Chapter 6

Dynamics

This section focuses on the dynamics of open-chain robots,i.e., we consider motion by taking into account the forces and torques that cause it. We consider the 3R spatial open-chain robot as an example to derive the dynamic equation (or equation of motion), which is a set of second-order differential equations of the form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau^{\top}, \tag{6.1}$$

$$\tau^{\top} = (\tau_g + \tau_f + \tau_i + \tau_o)^{\top},$$

where $q(t) \in \mathbb{R}^n$, $\dot{q}(t) \in \mathbb{R}^n$, $\ddot{q}(t) \in \mathbb{R}^n$ are vectors of the generalized joint position, velocity and acceleration respectively, $M(q) \in \mathbb{R}^{n\times n}$ is a symmetric positive-definite matrix which is called inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n\times 1}$ is called unprecisely the vector of Coriolis and centrifugal forces, $G(q) \in \mathbb{R}^{n\times 1}$ is the vector of potential forces, $\tau^{\top} \in \mathbb{R}^{n\times 1}$ is the generalized force/torque, which is applied to the robot.

6.1 Dynamics of a rigid body

A robot consists of links, e.g., rigid bodies. Since we consider a spatial motion, a rigid body has to be able to move with 6 degrees of freedom. Let us consider dynamics of a rigid body.

Let us start with a point mass, the momentum is defined as:

$$p = mv,$$

where p is momentum, m is mass, v is velocity of the mass. This equation is known as Eulers equation. By taking the derivative of both sides, we obtain Newton's second law

$$F = \dot{p} = m\dot{v} + \dot{m}v,$$

where $\dot{v}$ is the acceleration of the point mass. For most cases, where the mass has a constant value, the formula simplifies to

$$F = \dot{p} = m\dot{v}.$$

Taking into account rotation, the rotational momentum p_r is defined as:

$$p_r = J\omega,$$

where J is the rotational inertia, ω is angular velocity. By taking the derivative of both sides, we obtain Newton's second law for a rotational motion

$$\tau = \dot{p}_r = J\dot{\omega} + \dot{J}\omega,$$

where $\dot{J}$ is zero for most cases, so that formula simplifies to

$$\tau = \dot{p}_r = J\dot{\omega}.$$

We can generalize Euler's equation for a rigid body by

$$(\mathcal{P}^i)^{\top} := \mathcal{I}^{k,i} T_i^{i,0}$$

where $\mathcal{P}^i$ is the generalized **momentum** in the screw theory, $T_i^{i,0}$ is a twist of a body i, that moves in respect of frame Ψ_0, and expressed in a body frame Ψ_i, and $\mathcal{I}^{k,i}$ is a 6x6 matrix called the **inertia tensor** for the rigid body with fixed frame Ψ_i expressed in the frame Ψ_k. Finding or calculating this matrix can be simplified by using the **principal inertia frame** Ψ_k, centered in the center of mass and properly oriented such that the corresponding H_k^i gives

$$\mathcal{I}^{k,i} = \begin{pmatrix} J_i & 0 \\ 0 & m_i I, \end{pmatrix} \tag{6.2}$$

where $\mathcal{I}^{k,i}$ is constant if and only if H_k^i is constant, m_i is a body's mass, I is identity matrix, and $J_i \in \mathbb{R}^{3\times3}$ matrix called **moment of inertia**

$$J_i = \begin{pmatrix} j_x & 0 & 0 \\ 0 & j_y & 0 \\ 0 & 0 & j_z \end{pmatrix}.$$

By taking the derivative of both sides for the equation 6.3, we obtain Newton's second law for a spatial motion

$$\dot{\mathcal{P}}^{0,i} = W^{0,i}, \tag{6.3}$$

where $\mathcal{P}^{0,i}$ is the moment of a body i, expressed in the inertial frame Ψ_0. The time derivative of $\mathcal{P}^{0,i}$ is equal to wrench, so the moment is a convector. Let 's express the momentum in a body frame Ψ_i:

$$(\dot{\mathcal{P}}^{0,i})^\top = (W^{0,i})^\top$$

$$(\dot{Ad_{H_0^i}^\top (\mathcal{P}^i)^\top}) = (\dot{Ad}_{H_0^i}^\top)(\mathcal{P}^i)^\top + Ad_{H_0^i}^\top (\dot{\mathcal{P}}^i)^\top.$$

Considering that

$$(\dot{Ad}_{H_0^i}^\top) = ad_{T_0^{0,i}}^\top Ad_{H_0^i}^\top,$$

we get that

$$ad_{T_0^{0,i}}^\top Ad_{H_0^i}^\top (\mathcal{P}^i)^\top + Ad_{H_0^i}^\top (\dot{\mathcal{P}}^i)^\top = (W^{0,i})^\top.$$

Multiplying the left side by $Ad_{H_i^0}^\top$ on the left we get:

$$\underbrace{Ad_{H_i^0}^\top ad_{T_0^{0,i}}^\top Ad_{H_0^i}^\top}_{ad_{T_0^{i,i}}^\top}(\mathcal{P}^i)^\top + \underbrace{Ad_{H_i^0}^\top Ad_{H_0^i}^\top}_{I}(\dot{\mathcal{P}}^i)^\top = \underbrace{Ad_{H_i^0}^\top (W^{0,i})^\top}_{(W^i)^\top}$$

$$ad_{T_0^{i,i}}^\top (\mathcal{P}^i)^\top + I(\dot{\mathcal{P}}^i)^\top = (W^i)^\top,$$

from where we express the derivative of momentum

$$(\dot{\mathcal{P}}^i)^\top = (W^i)^\top - ad_{T_0^{i,i}}^\top (\mathcal{P}^i)^\top,$$

given that $-T_0^{i,i} = T_i^{i,0}$, we get

$$(\dot{\mathcal{P}}^i)^\top = ad_{T_i^{i,0}}^\top (\mathcal{P}^i)^\top + (W^i)^\top \ \rightarrow \ \dot{\mathcal{P}}^i = \mathcal{P}^i ad_{T_i^{i,0}} + W^i$$

Now we can multiply both sides of the equation on twist to get a power equation for a rigid body

$$\underbrace{\dot{\mathcal{P}}^i T_i^{i,0}}_{1} = \underbrace{\mathcal{P}^i ad_{T_i^{i,0}} T_i^{i,0}}_{2} + \underbrace{W^i T_i^{i,0}}_{3}, \tag{6.4}$$

The first term on the left (1) is the stored energy in a body. The second term on the right (3) is the supplied energy. We see that an extra term has appeared (2), that always has a power value of zero, due to the structure of $ad_{T_i^{i,0}}$. That term is needed to model the Dzhanibekov effect. Fig. 6.1 shows a bond-graph of rigid body dynamics, constructed according to eq. 6.4.

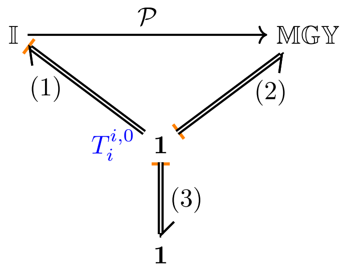

Figure 6.1: Body dynamics in bond graphs

6.2 Dynamics of a serial chain

The general dynamic equation (6.1) is typically derived by either the energy-based Euler-Lagrangian formalism or the force-balance-based Newton-Euler formalism. The kinetic energy $\mathcal{K}$ of a point mass m is a function of velocity v

$$\mathcal{K}(v) = \frac{1}{2} v^\top m v.$$

The kinetic energy $\mathcal{K}$ of a rigid body in Lie groups & Screw Theory is given as

$$\mathcal{K} = \frac{1}{2} T_i^{k,0\,\top} \mathcal{I}^{k,i} T_i^{k,0}. \tag{6.5}$$

The Euler-Lagrangian formulation derives the general equation of motion from the kinetic and potential energy of a robot

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) - \frac{\partial \mathcal{L}}{\partial q} = 0,$$

where $\mathcal{L}(q,\dot{q}) = \mathcal{K}(q,\dot{q}) - \mathcal{P}(q)$ is called Lagrangian, $\mathcal{K}(q,\dot{q})$ is kinetic energy, $\mathcal{P}(q)$ is potential energy, and $q = \begin{pmatrix} q_1 & \dots & q_n \end{pmatrix}^\top$ is a vector of generalized coordinates.

If the system is not conservative, then if there are inputs τ such that is the power $P = \tau \dot{q}$ supplied to the system, the equations become:

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\Big) - \frac{\partial \mathcal{L}}{\partial q} = \tau^\top. \tag{6.6}$$

If we find an expression of $\mathcal{K}(q,\dot{q})$ and $\mathcal{P}(q)$ for the robot with q generalized coordinates, we will be able to give the dynamic equation (6.6), where τ is a torque for a rotational joint or a force for a prismatic joint.

Total kinetic energy of the entire robot is the sum of the energies of each body

$$\mathcal{K} = \sum_i \mathcal{K}_i.$$

For a body i its energy can be expressed as:

$$\mathcal{K}_i = \frac{1}{2} (T_i^{i,0})^\top \mathcal{I}^i T_i^{i,0}. \tag{6.7}$$

We can consider the space Jacobian (5.11) as gives the twist of body i as function of $\dot{q}$:

$$T_i^{0,0} = J_{s_i}(q)\dot{q},$$

where the geometric space Jacobian for a body i equals

$$J_i(q) := (T_1 \dots T_i \; 0 \dots 0). \tag{6.8}$$

Taking into account (4.8) and (6.8) we can rewrite the kinetic energy (6.7) as

$$\mathcal{K}_i = \frac{1}{2} \dot{q}^\top \underbrace{J_i^\top(q) Ad^\top_{H_0^i} \mathcal{I}^i Ad_{H_0^i} J_i(q)}_{M_i(q)} \dot{q} = \mathcal{K}(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M(q) \dot{q},$$

where configuration dependent matrix $M(q)$ is a symmetric positive-definite inertia matrix for the manipulator and it is *fundamental* for the dynamics of the robot.

$$M(q) = \sum_i M_i(q) = \sum_i M_i(q) = J_i^\top(q) Ad_{H_0^i}^\top \mathcal{I}^i Ad_{H_0^i} J_i(q). \tag{6.9}$$

If we choose the $\hat{z}$ axis of an inertial reference frame fixed to the vertical, we have that for a body i the *potential energy* is

$$\mathcal{P}_i(q) = m_i g h = m_i g p_{g_z}^0 = m_i g \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} P_g^0 = m_i g \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} H_i^0(q) P_g^i,$$

where P_g^i is the location of the center of gravity g of the body i in the frame Ψ_i. Total potential energy of the entire robot is the sum of the energies of each body:

$$\mathcal{P}(q) = \sum_i \mathcal{P}_i(q). \tag{6.10}$$

If we derive the Lagrangian for a robot and apply the Euler-Lagrange equations we will get the equation of motion as

$$\sum_{j=1}^n M_{ij}(q)\ddot{q}^j + \sum_{j,k=1}^n \Gamma_{i,j,k}\dot{q}^j\dot{q}^k + \frac{\partial \mathcal{P}}{\partial q^i} = \tau_i,$$

where

$$\Gamma_{i,j,k} := \frac{1}{2}\Big(\frac{\partial M_{ij}}{\partial q^k} + \frac{\partial M_{ik}}{\partial q^j} - \frac{\partial M_{kj}}{\partial q^i}\Big)$$

are called Christoffels symbols for $M(q)$. We define the vector of Coriolis and centrifugal forces as

$$C_{i,j}(q,\dot{q}) = \Gamma_{i,j,k}\dot{q}^k, \tag{6.11}$$

and now we can write the final equation of dynamics

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau^\top, \tag{6.12}$$

where $N(q) = \frac{\partial \mathcal{P}}{\partial q}$, τ is a generalized co-vector of the applied load to the links, which is the equivalent joint torques due to all external forces like motors, interaction forces and friction.

6.3 Dynamics of 3R robot

The inertia matrix $M(q)$ is the sum of inertia masses of all three links

$$M(q) := \sum_i M_i(q) = M_1(q) + M_2(q) + M_3(q),$$

where M_i is the inertia mass of the link i:

$$M_i = J_i^\top Ad_{H_0^{c_i}}^\top \mathcal{I}_i Ad_{H_0^{c_i}} J_i.$$

All geometrical Jacobians J_i are presented in equation (5.13). Use equations (4.8) and (4.10) to calculate adjoints $Ad_{H_0^{c_i}}$ and transpose adjoints $Ad_{H_0^{c_1}}^\top$ of homogeneous transformation matrix $H_0^{c_i}$; c_i is the center of gravity of link i and $\mathcal{I}_i$ is the inertia tensor.

The vector of Coriolis and centrifugal forces $C(q,\dot{q}) \in \mathbb{R}^{nx1}$ needs to be calculated using equation (6.11).

The vector of gravity $N(q) \in \mathbb{R}^{n\times 1}$ is equal to the gradient of robots' potential energy $\mathcal{P}(q)$:

$$N(q) = \frac{\partial \mathcal{P}}{\partial q_i} = \begin{pmatrix} \frac{\partial \mathcal{P}}{\partial q_1} \\ \frac{\partial \mathcal{P}}{\partial q_2} \\ \frac{\partial \mathcal{P}}{\partial q_3} \end{pmatrix}.$$

Robot's potential energy $\mathcal{P}(q)$ is the sum of inertia potential energy of all three links

$$\mathcal{P}(q) = \sum_i \mathcal{P}_i(q) = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3,$$

where $\mathcal{P}_i$ is the potential energy of the link i

$$\mathcal{P}_i = m_i g \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} H_{c_i}^0 \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^\top,$$

if we choose the $\hat{z}$ axis as the vertical axis. The homogeneous transformation matrix $H_{c_i}^0$ describes the state of the CoM of the body i.

As a result, the matrix of inertia $M(q) \in \mathbb{R}^{nxn}$, the vector of Coriolis and centrifugal forces $C(q, \dot{q}) \in \mathbb{R}^{nx1}$ and the vector of gravity $N(q) \in \mathbb{R}^{nx1}$ must be calculated symbolically in a mathematical package, as a result, we get the equation of dynamics

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = \tau^\top.$$

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