Chapter 4

Rigid body spatial Kinematics

Multibody modeling using Lie Groups and Screw Theory and control are well presented in the literature [5, 3, 2, 4, 1, 7, 6]. Many books on Lie Groups and Screw Theory focus on fundamental theory and details of the calculation, resulting in losing the elegant and straightforward geometric interpretation.

In Chapter 3 we give only an intuition of Lie Groups and Screw Theory modeling. This section aims to recall some elements of the modeling of mechanical systems that are needed to construct mathematical models of the mechanical behavior of multibody systems. This section contains a brief description of the necessary ingredients to model mechanisms comprised of a finite number of rigid links interconnected by ideal joints: rotation matrices in $SO(3)$, homogeneous transformation matrices in $SE(3)$, twists and wrenches, and show how it all can be used to derive control algorithms for robots.

A rigid body or link is described as a compact volume of point masses, all of which have fixed relative distances. An ideal joint is described as a constraint between two rigid links that allow only specific relative motions and prevent others, independently of the forces and torques applied to the links [2].

4.1 Rotation

Angular position

Consider a physical point p in a 2 dimensional **Euclidean space** $p \in \varepsilon(2)$ (Fig. 4.1). In order to work with numbers, we need to express the physical point p in some **coordinate system** Ψ_i to get numbers describing p^i. Fig. 4.1 shows two coordinate systems Ψ_1 and Ψ_2 rotated with respect to each other of an angle θ, $o = o_1 = 0_2$ is the **origins** of the coordinate systems. Let us consider a pure rotation around o in the plane and derive what will be called **rotation matrices**, which happen to belong to the **matrix special orthogonal group** $SO(2)$.

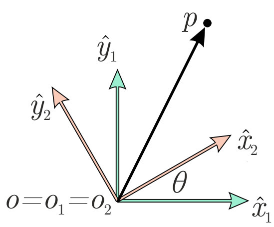

Figure 4.1: Planar rotation

The coordinate x_i of the p expressed in the frame Ψ_i is equal to the **inner product** of vector $(p - o_i)$ with the unit vector $\hat{x}_i$

$$x_i = \langle (p - o), \hat{x}_i \rangle \in \mathbb{R},\ \forall i.$$

The same is also true for coordinate y_i of the p^i

$$y_i = \langle (p - o), \hat{y}_i \rangle \in \mathbb{R},\ \forall i.$$

Now the task is to describe a change of coordinates of the rotated frames. For the physical point p expressed in the coordinate system Ψ_1 we have that

$$p^1 := \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \langle (p - o_1), \hat{x}_1 \rangle \\ \langle (p - o_1), \hat{y}_1 \rangle \end{pmatrix},$$

where p^1 is a representation of the physical point p in frame Ψ_1, x_1 and y_1 are coordinates of the point p in frame Ψ_1, o_1 is the origin expressed in Ψ_1, $\hat{x}_1$ and $\hat{y}_1$ are unit vectors. Thus, the basis vectors $\hat{x}_1$ and $\hat{y}_1$ with coefficients $x_1, y_2 \in \mathbb{R}$ respectively forms a **linear combination**

$$(p - o_1) = \begin{pmatrix} x_1 & y_1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{y}_1 \end{pmatrix} = x_1\hat{x}_1 + y_1\hat{y}_1, x_1, x_2 \in \mathbb{R}\ \hat{x}_1, \hat{y}_1 \in \varepsilon_*(2).$$

The same physical point p can be presented in the second frame Ψ_2 as

$$p^2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \langle (p - o_2), \hat{x}_2 \rangle \\ \langle (p - o_2), \hat{y}_2 \rangle \end{pmatrix},$$

since $o_1 = o_2$ we have that $(p - o_2) = (p - o_1)$. Thus, we can derive the relation between expressions of the physical point in both frames Ψ_2 and Ψ_1 and using the linearity of the inner product we have

$$p^2 = \begin{pmatrix} \langle x_1\hat{x}_1 + y_1\hat{y}_1, \hat{x}_2 \rangle \\ \langle x_1\hat{x}_1 + y_1\hat{y}_1, \hat{y}_2 \rangle \end{pmatrix} = \begin{pmatrix} x_1 \langle \hat{x}_1, \hat{x}_2 \rangle + y_1 \langle \hat{y}_1, \hat{x}_2 \rangle \\ x_1 \langle \hat{x}_1, \hat{y}_2 \rangle + y_1 \langle \hat{y}_1, \hat{y}_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle \hat{x}_1, \hat{x}_2 \rangle & \langle \hat{y}_1, \hat{x}_2 \rangle \\ \langle \hat{x}_1, \hat{y}_2 \rangle & \langle \hat{y}_1, \hat{y}_2 \rangle \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = R_1^2 p^1,$$

where p^2 and p^1 are representations of a physical point p in frames Ψ_2 and Ψ_1 respectively, R_1^2 is called a **rotation matrix** that represent the change of coordinates between the frames from Ψ_1 to Ψ_2. In this particular example R_1^2 equals to

$$R_1^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

It can be easily shown, generalizing the procedure just shown, that for two right-handed coordinate frames Ψ_i and Ψ_j in 3 dimensional Euclidean space $p \in \varepsilon(3)$ we get the following

$$p^j = \begin{pmatrix} x_j \\ y_j \\ z_j \end{pmatrix} = \begin{pmatrix} \langle \hat{x}_i, \hat{x}_j \rangle & \langle \hat{y}_i, \hat{x}_j \rangle & \langle \hat{z}_i, \hat{x}_j \rangle \\ \langle \hat{x}_i, \hat{y}_j \rangle & \langle \hat{y}_i, \hat{y}_j \rangle & \langle \hat{z}_i, \hat{y}_j \rangle \\ \langle \hat{x}_i, \hat{z}_j \rangle & \langle \hat{y}_i, \hat{z}_j \rangle & \langle \hat{z}_i, \hat{z}_j \rangle \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = R_i^j p^i, \tag{4.1}$$

where R_i^j is a rotation matrix. It can be seen that, if frames are right-oriented and orthogonal ($\langle e_i, e_j \rangle = \delta_{ij}$), then $R^\top = R^{-1}$ and $\det R = 1$.

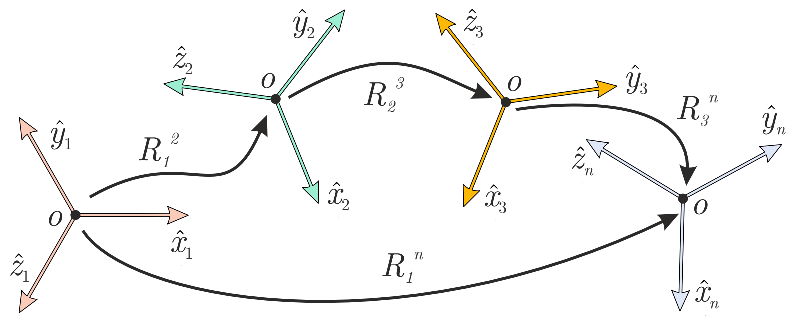

Figure 4.2: Chain rule

Proposition 1. *The set of 3×3 rotation matrices such that $R^{-1} = R^\top$ and det R=1 together with the matrix multiplication forms the* ***special orthogonal group*** *$SO(3)$:*

$$SO(3) := \{R \in \mathbb{R}^{3\times3}; R^{-1} = R^\top, detR = 1\}.$$

The set of 2×2 rotation matrices is a subgroup of $SO(3)$ and it is denoted $SO(2) := \{R \in \mathbb{R}^{2\times2}; R^{-1} = R^\top, \det R = 1\}$. The elements of $SO(2)$ represent planar orientations and the elements of $SO(3)$ represent spatial orientations. The sets of rotation matrices $SO(2)$ and $SO(3)$ with the matrix multiplication as operation are called **groups** since they satisfy the properties required for a mathematical group $(SO(3), \bullet)$:

- Closure: $\forall R_1, R_2 \in SO(3)\colon R_1R_2 \in SO(3)$;
- Associativity: $\forall R_1, R_2, R_3 \in SO(3)\colon (R_1R_2)R_3 = R_1(R_2R_3)$;

– Identity element existence: $\exists I \in SO(3) \quad \forall R \in SO(3)\colon IR = RI = R$;

– Inverse element existence: $\forall R \in SO(3) \quad \exists R^{-1} \in SO(3)\colon RR^{-1} = R^{-1}R = I$.

The rotation matrices are needed for representing an orientation, changing the reference frame, and as an operation of rotation of a vector or a frame. Fig. 4.2 show the chain rule that helps to describe the motion using a sequence of frames with the same origin o.

$$R^1_n = R^1_2 R^2_3 \ldots R^{n-1}_n,\ R^n_1 = {R^1_n}^\top = R^n_{n-1} \ldots R^3_2 R^2_1.$$

Angular velocity

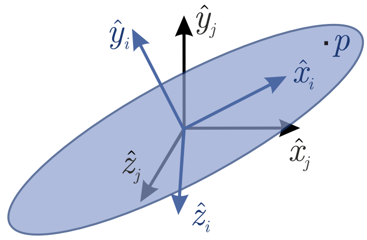

Figure 4.3: Rotation of a rigid body around $\hat{\omega}$

Let us consider a pure rotation. Fig. 4.3 shows a floating rigid body with attached fixed frame Ψ_i (unit axis $\hat{x}_i, \hat{y}_i$, and $\hat{z}_i$) and a physical point p fixed in Ψ_i. Considering eq. 4.1 it is possible to derive the position rate of the point p in the frame Ψ_j:

$$\dot{p}^j = (\dot{R^j_i p^i}) = \dot{R}^j_i p^i + R^j_i \dot{p}^i = \dot{R}^j_i p^i,$$

since $\dot{p}^i$=0. Taking into account that $p^i = R^i_j p^j$, we can come up with

$$\dot{p}^j = \dot{R}^j_i R^i_j p^j = \tilde{w}^{j,j}_i p^j, \implies \tilde{w}^{j,j}_i = \dot{R}^j_i R^i_j, \tag{4.2}$$

where the rotation matrix R^j_i describes the orientation of frame Ψ_i with respect to the Ψ_j frame, $\dot{R}^j_i$ is its time rate of change, and $\tilde{w}^{j,j}_i$ is the **skew-symmetric** representation of the angular velocity for a frame Ψ_i, with respect to Ψ_j, and expressed in Ψ_j.

Proposition 2. *If a rotation matrix $R(t) \in SO(3)$ is a differentiable function of time, in this case $\dot{R}R^\top$ and $R^\top\dot{R}$ are skew-symmetric and belonging to **Lie algebra** $so(3)$:*

$$so(3) := \{\tilde{w} \in \mathbb{R}^{3\times 3}\ s.t.\ -\tilde{w} = \tilde{w}^\top\}$$

In general case the angular velocity $\tilde{w}_a{}^{c,b}$ is given of for a body with fixed frame Ψ_a, with respect to the frame Ψ_b, and expressed in the frame Ψ_c. We have two natural choices to represent the angular velocity in the body frame Ψ_i or some frame of an another body Ψ_j (Table 4.1):

$$\exists w^{j,j}_i, w^{i,j}_i \in \mathbb{R}^3\ s.t.\ \tilde{w}^{j,j}_i = \dot{R}^j_i {R^j_i}^\top,\ \tilde{w}^{i,j}_i = {R^j_i}^\top \dot{R}^j_i.$$

To change the reference frame for angular velocity from Ψ_i to Ψ_j use the following

$$w^{i,j}_i = R^i_j w^{j,j}_i.$$

$\tilde{w}^{\bullet,\bullet}_{\bullet}$	$\dot{R}^j_i$	$\dot{p}^j$	Representation
$\tilde{w}^{j,j}_i := \dot{R}^j_i R^i_j$	$\dot{R}^j_i = \tilde{w}^{j,j}_i R^j_i$	$\dot{p}^j = (w^{j,j}_i \wedge R^j_i)p^i = \tilde{w}^{j,j}_i R^j_i p^i = \tilde{w}^{j,j}_i p^j$	Ψ_j frame
$\tilde{w}^{i,j}_i := R^i_j \dot{R}^j_i$	$\dot{R}^j_i = R^j_i \tilde{w}^{i,j}_i$	$\dot{p}^j = R^j_i(w^{i,j}_i \wedge p^i) = R^j_i(R^i_j \tilde{w}^{j,j}_i R^j_i)p^i = \tilde{w}^{j,j}_i p^j$	Ψ_i frame

Table 4.1: Two natural choices to express an angular velocity

4.2 General Motion

To describe any rigid body's orientation and position in 3-dimensional Euclidean space $\varepsilon(3)$ we need to introduce a homogeneous transformation matrix, an element the Special Euclidean group $SE(3)$.

Change of Cartesian coordinates in 2D

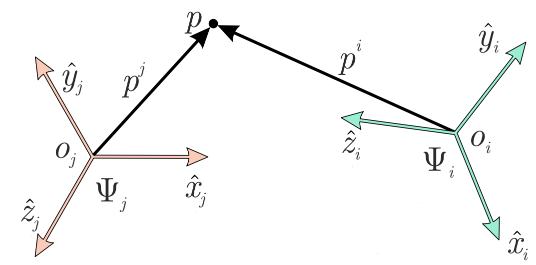

Figure 4.4: General planar motion of a physical point p

At the first step, let us consider a general planar motions to derive **homogeneous transformation matrices**, which belong to the **Special Euclidean group** $SE(2)$, and then at the second step proceed to the spatial case.

Fig. 4.4 shows a physical point p, which belongs to the 2-dimensional Euclidean space $p \in \varepsilon(2)$. It can be expressed in any coordinate frame. Here we consider two right-handed coordinate frames Ψ_i and Ψ_j. The task is to describe point p in both frames.

Suppose that the physical point p expressed in the frame Ψ_j can be described as function of p expressed in Ψ_i in as follows

$$p^j = Ap^i + B, \tag{4.3}$$

where A is some matrix, p^i is coordinate representation of the physical point p expressed in frame Ψ_i, and B is a numerical vector. We need to find both unknown parameters A and B.

To calculate B let's assume that point p is coincident with the origin o_i of the frame Ψ_i

$$p = o_i \implies p^i = o_i^i = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$p^j = Ap^i + B = A\begin{pmatrix} 0 \\ 0 \end{pmatrix} + B \implies B = o_i^j.$$

Thus, we get that $B = o_i^j$ is the position of the origin of frame Ψ_i expressed in frame Ψ_j.

Taking into account (4.3) let us calculate A. If we suppose $p = \hat{x}_i$, then $p^i = 1 \cdot \hat{x}_i + 0 \cdot \hat{y}_i = x_i^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$p^j = Ap^i + B \implies (p^j - B) = Ap^i \implies (x_i^j - B) = A_1,$$

where A_1 is the first column of A. If we suppose $p = y_i$, then $p^i = \hat{y}_i^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$(p^j - B) = Ap^i \implies (y_i^j - B) = A_2,$$

where is the second column of A. It could further be seen that if the frame are orthonormal we have

$$p^j = Ap^i + B \implies p^j = R_i^j p^i + o_i^j,$$

where $R_i^j \in SO(3)$ is a rotation matrix and $o_i^j \in \mathbb{R}^3$ is a position vector.

Change of Cartesian coordinates in 3D

We can consider a general change of **Cartesian coordinates** in 3-dimensional Euclidean space $\varepsilon(3)$. If we denote a vector $P \in \mathbb{R}^4$ define from in the following way $p \in \mathbb{R}^3$

$$p = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \implies P := \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix},$$

we can proceed to a general change of Cartesian coordinates in $\varepsilon(3)$ from a right handed frame Ψ_i to another right handed frame Ψ_j that can be expressed with what is called a **homogeneous transformation matrix** $H_i^j \in \mathbb{R}^{4\times 4}$

$$H_i^j := \begin{pmatrix} R_i^j & o_i^j \\ 0 & 1 \end{pmatrix}$$

$$p^j = R_i^j p^i + o_i^j \implies P^j = \begin{pmatrix} R_i^j & o_i^j \\ 0 & 1 \end{pmatrix} P^i = H_i^j P^i.$$

Proposition 3. *The set of 4×4 homogeneous transformation matrices $H \in \mathbb{R}^{4\times 4}$ together with the matrix multiplication forms the **Special Euclidean group** $SE(3)$:*

$$H_i^j := \left\{ \begin{pmatrix} R_i^j & p_i^j \\ 0 & 1 \end{pmatrix} s.t. R_i^j \in SO(3), p_i^j \in \mathbb{R}^3 \right\}, \tag{4.4}$$

where $R_i^j \in SO(3)$ is a rotation matrix and p_i^j is a vector in $\mathbb{R}^3$.

The set of homogeneous transformation matrices $SE(3)$ together with matrix multiplication is called a group since it satisfies the properties required of a mathematical group $(SE(3), \bullet)$:

- Closure: $\forall H_1, H_2 \in SE(3) \colon H_1 H_2 \in SE(3)$
- Associativity: $\forall H_1, H_2, H_3 \in SE(3) \colon (H_1 H_2) H_3 = H_1 (H_2 H_3)$;
- Identity element existence: $\exists I \in SE(3) \quad \forall H \in SE(3) \colon IH = HI = H$;
- Inverse element existence: $\forall H \in SE(3) \quad \exists H^{-1} \in SE(3) \colon HH^{-1} = H^{-1}H = I$.

It can be easily be proven that the transpose of a homogeneous transformation matrix is not equal to its inverse. The inverse homogeneous transformation matrix can be calculated using the formula

$$H_j^i = (H_i^j)^{-1} = \begin{pmatrix} (R_i^j)^\top & -(R_i^j)^\top o_i^j \\ 0 & 1 \end{pmatrix}.$$

The chain rule, thanks to the introduced construction and extension to 4-vectors, holds

$$H_n^1 = H_2^1 H_3^2 \dots H_n^{n-1},\ H_1^n = {H_n^1}^\top = H_{n-1}^n \dots H_2^3 H_1^2. \tag{4.5}$$

4.3 Twists & Wrenches

Twists

Consider a physical point p in the 3-dimensional Euclidean space $\varepsilon(3)$. It can be expressed in two random frames Ψ_i and Ψ_j and a homogeneous matrix H_i^j can be used for a change of Cartesian coordinates from P^i to P^j

$$P^j = H_i^j P^i.$$

$\tilde{T}_\bullet^{\bullet,\bullet}$	$\dot{H}_i^j$	$\dot{P}^j$	Representation
$\tilde{T}_i^{j,j} := \dot{H}_i^j H_j^i$	$\dot{H}_i^j = \tilde{T}_i^{j,j} H_i^j$	$\dot{P}^j = \tilde{T}(H_i^j P^i)$	Ψ_j frame
$\tilde{T}_i^{i,j} := H_j^i \dot{H}_i^j$	$\dot{H}_i^j = H_i^j \tilde{T} p_i^{i,j}$	$\dot{P}^j = H_i^j(\tilde{T} P^i)$	Ψ_i frame

Table 4.2: Two natural choices to express a twist

If the physical point P is fixed with respect to Ψ_i then we have that $\dot{P}^i = 0$ and time rate of change for P^j gives

$$\dot{P}^j = \dot{H}_i^j P^i + H_i^j \dot{P}^i = \dot{H}_i^j P^i,$$

where $P^k = (p^k \ 1)^\top, \ k = i, j, \ H_i^j \in SE(3)$.

Proposition 4. *If a homogeneous matrix $H(t) \in SE(3)$ is a differentiable function of time, it could be shown that then the products $\dot{H}H^\top$ and $H^\top \dot{H}$ belong to **Lie algebra** $se(3)$:*

$$se(3) := \left\{ \begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix} \ s.t. \ \Omega \in so(3), v \in \mathbb{R}^3 \right\}, \tag{4.6}$$

Therefore we define the generalization of velocity for a rigid body called **twist**, that have the following matrix form

$$\tilde{T}_i^{j,j} = \dot{H}_i^j H_j^i = \begin{pmatrix} \tilde{w}_i^{j,j} & v_i^{j,j} \\ 0 & 0 \end{pmatrix}, \tag{4.7}$$

$$\tilde{T}_i^{i,j} = H_j^i \dot{H}_i^j = \begin{pmatrix} \tilde{w}_i^{i,j} & v_i^{i,j} \\ 0 & 0 \end{pmatrix},$$

where $\tilde{T}_i^{j,j} \in se(3)$ is twist in matrix form for a body with the fixed frame Ψ_i, with respect to Ψ_j and expressed in Ψ_j, $\tilde{T}_i^{i,j} \in se(3)$ is the same twist but expressed in Ψ_i, $\tilde{w}_{\bullet}^{\bullet,\bullet} \in so(3)$ is the skew-symmetric matrix of the angular velocity of body, $v_i^{j,j} \in \mathbb{R}^3$ denotes the instantaneous lateral velocity of a point fixed in frame Ψ_i that passes through the origin of frame Ψ_j in respect of frame Ψ_j, and $v_i^{j,j} \in \mathbb{R}^3$ is the same velocity but in respect of frame Ψ_i (Table 4.2).

Proposition 5. *In a general case, the twist of a body fixed with a frame Ψ_i with respect to a frame Ψ_j and expressed in the frame Ψ_k can be represented as $T_i^{k,j} \in \mathbb{R}^6$ in a vector form*

$$T_i^{k,j} = \begin{pmatrix} w_i^{k,j} \\ v_i^{k,j} \end{pmatrix}$$

where $w_i^{k,j} \in \mathbb{R}^3$ is the angular velocity of body fixed with a frame Ψ_i relative to body fixed with a frame Ψ_j expressed in coordinate frame Ψ_k, and $v_i^{k,j} \in \mathbb{R}^3$ denotes the instantaneous lateral velocity of a point fixed in frame Ψ_i relative to frame Ψ_j that passes through the origin of frame Ψ_k.

Similar to the homogeneous matrix, describing the change of coordinates for two coordinate frames, a matrix called an **Adjoint matrix** $Ad_{H_j^k}$ of H_j^k can be used to change the coordinate frame in which the twist in vector form is expressed:

$$T_i^{k,j} = Ad_{H_j^k} T_i^{j,j} = \begin{pmatrix} R_j^k & 0 \\ \tilde{p}_j^k R_i^k & R_j^k \end{pmatrix} T_i^{j,j}. \tag{4.8}$$

For a change of the coordinate frames for twist $\tilde{T}_i^{k,j}$ in a matrix form we can use the following

$$\tilde{T}_i^{k,j} = H_l^k \tilde{T}_i^{l,j} H_k^l. \tag{4.9}$$

Remarks about Twists

Fig. 4.5 shows two floating rigid bodies. Let us consider a green body with attached frames Ψ_3 and Ψ_4. Since the body is considered as rigid we can assume H_3^4 is constant. The twist of Ψ_3 with respect to Ψ_1 expressed in the frame Ψ_1 is defined as

$$\tilde{T}_3^{1,1} = \dot{H}_3^1 H_1^3.$$

Consider the twist of Ψ_4 with respect to Ψ_1 expressed in the frame Ψ_1

$$\tilde{T}_4^{1,1} = \dot{H}_4^1 H_1^4 = (\dot{\overline{H_3^1 H_4^3}}) H_1^4 = (\dot{H}_3^1 H_4^3 + H_3^1 \dot{H}_4^3) H_1^4 =$$

$$\dot{H}_3^1 H_4^3 H_1^4 = \dot{H}_3^1 H_1^3 = \tilde{T}_3^{1,1} \implies \tilde{T}_4{}^{1,1} = \tilde{T}_3{}^{1,1}.$$

This shows that we can talk about the movement of the green body without taking into account which frames we use fixed to it since the twists $\tilde{T}_4^{1,1} = \tilde{T}_3^{1,1}$ are equal.

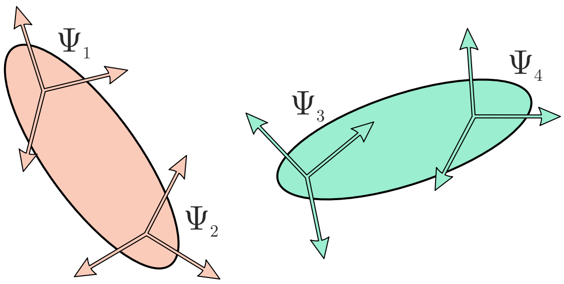

Figure 4.5: Two floating bodies

We can repeat that trick for twist of the frame Ψ_3 with respect to the frame Ψ_1 expressed in the frame Ψ_2 taking into account that H_1^2 is constant

$$\tilde{T}_3^{2,1} = H_i^2 \tilde{T}_3^{i,1} H_2^i = H_1^2 \tilde{T}_3^{1,1} H_2^1 = H_1^2 \dot{H}_3^1 H_1^3 H_2^1 = H_1^2 \dot{H}_3^1 H_1^3 H_2^1 =$$
$$= (\dot{H_1^2 H_3^1}) H_1^3 H_2^1 = \dot{H}_3^2 H_2^3 = \tilde{T}_3^{2,2}, \implies \tilde{T}_3^{2,1} = \tilde{T}_3^{2,2}.$$

This shows that the twists $\tilde{T}_3^{2,1} = \tilde{T}_3^{2,2}$ are equal.

Thus, we can consider the motion of any body without taking into account which frames we use fixed to it. To conclude, we use the following notation for a twist between rigid bodies A and B $T_A^{F,B}$, where F indicates a frame of reference.

Wrenches

Power is a scalar physical quantity equal to the amount of energy transferred or converted per unit of time. For a linear mechanical system, a power is the product of a force F applied to an object and its linear velocity v; for an angular mechanical system, a power is the product of a torque τ applied to an object and its angular velocity ω

$$\sum P = Fv + \tau\omega,$$

where F and τ are row vectors and v and ω are column vectors.

Let us consider a motion of a body with fixed frame Ψ_i with respect to Ψ_j expressed in a frame Ψ_k. A twist $T_i^{k,j}$ is the generalization of velocities that contains both $v_i^{k,j}$ and $\omega_i^{k,j}$. The wrench is analogous generalization that merges the forces F_i^k and torques τ_i^k into a single six-dimensional spatial force row vector.

Proposition 6. *In a general case, the wrench of a body fixed with a frame Ψ_i and expressed in the frame Ψ_k can be represented as $W_i^k \in \mathbb{R}^{1\times 6}$ in a vector form*

$$W_i^k = \begin{pmatrix} \tau_i^k & F_i^k \end{pmatrix},$$

where $\tau_i^k \in \mathbb{R}^{1\times 3}$ is a torque and $F_i^k \in \mathbb{R}^{1\times 3}$ denotes a force applied to a body fixed with a frame Ψ_i expressed in coordinate frame Ψ_k.

The product of Wrench and Twist equals power, if they are expressed in the same coordinate frame

$$P = W_i^k T_i^{k,j} = \begin{pmatrix} \tau_i^k & f_i^k \end{pmatrix} \begin{pmatrix} w_i^{k,j} \\ v_i^{k,j} \end{pmatrix} = \tau_i^k w_i^{k,j} + f_i^k v_i^{k,j}.$$

If more than one wrench acts on a rigid body with fixed frame Ψ_j, the total wrench on the body is simply the vector sum of the individual wrenches, provided the wrenches are expressed in the same frame Ψ_k

$$\sum_i^n {W_j^k}_i = {W_j^k}_1 + {W_j^k}_2 + \dots {W_j^k}_n.$$

Wrench with a zero linear component is called a pure torque, and Wrench with a zero angular component is called a pure force. A wrench expressed in frame Ψ_i can be converted to a different frame Ψ_j using a transpose adjoint $Ad_{H_i^j}^\top$ that can be easily be shown through product of Twists and Wrenches

$$W_j^j T_j^{j,i} = W_j^j Ad_{H_i^j} T_j^{i,i} = ({Ad_{H_i^j}}^\top {W_j^j}^\top) T_j^{i,i} = W_j^i T_j^{i,i},$$

$${W_j^i}^\top = Ad_{H_i^j}^\top {W^j}^\top = \begin{pmatrix} R_j^i & -R_j^i \tilde{p}_i^j \\ 0 & R_j^i \end{pmatrix} {W_j^j}^\top. \tag{4.10}$$

4.4 Geometric Interpretation

Theorem 1. *Mozzi's theorem*[1] *states that any element of $SE(3)$ can be described as a pure instantaneous rotation around an axis plus a pure instantaneous translation along the same axis:*

$$\forall \begin{pmatrix} w \\ v \end{pmatrix}, \exists r, \lambda \; s.t. \implies \underbrace{\begin{pmatrix} w \\ v \end{pmatrix}}_{twist} = \underbrace{\begin{pmatrix} w \\ r \wedge w \end{pmatrix}}_{rotation} + \underbrace{\lambda \begin{pmatrix} 0 \\ w \end{pmatrix}}_{translation}, \tag{4.11}$$

where w is an angular velocity, λ is a length called pitch, and r is a point on an axis of motion.

Examination of this equation together with Fig. 4.6a shows that v is the velocity of an imaginary point passing through the origin of the coordinate system in which the twist is expressed and moving together with the object. The twist can be associated with a geometrical line, namely the line passing through r and spanned by w, which is left invariant by the rotation.

Theorem 2. *The decomposition of twists using Mozzi's theorem, can be given for wrenches by its dual analogous theorem, called Poinsot's theorem. This theorem states that any element of the dual $SE^*(3)$ can be split as a sum of two terms:*

$$\forall \begin{pmatrix} \tau \\ F \end{pmatrix}, \exists r, \lambda \; s.t. \implies \begin{pmatrix} \tau \\ F \end{pmatrix} = \underbrace{\begin{pmatrix} r \wedge F \\ F \end{pmatrix}}_{force} + \underbrace{\lambda \begin{pmatrix} F \\ 0 \end{pmatrix}}_{torque} \tag{4.12}$$

Analog to the twist, Fig. 4.6b shows how a wrench can be visualized. The first element of the equation (4.12) is representing a pure force applied along the line passing through r. The second is instead a pure torque, which does not need to be associated with a point on an axis of motion.

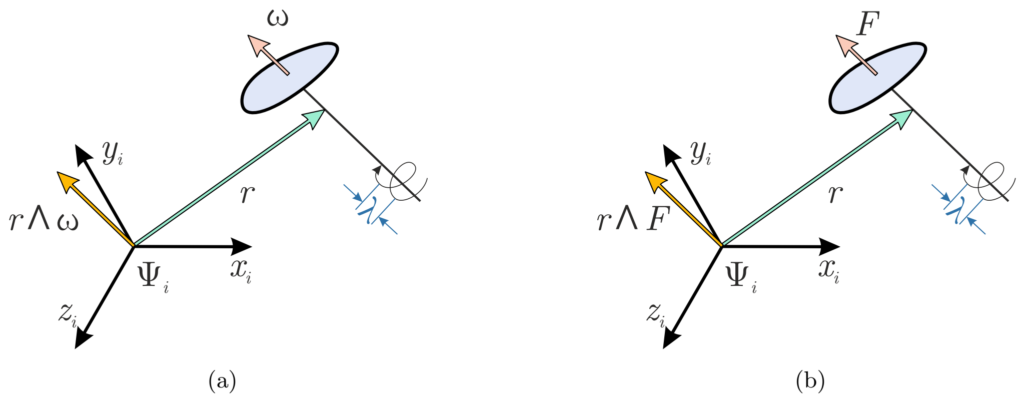

Figure 4.6: Geometric interpretation of **(a)** Twist and **(b)** Wrench

[1] Also referred as Chasles' decomposition theorem [5]

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