
   $(-1, -1/2)$: Local maximum
3. $f(x, y) = (x^2 + y^2 - 1)y$
   $(0, 1/\sqrt{3})$: Local minimum
   $(0, -1/\sqrt{3})$: Local maximum
   $(1, 0)$: Saddle point
   $(-1, 0)$: Saddle point
4. $f(x, y) = x\sin(x + y)$
   $(0, n\pi)$ where $n \in \mathbb{Z}$: Infinitely many saddle points
5. $f(x, y) = x^2 - 2x + y^3 - xy^2$
   $(1, 0)$: Saddle point
   $(3, 2)$: Saddle point
   $\left(\dfrac{3}{2}, 1\right)$: Local minimum
6. $f(x, y) = (x + y)(xy + 1)$
   $(a, a)$ where $a \in \mathbb{R}$: Infinitely many saddle points
   $(a, -a)$ where $a \in \mathbb{R}$: Infinitely many saddle points

7. $f(x, y) = x^2 + y^2 + x^2y + 4$

   No answers provided.

The following GeoGebra app allows you to explore the critical points.

**Instructions**

1. Select a function from the dropdown menu to see its 3D graph.
2. Rotate the image or zoom in on the points.
3. Click to reset the view.

External resource: https://www.geogebra.org/material/iframe/id/n3p8rzqh/

# Unit 10

# Optimization Problems

## 10.1 - The Extreme Value Theorem

### The Extreme Value Theorem for Functions of One Variable

As we saw in single-variable calculus, we are often interested in finding the largest or smallest possible value of a function $f$ on some specified interval $I$. In case of functions of several variables, we are interested in finding the largest or smallest possible value of a function $f$ on some specified set $S$.

Let's review the definition of absolute maximum and absolute maximum value of a real-valued function $f$.

Definition: **Absolute Maximum and Minimum,** Functions of One Variable
Given a real-valued function $f(x)$ and an interval $I \subseteq \mathbb{R}$,

1. a point $x = c \in I$ is called an **absolute maximum point** of $f$ on $I$ if

$$f(x) \leq f(c) \quad \text{for all } x \in I$$

   The value $f(c)$ is called the **absolute maximum value** of $f$ on $I$.

2. a point $x = c \in I$ is called an **absolute minimum point** of $f$ on $I$ if

$$f(x) \geq f(c) \quad \text{for all } x \in I$$

   The value $f(c)$ is called the **absolute minimum value** of $f$ on $I$.

Recall from single-variable calculus that the Extreme Value Theorem gives conditions that imply the existence of a maximum value and minimum value of $f$ on an interval $I$. Let's recall the Extreme Value Theorem.

Theorem 1: The Extreme Value Theorem for Functions of One Variable
If $f(x)$ is continuous on a finite closed interval $I$, then there exists $c_1, c_2 \in I$ such that

$$f(c_1) \leq f(x) \leq f(c_2) \quad \text{for all } x \in I$$

The Extreme Value Theorem tells us that on a closed interval, a continuous function of one variable will always have absolute maxima and minima.

For our purposes, the important thing is to be able to give counterexamples to show that the conclusion may not be valid if the hypotheses are not satisfied.

**Your Turn**

Give a function $f(x)$ and an interval $I$ such that:

- $I$ is closed, but $f$ does not have an absolute maximum on $I$.

A question appears in Mobius

- $I$ is finite and $f$ is continuous on $I$, but $f$ does not have an absolute maximum on $I$.

A question appears in Mobius

- $I$ is infinite and $f$ is continuous on $I$, but $f$ does not have an absolute minimum.

A question appears in Mobius

Now, we will introduce the definition of absolute extrema for the functions of two variables.

**Definition: Absolute Maximum and Minimum**
Given a function $f(x, y)$ and a set $S \subseteq \mathbb{R}^2$,

1. a point $(a, b) \in S$ is an **absolute maximum point** of $f$ on $S$ if

$$f(x, y) \leq f(a, b) \quad \text{for all } (x, y) \in S$$

   The value $f(a, b)$ is called the **absolute maximum value** of $f$ on $S$.

2. a point $(a, b) \in S$ is an **absolute minimum point** of $f$ on $S$ if

$$f(x, y) \geq f(a, b) \quad \text{for all } (x, y) \in S$$

   The value $f(a, b)$ is called the **absolute minimum value** of $f$ on $S$.

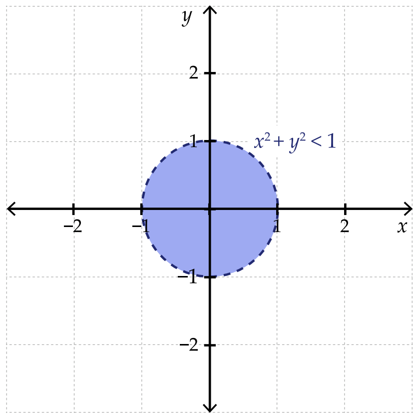
Whether or not a multivariable function $f$ has a maximum or minimum value on $S$ depends on $f$ and on the set $S$. So, before we generalize the Extreme Value Theorem, we will need to look at a few new definitions about sets.

## Topological Aspects of $\mathbb{R}^2$

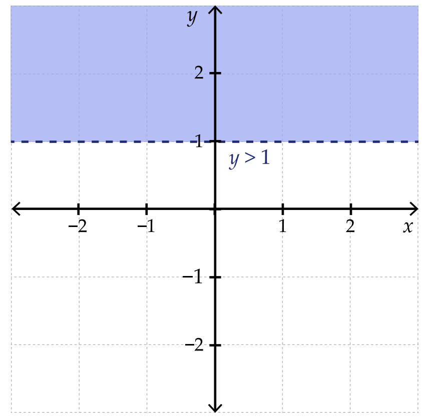
In order to generalize the Extreme Value Theorem to the functions of two variables, we need to generalize the concept of a finite closed interval to sets in $\mathbb{R}^2$. To do this, we will take a brief detour into some topological aspects of $\mathbb{R}^2$. Let's start by defining a bounded set.

**Definition: Bounded Set**
A set $S \subset \mathbb{R}^2$ is said to be **bounded** if and only if it is contained in some neighbourhood of the origin.

Observe that the definition implies that every point in $S$ must have finite distance from the origin. These sets are easy to visualize in 2D space.




The set of points $\{(x, y) \mid x^2 + y^2 < 1\}$ is an example of a bounded set since it can be contained in the neighbourhood of radius 2 around the origin.

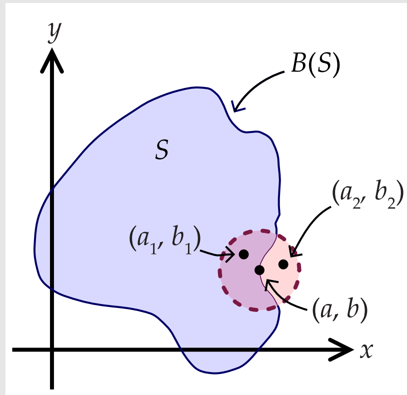



The set of points $\{(x, y) \mid y > 1\}$ is an example of an unbounded set. It cannot be contained in a neighbourhood of finite radius around the origin.

A question appears in Mobius

Intuitively, a "boundary point" of a set $S \subset \mathbb{R}^2$ is a point which lies on the "edge" of $S$. Here is the formal definition.

**Definition: Boundary Point**
Given a set $S \subset \mathbb{R}^2$, a point $(a, b) \in \mathbb{R}^2$ is said to be a **boundary point** of $S$ if and only if every neighbourhood of $(a, b)$ contains at least one point in $S$ and one point not in $S$.




**Example 1**
Consider $S = (0, 1) \subset \mathbb{R}$.
0 and 1 are the boundary points of $S$.

**Definition: Boundary of $S$**
The set $B(S)$ of all boundary points of $S$ is called the **boundary** of $S$.

**Example 2**
Consider $S = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$.
The boundary of $S$ is the $y$-axis which is in $S$.

A question appears in Mobius

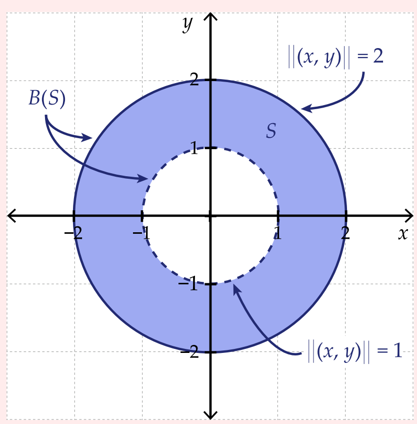
A question appears in Mobius

**Definition: Closed Set**
A set $S \subseteq \mathbb{R}^2$ is said to be **closed** if $S$ contains all of its boundary points.

**Example 3**
Consider $S = \{(x, y) \in \mathbb{R}^2 \mid 1 < \|(x, y)\| \leq 2\}$. The boundary of $S$ is the set of all boundary points. So, as indicated in the diagram, the boundary of $S$ is

$$B(S) = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1 \text{ or } \|(x, y)\| = 2\}$$

Since the points $(x, y)$ such that $\|(x, y)\| = 1$ are not in $S$, we have that $S$ is not closed.




**Example 4**
Consider $S = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$ again for which the boundary of $S$ was the $y$-axis. Since the $y$-axis is in $S$, $S$ is closed.

Observe that the concept of a "closed set" in $\mathbb{R}^2$ generalizes the idea of a closed interval in $\mathbb{R}$.

A question appears in Mobius

## The Extreme Value Theorem for Functions of Two Variables

With all of the definitions from the previous lesson, we can now state the generalization of the Extreme Value Theorem to $\mathbb{R}^2$.

Theorem 2: Extreme Value Theorem (EVT) for Functions of Two Variables
If $f(x, y)$ is continuous on a closed and bounded set $S \subset \mathbb{R}^2$, then there exist points $(a, b), (c, d) \in S$ such that

$$f(a, b) \leq f(x, y) \leq f(c, d) \quad \text{for all } (x, y) \in S$$

The proof of this theorem is beyond the scope of this course. Note that this theorem is telling us something quite similar to the single-variable Extreme Value Theorem: A continuous function of two variables on **a closed and bounded set** will always have absolute maxima and minima.

A question appears in Mobius

**Remark**
A function $f(x, y)$ may have an absolute maximum and/or an absolute minimum on a set $S \subseteq \mathbb{R}^2$ even if the conditions of the Extreme Value Theorem are not satisfied.

**Example 4**

Let $S = \{(x, y) \in \mathbb{R}^2 \mid x > -1, y \in \mathbb{R}\}$ and let $f(x, y) = \begin{cases} 1 & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

The function $f$ is not continuous on $S$ and $S$ is neither closed nor bounded. However, 1 is clearly the maximum value of $f$ on $S$ and 0 is clearly the minimum.

## 10.2 - Algorithm for Extreme Values

### Algorithm for Extreme Values

Recall that if a single variable function $f(x)$ is continuous, then the maximum value and minimum value of $f$ on an interval $[a, b]$ occur either at a critical point of $f$ (i.e. $f'(c) = 0$, or $f'(c)$ does not exist) or at an endpoint of the interval. Moreover, our algorithm for finding the maximum and/or minimum value for $f$ on $[a, b]$ was to find the values of $f$ at any critical points of $f$ in $[a, b]$ and compare them to the values of $f$ at the endpoints $x = a$ and $x = b$. This method was called the Closed Interval Method.

The Closed Interval Method can be generalized to functions of two variables $f(x, y)$ in the following way:

Let $S \subset \mathbb{R}^2$ be a closed and bounded set, with boundary $B(S)$ and suppose that $f$ is continuous on $S$. The maximum value and minimum value of $f$ on $S$ will occur either at a critical point of $f$ that is in $S$, or at a point on the boundary of $S$. Thus, we get the following procedure which corresponds to what we were doing for functions of one variable.

**Algorithm**
First, check to see if the given set $S \subset \mathbb{R}^2$ is closed and bounded.
Next, check to see if the given function $f(x, y)$ is continuous on $S$.
If both conditions above are satisfied, then to find the maximum and/or minimum value of a function $f(x, y)$:

1. Find all critical points of $f$ that are contained in $S$.
2. Evaluate $f$ at each such point.
3. Find the maximum and minimum values of $f$ on the boundary $B(S)$.
4. The maximum value of $f$ on $S$ is the largest value of the function found in steps 2 and 3.
   The minimum value of $f$ on $S$ is the smallest value of the function found in steps 2 and 3.

Note that the absolute maximum and/or minimum value may occur at more than one point in $S$. Furthermore, in this algorithm, it is not necessary to determine whether the critical points are local maximum or minimum points.

**Example 1**
Find the maximum value of $f(x,y)=xy$ on the set

$$S=\{(x,y)\mid x^2+y^2\le 1\}$$

**Solution:**
The set $S$ is closed and bounded and $f(x,y)$ is continuous on $S$, so we proceed with our algorithm.
First, we observe that $\nabla f(x,y)=(y,x)$, hence the only critical point of $f$ is $(0,0)$ which is in $S$. We have $f(0,0)=0$.
Second, we look for the maximum value of $f$ on the boundary $B(S)$ of $S$.
To do this, we describe the boundary (the unit circle $x^2+y^2=1$) in the parametric form:

$$x=\cos t, \qquad y=\sin t, \qquad 0\le t\le 2\pi$$

On $B(S)$, $f$ has the values

$$g(t)=f(\cos t,\sin t)=\cos t\sin t=\frac{1}{2}\sin 2t$$

The problem now is to find the maximum value of $g(t)$ on the interval $0\le t\le 2\pi$.
We use the method from single-variable calculus. We have

$$g'(t)=\cos 2t$$

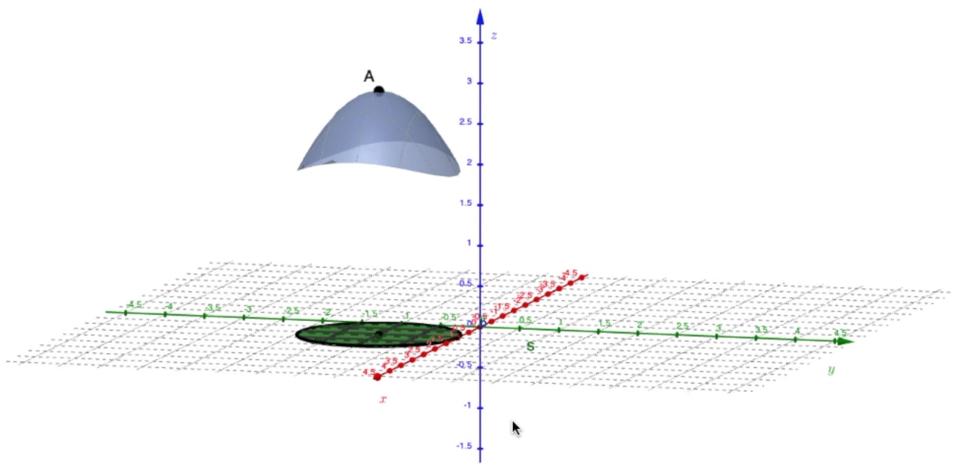
Hence, on $0\le t\le 2\pi$, the critical points of $g$ are at $t=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$. We have

$$g\left(\frac{\pi}{4}\right)=\frac{1}{2},\quad g\left(\frac{3\pi}{4}\right)=-\frac{1}{2},\quad g\left(\frac{5\pi}{4}\right)=\frac{1}{2},\quad g\left(\frac{7\pi}{4}\right)=-\frac{1}{2}$$

Finally, we have $g(0)=0$ and $g(2\pi)=0$.
So, the maximum value of $f$ on the boundary $B(S)$ is $\frac{1}{2}$ and occurs at $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$.
Comparing the values we found in the first and second steps, we see that the maximum value of $f$ on $S$ is $\frac{1}{2}$ and occurs on the boundary at $\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$.
Use the interactive applet to see 3D images of $f$ and the set $S$. Restricting $f(x,y)$ to the set $S$ shows the maximum points A: $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and B: $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$. Rotate or zoom the figure to see A and B from different angles. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/hugz98fd/

*A video appears here.*




Created with `https://www.geogebra.org`. CC BY-NC-SA 3.0.

*Additional content appears in Mobius.*

**Your Turn**

Find the maximum of $f(x, y) = x^2y - y$ on the set $S = \{(x, y) \mid 9x^2 + 4y^2 \leq 36\}$ if it exists.

Use the GeoGebra app to view the function $f$ and the set $S$ from different angles to get a sense of what the maximum is and the points at which it occurs. Then step through the following calculations to verify your intuition.

**Instructions:**

1. Click and drag the image to view it from different angles.
2. Work through the questions in the step by step solution that follows the GeoGebra app, then compare the final answer with what you saw in the app.

External resource: https://www.geogebra.org/material/iframe/id/fsvsc3ak/

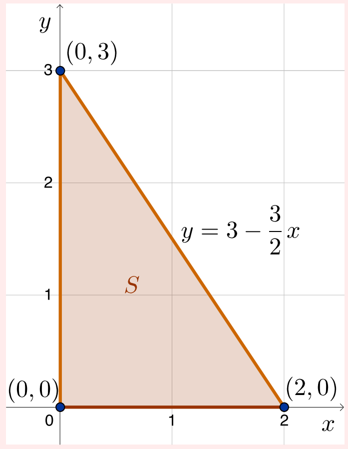
Created with `https://www.geogebra.org`. CC BY-NC-SA 3.0.

A question appears in Mobius

### Examples Continued

**Example 2**

Find the maximum and minimum value of $f(x,y) = xy - 2x - y + 2$ on the triangular region $S$ with vertices $(0,0)$, $(2,0)$ and $(0,3)$.




**Solution:**

The set $S$ is closed and bounded and $f(x,y)$ is continuous on $S$, so we proceed with our algorithm.
First, we observe that $\nabla f(x,y) = (y-2, x-1)$ so the only critical point of $f$ is $(1,2)$. Since $(1,2) \notin S$, this critical point plays no part in the solution.
The second step is to evaluate $f$ on the boundary $B(S)$ of $S$. This has to be done on the three straight line segments separately. The values of $f$ on $B(S)$ define a function of one variable, which we denote by $g$.

- **Case 1:** $x = 0$**,** $0 \le y \le 3$**.** Let $g(y) = f(0,y) = -y + 2$. By inspection, the maximum of $g$ on the interval $[0,3]$ occurs at the end point $y = 0$, and the minimum occurs at the end point $y = 3$. So, $(0,0)$ and $(0,3)$ are possible maximum and minimum points for $f$.
- **Case 2:** $y = 0$**,** $0 \le x \le 2$**.**

  Let $g(x) = f(x,0) = -2x + 2$. As in Case 1, this leads to $(0,0)$ and $(2,0)$ as possible maximum and minimum points for $f$.
- **Case 3:** $y = 3 - \frac{3}{2}x$**,** $0 \le x \le 2$**.**

  Let $g(x) = f(x, 3 - \frac{3}{2}x) = -\frac{3}{2}x^2 + \frac{5}{2}x - 1$, after simplifying. To find the critical points of $g$ we solve

  $$0 = g'(x) = -3x + \frac{5}{2}$$

  This gives $x = \frac{5}{6}$. Hence, $\left(\frac{5}{6}, \frac{7}{4}\right)$ and the end points $(0,3)$ and $(2,0)$ are possible maximum and minimum points of $f$.

Now evaluate $f$ at all points found above:

$$f(0,0) = 2, \quad f(0,3) = -1, \quad f(2,0) = -2, \quad f\left(\frac{5}{6}, \frac{7}{4}\right) = \frac{1}{24}$$

Consequently, the maximum value of $f$ on $S$ is $f(0,0) = 2$, and minimum value of $f$ on $S$ is $f(2,0) = -2$.
Use the applet below to confirm these calculations. Restricting $f(x,y)$ to the set $S$ shows the maximum point $A$ and the minimum point $B$. Click to reset to the original configuration.

External resource: https://www.geogebra.org/material/iframe/id/r39bta8u/

A question appears in Mobius

## 10.3 - Optimization with Constraints

### Optimization with Constraints

In many real world problems, we wish to find the maximum or minimum of a function $f(x, y)$ subject to a constraint $g(x, y) = k$.

For example, assume that a manufacturer has three product lines. Let $x, y, z$ denote the number of articles produced of each type and let $a, b, c$ denote the profit per article for the three product lines respectively. The total profit is given by

$$P(x, y, z) = ax + by + cz$$

Further, assume that the manufacturer wishes to maintain production costs at a constant level of $k$ dollars per day. The production costs $C$ depend on the number of articles $x, y, z$. That is, we require that

$$C(x, y, z) = k$$

The problem is to find the maximum profit $P(x, y, z)$ subject to the constraint $C(x, y, z) = k$.

Observe that in step 2 of our algorithm for finding extreme values, we also need to find the maximum and/or minimum of $f$ subject to a constraint, namely the boundary of the region. In the last lesson, we did this by finding a parametric representation for the boundary. Of course, in many cases this may be extremely difficult or impossible to do.

We now derive an algorithm which will allow us to find the maximum and/or minimum of a differentiable function $f$ on a smooth curve $g(x, y) = k$ without having to parameterize the curve.

### Method of Lagrange Multipliers

We want to find the maximum or minimum value of a differentiable function $f(x, y)$ subject to the constraint $g(x, y) = k$ where $g \in C^1$, or, more geometrically, to find the maximum or minimum value of $f(x, y)$ on the level curve $g(x, y) = k$.

If $f(x, y)$ has a local maximum or minimum at $(a, b)$ relative to nearby points on the curve $g(x, y) = k$ and $\nabla g(a, b) \neq$

$(0,0)$, then, by the Implicit Function Theorem, $g(x,y)=k$ can be described by parametric equations

$$x=p(t), \quad y=q(t) \tag{*}$$

with $p$ and $q$ differentiable, and $(a,b)=(p(t_0),q(t_0))$ for some $t_0$. Define

$$u(t)=f(p(t),q(t))$$

The function $u$ gives the values of $f$ on the constraint curve, and hence has a local maximum or minimum at $t_0$. It follows that

$$u'(t_0)=0 \tag{**}$$

Assuming that $f$ is differentiable, we can apply the Chain Rule to get

$$u'(t)=f_x(p(t),q(t))p'(t)+f_y(p(t),q(t))q'(t)$$

Evaluating this at $t_0$ and using $(**)$ gives

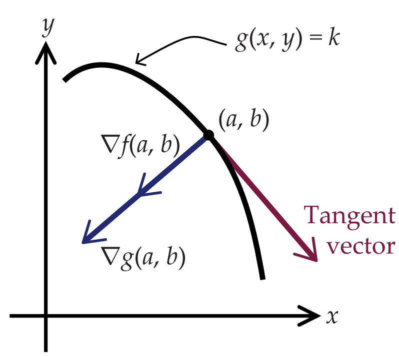
$$0=f_x(a,b)p'(t_0)+f_y(a,b)q'(t_0)$$

This can be written as

$$\nabla f(a,b)\cdot\big(p'(t_0),q'(t_0)\big)=0 \tag{***}$$

Recall the geometric interpretation of the gradient vector $\nabla g(a,b)$: if $\nabla g(a,b)$is non-zero, it is orthogonal to the level curve $g(x,y)=k$ at $(a,b)$. Thus, since $(p'(t_0),q'(t_0))$ is the tangent vector to the constraint curve $(*)$ we also have

$$\nabla g(a,b)\cdot\Big(p'(t_0),q'(t_0)\Big)=0 \tag{****}$$




Since we are working in two dimensions, equations $(***)$ and $(****)$ imply that $\nabla f(a,b)$ and $\nabla g(a,b)$ are scalar multiples of each other. That is, there exists a constant $\lambda$ such that

$$\nabla f(a,b)=\lambda\nabla g(a,b)$$

This leads to the following procedure, called the **Method of Lagrange Multipliers** which will be introduced on the next page.

## Lagrange Multiplier Algorithm

**Lagrange Multiplier Algorithm**
Assume that $f(x, y)$ is a differentiable function and $g \in C^1$. To find the maximum value and minimum value of $f$ subject to the constraint $g(x, y) = k$, evaluate $f(x, y)$ at all points $(a, b)$ which satisfy one of the following conditions.

1. $\nabla f(a, b) = \lambda \nabla g(a, b)$ and $g(a, b) = k$
2. $\nabla g(a, b) = (0, 0)$ and $g(a, b) = k$
3. $(a, b)$ is an end point of the curve $g(x, y) = k$

The maximum/minimum value of $f(x, y)$ is the largest/smallest value of $f$ obtained at the points found in conditions 1-3.

To find the points $(a, b)$ in condition 1 we have to solve the system of 3 equations in 3 unknowns

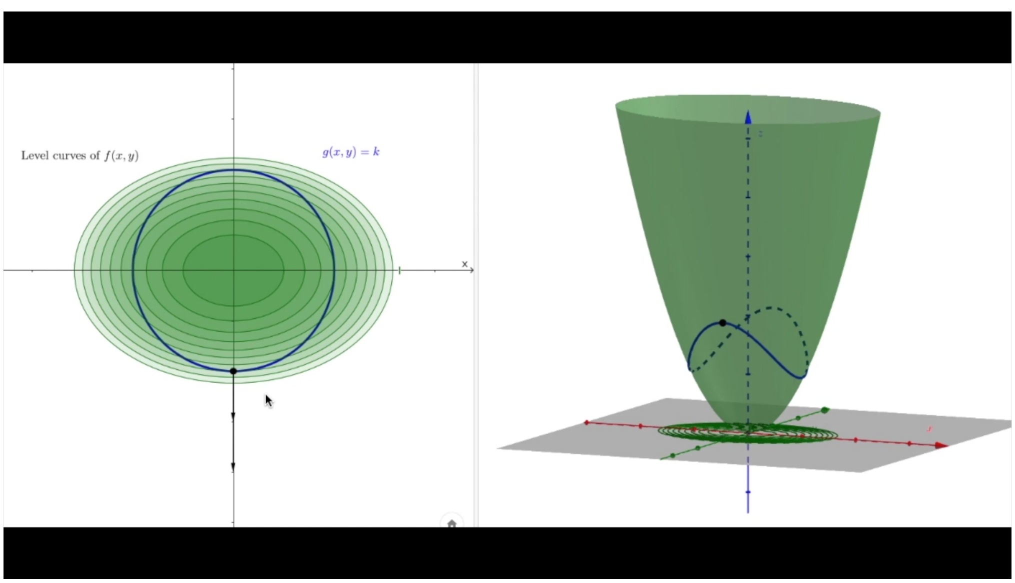
$$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y) \\ f_y(x, y) &= \lambda g_y(x, y) \\ g(x, y) &= k \end{aligned}$$

for $x$ and $y$. It is important that this is done systematically to ensure that we have found all possible points. The variable $\lambda$, called the **Lagrange multiplier**, is not required for our purposes and so should be eliminated. However, in some real world applications, the value of $\lambda$ can be extremely useful.

**Remarks**

1. Observe that condition 2 must be included since we assumed that $\nabla g(a, b) \neq (0, 0)$ in the derivation.
2. If the curve $g(x, y) = k$ is unbounded, then we must consider $\displaystyle\lim_{\|(x,y)\| \to \infty} f(x, y)$ for $(x, y)$ satisfying $g(x, y) = k$. We will not see cases like this in this course.

*A video appears here.*



*Additional content appears in Mobius.*

**Example 1**
Find the maximum value of $6x + 4y - 7$ on the ellipse $3x^2 + y^2 = 28$.
**Solution:**
We want to find the maximum of

$$f(x, y) = 6x + 4y - 7$$

subject to the constraint

$$g(x, y) = 3x^2 + y^2 = 28$$

1. First, we look at the conditions $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 28$.

   We check the algorithm's first condition in the following Your Turn exercise.

A question appears in Mobius

Example 1 (Continued)

1. Next, we check the conditions $\nabla g(x, y) = (0, 0)$ and $g(x, y) = 28$.

   We have $(0, 0) = \nabla g(x, y) = (6x, 2y)$ implies $x = y = 0$, which does not satisfy the constraint $(\dagger\dagger\dagger)$. Hence, there are no points in this step.

2. We check the end points.

   In this case, there are no endpoints since the constraint is a closed curve (an ellipse).

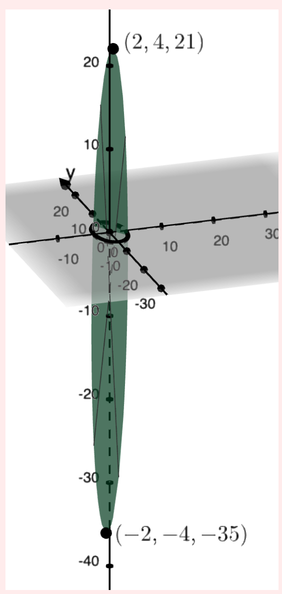
Finally, we evaluate $f$ at all the points found in the above 3 steps. We get

$$f(2, 4) = 21$$
$$f(-2, -4) = -35$$

So, the maximum value of $f$ on $3x^2 + y^2 = 28$ is 21 and occurs at $(2, 4)$.
We can view the result geometrically. The straight lines are the level curves

$$f(x, y) = 6x + 4y - 7 = k$$

Notice that $\nabla f$ and $\nabla g$ are parallel at the maximum point.
The image shows the function $f(x, y) = 6x + 4y - 7$ restricted to the ellipse $3x^2 + y^2 = 28$ with the maximum point, $(2, 4, 21)$, and minimum point, $(-2, -4, -35)$.



## Example 2

**Example 2**
Find the maximum and minimum values of $f(x,y)=y$ on the curve defined by $y^2+x^4-x^3=0$.
**Solution:**
We have $f(x,y)=y$ and constraint $g(x,y)=y^2+x^4-x^3=0$.

1. First, we look at the conditions $\nabla f(x,y)=\lambda\nabla g(x,y)$ and $g(x,y)=0$.

   We get $\nabla f(x,y)=(0,1)$ and $\nabla g(x,y)=(4x^3-3x^2,2y)$, so we need to solve

   $$\begin{aligned} 0 &= \lambda(4x^3-3x^2)=x^2(4x-3)\lambda \qquad (\star)\\ 1 &= \lambda(2y) \qquad (\star\star)\\ 0 &= y^2+x^4-x^3 \qquad (\star\star\star) \end{aligned}$$

   Clearly $\lambda\neq 0$ because of $(\star\star)$, so $(\star)$ gives $x=0$ or $x=\dfrac{3}{4}$.

   If $x=0$, then $(\star\star\star)$ gives $y=0$ which does not satisfy $(\star\star)$.

   If $x=\dfrac{3}{4}$, then $(\star\star\star)$ gives $0=y^2-\dfrac{27}{256}$ which implies $y=\pm\dfrac{3\sqrt{3}}{16}$.

   Hence, we get two points $\left(\dfrac{3}{4},\dfrac{3\sqrt{3}}{16}\right)$ and $\left(\dfrac{3}{4},-\dfrac{3\sqrt{3}}{16}\right)$.

2. Next, we check the conditions $\nabla g(x,y)=(0,0)$ and $g(x,y)=0$.

   We have
   $$(0,0)=\nabla g(x,y)=(4x^3-3x^2,2y)$$

   We check the algorithm's second condition in the following Your Turn exercise.

A question appears in Mobius

Example 2 (Continued)

1. Check end points.

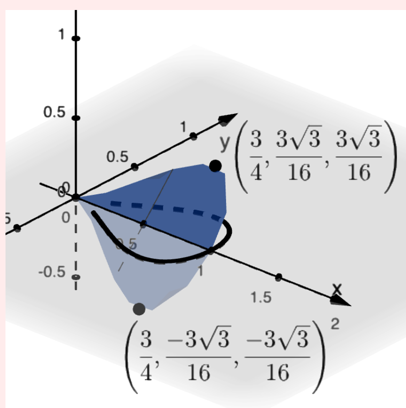
   Graphing the curve, we see that it is closed; therefore there are no end points.

Evaluating $f$ at all the points found above gives

$$f\left(\frac{3}{4}, -\frac{3\sqrt{3}}{16}\right) = -\frac{3\sqrt{3}}{16}$$
$$f\left(\frac{3}{4}, \frac{3\sqrt{3}}{16}\right) = \frac{3\sqrt{3}}{16}$$
$$f(0,0) = 0$$

Thus, the maximum value is $\frac{3\sqrt{3}}{16}$ at $\left(\frac{3}{4}, \frac{3\sqrt{3}}{16}\right)$ and the minimum value is $-\frac{3\sqrt{3}}{16}$ at $\left(\frac{3}{4}, -\frac{3\sqrt{3}}{16}\right)$.

The image shows the function $f(x,y) = y$ restricted to the curve $y^2 + x^4 - x^3 = 0$ with the maximum point, $\left(\frac{3}{4}, \frac{3\sqrt{3}}{16}, \frac{3\sqrt{3}}{16}\right)$, and minimum point, $\left(\frac{3}{4}, -\frac{3\sqrt{3}}{16}, -\frac{3\sqrt{3}}{16}\right)$.



## Example 3

**Example 3**
Let $R$ be the region bounded by the curve $x = \sqrt{1-y^2}$ and the $y$-axis. Find the maximum and minimum value of $f(x,y) = x^2 - \frac{1}{2}x + y^2$ on the region $R$.
**Solution:**
Observe that this is an extreme value on a *region* problem as seen in the previous lesson. Thus, we apply our algorithm from the previous lesson.
We first find critical points of $f$ inside the region $R$. We have

$$\nabla f = (2x - \frac{1}{2}, 2y) = (0,0) \Rightarrow x = \frac{1}{4}, y = 0$$

There is one critical point $\left(\frac{1}{4}, 0\right)$, which is inside the region and $f\left(\frac{1}{4}, 0\right) = -\frac{1}{16}$.
Next, we find the maximum and minimum of $f$ on the boundary of $R$. The boundary has two parts, the $y$-axis and the semi-circle $x = \sqrt{1-y^2}$.
For the $y$-axis, we have $x = 0$, $-1 \leq y \leq 1$, so on this line we have $f(0,y) = 0 + y^2$ which we know has minimum 0 at $(0,0)$ and maximum 1 and $(0, \pm 1)$.
For the semi-circle, instead of parameterizing, we will use the method of Lagrange multipliers. To make the calculations easier, we simplify the constraint to $x^2 + y^2 = 1$, $x \geq 0$. Hence, we take $g(x,y) = x^2 + y^2 = 1$, $x \geq 0$.

1. First, we check the conditions $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 1$.

   This gives the following system of equations:

$$\begin{aligned} 2x - \frac{1}{2} &= \lambda(2x) && (\ddagger) \\ 2y &= \lambda(2y) && (\ddagger\ddagger) \\ x^2 + y^2 = 1, &\quad x \geq 0 && (\ddagger\ddagger\ddagger) \end{aligned}$$

   From $(\ddagger\ddagger)$ we see that $y = 0$ or $\lambda = 1$.

   If $y = 0$, then $(\ddagger\ddagger\ddagger)$ gives $x = 1$ (since $x \geq 0$). With $\lambda = \frac{3}{4}$ $(\ddagger)$ is also satisfied. Thus, $(1,0)$ is a point.

   If $\lambda = 1$, then $(\ddagger)$ is $2x - \frac{1}{2} = 2x$, which has no solutions, so we get no points.

2. Next, we check the conditions $\nabla g(x,y) = (0,0)$ and $g(x,y) = 1$.

   We have $\nabla g(x,y) = (2x, 2y) = (0,0)$ only if $x = 0$ and $y = 0$, but this does not satisfy the constraint so there are no points here.

3. Check end points.

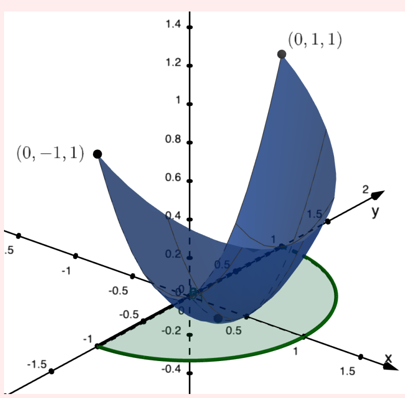
   We check the algorithm's 3rd condition in the following Your Turn exercise.

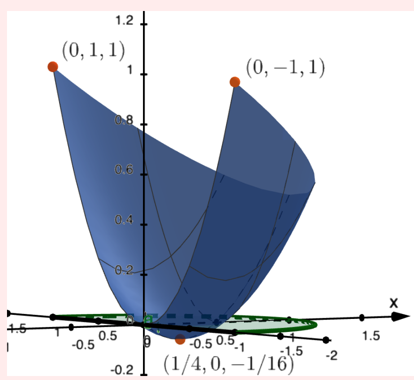
A question appears in Mobius

Example 3 (Continued)
Putting all the points into $f$ gives

$$f(1,0)=\frac{1}{2}, \quad f(0,1)=1, \quad f(0,-1)=1$$

Thus, on the semi-circle the maximum of $f$ is 1 at $(0,\pm 1)$ and the minimum of $f$ is $\frac{1}{2}$ at $(1,0)$.
Comparing the values of $f$ found in all steps, we find that the maximum of $f$ on $R$ is 1 at $(0,\pm 1)$ and the minimum of $f$ is $-\frac{1}{16}$ at $\left(\frac{1}{4},0\right)$.
The image shows the function $f(x,y)=x^2-\frac{1}{2}x+y^2$ restricted to the region $R$ with the maximum points, $(0,1,1)$ and $(0,-1,1)$, and minimum point, $\left(\frac{1}{4},0,-\frac{1}{16}\right)$.



A question appears in Mobius

## Method of Lagrange Multipliers for Functions of Three Variables

We can generalize the algorithm for $f(x, y)$ to work for functions of three variables $f(x, y, z)$.

**Algorithm**
To find the maximum/minimum value of a differentiable function $f(x, y, z)$ subject to $g(x, y, z) = k$ such that $g \in C^1$, we evaluate $f(x, y, z)$ at all points $(a, b, c)$ which satisfy one of the following:

1. $\nabla f(a, b, c) = \lambda \nabla g(a, b, c)$ and $g(a, b, c) = k$
2. $\nabla g(a, b, c) = (0, 0, 0)$ and $g(a, b, c) = k$
3. $(a, b, c)$ is a boundary point of the surface $g(x, y, z) = k$

The maximum/minimum value of $f(x, y, z)$ is the largest/smallest value of $f$ obtained from all points found in conditions 1-3.

**Remark**
If condition 1 in the algorithm holds, it follows that the level surface $f(x, y, z) = f(a, b, c)$ and the constraint surface $g(x, y, z) = k$ are tangent at the point $(a, b, c)$, since their normals coincide.

**Example 4**
Find the point on the sphere $x^2 + y^2 + z^2 = 1$ which is closest to the point $(1, 2, 2)$.
**Solution:**
We want to minimize the distance between the point $(1, 2, 2)$ and a point $(x, y, z)$ on the given sphere. To simplify things, we consider the square of this distance, which is given by the function

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 2)^2$$

The constraint is $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

1. First, we look at the conditions $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 1$.

$$2(x - 1) = 2\lambda x \qquad (*)$$
$$2(y - 2) = 2\lambda y \qquad (**)$$
$$2(z - 2) = 2\lambda z \qquad (***)$$
$$x^2 + y^2 + z^2 = 1 \qquad (****)$$

   Observe that $(*)$, $(**)$, and $(***)$ give that $x \neq 0$, $y \neq 0$, and $z \neq 0$. Hence, solving these equations for $\lambda$ and setting them equal to each other gives

$$\frac{x-1}{x} = \frac{y-2}{y} = \frac{z-2}{z}$$

   Looking at each pair, we find that $y = 2x$, $z = 2x$, and thus $y = z$. Putting these into the constraint $(****)$ gives two points, $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and $\left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

2. Next, we check the conditions $\nabla g(x, y, z) = (0, 0, 0)$ and $g(x, y, z) = 1$. We have $\nabla g(x, y, z) = (0, 0, 0)$ implies $x = y = z = 0$, which does not satisfy the constraint.

3. Check the endpoints.

   There are no boundary points since the constraint is a closed surface.

Evaluating $f$ at all the points found above gives

$$f\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = 4$$
$$f\left(-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right) = 16$$

Thus, the point $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ is the point on the sphere $x^2 + y^2 + z^2 = 1$ that is closest to the point $(1, 2, 2)$.

**Remark**
Keep in mind the geometric interpretation. The level sets $f(x, y, z) = k$ are spheres centered on the point $(1, 2, 2)$. The minimum distance occurs when one of the spheres touches (i.e. is tangent to) the constraint surface which is the sphere $g(x, y, z) = 1$. At the point of tangency, the normals are parallel, i.e. $\nabla f = \lambda \nabla g$.

### Your Turn

Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

A question appears in Mobius

### Generalization

The method of Lagrange multipliers can be generalized to functions of $n$ variables $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$ and with $r$ constraints of the form

$$g_1(\vec{x}) = 0, \quad g_2(\vec{x}) = 0, \quad \ldots, \quad g_r(\vec{x}) = 0 \tag{*}$$

In order to find the possible maximum and minimum points of $f$ subject to the constraints (*), we have to find all the points $\vec{a}$ such that

$$\nabla f(\vec{a}) = \lambda_1 \nabla g_1(\vec{a}) + \cdots + \lambda_r \nabla g_r(\vec{a}), \quad \text{and} \quad g_i(\vec{a}) = 0, \qquad 1 \leq i \leq r$$

The scalars $\lambda_1, \ldots, \lambda_r$ are the Lagrange multipliers. When $r = 1$, and $n = 2$ or 3, this reduces to the previous algorithms.

## 10.4 - Putting It All Together

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Worked Example 2

The steady-state temperature at position $(x, y)$ of a metal disc, $x^2 + y^2 \leq b^2$, where $b$ is a positive constant, is given by

$$f(x, y) = 100 + x^3 - 3xy^2$$

Find the maximum and the minimum temperature on the disc using the algorithm for Extreme Values.

(Note that since the disc $x^2 + y^2 \leq b^2$ is closed and bounded and $f$ is continuous on the disc, we can apply the EVT.)

For some parts you may need to enter an expression such as (symbolic form of $\sin(t)^\wedge x$). Recall that this is entered as 'sin$(t)^\wedge x$'.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

f. Find the critical points of $g(t)$.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Practice Problems

Try to answer the questions. If you are having trouble, check for a hint before looking at the solutions.

1. Find the maximum and minimum values of the function $f(x,y) = x + 2y$ on the disc $x^2 + y^2 \leq 4$.

2. Find the maximum and minimum values of the function $f(x,y) = xye^{-\frac{1}{2}x-\frac{1}{3}y}$ on the triangular set with vertices $(0,0)$, $(2,0)$ and $(0,3)$.

3. Find the maximum and minimum of the function $f(x,y) = x^3 - 3x + y^2 + 2y$ on the region bounded by the lines $x = 0$, $y = 0$, $x + y = 1$.

4. Assume the earth is located at $(x,y,z) = (0,0,0)$ and the path of a comet is given by
$$3x^2 + 8xy - 3y^2 = 5^3, \quad z = 0, \quad x > 0$$
Find the distance of closest approach to the centre of the earth. Units are in $km \times 10^5$. Illustrate your answer with a sketch. [*Hint provided below*]

5. Use the method of Lagrange multipliers to find the maximum and minimum values of $xy + z^2$ on the surface $x^2 + y^2 + z^2 = 1$.

6. Use Lagrange multipliers to find the maximum value of $x + y + z$ on the ellipsoid $x^2 + \dfrac{1}{4}y^2 + \dfrac{1}{9}z^2 = 1$. Discuss briefly a geometrical interpretation.

7. Let $f(x,y) = x^2 + y^2 - \dfrac{1}{2}y$.
   (a) Use the method of Lagrange multipliers to find the maximum and minimum points of $f(x,y)$ on the curve $y = \sqrt{1-2x^2}$.
   (b) Let $R$ be the region bounded by the curve $y = \sqrt{1-2x^2}$ and the $x$-axis. Find the maximum and minimum value of $f(x,y)$ on the region $R$.

8. Find the greatest and least distance of the surface $6x^2 + 4xy + 3y^2 + 14z^2 = 14$ from the origin.

9. Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x$ on the piriform curve defined by
$$y^2 + x^4 - x^3 = 0$$

10. Consider all pentagons which have a line of symmetry, two adjacent interior angles of $90^0$, and a perimeter of fixed length $L$. Find the shape that encloses the largest area.

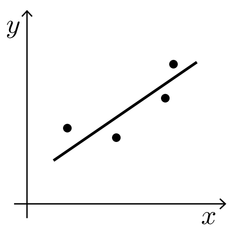
11. Find the maximum and minimum value of the function $f(x,y) = (x+1)^2 + y^2$ on the part of the graph of $y^2 - x^3 = 0$ from $(1,-1)$ to $(1,1)$.

12. Prove that $x^4 + y^4 - 4b^2xy \geq -2b^4$ for all $x, y \in \mathbb{R}$.

13. Consider $f(x,y) = (x^2 + y^2 + k)e^{-x^2-y^2}$ where $k$ is a constant. The properties of $f$ depend in a significant way on $k$. Analyse the function as regards local and global maxima and minima. Sketch/describe the surface $z = f(x,y)$. How many qualitatively different cases are there?

14. Consider a set of points $(x_i, y_i)$, $i = 1, 2, \ldots, n$, which are close to lying on a straight line $y = mx + b$. In order to find the straight line which "best fits" the points, we minimize the sum of the squares of the errors:
$$E(m,b) = \sum_{i=1}^{n} \big[y_i - (mx_i + b)\big]^2$$

In other words, we find the minimum value of $E(m,b)$, for all values of the slope $m$ and intercept $b$, i.e. for all $(m,b) \in \mathbb{R}^2$.




Apply this method to find the straight line which best fits the points $(0,1)$, $(2,3)$, $(3,6)$, and $(4,8)$.

**Suggestion:** Do not expand $E(m,b)$ before calculating the partial derivatives.

15. Suppose that a function $f(x,y)$ has exactly one critical point which is a local minimum. Does $f$ have a minimum on $\mathbb{R}^2$?

    Discuss with reference to the functions $f_1(x,y) = x^2 + y^2(1-x)^3$ and $f_2(x,y) = x^2 + y^2$.

16. a. Use the method of Lagrange multipliers to prove that if $x_1^2 + x_2^2 + x_3^2 = 1$, then $x_1^2 x_2^2 x_3^2 \leq \frac{1}{3^3}$

    b. Hence prove that for all positive real numbers $a_1, a_2$ and $a_3$,

    $$(a_1 a_2 a_3)^{\frac{1}{3}} \leq \frac{a_1 + a_2 + a_3}{3}$$

    c. Generalize (a) and (b) to deduce the *arithmetic-geometric mean inequality*:

    $$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

    for all positive real numbers $a_1, a_2, \ldots, a_n$ and any positive integer $n$.

17. Find the maximum and minimum value of $F(x,y) = x^2 + 2x + y^2$ subject to the constraint $x^2 + 4y^2 \leq 24$.

## Hint

*Additional content appears in Mobius.*

## Select Answers and Solutions

1. The maximum value is $2\sqrt{5}$ and the minimum value is $-2\sqrt{5}$.

2. No answer provided.

3. The maximum value is 3 at $(0,1)$ and the minimum value is $-2$ at $(1,0)$

4. The minimum is $5 \times 10^5$ km at $(2\sqrt{5}, \sqrt{5})$.

   Sketch not provided.

5. The maximum value is 1 at $(0,0,1)$ and the minimum value is $-\frac{1}{2}$ at points $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$.

6. The maximum value is $\sqrt{14}$ at $\left(\frac{1}{\sqrt{14}}, \frac{4}{\sqrt{14}}, \frac{9}{\sqrt{14}}\right)$

7. a. The maximum points are $(0,1)$ and $\left(\pm\sqrt{\frac{1}{2}}, 0\right)$. The minimum points are $\left(\pm\sqrt{\frac{3}{8}}, \frac{1}{2}\right)$.

   b. The maximum value is $\frac{1}{2}$ at $(0,1)$ and at $\left(\pm\sqrt{\tfrac{1}{2}}, 0\right)$ and the minimum value is $-\frac{1}{16}$ at $\left(0, \frac{1}{4}\right)$

8. No answer provided.

9. The maximum value is 0 at $(1,0)$ and the minimum value is 0 at $(0,0)$.

10. No answer provided.

11. No answer provided.

12. No answer provided.

13. No answer provided.

14. No answer provided.

15. No answer provided.

16. No answer provided.

17. The maximum value is $24 + 4\sqrt{6}$ at $(2\sqrt{6}, 0)$ and the minimum value is $-1$ at $(-1, 0)$.

# Unit 11

# Coordinate Systems

## 11.1 - Polar Coordinates

### Coordinate Systems

A **coordinate system** is a system for representing the location of a point in a space by an ordered $n$-tuple. We call the elements of the $n$-tuple the **coordinates** of the point.

We are used to using the Cartesian coordinate system in which the location of the point is represented by the directed distance from a set of perpendicular axes which all intersect at a point $O$ which we call the origin. However, you may also be used to other coordinate systems. For example, the geographic coordinate system represents location on the earth by longitude, latitude, and altitude.

We will now look at three other important coordinate systems:

- polar coordinates
- cylindrical coordinates
- spherical coordinates

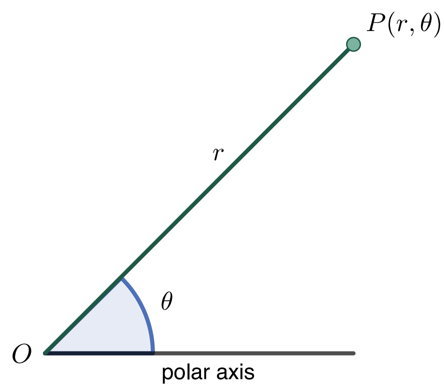
These coordinate systems will be very useful to us in the coming lessons on multiple integrals. As we will see, a common technique for solving multiple integrals is to choose a coordinate system which simplifies the calculations. The choice of coordinates often depends on the type of symmerty involved in the problem to be solved.

Let's begin with the study of polar coordinates.

### Polar Coordinates

As in all coordinate systems, we must have a frame of reference for our coordinate system. So, in a plane we choose a point $O$ called the **pole** (or origin). From $O$ we draw a ray called the **polar axis**. Generally, the polar axis is drawn horizontally to the right to match the positive $x$-axis in Cartesian coordinates.

Let $P$ be any point in the plane. We will represent the position of $P$ by the ordered pair $(r, \theta)$ where $r \geq 0$ is the length of the line $OP$ and $\theta$ is the angle between the polar axis and $OP$. We call $r$ and $\theta$ the polar coordinates of $P$.




We will use the following conventions in this course:

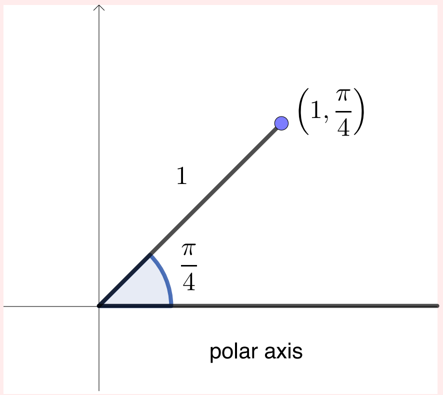
- An angle $\theta$ is considered positive if measured in the counterclockwise direction from the polar axis and negative if measured in the clockwise direction.
- The origin $O$ is represented by the polar coordinates $(0, \theta)$ for any value of $\theta$.
- The value of $r$ is non-negative to coincide with the interpretation of $r$ as distance.

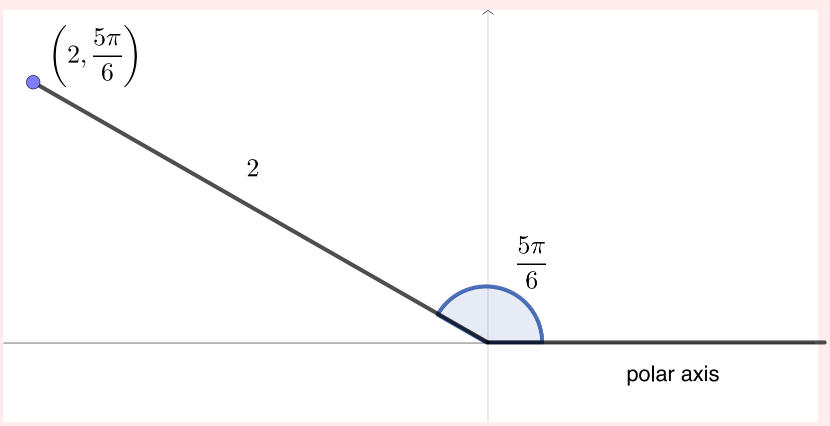
**Remark**
Since we use the distance $r$ from the pole in our representation, polar coordinates are suited for solving problems in which there is symmetry about the pole.

**Example 1**
Plot the following points in polar coordinates:

1. $\left(1, \dfrac{\pi}{4}\right)$
2. $\left(2, \dfrac{5\pi}{6}\right)$

**Solution:**

1. 


2. 


**Your Turn 1**

The following app is designed to give you an intuition for polar coordinates.

**Instructions**

1. Use the sliders to enter the polar coordinates $(r, \theta)$ of the point that you wish to draw. The value of $\theta$ must fall between 0 and $2\pi$.
2. Click 'Show Construction' to watch the point get sketched on the plane.

External resource: https://www.geogebra.org/material/iframe/id/jxxknweh/



There is one important difference between polar coordinates and Cartesian coordinates: in Cartesian coordinates, each point has a unique representation $(x, y)$. However, observe that a point $(r, \theta)$ in polar coordinates can have infinitely many representations. In particular,

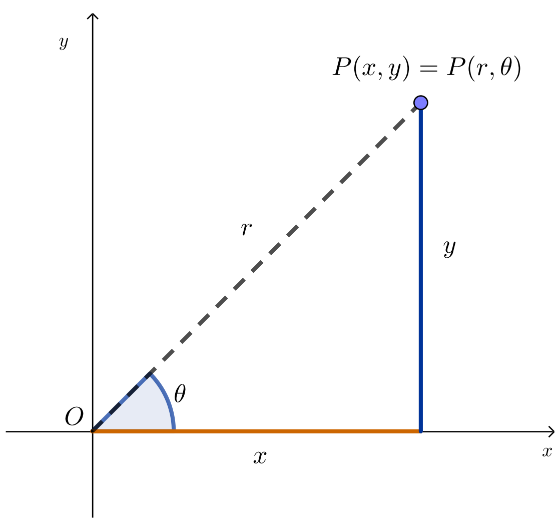
$$(r, \theta) = (r, \theta + 2\pi k), \quad k \in \mathbb{Z}$$

A question appears in Mobius

## Relationship to Cartesian Coordinates

If we now place the pole $O$ at the origin of the Cartesian plane and lie the polar axis along the positive $x$-axis, we can find a relationship between the coordinates of a point $P$ in the two coordinate systems. In particular, we see from the diagram that

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \tan\theta &= \frac{y}{x} \end{aligned}$$
$$x^2 + y^2 = r^2$$



**Example 2**
Convert the following points from polar coordinates to Cartesian coordinates.

1. $\left(2, -\frac{\pi}{3}\right)$
2. $\left(1, \frac{3\pi}{4}\right)$

**Solution:**

1. For the point $\left(2, -\frac{\pi}{3}\right)$ in polar coordinates, we have $x = 2\cos\left(-\frac{\pi}{3}\right) = 1$ and $y = 2\sin\left(-\frac{\pi}{3}\right) = -\sqrt{3}$. Hence, the point is $(1, -\sqrt{3})$ in Cartesian coordinates.
2. For the point $\left(1, \frac{3\pi}{4}\right)$ in polar coordinates, we have $x = \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $y = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$. So the point has Cartesian coordinates $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

**Example 3**
Convert the point $(1, 1)$ from Cartesian coordinates to polar coordinates.
**Solution:**
We have $x = 1$ and $y = 1$, so $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = 1$. Since $x$ and $y$ are both positive, the point is in the first quadrant, and hence

$$\theta = \frac{\pi}{4} + 2\pi k, \quad k \in \mathbb{Z}$$

Therefore, we get the polar coordinate representations $\left(\sqrt{2}, \frac{\pi}{4} + 2\pi k\right)$, $k \in \mathbb{Z}$.

Often, we do not need to find all possible polar representations for a point. Thus, we further restrict ourselves to a range of $\theta$ (such as $0 \le \theta < 2\pi$ or $-\pi < \theta \le \pi$) which gives unique representation.

**Remark**
The equation $\tan\theta = \frac{y}{x}$ does not uniquely determine $\theta$, since over $0 \le \theta \le 2\pi$ each value of $\tan\theta$ occurs twice. We must be careful to choose the $\theta$ which lies in the correct quadrant.

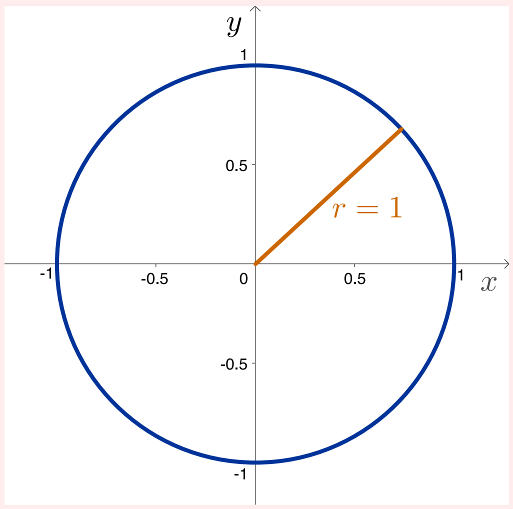
A question appears in Mobius

Now that we know we know what a point looks like in polar coordinates, we can next take a look at what the graphs of polar equations look like. On the next page, we will introduce graphs in polar coordinates.

## Graphs in Polar Coordinates

The graph of an explicitly defined polar equation is of the form $r = f(\theta)$ or $\theta = f(r)$. Alternatively, we can define a polar equation implicitly as $f(r, \theta) = 0$. This is a curve that consists of all points that have at least one polar representation $(r, \theta)$ which satisfies the equation of the curve.

**Example 4**
Sketch the polar equation $r = 1$.
**Solution:**
This is the curve which consists of all points $(r, \theta) = (1, \theta)$, $\theta \in \mathbb{R}$. Observe that this is all points that have distance 1 from the origin. Hence, we get a circle of radius 1 centred at the origin.




**Your Turn 1**

Sketch the polar equation $\theta = \dfrac{\pi}{4}$.

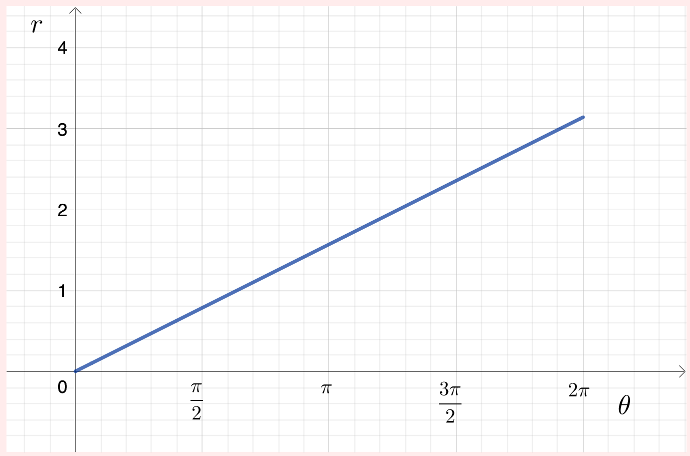
A question appears in Mobius

**Example 5**

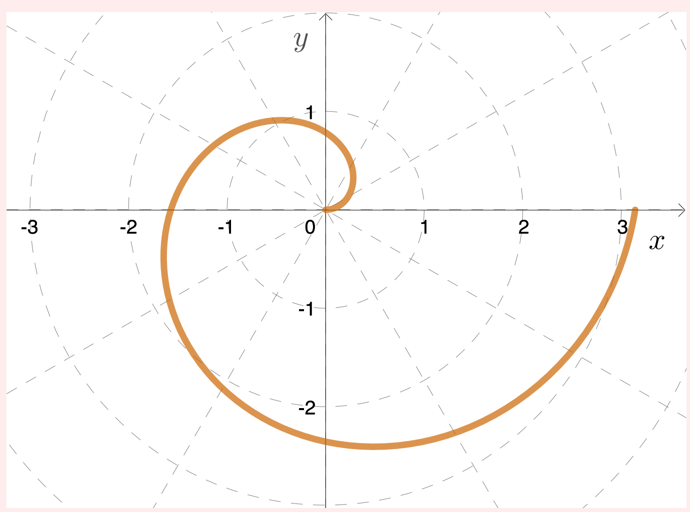
Sketch the polar equation $r = \frac{1}{2}\theta$, $0 \leq \theta \leq 2\pi$.

**Solution:**

One way to try to sketch a curve is to make a table of values and plot the points for various $\theta$, however, this is quite tedious. Instead, let's consider sketching the curve as if it was given in Cartesian coordinates in the $r\theta$-plane:




Essentially we have created a table of infinitely many values which allows us to see how $r$ grows as $\theta$ increases from 0 to $2\pi$. Finally, we sketch the given curve in the $xy$-plane where $r$ is the distance to the origin and $\theta$ is the angle measured counterclockwise from the $x$-axis. We see that the distance from the origin grows linearly as we increase the angle; we get a spiral:



Remark

The polar equation $r = e^{\theta}$ gives a **logarithmic spiral** which often appears in nature. The nautilus shell is a nice example of this spiral.

FlamingPumpkin/E+/Getty Images

**Your Turn 2**

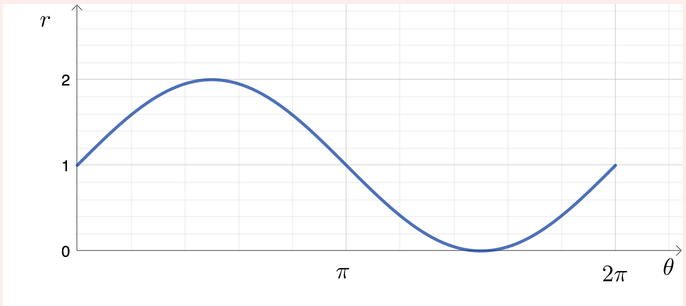
The interactive app below is to help you understand how to graph polar curves. Note that in this app, we are using $t$ to represent $\theta$.

**Instructions**

1. Enter a curve in the input box.
2. Change the bounds of $t$ to observe how the sketch of the curve changes.
3. Use the slider to watch the curve get traced on the $xy$- and $rt$-planes.

External resource: https://www.geogebra.org/material/iframe/id/ekbk2zwf/

**Example 6**
Sketch the polar equation $r = 1 + \sin\theta$.
**Solution:**
To sketch this equation, we first sketch the curve in Cartesian coordinates in the $r\theta$-plane and use this graph to plot points in the $xy$-plane.




Observe from the diagram that as $\theta$ increases from 0 to $\frac{\pi}{2}$ the radius increases from 1 to 2.

- In the applet below, use the slider to change $\theta$ from 0 to $\frac{\pi}{2}$ to observe the shape of the graph on this interval.

Next, observe that when $\theta$ increases from $\frac{\pi}{2}$ to $\pi$ the radius decreases from 2 to 1.

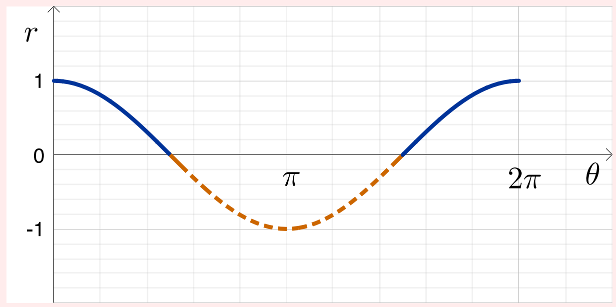
- Again, in the applet below, use the slider to change $\theta$ from $\frac{\pi}{2}$ to $\pi$ to observe the shape of the graph on this interval.

Finally, when $\theta$ increases from $\pi$ to $\frac{3\pi}{2}$ we get the radius decreases from 1 to 0, and as $\theta$ increases from $\frac{3\pi}{2}$ to $2\pi$ the radius increases from 0 to 1.

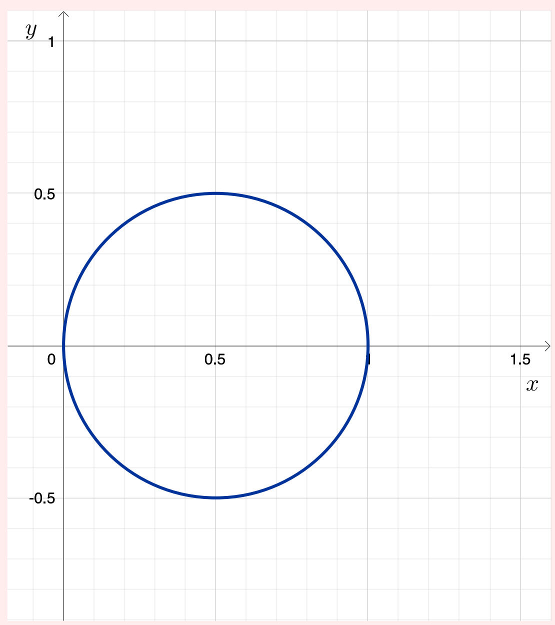
- Use the slider in the applet below to complete the graph. The final curve is called a **cardioid**.

External resource: https://www.geogebra.org/material/iframe/id/djgydx3r/

**Example 7**
Sketch the polar equation $r = \cos\theta$.
**Solution:**
We first sketch the curve in Cartesian coordinates in the $r\theta$-plane.




We see that as $\theta$ increases from 0 to $\frac{\pi}{2}$ the radius decreases from 1 to 0. This forms the upper half of the circle we get in the figure below.
For values of $\theta$ from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ the radius is negative, thus we do not draw any points since we have made the restriction that $r \geq 0$.
As $\theta$ moves from $\frac{3\pi}{2}$ to $2\pi$ the radius increases from 0 to 1. This forms the lower half of the circle in the figure below.




**Your Turn 3**

Sketch the polar equation $r = \sin\theta$ .

A question appears in Mobius

Sketch the polar equation $r = 1 - 2\cos\theta$.

A question appears in Mobius

## Equations in Polar Coordinates

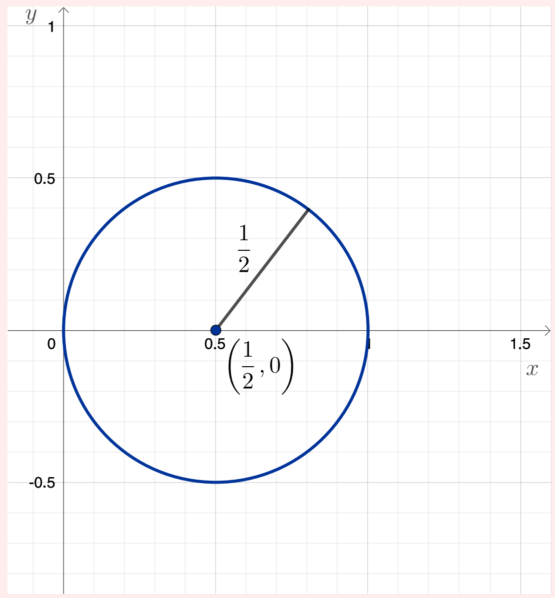
We have seen that we can use the following equations to convert points between the Cartesian coordinate system and the polar coordinate system:

$$\begin{aligned} x &= r\cos\theta & \qquad r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \qquad \tan\theta &= \frac{y}{x} \end{aligned}$$
$$x^2 + y^2 = r^2$$

Thus, we can also use these equations to convert equations of curves between the two coordinate systems.

**Example 8**
Convert the equation $r = \cos\theta$ to Cartesian coordinates.
**Solution:**
Since $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we get

$$\begin{aligned} r &= \cos\theta \\ r^2 &= r\cos\theta \\ x^2 + y^2 &= x \\ \left(x - \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4} \end{aligned}$$




This is an equation of a circle of radius $\frac{1}{2}$ centered at $\left(\frac{1}{2}, 0\right)$ as we drew in the previous example.

**Example 9**
Convert the equation of the curve $(x^2 + y^2)^{3/2} = 2xy$ to polar coordinates.
**Solution:**
Since $x = r\cos\theta$ and $y = r\sin\theta$ we get

$$\begin{aligned} (x^2 + y^2)^{3/2} &= 2xy \\ r^3 &= 2(r\cos\theta)(r\sin\theta) \\ r^3 &= r^2 \sin 2\theta \\ r &= \sin 2\theta \end{aligned}$$

Notice that the last simplification is only valid since the pole, $r = 0$, is still included in the graph (the case where $\theta = \pi$).
Observe that since we have the restriction $r \geq 0$ we must also have $\sin 2\theta \geq 0$. Hence, we find that a domain of the function is

$$0 \leq \theta \leq \frac{\pi}{2}, \quad \pi \leq \theta \leq \frac{3\pi}{2}$$

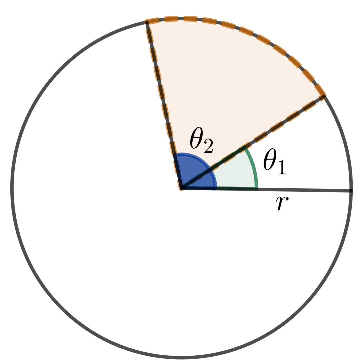
A question appears in Mobius

## Area in Polar Coordinates

We now wish to derive the formula for computing area between curves in polar coordinates. This will be different from our usual approach as it does not make sense to use rectangles to find our area. In polar coordinates, it is natural to use sectors of a circle.

Recall that if $\theta_1$ and $\theta_2$ with $\theta_2 > \theta_1$ are two angles in a circle of radius $r$, then the area between them is

$$\text{Area of sector} = \frac{\theta_2 - \theta_1}{2\pi}\pi r^2 = \frac{1}{2}r^2(\theta_2 - \theta_1)$$

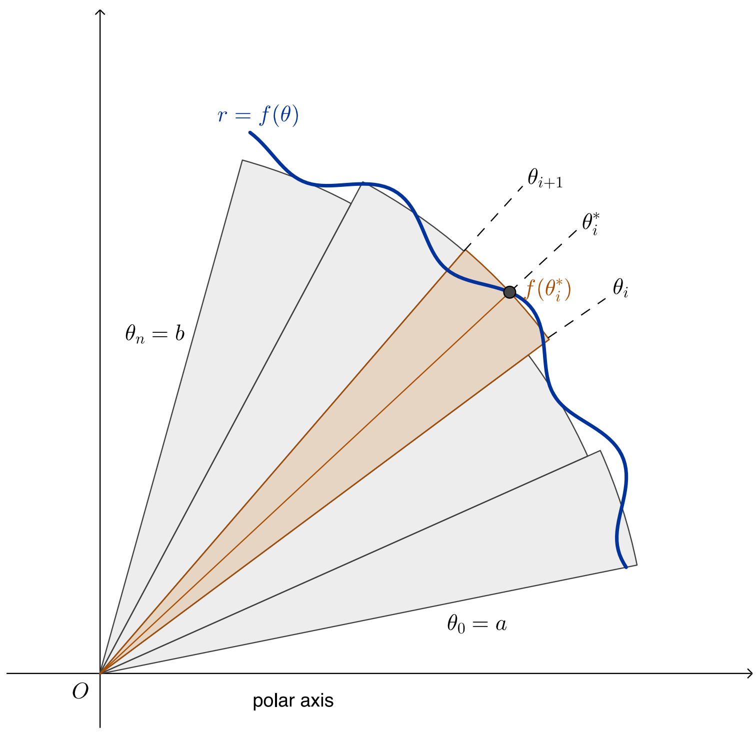



We now derive the area formula.

First, we divide the region bounded by $\theta = a$, $\theta = b$ and $r = f(\theta)$ into subregions $\theta_0, \ldots, \theta_n$ of equal difference $\Delta\theta$. Let's call $\theta_0 = a$ and $\theta_n = b$.

Then, for each subregion bounded by $\theta_i$ and $\theta_{i+1}$, $0 \leq i < n-1$, we pick some point $\theta_i^*$ with $\theta_i \leq \theta_i^* \leq \theta_{i+1}$.

We then form the sector between $\theta_i$ and $\theta_{i+1}$ with radius $f(\theta_i^*)$.




The area of this sector is

$$\frac{1}{2}[f(\theta_i^*)]^2\Delta\theta$$

Hence, the area is approximately

$$\sum_{i=0}^{n-1}\frac{1}{2}[f(\theta_i^*)]^2\Delta\theta$$

Thus, as we let the number of subdivisions go to infinity and hence letting each of the $\Delta\theta_i$ tend to 0 we get

$$\begin{aligned} A &= \lim_{\|\Delta\theta_i\|\to 0}\sum_{i=0}^{n-1}\frac{1}{2}[f(\theta_i^*)]^2\Delta\theta \\ &= \int_a^b \frac{1}{2}[f(\theta)]^2\ d\theta \end{aligned}$$

**Your Turn 1**

The interactive applet shows how the area below a curve is approximated using sectors of a circle.

**Instructions**

1. Use the slider to increase the number of subdivisons $n$.
2. Observe how the approximation for the area improves as $n$ increases.

External resource: https://www.geogebra.org/material/iframe/id/mgdew295/

**Example 10**
Find the area inside the circle $r = a$.
**Solution:**
Using the formula for the area of a circle, we immediately see that the answer is $A = \pi r^2 = \pi a^2$.
Now, let's verify this result using the new formula we derived above.
We need $\theta$ to range from 0 to $2\pi$ to make the whole circle so we have

$$A = \int_0^{2\pi} \frac{1}{2} a^2 \, d\theta = \frac{1}{2} a^2 [2\pi - 0] = \pi a^2$$

A question appears in Mobius

**Your Turn 3**

In this app, you will be able to trace and calculate the area inside polar curves drawn on the $r\theta$-plane. Note that in this app, we are using $t$ to represent $\theta$.

**Instructions**

1. Enter a curve (as a function of $t$) in the input box and change the bounds of $t$.
2. Use the $t$ slider to watch the curve get traced on the $xy$-plane and to see how the area $A$ changes.

External resource: https://www.geogebra.org/material/iframe/id/dpgwx8s4/



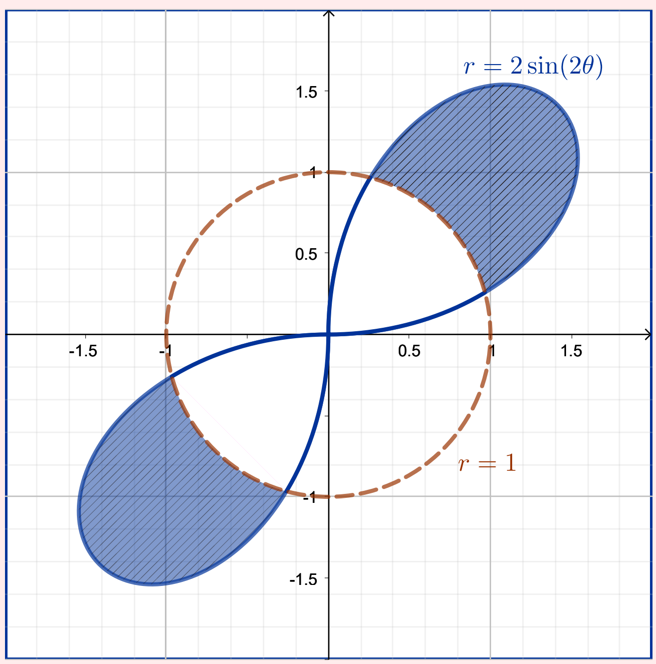
## Area Between Curves in Polar Coordinates

Next, we introduce an algorithm to find the area between two curves in polar coordinates.

**Algorithm**
To find the area between two curves in polar coordinates, we use the same method we used for doing this in Cartesian coordinates.

1. Find the points of intersections.
2. Graph the curves and split the desired region into easily integrable regions.
3. Integrate.

Example 11
Find the area inside $r = 2\sin(2\theta)$, but outside $r = 1$.
**Solution:**
Setting the curves equal to each other we get $1 = 2\sin(2\theta)$, hence $2\theta = \frac{\pi}{6}$ or $2\theta = \frac{5\pi}{6}$.
Therefore, we want to integrate over the region $\frac{\pi}{12}$ to $\frac{5\pi}{12}$.
To find the shaded area, we will find the area inside the curve in the first quadrant and subtract the area of the region that is inside both the circle and the curve.
Finally, we will multiply by 2 for the symmetric region in the third quadrant.




Note that the shape formed by $r = 2\sin(2\theta)$is called a **lemniscate**.

We get

$$
\begin{aligned}
A &= 2\left(\int_{\pi/12}^{5\pi/12} \frac{1}{2}(2\sin(2\theta))^2 \; d\theta - \int_{\pi/12}^{5\pi/12} \frac{1}{2}(1)^2 \; d\theta\right) \\
&= 2\left(\int_{\pi/12}^{5\pi/12} 2\sin^2(2\theta) \; d\theta - \int_{\pi/12}^{5\pi/12} \frac{1}{2} \; d\theta\right) \\
&= 2\left(\underbrace{\int_{\pi/12}^{5\pi/12} 2\frac{1-\cos 4\theta}{2} \; d\theta}_{} - \underbrace{\int_{\pi/12}^{5\pi/12} \frac{1}{2} \; d\theta}_{}\right) \\
&= 2\left(\qquad \frac{\pi}{3} + \frac{\sqrt{3}}{4} \qquad - \qquad \frac{\pi}{6}\right) \\
&= \frac{\pi}{3} + \frac{\sqrt{3}}{2}
\end{aligned}
$$

Remark

Finding points of intersection can be tricky, especially at the pole/origin which does not have a unique representation: $(0,\theta)$ for any $\theta$ represents the origin, so simply setting expressions equal to each other may "miss" that point. It is essential to sketch the region when finding points of intersection. You can use the app provided below to check the intersection points.

**Your Turn 1**

In this app, you can sketch two curves and observe the points at which they intersect. Try entering the curves from the next Your Turn exercise to check whether you found all of their intersections. Note that in this app, we are using $t$ to represent $\theta$ .

**Instructions**

1. Enter two curves (as a function of $t$) in the input boxes.
2. Change the bounds of the angle $t$.
3. Use the slider to sketch the curves on the $xy$-plane and to see at which point(s) the curves intersect.

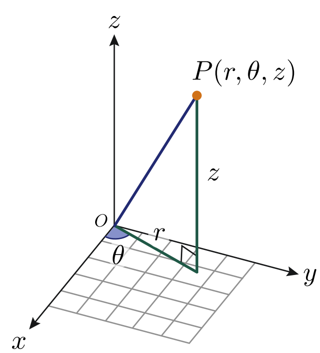
External resource: https://www.geogebra.org/material/iframe/id/c2fhbmg5/

A question appears in Mobius

## 11.2 - Cylindrical Coordinates

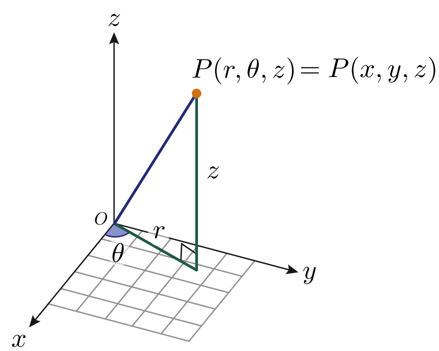
### Cylindrical Coordinates

Observe that we can extend polar coordinates to 3-dimensional space by introducing another axis, called the **axis of symmetry**, through the pole perpendicular to the polar plane. We then represent any point $P$ in the space by the cylindrical coordinates $(r, \theta, z)$ where $r$ and $\theta$ are as in polar coordinates and $z$ is the height above (or below) the polar plane. Thus, as in polar coordinates, we have the restrictions $r \geq 0$, $0 \leq \theta < 2\pi$ (or $-\pi < \theta \leq \pi$).




**Remark**
Notation for cylindrical coordinates may vary from author to author. In particular, in the sciences they generally use the Standard ISO 31-11 notation which gives the cylindrical coordinates as $(\rho, \varphi, z)$.

If we place the pole at the origin and the polar axis along the positive $x$-axis as in polar coordinates and place the axis of symmetry along the $z$-axis we then can relate a point $P\ (r, \theta, z)$ in cylindrical and a point $P\ (x, y, z)$ Cartesian coordinates by




$$
\begin{aligned}
x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\
y &= r\sin\theta & \tan\theta &= \frac{y}{x} \\
z &= z & z &= z
\end{aligned}
$$

$$x^2 + y^2 = r^2$$

**Your Turn 1**

The following app is designed to give you an intuition for cylindrical coordinates.

**Instructions**

1. Use the sliders to choose the cylindrical coordinates $(r, \theta, z)$ of the point that you wish to draw. Note that this applet uses the unit of radians for $\theta$.
2. Click '"Show Construction"' to watch the point get sketched in 3D.
3. Check "Show Cylinder" to display the cylinder corresponding to the point you picked and study each component of your choice for $(r, \theta, z)$ in 3D.

External resource: https://www.geogebra.org/material/iframe/id/qskcjfvm/



**Remark**
Cylindrical coordinates are useful when there is symmetry about an axis. Thus, it is sometimes desirable to lie the polar axis and axis of symmetry along different axes.

**Example 1**
Convert the following points from cylindrical coordinates to Cartesian coordinates and verify the results using the app above:

1. $(2, 0, 0)$
2. $(0, \pi, 2)$

**Solution:**

1. The point $(2, 0, 0)$ in cylindrical coordinates has coordinates $r = 2$, $\theta = 0$ and $z = 0$. Hence, $x = 2\cos(0) = 2$, $y = 2\sin(0) = 0$ and $z = 0$, so the point in Cartesian coordinates is $(2, 0, 0)$.
2. The point $(0, \pi, 2)$ in cylindrical coordinates has coordinates $r = 0$, $\theta = \pi$, and $z = 2$. Since $r = 0$, we get $x = y = 0$ and so the point in Cartesian coordinates is $(0, 0, 2)$.

**Example 2**
Convert the following points from Cartesian coordinates to cylindrical coordinates and verify the results using the app above:

1. $(1, 1, 3)$
2. $(1, -\sqrt{3}, 1)$

**Solution:**

1. For the Cartesian point $(1, 1, 3)$, we have $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\tan\theta = 1$ which gives $\theta = \dfrac{\pi}{4}$ and $z = 3$. Thus, in cylindrical coordinates, the point is $\left(\sqrt{2}, \dfrac{\pi}{4}, 3\right)$.
2. For the Cartesian point $(1, -\sqrt{3}, 1)$, we have $z = 1$, $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$, and $\tan\theta = \dfrac{-\sqrt{3}}{1}$ which gives $\theta = \dfrac{5\pi}{3}$ since $\theta$ is in the fourth quadrant. Hence, in cylindrical coordinates, the point is $\left(2, \dfrac{5\pi}{3}, 1\right)$.

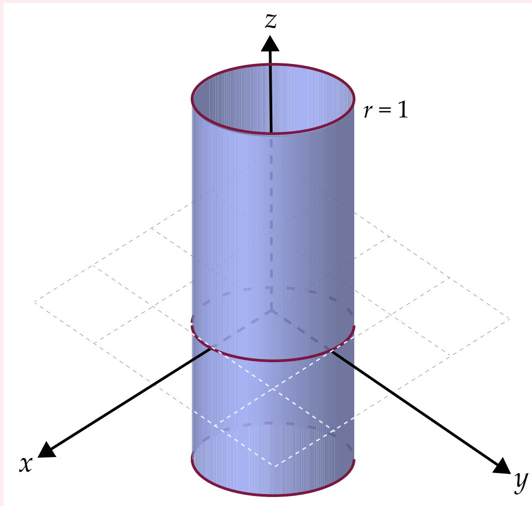
A question appears in Mobius

As we did with polar coordinates, the next thing we look at is the graphs in cylindrical coordinates.

## Graphs in Cylindrical Coordinates

As with functions $z = f(x, y)$, the graphs of functions $z = f(r, \theta)$, or more generally, $f(r, \theta, z) = 0$ are surfaces in $\mathbb{R}^3$.

**Example 3**
Sketch the graph of $r = 1$ in cylindrical coordinates.
**Solution:**
We know that $r = 1$ gives a circle of radius 1 in polar coordinates. Thus, in cylindrical coordinates, we have a circle of radius 1 at any value of $z$. Hence, we have an infinite cylinder of radius 1.




**Your Turn 1**

Sketch the graph of $z = r^2$ in cylindrical coordinates.

A question appears in Mobius

As we did in polar coordinates, we can also transform the equations of curves between the coordinate systems.

**Example 4**
Convert the equation $z = r^2 \cos\theta$ to Cartesian coordinates.
**Solution:**
Recall how we related a point $P(r, \theta, z)$ in cylindrical coordinates and a point in Cartesian coordinates $P(x, y, z)$:

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \tan\theta &= \frac{y}{x} \\ z &= z & z &= z \end{aligned}$$
$$x^2 + y^2 = r^2$$

Using these we get $z = r^2\cos\theta = r(r\cos\theta) = \sqrt{x^2+y^2}(x) = x\sqrt{x^2+y^2}$.

A question appears in Mobius

# 11.3 - Spherical Coordinates

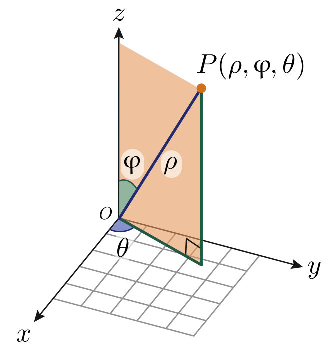
## Spherical Coordinates

In 2-dimensional space, we mentioned that polar coordinates were useful for problems that were symmetric about the origin. We now extend this idea to another 3-dimensional coordinate system called **spherical coordinates**.

As we did in cylindrical coordinates, we will use the pole $O$ and polar axis from polar coordinates and draw another axis $z$ perpendicular to the polar plane.

Let $P$ be any point in 3-dimensional space.

We will represent $P$ by the coordinates $(\rho, \varphi, \theta)$ where $\rho \geq 0$ is the length of the line $OP$, $\theta$ is the same angle as in polar and cylindrical coordinates, and $\varphi$ is the angle between the positive $z$-axis and the line $OP$.




**Your Turn 1**

The following app is designed to give you an intuition for spherical coordinates.

**Instructions**

1. Use the sliders to choose the spherical coordinates $(\rho, \varphi, \theta)$ of the point that you wish to draw. Note that this applet uses the unit of radians for angles $\theta$ and $\varphi$. Try a $\theta$ value between 0 and $\dfrac{\pi}{2}$ to start.
2. Click "Show Construction" to watch the point get sketched in 3D.
3. Check "Show Sphere" to display the sphere corresponding to the point you picked and study each component of your choice for $(\rho, \varphi, \theta)$ in 3D.

External resource: https://www.geogebra.org/material/iframe/id/tgzfzdzm/



Since we are keeping the same interpretation of $\theta$ from cylindrical coordinates, $\theta$ tells us the orientation of $P$ around the $z$-axis. Therefore, we only want $\varphi$ to indicate the "tilt" of the point with the $z$-axis. So, we restrict $0 \leq \varphi \leq \pi$.

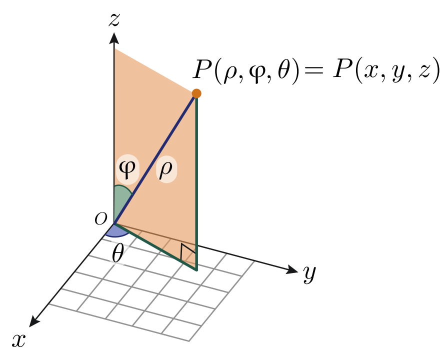
Thus, our restrictions in spherical coordinates are $\rho \geq 0$, $0 \leq \theta < 2\pi$ (or $-\pi < \theta \leq \pi$) and $0 \leq \varphi \leq \pi$.

**Remark**
The symbols used for spherical coordinates also vary from author to author as does the order in which they are written. In mathematics, it is not uncommon to find $\rho$ replaced by $r$. The standard ISO 31-11 convention uses $\varphi$ as the polar angle and $\theta$ as the angle with the positive $z$-axis. Therefore, it is very important to understand which notation is being used when reading an article.

From the diagram, we see that we can convert a point $P\ (x, y, z)$ from Cartesian coordinates to a point $P\ (\rho, \varphi, \theta)$ spherical coordinates with the following equations:

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta & \qquad \rho &= \sqrt{x^2 + y^2 + z^2} \\ y &= \rho \sin\varphi \sin\theta & \qquad \tan\theta &= \frac{y}{x} \\ z &= \rho \cos\varphi & \qquad \cos\varphi &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

$$x^2 + y^2 + z^2 = \rho^2$$




**Example 1**
Convert the following points from spherical coordinates to Cartesian coordinates.

1. $\left(1, \frac{\pi}{4}, \frac{\pi}{4}\right)$
2. $\left(1, \frac{\pi}{4}, \frac{5\pi}{4}\right)$

**Solution:**

1. We get
   - $x = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = \frac{1}{2}$,
   - $y = \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}$, and
   - $z = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

   Therefore, the point has Cartesian coordinates $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$.

2. We get
   - $x = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{4}\right) = -\frac{1}{2}$,
   - $y = \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{2}$, and
   - $z = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

   Therefore, the point has Cartesian coordinates $\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$.

A question appears in Mobius

**Example 2**
Convert the following points from Cartesian coordinates to spherical coordinates.

1. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{3}\right)$
2. $(-1,-1,-1)$

**Solution:**

1. We have $\rho = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\sqrt{3}\right)^2} = 2$, $\cos\varphi = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{\pi}{6}$.

   Since $\theta$ is in the first quadrant and $x, y > 0$, we get $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$.

   Hence, in spherical coordinates, the point is $\left(2, \frac{\pi}{6}, \frac{\pi}{4}\right)$.

2. We get $\rho = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$, $\cos\varphi = \frac{-1}{\sqrt{3}}$.

   Since $\theta$ is the third quadrant and $x, y < 0$, we get $\tan\theta = 1 \Rightarrow \theta = \frac{5\pi}{4}$.

   Thus, the point in spherical coordinates is $\left(\sqrt{3}, \arccos\left(\frac{-1}{\sqrt{3}}\right), \frac{5\pi}{4}\right)$.

A question appears in Mobius

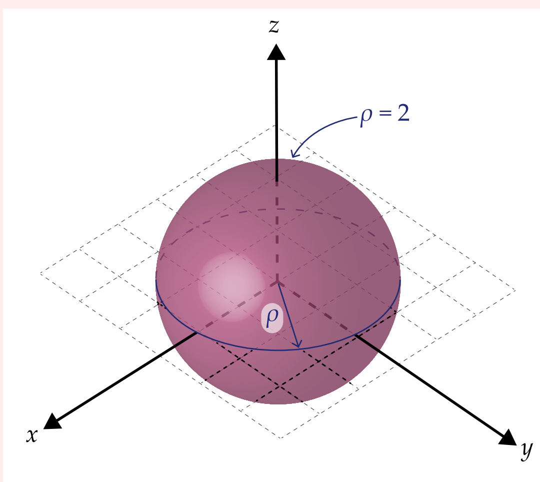
Observe from the above examples how $\theta$ determines which quadrant the point is in (its rotation around the $z$-axis) and how $\varphi$ determines whether the point will be above or below the $xy$-plane.

As usual, we will look at the graphs in spherical coordinates next.

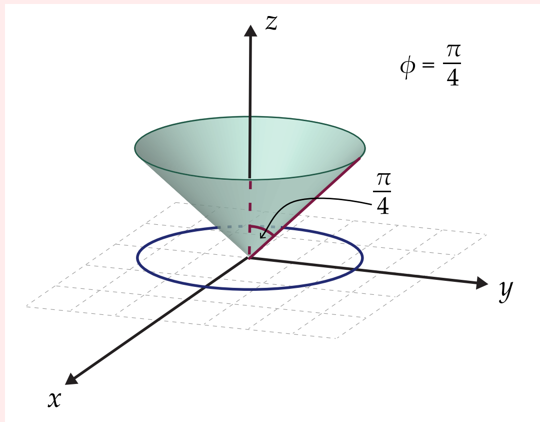
## Graphs in Spherical Coordinates

As with cylindrical coordinates, the graph of a function $f(\rho, \varphi, \theta) = 0$ in spherical coordinates gives a surface in $\mathbb{R}^3$.

**Example 3**
Sketch $\rho = 2$.
**Solution:**
Observe that this is the graph with all points 2 units from the origin. Hence, it is a sphere of radius 2.



**Example 4**
Sketch $\varphi = \dfrac{\pi}{4}$.
**Solution:**
First, imagine a line that makes a $\dfrac{\pi}{4}$ angle with the positive $z$-axis. Since there is no restriction on $\theta$, the graph of the surface will be this line rotated around the positive $z$-axis. Hence, we get a cone.




As with the other coordinate systems, we also want to convert equations between Cartesian and spherical coordinates.

**Example 5**
Convert $\rho = \sin\varphi\cos\theta$ to Cartesian coordinates.
**Solution:**
We first multiply both sides of the equation by $\rho$ to get

$$\rho^2 = \rho\sin\varphi\cos\theta$$

Hence, we can apply our conversion equations to get

$$\begin{aligned} x^2 + y^2 + z^2 &= x \\ \left(x - \frac{1}{2}\right)^2 + y^2 + z^2 &= \frac{1}{4} \end{aligned}$$

**Example 6**
Convert $z^2 = x^2 + y^2$ to spherical coordinates.
**Solution:**
We have

$$\begin{aligned} \rho^2\cos^2\varphi &= \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta \\ \cos^2\varphi &= \sin^2\varphi(\cos^2\theta + \sin^2\theta) \\ \tan^2\varphi &= 1 \end{aligned}$$

Thus, $\tan\varphi = \pm 1$, so we get $\varphi = \dfrac{\pi}{4}$ or $\varphi = \dfrac{3\pi}{4}$. Graphically, $z^2 = x^2 + y^2$ represents two cones with their peaks touching the origin and centred on the $z$-axis. The cone centred around the positive $z$-axis corresponds to $\varphi = \dfrac{\pi}{4}$ and the cone centred around the negative $z$-axis corresponds to $\varphi = \dfrac{3\pi}{4}$.

A question appears in Mobius

# 11.4 - Putting It All Together

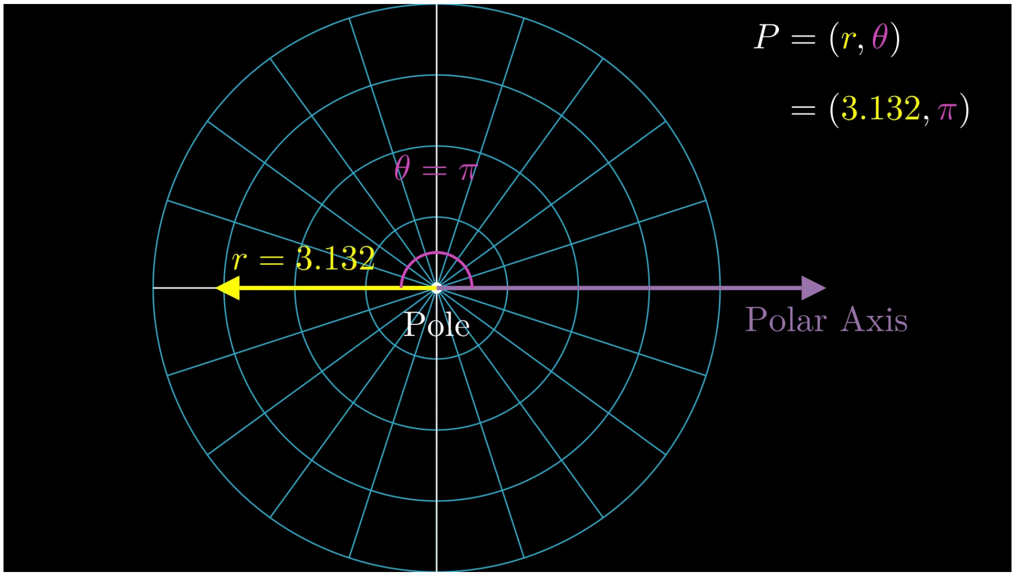
## Summary of Coordinate Systems

Let's briefly summarize the different coordinate systems that we've seen in 2 and 3 dimensions.

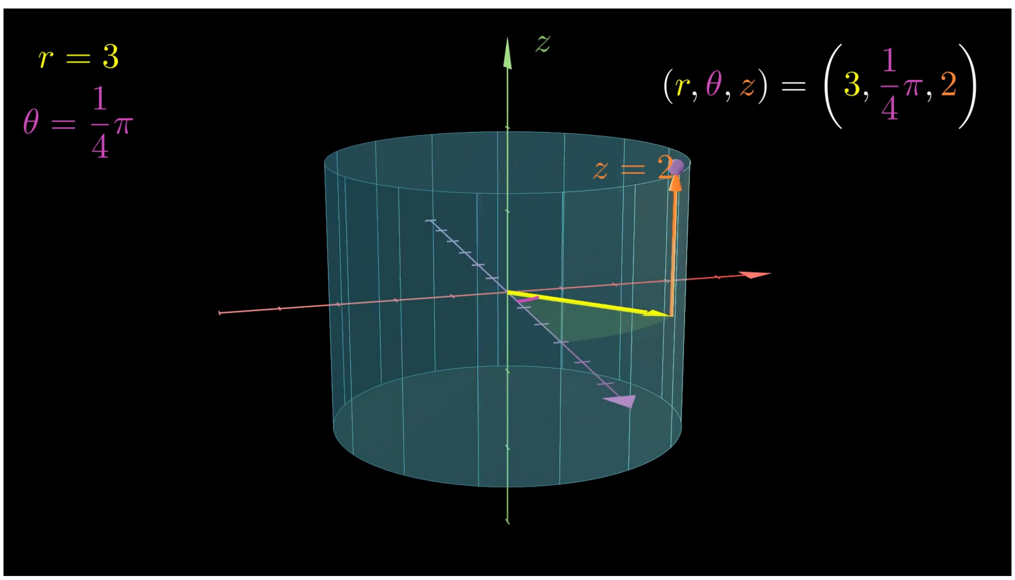
| Coordinate System | Coordinates | From Cartesian | To Cartesian | When to use |
|---|---|---|---|---|
| Polar | $(r,\theta)$ | $r=\sqrt{x^2+y^2}$<br>$\tan(\theta)=\frac{y}{x}$ | $x=r\cos(\theta)$<br>$y=r\sin(\theta)$ | When there is symmetry about the origin in $2D$ |
| Cylindrical | $(r,\theta,z)$ | $r=\sqrt{x^2+y^2}$<br>$\tan(\theta)=\frac{y}{x}$<br>$z=z$ | $x=r\cos(\theta)$<br>$y=r\sin(\theta)$<br>$z=z$ | When there is symmetry about the $z$-axis in $3D$ |
| Spherical | $(\rho,\varphi,\theta)$ | $\rho=\sqrt{x^2+y^2+z^2}$<br>$\tan(\theta)=\frac{y}{x}$<br>$\cos(\varphi)=\frac{z}{\sqrt{x^2+y^2+z^2}}$ | $x=\rho\sin(\varphi)\cos(\theta)$<br>$y=\rho\sin(\varphi)\sin(\theta)$<br>$z=\rho\cos(\varphi)$ | When there is symmetry about the origin in $3D$ |

In the following videos, we will revisit the construction of points in different coordinates. Pause the video as needed to make connections between the parameters for each coordinate system and their geometric interpretations.

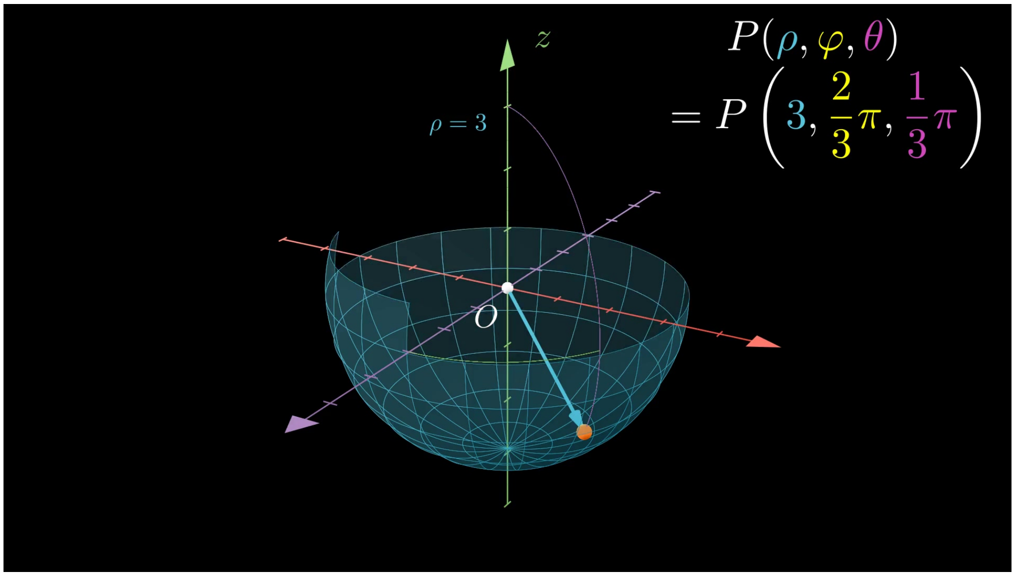
*A video appears here.*




*A video appears here.*



*A video appears here.*



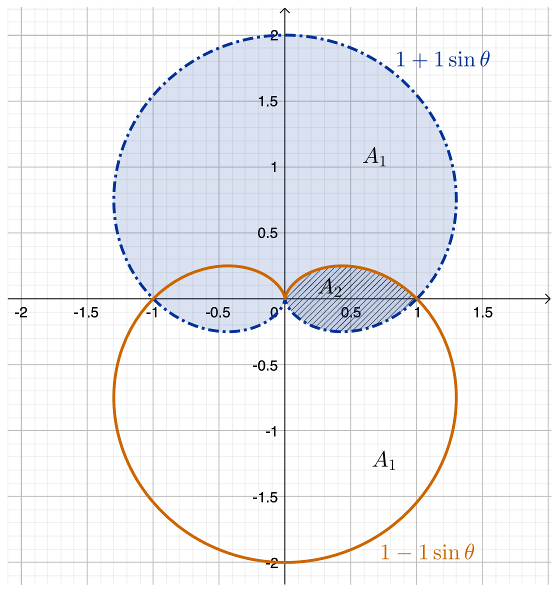

A question appears in Mobius

A question appears in Mobius

## Worked Example 2

Find the total area of the region inside $r = 1 + 1\sin\theta$ and $r = 1 - 1\sin\theta$.

The curves are illustrated below for $0 \leq \theta \leq 2\pi$:




We will find the total area of each region, then subtract the area of the intersection regions.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Worked Example 3

Let $C = \{(x, y, z) \mid z \geq \sqrt{x^2 + y^2},\, x^2 + y^2 + (z-1)^2 \leq 1\}$ be a region in $\mathbb{R}^3$.

a. Give a description in spherical coordinates.

A question appears in Mobius

b. Give a description in cylindrical coordinates.

A question appears in Mobius

## Practice Problems

1. Convert the following points from Cartesian coordinates to polar coordinates with $0 \leq \theta < 2\pi$.

   (a) $(-2, 2)$
   (b) $(\sqrt{3}, -1)$
   (c) $(-1, -\sqrt{3})$
   (d) $(2, 1)$

2. Convert the following points from polar coordinates to Cartesian coordinates.

   (a) $(2, \pi/3)$
   (b) $(3, 5\pi/6)$
   (c) $(3, 2\pi/3)$
   (d) $(2, -\pi/6)$

3. Sketch the region enclosed by $r = \sin\theta$ and find the area.

4. For each of the indicated regions in polar coordinates, sketch the region and find the area.
   (a) Inside both $r = 1 + 1\sin\theta$ and $r = 1 - 1\sin\theta$.
   (b) Inside $r = \sin\theta$ and outside $r = \sin 2\theta$.

5. For each of the indicated regions in polar coordinates, sketch the region and find the area.
   (a) The region enclosed by $r = \sin 3\theta$.
   (b) Inside both $r = 2 + 2\cos\theta$ and $r = 2 - 2\cos\theta$.

6. Convert the following equations in Cartesian coordinates to cylindrical coordinates.
   (a) $z = \sqrt{2x^2 + 2y^2}$
   (b) $x = y$
   (c) $z^2 = x^2 - y^2$

7. Convert the following equations in Cartesian coordinates to cylindrical coordinates.
   (a) $z = x^2 + y^2$
   (b) $1 = x^2 - y^2$

8. Convert the following equations in Cartesian coordinates to spherical coordinates.
   (a) $x^2 + y^2 = 4$
   (b) $x^2 + y^2 + z^2 = 2x$
   (c) $z = -\sqrt{x^2 + y^2}$
   (d) $z^2 = x^2 - y^2$

9. Convert the following equations in Cartesian coordinates to spherical coordinates.
   (a) $x = y$
   (b) $(x^2 + y^2 + z^2)^2 = z$

10. For each of the following regions in $\mathbb{R}^3$, given in Cartesian coordinates, give a description in cylindrical coordinates, and give a description in spherical coordinates.
   (a) $R = \{(x, y, z) : z \geq \sqrt{x^2 + y^2},\ z \leq 1\}$
   (b) $L = \{(x, y, z) : z \geq \sqrt{x^2 + y^2},\ x^2 + y^2 + z^2 \leq 2\}$

## Select Answers and Solutions

1. (a) $(\sqrt{8}, 3\pi/4)$
   (b) $(2, 11\pi/6)$
   (c) $(2, 4\pi/3)$
   (d) $(5, \arctan(1/2))$

2. (a) $(1, \sqrt{3})$
   (b) $(-3\sqrt{3}/2, 3/2)$
   (c) $(-3/2, 3\sqrt{3}/2)$
   (d) $(\sqrt{3}, -1)$

3. The area equals $\pi/4$.

4. (a) $(3\pi/2 + 4)$

   (b) $(\pi/8 + 3\sqrt{3}/32)$

5. No answer provided.

6. (a) $z = \sqrt{2x^2 + 2y^2} \Rightarrow z = \sqrt{2}r$
   (b) $x = y \Rightarrow \theta = \pi/5, \theta = 5\pi/4$
   (c) $z^2 = x^2 - y^2 \Rightarrow z^2 = r^2\cos(2\theta)$

7. No answer provided.

8. (a) $x^2 + y^2 = 4 \Rightarrow \rho^2 \sin^2\varphi = 4$
   (b) $x^2 + y^2 + z^2 = 2x \Rightarrow \rho = 2\sin\varphi\cos\theta$
   (c) $z = -\sqrt{x^2 + y^2} \Rightarrow \cos\varphi = -|\sin\varphi|$
   (d) $z^2 = x^2 - y^2 \Rightarrow 1 = \tan^2\varphi\cos(2\theta)$

9. No answer provided.

10. (a) $L = \{(x, y, z) : z \geq \sqrt{x^2 + y^2},\ x^2 + y^2 + z^2 \leq 2\}$
    Description in spherical coordinates: $\{(\rho, \varphi, \theta) : 0 \leq \varphi \leq \pi/4\ ,\ \rho \leq \sqrt{2}\}$
    Description in cylindrical coordinates: $\{(z, r, \theta) : z \geq r, r^2 + z^2 \leq 2\}$
    (b) $R = \{(x, y, z) : z \geq \sqrt{x^2 + y^2},\ z \leq 1\}$
    Description in spherical coordinates: $\{(\rho, \varphi, \theta) : 0 \leq \varphi \leq \pi/4\ ,\ \rho \leq 1/\cos\varphi\}$
    Description in cylindrical coordinates: $\{(z, r, \theta) : r \leq z \leq 1\}$

(Special thanks to Milad Farsi for providing the solutions to these exercises.)

# Unit 12

# Mappings of $\mathbb{R}^2$ into $\mathbb{R}^2$

## 12.1 - The Geometry of Mappings

### The Geometry of Mappings

### Introduction

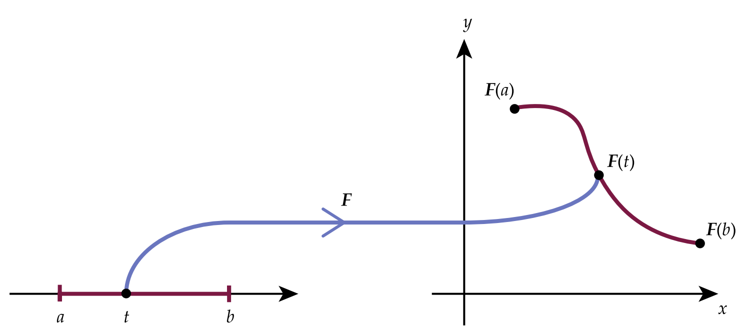
So far we have studied scalar functions, that is, functions which map a subset of $\mathbb{R}^n$ into $\mathbb{R}$. We now extend the ideas of differential calculus to more general functions.

**Definition: Vector-Valued Function**
A function whose domain is a subset of $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^m$ is called a **vector-valued** function.

You have already worked with the simplest type of vector-valued functions. Consider parametric equations $x = f(t)$, $y = g(t)$ for a curve in $\mathbb{R}^2$.

These two scalar equations can be written as a vector equation:

$$(x, y) = F(t) = (f(t), g(t))$$




The function $F$ maps $t$ to $F(t)$, so the domain of $F$ is a subset of $\mathbb{R}$ and its codomain is $\mathbb{R}^2$. Consequently, $F$ is a vector-valued function.

While we represent $(f(t), g(t))$ as a point in $\mathbb{R}^2$, remember that it can also be thought of as a position vector.

**Definition: Mapping**
A vector-valued function whose domain is a subset of $\mathbb{R}^n$ and whose codomain is a subset of $\mathbb{R}^n$ is called a **mapping** (or transformation).

You might recognize the term “mapping” from your linear algebra courses. As we will see, linear algebra plays an important role in our study of mappings.

## The Geometry of Mappings

A pair of equations

$$u = f(x, y)$$
$$v = g(x, y)$$

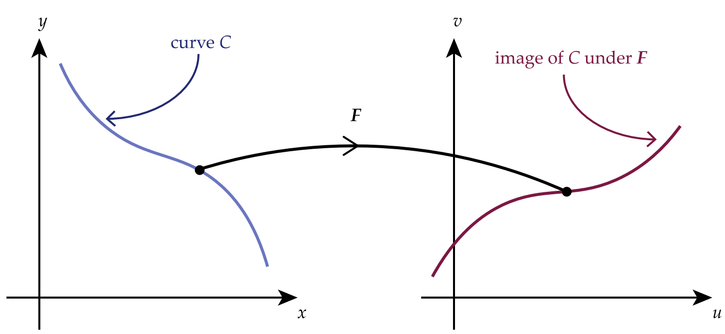
associates with each point $(x, y) \in \mathbb{R}^2$ a single point $(u, v) \in \mathbb{R}^2$, and thus defines a vector-valued function

$$(u, v) = F(x, y) = (f(x, y), g(x, y))$$

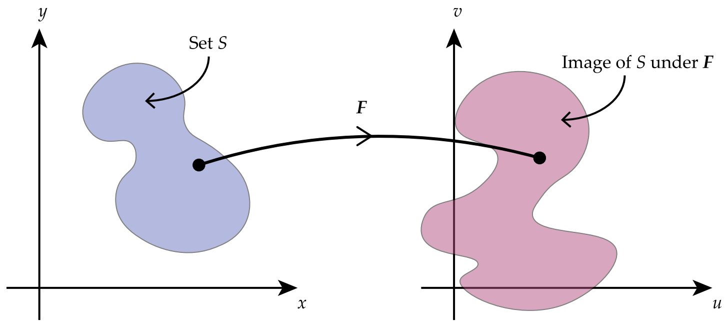
The scalar functions $f$ and $g$ are called the **component functions** of the mapping.

Mappings of $\mathbb{R}^n$ into $\mathbb{R}^n$ have many applications, such as defining curvilinear coordinate systems (e.g. polar coordinates), and performing a change of variables in multiple integrals (which we will see in Units 14 and 15). Beyond this course, mappings are used in applied mathematics, in statistics, and in computer graphics for simplifying problems in two or more variables.

In general, if a mapping $F$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ acts on a curve $C$ in its domain, it will determine a curve in its range, denoted by $F(C)$ and called the **image of $C$ under $F$**.




More generally, if a mapping $F$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ acts on any subset $S$ in its domain it will determine a set $F(S)$ in its range, called the **image of $S$ under $F$**.




In order to develop an intuitive geometric understanding of a mapping, it is helpful to determine the images of different curves and sets under the mapping. In general, a mapping will deform a given curve or set.

**Example 1**

Consider the mapping defined by $(u, v) = F(x, y) = \left(\frac{1}{2}(x + y), \frac{1}{2}(-x + y)\right)$.

1. Find the images of the lines $x = k$ and $y = \ell$ under $F$.

   **Solution:** We are given that $u = \frac{1}{2}(x + y)$ and $v = \frac{1}{2}(-x + y)$. We need to use these equations to convert the equations $x = k$ and $y = \ell$ in terms of $u$ and $v$.

   One way we can do this is to first solve for $x$ and $y$ in terms of $u$ and $v$.

   Observe that we have
   $$x = u - v \quad \text{and} \quad y = u + v$$

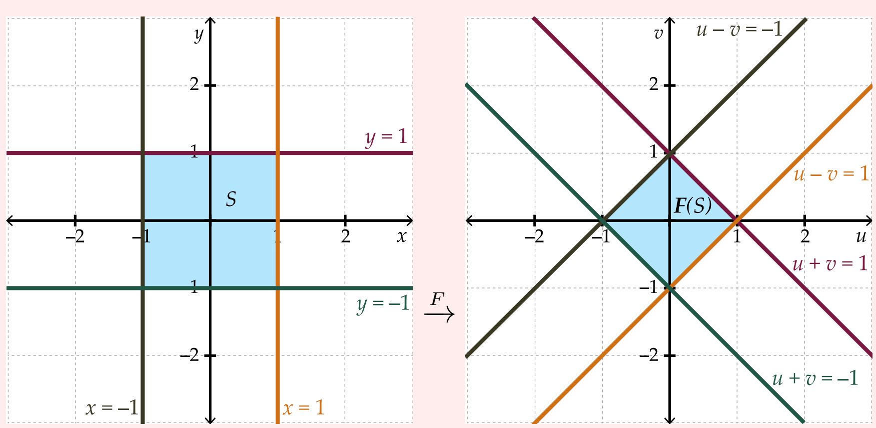
   Thus, a line $x = k$ under the mapping becomes
   $$u - v = k$$

   Similarly, a line $y = \ell$ is transformed into
   $$u + v = \ell$$

2. Find the image of the square $S = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$ under $F$.

   **Solution:**

   To determine the image of $S$ under $F$, we find the image of each of the boundary lines. In particular, by choosing $k = \pm 1$ and $\ell = \pm 1$, we obtain the images of the sides of the square $S$.



Observe that the mapping in the example above is linear. For any linear mapping, the image of a straight line in the $xy$-plane is a straight line in the $uv$-plane. However, we see from the image of $S$ under $F$ that the lines are contracted and rotated by $F$.

**Your Turn 1**

Find the image of the circle $(x-1)^2 + y^2 = 1$ under the mapping $F$ defined in the example above.

A question appears in Mobius

**Example 2**
Find the image of $D = \{(x, y) \mid -1 \le x \le 3, 0 \le y \le 2\}$ under the mapping

$$(u, v) = T(x, y) = (x^2 - y^2, xy)$$

**Solution:**
To determine the image of $D$ under $T$, we find the image of each of the boundary lines. In this case, it is not so easy to solve for $x$ and $y$ in terms of $u$ and $v$. We instead substitute the equation of each boundary line directly into the mapping.
For the line $x = -1$, $0 \le y \le 2$, we get

$$\begin{aligned} u &= (-1)^2 - y^2 = 1 - y^2 \\ v &= (-1)y = -y \end{aligned}$$

We want equations of curves in the $uv$-plane, so we eliminate $y$ to obtain

$$u = 1 - (-v)^2 = 1 - v^2$$

Since $v = -y$, the condition $0 \le y \le 2$ gives

$$0 \le -v \le 2 \Rightarrow -2 \le v \le 0$$

For the line $x = 3$, $0 \le y \le 2$, we get $v = 3y$, so

$$u = (3)^2 - y^2 = 9 - y^2 = 9 - \left(\frac{1}{3}v\right)^2 = 9 - \frac{1}{9}v^2$$

with

$$0 \le \frac{1}{3}v \le 2 \Rightarrow 0 \le v \le 6$$

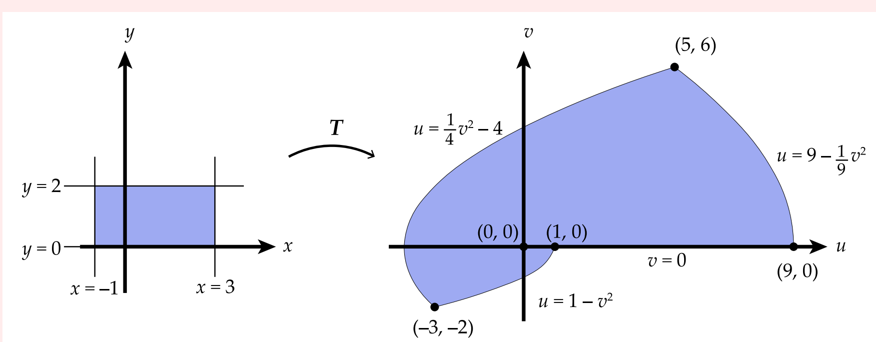
For the line $y = 2$, $-1 \le x \le 3$, we get $v = 2x$, so

$$u = x^2 - 2^2 = x^2 - 4 = \frac{1}{4}v^2 - 4, \quad -2 \le v \le 6$$

For the line $y = 0$, $-1 \le x \le 3$, we get $v = 0$ and

$$u = x^2 - 0^2 = x^2$$

Since $x$ runs from $-1$ to 3 and $u = x^2$, we get that $u$ starts at 1 (when $x = -1$), moves to $u = 0$ (when $x = 0$) and then $u$ moves from 0 to 9 (as $x$ changes from 0 to 3).



**Example 3**
Find the image of the rectangle

$$R = \left\{(r,\theta) \mid 1 \le r \le 2, \quad \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$$

under the mapping from polar coordinates to Cartesian coordinates defined by

$$(x,y) = F(r,\theta) = (r\cos\theta, r\sin\theta)$$

**Solution:**
To find the image of the rectangle, we will find the image of each of the boundary lines under $F$. For the line $r = 1$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$ we get

$$x = \cos\theta, \qquad y = \sin\theta$$

for $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$. In this case, we don't need to eliminate $\theta$ since we recognize these are parametric equations of a circle of radius 1, since they imply

$$x^2 + y^2 = 1$$

Thus, the image is the part of the unit circle with $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$.
Similarly, we see that the line $r = 2$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$ gives the part of the circle of radius 2 for which $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$.
The image of a line $\theta = \frac{\pi}{4}$, $1 \le r \le 2$ is

$$x = r\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}r, \qquad y = r\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}r$$

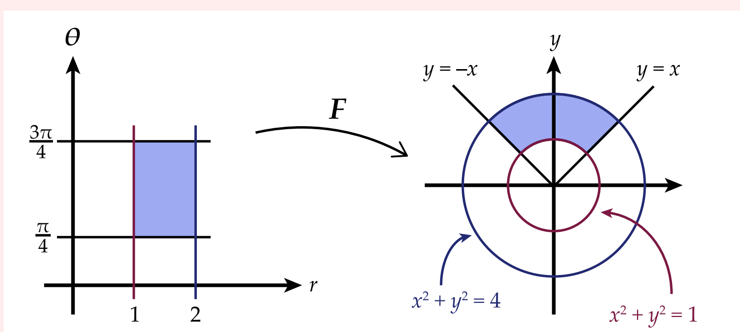
for $1 \le r \le 2$. Eliminating $r$ gives $y = x$. Moreover, we have that $r = \sqrt{2}x$ and hence $1 \le r \le 2$ gives that $x$ has values from

$$1 \le \sqrt{2}x \le 2 \Rightarrow \frac{1}{\sqrt{2}} \le x \le \sqrt{2}$$

Similarly, for the line $\theta = \frac{3\pi}{4}$, $1 \le r \le 2$ we get

$$x = r\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}r, \qquad y = r\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}r$$

for $1 \le r \le 2$. Thus, the image is the line $y = -x$ with $x$ values $-\sqrt{2} \le x \le -\frac{1}{\sqrt{2}}$.



**Remark**
The mapping from polar coordinates to Cartesian coordinates is non-linear. The image of a straight line is not necessarily a straight line.

### Your Turn 2

In the interactive applet below, you will be able to see how the mappings from Examples 1 to 3 change a polygon in $\mathbb{R}^2$.

### Stop and Think

Note that the labeling of the axes will change as the variables change. Try to label the axes correctly for the exercises given in Your Turn.

**Instructions**

1. Use the dropdown menu to select a mapping.
2. You can click and drag the corners of the polygon to experiment with different shapes.
3. Depending on the mapping that you selected, you may need to zoom in or out to view its image.

External resource: https://www.geogebra.org/material/iframe/id/bmjjxwwd/



### Your Turn 3

Find the image of the square

$$S = \{(x, y) \mid 1 \leq x \leq 2, 2 \leq y \leq 3\}$$

under the mapping defined by

$$(u, v) = F(x, y) = (xy, y)$$

A question appears in Mobius

# 12.2 - The Linear Approximation of a Mapping

## The Linear Approximation of a Mapping

### Linear Approximation of a Mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$

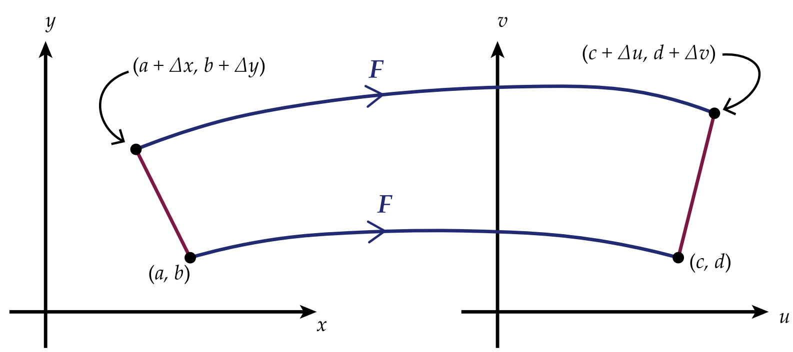
Consider a mapping $F$ defined by $u = f(x, y)$, $v = g(x, y)$.

We assume that $F$ has continuous partial derivatives. By this, we mean that the component functions $f$ and $g$ have continuous partial derivatives.

The image of a point $(a, b)$ in the $xy$-plane is the point $(c, d)$ in the $uv$-plane, where

$$c = f(a, b), \qquad d = g(a, b)$$

As usual, we want to approximate the image $(c + \Delta u,\ d + \Delta v)$ of a nearby point $(a + \Delta x,\ b + \Delta y)$.




We do this by using the linear approximation for $f(x, y)$ and $g(x, y)$ separately. We get

$$\Delta u \approx \frac{\partial f}{\partial x}(a, b)\Delta x + \frac{\partial f}{\partial y}(a, b)\Delta y$$
$$\Delta v \approx \frac{\partial g}{\partial x}(a, b)\Delta x + \frac{\partial g}{\partial y}(a, b)\Delta y$$

for $\Delta x$ and $\Delta y$ sufficiently small. This can be written in matrix form as:

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} \frac{\partial f}{\partial x}(a, b) & \frac{\partial f}{\partial y}(a, b) \\ \frac{\partial g}{\partial x}(a, b) & \frac{\partial g}{\partial y}(a, b) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

where the product on the right side of the equation is **matrix multiplication.**

Observe that this resembles our usual form of the linear approximation where the $2 \times 2$ matrix is taking the place of the "derivative". Thus, we make the following definition.

**Definition: Derivative Matrix**
The **derivative matrix** of a mapping defined by

$$F(x,y) = (f(x,y), g(x,y))$$

is denoted $DF$ and defined by

$$DF = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}$$

**Example 1**
Find the derivative matrix of the mapping

$$(u,v) = F(x,y) = (x^2 \sin y, y^2 \cos x)$$

**Solution:**
We have $f(x,y) = x^2 \sin y$ and $g(x,y) = y^2 \cos x$. So,

$$\begin{aligned} DF(x,y) &= \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} 2x \sin y & x^2 \cos y \\ -y^2 \sin x & 2y \cos x \end{bmatrix} \end{aligned}$$

If we introduce the column vectors

$$\Delta\vec{u} = \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}, \qquad \Delta\vec{x} = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

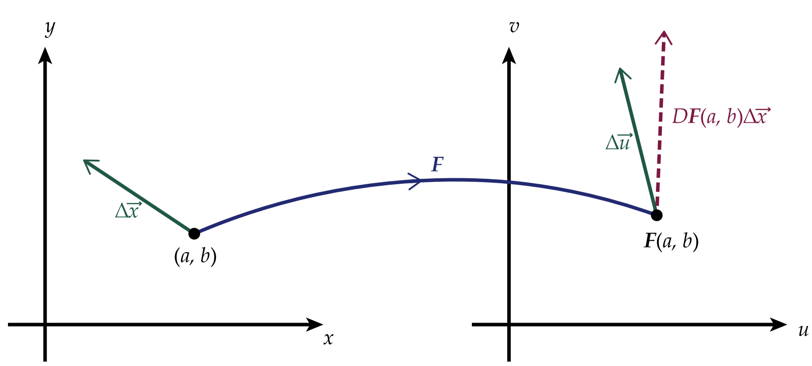
then the **increment form of the linear approximation for mappings** becomes

$$\Delta\vec{u} \approx DF(a,b)\Delta\vec{x}$$

for $\Delta\vec{x}$ sufficiently small. Thus, the **linear approximation for mappings** is

$$F(x,y) \approx F(a,b) + DF(a,b)\Delta\vec{x}$$

The geometrical interpretation of the linear approximation for mappings is this: the derivative matrix $DF(a,b)$ acts as a linear mapping on the displacement vector $\Delta\vec{x}$ to give an approximation of the image $\Delta\vec{u}$ of the displacement under $F$.



A question appears in Mobius

**Example 2**
Consider the mapping defined by

$$(u,v) = F(x,y) = \left(-x + \sqrt{x^2+y^2}, x + \sqrt{x^2+y^2}\right)$$

as in the previous Your Turn exercise. Use the linear approximation to estimate the image of the point $(3.02, 3.99)$ under $F$.
**Solution:**
As we have seen in the previous your turn exercise, the derivative matrix of $F$ is

$$DF(x,y) = \begin{bmatrix} -1 + \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ 1 + \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

As a reference point choose $(3,4)$ since 3 is close to 3.02 and 4 is close to 3.99. Then $F(3,4) = (2,8)$ and

$$DF(3,4) = \begin{bmatrix} -\dfrac{2}{5} & \dfrac{4}{5} \\ \dfrac{8}{5} & \dfrac{4}{5} \end{bmatrix}$$

The displacement in the $uv$-plane is approximated by

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx DF(3,4) \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{5} & \dfrac{4}{5} \\ \dfrac{8}{5} & \dfrac{4}{5} \end{bmatrix} \begin{bmatrix} 0.02 \\ -0.01 \end{bmatrix} = \begin{bmatrix} -0.016 \\ 0.024 \end{bmatrix}$$

Thus, the linear approximation gives

$$F(3.02, 3.99) \approx (2,8) + (-0.016, 0.024) = (1.984, 8.024)$$

Note: The calculator value is $(1.98405, 8.02405)$.

A question appears in Mobius

### Generalization

A mapping $F$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is defined by a set of $m$ component functions:

$$\begin{aligned} u_1 =& f_1(x_1, \dots, x_n) \\ \vdots& \\ u_m =& f_m(x_1, \dots, x_n) \end{aligned}$$

Or, in vector notation

$$\vec{u} = F(\vec{x}) = (f_1(\vec{x}), \cdots f_m(\vec{x})), \quad \vec{x} \in \mathbb{R}^n$$

If we assume that $F$ has continuous partial derivatives, then the derivative matrix of $F$ is the $m \times n$ matrix defined by

$$DF(\vec{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

As expected, the linear approximation for $F$ at $\vec{a}$ is

$$F(\vec{x}) \approx F(\vec{a}) + DF(\vec{a})\Delta\vec{x}$$

where

$$\Delta\vec{u} = \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_m \end{bmatrix} \in \mathbb{R}^m, \qquad \Delta\vec{x} = \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} \in \mathbb{R}^n$$

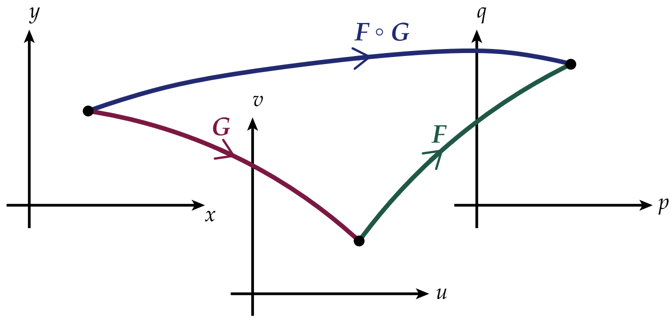
## 12.3 - Composite Mappings and the Chain Rule

### Composite Mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ and the Chain Rule

The next step in developing the theory of mappings is to study the composition of two mappings.

Consider successive mappings $F$ and $G$ of $\mathbb{R}^2$ into $\mathbb{R}^2$, defined by

$$F : \begin{cases} p = p(u,v) \\ q = q(u,v) \end{cases} \qquad G : \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases} \tag{*}$$




The composite mapping $F \circ G$, defined by

$$\begin{cases} p = p\Big(u(x,y), v(x,y)\Big) \\ q = q\Big(u(x,y), v(x,y)\Big) \end{cases} \tag{**}$$

maps the $xy$-plane directly into the $pq$-plane.

We want to know how the derivative matrix $D(F \circ G)$ of the composite mapping is related to the derivative matrices $DF$ and $DG$. As it turns out, $D(F \circ G)(x,y)$ is the matrix product of $DF(u,v)$ and $DG(x,y)$, where $(u,v) = G(x,y)$.

We state this formally in the following theorem.

**Theorem 1: Chain Rule in Matrix Form** for mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$
Let $F$ and $G$ be mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. If $G$ has continuous partial derivatives at $(x,y)$ and $F$ has continuous partial derivatives at $(u,v) = G(x,y)$, then the composite mapping $F \circ G$ has continuous partial derivatives at $(x,y)$ and

$$D(F \circ G)(x,y) = DF(u,v)DG(x,y)$$

**Proof:** Define the component functions for $F$, $G$, and $F \circ G$ as in equations $(*)$ and $(**)$. Then, the Chain Rule for scalar functions gives

$$\begin{aligned} DF(u,v)DG(x,y) &= \begin{bmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial p}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial p}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial q}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial q}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial q}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial q}{\partial v}\frac{\partial v}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{bmatrix} \\ &= D(F \circ G)(x,y) \end{aligned}$$

as required. $\square$

**Example 1**
Consider the mappings $G$ and $F$ defined by

$$(u, v) = G(x, y) = (xy, x + y)$$
$$(p, q) = F(u, v) = (u - v, u^2)$$

1. Find the composite mapping $F \circ G$
2. Find the derivative matrices $DG$, $DF$, $D(F \circ G)$, and verify the Chain Rule formula.

**Solution:**

1. The composite mapping is

$$(p, q) = F(G(x, y)) = F(xy, x + y) = (xy - x - y, x^2y^2)$$

2. The derivative matrices are:

$$DG(x, y) = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}, \quad DF(u, v) = \begin{bmatrix} 1 & -1 \\ 2u & 0 \end{bmatrix}, \quad D(F \circ G)(x, y) = \begin{bmatrix} y-1 & x-1 \\ 2xy^2 & 2x^2y \end{bmatrix}$$

Form the matrix product,

$$\begin{aligned} DF(u, v)DG(x, y) &= \begin{bmatrix} 1 & -1 \\ 2u & 0 \end{bmatrix} \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} y-1 & x-1 \\ 2uy & 2ux \end{bmatrix} \\ &= \begin{bmatrix} y-1 & x-1 \\ 2xy^2 & 2x^2y \end{bmatrix}, \quad \text{on substituting } u = xy \\ &= D(F \circ G)(x, y), \quad \text{as required.} \end{aligned}$$

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## 12.4 - Putting It All Together

### Worked Example 1

Find the image of the square $D = \{(x, y) \mid 1 \leq x \leq 2, 2 \leq y \leq \ 3\}$ under the image of $T(x, y) = (e^{x+y}, e^{x-y})$.

**Solution:**

We begin by setting $u = e^{x+y}$ and $v = e^{x-y}$ .

Note that with this setting we have $u \neq 0, v \neq 0$.

Then, we substitute the equation of each line directly into the mapping.

We have 4 cases to look at. Below, we'll study each case separately.

Try each case yourself, and verify your results by clicking and revealing the conclusion for each case.

- **Case 1:** $x = 1,\ 2 \leq y \leq \ 3$

A question appears in Mobius

- **Case 2:** $x = 2,\ 2 \leq y \leq\ 3$

A question appears in Mobius

- **Case 3:** $y = 2,\ 1 \leq x \leq\ 2$

A question appears in Mobius

- **Case 4:** $y = 3,\ 1 \leq x \leq\ 2$

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

## Application

A Möbius transformation has the form $f(z) = \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

We can illustrate this transformation on the complex plane, where a complex number $z = x + iy$ has coordinates $(x, y)$. Notice that the complex plane looks a lot like $\mathbb{R}^2$. (You might remember from your linear algebra course that $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic vector spaces over $\mathbb{R}$.)

Notice that when $c \neq 0$, the denominator will equal zero when $z = \dfrac{-d}{c}$. This means that the transformation is only well-defined on the complex plane when $c = 0$. However, we get around this by making infinity a point and mapping $f\left(\dfrac{-d}{c}\right) = \infty$. Adding this point is called one-point compactification of $\mathbb{C}$, and it "transforms" $\mathbb{C}$ from a plane into a sphere with the infinity point at the North pole.

This means that the Möbius transformation maps the sphere onto itself. The exact result of the mapping depends on the values of $a, b, c, d$: a Möbius transformation is a composition of translations, dilations, rotations, and inversion. Möbius transformations have applications in physics, computer science, and image processing. We are only scratching the surface of these transformations here.

The applet below illustrates how a grid on the complex plane is transformed by Möbius transformations.

**Instructions**

1. Change the values of $a$, $b$, $c$, and $d$ by dragging them in the complex plane.
2. Observe how the values of $a, b, c$, and $d$ transform the grid.
   - Start by setting $b$ and $c$ to zero. The quotient $\dfrac{a}{d}$ determines how the grid is scaled and rotated.
   - Next, move $b$ away from zero. Notice how the grid gets translated.
   - Finally, move $c$ away from zero: this distorts the grid. Now, notice that the behaviour of the grid when moving $a, b$, and $d$ is no longer as predictable as it was when $c$ was equal to zero.

External resource: https://www.geogebra.org/material/iframe/id/jmxdqpgq/



To learn more about Möbius transformations, have a look at this video: `https://www.youtube.com/watch?v=0z1fIsUNhO4`

## Practice Problems

1. Find the image of $D = \{(x, y) \in \mathbb{R}^2 | -1 \leq x \leq 2, 0 \leq y \leq 2\}$ under $T(x, y) = (x + 2y, 3x - y)$.
2. Find the image of $D = \{(x, y) \in \mathbb{R}^2 | 1 \leq x \leq 2, 0 \leq y \leq 2\}$ under $T(x, y) = (x^2 + y^2, x^2 - y^2)$.
3. Use the linear approximation in matrix form to find the approximate image of the point (3.1, 3.9) under the map defined by
$$(u, v) = F(x, y) = \left(\sqrt{x^2 + y^2}, \frac{x}{\sqrt{x^2 + y^2}}\right)$$
4. Consider the maps $F$ and $G$ defined by
$$F(u, v) = (v + u^2, u), \qquad G(x, y) = (e^x y, 2e^{-x} y)$$

State the Chain Rule in matrix form, and use it to calculate the derivative $D(F \circ G)(0,1)$ of the composite map.

5. Let $(u,v) = F(x,y) = \left(x\ln(y-x^4), (2+\frac{y}{x})^{3/2}\right)$. Suppose that $G(u,v)$ has continuous partial derivatives with $G(0,8) = (1,-1)$ and $DG(0,8) = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$. Use the linear approximation to approximate $(G \circ F)(0.9, 2.1)$.

6. Sketch the image of the square

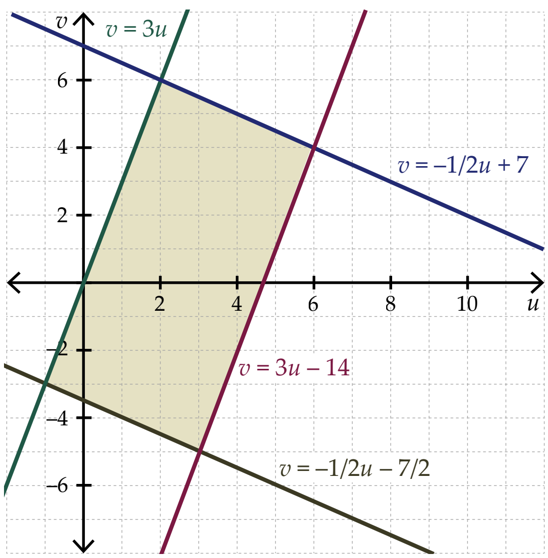
$$D = \{(x,y) \mid 1 \le x \le 2, 2 \le y \le 3\}$$

under the map

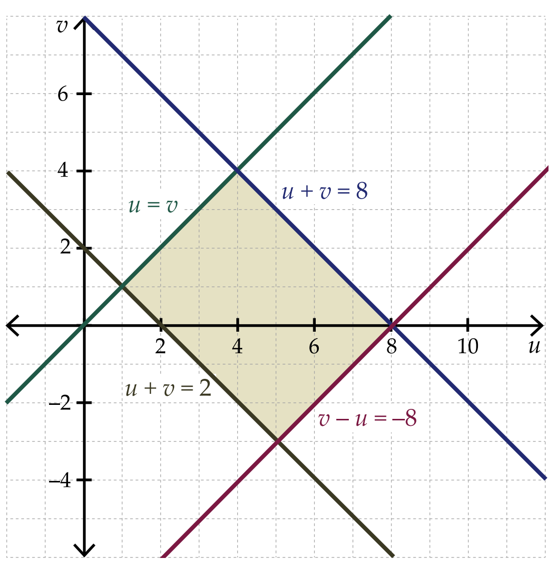
$$T(x,y) = \left(x\cos\left(\frac{\pi}{3}xy\right), x\sin\left(\frac{\pi}{3}xy\right)\right)$$

## Select Answers and Solutions

1.




2.

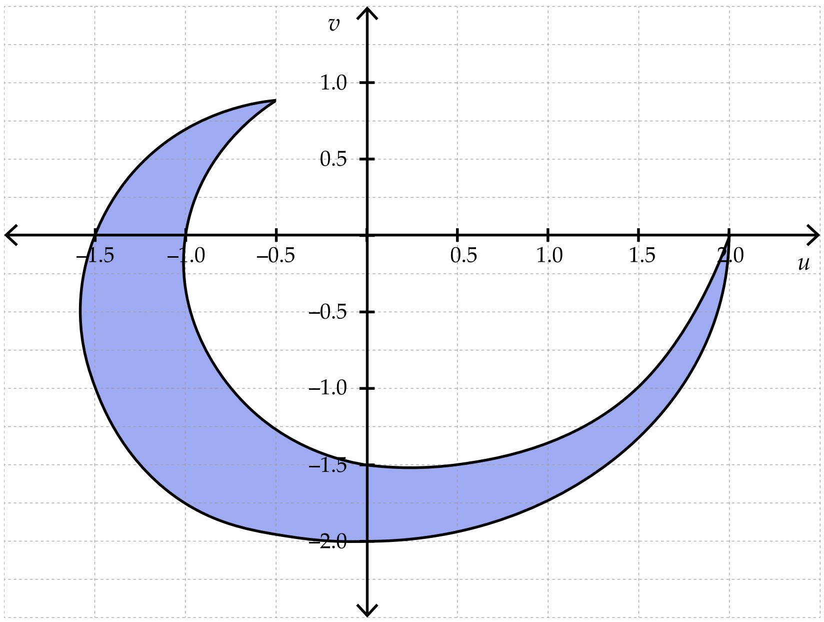


3. $F(3.1, 3.9) \approx (4.98, 0.6224)$

4. $D(F \circ G)(0, 1) = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$

5. $(G \circ F)(0.9, 2.1) \approx (0.9, -0.3)$.

6.



# Unit 13

# Jacobians and Inverse Mappings

## 13.1 - The Inverse Mapping Theorem

### Invertible and Inverse Mappings

Our goal now is to find a condition which will guarantee that a mapping $(u, v) = F(x, y)$ has an inverse. We start by defining inverse mappings in the expected way.

**Definition: Invertible Mapping** and **Inverse Mapping**
Let $F$ be a mapping from a set $D_{xy}$ onto a set $D_{uv}$. If there exists a mapping $F^{-1}$, called the **inverse of** $F$ which maps $D_{uv}$ onto $D_{xy}$ such that

$$(x, y) = F^{-1}(u, v) \quad \text{if and only if} \quad (u, v) = F(x, y)$$

then $F$ is **invertible** on $D_{xy}$.

As usual, we have

$$\begin{aligned} (F^{-1} \circ F)(x, y) &= (x, y) \qquad \text{for all } (x, y) \in D_{xy} \\ (F \circ F^{-1})(u, v) &= (u, v) \qquad \text{for all } (u, v) \in D_{uv} \end{aligned} \tag{*}$$

Recall that a function being invertible is related to it being one-to-one.

**Definition: One-to-One**
A mapping $F$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is said to be **one-to-one** (or **injective**) on a set $D_{xy}$ if and only if $F(a, b) = F(c, d)$ implies $(a, b) = (c, d)$, for all $(a, b), (c, d) \in D_{xy}$.

A one-to-one mapping sends each element of the domain to a unique element of the codomain. The next theorem relates a mapping being one-to-one to its invertibility:

**Theorem 1:** One-to-One Implies Invertible
Let $F$ be a mapping from a set $D_{xy}$ onto a set $D_{uv}$. If $F$ is one-to-one on $D_{xy}$, then $F$ is invertible on $D_{xy}$.

Now, recall from single-variable calculus that if $f'(x) > 0$ for all $x \in [a, b]$, then $f$ is one-to-one on $[a, b]$ and hence has an inverse on $[a, b]$. Thus, for a mapping $F$, it makes sense to investigate the relation between the derivative matrix $DF$ of $F$ and $F$ being invertible. We start with the following theorem.

**Theorem 2: Inverse of th**e Derivative Ma**trix**
Consider a mapping $F$ which maps $D_{xy}$ onto $D_{uv}$.
If $F$ has continuous partial derivatives at $\vec{x} \in D_{xy}$ and there exists an inverse mapping $F^{-1}$ of $F$ which has continuous partial derivatives at $\vec{u} = F(\vec{x}) \in D_{uv}$, then

$$DF^{-1}(\vec{u})DF(\vec{x}) = I$$

**Proof:** By the Chain Rule in Matrix Form we get

$$DF^{-1}(\vec{u})DF(\vec{x}) = D(F^{-1} \circ F)(\vec{x})$$

Then, by equation $(*)$ we have

$$D(F^{-1} \circ F)(\vec{x}) = D\vec{x} = \begin{bmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial y} \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

as required.

□

The above theorem gives us a way of finding the inverse of the mapping $DF(\vec{x})$:

If $F$ has continuous partial derivatives and has an inverse mapping with continuous partial derivatives, then $DF^{-1}(\vec{x}) = DF^{-1}(\vec{u})$.

Let's see how this works in practice.

**Example 1**
Consider the mapping defined by

$$(u, v) = F(x, y) = (y + x^2, x)$$

Solve for the inverse mapping $F^{-1}$, find the derivative matrices $DF$ and $DF^{-1}$, and verify that $DF^{-1}(u, v)$ is the matrix inverse of $DF(x, y)$.
**Solution:**
The inverse mapping is obtained by solving

$$\begin{aligned} u &= y + x^2 \\ v &= x \end{aligned}$$

for $x$ and $y$. We obtain

$$\begin{aligned} x &= v \\ y &= u - v^2 \end{aligned}$$

Hence, the inverse mapping is

$$\begin{aligned} (x, y) &= F^{-1}(u, v) \\ &= (v, u - v^2) \end{aligned}$$

The derivative matrices are:

$$\begin{aligned} DF(x, y) &= \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ DF^{-1}(u, v) &= \begin{bmatrix} 0 & 1 \\ 1 & -2v \end{bmatrix} \end{aligned}$$

Form the matrix product,

$$\begin{aligned} DF^{-1}(u, v)DF(x, y) &= \begin{bmatrix} 0 & 1 \\ 1 & -2v \end{bmatrix} \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 2x - 2v & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} && \text{on substituting } v = x \end{aligned}$$

The fact that we could solve and obtain a unique solution for $x$ and $y$ in the preceding example proves that $F$ has an inverse mapping on $\mathbb{R}^2$. However, we can only carry out this step in simple examples. Hence, it is useful to develop a test to determine if a mapping $F$ has an inverse mapping.

## The Jacobian of a Mapping

The determinant of the derivative matrix plays an important role in the study of mappings and in their application to multiple integrals.

**Definition:** The Jac**obian**
The **Jacobian** of a mapping

$$(u, v) = F(x, y) = (u(x, y), v(x, y))$$

is denoted $\dfrac{\partial(u, v)}{\partial(x, y)}$, and is defined by

$$\frac{\partial(u, v)}{\partial(x, y)} = \det[DF(x, y)] = \det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

A question appears in Mobius

A question appears in Mobius

We can interpret the Inverse of the Derivative Matrix Theorem as asserting that if a mapping $F$ is invertible, then its derivative matrix $DF(x, y)$ is invertible, and its inverse matrix is the derivative matrix $DF^{-1}(u, v)$ of the inverse map.

Recall from linear algebra that a square matrix has an inverse matrix if and only if its determinant is non-zero. Thus, it follows from the Inverse of the Derivative Matrix Theorem that if a mapping $F$ has an inverse mapping $F^{-1}$ (and both mappings have continuous partial derivatives), then the Jacobian of $F$ is non-zero. This is stated as a corollary to the Inverse of the Derivative Matrix Theorem.

**Corollary 3**
Consider a mapping defined by

$$(u, v) = F(x, y) = (f(x, y), g(x, y))$$

which maps a subset $D_{xy}$ onto a subset $D_{uv}$. Suppose that $f$ and $g$ have continuous partial derivatives on $D_{xy}$. If $F$ has an inverse mapping $F^{-1}$, with continuous partial derivatives on $D_{uv}$, then the Jacobian of $F$ is non-zero:

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0, \quad \text{on } D_{xy}$$

The notation $\frac{\partial(u, v)}{\partial(x, y)}$ for the Jacobian reminds us of which partial derivatives have to be calculated. Thus, if $F$ maps $(x, y) \to (u, v)$ and is one-to-one, then the inverse mapping $F^{-1}$ maps $(u, v) \to (x, y)$, and the Jacobian of the inverse mapping is denoted by

$$\frac{\partial(x, y)}{\partial(u, v)} = \det[DF^{-1}(u, v)] = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Recall from linear algebra that $\det(AB) = \det A \det B$ for all $n \times n$ matrices $A, B$.

Thus, we can deduce from the Inverse of the Derivative Matrix Theorem a simple relationship between the Jacobian of a mapping and the Jacobian of the inverse mapping. We state this as a corollary to the Inverse of the Derivative Matrix Theorem.

**Corollary 4: Inverse Property of the Jacobian**
Consider a mapping $F$ which maps $D_{xy}$ onto $D_{uv}$. If $F$ has continuous partial derivatives at $\vec{x} \in D_{xy}$ and there exists an inverse mapping $F^{-1}$ of $F$ which has continuous partial derivatives at $\vec{u} = F(\vec{x}) \in D_{uv}$, then

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}$$

**Proof:** By the Inverse of the Derivative Matrix Theorem, we have

$$I = DF^{-1}(u, v)DF(x, y)$$

Taking the determinant of this equation gives

$$\begin{aligned} \det I &= \det(DF^{-1}(u, v)DF(x, y)) \\ 1 &= \det(DF^{-1}(u, v)) \det(DF(x, y)) \end{aligned}$$

Thus, by definition of the Jacobian,

$$1 = \frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(x, y)}$$

Since $DF(x, y)$ is invertible, we have $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$. Therefore, we get

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}$$

$\square$

The Inverse Property of the Jacobian tells us that, under the conditions of the Inverse of the Derivative Matrix Theorem, the Jacobian of $F^{-1}$ is the inverse of the Jacobian of $F$.

Since we are interested in being able to test whether $F^{-1}$ exists, we ask whether Corollary 3 admits a converse, i.e., does $\dfrac{\partial(u,v)}{\partial(x,y)} \neq 0$ on $D_{xy}$ imply that $F^{-1}$ exists? Unfortunately, the answer is **no**, unless we formulate the question more carefully.

The following example shows what can go wrong.

**Example 2**
Consider the mapping defined by

$$(u,v) = F(x,y) = (e^x \cos y, e^x \sin y)$$

Show that $\dfrac{\partial(u,v)}{\partial(x,y)} \neq 0$ on $\mathbb{R}^2$, but that $F^{-1}$ does not exist on $\mathbb{R}^2$.
**Solution:**
Observe that

$$\frac{\partial(u,v)}{\partial(x,y)} = e^{2x} > 0 \quad \text{for all } (x,y) \in \mathbb{R}^2$$

However, $F$ is not one-to-one on $\mathbb{R}^2$, since, for example

$$F(0,0) = F(0,2\pi) = (1,0)$$

Thus, $F^{-1}$ does not exist on $\mathbb{R}^2$.

The mapping in the previous example is not invertible because of the periodic behavior of $\sin y$ and $\cos y$. However, we know that we can create inverse functions for these by restricting their domain to a neighborhood where they are one-to-one. Similarly, in the previous example, if we restrict the domain to a neighborhood $N(0,0)$ of radius less than $2\pi$, it will be possible to solve uniquely for $x$ and $y$ in terms of $u$ and $v$; in particular, an inverse mapping does exist.

We generalize this idea on the next page with a new theorem.

## The Inverse Mapping Theorem

**Theorem 5: The Inverse Mapping Theorem**
If a mapping $(u,v) = F(x,y)$ has continuous partial derivatives in some neighborhood of $(a,b)$ and $\dfrac{\partial(u,v)}{\partial(x,y)} \neq 0$ at $(a,b)$, then there is a neighborhood of $(a,b)$ in which $F$ has an inverse mapping $(x,y) = F^{-1}(u,v)$ which has continuous partial derivatives.

The Inverse Mapping Theorem tells us that if we restrict our attention to a neighbourhood of a point $(a,b)$ in which the mapping $F$ has continuous partial derivatives and a non-zero Jacobian, then $F$ has an inverse mapping with continuous partial derivatives in a neighbourhood of $(a,b)$. The proof of this theorem is beyond the scope of this course.

**Example 3**
Consider the mapping defined by

$$(u,v) = F(x,y) = (xy - x^2, x + y)$$

Show that $F$ has an inverse mapping in a neighborhood of $(1,-2)$.
**Solution:**
The Jacobian of $F$ is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \det \begin{bmatrix} y - 2x & x \\ 1 & 1 \end{bmatrix} = y - 3x$$

Hence at $(x,y) = (1,-2)$, the Jacobian is non-zero. The partial derivatives of $F$ are continuous by the Continuity Theorems. Thus, by the Inverse Mapping Theorem, there is a neighborhood of $(1,-2)$ in which $F$ has an inverse mapping.

**Your Turn 1**

Show that the inverse mapping of

$$(u,v) = F(x,y) = (xy - x^2, x + y)$$

from the previous example, is given by

$$(x,y) = F^{-1}(u,v) = \left(\frac{1}{4}(v + \sqrt{v^2 - 8u}), \frac{1}{4}(3v - \sqrt{v^2 - 8u})\right)$$

A question appears in Mobius

**Your Turn 2**

Consider the mapping defined by

$$(u,v) = F(x,y) = (x + y, 2xy^2)$$

Show that $F$ has an inverse mapping in a neighbourhood of $(0,1)$.

A question appears in Mobius

# 13.2 - Geometrical Interpretation of the Jacobian

## Geometrical Interpretation of the Jacobian

Geometrical Interpretation of the Jacobian in 2D

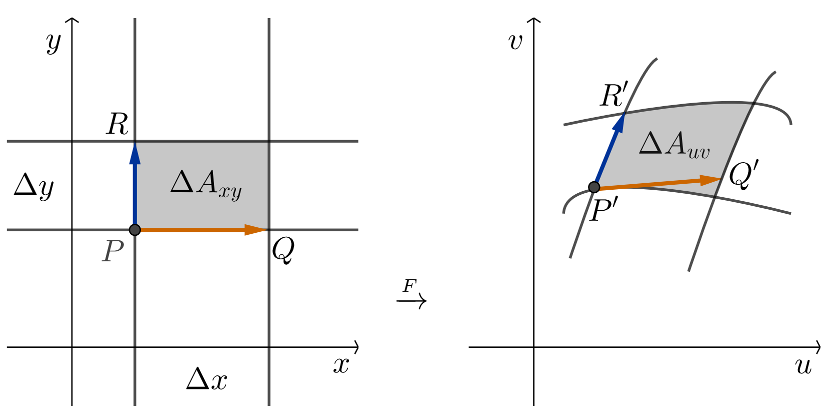
In this section, we explain the geometrical interpretation of the Jacobian of a mapping. This interpretation is based on the following result from linear algebra.

The area of a parallelogram defined by the vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is given by

$$\text{Area } = \left| \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \right| \tag{*}$$

Note that area must always be non-negative, so we take the absolute value of the determinant in equation (*).

Given a parallelogram induced by vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in the $xy$-plane, the mapping $F$ sends it to some other region of space in the $uv$-plane. Depending on the mapping $F$, this region of space may or may not be a parallelogram.




We approximate the image under $F$ of the rectangle defined by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as a parallelogram defined by the vectors $\overrightarrow{P'Q'}$ and $\overrightarrow{P'R'}$, and use the linear approximation to approximate $\overrightarrow{P'Q'}$ and $\overrightarrow{P'R'}$.

Since $\overrightarrow{PQ} = \begin{bmatrix} \Delta x \\ 0 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 0 \\ \Delta y \end{bmatrix}$, we obtain

$$\overrightarrow{P'Q'} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ 0 \end{bmatrix} = \begin{bmatrix} u_x \Delta x \\ v_x \Delta x \end{bmatrix}$$

$$\overrightarrow{P'R'} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} 0 \\ \Delta y \end{bmatrix} = \begin{bmatrix} u_y \Delta y \\ v_y \Delta y \end{bmatrix}$$

for $\Delta x$ and $\Delta y$ sufficiently small. Note that the partial derivatives are evaluated at $P$. We have

$$\Delta A_{xy} = \Delta x \Delta y$$

and so, by $(*)$,

$$\Delta A_{uv} \approx \left| \det \begin{bmatrix} u_x \Delta x & u_y \Delta y \\ v_x \Delta x & v_y \Delta y \end{bmatrix} \right| = \left| \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \right| \Delta x \Delta y$$

since $\Delta x$ and $\Delta y$ are positive.

Thus, by definition of the Jacobian,

$$\Delta A_{uv} \approx \left| \frac{\partial(u,v)}{\partial(x,y)} \right| \Delta A_{xy} \tag{**}$$

where the Jacobian is evaluated at $P$.

In words, the Jacobian of a mapping $F$ describes the extent to which $F$ increases or decreases areas. We can think of the Jacobian of $F$ as a scaling factor for (very small) areas that are mapped by $F$. This local scaling factor may or may not be linear, depending on $F$. Keep in mind that the basic relation $(**)$ is an approximation, which is valid only for small areas, and which becomes increasingly accurate as $\Delta x$ and $\Delta y$ tend to zero.

**Your Turn 1**

In this interactive activity, you can observe how the transformation $x = u^2 - v^2$ and $y = 2uv$ maps a square in the $xy$-plane and scales the area of its image. From the starting configuration, notice that transformation sends the blue square in the $xy$-plane to the orange region in the $uv$-plane. The area of the blue square ($A_{xy}$) and the value of the Jacobian are shown in the $xy$-plane. The $uv$-plane displays the area of the image of the square ($A_{uv}$ i.e., the area of the orange region) and the value of the linear approximation of the area, $\Delta A_{uv}$.

**Instructions:**

1. Using the slider labelled $\Delta x$, make the square in the $xy$-plane smaller. In the $uv$-plane, observe how the area of the region, $A_{uv}$, and the linear approximation of its area, $\Delta A_{uv}$, behave as the value of $\Delta x$ goes to zero.
2. Click and drag green dot in the $xy$-plane to move the both the blue square in the $xy$-plane and the orange region in the $uv$-plane to repeat the experiment with different squares.

External resource: https://www.geogebra.org/material/iframe/id/vbkdysn7/



As $\Delta x$ goes to zero, we notice that the area of the image of the square $A_{uv}$ and of its approximation, $\Delta A_{uv}$ get closer and closer. This tells us that for small areas, the approximation $\Delta A_{uv} \approx \left| \frac{\partial(u,v)}{\partial(x,y)} \right| \Delta A_{xy}$ is indeed a good one.

Note that for a linear mapping $(u,v) = F(x,y) = (ax+by, cx+dy)$ where $a, b, c, d$ are constants, the derivative matrix is

$$DF(x,y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and thus the linear approximation is exact by Taylor's Theorem since all second partial derivatives are zero. Therefore, for a linear mapping the approximation $(**)$ becomes an exact relation.

**Example 1**
Calculate the approximate area of the image of a small rectangle of area $\Delta x \Delta y$, located at the point $(3,4)$, under the mapping $F$ defined by

$$(u,v) = F(x,y) = \left(-x + \sqrt{x^2+y^2}, \quad x + \sqrt{x^2+y^2}\right)$$

**Solution:**
Differentiation and evaluation at $(3,4)$ give the derivative matrix at $(3,4)$:

$$DF(3,4) = \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5} \end{bmatrix}$$

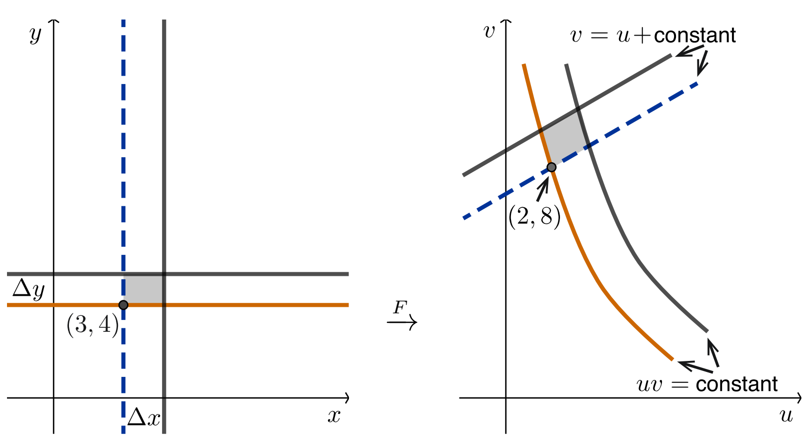
At $(3,4)$ the Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5} \end{bmatrix} = -\frac{8}{5}$$

Therefore, the area of the image is approximately

$$\Delta A_{uv} \approx \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \Delta A_{xy} \approx \frac{8}{5} \Delta A_{xy}$$

We can use a diagram to demonstrate what is happening geometrically in the example. Under $F$, the rectangle bounded below by the line $y = 4$ and on the left by the line $x = 3$ was slightly deformed by $F$ to form a quadrilateral. In the $uv$-plane, the image of the line $y = 4$ runs down the left side of the quadrilateral and the image of the line $x = 3$ runs along the bottom side of the quadrilateral. Notice that the two horizontal lines in the $xy$-plane are sent by $F$ to hyperbolae with general form $uv = C$ for some constant $C$. The two vertical lines in the $xy$-plane are sent by $F$ to lines having general form $v = u + D$ for some constant $D$.

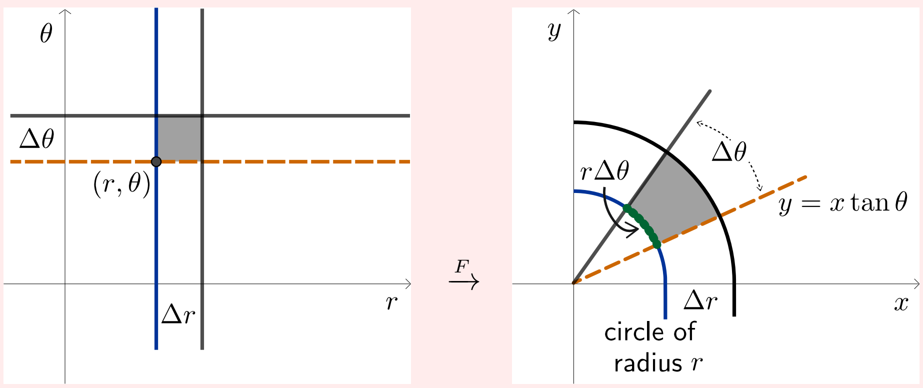



The value of $\Delta A_{uv}$ tells us approximately how much the area of the original rectangle was scaled by $F$. In this case, the scaling factor is $\frac{8}{5}$: the area of the image of the rectangle under $F$ is approximately $\frac{8}{5}$ times the area of the original rectangle. Notice that the area was scaled linearly i.e., by a constant factor.

**Example 2**
Consider the mapping $F$ defined by

$$(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta)$$

Find the image in the $xy$-plane, of a rectangle in the $r\theta$-plane, and verify directly that the Jacobian gives the magnification factor for area.
**Solution:**
Geometrically, the rectangle with the bottom left corner at the arbitrary point $(r, \theta)$ is mapped by $F$ to a sector of an annulus. The vertical line passing through $(r, \theta)$ in the $r\theta$-plane is sent by $F$ to a circle of radius $r$ from the origin. The horizontal line passing through $(r, \theta)$ in the $r\theta$-plane is sent by $F$ to the line $y = x\tan\theta$. The quantity $\Delta r$ gives the width of the annulus in the $xy$-plane, while the quantity $\Delta\theta$ gives the angle of the sector.




The area of the rectangle in the $r\theta$-plane is

$$\Delta A_{r\theta} = \Delta r \Delta\theta$$

The image of this rectangle in the $xy$-plane can be approximated by a rectangle with sides of length $r\Delta\theta$ and $\Delta r$, for sufficiently small $\Delta r$ and $\Delta\theta$ .
So,

$$\Delta A_{xy} \approx r\Delta r\Delta\theta = r\Delta A_{r\theta}$$

However, the Jacobian of the mapping is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r > 0$$

Consequently,

$$\Delta A_{xy} \approx \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| \Delta A_{r\theta}$$

which verifies the area transformation formula $(**)$.

A question appears in Mobius

A question appears in Mobius

**Your Turn 4**

Use the Jacobian to verify the well-known result that any linear mapping $F$ which is a rotation,

$$(u, v) = F(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$$

where $\theta$ is a constant, preserves areas.

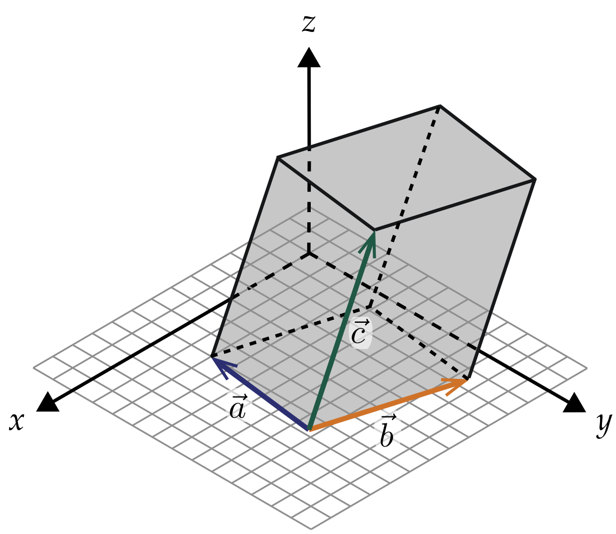
A question appears in Mobius

## Geometrical Interpretation of the Jacobian in 3D

The interpretation of the Jacobian in 3D is based on the following result from linear algebra.

The volume of a parallelepiped which is defined by three vectors $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is given by

$$\text{Volume } = \left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right|$$




Again, note that volume must always be non-negative, so we take the absolute value of the determinant.

Consider a mapping defined by

$$(u, v, w) = F(x, y, z) = \Big( f(x, y, z), g(x, y, z), h(x, y, z) \Big)$$

The image of a small rectangular block of volume $\Delta V_{xyz} = \Delta x \Delta y \Delta z$ in $xyz$-space under this mapping can be approximated by a small parallelepiped in $uvw$-space. As in the 2D case, we can use the linear approximation and the formula above to approximate the volume $\Delta V_{uvw}$ of the image. The result is

$$\Delta V_{uvw} \approx \left| \frac{\partial(u,v,w)}{\partial(x,y,z)} \right| \Delta V_{xyz}$$

where $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ is the Jacobian of the mapping $F$ evaluated at $(x,y,z)$.

A question appears in Mobius

## Generalization

In the unit on mappings of $\mathbb{R}^2$ into $\mathbb{R}^2$, we generalized the concept of a mapping $F$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ to a mapping $F$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, and defined the $m \times n$ derivative matrix $DF(\vec{x})$. If $m = n$, then we can define the Jacobian of the mapping as follows.

**Definition:** The Ja**cobian - General Form**
For a mapping defined by

$$\vec{u} = F(\vec{x}) = \Big( f_1(\vec{x}), \dots, f_n(\vec{x}) \Big)$$

where $\vec{u} = (u_1, \dots, u_n)$ and $\vec{x} = (x_1, \dots, x_n)$, the **Jacobian of** $F$ is

$$\frac{\partial(u_1,\dots,u_n)}{\partial(x_1,\dots,x_n)} = \det[DF(\vec{x})] = \det \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

We note that the inverse property of the Jacobian also generalizes:

$$\frac{\partial(x_1,\dots,x_n)}{\partial(u_1,\dots,u_n)} = \frac{1}{\dfrac{\partial(u_1,\dots,u_n)}{\partial(x_1,\dots,x_n)}}$$

where $\dfrac{\partial(x_1,\dots,x_n)}{\partial(u_1,\dots,u_n)}$ is the Jacobian of the inverse mapping of $F$.

## 13.3 - Constructing Mappings

### Constructing Mappings

When working with double and triple integrals, it will be very important to be able to invent an invertible mapping which transforms one region to another, simpler region.

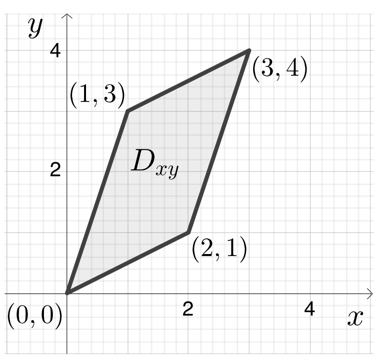
We demonstrate this with some examples.

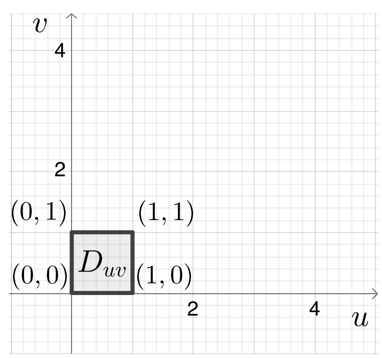
*A slideshow appears in Mobius.*

Slide

## Example 1 - Part A

Find a mapping $F$ which transforms the parallelogram, $D_{xy}$, with vertices $(0,0)$, $(2,1)$, $(3,4)$, and $(1,3)$ in the $xy$-plane into the unit square, $D_{uv}$, $0 \leq u \leq 1$, $0 \leq v \leq 1$ in the $uv$-plane.
**Solution:**




We use the vertices to find the boundary lines in the $xy$-plane. They are:

- $2y - x = 0$
- $2y - x = 5$
- $3x - y = 0$
- $3x - y = 5$

We want $u = f(x,y)$ and $v = g(x,y)$, so that the images of the boundary lines are:

- $u = 0$
- $u = 1$
- $v = 0$
- $v = 1$

Slide

## Example 1 - Part A

Find a mapping $F$ which transforms the parallelogram, $D_{xy}$, with vertices $(0, 0)$, $(2, 1)$, $(3, 4)$, and $(1, 3)$ in the $xy$-plane into the unit square, $D_{uv}$, $0 \leq u \leq 1$, $0 \leq v \leq 1$ in the $uv$-plane.
**Solution:**

We use the vertices to find the boundary lines in the $xy$-plane. They are:

- $2y - x = 0$
- $3x - y = 0$
- $2y - x = 5$
- $3x - y = 5$

We want $u = f(x, y)$ and $v = g(x, y)$, so that the images of the boundary lines are :

- $u = 0$
- $v = 0$
- $u = 1$
- $v = 1$

Observe that the bounding lines come in pairs.
To get the first pair to have images $u = 0$ and $u = 1$, we can take $u = \dfrac{2y - x}{5}$.
For the second pair to have images $v = 0$ and $v = 1$ we can take $v = \dfrac{3x - y}{5}$.
Thus, a desired mapping is

$$(u, v) = F(x, y) = \left(\frac{2y - x}{5}, \frac{3x - y}{5}\right)$$

Slide

## Example 1 - Part B

Calculate the Jacobian of $F$ and hence find the area of the parallelogram in the $xy$-plane.
**Solution:**
In part a we found the mapping

$$(u, v) = F(x, y) = \left(\frac{2y - x}{5}, \frac{3x - y}{5}\right)$$

The Jacobian is

$$\frac{\partial(u, v)}{\partial(x, y)} = \det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \det \begin{bmatrix} -\dfrac{1}{5} & \dfrac{2}{5} \\ \dfrac{3}{5} & -\dfrac{1}{5} \end{bmatrix} = -\frac{1}{5}$$

Since the mapping is linear, we have the exact relation

$$A_{uv} = \left|\frac{\partial(u, v)}{\partial(x, y)}\right| A_{xy} = \frac{1}{5} A_{xy}$$

Hence, the area of the parallelogram in the $xy$-plane is 5 square units.

**Example 2**

Find a linear mapping which transforms the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ into the unit circle $u^2 + v^2 = 1$.

**Solution:**

We want to pick $u = f(x, y)$ and $v = g(x, y)$, such that we turn $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ into $u^2 + v^2 = 1$. If we write the ellipse as $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = 1$, then it is clear that we want to take $u = \dfrac{x}{a}$ and $v = \dfrac{y}{b}$. Hence, the desired mapping is

$$(u, v) = F(x, y) = \left(\frac{x}{a}, \frac{y}{b}\right)$$

**Your Turn 1**

Find a linear mapping $F$ which transforms the ellipse $3x^2 + 2xy + y^2 = 4$ into the circle $u^2 + v^2 = 4$.

A question appears in Mobius

**Example 3**

Find an invertible mapping which will transform the region $D_{xy}$ in the first quadrant bounded by the hyperbolae $xy = 1$, $xy = 3$, $x^2 - y^2 = 2$, and $x^2 - y^2 = 4$ into a square in the $uv$-plane.

**Solution:**

We again see that we have pairs of equations. Thus, if we take $u = xy$ and $v = x^2 - y^2$ we see that the images of the hyperbolae $xy = 1$, $xy = 3$, $x^2 - y^2 = 2$, and $x^2 - y^2 = 4$ are $u = 1$, $u = 3$, $v = 2$, $v = 4$. Hence, the mapping

$$(u, v) = F(x, y) = (xy, x^2 - y^2)$$

gives the desired transformation. Observe that it would be difficult to solve for the inverse explicitly, however, we can at least show that the mapping is locally invertible by applying the Inverse Mapping Theorem. The Jacobian of $F$ is

$$\det DF(x, y) = \det \begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix} = -2x^2 - 2y^2$$

which is non-zero on the region $D_{xy}$ and $F$ has continuous partial derivatives, so $F$ is invertible in a neighborhood of every point in $D_{xy}$ by the Inverse Mapping Theorem.

**Remark**
In the previous example, the solution does not actually prove that the mapping is invertible on the entire region. In practice, we often assume that "invertible in a neighborhood of each point" implies "invertible over the entire region", but this is not always true. For example, $F(r, \theta) = (r\cos\theta, r\sin\theta)$ on $1 \leq r \leq 2$, $0 \leq \theta \leq 4\pi$ has non-zero Jacobian at each point but is not one-to-one due to periodicity.

**Your Turn 2**

Find an invertible mapping which will transform the region $D_{xyz}$ in the first octant bound by $xy = 1$, $xy = 3$, $xz = 1$, $xz = 3$, $yz = 2$, and $yz = 4$ into a cube in the $uvw$-space.

A question appears in Mobius

## 13.4 - Putting It All Together

### Worked Example 1

Find the Jacobian of the mapping

$$(u, v) = F(x, y) = (x^2 \sin y, y^2 \cos x)$$

A question appears in Mobius

**Step 2**: Find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

A question appears in Mobius

### Worked Example 2

Calculate the approximate area of the image of a small rectangle of area $\Delta x \Delta y$ located at the point $(a,b) = \left(1, \dfrac{1}{2}\right)$ under the map $T$ defined by

$$T(x,y) = (xy, x^2 - y^2)$$

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

### Worked Example 3

Invent an invertible transformation that maps the ellipsoid $x^2+2y^2+2z^2+2xy+2xz+2yz=1$ onto the unit sphere centered at the origin.

A question appears in Mobius

**Step 2:** Write $x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz = 1$ in the form $(\_\_)^2 + (\_\_)^2 + (\_\_)^2 = 1$

A question appears in Mobius

**Step 3:** Make a guess for $u, v, w$:

A question appears in Mobius

**Step 4:** Show that $F$ is invertible.

A question appears in Mobius

A question appears in Mobius

## Practice Problems

Try to answer the questions. If you are having trouble, check for a hint before looking at the solutions.

1. Consider the map defined by
$$(u, v) = F(x, y) = (y + xy, y - xy)$$
   (a) Show that $F$ has an inverse map by finding $F^{-1}$ explicitly.
   (b) Find the derivative matrices $DF(x, y)$ and $DF^{-1}(u, v)$ and verify that
$$DF(x, y)DF^{-1}(u, v) = I$$
   (c) Verify that the Jacobians satisfy
$$\frac{\partial(x, y)}{\partial(u, v)} = \left[\frac{\partial(u, v)}{\partial(x, y)}\right]^{-1}$$

2. Calculate the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$ for the following map $T$. Find all points at which the Jacobian is zero. Use

the Inverse Mapping Theorem to prove that $T^{-1}$ exists in a neighbourhood of the indicated point:$(u,v) = T(x,y) = (\cos(x+y), \sin(x-y))$; $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

3. Invent a transformation that maps the parallelogram bounded by the lines $y = 3x-4, y = 3x, y = \frac{1}{2}x$ and $y = \frac{1}{2}(x+4)$ onto the unit square in the first quadrant.
4. Invent a transformation that maps the ellipse $x^2 + 4xy + 5y^2 = 4$ onto the unit circle.
5. Invent an invertible transformation that transforms the ellipse $x^2 + 4xy + 5y^2 = 5$ onto the unit circle and determine the inverse map. [*Hint provided below*]
6. Invent an invertible transformation that transforms the ellipse $3x^2 + 6xy + 4y^2 = 4$ onto the unit circle and determine the inverse map.
7. Invent a transformation that maps the ellipsoid $x^2 + 8y^2 + 6z^2 + 4xy - 2xz + 4yz = 9$ onto the unit sphere.

## Hint

*Additional content appears in Mobius.*

## Select Answers and Solutions

1. Consider the map $(x,y) = F^{-1}(u,v) = \left(\frac{u-v}{u+v}, \frac{u+v}{2}\right)$.
2. No answer provided.
3. One such mapping is $(u,v) = \left(\frac{3x-y}{4}, \frac{2y-x}{4}\right)$.
4. One such mapping is $(u,v) = \left(\frac{x+2y}{2}, \frac{y}{2}\right)$.
5. No answer provided.
6. One such mapping is $(u,v) = \left(\frac{\sqrt{3}x}{4}, \frac{3x}{4} + y\right)$.
7. No answer provided.

# Unit 14

# Double Integrals

## 14.1 - Definition of Double Integrals

### Definition of Double Integrals

In single-variable calculus, we used integrals to find the area under a continuous curve $y = f(x)$ over a closed interval $[a, b]$. Formally, the definite integral is defined as a limit of Riemann sums:

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

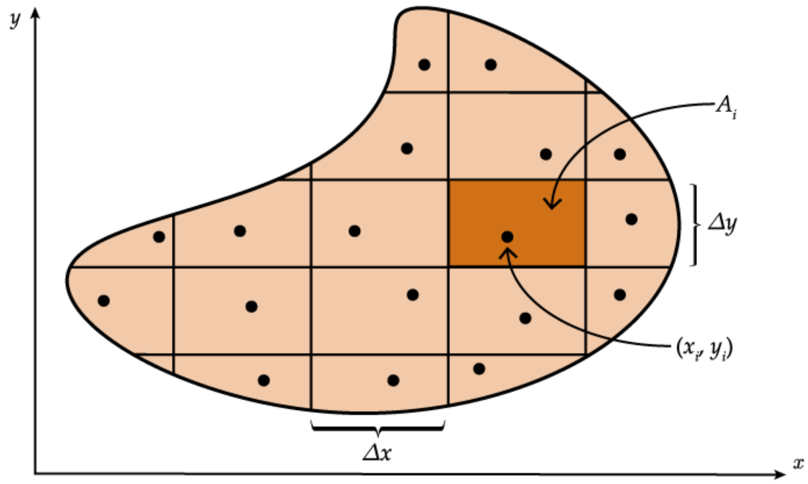
where $\Delta x_i$ is the length of the $i$-th subinterval in some decomposition (i.e. partition) of the interval $[a, b]$ and $x_i$ is some reference point in the $i$-th subinterval. Note that we used a single integral sign for these calculations.

We have also seen that the single integral had many applications besides calculating areas under curves. We can use single integrals for finding the mass of thin rods, for calculating work, and for finding volumes of revolution. What if we want to calculate the mass of a thin plate, or to find the volume of more complicated regions? For these calculations, we will use **double integrals**.

Let $D$ be a closed and bounded set in $\mathbb{R}^2$ whose boundary is a piecewise smooth closed curve. Let $f(x, y)$ be a function which is bounded on $D$, that is, there exists a number $M$ such that $|f(x, y)| \leq M$ for all $(x, y) \in D$.

Subdivide $D$ by means of straight lines parallel to the axes, forming a partition $P$ of $D$. Label the $n$ rectangles that lie completely in $D$, in some specific order, and denote their areas by $\Delta A_i$, $i = 1, \ldots, n$. Choose a point $(x_i, y_i)$ in the $i$-th rectangle and form the Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i = \sum_{i=1}^{n} f(x_i, y_i)\Delta x_i \Delta y_i \qquad (*)$$




Functions whose Riemann sums over a closed and bounded set $D$ behave nicely are given a special name:

**Definition: Integrable function**
Let $D \subset \mathbb{R}^2$ be closed and bounded. Let $P$ be a partition of $D$ as described above, and let $|\Delta P|$ denote the length of the longest side of all rectangles in the partition $P$. A function $f(x, y)$ which is bounded on $D$ is **integrable** on $D$ if all Riemann sums approach the same value as $|\Delta P| \to 0$.

Now that we have defined integrable functions, we can define the double integral:

**Definition: Double Integral**
If $f(x, y)$ is integrable on a closed bounded set $D$, then we define the **double integral** of $f$ on $D$ as

$$\iint_D f(x, y)\, dA = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

After reading this definition, we might ask: Is there any guarantee that the limiting process in the definition of the double integral actually leads to a unique value, i.e., that the limit exists? It is possible to define weird functions for which the limit does not exist, i.e. which are not integrable on $D$?

If $f$ is continuous on $D$, it can be proved that $f$ is integrable on $D$, that is, that the double integral of $f$ exists. However, functions which are discontinuous on $D$ may still be integrable on $D$. For example, if $f$ is piece-wise continuous and continuous on $D$ except at points which lie on a curve $C$ then $f$ is integrable. The proofs of these results are beyond the scope of this course.

## Interpretations of the Double Integral

The double integral symbol

$$\iint_D f(x, y)\, dA$$

stands for the limit of a Riemann sum. In itself, the double integral is a mathematically defined object. It has many interpretations depending on the meaning that we assign to the integrand $f(x, y)$. The "$dA$" in the double integral symbol should remind us of the area of a rectangle in a partition of $D$.

## Double Integral as Area

The simplest interpretation is when we specialize $f$ to be the constant function with value one:

$$f(x,y) = 1, \quad \text{for all } (x,y) \in D$$

Then the Riemann sum $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$ simply sums the areas of all rectangles in $D$, and the double integral serves to define the area $A(D)$ of the set $D$:

$$A(D) = \iint_D 1 \; dA$$

## Double Integral as Volume

If $f(x,y) \geq 0$ for all $(x,y) \in D$, then the double integral

$$\iint_D f(x,y) \; dA$$

can be interpreted as the volume $V(S)$ of the region defined by

$$S = \Big\{ (x,y,z) \mid 0 \leq z \leq f(x,y), (x,y) \in D \Big\}$$

which represents the solid below the surface $z = f(x,y)$ and above the set $D$ in the $xy$-plane. The justification is as follows.

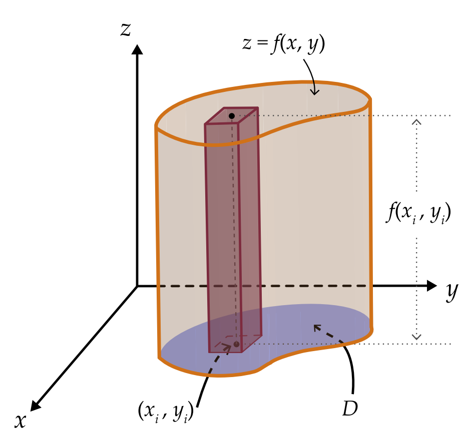
The partition $P$ of $D$ decomposes the solid $S$ into vertical "columns".

The height of the column above the $i$-th rectangle is approximately $f(x_i, y_i)$, and so its volume is approximately

$$f(x_i, y_i)\Delta A_i$$

The Riemann sum $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$ thus approximates the volume $V(S)$:

$$V(S) \approx \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$



As $|\Delta P| \to 0$ the partition becomes increasingly fine, so the error in the approximation will tend to zero. Thus, the volume $V(S)$ is

$$V(S) = \iint_D f(x,y)\ dA$$

We will see in the next unit that triple integrals can also be interpreted as volumes.

**Your Turn**

This interactive applet demonstrates the volume interpretation of double integrals. Notice how the approximation for the volume under the function improves as the partition gets finer.

**Instructions:**

1. Enter a function of $x$ and $y$ in the input box.
2. Change the domain of integration by entering ranges for $x$ and $y$ in the input boxes or by manually dragging the dots in the $xy$ plane.
3. Change the number of partitions of the domain using the sliders for the number of subintervals $\Delta x_i$ and $\Delta y_i$.
4. Observe how increasing the number of partitions (increasing the number of $\Delta x_i$ and of $\Delta y_i$) improves the volume approximation.
5. Repeat the experiment with different functions and domains. A few ideas to try are:
   (a) $f(x,y) = \sin(x+y)$ with $x \in [-\pi, \pi]$ and $y \in [-\pi, \pi]$
   (b) $f(x,y) = e^{xy}$ with $x \in [-1, 1]$ and $y \in [-2, 2]$

External resource: https://www.geogebra.org/material/iframe/id/kzufeays/



## Double Integral as Mass

Think of a thin flat plate of metal whose density varies with position. Since the plate is thin, it is reasonable to describe the varying density by an "area density", that is a function $f(x,y)$ that gives the mass per unit area at position $(x,y)$. In other words, the mass of a small rectangle of area $\Delta A_i$ located at position $(x_i, y_i)$ will be approximately

$$\Delta M_i \approx f(x_i, y_i)\Delta A_i$$

The Riemann sum $(*)$ corresponding to a partition $P$ of $D$ will approximate the total mass $M$ of the plate $D$, and the double integral of $f$ over $D$, being the limit of the sum, will represent the total mass:

$$M = \iint_D f(x,y)\ dA$$

## Double Integral as Probability

Let $f(x,y)$ be the probability density of a continuous 2D random variable $(X,Y)$. The probability that $(X,Y) \in D$, a given subset of $\mathbb{R}^2$, is

$$P((X,Y) \in D) = \iint_D f(x,y)\ dA$$

## Average Value of a Function

The double integral is also used to define the average value of a function $f(x, y)$ over a set $D \subset \mathbb{R}^2$.

Recall for a function of one variable, $f(x)$, the average value of $f$ over an interval $[a, b]$, denoted $f_{av}$, is defined by

$$f_{av} = \frac{1}{b-a} \int_a^b f(x)\ dx$$

Similarly, for a function of two variables $f(x, y)$, we can define the average value of $f$ over a closed and bounded subset $D$ of $\mathbb{R}^2$ by

$$f_{av} = \frac{1}{A(D)} \iint\limits_D f(x, y)\ dA$$

## Properties of the Double Integral

The basic properties of single integrals can be generalized to double integrals.

**Theorem 1: Linearity**
If $D \subset \mathbb{R}^2$ is a closed and bounded set and $f$ and $g$ are two integrable functions on $D$, then for any constant $c$:

$$\iint\limits_D (f+g)\ dA = \iint\limits_D f\ dA + \iint\limits_D g\ dA$$

and

$$\iint\limits_D cf\ dA = c \iint_D f\ dA$$

Linearity tells us that the double integral of a sum of two functions is the sum of the double integral of each function, and that the double integral of a scaled function is the scalar multiple of the double integral of the function. As the next result shows, double integrals also behave nicely with inequalities.

**Theorem 2: Basic Inequality**
If $D \subset \mathbb{R}^2$ is a closed and bounded set and $f$ and $g$ are two integrable functions on $D$ such that $f(x, y) \leq g(x, y)$ for all $(x, y) \in D$, then

$$\iint\limits_D f\ dA \leq \iint\limits_D g\ dA$$

The Basic Inequality property is often used to obtain an estimate for a double integral that cannot be evaluated exactly.

Another useful property satisfied by the double integral is the absolute value inequality.

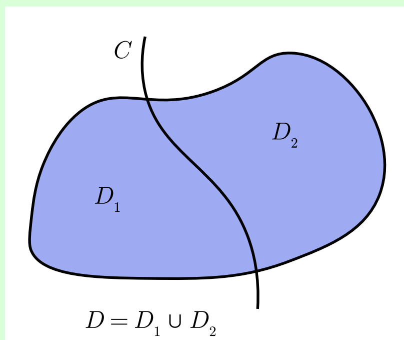
**Theorem 3: Absolute Value Inequality**
If $D \subset \mathbb{R}^2$ is a closed and bounded set and $f$ is an integrable function on $D$, then

$$\left| \iint\limits_D f \; dA \right| \leq \iint\limits_D |f| \; dA$$

The last property we mention here is the decomposition property:

**Theorem 4: Decomposition**
Let $D \subset \mathbb{R}^2$ be a closed and bounded set and let $f$ be an integrable function on $D$. If $D$ is decomposed into two closed and bounded subsets $D_1$ and $D_2$ by a piecewise smooth curve $C$, then

$$\iint\limits_D f \; dA = \iint_{D_1} f \; dA + \iint_{D_2} f \; dA$$




This property is essential for dealing with complicated regions of integration and with discontinuous integrands.

A question appears in Mobius

# 14.2 - Iterated Integrals

## Iterated Integrals

Double integrals can be evaluated approximately by using a computer to evaluate a suitable Riemann sum. The accuracy of the approximation would depend on how fine a partition was chosen.

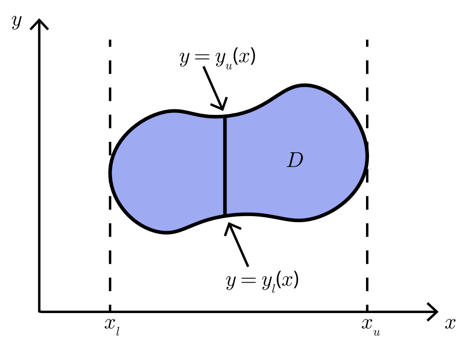
But it is natural to ask: is it possible to calculate double integrals exactly, using methods that work for single integrals? For sufficiently simple functions and regions of integration, the answer is **yes**. The idea is to write the double integral as a succession of two single integrals, called an **iterated integral**. In this lesson, we will derive a method for doing this by using the interpretation of the double integral as volume.

Let $D$ be a region in the $xy$-plane and let $f$ be a function such that $f(x,y) \geq 0$ for all $(x,y) \in D$. If $V$ denotes the volume of the solid above $D$ and below the surface $z = f(x,y)$, then we have

$$V = \iint\limits_D f(x,y)\ dA$$

Assume that the region $D$ lies between vertical lines $x = x_\ell$ and $x = x_u$ with $x_\ell < x_u$ and has top curve $y = y_u(x)$ and bottom curve $y = y_\ell(x)$. That is, $D$ is described by the inequalities

$$y_\ell(x) \leq y \leq y_u(x), \quad \text{and} \quad x_\ell \leq x \leq x_u$$




Note that in the figure above, the region $D$ is bounded by two straight lines $x = x_\ell$ on the left and and $x = x_u$ on the right. We can find the volume of a region by integrating over all possible cross-sectional areas. That is,

$$V = \int_{x_\ell}^{x_u} A(x)\ dx$$

where $A(x)$ is the cross-sectional area of the solid for any fixed value of $x$. However, we know that the cross-sectional area $A(x)$ is the area under the cross-section $z = f(x,y)$, and thus is given by a single integral

$$A(x) = \int_{y_\ell(x)}^{y_u(x)} f(x,y)\ dy$$

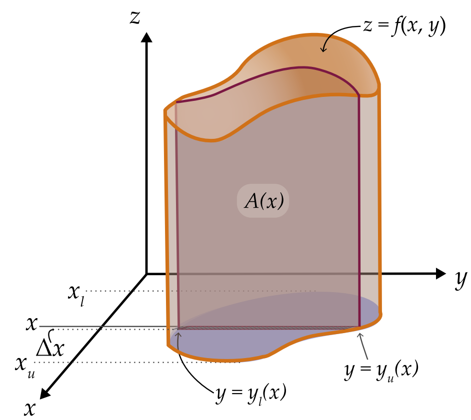
Hence, the volume of the region is

$$V = \int_{x_\ell}^{x_u} \left( \int_{y_\ell(x)}^{y_u(x)} f(x,y)\ dy \right)\ dx$$

Thus, we have

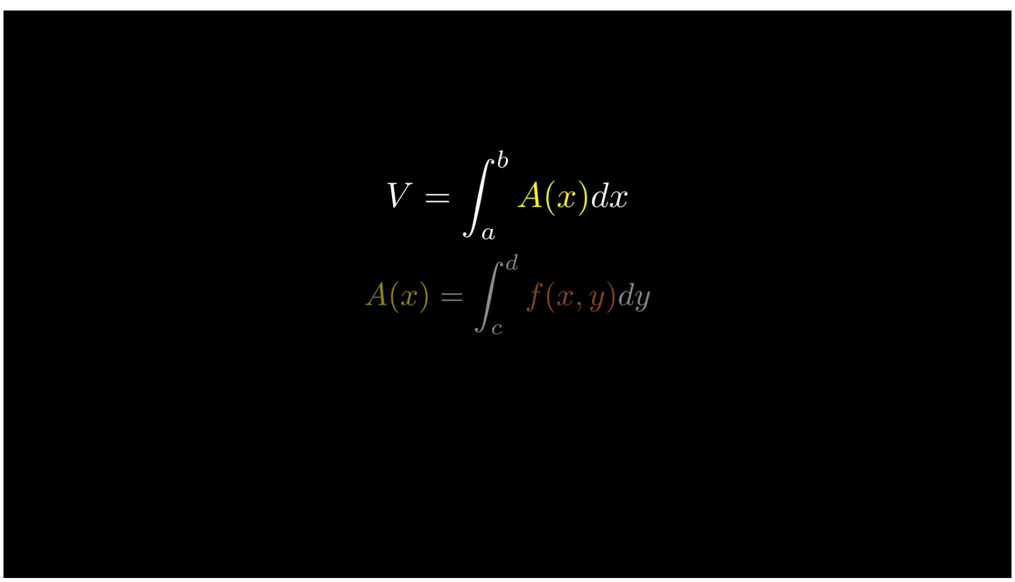
$$\iint\limits_D f(x,y)\ dA = \int_{x_\ell}^{x_u} \int_{y_\ell(x)}^{y_u(x)} f(x,y)\ dy\ dx$$

as desired.




In the following video, we will construct the figure above one step at a time. We will choose the region of integration $D$ to be a rectangle for convenience. Pause the video as needed to make connections to between the cross-sectional area $A(x)$ and the volume of the solid above$D$and below the surface$z = f(x, y)$.

*A video appears here.*




Let's state what we outlined more formally:

**Theorem 1: Iterated Integrals**
Let $D \subset \mathbb{R}^2$ be defined by

$$y_\ell(x) \le y \le y_u(x), \quad \text{and} \quad x_\ell \le x \le x_u$$

where $y_\ell(x)$ and $y_u(x)$ are continuous for $x_\ell \le x \le x_u$. If $f(x, y)$ is continuous on $D$, then

$$\iint_D f(x, y)\; dA = \int_{x_\ell}^{x_u} \int_{y_\ell(x)}^{y_u(x)} f(x, y)\; dy\; dx$$

The proof of this result is beyond the scope of this course.

**Remark**
Although the parentheses around the inner integral are usually omitted, we must evaluate it first. Moreover, as in our interpretation of volume above, when evaluating the inner integral, we are integrating with respect to $y$ while holding $x$ constant. That is, we are using **partial integration**.

## Examples Part 1

*A slideshow appears in Mobius.*

Slide

# Example 1 - Question

Evaluate $\displaystyle\iint_D xy\; dA$ where $D$ is the triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$.

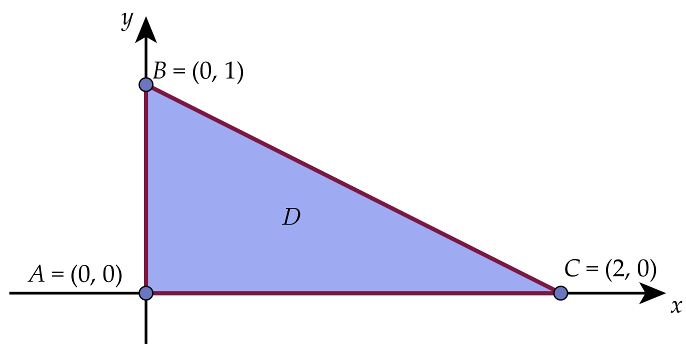
Slide

## Example 1 - Step 1

Evaluate $\displaystyle\iint_D xy\, dA$ where $D$ is the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.

**Solution:**

**Step 1: Sketch the region**



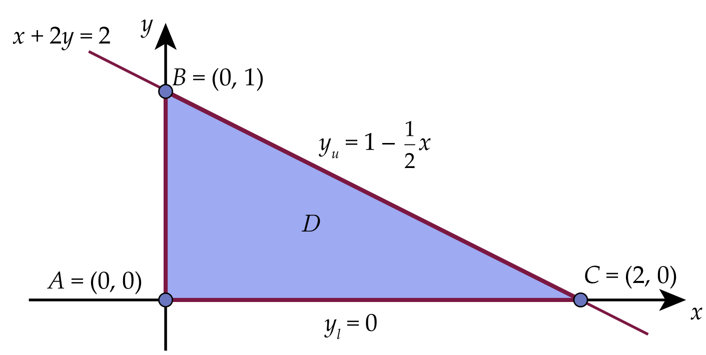

Slide

## Example 1 - Step 2

Evaluate $\displaystyle\iint_D xy\, dA$ where $D$ is the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.

**Solution:**

**Step 2: Set up inequalities**




From the sketch of $D$ we see that the value of $x$ is between 0 and 2.
The value of $y$ is bounded below by the function $y_l = 0$.
The value of $y$ is bounded above by the line passing through the points $B$ and $C$, this line is $x + 2y = 2$

Slide

## Example 1 - Step 2 (Continued)

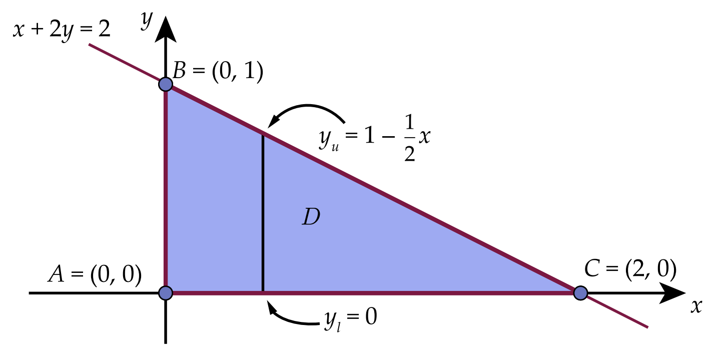
Evaluate $\displaystyle\iint_D xy\, dA$ where $D$ is the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.

**Solution:**

**Step 2: Set up inequalities**

The set $D$ is defined by

$$0 \le x \le 2 \quad \text{and} \quad 0 \le y \le 1 - \frac{1}{2}x$$




Slide

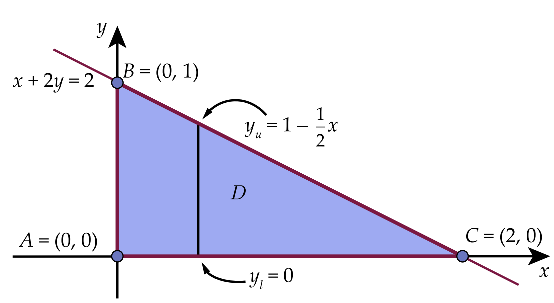
## Example 1 - Step 3

Evaluate $\displaystyle\iint_D xy\, dA$ where $D$ is the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.

**Solution:**

**Step 3: Set up and evaluate the integral**

$$\begin{aligned}
\iint_D xy\, dA &= \int_{x=0}^{2} \underbrace{\int_{y=0}^{1-\frac{1}{2}x} xy\, dy}\, dx \\
&= \int_{x=0}^{2} x\left(\frac{1}{2}y^2\right)\Bigg|_0^{1-\frac{1}{2}x} dx \\
&= \frac{1}{2}\int_0^2 x\left(1-\frac{1}{2}x\right)^2 dx \\
&= \frac{1}{4}x^2 - \frac{1}{6}x^3 + \frac{1}{32}x^4\Bigg|_0^2 \\
&= \frac{1}{6}
\end{aligned}$$

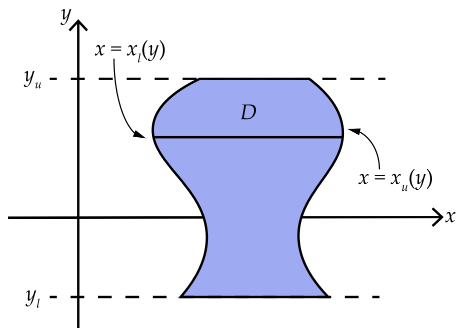


Suppose now that the set $D$ can be described by inequalities of the form

$$x_\ell(y) \le x \le x_u(y), \quad \text{and} \quad y_\ell \le y \le y_u$$

where $y_\ell$, $y_u$ are constants and $x_\ell(y)$, $x_u(y)$ are continuous functions of $y$ on the interval

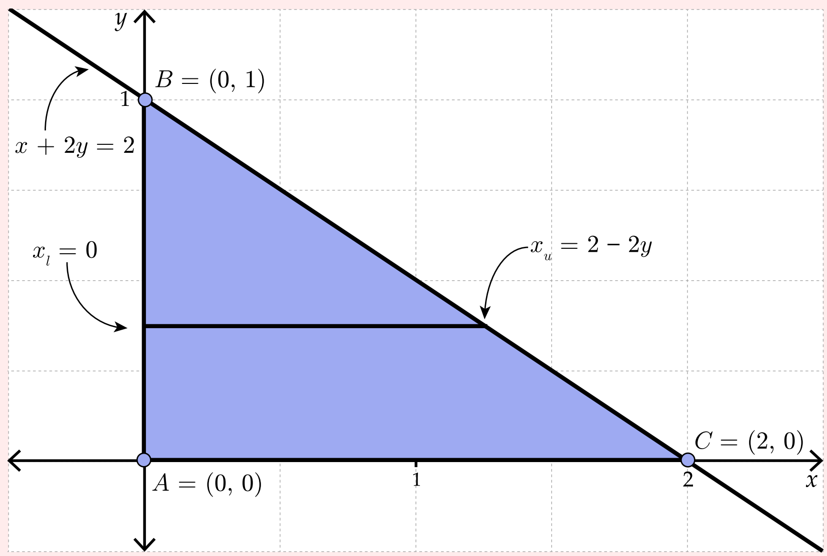
$$y_\ell \le y \le y_u$$




This time, the region $D$ is bounded by two straight lines: $y = y_u$ from above and $y = y_\ell$ from below. Then, by reversing the roles of $x$ and $y$ in Iterated Integrals Theorem, the double integral $\iint\limits_D f(x,y)\ dA$ can be written as an iterated integral in the order "$x$ first, then $y$":

$$\iint\limits_D f(x,y)\ dA = \int_{y_\ell}^{y_u} \int_{x_\ell(y)}^{x_u(y)} f(x,y)\ dx\ dy \tag{**}$$

**Example 2**
Evaluate the integral in the previous example by integrating with respect to $x$ first.
**Solution:**
In order to integrate with respect to $x$ first, we describe the set $D$ by $0 \le x \le 2(1-y)$, $0 \le y \le 1$.




So, by equation (∗∗) we get

$$\begin{aligned}\iint_D xy\ dA &= \int_{y=0}^{1} \underbrace{\int_{x=0}^{2(1-y)} xy\ dx}\ dy \\ &= \int_{y=0}^{1} \underbrace{y\left(\frac{1}{2}x^2\right)\Bigg|_0^{2(1-y)}}\ dy \\ &= 2\int_0^1 y(1-y)^2\ dy \\ &= \frac{1}{6}\end{aligned}$$

A question appears in Mobius

**Example 3**

Using your answer from the previous Your Turn question, evaluate $\iint_D (x+2y)\ dA$.

**Solution:**

$$\begin{aligned}\iint_D (x+2y)\ dA &= \int_{-1}^{2} \underbrace{\int_{x^2}^{x+2} x+2y\ dy}\ dx \\ &= \int_{-1}^{2} \underbrace{\Big[xy+y^2\Big]_{x^2}^{x+2}}\ dx \\ &= \int_{-1}^{2} (2x^2+6x+4-x^3-x^4)\ dx \\ &= \frac{333}{20}\end{aligned}$$

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

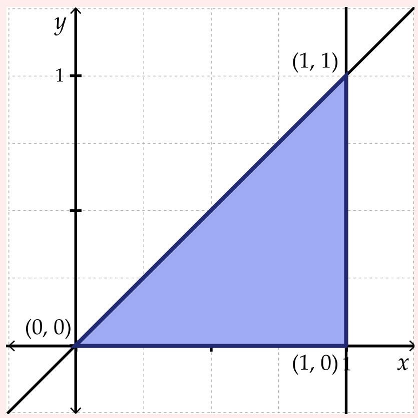
## Examples Part 2

**Example 4**

Let $D$ be the region bounded by the lines $y = 0$, $x = 1$, and $y = x$. Find $\iint_D e^{x^2}\ dA$.

**Solution:**

Although we can easily write the region so that we could integrate with respect either variable first, we see that choosing to integrate with respect to $x$ first would be a bad choice since there is no known anti-derivative of $e^{x^2}$. Thus, we write the region as $0 \le y \le x$, $0 \le x \le 1$.



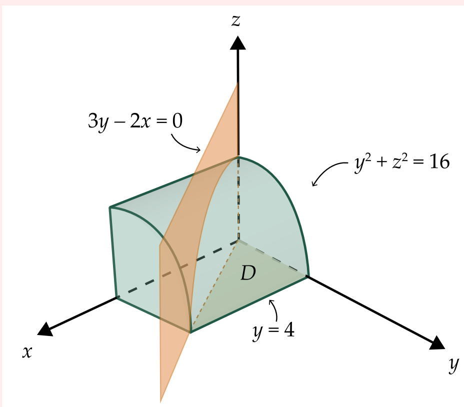

Then, we get

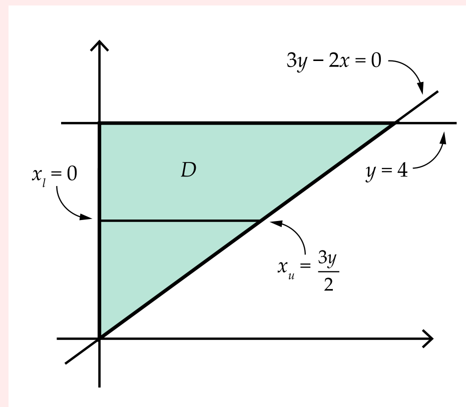
$$\begin{aligned}
\iint_D e^{x^2}\ dA &= \int_0^1 \underbrace{\int_0^x e^{x^2}\ dy}\ dx \\
&= \int_0^1 \underbrace{\left(ye^{x^2}\right)\Big|_0^x}\ dx \\
&= \int_0^1 xe^{x^2}\ dx \\
&= \frac{1}{2}(e-1)
\end{aligned}$$

**Example 5**
Find the volume of the solid $S$ in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) bounded by the cylinder $y^2 + z^2 = 16$, and the planes $3y - 2x = 0$, $x = 0$, $z = 0$.
**Solution:**
The cylinder $y^2 + z^2 = 16$ runs parallel to the $x$-axis (since there is no $x$-dependence). The plane $3y - 2x = 0$ is vertical (since there is no $z$-dependence). The solid is described by

$$0 \leq z \leq \sqrt{16 - y^2} \quad \text{and } (x, y) \in D$$

where $D$ is the region in the $xy$-plane bounded by $3y - 2x = 0$, $x = 0$, and $y = 4$.




Hence, the volume of the solid is

$$\iint_D \sqrt{16 - y^2}\, dA$$

Observe that we can represent the set $D$ as $0 \leq x \leq \dfrac{3y}{2}$, and $0 \leq y \leq 4$. Thus, the volume is

$$\begin{aligned}
\iint_D \sqrt{16 - y^2}\, dA &= \int_0^4 \underbrace{\int_0^{3y/2} \sqrt{16 - y^2}\, dx}\; dy \\
&= \int_0^4 \underbrace{\sqrt{16 - y^2}(x)\Big|_0^{3y/2}}\; dy \\
&= \int_0^4 \frac{3y}{2}\sqrt{16 - y^2}\, dy \\
&= -\frac{1}{2}(16 - y^2)^{3/2}\Big|_0^4 \\
&= 32 \quad \text{cubic units.}
\end{aligned}$$

Observe that the region in Example 5 could have also been represented by $\dfrac{2x}{3} \leq y \leq 4$, $0 \leq x \leq 6$. Hence, we could have applied Theorem 1, instead of using equation ($**$). However, notice that if we had applied the Iterated Integral Theorem instead, our inner integral would have been

$$\int_{2x/3}^4 \sqrt{16 - y^2}\, dy$$

which would have been more difficult.

## General Advice

When evaluating a double integral

$$\iint\limits_D f(x,y)\ dA$$

we must take into account two factors:

- the shape of the region $D$.
- the form of the integrand $f(x,y)$.

Either of these factors may make it desirable or even essential to use one order of integration instead of the other.

A question appears in Mobius

A question appears in Mobius

Remark

You may have noticed that the feedback to the Your Turn question contained some new notation:

$$\int_{-2}^{2} dy \int_{0}^{4-y^2} (x+y)dx$$

Moving the differentials to "match" with their corresponding integral can help to avoid confusion about the order of integration. We will sometimes use this notation in Units 14 and 15.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

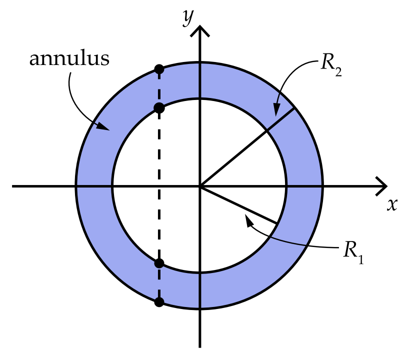
A question appears in Mobius

## Using the Decomposition Theorem

For more complicated regions we may not be able to apply our previous methods so easily. For example, an annulus cannot be described by the usual inequalities since a vertical or a horizontal line may intersect the boundary of $D$ in more than two points. A simple approach to evaluating the double integral

$$\iint_D f(x,y)\ dA$$

where $D$ is the annulus is to let $D_1, D_2$ denote the discs of radius $R_1$ and $R_2$ respectively.




Then, by the Decomposition Theorem,

$$\iint_{D_2} f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_D f(x,y)\ dA$$

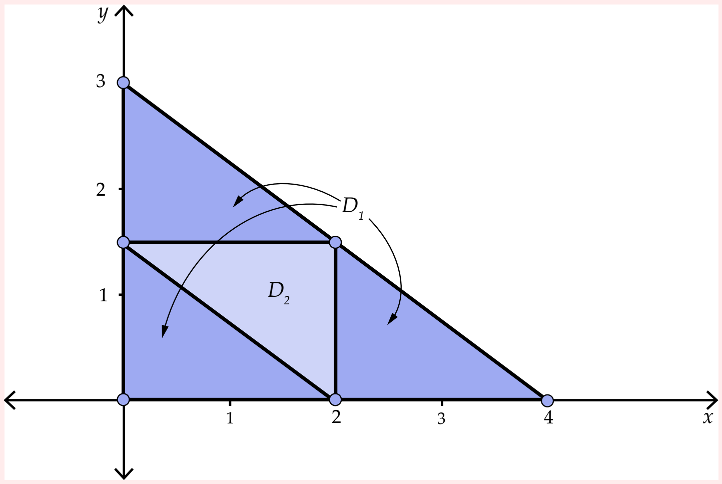
and so the required integral is

$$\iint_D f(x,y)\ dA = \iint_{D_2} f(x,y)\ dA - \iint_{D_1} f(x,y)\ dA$$

Both integrals on the right can be written as iterated integrals in the usual way.

Example 6

Evaluate the integral $\iint_D (x^2+y^2)dA$ where $D$ is the region illustrated below. $D = D_1 - D_2$ where $D_1$ is the outer triangle and $D_2$ is the inner triangle.




**Solution:**

Note that we can write this integral as a difference of integrals where $D_1$ is the outer triangle and $D_2$ is the inner triangle. By the Decomposition Theorem,

$$\iint_D (x^2+y^2)dA = \iint_{D_1} (x^2+y^2)dA - \iint_{D_2} (x^2+y^2)dA$$

The region $D_1$ is described by the inequalities $0 \leq x \leq 4$ and $0 \leq y \leq -\frac{3}{4}x + 3$ .

The region $D_2$ is described by the inequalities $0 \leq x \leq 2$ and $-\frac{3}{4}x + \frac{3}{2} \leq y \leq \frac{3}{2}$ .

Writing each integral as an iterated integral, we have

$$\begin{aligned}
\iint_D (x^2+y^2)dA &= \iint_{D_1} (x^2+y^2)dA - \iint_{D_2} (x^2+y^2)dA \\
&= \int_0^4 \int_0^{-\frac{3}{4}x+3} (x^2+y^2)\ dy\ dx - \int_0^2 \int_{-\frac{3}{4}x+\frac{3}{2}}^{\frac{3}{2}} (x^2+y^2)\ dy\ dx \\
&= \int_0^4 -\frac{3}{4}(x^3-4x^2) - \frac{9}{64}(x-4)^3\ dx - \int_0^2 \frac{3}{64}19x^3 - 18x^3 + 36x\ dx \\
&= 25 - \frac{75}{16} \\
&= \frac{325}{16}
\end{aligned}$$

For this or even more complicated regions, we can often make things simpler by applying a change of variables. This will be the topic of our next lesson.

# 14.3 - The Change of Variable Theorem

## The Change of Variable Theorem

As we saw at the end of the last lesson, there are cases where we are integrating over complicated regions which are not easily describable in Cartesian coordinates. In these cases, a well-chosen mapping $F$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ can be used to simplify a double integral

$$\iint_{D_{xy}} G(x, y)\ dA$$

either by changing the integrand $G(x, y)$, or by deforming the set $D_{xy}$ in the $xy$-plane into a simpler shape $D_{uv}$ in the $uv$-plane.

The process is called a **change of variables** in the double integral. In this type of calculation, it is convenient to replace the symbol "$dA$" in the double integral by "$dx\ dy$" when working in the $xy$-plane, and by "$du\ dv$" when working in the $uv$-plane.

In order to derive the change of variable formula for double integrals, we need the formula which describes how areas are related under a mapping $F$ given by

$$(x, y) = F(u, v) = (f(u, v), g(u, v)) \tag{*}$$

The geometric interpretation of the Jacobian gives us

$$\Delta A_{xy} \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta A_{uv} \tag{**}$$

for $\Delta u$, $\Delta v$ sufficiently small where the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ is evaluated at a point in the region. Notice that we have interchanged the roles of $(x, y)$ and $(u, v)$ in equations $(*)$ and $(**)$, as compared to Unit 13. We do this because we want $F$ to be a mapping that takes the $uv$-plane into the $xy$-plane, which will make the statement of the Change of Variable Theorem simpler.

**Theorem 1: Change of Variable Theorem**
Let each of $D_{uv}$ and $D_{xy}$ be a closed bounded set whose boundary is a piecewise-smooth closed curve. Let

$$(x, y) = F(u, v) = (f(u, v), g(u, v))$$

be a one-to-one mapping of $D_{uv}$ onto $D_{xy}$, with $f, g \in C^1$, and $\dfrac{\partial(x, y)}{\partial(u, v)} \neq 0$ except for possibly on a finite collection of piecewise-smooth curves in $D_{uv}$. If $G(x, y)$ is continuous on $D_{xy}$, then

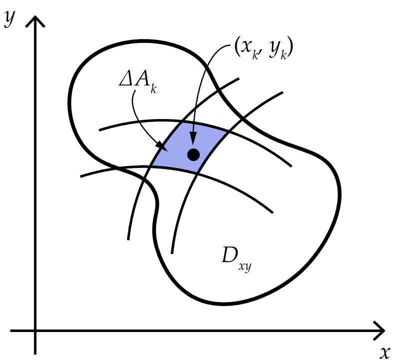
$$\iint_{D_{xy}} G(x, y)\ dx\ dy = \iint_{D_{uv}} G\Big(f(u, v), g(u, v)\Big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\ du\ dv$$

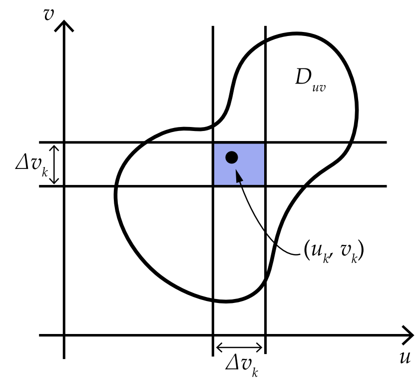
At first glance, this theorem can be intimidating. Essentially, it is telling us that, under certain reasonable conditions, we can change variables in such a way as to simplify a double integral. When we do this change, however, we have to take the local scaling factor into account, hence the need for the Jacobian factor.

While proof of this theorem is beyond the scope of this course, we can make the result plausible by the following argument:

Consider a partition $P$ of $D_{uv}$ into rectangles by means of straight lines parallel to the coordinate axes. The images of these lines under the given transformation will, in general, be two families of curves which will define a partition

$P^*$ of $D_{xy}$ into elements of the area which are approximately parallelograms. We can use this partition, instead of a rectangular partition, to define $\displaystyle\iint_{D_{xy}} F(x,y)\ dx\ dy$.




Thus,

$$\begin{aligned}
\iint_{D_{xy}} G(x,y)\ dx\ dy &= \lim_{\Delta P^* \to 0} \sum_{i=1}^{n} G(x_i, y_i)\Delta A_i \\
&= \lim_{\Delta P^* \to 0} \sum_{i=1}^{n} G\Big(f(u_i,v_i), g(u_i,v_i)\Big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u_i,v_i)} \Delta A_i \\
&= \iint_{D_{uv}} G\Big(f(u,v), g(u,v)\Big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dA
\end{aligned}$$

by using the definition of double integral relative to the rectangular partition of $D_{uv}$.

The lack of rigor occurs when we use the approximation $(**)$ in the second line.

For another take on the Change of Variable Theorem, see the `https://sequentialmath.com/comic/change-of-variables-theorem`.

**Example 1**

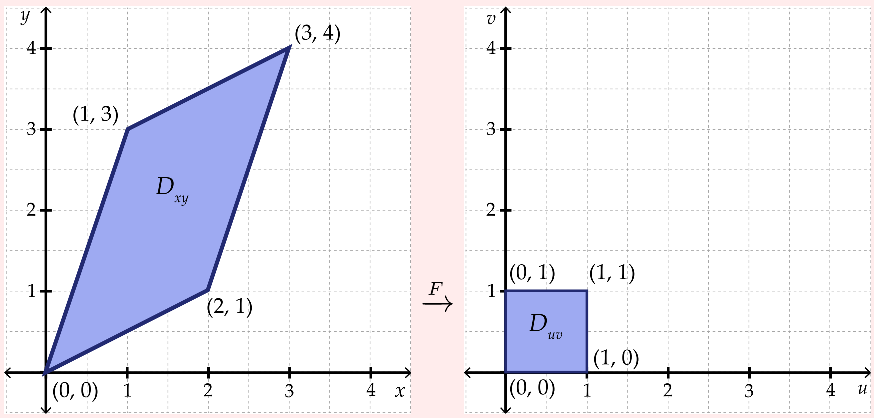
Evaluate $\displaystyle\iint\limits_{D_{xy}} (x+y)\ dA$, where $D_{xy}$ is the set bounded by the parallelogram with vertices $(0,0)$, $(2,1)$, $(1,3)$, and $(3,4)$.

**Solution:**

In a previous example, we found that the mapping

$$(u,v) = F(x,y) = \left(\frac{1}{5}(2y-x), \frac{1}{5}(3x-y)\right)$$

maps $D_{xy}$ onto $D_{uv}$, the unit square in the $uv$-plane.




The Jacobian of $F$ is

$$\frac{\partial(u,v)}{\partial(x,y)} = -\frac{1}{5}$$

Observe that our mapping $F$ maps $D_{xy}$ to $D_{uv}$, but the Change of Variable Theorem requires a mapping which maps $D_{uv}$ to $D_{xy}$. In particular, we actually require the inverse of our mapping. Solving for $x$ and $y$ we find that

$$(x,y) = F^{-1}(u,v) = (u+2v, 3u+v)$$

Hence, $\dfrac{\partial(x,y)}{\partial(u,v)} = -5$, and the integrand becomes $x+y = 4u+3v$. Then, the Change of Variable Theorem gives

$$\iint\limits_{D_{xy}} (x+y)\ dx\ dy = \iint\limits_{D_{uv}} (4u+3v)|-5|\ du\ dv$$

It is straightforward to write this double integral as an iterated integral and evaluate it.

$$\iint\limits_{D_{uv}} (4u+3v)|-5|\ du\ dv = \int_0^1 \int_0^1 5(4u+3v)\ du\ dv = \int_0^1 5(3v+2)\ dv = \frac{35}{2}$$

The final result is

$$\iint\limits_{D_{xy}} (x+y)\ dA = \frac{35}{2}$$

A question appears in Mobius

**Example 2**

Use the mapping $(u, v) = F(x, y) = (x + y, -x + y)$ to evaluate

$$\int_0^{\pi} \int_0^{\pi - y} (x + y)\cos(x - y)\ dx\ dy$$

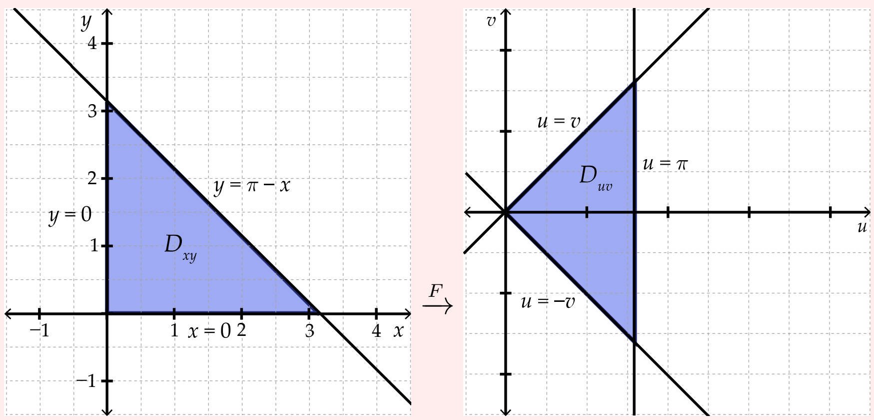
**Solution:**

The region $D_{xy}$ of integration is $0 \leq x \leq \pi - y$ and $0 \leq y \leq \pi$. Thus, the region is bounded by the lines $x = 0$, $y = 0$ and $x = \pi - y$. Since we want to integrate with respect to $u$ and $v$, we use the mapping $F$ to determine the new region $D_{uv}$.

For the line $x = 0$, $0 \leq y \leq \pi$, we get $v = y = u$ with $0 \leq u \leq \pi$.

For the line $y = 0$, $0 \leq x \leq \pi$, we get $v = -x = -u$ with $0 \leq u \leq \pi$.

For the line $x + y = \pi$, $0 \leq x \leq \pi$, we get $u = \pi$. We also have $u - v = 2x$. Thus, $v = u - 2x = \pi - 2x$, which implies $-\pi \leq v \leq \pi$.




To apply the Change of Variables Theorem, we also require the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$. Rather than finding the inverse mapping, we can instead use the inverse property of the Jacobian. We have

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

Thus,

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{2}$$

Since the Jacobian is non-zero and the mapping has continuous partial derivatives, we can apply the Change of Variables Theorem to get

$$\int_0^{\pi} \int_0^{\pi - y} (x + y)\cos(x - y)\ dx\ dy = \iint_{D_{uv}} u\cos(-v) \left|\frac{1}{2}\right|\ dA$$

From the diagram, we observe that we can write the region $D_{uv}$ as $-u \leq v \leq u$ and $0 \leq u \leq \pi$. Thus,

$$\begin{aligned}
\int_0^{\pi} \int_0^{\pi - y} (x + y)\cos(x - y)\ dx\ dy &= \int_0^{\pi} \int_{-u}^{u} u\cos(-v) \overbrace{\left|\frac{1}{2}\right|}^{Jacobian}\ dv\ du \\
&= \frac{1}{2} \int_0^{\pi} -u\sin(-v)\Big|_{-u}^{u}\ du \\
&= \frac{1}{2} \int_0^{\pi} 2u\sin u\ du = -u\cos u + \sin u\Big|_0^{\pi} \\
&= \pi
\end{aligned}$$

A question appears in Mobius

## Double Integrals in Polar Coordinates

If the boundary of the region is a circle centered on the origin or a circle that passes through the origin, it will often help to transform from polar to Cartesian coordinates. Recall that the mapping from polar to Cartesian coordinates is

$$(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta)$$

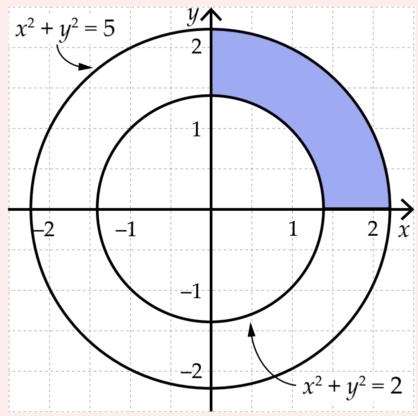
which has Jacobian,

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r$$

Hence, we must restrict $r > 0$ so that the mapping is one-to-one and the Jacobian is non-zero in order to apply the Change of Variable Theorem. Note that we can make this restriction even if the origin is in the region as the integral over a single point is 0.

Example 3

Evaluate $\displaystyle\iint_{D_{xy}} (2x+y)\ dA$ where $D_{xy}$ is the quarter annulus shown below.




**Solution:**

We first convert from Cartesian coordinates to polar coordinates.
Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$2x + y = 2r\cos(\theta) + r\sin(\theta) = r(2\cos(\theta) + \sin(\theta))$$

By the Change of Variables Theorem,

$$\iint_{D_{xy}} (2x+y)\ dA = \iint_{D_{r\theta}} r(2\cos(\theta) + \sin(\theta))\ \overbrace{|r|}^{Jacobian}\ \ dr\ d\theta$$

Note that the annulus is described by the inequalities

$$0 \leq \theta \leq \frac{\pi}{2} \quad \text{and} \quad 2 \leq r \leq 5$$

We can thus write the integral over $D_{r\theta}$ as an iterated integral:

$$\begin{aligned}
\iint_{D_{r\theta}} r^2(2\cos(\theta) + \sin(\theta))\ dr\ d\theta &= \int_0^{\frac{\pi}{2}} \underbrace{\int_2^5 r^2(2\cos(\theta)+\sin(\theta))\ dr}\ d\theta \\
&= \underbrace{\int_0^{\frac{\pi}{2}} 39(\sin(\theta) + 2\cos(\theta))\ d\theta} \\
&= 117
\end{aligned}$$

**Example 4**

Evaluate $\displaystyle\iint\limits_{D_{xy}} \frac{x}{x^2+y^2}\, dA$ where $D_{xy}$ is the half disc $(x-1)^2+y^2 \leq 1$, $x \geq 1$.

**Solution:**

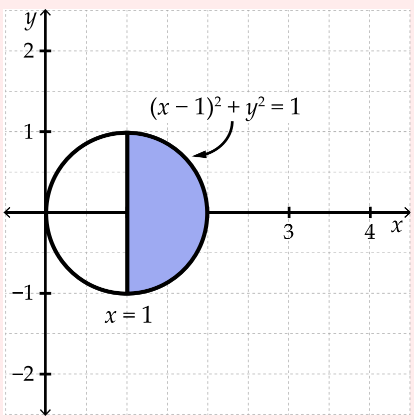
We first convert the equations from Cartesian coordinates to polar coordinates. Since $x = r\cos\theta$ we get that $x = 1$ becomes

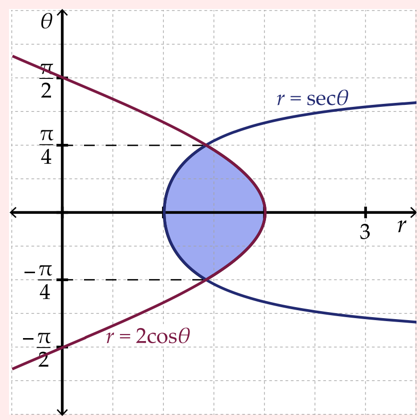
$$\begin{aligned} r\cos\theta &= 1 \\ r &= \sec\theta \end{aligned}$$

Similarly, $x^2+y^2 = 2x$ becomes

$$\begin{aligned} r^2 &= 2r\cos\theta \\ r &= 2\cos\theta \end{aligned}$$

assuming $r \neq 0$. The image $D_{r\theta}$ is shown in the figure below. The values of $\theta$ at the points of intersection are obtained by solving $\sec\theta = 2\cos\theta$, giving $\theta = \pm\dfrac{\pi}{4}$.




The Change of Variable Theorem thus implies

$$\iint\limits_{D_{xy}} \frac{x}{x^2+y^2}\, dx\, dy = \iint\limits_{D_{r\theta}} \frac{r\cos\theta}{r^2} \overbrace{|r|}^{Jacobian}\, dr\, d\theta = \iint\limits_{D_{r\theta}} \cos\theta\, dr\, d\theta.$$

The set $D_{r\theta}$ is described by the inequalities

$$\sec\theta \leq r \leq 2\cos\theta, \quad \text{and} \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

We can thus write the integral over $D_{r\theta}$ as an iterated integral,

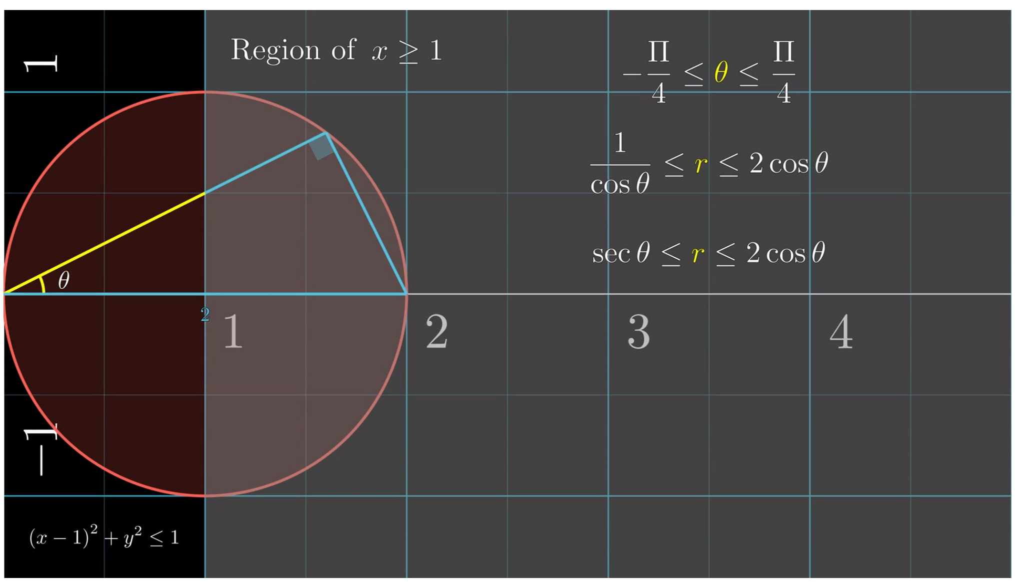
$$\begin{aligned}
\iint\limits_{D_{r\theta}} \cos\theta \ dr \ d\theta &= \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \underbrace{\int\limits_{\sec\theta}^{2\cos\theta} \cos\theta \ dr}\ d\theta \\
&= \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \underbrace{\cos\theta(2\cos\theta - \sec\theta)}\ d\theta \\
&= \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(2\theta)\ d\theta \\
&= \frac{1}{2}\sin(2\theta)\Big|_{-\pi/4}^{\pi/4} \\
&= 1
\end{aligned}$$

After calculating this integral, we find

$$\iint\limits_{D_{xy}} \frac{x}{x^2+y^2}\ dx\ dy = 1$$

The following video gives more detail about how we found the bounds of integration for the set $D_{r\theta}$.

*A video appears here.*




*Additional content appears in Mobius.*

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

### Tips for Solving Double Integrals Over a Region $D$

When solving double integrals, the goal is always to write the integral as an iterated integral. This is done by converting the region $D$.

If $D$ does not allow us to write an iterated integral, we can try to:

- change variables using the Change of Variables Theorem
- split $D$ into a disjoint union using the Decomposition Theorem

If $D$ allows us to write an iterated integral, but we are still getting stuck, we can try to:

- change the order of integration
- rewrite $D$ to set up a different iterated integral
- perform a change of variables to set up a different iterated integral

## 14.4 - Putting It All Together

### Worked Example 1

Evaluate $\iint_D \sin(x+y)\ dx\ dy$, where $D$ is the triangular region with vertices $(0,0)$, $(\pi,0)$ and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

a. Sketch the region.

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

**A question appears in Mobius**

**A question appears in Mobius**

**A question appears in Mobius**

d. At what points of the disc does the temperature equal the average temperature?

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

c. The number of bacteria per unit area in $D_{xy}$ is given by

$$c(x,y) = \frac{10}{9\pi}(x^2 + 4xy + 13y^2)^2$$

Use the Change of Variable Theorem to write an expression for the number of bacteria in $D_{xy}$ as double integral over $D_{uv}$.

A question appears in Mobius

A question appears in Mobius

## Practice Problems

Try to answer the questions. If you are having trouble, check for a hint before looking at the solutions.

1. Show that $\displaystyle\iint_D (ax+by)\ dx\ dy = \frac{1}{3}(a+b)$, where $D$ is the region in the first quadrant bounded by the circle $x^2+y^2=1$ and the lines $x=0$, $y=0$; $a, b$ are constants.

2. Find the volume of the solid with height $h(x,y)=xy$ and base $D$ where $D$ is bounded by $y=\frac{1}{2}x$, $y=\sqrt{x}$, $x=2$ and $x=4$.

3. Evaluate the following integrals.

   (a) $\displaystyle\iint_D xy^2\ dA$ where $D$ is the region bounded by $y=x$, $y=2x$ and $x=3$.

   (b) $\displaystyle\int_0^1\int_x^1 y\sqrt{1-y^3}\ dy\ dx.$

   (c) $\displaystyle\iint_{D_{xy}} x^2\ dA$, where $D_{xy}$ is bounded by the ellipse $3x^2+6xy+4y^2=4$.

4. Evaluate $\displaystyle\iint_D e^{-y^2}\ dx\ dy$, where $D$ is the triangular region with vertices $(0,0)$, $(0,1)$ and $(1,1)$.

5. Evaluate $\displaystyle\iint_D x+y^2\ dA$ where $D$ is the region bounded by $x=y$, $x=-y$ and $y=-1$.

6. For the following iterated integral sketch the region of integration, and evaluate the integral by reversing the order of integration:

$$\int_0^1\left(\int_{y=x}^{\sqrt{x}}\frac{\sin y}{y}\ dy\right)\ dx$$

7. Let $D$ be the quarter disc in the first quadrant defined by $x^2+y^2\leq 1$. Use the inequality

$$x - \frac{1}{6}x^3 \le \sin x \le x, \text{ for } x \ge 0$$

to show that

$$\frac{14}{45} \le \iint_D \sin x \, dA \le \frac{15}{45}$$

**Note:** You will not succeed in evaluating this integral exactly.

8. Let $D$ be the unit square $0 \le x \le 1$, $b \le y \le b+1$. Show that

$$\iint_D x^y \, dA = \ln\left(\frac{b+2}{b+1}\right)$$

9. Evaluate $\displaystyle\iint_D e^{-|x+y|} \, dA$, where $D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$.

10. Evaluate
    (a) $\displaystyle\iint_D xy \, dA$
    (b) $\displaystyle\iint_D \sin x \, dA$,

    where $D$ is the unit disc centered at the origin. [*Hint provided below*]

## Hint

*Additional content appears in Mobius.*

## Select Answers and Solutions

1. $\frac{a}{3} + \frac{b}{3}$
2. $\frac{11}{6}$
3. (a) $\frac{567}{5}$
   (b) $\frac{2}{9}$
   (c) No answer provided.
4. $\frac{1 - e^{-1}}{2}$
5. $\frac{1}{2}$
6. $1 - \sin(1)$
7. No answer provided.
8. $\ln\left(\frac{b+2}{b+1}\right)$
9. $2 + 2e^{-2}$
10. (a) 0
    (b) 0

# Unit 15

# Triple Integrals

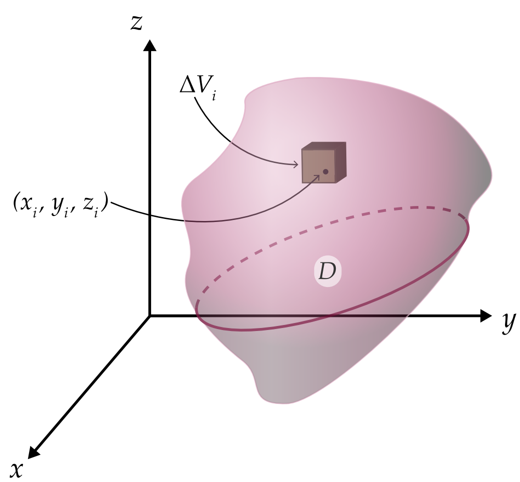
## 15.1 - Definition of Triple Integrals

### Definition of Triple Integrals

In this unit, we will work on defining triple integrals in a manner analogous to our definition of double integrals in the previous unit.

Let $D$ be a closed bounded set in $\mathbb{R}^3$ whose boundary consists of a finite number of surface elements which are smooth except possibly at isolated points. Let $f(x, y, z)$ be a function which is bounded on $D$. Subdivide $D$ by means of three families of planes which are parallel to the $xy$-, $yz$-, and $xz$-planes respectively, forming a partition $P$ of $D$.
Label the $N$ rectangular blocks that lie completely in $D$ in some specific order, and denote their volumes by $\Delta V_i, i = 1, \ldots, n$. Choose an arbitrary point $(x_i, y_i, z_i)$ in the $i$-th block, $i = 1, \ldots, n$, and form the Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i \qquad (*)$$




Let $\Delta P$ denote the maximum of the dimensions of all rectangular blocks in the partition $P$.

We define integrable functions on closed and bounded sets in $\mathbb{R}^3$ in the same way as we did for $\mathbb{R}^2$.

> **Definition: Integrable**
> A function $f(x, y, z)$ which is bounded on a closed bounded set $D \subset \mathbb{R}^3$ is said to be **integrable** on $D$ if and only if all Riemann sums approach the same value as $\Delta P \to 0$.

The triple integral is defined analogously to the double integral:

**Definition: Triple Integral**
If $f(x, y, z)$ is integrable on a closed bounded set $D$, then we define the **triple integral** of $f$ over $D$ as

$$\iiint\limits_D f(x, y, z)\; dV = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

This definition might prompt us to ask whether there is any guarantee that the limiting process in the definition of the triple integral actually leads to a unique value, i.e. that the limit exists. It is possible to define weird functions for which the limit does not exist, i.e. which are not integrable on $D$?

If $f$ is continuous on $D$, it can be proved that $f$ is integrable on $D$. However, functions which are discontinuous in $D$ may still be integrable on $D$. For example, if $f$ is continuous on $D$ except at points which lie on a surface or curve in $D$, then $f$ is integrable on $D$. The proofs of these results are beyond the scope of this course.

## Interpretations of the Triple Integral

The triple integral symbol

$$\iiint\limits_D f(x, y, z)\; dV$$

stands for the limit of a Riemann sum. In itself, the triple integral is a mathematically defined object. It has many interpretations, depending on the meaning that we assign to the integrand $f(x, y, z)$. The "$dV$" in the triple integral symbol should remind us of the volume of a rectangular block in a partition of $D$.

## Triple Integral as Volume

The simplest interpretation is when we specialize $f$ to be the constant function with value one:

$$f(x, y, z) = 1, \quad \text{for all } (x, y, z) \in D$$

Then the Riemann sum $\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$ simply sums the volumes of all rectangular blocks in $D$, and the triple integral over $D$ serves to define the volume $V(D)$ of the set $D$:

$$V(D) = \iiint\limits_D 1\; dV$$

## Triple Integral as Mass

Think of a planet or star whose density varies with position. Let $D$ denote the subset of $\mathbb{R}^3$ occupied by the star. Let $f(x, y, z)$ denote the density (mass per unit volume) at position $(x, y, z)$. The mass of a small rectangular block located within the star at position $(x_i, y_i, z_i)$ will be approximately

$$\Delta M_i \approx f(x_i, y_i, z_i)\Delta V_i$$

Thus, the Riemann sum corresponding to a partition $P$ of $D$

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

will approximate the total mass $M$ of the star, and the triple integral of $f$ over $D$, being the limit of the Riemann sum, will represent the total mass:

$$M = \iiint\limits_D f(x, y, z)\ dV$$

## Average Value of a Function

By analogy with functions of one and two variables, we can use the triple integral to define the average value of a function $f(x, y, z)$ over a closed and bounded set $D \subset \mathbb{R}^3$.

**Definition: Average Value**
Let $D \subset \mathbb{R}^3$ be closed and bounded with volume $V(D) \neq 0$, and let $f(x, y, z)$ be a bounded and integrable function on $D$. The **average value** of $f$ over $D$ is defined by

$$f_{avg} = \frac{1}{V(D)} \iiint\limits_D f(x, y, z)\ dV$$

**Remark**
If you have the impression that you have read this section someplace else, you're right. Compare it with the Definition of Double Integrals. The only essential change here is to replace "area" by "volume".

## Properties of the Triple Integral

The triple integral satisfies the same basic properties as the double integral.

**Theorem 1: Linearity**
If $D \subset \mathbb{R}^3$ is a closed and bounded set, $c$ is a constant, and $f$ and $g$ are two integrable functions on $D$, then

$$\iiint\limits_D (f + g)\ dV = \iiint\limits_D f\ dV + \iiint\limits_D g\ dV$$

and

$$\iiint\limits_D cf\ dV = c \iiint_D f\ dV$$

Linearity tells us that the triple integral of a sum of two functions is the sum of the triple integral of each function and that the triple integral of a scaled function is the scalar multiple of the triple integral of the function. As the next result shows, triple integrals also behave nicely with inequalities.

**Theorem 2: Basic Inequality**
If $D \subset \mathbb{R}^3$ is a closed and bounded set and $f$ and $g$ are two integrable functions on $D$ such that $f(x, y, z) \leq g(x, y, z)$ for all $(x, y, z) \in D$, then

$$\iiint\limits_D f\ dV \leq \iiint\limits_D g\ dV$$

The Basic Inequality property is often used to obtain an estimate for a triple integral that cannot be evaluated exactly.

Another useful property satisfied by the triple integral is the absolute value inequality.

**Theorem 3: Absolute Value Inequality**
If $D \subset \mathbb{R}^3$ is a closed and bounded set and $f$ is an integrable function on $D$, then

$$\left| \iiint\limits_D f \, dV \right| \leq \iiint\limits_D |f| \, dV$$

The last property we mention here is the decomposition property:

**Theorem 4: Decomposition**
Assume $D \subset \mathbb{R}^3$ is a closed and bounded set and $f$ is an integrable function on $D$. If $D$ is decomposed into two closed and bounded subsets $D_1$ and $D_2$ by a piecewise smooth surface $C$, then

$$\iiint\limits_D f \, dV = \iiint_{D_1} f \, dV + \iiint_{D_2} f \, dV$$

This property is essential for dealing with complicated regions of integration and with discontinuous integrands.

# 15.2 - Iterated Integrals

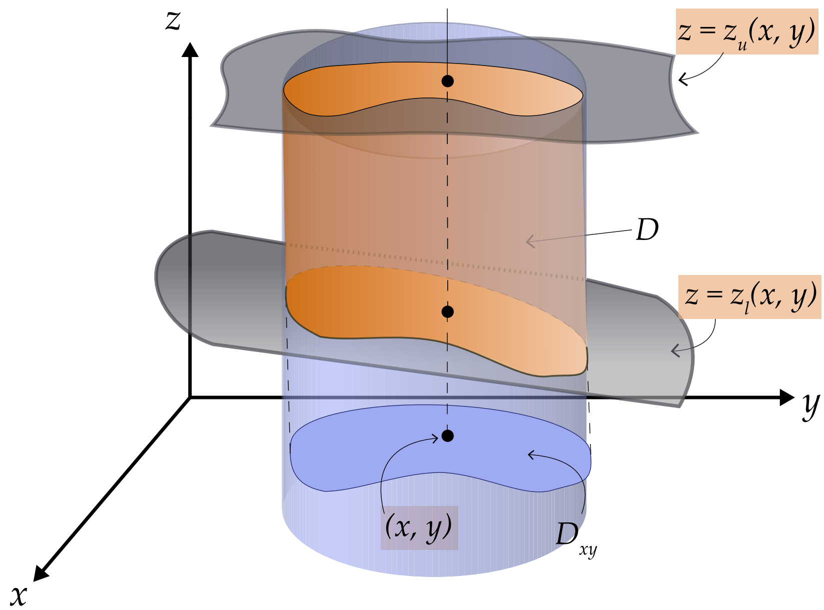
## Iterated Integrals

We generalize the method used in the unit on double integral, and show how to express a triple integral as a 3-fold iterated integral. This enables us to evaluate triple integrals exactly for sufficiently simple functions and integration sets.

*A slideshow appears in Mobius.*

Slide

## Iterated Integrals




In order to write a triple integral as an iterated integral, we take an arbitrary point $(x, y) \in D_{xy}$. Then we integrate $f(x, y, z)$ with respect to $z$ from $z_\ell(x, y)$ to $z_u(x, y)$, and integrate the result over $D_{xy}$, as a double integral.

This procedure essentially sums over all the rectangular blocks in a partition of $D$, and hence gives the triple integral of $f(x, y, z)$ over $D$.

**Theorem 1:** Iterated Integrals
Let $D$ be the subset of $\mathbb{R}^3$ defined by

$$z_\ell(x, y) \leq z \leq z_u(x, y) \quad \text{and} \quad (x, y) \in D_{xy}$$

where $z_\ell$ and $z_u$ are continuous functions on $D_{xy}$, and $D_{xy}$ is a closed bounded subset in $\mathbb{R}^2$, whose boundary is a piecewise smooth closed curve. If $f(x, y, z)$ is continuous on $D$, then

$$\iiint\limits_D f(x, y, z)\ dV = \iint\limits_{D_{xy}} \int\limits_{z_\ell(x,y)}^{z_u(x,y)} f(x, y, z)\ dz\ dA$$

**Remark**
As with double iterated integrals, we are doing partial integration. That is, to evaluate the inner integral of

$$\iint\limits_{D_{xy}} \int\limits_{z_\ell(x,y)}^{z_u(x,y)} f(x, y, z)\ dz\ dA$$

we hold $x$ and $y$ constant and integrate with respect to $z$.

Keep in mind that when evaluating a triple integral, it is not essential to integrate first with respect to $z$. We choose the order of integration that is most convenient.

That is, if we describe $D$ by inequalities of the form

$$x_\ell(y,z) \leq x \leq x_u(y,z)$$

with $(y,z) \in D_{yz}$, then we would get

$$\iiint\limits_D f(x,y,z)\ dV = \iint\limits_{D_{yz}} \int\limits_{x_\ell(y,z)}^{x_u(y,z)} f(x,y,z)\ dx\ dA$$

On the other hand, if we describe $D$ by inequalities of the form

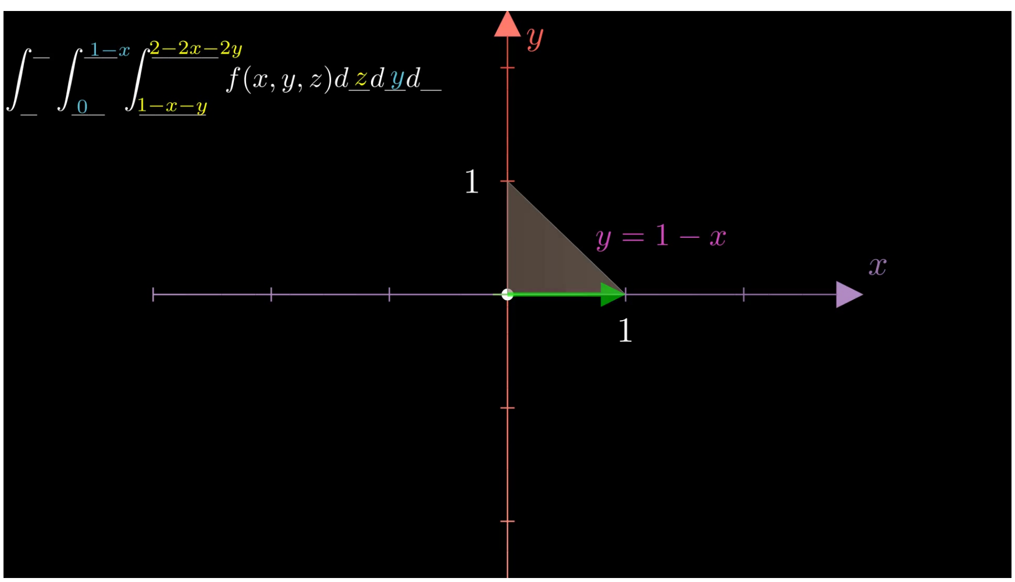
$$y_\ell(x,z) \leq y \leq y_u(x,z)$$

with $(x,z) \in D_{xz}$, then we would get

$$\iiint\limits_D f(x,y,z)\ dV = \iint\limits_{D_{xz}} \int\limits_{y_\ell(x,z)}^{y_u(x,z)} f(x,y,z)\ dy\ dA$$

In the following video, you will see a quick demonstration of how to determine the upper and lower limits for each integral in the triple integral. Pause the video as needed to better understand the order of integration and how we switch the view from 3D to 2D.

*A video appears here.*




We will demonstrate this in the example and exercises below.

**Example 1**

Evaluate $\iiint_D z\, dV$, where $D$ is the solid tetrahedron with vertices $(1,0,0)$, $(0,2,0)$, $(0,0,3)$, and $(0,0,0)$.

**Solution:**

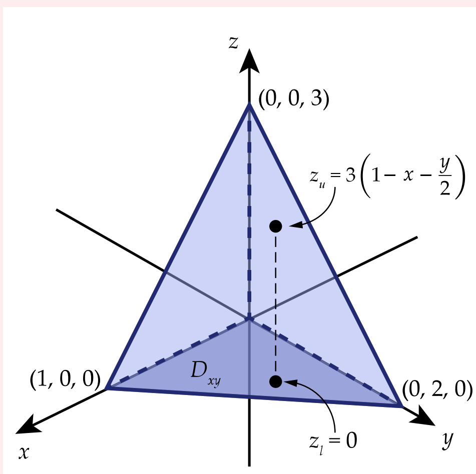
The tetrahedron is bounded by the planes $x=0$, $y=0$, $z=0$, and $x+\dfrac{y}{2}+\dfrac{z}{3}=1$.

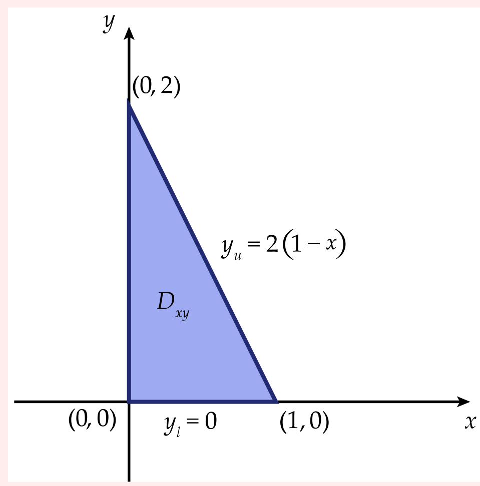
Thus, the region $D$ can be described by

$$0 \le z \le 3\left(1-x-\frac{y}{2}\right), \quad \text{and} \quad (x,y)\in D_{xy}$$

where $D_{xy}$ is bounded by $x=0$, $y=0$, and the intersection of the inclined face with $z=0$. The intersection is

$$x+\frac{y}{2}+\frac{0}{3}=1 \Rightarrow y=2\,(1-x)$$




Thus, by Theorem 1,

$$\iiint_D z\, dV = \iint_{D_{xy}} \int_0^{3\left(1-x-\frac{y}{2}\right)} z\, dz\, dA = \int_0^1 \int_0^{2(1-x)} \int_0^{3\left(1-x-\frac{y}{2}\right)} z\, dz\, dy\, dx$$

on writing the outer double integral over $D_{xy}$ as a double iterated integral.

We have

$$\begin{aligned}
\iiint_D z\, dV &= \int_0^1 dx \int_0^{2(1-x)} dy \underbrace{\int_0^{3\left(1-x-\frac{y}{2}\right)} z\, dz} \\
&= \int_0^1 dx \int_0^{2(1-x)} \frac{3^2(1-x-\frac{y}{2})^2}{2} dy \\
&= \int_0^1 -3(x-1)^3\, dx \\
&= \frac{3}{4}
\end{aligned}$$

**Your Turn 1**

Write the triple integral in Example 1 as an iterated integral taking the variables in the order $y, x, z$. Evaluate the iterated integral and verify you get the same answer as in Exercise 1.

A question appears in Mobius

A question appears in Mobius

**Example 2**

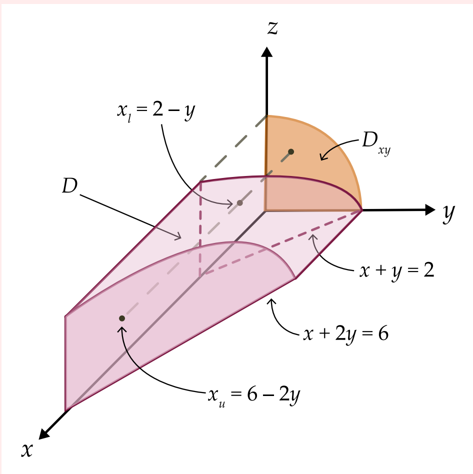
Evaluate $\iiint\limits_D \frac{z}{4-y}\, dV$, where $D$ is the region bounded by the cylinder $y^2+z^2=4$, and the planes $x+y=2$, $x+2y=6$, $z=0$, $y=0$, and lying in the first octant.

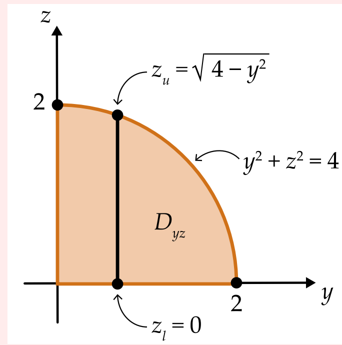
**Solution:**

Since $x$ only occurs in two of the equations, it is convenient to integrate first with respect to $x$, and describe $D$ by the inequalities

$$2-y \le x \le 6-2y \quad \text{and} \quad (y,z) \in D_{yz}$$

where $D_{yz}$ is the region in the first quadrant bounded by $y^2+z^2=4$, $y=0$, and $z=0$.




Thus,

$$\begin{aligned}
\iiint\limits_D \frac{z}{4-y}\, dV &= \iint\limits_{D_{yz}} \overbrace{\int\limits_{2-y}^{6-2y} \frac{z}{4-y}\, dx}^{\frac{z}{4-y}(x)\big|_{2-y}^{6-2y}=z}\; dA \\
&= \iint\limits_{D_{yz}} z\, dA, \\
&= \int\limits_0^2 \int\limits_0^{\sqrt{4-y^2}} z\, dz\, dy \\
&= \frac{1}{2}\int\limits_0^2 (4-y^2)\, dy \\
&= \frac{8}{3}
\end{aligned}$$

A question appears in Mobius

A question appears in Mobius

A question appears in Mobius

# 15.3 - The Change of Variable Theorem

## The Change of Variable Theorem

A mapping $F$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ can be used to simplify a triple integral

$$\iiint\limits_{D_{xyz}} G(x, y, z)\ dV$$

either by changing the integrand $G(x, y, z)$ or by deforming the set $D_{xyz}$ in $xyz$-space into a simpler shape $D_{uvw}$ in $uvw$-space, thereby simplifying the limits of integration. In this type of calculation, it is convenient to replace the symbol "$dV$" in the triple integral by "$dx\ dy\ dz$" if we are working in $xyz$-space, and by "$du\ dv\ dw$" if we are working in $uvw$-space.

**Theorem 1: Change of Variable Theorem**
Let

$$x = f(u, v, w), \quad y = g(u, v, w), \quad z = h(u, v, w)$$

be a one-to-one mapping of $D_{uvw}$ onto $D_{xyz}$, with $f, g, h$ having continuous partials, and

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \neq 0 \quad \text{on} \quad D_{uvw}$$

If $G(x, y, z)$ is continuous on $D_{xyz}$, then

$$\iiint\limits_{D_{xyz}} G(x, y, z)\ dV = \iiint\limits_{D_{uvw}} G\Big(f(u, v, w), g(u, v, w), h(u, v, w)\Big) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|\ dV$$

A proof is beyond the scope of this course, but the volume transformation formula using the Jacobian makes the theorem plausible, as in the case of the double integral.

**Example 1**

Evaluate $I = \iiint\limits_{D_{xyz}} x^2 \, dV$, where $D_{xyz}$ is the subset of $\mathbb{R}^3$ bounded by the surfaces $xy = 1$, $xy = 3$, and the planes $y + z = -1$, $y + z = 0$, $x + y + z = 1$ and $x + y + z = 2$.

**Solution:**

This solid is bounded by level surfaces of three functions, namely

$$xy, \quad y + z, \quad \text{and} \quad x + y + z$$

Thus, the solid $D_{xyz}$ is described by the inequalities

$$1 \leq xy \leq 3, \quad -1 \leq y + z \leq 0, \quad 1 \leq x + y + z \leq 2 \tag{*}$$

This suggests that we define a mapping

$$u = xy, \quad v = y + z, \quad w = x + y + z \tag{**}$$

The Jacobian is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \det \begin{bmatrix} y & x & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = x$$

By the Change of Variable Theorem,

$$I = \iiint\limits_{D_{xyz}} x^2 \, dx \, dy \, dz = \iiint\limits_{D_{uvw}} x^2 \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw$$

By the inverse property of the Jacobian,

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \left[ \frac{\partial(u, v, w, )}{\partial(x, y, z)} \right]^{-1} = \frac{1}{x}$$

It follows from the inequalities $(*)$ that $x > 0$ on $D_{xyz}$. Thus, equation $(**)$ gives

$$I = \iiint\limits_{D_{uvw}} x \, du \, dv \, dw$$

The next step is to express the integrand $x$ in terms of $u$, $v$, $w$. It follows from equations $(**)$ that $x = w - v$. Hence,

$$I = \iiint\limits_{D_{uvw}} (w - v) \, du \, dv \, dw \tag{***}$$

The inequalities $(*)$ imply that the image of the set $D_{xyz}$ under the mapping $(**)$ is the rectangular block $D_{uvw}$ defined by

$$1 \leq u \leq 3, \quad -1 \leq v \leq 0, \quad 1 \leq w \leq 2$$

Therefore, we can write the triple integral $(***)$ as an iterated integral, and since $D_{uvw}$ is rectangular, we can choose any order of integration that we like:

$$
\begin{aligned}
I &= \int_1^2 dw \int_{-1}^0 dv \int_1^3 (w - v)\ du \\
&= \int_1^2 dw \int_{-1}^0 (u(w - v))\Big|_1^3\ dv \\
&= \int_1^2 dw \int_{-1}^0 2(w - v)\ dv \\
&= 2\int_1^2 \left(wv - \frac{v^2}{2}\right)\Bigg|_{-1}^0\ dw \\
&= 2\int_1^2 w + \frac{1}{2}\ dw \\
&= 2\left(\frac{w^2}{2} + \frac{w}{2}\right)\Bigg|_1^2 \\
&= 4
\end{aligned}
$$

**Your Turn 1**

Verify the result $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = x$ in Example 1.

A question appears in Mobius

A question appears in Mobius

In double integrals, we saw that if there is symmetry about the origin it may be helpful to evaluate the double integral using polar coordinates. Similarly, if we have symmetry about the $z$-axis or the origin in $\mathbb{R}^3$ it may be helpful to use our mappings to cylindrical coordinates or spherical coordinates.

## Triple Integrals in Cylindrical Coordinates

Recall that the mapping from Cartesian coordinates to cylindrical coordinates is

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

with $r \geq 0$, $0 \leq \theta < 2\pi$. We can verify that the Jacobian is $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$ .

Since we need $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} \neq 0$, we must again restrict $r > 0$.

So for cylindrical coordinates, the formula in the Change of Variable Theorem reads

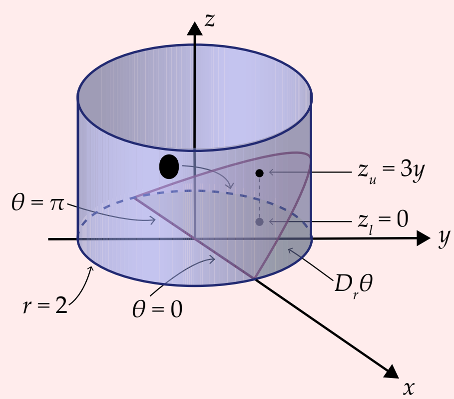
$$\iiint\limits_{D_{xyz}} G(x,y,z)\ dx\ dy\ dz = \iiint\limits_{D_{r\theta z}} G(r\cos\theta, r\sin\theta, z) r\ dr\ d\theta\ dz$$

**Example 2**
A wedge is cut from the cylinder $x^2 + y^2 = 2^2$, by the planes $z = 0$ and $z = 3y$, where $y$ is assumed to be non-negative. Find the volume of the wedge.
**Solution:**
The volume $V$ is given by

$$V = \iiint_R 1 \; dV$$

In cylindrical coordinates, we have the cylinder $r = 2$, the plane $z = 0$ and the plane $z = 3r\sin\theta$. Hence, the solid is described by

$$0 \le z \le 3r\sin\theta, \quad 0 \le r \le 2, \quad 0 \le \theta \le \pi$$




Using the Change of Variable Theorem gives

$$\begin{aligned} V &= \int_0^{\pi}\int_0^{2}\int_0^{3r\sin\theta} r \; dz \; dr \; d\theta \\ &= \int_0^{\pi}\int_0^{2} 3r^2\sin\theta \; dr \; d\theta \\ &= \int_0^{\pi} 8\sin\theta \; d\theta \\ &= 16 \end{aligned}$$

A question appears in Mobius

A question appears in Mobius

## Triple Integrals in Spherical Coordinates

Recall that the mapping from spherical coordinates to Cartesian coordinates is

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho \cos\varphi$$

with $\rho \geq 0$, $0 \leq \varphi \leq \pi$, $0 \leq \theta < 2\pi$.

The Jacobian is

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} \right| = \rho^2 \sin\varphi$$

**Your Turn 1**

Verify that $\dfrac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} = \rho^2 \sin\varphi$.

A question appears in Mobius

Thus, for spherical coordinates, we must restrict $\rho > 0$ and $0 < \varphi < \pi$ so that the Jacobian is non-zero and the mapping is one-to-one. Observe that this means we are not just removing one point, but the entire $z$-axis. However, this still will not affect our result as the triple integral over the $z$-axis is 0. Hence, the Change of Variable Theorem in spherical coordinates reads:

$$\iiint\limits_{D_{xyz}} G(x,y,z)dV = \iiint\limits_{D_{\rho\theta\varphi}} G(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi \; d\rho d\theta d\varphi$$

**Example 3**

Evaluate $\iiint\limits_{D} \frac{1}{x^2+y^2+z^2} dV$ where $D$ is the spherical shell between the spheres of radius 3 and radius 5 centered on the origin .

**Solution:**

We would not succeed in evaluating this triple integral as an iterated integral in terms of $x$, $y$, and $z$. However, if we use spherical coordinates, the calculation is simple.

In terms of spherical coordinates $\rho, \varphi, \theta$, the set $D$ is defined by

$$3 \le \rho \le 5, \quad 0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$$

Using the Change of Variable Theorem gives

$$\iiint\limits_{D} \frac{1}{x^2+y^2+z^2} \; dV = \int_0^{2\pi}\int_0^{\pi}\int_3^{5} \frac{1}{\rho^2}(\rho^2\sin\varphi) \; d\rho \; d\varphi \; d\theta = 8\pi$$

A question appears in Mobius

A question appears in Mobius

## 15.4 - Putting It All Together

A question appears in Mobius

### Worked Example 2

Find the volume of the region inside $x^2 + y^2 + z^2 = 2$ but outside $x^2 + y^2 = 1$. That is, the region of the sphere that lies outside of the cylinder.

Use the GeoGebra applet to study the region.

External resource: https://www.geogebra.org/material/iframe/id/sngnrdve/

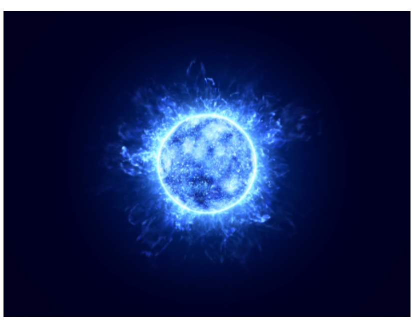


From the shape of these two surfaces, it is a natural thing to try either cylindrical coordinates or spherical coordinates. Indeed, it is possible to solve this problem in two different ways. Is one easier than the other? We will go over both methods, so you can answer this question.

A question appears in Mobius

A question appears in Mobius

## Application



aryos/E+/Getty Images

A spherical star of radius $b$ has a core of radius $\frac{1}{2}b$ with constant density $\rho_0$ kg/m$^3$. The density of the outer shell is proportional to $\frac{1}{r}$, where $r$ is the distance to the centre.
If the density is a continuous function of $r$, for $0 \leq r \leq b$, find the total mass of the star.

**Step 1:** Find the mass for the inner core of the star.

Recall that Mass = Density $\times$ Volume.

A question appears in Mobius

Now that we know the mass for the inner core, let's take the steps necessary to find the mass of the outer shell.

**Step 2:** If the density of the outer shell is proportional to $\frac{1}{r}$, where $r$ is the distance to the centre, write the density function of $r$ using a proportionality constant $k$.

A question appears in Mobius

**Step 3:** If the density is a continuous function of $r$, how can we relate $\rho_0$ with $\rho(r)$?

A question appears in Mobius

**Step 4:** Use the relation between $\rho_0$ with $\rho(r)$ from the previous step to find the proportionality constant $k$ and then rewrite $\rho(r)$.

A question appears in Mobius

Now that we know the function for the density of the outer shell $\rho(r)$, we can find the mass of the outer shell using a triple integral.

**Step 5:** Find the mass of the outer shell.

A question appears in Mobius

**Step 6:** Find the total mass of the star.

A question appears in Mobius

## Practice Problems

1. Write $\displaystyle\iiint\limits_D f(x,y,z)\ dV$ as an iterated integral, for each 3-D region $D$.

   (a) $D$ is the rectangular box defined by $|x-1| \leq 2, |y| \leq 3, |z+1| \leq 1$.

   (b) $D$ is the cylindrical solid defined by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1, |z-2| \leq 1$.

   (c) $D$ is the tetrahedron with vertices $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ and $(0,0,-c)$.

   (d) $D$ is the "ice-cream cone" bounded by $x^2+y^2 = \dfrac{1}{4}z^2$, $z \geq 0$ and the hemisphere defined by $x^2+y^2+z^2 = 25$, $z > 0$.

   (e) $D$ is the solid bounded by the paraboloid $y = 1-x^2-z^2$, and the hemisphere defined by $x^2+y^2+z^2 = 3$, $y < 0$ in $D$, $y < 1-x^2-z^2$.

2. Consider the triple integral $\displaystyle\iiint\limits_D e^x\ dV$, where $D$ is the 3-d region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 1$. Write it as an iterated integral in the order $z, y, x$. Notice that the order $x, z, y$ will give a simpler integration. Evaluate the integral using this order.

3. Evaluate $\iiint\limits_D x^2 + y \, dV$ where $D$ is the region bounded by $x + y + z = 2$, $z = 2$, $x = 1$ and $y = x$.

4. Describe the 3-d region of integration for the iterated integral

$$\int_{y=0}^{1} \int_{x=y-1}^{1-y} \int_{z=-\sqrt{(1-y)^2-x^2}}^{\sqrt{(1-y)^2-x^2}} f(x, y, z) \, dz \, dx \, dy$$

and find the limits when the order of integration is $y$, $x$, $z$.

5. The temperature at points in the cube

$$C = \{(x, y, z) \mid |x| \le 1, |y| \le 1, |z| \le 1\}$$

is $100r^2$, where $r$ is the distance to the origin. Find the average temperature. At what points of the cube does the temperature equal the average temperature?

6. Determine the volume bounded by the cone $z = 2\sqrt{x^2 + y^2}$ and the paraboloid $z = 1 - 8(x^2 + y^2)$.

7. Evaluate the following triple integrals by transforming to spherical coordinates:

(a) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz \, dy \, dx$

(b) $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{3}} dz \, dx \, dy$

8. Let $V$ denote the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Find $V$ by performing a transformation, but without integrating.

9. Suppose that a hemispherical tank (with the flat part on the ground) with radius $R$ is partially filled with water, so that the depth of the water is $h$. Find the volume of water in the tank. Consider spherical and cylindrical coordinates before trying to evaluate; one of them will be easier to use.

10. A glacier which occupies the region

$$-\sqrt{10^{-2} - x^2} < z < 0$$

moves parallel to the $y$-axis with velocity in km/year given by

$$v(x, z) = 10^{-3}[1 - 10^2(x^2 + z^2)]$$

Find the volume of ice $V$ moved through the $xz$-plane in a year (distances are in kilometers).

## Select Answers and Solutions

1. (a) $\displaystyle\int_{-2}^{0} \int_{-3}^{3} \int_{1}^{3} f \, dx \, dy \, dz$

(b) $\displaystyle\int_1^3 \int_0^{2\pi} \int_0^1 f(x(r,\theta), y(r,\theta), z)|ab|r \, dr \, d\theta \, dz$

(c) $\displaystyle\int_0^a \int_0^{b-\frac{b}{a}x} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} f \, dz \, dy \, dx + \int_0^a \int_0^{b-\frac{b}{a}x} \int_{-c(1-\frac{x}{a}-\frac{y}{b})}^{0} f \, dz \, dy \, dx$

(d) $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{20}} \int_0^{z/2} f(x(r,\theta), y(r,\theta), z)r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_{\sqrt{20}}^{5} \int_0^{\sqrt{25-z^2}} f(x(r,\theta), y(r,\theta), z)r \, dr \, dz \, d\theta$

(e) $\displaystyle\int_0^{2\pi}\int_{-1}^{1}\int_0^{\sqrt{1-y}} f(x(r,\theta),y(r,\theta),z)r\ dr\ dy\ d\theta + \int_0^{2\pi}\int_{-\sqrt{3}}^{-1}\int_0^{\sqrt{3-y^2}} f(x(r,\theta),y(r,\theta),z)r\ dr\ dy\ d\theta$

2. Start with $\displaystyle\iiint\limits_D e^x\ dV = \int_0^1\int_0^{1-x}\int_1^{1-x-y} e^x\ dz\ dy\ dx.$

3. $\dfrac{17}{30}$

4. The integration region is the region enclosed by line $x+y=1$, $y=x+1$ and saddle $z^2=(1-y)^2-x^2$ with $y$ between 0 and 1

5. No answer provided.

6. No answer provided.

7. No answer provided.

8. $\dfrac{4\pi^3 abc}{3}$

9. $\pi\left(R^3h-\dfrac{h^3}{3}\right)$