CO 342

Introduction to Graph Theory

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Spring 2023

Contents

1 Connectivity

1.1 Terminology

Definition 1.1: Graph

A graph is a triple $G : (V, E, i)$ where V and E are finite sets and i is a function from $V \times E$ to $\{0, 1\}$ such that for each $e \in E$, there are exactly 2 $v \in V$, for which $i(v, e) = 1$.

We provide some basic terminologies of graph theory before we start:

- e and v are **incident** in G if $i(v, e) = 1$
- v_1 and v_2 are **adjcant** in G if $\exists e$ incident to both v_1 and v_2
- G is **simple** if for each pair v_1, v_2, at most one pair is incident to both (no multi egdes)

Definition 1.2: Walk

A **walk** of a graph G is an alternating sequence

$$v_0, e_1, v_1, \ldots, e_k, v_k$$

where $v_0, v_1, \ldots, v_k$ are vertices of G (not necessarily disjoint), and $e_1, \ldots, e_k$ are edges of G such that each e_i is an edge from v_{i-1} to v_i.

Definition 1.3: Path

A **path** is a walk where $v_0, \ldots, v_k$ are distinct vertices (and therefore edges are distinct). The length of a path is defined as the number of edges.

Note that vertex disjoint implies edge disjoint, but not vice versa.

Definition 1.4: Connectivity

Two vertices v_1, v_2 of G are connected if there exists a **walk**.

Proposition 1.1

Consideronnectedness is an **equivalence relation** on vertices of G. That is, it follows reflexivity, symmetry, and transitivity.

Lemma 1.2: I

u and v are connected, there is a path from u to v.

Proof:

Consider a shortest walk from u to v. □

Definition 1.5: Component

Since connectedness is an equivalence relation, the vertex set can be partitioned into equivalence class of vertices where v_1, v_2 are connected iff they belong to the same equivalence class. Each subgraph of G obtained by taking one of these classes all the edges within it is a component of G.
Hence the component have pairwise disjoint vertex/edge sets and their union is the whole graph.

A graph is **connected** if it has precisely **one component**.

1.2 K-Connectedness

Definition 1.6: K-Connect

A graph is **k-connected** if there does not exists a set $X \subseteq V(G)$ with $|X| < k$ such that $G - X$ is disconnected.
Here $G - X$ is the subgraph of G obtained by removing all vertices in X and all thier incident edges.

Definition 1.7: AB-Path

Given A set A, B of vertices in G, an **AB-path** is a path P from a vertex in A to a vertex in B, so that P intersects A only at its frist vertex and B only at its last.

Note that if $A \cap B \neq \emptyset$, then every $v \in A \cap B$ gives a AB-path with no edges. For a vertex a and a set B, an aB-path means a $\{a\}B$-path.

2 Planarity

2.1 Terminology

Let G be a graph with vertices V, edges E, an embedding of G in $\mathbb{R}^2$ is a function φ such that:

- For each vertex v of G, φ is a point in $\mathbb{R}^2$, and no 2 vertices are mapped to the same point by φ (injective).
- For each edge e with ends u, v, $\varphi(e)$ is a curve from $\varphi(u)$ to $\varphi(v)$.
- For distinct edges e, f of G, the images of $\varphi(e)$ and $\varphi(f)$ are disjoint (as subsets of $\mathbb{R}^2$) except where e and f intersect at a vertex.
- For all $u \in V, e \in E$, u is an $\varphi(e)$ iff u is an end of e.

Definition 2.1: Planar graphs

A graph is **planar** if it has an embedding in $\mathbb{R}^2$, otherwise it's non-planar.
If φ is an embedding of G in $\mathbb{R}^2$, then $\varphi(G)$ for the union of the images of vertices and edges, are subsets of $\mathbb{R}^2$.

We can simplify this definition by the following propositions:

Proposition 2.1

If u is an open set, then x, y are connected in u iff they are polygonally connected in u.

Corollary 2.2

Given $u \subseteq \mathbb{R}^2$ open, we have

$$x, y \text{ connected in } u \Leftrightarrow x, y \text{ polygonally connected in } u$$

Corollary 2.3

If G has a planar embedding φ, then it has a planar embedding where all arcs are polygonal.

Now we have the Polygonal Jordan Curve Theorem

Theorem 2.4: PJCT

If C is a polygon, then $\mathbb{R}^2 \backslash C$ has exactly two regions.

Definition 2.2: Frontier

Given a set $S \subset \mathbb{R}^2$, the frontier of S is the the boundary set of S.

Lemma 2.5

If x_1, y_1, x_2, y_2 occur in cyclic order around some polygon C, and P is a polygonal curve from x_1 to x_2 with $P \subseteq \mathbb{R}^2 \backslash C$. Then $\mathbb{R}^2 \backslash (C \cup P)$ has three regions $f_0.f_1, f_2$ such that f_0 is a region of $\mathbb{R}^2 \backslash$, and $f_1 \cup f_2 \cup P$ is the other region of $\mathbb{R}^2 \backslash C$, and y_1 is not in the frontier of f_2, y_2 is not in the frontier of f_1.

Definition 2.3: Topological Minor

A graph H is a **topological minor** of a graph G if there is a function φ such that

1. for each vertex v of H, $\varphi(v)$ is a vertex of G.
2. for every edge e of H with ends u, v, $\varphi(e)$ is a $\varphi(u)\varphi(v)$-path in G.

2.2 Euler's Formula

We study the Euler's Formula in this section.

Theorem 2.6: Euler's Formula

If φ is an embedding of $G = (V, E)$ in the plane, and F is the set of faces of φ, then $|V| - |E| + |F| = 1 + c$, where c is the number of components of G.

Proof:

Let H be a maximal spanning forest for G, H consists of a spanning tree H_i for each component G_i of G, and $|E(H_i)| = |V(H_i)| - 1$, so $|E(H)| = \sum_i |E(H_i)| = (\sum_i |V(H_i)| - 1) = (\sum_i |V(H_i)|) - c = |V| - c$. So the embedding of H, given by φ has one face (H is a forest), $|V|$ vertices, and $|V| - c$ edges. So $|V(H)| - |E(H)| + |F(H)| = |V| - (|V| - c) + 1 = 1 + c$. Let H' be a maximal subgraph of G such that H is a subgraph of H' and $|V| - |E(H')| + |F(H')| = 1 + c$. If $H' = G$, then G satisfies Euler's formula, as required. Otherwise, G has an edge e outside $E(H')$, Let $H' + e$ be the subgraph of G outlined from

H' by adding e. Since e is in a cycle of $H'+e$, we know that $|F(H'+e)| = |F(H)|+1$. Clearly $E(H'+e) = |E(H')|+1$. So

$$\begin{aligned} |V| - |E(H'+e)| + |F(H'+e)| &= |V| - |E(H')| - 1 + |F(H')| + 1 \\ &= |V| - |E(H')| + |F(H')| \\ &= 1 + c \end{aligned}$$

so $H'+e$ contradicts the maximality of H. $\square$

Lemma 2.7

If φ is an embedding of a graph G that contains a cycle, then the bounding of every face of G contains a cycle.

Proof: Exercise

Lemma 2.8

Each edge in a planer embedding is in ≤ 2 faces boundaries.

Proposition 2.9

If G is a simple planar graph on ≥ 3 vertices, then

$$|E(G)| \leq 3|V(G)| - 6$$

Proof:

We combine Euler's formula with an inequality relating the # edges and # faces in an embedding.

Let $V = V(G), E = E(G)$, let F be the set of faces in some planar embedding of G, and $c = \#$ components of G.

If G is a forest, then $|E| \leq |V| - 1 \leq 3|V| - 6$. Otherwise, every face boundary contains a cycle, so has ≥ 3 edges. Let $A = \{(e, f) : f \in F$, and e is the boundary of F$\}$. Since each e is in the boundary of ≤ 2 faces, we know $|A| \leq 2|E|$. Since each $f \in F$ has ≥ 3 edges in its boundary, we know $|A| \geq 3|F|$. So $3|F| \leq 2|E|$, that is, $|F| \leq \frac{2}{3}|E|$. By Euler's Formula,

$$1 + c = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E|$$

so $|E| \leq 3(|V| - 1 - c) \leq 3|V| - 6$ (since $c \geq 1$). $\square$

By applying this proposition we can show that K_5 is non-planar. Note that

$$|E| \leq \binom{|V|}{2}$$

which is bounded by quadratic formula of vertices, while simple planar graphs are bounded by linear formula of vertices.

2.3 2-connected planar graphs

We want to investigate under what conditions will every face boundary is a face (instead of containing a cycle). It turns out that every 2-connected graph satisfies this property:

Proposition 2.10

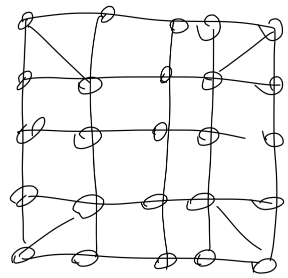
If φ is an embedding of a 2-connected graph G, then every face bounding of G is a cycle.

Proof:

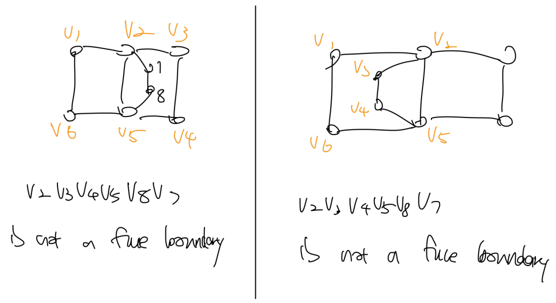
Induction with ear-decompostion: adding a path splits one face into 2 faces bounded by cycles and doesn't change any other face boundary. $\square$

2.4 3-connected planar graphs

Given a graph G that is known to be planar, can we determine which cycles appear as face boundary in an embedding of G, without knowing the embedding?



(We can tell when embedding is given)



(But we can't tell for general graphs without embedding)

The problem is **lack of 3-connectedness**.
A cycle C of G is **non-separating** if $G - V(C)$ is connected. C is **induced** in G if there is no edge of $G \backslash E(C)$ with both ends in C (there is no chord of C).

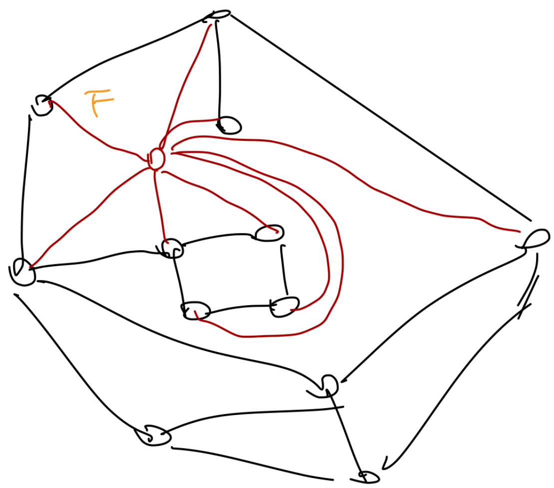
Proposition 2.11

If φ is an embedding of a 3-connected graph G, then C is a face (facial cycle) boundary iff C is non-seperating and induced by G.

We won't prove this, but give a useful lemma for the proof:

Lemma 2.12

Let G be a planar graph, and F be a face bounding in some embedding of G. Let G' be the graph obtained from G by adding a vertex v, and joining v to each vertex of F. Then *varphi* extends to an embedding of G'.



A graphical illustration

There are more facts about 3-connected planar graphs:

1. They have a unique embedding in the plane/sphere (up to homomorphism)
2. They have a embedding in the plane with all edges are straight line segment <u>and</u> all faces are convex polygons.
3. They are exactly the stretch of polygons.

2.5 Kuratowski's Theorem

In this section we will study Kuratowski's Theorem, there are 2 versions of Kuratowski's Theorem, one in terms of **minor** and one in terms of **topological minor**. Minor is stronger than topological minor: every topological minor is a minor, but not vice versa.

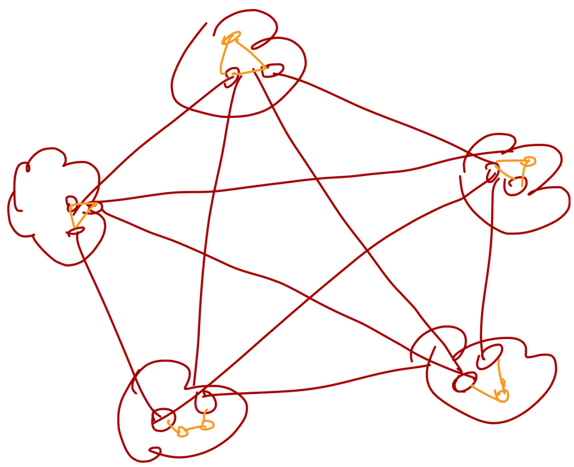
Definition 2.4: Minor

A graph H is a **minor** of a graph G if H can be botained from a subgraph G' of G by a sequence of edge-contractions.

A few notes:

- H is a minor of G iff H is obtained from G by vertex deletion, edge deletion, and edge-contractions.
- H is a minor of G iff there is a "model" of H in G
  - vertices of H: disjoint connected subgraphs of G.

  – edges of H: edges of G between subgraphs

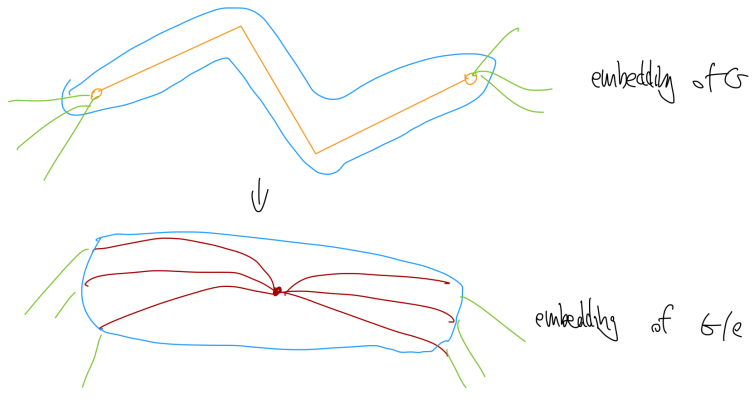


Proposition 2.13

If G is planar, and $H \leq G$, then H is planar.

Proof:

Since subgraphs of planar graphs are planar, it is enough to show that contracting a single edge in a planar graph keeps it planar.



(How to contract edges)

□

The forward direction of Kuratowski's Theorem is obvious:

Corollary 2.14

If G has $K_{3,3}$ or K_5 as a **minor**, G is non-planar.

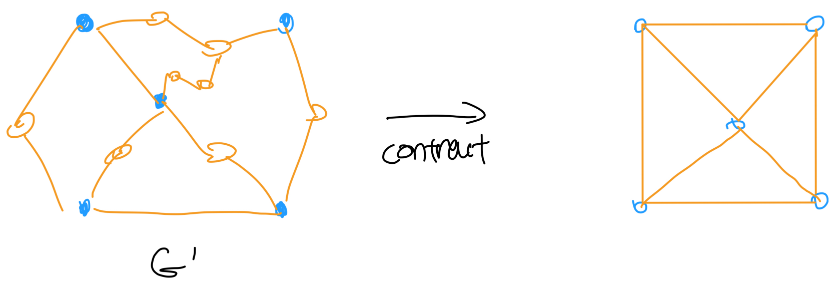
To prove the backward direction, we need the following proposition

Proposition 2.15

If G has H as a topological minor, then G has H as minor.

Proof:

For each edge e of H, let P_e be the corresponding path of G. Let G' be the subgraph of G that is the union of all P_e.
Now H is obtained from G' by contracting all but one edge in each path P_e:



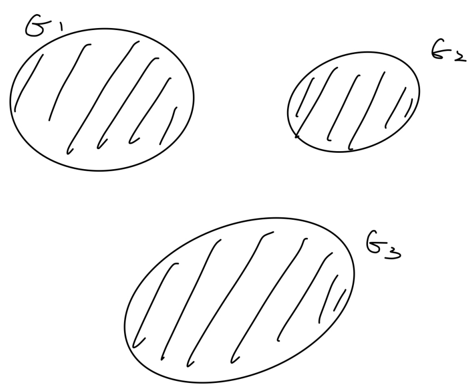
□

Now we give the proof

Theorem 2.16: Kuratowski' Thoerem

If G has no $K_{3,3}$-minor or K_5-minor, then G is planar.

The idea of proof is by contradiction to pick a minimal counter-example, with the observation that the example must have one component by minimality.

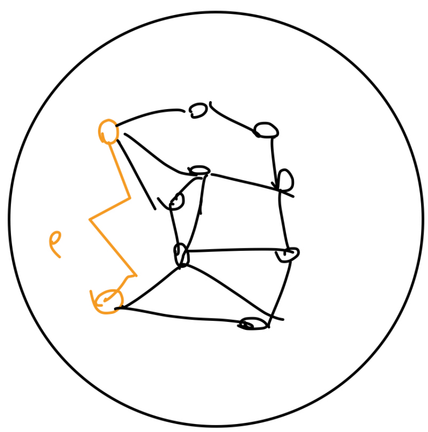


Proof:

Suppose for contradiction that G has no $K_{3,3}$ or K_5 minor, but is nonplanar. Choose G to have as few edges as possible, that is, $|V(G)| + |E(G)|$ is as small as possible.

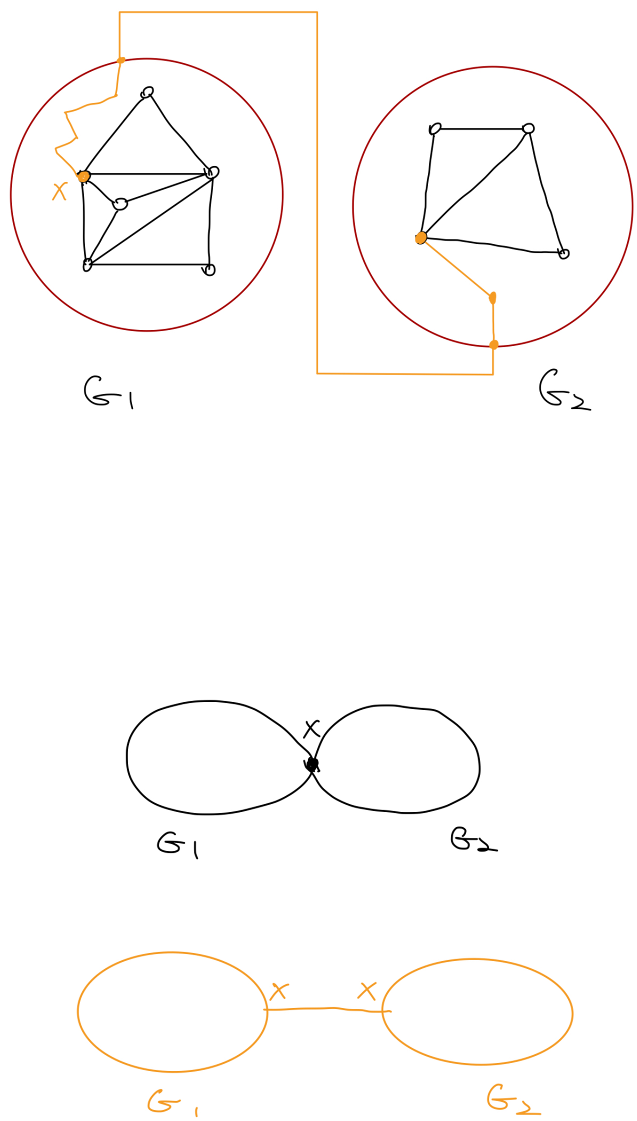
Claim 1: G is connected.
Suppose not, let $G_1, \ldots, G_k$ be its components. Since there are ≥ 2 components, we have $|V(G_i)| + |E(G_i)| < |V(G)| + |E(G)|$, so none of the G_i has $K_{3,3}$ or K_5 as a minor. Therefore, by the minimality in the choice of G, all G_i are planar. Then we can combine planar embedding of G_i to make a planar embedding of G, giving a contradiction.

To continue, we use (but not prove) the following:
Claim 2: For any embedding φ of G and any edge e (or vertex v) of G, and any disc $D \subseteq \mathbb{R}^2$, there is an embedding φ' of G such that $\varphi'(G) \subseteq D$, and e (or v) is contaiend in the boundary of the unbounded face (outside face) of φ'.



We want to show that every graph with no $K_{3,3}$-minor or K_5-minor is planar. We chose G to be a minimal counterexample. We already proved that G is connected.

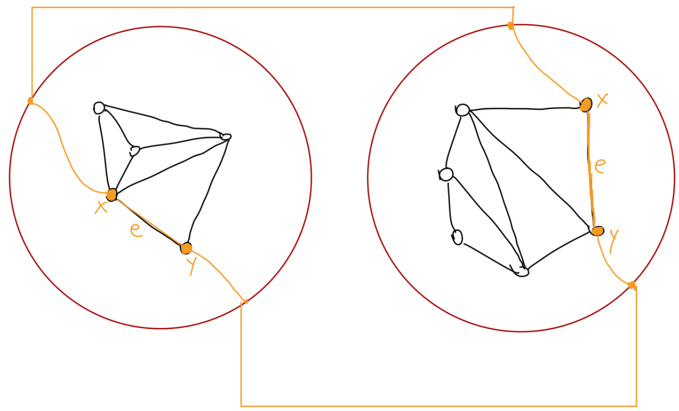
Claim 3: G is 2-connected.
If not, then G has a cut vertex x. Let G_1, G_2 be a proper subgraphs of G such that $G = G_1 \cup G_2$, and $V(G_1) \cap V(G_2) = \{x\}$. Since both G_i are smaller than G, and have no $K_{3,3}$ or K_5 minor, both are planar. Consider embeddings of G_1, G_2 in disjoint discs D_1, D_2 in the plane where x is embedded by both in the boundary of the outer face (true by Claim 2). We can find an arc e between the two copies of x in the resulting drawing to get an embedding of a graph G' such that $G'/e \cong G$ for some edge. Since G' is planar, and $G \cong G'/e$, G is also planar, a contradiction.



A graphical illustration

Claim 4: G is 3-connected.

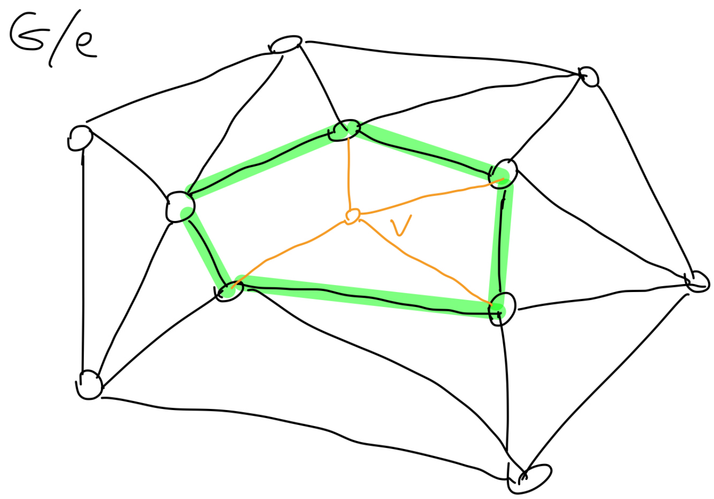
Suppose not, then there are vertices x, y of G and subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{x, y\}$. By a similar argument to the previous claim, G_1, G_2 are planar.



Let G_1' and G_2' be obtained from G_1, G_2 respectively by adding a new edge f from x to y (choose a vertex w of $G_2 - \{x, y\}$ and take a $(w, \{x, y\})$-fan to get this path, disclaimer: applying fam lemma to G, not G_2 because G_2 is not necessarily 2-connected). Since G is 2-connected, there is a path P with ≥ 2 edges in G_2 from x to y, now G' is obtained from the subgraph $G_1 \cup P$ by contracting all but one edge of P, since $K_{3,3}, K_5 \not\leq G$, and G_1' is a minor of G with fewer vertices with $K_{3,3}, K_5 \not\leq G$, so G_1' is planar. Similarly, G_2' is planar. Now consider drawings of G_1', G_2' in disjoint discs in $\mathbb{R}^2$, where e is on the outer face. We can now combine these drawing and use connectedness of the unbounded face to obtain a planar drawing of the following graph G'. So G' is planar, so $G = G' \backslash \{e_1, e_2\} / \{f_1, f_2\}$ is planar (by contracting), a contradiction.

Now we have $K_{3,3}, K_5 \not\leq G$, G nonplanar, 3-connected. For every $e \in E(m)$, G/e and $G \backslash e$ are planar (because they are minor of G, so have no $K_{3,3}, K_5$-minor, and they are smaller than G, so are not counterexample).

Claim 5: G is simple.
If not, delete an edge e parallel to some other edge, draw $G \backslash e$, and add e back to the drawing.

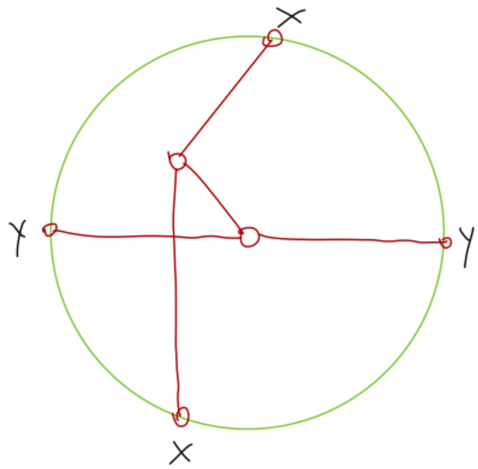


Also note that, since G nonplanar, $|V(G)| \geq 4$. By lemma from (much) earlier, G has an edge $e = xy$ such that G/e is 3-connected. We also know that G/e is planar. Let u be the vertex of G/e corresponding to e, and consider a planar embedding φ of G/e.
$(G/e) - v$ is a 2-connected planar graph, so every face bounding is a cycle, so every face bounding is a cycle, now v is embedded in some face of $(G/e) - v$ where bondary is a cycle C, and all nbrs of v in G/e lie in C.
We need a lemma

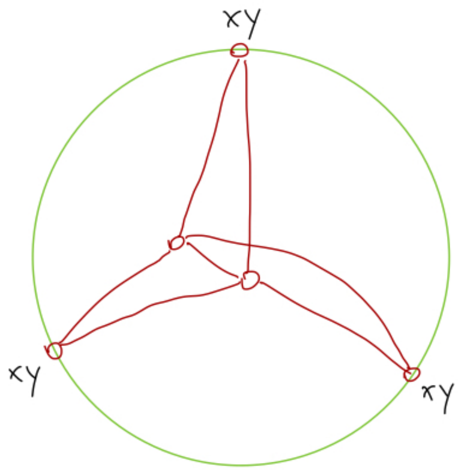
Lemma 2.17

Given sets X, Y of vertices in a cycle C, either

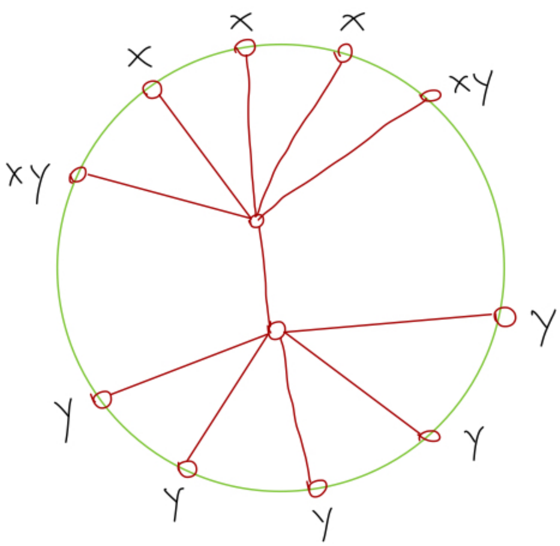
1. there exists $x, x' \in X$, $y, y' \in Y$ such that y, y' are in different components of $C - \{x, x'\}$.
2. there are paths P_x, P_y of C such that $E(P_x) \cap E(P_y) = \emptyset$, $P_x \cup P_y = C$, and $X \subseteq V(P_x), Y \subseteq V(P_y)$.
3. $|X \cap Y| \geq 3$



$\Rightarrow$ G has $K_{3,3}$-minor



$\Rightarrow$ G has K_5-minor



$\Rightarrow$ G planar

(There are only 3 possible cases)

Proof:

We may assume by symmetry that $|X| \leq |Y|$.

If $|X| \leq 1$, choose P_x to be a path with one edge containing all vertices in X, and choose P_y to be $C - p_x$, then (2) holds.

So $|X| \geq 2$. If $Y \backslash X = \emptyset$, then $X = Y$, suppose this holds, if $|X| = |Y| = 2$, then let $\{a, b\} = X = Y$, then choose P_x and P_y to be the two distinct ab-path in C. Now (2) holds.

Otherwise $|X| = |Y| \geq 3$, so $|X \cap Y| = |X| \geq 3$, so (3) holds.

So we may assume there exists $b \in Y \backslash X$. Since $b \notin X$, C is 2-connected, and $|X| \geq 2$, there is a bX-fan in C of size 2. Let P_1, P_2 be the paths in this fan. Let $P_y = P_1 \cup P_2$, since P_1, P_2 form a fan, P_y has no internal vertices in X. Let P_x be the other path in C between the ends of P_y. Since P_y has no internal vertices in X, we know that $X \subseteq V(P_x)$.

If $Y \subseteq V(P_y)$, then (2) holds. Otherwise, there is some $b' \in Y$ in a different component of $C-$ {ends of P_x}, so (1) holds. $\square$

(Back to Kuratowski) Let $X = \{\text{nbrs of } x \text{ in } C\}$, and $Y = \{\text{nbrs of } y \text{ in } C\}$. We now apply the lemma to X, Y and C.

If (3) holds, then x, y have 3 common neighbors a, b, c in C (in G). Now the vertices a, b, c, x, y are the terminals of a topological K_5-minor of G. Therefore G has a K_5-minor, a contradiction.

If (1) holds, then there exists a, b, a', b' (in that order) around C such that a, a' are neighbors of x, and b, b' are neighbors of Y. Now x, y, a, b, a', b' are the terminals of a topological $K_{3,3}$-minor of G, so G has $K_{3,3}$ as a minor ,a contradiction.

If (2) holds, we use the fact that for any polygon $C \subseteq \mathbb{R}^2$ with vertices in cyclic order $a_1, \ldots, a_t$, and any x in the interior of C, we can find arcs $A_1, A_2, \ldots, A_t$ from x to the a_i, intersecting only at x, and leaving x in the same cyclic order as the a_i occur around C (the proof is by inductivly draw the arcs one by one). Using the lemma, construct a panar embedding of G as follows:

1. take the drawing pf $G/e - u$ we were considering
2. add u back, and use the lemma to construct arcs from u to all vertices in $X \cup Y$.
3. let D be a small disc centred at u within D, split u into two vertices x, y, and use straight line segments to alter the drawing of G/e to a drawing of G. This contradicts the nonplanarity of G.

$\square$

The quickest way to say Kuratowski thoerem is: $K_{3,3}$ and K_5 are the **excluded minors** for planarity $\Leftrightarrow$ $K_{3,3}$ and K_5 are the unique minor-minimal nonplanar graphs.

There are other interesting theorems

Theorem 2.18

G is toroidal iff G does not have S (finite set of graphs, ≥ 16000) as a minor

Theorem 2.19

G is linkless-embeddable in $\mathbb{R}^3$ iff G does not contain S (set of graphs) as a minor.

The topological Kuratowski is equivalent to the Kuratowski theorem:

Proposition 2.20

For a graph G, the following are equivalent:

1. $K_{3,3}$ or K_5 is a minor of G
2. $K_{3,3}$ or K_5 is a topological minor of G

This follows from 3 statements:

1. For all H, if H is a topological minor of G, then H is a minor of G (proved already)
2. For all H of max degree ≤ 3 (e.g. $K_3, 3$), if H is a minor of G, then H is a topological minor of G. (A3Q5)
3. If G has K_5 as a minor, it has K_5 or $K_{3,3}$ as a topological minor. (A3Q5)

Also, one can adapt one proof of Kuratowski to show that every planar graph can be drawn with all edges as straight line segments.

3 Matchings

3.1 Terminology

Recall that

Definition 3.1: Matchings

A matching in a graph G is a set M of edges of G so that no 2 edges share an end.

In planarity section, we didn't state graph has to be simple because having multigraph can make our life easier, but it is not the case for matchings. From now on, we will assume that all our graphs are simple in this chapter.
We also talked about covers in MATH 239:

Definition 3.2: Covers

A cover of G is a set $W \subseteq V(G)$ so that every edge of G has an end in W.

Observe that if M is a matching of G, and W is a cover of G, then

$$|M| \leq |W|$$

because each edge in M has an end in W, and no two have a common end. This observation leads to following corollaries:

Corollary 3.1

If M is a matching and w is a cover such that

$$|M| = |W|$$

then W contains exactly one end of each edge in M, and no other vertices

Corollary 3.2

If $\nu(G)$ is the size of maximum matching of G, and $\tau(G)$ is the size of a minimum cover of G, then $\nu(G) \leq \tau(G)$.

3.2 König's Theorem

The famous König's theorem on bibartitle graph states that $\nu(G) = \tau(G)$. The above corollary shows that one direction holds for arbitrary G. The proof we showed for König' theorem is a bit sneaky, before that, we give some useful lemmas.

Proposition 3.3

If C is an even cycle on n vertices, then $\nu(C) = \tau(C) = n$

(insert figure here) Idea of proof: {every other edge} and {every other vertices} are a matching and a cover respectively, each of size n.

Proposition 3.4

If P is a path on n vertices, then

$$\nu(P) = \tau(P) = \lfloor \frac{n}{2} \rfloor$$

Proof:

If P has vertices $v_1, \ldots, v_n$, then

- $\{v_2, v_4, \ldots, v_{2\lfloor \frac{n}{2} \rfloor}\}$ is a cover
- $\{v_1 v_2, v_3 v_4, \ldots\}$ is a matching

both with size $\lfloor \frac{n}{2} \rfloor$. □

However, note that statement fails for odd cycles because $\tau(C_{2n+1}) = n + 1$, $\nu(C_{2n+1}) = n$.

Proposition 3.5

If $G_1, \ldots, G_k$ are the components of G, then

$$\nu(G) = \sum_{i=1}^{k} \nu(G_i)$$

and

$$\tau(G) = \sum_{i=1}^{k} \tau(G_i)$$

Proof:

Easy. □

Now we can prove the König's theorem:

Theorem 3.6: König's Theorem

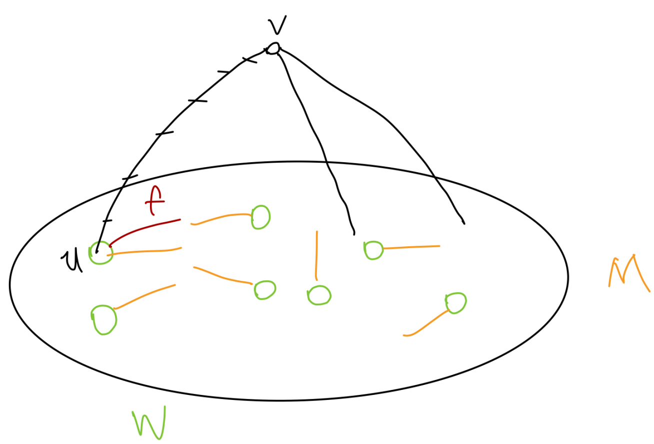
If G is bipartite, then $\nu(G) = \tau(G)$.

That is, there is a matching M and a cover W such that $|M| = |W|$.

Proof:

We need to show that $\tau(G) \leq \nu(G)$ for bipartite G. Let G be a counterexample on as few edges as possible.

Claim: G has a vertex of degree ≥ 3.
If not, then every component is a path or a cycle, so König's thoeorem holds for each component, so it holds for G, since τ and ν are additive over components.



Let u be a vertex of degree ≥ 3, and v be a neighbor of u. We split into cases depending on whether $\nu(G - v) = \nu(G)$.
If $\nu(G - v) \leq \nu(G) - 1$, then let W_0 be a min vertex cover of $G - v$, since $G - v$ is not a counterexample, we know that

$$|W_0| = \nu(G - v) \leq \nu(G) - 1$$

Since W_0 is a cover of $G - v$, $W_0 \cup \{v\}$ is a cover of G, so

$$\tau(G) \leq |W_0 \cup \{v\}| \leq (\nu(G) - 1) + 1 = \nu(G)$$

This contradicts that G being a counterexample.
Otherwise, $\nu(G - v) = \nu(G)$. In other words, each maximum matching of $G - v$ is also a

maximum matching of G. Let M be a maximum matching of both $G - v$ and G. Since $\deg(u) \geq 3$, there is an edge f incident with u but not v, such that $f \notin M$. Since $f \notin M$, the set M is a matching of $G - f$, so

$$\nu(G - f) \geq |M| = \nu(G) \geq \nu(G - f)$$

so $\nu(G-f) = |M|$. Since $|E(G-f)| < |E(G)|$, we know that $\tau(G-f) = \nu(G-f) = |M|$. Let W be a cover of $G - f$ with $|W| = |M|$. We know that W contains exactly one end of edge in M, and nothing else. In particular, $v \notin W$. Since W is a cover, it contains at least one end of the edge uv, so we must have $u \in W$. By choice of W, W contains an end of every edge in $G - f$ and since $u \in W$, W also contains an end of f. Therefore, W is a cover of G. So

$$\tau(G) \leq |W| = |M| = \nu(G)$$

which contradicts $\tau(G) > \nu(G)$. □

König's Theorem can be thought of in different ways. For bipartite G:

1. Either G has a t-edge matching, or there is a good reason it doesn't, a cover of size $< t$.

2. There is a maximum matching M of G, either with a cover W (a certificate or witness) of G that "proves" there is no larger matching.

3.3 Perfect Matching

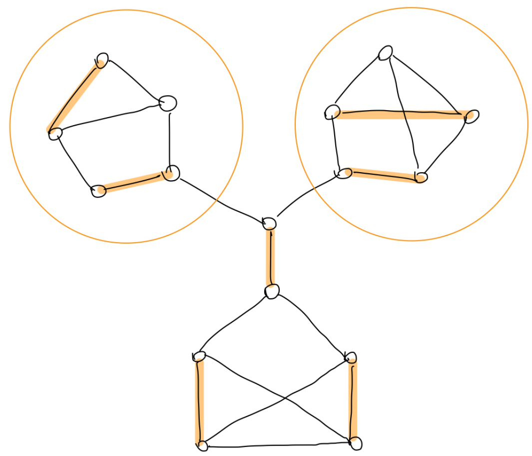
Definition 3.3: Perfect Matching

A perfect matching is a matching M such that $|M| = \frac{1}{2}|V(G)|$.

Note that $\nu(G) \leq \lfloor \frac{V(G)}{2} \rfloor$. The wrong conjecture one can make is every 3-regular graph has a perfect matching. Let's illustrate a counter example: consider the following 3-regular graph with 16 vertices, where one vertex is the cut vertex. Note that there are 3 bridges connecting the cut vertex, so exactly one of these 3 bridges has to be contained in a perfect matching. However, if one bridge is in the macthing, the two bridges connecting two other 5-vertex components can't have a perfect matching given their number of vertices is odd. So this is a counterexample of 3-regular graph without perfect matching. Although it is true that

Theorem 3.7: Petersen Theorem (1891)

Every bridgeless 3-regular graph has a perfect matching.



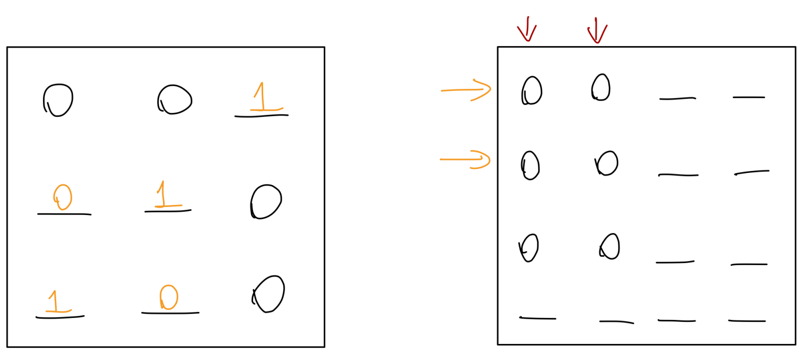
Counterexample: 3-regular graph with 16 vertices

Conjecture 1970: There exists $\beta > 0$ such that every 3-regular bridgeless graph has $\leq (1+\beta)|V(G)|$ perfect matching.

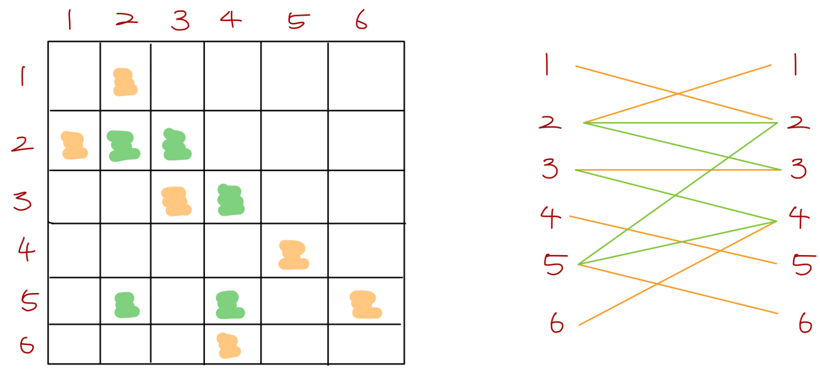
Theorem 3.8: Keperet, King, Kardos, Norine, Kral (2012)

Conejecture 1970 is true with $\beta \geq 0.0001$.

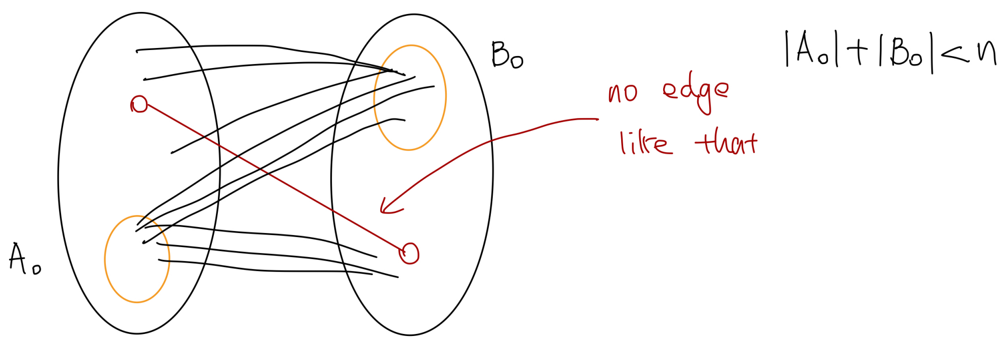
Q: Given an $n \times n$ matrix where some entries are given to be zero, can we fill in the other entries so that the matrix has nonzero determinant?



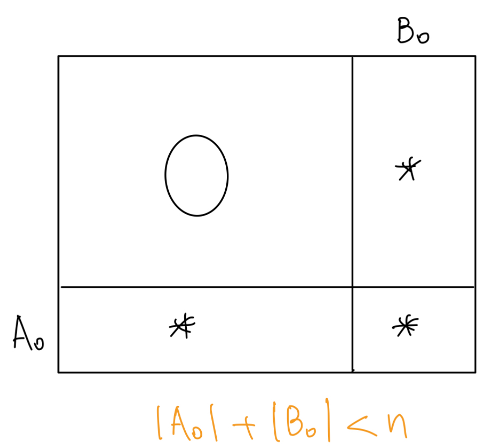
Observation: If there exists $a_1, a_2, \ldots, a_n$ permutation of $\{1, \ldots, n\}$ such that entries i, a_i are allowed to be nonzero, then the answer is yes. Encoding matrix as a bp graph with both sides of size n, we see that if the graph has a perfect matching, answer is yes.



By König' Theorem, if there is no perfect matching, then there is cover in G of size $< n$.



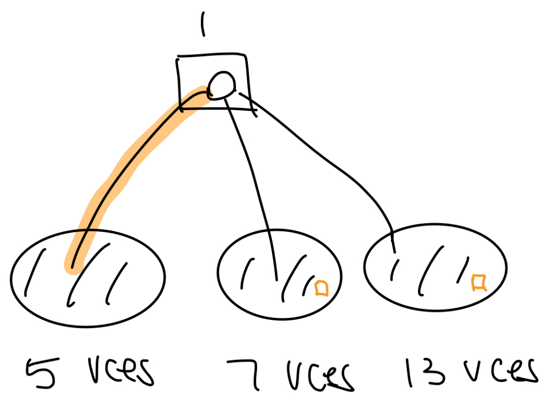
Let A_0 and B_0 be the subset of covers in two partitions of G. Every edge has one end in A_0 or one end in B_0. Converting the bipartition graph back to matrix, we can swap order of vertices so that the matrix looks like the following:



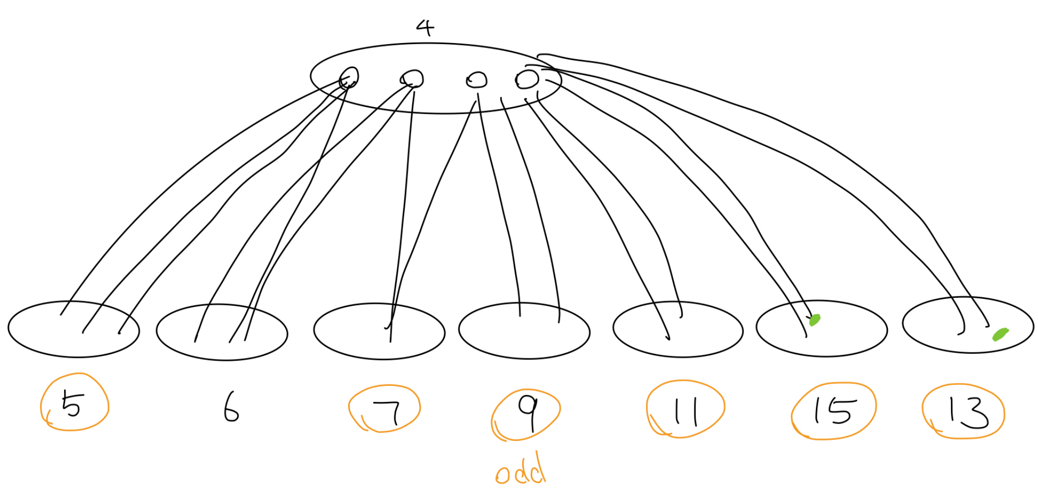
Since $|A_0| + |B_0| < n$, we can make argument about the rank of vertices showing that the system is unsolvable, so no such $a_1, a_2, \ldots, a_n$ exists.

3.4 Tutte-Berge Formula

We continue with our discussion about 3-regular graphs. Now consider a more general graph



where a vertex connecting to 3 compoennts of odd vertices. At most one component can has an edge connecting the cut vertex and a vertex in that component, and thus two other components do not have perfect macthings. Similarly,



the above graph has separator of size 4, and 6 components of odd number of vertices, so at least 2 compoennts do not have an edge connecting one of their vertices and one of 4 vertices, and thus can't have a perfect matching.

Idea: If G has a "small" set of vertices whose deletion gives a graph with a "large" number of odd components, then matchings in G cannot be too big.

Definition 3.4: M-saturated vertices

Given a matching M of G, the vertices of G that are an end of an edge M are called **M-saturated** vertices. Otherwise, the vertices are unsaturated (aka **M-exposed**).

We say M **saturates** its saturated vertices and **avoids** its unsaturated vertices.

Proposition 3.9

If M is a matching of G and X is a set of vertices of G, then there are at least

$$\mathrm{oc}(G - X) - |X|$$

M-unsaturated vertices in G where $\mathrm{oc}(G - X) = \#$ of components of $G - X$ with an odd number of vertices.

The bound the proposition gives is often useless (when $\mathrm{oc}(G - X) \leq |X|$)

Proof:

Let $\mathcal{C}$ be the set of odd components of $G - X$ that contains a vertex that is matched by M to a vertex in X. Since no two edges of M have the same end in X, there are at most $|X|$ edges of M from X to $V - X$, so at most $|X|$ components of $G - X$ contains a vertex matched by M to a vertex in X. Therefore, $|\mathcal{C}| \leq |X|$. So, there are at least $\mathrm{oc}(G - X) - |\mathcal{C}| \geq \mathrm{oc}(G - X) - |X|$ odd components of $G - X$ that contain no verex matched to anything in X.

For each such component H, no edge of M has exactly one end in H, so the number of saturated vertices in H is even. Since H is odd, it must contain ≥ 1 unsaturated vertex. There are $\geq \mathrm{oc}(G - X) - |X|$ different H, G has this many M-unsaturated vertices.

□

For every set $X \subseteq V$ and matching M, there are $\geq \mathrm{oc}(G - X) - |X|$ M-saturated vertices.

Corollary 3.10

If there is a set X such that $\mathrm{oc}(G - X) > |X|$ then G has no perfect matching.

Proof:

By the bound, every mathcing avoids $\geq \mathrm{oc}(G - X) - X > 0$ vertices, so there is no perfect mathcing.

□

Corollary 3.11

If $X \subseteq V(G)$, then

$$\nu(G) \leq \frac{1}{2}(|V| - \text{oc}(G - X) + |X|)$$

Proof:

Let M be a matching of G. Then there are at least $\text{oc}(G - X) - X$ M-saturated vertices, so there are at most $|V| - \text{oc}(G - X) + |X|$ saturated vertices.
Therefore

$$|M| \leq \frac{1}{2}(|V| - \text{oc}(G - X) + |X|)$$

This holds for all M, so we get the bound on ν. $\square$

Corollary 3.12

If $X \subseteq V(G)$ and M is a mathcing of size $\geq \frac{1}{2}(|V| - \text{oc}(G - X) + |X|)$, then

- every odd component of $G - X$ has a macthing (i.e. M) saturating all but one vertex
- exactly $|X|$ odd components of $G - X$ contain a vertex matched to a vertex of X
- every even component of $G - X$ has a perfect matching (contained in M).

Corollary 3.13

$$\nu(G) \leq \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| - \text{oc}(G - X) + |X|)$$

(true because the RHS is an upper bound for every X).

Tutte-Berge Formula
Idea: Try to understand the matching # of graphs with the property that deleting any one vertex doesn't change the matching #. Generalize this to make it work for general graph.

Definition 3.5: Hypomatchable

A graph H is hypomatchable if H is connected, and $\nu(H - v) = \nu(H)$ for all $v \in V(H)$.

Proposition 3.14

If G is a hypomatchable graph, then H has an odd number of vertices, and $\nu(H) = \frac{1}{2}(|V(H)| - 1)$ (H has a matching saturating all but one vertex).

Proof:

Define a relation $\sim$ on $V(H)$ by $u \sim v$ iff $u = v$ or $\nu(H - \{u, v\}) < \nu(H)$.

Claim 1 and 2: $\sim$ is symmetric and reflexive.
Easy.

Claim 3: If u, v are adjacent in H, then $u \sim v$.
Since uv is an edge, $\nu(H) \geq \nu(H - \{u, v\}) + 1$ because, for each matching M of $H - \{u, v\}$, $M \cup \{u, v\}$ is a matching of H. Therefore $\nu(H) > \nu(H - \{u, v\})$.

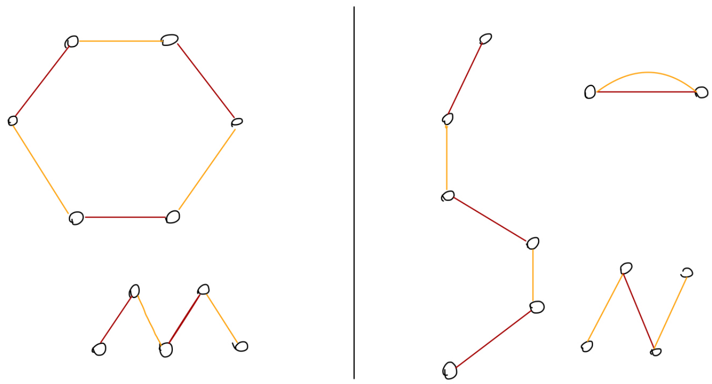
Claim 4: If $u \sim v$, $v \sim w$, then $u \sim w$.
Suppose $u \sim v$, $v \sim w$, $u \not\sim w$. Since H is hypomatchable, there is a max matching M_v that avoids v. Since $u \not\sim w$, we have $\nu(G - \{u, w\}) = \nu(G)$, so there is a maximum matching M_{uw} avoiding both u and w.
Idea: Analyze the structure of M_{uw} and M_v to find either

- A larger matching (contradicting maximality)
- A matching avoiding v and one of u and w (contradicting $u \sim v$ or $v \sim v$ or $v \sim w$).

Since no matching avoids v and one of u and w, we must have that M_v saturates u and w. Similarly, M_{uw} saturates v. Consider the subgraph H_0 of H with $V(H_0) = V(H)$, and

$$E(H_0) = (E(M_{uw}) \cup E(M_v)) \backslash E(M_{uw}) \cap E(M_v) = E(M_{uw}) \triangle E(M_v)$$

Since the subgraph on M_{uw} and M_v have max degree ≤ 1, H_0 has maximum degree ≤ 2, so every component is a path or a cycle. Since no vertex is incident with two edges in the same matching, the paths must alternate between edges in M_{uw} and egdes in M_v.(figure) If some path component of H_0 has an odd number of edges, then it contains more edges in one matching than in the other. Let M be the matching in the path with fewer edges than the other, and the larger matching be M'. Replacing the dges in $M \cap E(P)$ with the edges in $M' \cap E(P)$ gives a larger matching than M in H, contradicting the maximality of M. So every path of H_0 has an even number of edges.



(in H_0) (not in H_0)

Since M_{uw} saturates v but not u or w, and M_v saturates u and w but not v, each of u, v, w has degree 1 in H_0. Since every component of H_0 is a path or a cycle, each of u, v, w is an end of a path component in H_0. So there is some path component P of H_0 having one end in $x \in \{u, v, w\}$ and where other end is not in $\{u, v, w\}$. Let $M \in \{M_{uw}, M_v\}$ be the matching that saturates x, and M' be the other matching. Taking M removing the edges in $M \cap E(P)$, and adding back the edges in $M' \cap E(P)$ gives a matching that saturates strictly fewer vertices in $\{u, v, w\}$ than M does. In all three cases ($x = u, x = v, x = u$), this contradicts either $u \sim v$ or $v \sim w$.

We show that $u \sim v$ for all $u, v \in V(H)$. Let $u = x_1, x_2, \dots, x_k = v$ be a uv-path. Let i be the maximal such that $u \sim x_i$. If $i = k$ we have the result (we know i makes sense, since $u \sim u = x_1$).

If $i < k$, then we know $u \sim x_i$ and $x_i \sim x_{i+1}$ (they are adjacent) so $u \sim x_{i+1}$ by (4). This contradicts the maximality of i.

Now consider a maximum matching M of H. Since $\nu(H - x) = \nu(H)$ for all x, M cannot be a perfect matching, so $2|M| \leq |V(H)| - 1$. If there are two M-unsaturated vertices u, v, then $\nu(H - \{u, v\}) \geq |M| = \nu(H)$, this contradicts $u \sim v$. Therefore,

$$2|M| \geq |V(H)| - 1$$

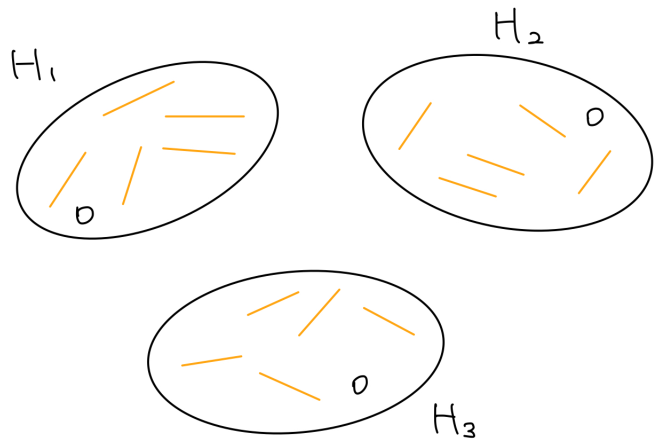
Therefore, $\nu(H) = |M| = \frac{1}{2}(|V(H)| - 1)$ and $|V(H)|$ is odd as required. $\square$

Corollary 3.15

If H is any graph such that $\nu(H-x) = \nu(H)$ for all x, then every component of H is odd, and $\nu(H) = \frac{1}{2}(|V(G)| - \text{oc}(H))$.

Proof:

Let $H_1, \ldots, H_k$ be the components of H. We argue that each H_t is hypomatchable.



Let $x \in V(H_t)$,

$$\begin{aligned}
\nu(H-x) &= \sum_{i=1, i\neq t}^{k} \nu(H_i) + \nu(H_t - x) \\
&= \sum_{i=1}^{k} \nu(H_i) + \nu(H_t - x) - \nu(H_t) \\
&= \nu(H) + \nu(H_t - x) - \nu(H_t)
\end{aligned}$$

so $\nu(H_t - x) = \nu(H_t)$, so H_t is hypomatchable. So each H_i has an odd number of vertices, and $\nu(H_i) = \frac{1}{2}(|V(H_i)| - 1)$. Thus

$$\begin{aligned}
\nu(H) = \sum_{i=1}^{k} \nu(H_i) &= \sum_{i=1}^{k} \frac{1}{2}(|V(H_i)| - 1) \\
&= \frac{1}{2}\sum_{i=1}^{k}(|V(H_i)|) - \frac{1}{2}k
\end{aligned}$$

$$= \frac{1}{2}(|V(H)| - \mathrm{oc}(H))$$

$\square$

Now we can prove the Tutte-Berge Formula

Theorem 3.16: Tutte-Berge Formula

$$\nu(G) = \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| - \mathrm{oc}(G - X) + |X|)$$

(equivalently, there exists $X \subseteq V(G)$ such that $\nu(G) = \frac{1}{2}(|V(G)| - \mathrm{oc}(G - X) + |X|)$).

Proof:

We proved if H is a graph such that $\nu(H - v) = \nu(H)$ for all $v \in V(H)$, then every component of H is hypomatchable, and $\nu(H) = \frac{1}{2}(|V(H)| - \mathrm{oc}(H))$.

Let G be a graph, and let $X \subset V(G)$ be maximal such that $\nu(G - X) = \nu(G) - X$ (This is well-defined because $X = \emptyset$ satisfies this condition). If there is a vertex u in $G - X$ such that $\nu(G - X - u) < \nu(G - X)$, then

$$\nu(G - (X \cup \{u\})) < \nu(G - X) = \nu(G) - |X|$$

so $\nu(G - (X \cup \{u\})) \leq \nu(G) - |X \cup \{u\}|$, but also $\nu(G - X \cup \{u\}) \geq \nu(G - X) - 1 = \nu(G) - |X \cup \{u\}|$ (because deleting one vertex drops ν by ≤ 1). So $\nu(G - \{X \cup \{u\}\}) = \nu(G) - |X \cup \{u\}|$, which contradicts the fact that X is maximal. Therefore, we have that $\nu((G - X) - u) = \nu(G - X)$ for all $u \in \nu(G - X)$. By the prop, we have $\nu(G - X) = \frac{1}{2}(|V(G - X)| - \mathrm{oc}(G - X))$. So

$$\begin{aligned}
\nu(G) &= \nu(G - X) + |X| \\
&= \frac{1}{2}(|V(G - X)| - \mathrm{oc}(G - X)) + |X| \\
&= \frac{1}{2}(|V(G)| + |X| - \mathrm{oc}(G - X))
\end{aligned}$$

We proved earlier that $\nu(G) \leq \frac{1}{2}(|V(G)| + |Y| - \mathrm{oc}(G - Y))$ for all Y, so we have found some particular X where equality holds. Thus,

$$\nu(G) = \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| + |X| - \mathrm{oc}(G - X))$$

as desired. $\square$

3.5 Results of Tutte-Berge Formula

Definition 3.6: Berge Witness

A **Berge Witness** is a set $X_0 \subseteq V(G)$ such that $\nu(G) = \frac{1}{2}(|V(G)|+|X_0|-\mathrm{oc}(G-X_0))$.

Our proof for Tutte-Berge formula showed that if X is a maximal set such that $\nu(G-X) = \nu(G) - |X|$, then X is a Berge Witness.

Theorem 3.17: Tutte's Theorem

G has a perfect matching iff $|\mathrm{oc}(G-X)| \leq |X|$ for all $X \subseteq V(G)$.

Proof:

G has a perfect matching iff $\frac{1}{2}|V(G)| = \nu(G) = \min_{X \subseteq V(G)} \frac{1}{2}(|V(G)| - |X| - \mathrm{oc}(G - X))$ iff $\frac{1}{2}|V(G)| = \min_{X \subseteq \frac{1}{2}V(G)}(|V(G)| + |X| - \mathrm{oc}(G-X))$ iff $\frac{1}{2}|V(G)| = \frac{1}{2}|V(G)| + \min_{X \subseteq V(G)} \frac{1}{2}(|X| - \mathrm{oc}(G-X))$ iff $\min_{X \subseteq V(G)}(|X| - \mathrm{oc}(G-X)) = 0$.
If $\mathrm{oc}(G-X) \leq |X|$ for all X, then this minimum is clearly equal to zero (it is always non-negative, but is also ≤ 0, by choosing $X = \emptyset$). Conversely, if the minimum is zero, then clearly $|X| - \mathrm{oc}(G-X) \geq \min_{X \subseteq V(G)}(|X| - \mathrm{oc}(G-X)) = 0$ for all X, so $|X| \geq \mathrm{oc}(G-X)$.

□

Now, we can back to the 3-regular graphs discussed before.

Theorem 3.18: Petersen Theorem (1891)

If G is a 3-regular connected graph with no cut edge (bridge), then G has a perfect matching.

Proof:

By Tutte's Theorem, we need to show that $|\mathrm{oc}(G-X)| \leq |X|$ for all $X \subseteq V(G)$. Suppose that this fails, so there is some $X \subseteq V(G)$ such that $G - X$ has more than $|X|$ odd components.

Claim: For each odd component H of $G - X$, there are at least 3 edges from X to H.
Proof: If there is only one edge from X to H, then H is a component of $G - e$, so e is a cut edge, a contradiction. If there are two edges e, f from X to H, then H is a component of $G - \{e, f\}$, and in $G - \{e, f\}$, the degress of vertices in H sum to $3|V(H)| - 2$, which is odd. This contradicts handshaking.

So number of edges from $G - X$ to $X \geq 3(\#$ odd components of $G - X) > 3|X|$ by the choice of X. But since G is 3-regular, # edges from $G - X$ to X is $\leq 3|X|$, a contradiction.

□

Proposition 3.19

For $n \equiv 2(\text{mod} 4)$, there is an $(\frac{n}{2} - 1)$-regular graph with no perfect matching,

Proof:

$K_{\frac{n}{2}} + K_{\frac{n}{2}}$ □

Proposition 3.20

For n odd, there is graph with min degree $\frac{n-1}{2}$, and no perfect matching.

Proposition 3.21: Folklore

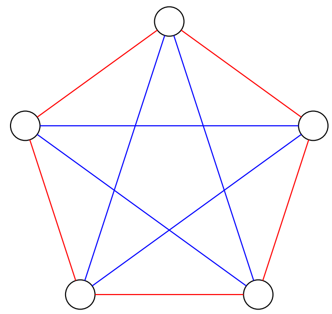
If $|V(G)|$ is even, every vertex of G has degree $\geq \frac{1}{2}|V(G)|$ then G has a perfect matching.

Proof:

Let M be a largest matching. Since $|V(M)|$ is even, if M is not a perfect matching, then there exists $u, v \notin V(M)$. If there is some $xy \in M$ such that there are ≥ 3 edges from uv to xy, make M bigger. So for each $e \in M$, there are ≤ 2 edges from uv to e. So # edges from uv to M is $\leq 2|M|$. But $\deg(u), \deg(v) \geq \frac{n}{2}$ so there are n edges from uv to M. So $2|M| \geq$ # edges from uv to $M \geq n$ so $|M| \geq \frac{n}{2}$, so M is perfect. □

4 Guest Talk

Your're planning a party and you want to make sure either 3 attendeces know each other or 3 attendees do not know each other.
Q: How many people do you need to invite?
Is 5 enough?



Is 6 enough?
The answer is yes. K_4 is a subgraph of K_6.

Definition 4.1

Let $m, n \in \mathbb{N}$, $R(m, n) =$ minimum int r (if it exists) such that every coloring of the edges of K_r with red and blue has either

1. a K_m with all edges red or
2. a K_n with all edges blue

In the above example, we showed that $R(3, 3) = 6$.
Observations:

1. $R(n, m) = R(m, n)$
2. $R(1, n) = 1$
3. $R(2, n) = n$

Proposition 4.1

$R(3, 4) \leq R(2, 4) + R(3, 3) = 4 + 6 = 10$.

Proof:

If there are $\geq R(2, 4)$ red, we find a blue K_4 or a red K_2 which makes a red K_3 with x.
If $\geq R(3, 3)$ blue, we find a red K_3 or a blue K_3 which makes a blue K_4 with x. $\square$

Proposition 4.2

$R(m,n) \le R(m-1,n) + R(m,n-1)$.

Proof:

Replace 3's with m's and 4's with n's and use reduction. $\square$

Theorem 4.3: Ramsey's Theorem (1930)

for all $m,n \in \mathbb{N}$, $R(m,n)$ exists.

$R(n) = R(n,n) \leftarrow$ diagonal Ramsey #'s.

Theorem 4.4

Let $m,n \ge 2$. Then $R(m,n) \le \binom{m+n-2}{m-1}$

Proof:

$R(2,n) = n = \binom{n}{1} = \binom{2rn-2}{2-1}$. Assume $m,n \ge 3$. Suppose not true for $m,n \in \mathbb{N}$ where $m+n$ is minimum. $R(m-1,n) \le \binom{m+n-3}{m-2}$, $R(m,n-1) \le \binom{m+n-3}{m-1}$. By proposition, $R(m,n) \le \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}$. $\square$

Theorem 4.5

If $n \ge 2$, then $R(n) > 2^{n/2}$.

Theorem 4.6: Campos, Griffiths, Morris, Sahasrabudhe (March 2023)

$R(n) \le 3.992^n$ for any sufficiently large n.

5 William Tutte

Theorem 5.1: Tutte's Convex Embedding Theorem

Every 3-connected graph has a planar drawing in which every interio face is a convex polygon.

Definition 5.1: Spring Embedding

Let C be a cycle in a graph G. A spring embedding of (G, C) is a mapping $\varphi : V \to \mathbb{R}^2$ s that

1. The vertices in C are mapped to a prescribed convex polygon.
2. Each vertex outside C is mapped to the average position (barycentre) of its neighbours.

The system $\varphi(v) = \frac{1}{\deg(v)} \sum_{u \sim v} \varphi(u) : v \in V - C$ is a pair of system of $|V - C|$ linear equations in $|V - C|$ unknowns. The coefficient matrix is a principal submatrix of the Laplacian matrix $L(G)$, so is known to be nonsingular. Therefore, spring embeddings are unique.