Chapter 5

Exponential coordinate representation

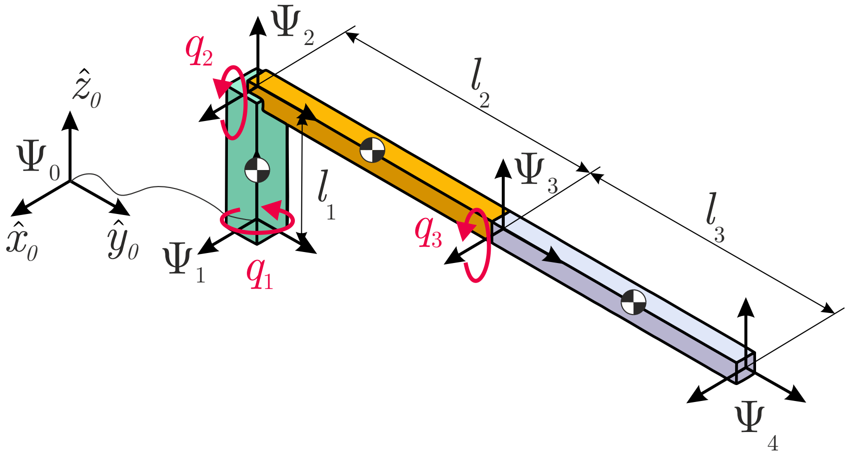

Figure 5.1: Spatial robot with three rotational joints

In this section, we introduce the main concepts for the mathematical modeling of an abstract spatial 3R robot's kinematics using Lie Groups and Screw Theory.

The spatial robot manipulator is shown in Fig. 5.1. The coordinate system Ψ_0 is a frame fixed is space and each coordinate system Ψ_i is rigidly connected to a link i. The first link with length l_1 rotates about the z_0 axis of Ψ_0; the second and the third links with lengths l_2 and l_3, respectively, rotate about the y_0 of Ψ_0.

Consider a simple vector linear ordinary differential equation (ODE) in the following form

$$\dot{x}(t) = Ax(t), \tag{5.1}$$

where $x(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$ is constant, and the initial condition $x(0) = x_0$ is given. The equation (5.1) has a solution

$$x(t) = e^{At}x(0).$$

The equation for angular velocity $\tilde{w}_i^{j,j}$ of a rigid body (4.2) can be treated as differential equation if it is constant for a moment of time that is considered

$$\tilde{w}_i^{j,j} = \dot{R}_i^j R_j^i \implies \dot{R} = \tilde{w}_i^{j,j} R_i^j, \tag{5.2}$$

The solution of eq. (5.2) is the solution of ODE eq. (5.1):

$$R_i^j(t) = e^{\tilde{w}_i^{j,j}t} R_i^j(0), \tag{5.3}$$

where $R_i^j(t)$ indicates the rotation matrix in the current state, $R_i^j(0) = I$ indicates the initial conditions. Thus, the rotation matrix can be calculated using an exponent with the degree of the skew-symmetric angular velocity matrix

$$R_i^j(t) = e^{\tilde{w}_i^{j,j}t} \iff R_i^j(q) = e^{\tilde{w}_i^{j,j}q} R_i^j(0), \tag{5.4}$$

where $R_i^j(t)$ and $R_i^j(q)$ are notation of rotation matrix in time and joint configuration.

Proposition 7. *For a given angular velocity $\tilde{w} = \tilde{\hat{w}}\theta$ such that θ is scalar angle of rotation and $\tilde{\hat{w}}$ is a skew-symmetric representation of a unit vector of angular velocity $\hat{w} \in \mathbb{R}^3$, the rotation matrix is equal to matrix exponential of $\tilde{w} \in so(3)$*

$$R = e^{\tilde{w}} = I + \tilde{w}\sin\theta + \tilde{w}^2(1 - \cos\theta). \tag{5.5}$$

Equation (5.5) is known as Rodrigues' formula for rotations. It is a tool to move from the Lie algebra $\tilde{w} \in so(3)$ to the Lie group $e^{\tilde{w}} \in SO(3)$.

Similarly, a Twist of a body (4.6) can be considered also as a linear differential equation

$$\dot{H}_i^j = \tilde{T}_i^{j,j} H_i^j. \tag{5.6}$$

where $\tilde{T}_i^{j,j}$ is a constant twist in matrix form. The solution of (5.6) is also a solution of ODE (5.1):

$$H_i^j(t) = e^{\tilde{T}_i^{j,j} t} H_i^j(0), \tag{5.7}$$

where $H_i^j(t)$ indicates the homogeneous transformation matrix in the current state, $H_i^j(0) = I$ indicates the initial conditions. Thus, the homogeneous transformation matrix also can be calculated using an exponent with the degree of the twist in matrix form

$$H_i^j(t) = e^{\tilde{T}_i^{j,j} t} \iff H_i^j(q_i) = e^{\tilde{\hat{T}}_i^{j,j} q_i} H_i^j(0), \tag{5.8}$$

where $H_i^j(t)$ and $H_i^j(q)$ are notation of the homogeneous transformation matrix in time and joint configuration.

Proposition 8. *For a given twist $\tilde{T} = \tilde{\hat{T}}\theta$ such that is $\tilde{\hat{T}}$ twist and $\theta \in \mathbb{R}$ is any distance traveled along the axis, the homogeneous transformation matrix is equal to matrix exponential of $\tilde{T} \in se(3)$*

$$H = e^{\tilde{T}} = e^{\begin{pmatrix} \tilde{w} & v \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} e^{\tilde{w}} & \frac{1}{||w||^2}\left((I - e^{\tilde{w}})(w \wedge v) + w^\top v w\right) \\ 0 & 1 \end{pmatrix} \tag{5.9}$$

Equation (5.9) is a tool to move from Lie algebra $\tilde{T} \in se(3)$ to the Lie groups $e^{\tilde{T}} \in SE(3)$.

Proposition 9. *Combining the chain rule concepts (4.5) together with matrix exponential (5.8) we introduce Brockett' Product of Exponentials (PoE) formula to describe the direct kinematics of open chains*

$$H_n^0(q_1, q_2, \dots q_n) = e^{\tilde{\hat{T}}_1^{0,0} q_1} e^{\tilde{\hat{T}}_2^{0,1} q_2} \dots e^{\tilde{\hat{T}}_n^{0,(n-1)} q_n} H_n^0(0), \tag{5.10}$$

where H_n^0, that depends on joints' state $q_1, q_2, \dots q_n$, describes configuration of a link with a frame Ψ_n with respect to a fixed frame Ψ_0, $H_n^0(0)$ indicates initial configuration.

The deriving of expressions for exponential coordinates of a homogeneous transformation and Brockett' Product of Exponentials formula can be found in number of books in references, i.e. [5].

5.1 Direct kinematics of a 3R robot

We are ready to consider a direct kinematic task for a spatial 3R robot, depicted in Fig. 5.1. The direct (or forward) kinematics of a robot refers to calculating its end-effector frame's position and orientation from its joint coordinates. Besides calculating end-effector configuration, we need to calculate direct kinematics for centers of masses (CoM) for all the links.

To solve the direct kinematic task for the end-effect and links' CoMs, we are going to use Brockett' Product of Exponentials (PoE) formula (5.10). We need to find screw axes $\hat{T}_i \in \mathbb{R}^6$ for all the joints and initial configuration of a robot, since they are necessary ingredient for the PoE formula.

The 3R robot's first link with a frame Ψ_1 rotates around $\hat{z}_0$ axis, while the second Ψ_2 and the third Ψ_3 rotate rotates around $\hat{x}_0$ axis. The unit angular velocities as follow

$$\hat{w}_1^{0,0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \hat{w}_2^{0,1} = \hat{w}_3^{0,2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

From Mozzi's decomposition theorem (4.11) we know how to calculate the linear velocity at the initial link's position using the vector product of the radius vector from Ψ_0 to Ψ_i and relevant unit angular velocity vector

$$v_i = r_i \wedge \hat{w}_i = -\hat{w}_i \wedge r_i = -\tilde{\hat{w}}_i r_i.$$

where $\tilde{\hat{w}}_i$ is the angular velocity in skew-symmetric matrix form. The radius vectors r_i for the three links respectively

$$r_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; r_2 = \begin{pmatrix} 0 \\ 0 \\ l_1 \end{pmatrix}, \; r_3 = \begin{pmatrix} 0 \\ l_2 \\ l_1 \end{pmatrix}.$$

Thus the desired Twists

$$\hat{T}_i^{0,i-1} = \begin{pmatrix} \hat{w}_i^{0,i-1} \\ v_i^{0,i-1} \end{pmatrix}$$

are found with the following linear velocities v_i in the initial position for all links

$$v_1^{0,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; v_2^{0,1} = \begin{pmatrix} 0 \\ l_1 \\ 0 \end{pmatrix}, \; v_3^{0,2} = \begin{pmatrix} 0 \\ l_1 \\ -l_2 \end{pmatrix}.$$

The following homogeneous transformation matrices indicate the initial configurations for the links' CoM c_i for each link i and the end-effector

$$H_{c_1}^0(0) = \begin{pmatrix} I^{3\times3} & \begin{matrix} 0 \\ 0.5l_1 \\ 0 \end{matrix} \\ 0^{1\times3} & 1 \end{pmatrix}, H_{c_2}^0(0) = \begin{pmatrix} I^{3\times3} & \begin{matrix} 0.5l_2 \\ l_1 \\ 0 \end{matrix} \\ 0^{1\times3} & 1 \end{pmatrix},$$

$$H_{c_3}^0(0) = \begin{pmatrix} I^{3\times3} & \begin{matrix} l_2 + 0.5l_3 \\ l_1 \\ 0 \end{matrix} \\ 0^{1\times3} & 1 \end{pmatrix}, H_4^0(0) = \begin{pmatrix} I^{3\times3} & \begin{matrix} l_2 + l_3 \\ l_1 \\ 0 \end{matrix} \\ 0^{1\times3} & 1 \end{pmatrix}.$$

To derive the dynamics equation on the later steps, we need to calculate the configurations of the end-effector's frame Ψ_4 and the links' CoMs c_i with respect to the frame fixed in space Ψ_0 using the Brockett' PoE formula (5.10):

$$H_{c_1}^0(q_1, q_2) = e^{\tilde{\hat{T}}_1^{0,0} q_1} H_{c_1}^0(0),$$

$$H_{c_2}^0(q_1, q_2) = e^{\tilde{\hat{T}}_1^{0,0} q_1} e^{\tilde{\hat{T}}_2^{0,1} q_2} H_{c_2}^0(0),$$

$$H_{c_3}^0(q_1, q_2, q_3) = e^{\tilde{\hat{T}}_1^{0,0} q_1} e^{\tilde{\hat{T}}_2^{0,1} q_2} e^{\tilde{\hat{T}}_3^{0,2} q_3} H_{c_3}^0(0),$$

$$H_4^0(q_1, q_2, q_3) = e^{\tilde{\hat{T}}_1^{0,0} q_1} e^{\tilde{\hat{T}}_2^{0,1} q_2} e^{\tilde{\hat{T}}_3^{0,2} q_3} H_4^0(0).$$

Matrix exponential from the Brockett' PoE formula can be calculated using the equation (5.9)

$$e^{\tilde{\hat{T}}_1^{0,0} q_1} = \begin{pmatrix} R_1^0 & (Iq_1 + (1 - \cos q_1)\tilde{\hat{w}}_1^{0,0} + (q_1 - \sin q_1)(\tilde{\hat{w}}_1^0)^2) v_1^{0,0}(0) \\ 0 & 1 \end{pmatrix},$$

$$e^{\tilde{\hat{T}}_2^{0,1} q_2} = \begin{pmatrix} R_2^1 & (Iq_2 + (1 - \cos q_2)\tilde{\hat{w}}_2^{0,1} + (q_2 - \sin q_2)(\tilde{\hat{w}}_2^1)^2) v_2^{0,1}(0) \\ 0 & 1 \end{pmatrix},$$

$$e^{\tilde{\hat{T}}_3^{0,2} q_3} = \begin{pmatrix} R_3^2 & (Iq_3 + (1 - \cos q_3)\tilde{\hat{w}}_3^{0,2} + (q_3 - \sin q_3)(\tilde{\hat{w}}_3^2)^2) v_3^{0,2}(0) \\ 0 & 1 \end{pmatrix},$$

where R_i^j is rotation matrix that describe orientation of link with fixed frame Ψ_i with respect to a frame Ψ_j, q_i is angular or linear displacement regarding the joint type, $\tilde{\hat{w}}_i^{0,i-1}$ is the unit angular velocity is skew-symmetric form, $v_i^{0,i-1}(0)$ is the initial configuration of the vector $v_i^{0,i-1}$ in initial configuration.

5.2 Differential Kinematics

In the previous subsection we have considered how to calculate the robot end-effector frame's position and orientation for a given set of joint positions. In this subsection we discuss the relationship between the Cartesian velocities of the end-effector and joint velocities of a serial 3R robot via the Jacobian. There are two ways to derive the differential kinematics: by inspection to get the **geometric Jacobian** or by taking the time derivative to get the most commonly known **analytic Jacobian**.

These are two fundamentally different ways to derive the differential kinematics. To derive the geometric Jacobian, we need to look at the robot's geometrical structure, which is modeled using the Lie groups and Screw Theory. On the other side, the analytic Jacobian is derived by taking the time-derivative of the joint coordinates, which are found by differentiating the direct kinematics of the manipulator.

Here we use the first way, which is based on the geometric approach. The geometric Jacobian can be split into two different kinds: **space Jacobian** J_s and **body Jacobian** J_b. The body Jacobian describes the links velocity with respect to its body frame, and the spatial Jacobian describing the links spatial velocity with respect to the base frame.

Proposition 10. *The Jacobian in coordinates of a frame fixed in space, or more simply the* ***space Jacobian*** $J_s \in \mathbb{R}^{6\times n}$ *relates the joint rate vector* $\dot{q} \in \mathbb{R}^n$ *to the end-effector's spatial twist* $T_n^{0,0}$

$$T_n^{0,0} = J_s(q)\dot{q}. \tag{5.11}$$

The number of rows of the Space Jacobian corresponds to the twist dimension, and the number of columns is equal to the number of links n

$$J_s(q) = \begin{pmatrix} T_{s_1} & T_{s_2} & \dots & T_{s_n} \end{pmatrix}, \tag{5.12}$$

where ith column of $J_s(q)$ is the ith joint twist, mapped to the current configuration of the serial-manipulator

$$T_{s_i} = T_i^{0,i-1} = Ad_{H_{i-1}^0} \hat{T}_i^{(i-1),(i-1)},$$

for $i = 2, \dots, n$ with the first column $T_{s_1} = \hat{T}_1$ equal to the screw axis of the first link, where $Ad_{H_{i-1}^0}$ is the adjoint (4.8).

Proposition 11. *The Jacobian in coordinates of a frame fixed with a body, or more simply the* ***body Jacobian*** $J_b \in \mathbb{R}^{6\times n}$ *relates the joint rate vector* $\dot{q} \in \mathbb{R}^n$ *to the end-effector's spatial twist* $T_n^{n,0}$

$$T_n^{n,0} = J_b(q)\dot{q}.$$

As the body Jacobian J_b as well as the space Jacobian J_s describe the instantaneous Cartesian velocity of the end-effector just from a different reference frame, it is possible to derived body Jacobian from the space Jacobian. This means that the space and body Jacobians can be related to each other with an Adjoint transformation

$$J_b(q) = Ad_{H_0^n(q)} J_s(q),$$

for

$$T_n^{n,0} = Ad_{H_0^n(q)} T_n^{0,0}.$$

From the definition of a Twist eq. (4.7) and chain rule for homogeneous transformation matrix $SE(3)$ eq. (4.5) a chain rule for Twists can be derived

$$T_n^{0,0} = T_1^{0,0} + T_2^{0,1} + \dots + T_n^{0,(n-1)}.$$

Multiplying the angular velocity vector of the links with the geometric Jacobian, we get the Twist of the end-effector

$$\dot{q} \xrightarrow{J(q)} T_n^{0,0}$$

and it is possible to map a Wrench into joints torques, which can be described with the help of the transposed Jacobian

$$\tau \xleftarrow{J^\top(q)} (W^n)^\top.$$

For the spatial open chain 3R robot the space geometrical Jacobian is the following

$$J(q) = \begin{pmatrix} T_1 & T_2 & T_3 \end{pmatrix},$$

where the columns equal to

$$T_1 = \hat{T}_1^{0,0},$$

$$T_2 = Ad_{e^{\tilde{\hat{T}}_1^{0,0} q_1} e^{\tilde{\hat{T}}_2^{0,1} q_2}} \hat{T}_2^{1,1},$$

$$T_3 = Ad_{e^{\tilde{\hat{T}}_1^{0,0} q_1} e^{\tilde{\hat{T}}_2^{0,1} q_2} e^{\tilde{\hat{T}}_3^{0,2} q_3}} \hat{T}_3^{2,2}.$$

The space Jacobians for all three links

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & \cos q_1 & 0 \\ 0 & \sin q_1 & 0 \\ 1 & 0 & 0 \\ 0 & -l_1 \sin q_1 & 0 \\ 0 & l_1 \cos q_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; J = J_3 = \begin{pmatrix} 0 & \cos q_1 & \cos q_1 \\ 0 & \sin q_1 & \sin q_1 \\ 1 & 0 & 0 \\ 0 & -l_1 \sin q_1 & -\sin q_1 (l_1 + l_2 \sin q_2) \\ 0 & l_1 \cos q_1 & \cos q_1 (l_1 + l_2 \sin q_2) \\ 0 & 0 & -l_2 \cos q_2 \end{pmatrix}. \tag{5.13}$$

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