607 LECTURE NOTES IN ECONOMICS AND MATHEMATICAL SYSTEMS

Markus Bouziane

# Pricing Interest-Rate Derivatives

A Fourier-Transform Based Approach

Springer

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## A Fourier-Transform Based Approach

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To Sabine

# Foreword

In a hypothetical conversation between a trader in interest-rate derivatives and a quantitative analyst, Brigo and Mercurio (2001) let the trader answer about the pros and cons of short rate models: "... we should be careful in thinking market models are the final and complete solution to all problems in interest rate models ... and who knows, maybe short rate models will come back one day..."

In his dissertation Dr. Markus Bouziane contributes to this comeback of short rate models. Using Fourier Transform methods he develops a modular framework for the pricing of interest-rate derivatives within the class of exponential-affine jump-diffusions. Based on a technique introduced by Lewis (2001) for equity options, the payoffs and the stochastic dynamics of interest-rate derivatives are transformed separately. This not only simplifies the application of the residue calculus but improves the efficiency of numerical evaluation schemes considerably. Dr. Bouziane introduces a refined Fractional Inverse Fast Fourier Transformation algorithm which is able to calculate thousands of prices within seconds for a given strike range. The potential of this method is demonstrated for several one- and two-dimensional models.

As a result the application of jump-enhanced short rate models for interest-rate derivatives is on the agenda again. I hope, Dr. Bouziane's monograph will stimulate further research in this direction.

Tübingen, November 2007 Rainer Schöbel

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Tübingen, November 2007 Markus Bouziane

# Contents

# List of Abbreviations and Symbols

| | |
|---|---|
| $\delta(x)$ | Dirac delta function |
| $\Gamma(x)$ | Gamma function |
| $\imath$ | imaginary unit, $\sqrt{-1}$ |
| $\iota_M$ | diag$[I_M]$ |
| $\mathcal{F}^x[\ldots], \mathcal{F}^{-1}[\ldots]$ | Fourier Transformation w.r.t. $x$ and inverse Transformation operator |
| $\mathbb{1}_{\mathcal{A}}$ | indicator function for the event $\mathcal{A}$ |
| $\mathbb{C}$ | the set of complex-valued numbers |
| $\mathbb{E}[\ldots], \mathbb{VAR}[\ldots]$ | expectation and variance operator |
| $\mathbb{P}, \mathbb{Q}$ | real-world and equivalent martingale measure |
| $\mathbb{R}$ | the set of real-valued numbers |
| $\mathfrak{F}_t$ | information set available up to time t |
| diag$[\ldots]$ | operator returning the diagonal elements of a quadratic matrix |
| FFT$[\ldots]$ | Fast Fourier Transformation operator |
| FRFT$[\ldots, \zeta]$ | Fractional Fourier Transformation operator with parameter $\zeta$ |
| IFFT$[\ldots]$ | inverse Fast Fourier Transformation operator |
| Res$[\ldots]$ | residue operator |
| Re$[z]$, Im$[z]$ | real and imaginary part of the complex-valued variable z |
| RMSE | root mean-squared error |
| RMSE$^a$ | approximate root mean-squared error |
| tr$[\ldots]$ | trace operator |

| | |
|---|---|
| $\psi(\ldots), \phi(\ldots)$ | characteristic function and its logarithm |
| $\boldsymbol{\lambda}$ | vector of jump intensities governing the Poisson vector process |
| $\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t), \boldsymbol{\Lambda}_{\boldsymbol{\lambda}}$ | vectors containing risk-compensating factors for the diffusion and jump parts, respectively |
| $\boldsymbol{\mu}_0, \boldsymbol{\mu}_1$ | constant coefficients determining the drift component of $\mathbf{x}_t$ |
| $\boldsymbol{\nu}(\mathbf{J})$ | matrix of jump-size distributions for the random variables $\mathbf{J}$ |
| $\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}_1$ | constant coefficients determining the volatility component of $\mathbf{x}_t$ |
| $\mathbf{J}$ | matrix of random jump sizes of $\mathbf{x}_t$ |
| $\mathbf{j}_n$ | $n$th row of the matrix $\mathbf{J}$ |
| $\mathbf{N}(\boldsymbol{\lambda} t)$ | vector of independent Poisson processes acting with an intensity $\boldsymbol{\lambda}$ |
| $\mathbf{W}_t$ | vector of independent Brownian motions |
| $\mathbf{x}_t$ | general stochastic vector process |
| $\xi(\mathbf{x}_t, t, T)$ | state-price kernel |
| $a(z,\tau), \mathbf{b}(z,\tau)$ | complex-valued coefficient functions of the general characteristic function |
| $AD(\ldots)$ | Arrow-Debreu price |
| $ARC_r(\ldots)$ | level-based, average-rate contract |
| $CAP_r(\ldots), CAP_Y(\ldots)$ | level- and yield-based cap contract |
| $CBC(\ldots), CBP(\ldots)$ | coupon-bond call and put option |
| $FLR_r(\ldots), FLR_Y(\ldots)$ | level- and yield-based floor contract |
| $FRA_r(\ldots), FRA_Y(\ldots)$ | level- and yield-based forward rate agreement |
| $g_0, \mathbf{g}_1$ | constant coefficients determining the characteristic payoff part of $\mathbf{x}_t$ |
| $I_M$ | identity matrix of rank $M$ |
| $K$ | strike value of an option contract |
| $Nom$ | Nominal Value |
| $P(\ldots), CB(\ldots)$ | zero bond, coupon bond |
| $p(\ldots)$ | probability density function |
| $p_{Ex}(J,\eta)$ | density function of an exponentially distributed random variable $J$ with mean and volatility $\eta$ |

| | |
|---|---|
| $p_{Ga}(J, \eta, p)$ | density function of a gamma distributed random variable $J$ with mean $\eta p$ and volatility $\eta\sqrt{p}$ |
| $p_{No}(J, \mu_J, \sigma_J)$ | density function of a normally distributed random variable $J$ with mean $\mu_J$ and volatility $\sigma_J$ |
| $SWA_r(\ldots), SWA_Y(\ldots)$ | level- and yield-based receiver swap |
| $SWP_Y(\ldots)$ | yield-based swaption contract |
| $t, \tau, T$ | calendar time, time to maturity and time of contract expiry |
| $UARC_r(\ldots)$ | level-based, unconditional average-rate contract |
| $w_0^A(z), \mathbf{w}_1^A(z)$ | complex-valued coefficient functions determining the short rate for an average-rate contract |
| $w_0, \mathbf{w}_1$ | constant coefficients determining the short rate |
| $x_t^{(m)}$ | $m$th element of the vector $\mathbf{x}_t$ |
| $Y(\ldots)$ | simple yield to maturity |
| $z$ | Fourier-transform variable with real part $z_r$ and imaginary part $z_i$, respectively |
| $ZBC(\ldots), ZBP(\ldots)$ | zero-bond call and put option |
| ITM, ATM, OTM | in the money, at the money, and out of the money |
| ODE, PDE and SDE | ordinary, partial, and stochastic differential equation |
| OU | Ornstein-Uhlenbeck (process) |
| SR | Square-Root (process) |

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# 1

# Introduction

## 1.1 Motivation and Objectives

In the last few years the demand for sophisticated term-structure models, capable of reflecting the market behavior more realistically, e.g. models which can reproduce the feature of market shocks, has dramatically increased. For example, according to the results of their empirical study, Brown and Dybvig (1986) and Aït-Sahalia (1996) question among others the use of pure diffusion models, such as the popular interest-rate models of Vasicek (1977) and Cox, Ingersoll and Ross (1985b), to describe the behavior of interest rates. Moreover, recent studies support the assumption of jump components in the term structure of interest rates. In the study of Hamilton (1996), Fed Funds rates on a daily base are analyzed. The author finds that settlement days and quarter-ends induce statistically significant jumps in the term structure of interest rates. Das (2002) analyzed Fed Funds rates on daily bases over the period 1988-1997. As a result of this study, the proposed jump models show a substantially better fit of the empirical data compared to the pure diffusion model. Durham (2005) also examined Fed Funds rates for the period 1988-2005. The model-generated yields of zero-bond prices are then calibrated to the Fed Funds Rate and one- and three-month U.S. Treasury bill rates. The author concludes that the so-called jump-diffusion models produce more accurate estimates of the interest-rate curves than the pure diffusion model[1].

[1] Additional studies examining the empirical performance of jump-diffusion models are given in, e.g. Lin and Yeh (1999), Zhou (2001), Wilkens (2005), and Chan (2005).

Thus, the ability of a term-structure model to reproduce these discount rate shocks, based e.g. on the adjustment of the discount rate by the European Central Bank, on an economic crisis, and quarter-end effects, is highly appreciated. Accordingly, jump-diffusion interest-rate models were developed to cover this issue. Ahn and Thompson (1988) introduced one of the first jump-diffusion models for the term structure of interest rates. In their study, the interest-rate dynamics are derived within an equilibrium framework similar to the one used in Cox, Ingersoll and Ross (1985b) and particular approximate closed-form zero-bond prices are obtained. Das and Foresi (1996) also derived zero-bond prices for a jump-enhanced Vasicek (1977) model. The authors apply an exponentially distributed jump component, where the absolute value of the jump sign is drawn from a Bernoulli distribution. An alternative jump specification for the mean-reverting normally distributed short rate is given in Baz and Das (1996)[2]. In their approach, the jump-size distribution is given by a normal distribution and approximate zero-bond prices are derived. An empirical test of a Square-Root interest-rate model enhanced with uniformly distributed jumps is given in Zhou (2001). The author fits the particular jump-diffusion model to weekly three-month Treasury bill yields. In Durham (2005), the author states an alternative approximation technique for zero-bond prices when the short rate follows the same dynamics as in Baz and Das (1996). Additionally, a bimodal normally distributed jump version of the Vasicek (1977) model together with a jump-enhanced two-factor model is presented[3].

In addition, for derivatives research purposes, an important feature such interest-rate models should exhibit is the ability to generate analytical solutions for the derivatives contracts to be priced. If this can be accomplished, the interest-rate instrument can be examined in depth, e.g. doing some sensitivity analysis. However, dealing with jump components, we often have to rely on time-consuming Monte-Carlo methods in order to price interest-rate derivatives. Thus, more ambitious pricing approaches are needed. Recently, integral transformations have been found to be reliable in deriving semi closed-form

[2] The same model specification is used in Das (2002).

[3] The approximation technique is also discussed in depth in Durham (2006) for the bimodal normally and exponentially distributed jump extension of a Vasicek (1977) short-rate model.

solutions of derivatives contracts under more complicated stochastic dynamics. The term *semi closed-form solutions* in this case refers to closed-form solutions in the image space, according to the particular transformation rule. Especially the subclass of Fourier Transformations have been proven to be useful for pricing problems in financial disciplines[4]. Basically, the main advantage of this transform technique consists in providing distribution independent pricing formulae. However, even semi closed-form pricing formulae are hard to obtain, dealing with jump-size distributions such as the normal and the gamma.

Accordingly, it is our objective to derive an efficient and accurate pricing tool for interest-rate derivatives within a Fourier-transform pricing approach, which is generally applicable to exponential-affine jump-diffusion models. This objective can be achieved within four steps. Firstly, we want a flexible short-rate process, which is able to integrate both diffusion and jump components. Thus, we extend the exponential-affine model presented in Duffie and Kan (1996) by introducing jump components. The second step is to refine the concept of a modular option pricing as proposed in Zhu (2000) by applying the pricing methodology explained in Lewis (2000) and Lewis (2001)[5]. Therefore, we want to formulate a distribution-independent pricing framework, where the particular interest-rate contract price can be clearly separated into stochastic and payoff specific parts. Apart from the pricing theory, we also need a tool to obtain numerical values of the contracts to be priced. A very popular strategy to price derivatives is the Monte-Carlo approach. However, being generally applicable, this numerical pricing approach suffers from its time-consuming calculations and its poor convergence to true solutions. The third objective of this thesis is to develop an algorithm, which appropriately computes option prices in the Lewis (2001) pricing approach. In contrast to the Fast Fourier Transformation (FFT), as used in Carr and Madan (1999) for the pricing of

[4] Heston (1993) is the seminal paper on this topic, where semi closed-form solutions for options on equities in a stochastic volatility model are derived for the first time. Among others, we mention the influential work of Bakshi and Madan (2000) and Duffie, Pan and Singleton (2000) in deriving option prices using Fourier Transformations.

[5] Even though this pricing method is mentioned for the first time in Lewis (2000), we henceforth refer to Lewis (2001) as the source, because of the detailed discussion and derivation of the pricing methodology.

equity options, we base our computations on the Inverse Fast Fourier Transform (IFFT). Consequently, we introduce in this thesis a new, IFFT-based pricing algorithm, which is able to calculate thousands of option prices within fractions of a second and is a straightforward application to option pricing in the Lewis (2001) framework. The last step is then to examine density functions and contract prices of some popular interest-rate diffusion models enhanced with three different jump candidates.

## 1.2 Structure of the Thesis

This thesis is organized as follows. We start in *chapter two* with the formulation of a general term-structure model, which is governed by a multivariate jump-diffusion process. After introducing some general concepts in stochastic calculus we demonstrate how the relevant risk-neutral coefficients of the instantaneous interest-rate process can be obtained. Afterwards, we discuss the technique of performing a Fourier Transformation and its inverse and state the system of ordinary differential equations the general characteristic function has to solve. In *chapter three* we discuss a representative collection of some interest-rate derivative contracts which can be solved within the Fourier-based pricing mechanism. We distinguish between contracts with conditional and unconditional exercise rights, because of the different pricing procedure.

Subsequently, in *chapter four* we discuss three Fourier-based pricing approaches. We begin our summary with the pricing technique using Fourier-transformed Arrow-Debreu state prices. Since this type of valuation was first applied by Heston (1993) and further discussed by Bakshi and Madan (2000), we henceforth refer to this approach as the Heston transform approach. Subsequently, we discuss the pricing procedure introduced by Carr and Madan (1999). In this thesis the authors exploit the Fourier Transformation applied not only to the state price densities but to the entire option price. They introduce a valuation approach where theoretical option prices can be subsequently recovered applying a highly efficient algorithm, namely the Fast Fourier Transform, hereafter denoted as FFT. Finally, we discuss the valuation methodology applied by Lewis (2001). This approach features several advantages. Firstly, its composition is highly modularized. Secondly, employing Cauchy's residue theorem, the approach can be consistently used both for interest-rate derivatives with unconditional and conditional exercise rights. Fortunately, this methodology enables the application of an refined IFFT algorithm which we implement in our pricing procedure.

In *chapter five*, we derive the particular Fourier Transformations of payoff functions needed in pricing the contract forms previously presented. Additionally, we derive in case of a one-factor term-structure model the Fourier representation of a swaption and a coupon-bond option, respectively. *Chapter six* gives an outline of the numerical algorithm used for pricing purposes. Again, we distinguish between the computation of derivatives with conditional and unconditional exercise rights. Subsequently, we present a further refinement of the pricing algorithm for option contracts by the application of the Fractional Fourier Transformation according to the article of Bailey and Swarztrauber (1994). The last part of the chapter discusses the issue of finding the optimal parameter constellation of the numerical algorithm.

In *chapter seven* we briefly discuss three different jump-size specifications and derive their general jump transforms. In *chapters eight* and *nine* we examine both jump-enhanced one-factor and two-factor interest-rate models and focus on the impact of different jump specifications. The particular one-factor models we enhance with jump components are the prominent interest-rate models introduced in Vasicek (1977) and Cox, Ingersoll and Ross (1985b). For the class of two-factor models we exemplarily discuss an additive model used in Schöbel and Zhu (2000) and a subordinated model according to Fong and Vasicek (1991a). To our knowledge, in case of the Fong and Vasicek (1991a) model, option prices are presented for the first time.

In *chapter ten*, we give a perspective of model extensions for which the pricing procedure is also capable in deriving numerical solutions. The first extension is to consider a special model class of non-affine interest-rate models. Another extension of our interest-rate model is to consider stochastic jump intensities. Since it fits into the exponential-affine model setup of Duffie, Pan and Singleton (2000), the implementation in our pricing procedure presents no greater difficulties. However, due to the non-existence of closed-form solutions in any case, we briefly discuss these extensions. In the last chapter, we review the results of our study and give some concluding remarks.

# 2

# A General Multi-Factor Model of the Term Structure of Interest Rates and the Principles of Characteristic Functions

## 2.1 An Extended Jump-Diffusion Term-Structure Model

The evolution of the yield curve can be described in various ways. For instance, it is possible to use such quantities as zero-bond prices, instantaneous forward rates and short interest rates, respectively, to build the term structure of interest rates. If the transformation law from one quantity to the other is known, the choice of the independent variable is just a matter of convenience.

In this thesis, we attempt to model the dynamics of the instantaneous interest rate, denoted hereafter by $r(\mathbf{x}_t)$, in order to construct our derivatives pricing framework. This instantaneous interest rate $r(\mathbf{x}_t)$ is also often referred to as the short-term interest rate or short rate, respectively, and characterizes the risk-free rate for borrowing or lending money over the infinitesimal time period $[t, t + \mathrm{d}t]$. Since we model the dynamics in a continuous trading environment, the relevant processes are described via stochastic differential equations.

The economy we consider has the trading interval $[0, T]$. The uncertainty under the physical probability measure is completely specified by the filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. In this formulation $\Omega$ denotes the complete set of all possible outcome elements $\omega \in \Omega$. The information available in the economy is contained within the filtration $(\mathfrak{F})_{t\geq 0}$, such that the level of uncertainty is resolved over the trading interval with respect to the information filtration. The last term, completing the probability space, is called the *real-world* probability measure $\mathbb{P}$ on $(\Omega, \mathfrak{F})$, since it reflects the *real-world* probability law of the data.

We model the dynamic behavior of the term structure in the spirit of Duffie and Kan (1996) and Duffie, Pan and Singleton (2000), to preserve an exponential-affine structure of the characteristic function. However, we extend the framework in Duffie and Kan (1996) to allow for $N$ different trigger processes[6], which offers more flexibility. The term structure is then modeled by a multi-factor structural Markov model of $M$ factors, represented by a random vector $\mathbf{x}_t$, which solves the multivariate stochastic differential equation,

$$\mathrm{d}\mathbf{x}_t = \begin{pmatrix} \mathrm{d}x_t^{(1)} \\ \mathrm{d}x_t^{(2)} \\ \vdots \\ \mathrm{d}x_t^{(M-1)} \\ \mathrm{d}x_t^{(M)} \end{pmatrix} = \boldsymbol{\mu}^{\mathbb{P}}(\mathbf{x}_t)\,\mathrm{d}t + \boldsymbol{\Sigma}(\mathbf{x}_t)\,\mathrm{d}\mathbf{W}_t^{\mathbb{P}} + \mathbf{J}\,\mathrm{d}\mathbf{N}(\boldsymbol{\lambda}^{\mathbb{P}}t). \tag{2.1}$$

The coefficient vector $\boldsymbol{\mu}^{\mathbb{P}}(\mathbf{x}_t)$ has the affine structure

$$\boldsymbol{\mu}^{\mathbb{P}}(\mathbf{x}_t) = \boldsymbol{\mu}_0^{\mathbb{P}} + \boldsymbol{\mu}_1^{\mathbb{P}}\mathbf{x}_t \tag{2.2}$$

with $(\boldsymbol{\mu}_0^{\mathbb{P}}, \boldsymbol{\mu}_1^{\mathbb{P}}) \in \mathbb{R}^M \times \mathbb{R}^{M\times M}$ and the variance-covariance matrix $\boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Sigma}(\mathbf{x}_t)'$ suffices the relation

$$\boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Sigma}(\mathbf{x}_t)' = \boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_1\mathbf{x}_t, \tag{2.3}$$

where $\boldsymbol{\Sigma}_0 \in \mathbb{R}^{M\times M}$ is a matrix and $\boldsymbol{\Sigma}_1 \in \mathbb{R}^{M\times M\times M}$ is a third order tensor. The vector $\mathbf{W}_t^{\mathbb{P}}$ in equation (2.1) represents $M$ orthogonal Wiener processes. Thus, we have[7]

$$\mathbb{E}^{\mathbb{P}}(\,\mathrm{d}\mathbf{W}_t^{\mathbb{P}}\,\mathrm{d}\mathbf{W}_t^{\mathbb{P}'}) = \mathbf{I}_M\,\mathrm{d}t$$

with $\mathbf{I}_M$ as the $M \times M$ identity matrix.

As mentioned above, we extend the ordinary diffusion model[8] with $N$ independent Poisson processes, condensed in the vector $\mathbf{N}(\boldsymbol{\lambda}^{\mathbb{P}}t)$. This vector process acts with constant and positive intensities[9] $\boldsymbol{\lambda}^{\mathbb{P}}$. We allow for every

[6] Chacko and Das (2002) model also the term structure with help of different Poisson processes. However, their approach consider a subordinated short rate.

[7] If not indicated otherwise, we subsequently use the shorthand notation $\mathbb{E}[\,\cdot\,]$ for the expression $\mathbb{E}[\,\cdot\,|\mathfrak{F}_t]$.

[8] This would be the original model approach presented in Duffie and Kan (1996).

[9] This exponential-affine model can be easily extended to stochastic jump intensities of the form $\boldsymbol{\lambda}^{\mathbb{P}}(\mathbf{x}_t) = \boldsymbol{\lambda}_0^{\mathbb{P}} + \boldsymbol{\lambda}_1^{\mathbb{P}}\mathbf{x}_t$. See Chapter 10.

particular factor in $\mathbf{x}_t$ an amount of $N$ different jumps drawn from a jump amplitude matrix $\mathbf{J} \in \mathbb{R}^{M \times N}$. Hence, the distribution functions of the particular jump amplitudes are given within the matrix $\boldsymbol{\nu}(\mathbf{J})$. Finally, all jump amplitudes in $\mathbf{J}$ are independent of the state of the vector $\mathbf{x}_t$[10].

To preserve the exponential-affine structure of any derivatives contract based on $r(\mathbf{x}_t)$ and $\mathbf{x}_t$, respectively, all random sources, the Brownian motions $\mathbf{W}_t^{\mathbb{P}}$, intensities $\boldsymbol{\lambda}^{\mathbb{P}}$ and jump amplitudes $\mathbf{J}$ are mutually independent. As a direct consequence of the independence of $\mathbf{J}$ and $\mathbf{x}_t$, there is no chance to generate an arbitrage opportunity according to available information before the particular jump occurs. Hence, given a jump time $t^*$, we have formally $\mathbf{J} \in \mathfrak{F}_{t^*-}$. Therefore, if a jump occurs at time $t^*$, nobody is able to predict the exact jump amplitude and cannot gain an arbitrarily large profit with certainty.

In this thesis, the choice of jump amplitudes in $\mathbf{J}$ can draw on three different types of distribution. These are:

- Exponentially distributed jumps.
- Normally distributed jumps.
- Gamma distributed jumps.

These jump distributions and the resulting jump transforms, which are used in our pricing mechanism, are covered in Chapter 7.

Basically, we prefer to model the term structure in terms of the instantaneous short interest rate $r(\mathbf{x}_t)$[11], because in this framework all fundamental quantities are properly defined as the expectation of some functionals on the underlying process $r(\mathbf{x}_t)$. Accordingly, we are able to construct an arbitrage-free economy and simultaneously guarantee a consistent pricing methodol-

[10] From a technical point of view, it is either possible to introduce a dependence on $\mathbf{x}_t$ for the jump intensity together with independent random jump amplitudes or a dependence on $\mathbf{x}_t$ for the jump amplitude together with constant jump intensities. See Zhou (2001), p. 4.

[11] Other approaches are possible, e.g. the direct approach as used in Schöbel (1987) and Briys, Crouhy and Schöbel (1991) or modeling the forward-rate process as done in Heath, Jarrow and Morton (1992).

ogy[12]. The drawback of this approach is that we might not be able to explain perfectly the entire term structure extracted from observed bond market prices and therefore must content ourselves with a best fit scenario.

The literature distinguishes between two approaches in modeling the short interest rate in a multidimensional framework. Firstly, we can identify a strategy, which we call henceforth the *subordinated* modeling approach. Here, the short rate is modeled as

$$r\left(\mathbf{x}_t\right) = w_0 + w_1 x_t^{(1)} (x_t^{(2)}, \ldots, x_t^{(M)}).$$

Consequently, the other $M-1$ stochastic factors are subordinated loadings, containing e.g. a stochastic volatility and/or a stochastic mean[13]. Apart from the stochastic variable $x_t^{(1)}$, we also consider the deterministic parameters $w_0$ and $w_1$ in modeling the short rate. Indeed, there are other factors, which can possibly have some other economic meaning worth to be included in the interest-rate model.

The second method in modeling short rates, which we call the *additive* modeling approach, is to represent $r_t$ as a weighted sum over $\mathbf{x}_t$, formally given by

$$r\left(\mathbf{x}_t\right) = w_0 + \mathbf{w}'\mathbf{x}_t,$$

[12] This means that all derivative prices are based on the same price of risk. See Culot (2003), Section 2.1.

[13] In Brennan and Schwartz (1979), Brennan and Schwartz (1980), and Brennan and Schwartz (1982) the short-rate process is subordinated by a stochastic long-term rate. Beaglehole and Tenney (1991) discuss a two-factor interest-rate model with a stochastic long-term mean component and Fong and Vasicek (1991a) introduce a short-rate model with stochastic volatility. A model where the short rate depends on a stochastic inflation factor is modeled in Pennacchi (1991). Kellerhals (2001) analyzes an interest-rate model with a stochastic market price of risk component. In Balduzzi, Das, Foresi and Sundaram (1996), the authors present a short-rate model with a stochastic mean and volatility component.

where $\mathbf{w}$ is a $M \times 1$ vector containing separate weights for the corresponding factor loadings in $\mathbf{x}_t$[14]. However, this model approach possibly entails difficulties in explaining the economic meaning of the variables $\mathbf{x}_t$[15].

## 2.2 Technical Preliminaries

Before we proceed any further, we have to discuss some general results and principles of stochastic analysis, which are commonly used in financial engineering, namely the prominent Itô's Lemma and the equally famous Feynman-Kac Theorem. These two principles play a major role in diffusion theory and are well connected. Since we consider discontinuous jumps in our model setup, we have to use extended versions of these two results. At first we have to state some regularity conditions on the jump-diffusion process, in order to guarantee their application.

**Definition 2.2.1 (Regularity Conditions for Jump-Diffusion Processes).** *If the vector process $\mathbf{x}_t$ represents a multivariate jump-diffusion, the parameter coefficients $\boldsymbol{\mu}(\mathbf{x}_t), \boldsymbol{\Sigma}(\mathbf{x}_t)$ have to satisfy the following technical conditions*[16] *for all $t \geq 0$*

- $\|\boldsymbol{\mu}(\mathbf{x}_t^a) - \boldsymbol{\mu}(\mathbf{x}_t^b)\| \leq A_1 \|\mathbf{x}_t^a - \mathbf{x}_t^b\|$
- $\|\boldsymbol{\Sigma}(\mathbf{x}_t^a)) - \boldsymbol{\Sigma}(\mathbf{x}_t^b)\| \leq A_2 \|\mathbf{x}_t^a - \mathbf{x}_t^b\|$
- $\|\boldsymbol{\mu}(\mathbf{x}_t^a)\| \leq A_1 \left(1 + \|\mathbf{x}_t^a\|\right)$
- $\|\boldsymbol{\Sigma}(\mathbf{x}_t^a))\| \leq A_2 \left(1 + \|\mathbf{x}_t^a\|\right)$

*where $\mathbf{x}_t^a, \mathbf{x}_t^b \in \mathbb{R}^M$ are two vectors containing different realizations of $\mathbf{x}_t$ and the constants $A_1, A_2 < \infty$ denote some scalar barriers. Additionally, we need*

[14] Langetieg (1980) models the short rate as an additive process consisting of two correlated Ornstein-Uhlenbeck processes. In Beaglehole and Tenney (1991) an additive, multivariate quadratic Gaussian interest-rate model is given. Longstaff and Schwartz (1992) and Chen and Scott (1992) model the interest-rate process as the sum of two uncorrelated Square-Root processes.

[15] A comprehensive discussion on this topic is given in Piazzesi (2003).

[16] The first two conditions are known as the Lipschitz conditions, the latter two represent the growth or polynomial growth conditions. See, for example, Karlin and Taylor (1981).

*for the jump components the integral* $\int_{\mathbb{R}} e^{cJ_{mn}} \, \mathrm{d}\nu(J_{mn})$ *to be well defined for every* $J_{mn} \in \mathbf{J}$ *and some constant* $c \in \mathbb{C}$.

If the conditions posed above are met, we are able to apply both Itô's Lemma and the Feynman-Kac Theorem.

We start with Itô's Lemma. This lemma enables us to determine the stochastic process driving some function $f(\mathbf{x}_t, t, T)$, depending on time $t$ and a stochastic (vector) variable, e.g. the process $\mathbf{x}_t$ given in equation (2.1). The variables $t$ and $\mathbf{x}_t$, respectively, are hereafter denoted as the independent variables. The coefficients $\boldsymbol{\mu}(\mathbf{x}_t)$ and $\boldsymbol{\lambda}$ used in this section have no superscripts, because the principles introduced here hold in general.

**Theorem 2.2.2 (Itô Formula for Jump-Diffusion Processes[17]).** *Assume the function* $f(\mathbf{x}_t, t, T)$ *is at least twice differentiable in* $\mathbf{x}_t$ *and once differentiable in* $t$. *Then the canonical decomposition of the stochastic differential equation for* $f(\mathbf{x}_t, t, T)$ *is given by*

$$\begin{aligned} \mathrm{d}f(\mathbf{x}_t, t, T) = & \Bigg( \frac{\partial f(\mathbf{x}_t, t, T)}{\partial t} + \boldsymbol{\mu}(\mathbf{x}_t)' \frac{\partial f(\mathbf{x}_t, t, T)}{\partial \mathbf{x}_t} \\ & + \frac{1}{2} \operatorname{tr} \left[ \boldsymbol{\Sigma}(\mathbf{x}_t) \boldsymbol{\Sigma}(\mathbf{x}_t)' \frac{\partial^2 f(\mathbf{x}_t, t, T)}{\partial \mathbf{x}_t \partial \mathbf{x}_t'} \right] \Bigg) \mathrm{d}t \\ & + \frac{\partial f(\mathbf{x}_t, t, T)}{\partial \mathbf{x}_t'} \boldsymbol{\Sigma}(\mathbf{x}_t) \, \mathrm{d}\mathbf{W}_t \\ & + \left( \mathbf{f}(\mathbf{x}_t, \mathbf{J}, t, T)' - f(\mathbf{x}_t, t, T) \right) \mathrm{d}\mathbf{N}(\boldsymbol{\lambda} t), \end{aligned} \tag{2.4}$$

*where the function* $\mathbf{f}(\mathbf{x}_t, \mathbf{J}, t, T)$ *contains all jump components with elements* $(\mathbf{f}(\mathbf{x}_t, \mathbf{J}, t, T))_n = f(\mathbf{x}_t + \mathbf{j}_n, t, T)$ *and* $\mathbf{j}_n \in \mathbb{R}^M$ *contains as* m*th element* $J_{mn}$ *of the amplitude matrix* $\mathbf{J}$.

Another key result which we use extensively is the Feynman-Kac theorem. This theorem provides us with a tool to determine the system of partial differential equations (PDEs), given an expectation.

[17] See, Kushner (1967), p. 15, for the jump-extended version of Itô's lemma.

**Theorem 2.2.3 (Feynman-Kac).** *If the restrictions in definition 2.2.1 hold, we have the expectation*

$$f(\mathbf{x}_t, t, T) = \mathbb{E}\left[ e^{-\int_t^T h(\mathbf{x}_s, s)\,\mathrm{d}s} f(\mathbf{x}_T, T, T) \right], \tag{2.5}$$

*solving the partial differential equation*

$$\begin{aligned} \frac{\partial f(\mathbf{x}_t, t, T)}{\partial t} + \boldsymbol{\mu}(\mathbf{x}_t)' \frac{\partial f(\mathbf{x}_t, t)}{\partial \mathbf{x}_t} + \frac{1}{2} \operatorname{tr}\left[ \boldsymbol{\Sigma}(\mathbf{x}_t) \boldsymbol{\Sigma}(\mathbf{x}_t)' \frac{\partial^2 f(\mathbf{x}_t, t, T)}{\partial \mathbf{x}_t \partial \mathbf{x}_t'} \right] \\ + \mathbb{E}_{\mathbf{J}} \left[ \mathbf{f}(\mathbf{x}_t, \mathbf{J}, t, T)' - f(\mathbf{x}_t, t, T) \right] \boldsymbol{\lambda} = h(\mathbf{x}_t, t) f(\mathbf{x}_t, t, T), \end{aligned} \tag{2.6}$$

*with boundary condition*[18]

$$f(\mathbf{x}_T, T, T) = G(\mathbf{x}_T) \tag{2.7}$$

*and* $\mathbf{f}(\mathbf{x}_t, \mathbf{J}, t, T)$ *as defined in theorem 2.2.2.*

In diffusion theory, the function $h(\mathbf{x}_t, t)$ is commonly addressed to as the killing rate of the expectation[19] and can be interpreted as some short rate. Since we use equivalently as killing rate a short rate characterized by the time constant coefficients $w_0$ and $\mathbf{w}$ we set the relation

$$h(\mathbf{x}_t, t) = r(\mathbf{x}_t).$$

As we will see, these two principles are the fundamental tools in obtaining the solutions for our upcoming valuation problems, especially in calculating the general characteristic function of a stochastic process, which is discussed in the next sections.

## 2.3 The Risk-Neutral Pricing Approach

So far, the stochastic behavior of the state vector $\mathbf{x}_t$ was assumed to be modeled under the *real-world* probability measure $\mathbb{P}$. This probability measure depends on the investor's assessment of the market and therefore cannot be

[18] The operator $\mathbb{E}_{\mathbf{J}}[\,\cdot\,]$ denotes the expectation with respect to the jump sizes $\mathbf{J}$.
[19] See, for example, Øksendal (2003), p. 145.

used in calculating unique derivatives prices[20]. However, for valuation purposes we need to derive contract prices under the condition of an arbitrage-free market[21], which will be shown in this section.

According to the seminal papers of Harrison and Kreps (1979) and Harrison and Pliska (1981), it is a well known and rigorously proved fact, if one can find at least one equivalent martingale measure with respect to $\mathbb{P}$, then the observed market is arbitrage-free and therefore a derivatives pricing framework can be established. Thus, we establish the link between this equivalent martingale measure $\mathbb{Q}$, also known as the risk-neutral probability measure[22], and the probability measure $\mathbb{P}$ in this section.

Since we are dealing with $M$ stochastic factors, primarily integrated in the short rate $r(\mathbf{x}_t)$, which are all non-tradable goods, we are confronted with an incomplete market. In contrast to other model frameworks in which factors represent prices of tradable goods, we encounter a somewhat more difficult situation to end up in a consistent arbitrage-free pricing approach[23]. Foremost, we need to introduce for every source of uncertainty a market price of risk reflecting the risk aversion of the market. The common procedure in this case is to choose a particular equivalent martingale measure, sometimes also called the pricing measure which determines the appropriate numeraire to be applied[24]. Having chosen the numeraire, which has the function of a denominator of the expected contingent claim and determines the martingale condition for the expectation, we afterwards have to extract yields for different maturities of zero-bond prices. In the next step the model prices of zero bonds

[20] See, for example, Musiela and Rutkowski (2005), p. 10.

[21] The arbitrage-free approach is also known as the partial equilibrium approach. Including preferences of investors, i.e. working with utility functions would be a general equilibrium approach. Schöbel (1995) gives a detailed overview of both approaches.

[22] The terminology can be justified, since in a risk-neutral world, where all market participants act under a risk-neutral utility behavior, the probability measures $\mathbb{P}$ and $\mathbb{Q}$ coincide. See, for example, Duffie (2001), p. 108.

[23] This statement holds only for tradable goods modeled by pure diffusion processes. Otherwise, due to the jump uncertainty one has again to implement some variable compensating jump risk. See Merton (1976).

[24] This can be for example the money market account or zero-coupon bond prices. See Dai and Singleton (2003), pp. 635-637.

are calibrated with respect to this empirical yield curve. In the calibration process for these parameters, two separate approaches can be utilized[25]. In the first approach one computes the particular model parameters under the $\mathbb{P}$ measure together with the different market prices of risk. The other method would be to calibrate the model onto the parameters under the objective measure $\mathbb{Q}$. A problem which is common to all model frameworks, where the instantaneous interest rate $r(\mathbf{x}_t)$ is used to describe the term structure of interest rates is that in general the given yield curve is not matched perfectly. Hence, we rather want an arbitrage-free model, which might not be able to explain perfectly all observed yields, but to state a model with an internally consistent stochastic environment.

In the upcoming subsections, we will first give an outline how the risk-neutral measure is defined and how the particular coefficients under this probability measure $\mathbb{Q}$ can be derived for our affine term-structure model. Due to the jump-diffusion framework, we also focus on the topic that our martingale measure should consider for discontinuous price shocks.

### 2.3.1 Arbitrage and the Equivalent Martingale Measure

Before we start with the formulation of our option-pricing methodology, we need to ensure the existence of an arbitrage-free pricing system. A very useful insight for this delicate matter is given in the above mentioned work of Harrison and Kreps (1979) and Harrison and Pliska (1981). Using measure theory, they judge the market to be arbitrage free enabling the consistent calculation of derivative prices if at least one equivalent martingale measure can be found, corresponding to the physical measure $\mathbb{P}$. Hence, using the money market account as numeraire in order to derive $\mathbb{Q}$, the price of a derivative contract would be just the discounted expectation of its terminal payoff $G(\mathbf{x}_T)$[26]. So our first step is to define the relevant conditions for an equivalent martingale measure.

[25] See Duffie, Pan and Singleton (2000), p. 1354.

[26] See, for example, Geman, Karoui and Rochet (1995) and Dai and Singleton (2003), p. 635.

**Definition 2.3.1 (Equivalent Probability Measure).** *Two probability measures* $\mathbb{P}$ *and* $\mathbb{Q}$ *are equivalent, if for any event* $\mathcal{A}$, $\mathbb{P}(\mathcal{A}) > 0$ *if and only if* $\mathbb{Q}(\mathcal{A}) > 0$.

According to definition 2.3.1, the equivalent probability measure $\mathbb{Q}$ must only agree on the same null sets given by $\mathbb{P}$. The next property we need, in order to obtain the probability measure $\mathbb{Q}$, is the martingale property.

**Definition 2.3.2 (Martingale Property).** *A stochastic process* $f(\mathbf{x}_t, t)$ *is a martingale under the probability measure* $\mathbb{Q}$ *if and only if the equality*

$$f(\mathbf{x}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left[f(\mathbf{x}_T, T, T)\right] \tag{2.8}$$

*holds for any* $t \leq T$.

This last definition ensures the fair game ability of our interest-rate market. Combining definitions 2.3.1 and 2.3.2 lead us to the equivalent martingale measure $\mathbb{Q}$ with respect to $\mathbb{P}$. Thus, to be a fair game, respectively a martingale, the probability measure $\mathbb{Q}$ transforms the probability law for $\mathbf{x}_t$, leaving the null sets of $\mathbb{P}$ untouched. In the next subsection we show the transition of the probability law from the real-world measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$.

### 2.3.2 Derivation of the Risk-Neutral Coefficients

Having found the formal conditions of an equivalent martingale measure, we now want to derive the transformation rule from measure $\mathbb{P}$ to $\mathbb{Q}$. This rule, also called the Radon-Nikodym derivative $\xi(\mathbf{x}_t, t, T)$, is represented by

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathfrak{F}_t} = \frac{\xi(\mathbf{x}_T, T, T)}{\xi(\mathbf{x}_t, t, T)}. \tag{2.9}$$

In order to derive the risk-neutral coefficients, we adopt the corresponding pricing-kernel methodology. Doing this, the pricing kernel or Radon-Nikodym derivative $\xi(\mathbf{x}_t, t, T)$, belongs itself to the class of exponential-affine functions of $\mathbf{x}_t$[27]. The principle of risk-neutrality implies for the state-price kernel an

[27] See, for example, Dai and Singleton (2003), p. 642.

expected discount rate equal to the instantaneous risk-free rate $r(\mathbf{x}_t)$. Thus, we need the equation

$$\mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\xi(\mathbf{x}_t,t,T)}{\xi(\mathbf{x}_t,t,T)}\right] = -r(\mathbf{x}_t)\,\mathrm{d}t, \tag{2.10}$$

to hold. Using this type of state-price kernel, we have the discounted expectation of an interest-rate derivatives price to fulfill the definition of a martingale as described in theorem 2.3.2. Consequently, ensuring the expectation made above holds and considering the systematic risk factors, we choose the specific form of $\xi(\mathbf{x}_t,t,T)$ to satisfy

$$\frac{\mathrm{d}\xi(\mathbf{x}_t,t,T)}{\xi(\mathbf{x}_t,t,T)} = -r(\mathbf{x}_t)\,\mathrm{d}t - \boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)'\,\mathrm{d}\mathbf{W}^{\mathbb{P}} - \boldsymbol{\Lambda}_{\boldsymbol{\lambda}}'\left(\mathrm{d}\mathbf{N}(\boldsymbol{\lambda}^{\mathbb{P}}t) - \boldsymbol{\lambda}^{\mathbb{P}}\,\mathrm{d}t\right). \tag{2.11}$$

The vectors $\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)$ and $\boldsymbol{\Lambda}_{\boldsymbol{\lambda}}$ compensate the sources of risk under the risk-neutral measure $\mathbb{Q}$ for the vector of Brownian motions and the vector of Poisson processes, respectively. The vector $\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)$ is characterized by the two relations[28]

$$\begin{aligned}\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)'\,\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t) &= l_0 + \mathbf{l}_1'\mathbf{x}_t \\ \boldsymbol{\Sigma}(\mathbf{x}_t)\,\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t) &= \mathbf{s}_0 + \mathbf{s}_1\mathbf{x}_t\end{aligned}$$

with $l_0 \in \mathbb{R}$, $\mathbf{l}_1, \mathbf{s}_0 \in \mathbb{R}^M$, and $\mathbf{s}_1 \in \mathbb{R}^{M\times M}$. Defining $\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)$ like this, we ensure the exponential-affine structure in the pricing kernel $\xi(\mathbf{x}_t,t,T)$. In contrast to the constant, $N$-dimensional vector $\boldsymbol{\Lambda}_{\boldsymbol{\lambda}}$, we need to establish in $\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)$ a dependence on the state vector $\mathbf{x}_t$ because of a possibly non-zero matrix $\boldsymbol{\Sigma}_1$[29]. Thus, if a particular factor $x_t^{(m)}$ has a constant volatility coefficient, meaning its volatility does not depend on any element in $\mathbf{x}_t$, there is either no dependence on $\mathbf{x}_t$ for the respective element in the the vector $\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)$ and vice versa. Since $\boldsymbol{\lambda}^{\mathbb{P}}$ is the vector of expected arrival rates, we have with

$$\mathbb{E}^{\mathbb{P}}\left[\mathrm{d}\mathbf{N}(\boldsymbol{\lambda}^{\mathbb{P}}t) - \boldsymbol{\lambda}^{\mathbb{P}}\,\mathrm{d}t\right] = \mathbf{0}_N,$$

a $\mathbb{P}$-martingale, representing a vector of compensated Poisson processes[30].

[28] Compare, for example, with Duffie, Pan and Singleton (2000), Culot (2003), and Dai and Singleton (2003).

[29] Dealing with a Square-Root process, we cannot set the particular market price of risk to a constant value, see Cox, Ingersoll and Ross (1985b), Section 5.

[30] A compensated Poisson process can be roughly seen as a discontinuous equivalent of a Brownian motion. See, for example, Karatzas and Shreve (1991), p. 12.

As a consequence of this incomplete market, the vectors $\mathbf{\Lambda}_{\mathbf{\Sigma}}(\mathbf{x}_t)$ and $\mathbf{\Lambda}_{\boldsymbol{\lambda}}$ are not uniquely defined. Therefore, the pricing kernel itself is not uniquely defined either and we have to determine these risk price vectors with a calibration of yields generated by the model to the empirical yield curve as mentioned earlier. We assume this calibration to depend on the yields of traded zero-coupon bonds $P(\mathbf{x}_t, t, T)$ with different times to maturities[31]. Suppressing unnecessary notations for convenience and applying Itô's Lemma, we get the following SDE for the $\mathbb{P}$-dynamics of a zero-coupon bond

$$\mathrm{d}P(\mathbf{x}_t, t, T) = \mu_P \, \mathrm{d}t + \boldsymbol{\sigma}'_P \, \mathrm{d}\mathbf{W}^{\mathbb{P}} + \mathbf{J}_P \, \mathrm{d}\mathbf{N}(\lambda^{\mathbb{P}} t) \tag{2.12}$$

with drift, diffusion and jump components[32]

$$\begin{aligned} \mu_P = & \frac{\partial P(\mathbf{x}_t, t, T)}{\partial t} + \boldsymbol{\mu}^{\mathbb{P}}(\mathbf{x}_t)' \frac{\partial P(\mathbf{x}_t, t, T)}{\partial \mathbf{x}_t} \\ & + \frac{1}{2} \mathrm{tr} \left[ \mathbf{\Sigma}(\mathbf{x}_t) \mathbf{\Sigma}(\mathbf{x}_t)' \frac{\partial^2 P(\mathbf{x}_t, t, T)}{\partial \mathbf{x}_t \partial \mathbf{x}'_t} \right], \end{aligned} \tag{2.13}$$

$$\boldsymbol{\sigma}_P = \mathbf{\Sigma}(\mathbf{x}_t) \frac{\partial P(\mathbf{x}_t, t, T)}{\partial \mathbf{x}_t}, \tag{2.14}$$

$$\mathbf{J}_P = \mathbf{P}(\mathbf{x}_t, \mathbf{J}, t, T)' - P(\mathbf{x}_t, t, T). \tag{2.15}$$

On the other hand, we impose the martingale condition for traded contracts, which is due to the chosen numeraire,

$$\begin{aligned} P(\mathbf{x}_t, t, T) = & \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(\mathbf{x}_s) \, \mathrm{d}s} P(\mathbf{x}_T, T, T) \right] \\ = & \mathbb{E}^{\mathbb{P}} \left[ \frac{\xi(\mathbf{x}_T, T, T)}{\xi(\mathbf{x}_t, t, T)} P(\mathbf{x}_T, T, T) \right]. \end{aligned} \tag{2.16}$$

Multiplying this last equation with $\xi(\mathbf{x}_t, t, T)$, which is known at time $t$ and therefore a certain quantity, we consequently have $\xi(\mathbf{x}_t, t, T) P(\mathbf{x}_t, t, T)$ to be a martingale and the infinitesimal increment $\mathrm{d}(\xi(\mathbf{x}_t, t, T) P(\mathbf{x}_t, t, T))$ to be a local martingale[33]. According to Theorem 2.2.2 we have

---

[31] Since coupon bonds are commonly traded, zero-bond values can be synthetically generated by coupon stripping.

[32] $\mathbf{P}(\mathbf{x}_t, \mathbf{J}, t, T)$ has the equivalent definition as $\mathbf{f}(\mathbf{x}_t, \mathbf{J}, t, T)$ with all calculations made with respect to $P(\mathbf{x}_t, t, T)$. See Theorem 2.2.2.

[33] The existence of a *local* martingale under the new measure $\mathbb{Q}$ is sufficient for the no-arbitrage condition. See Delbaen and Schachermayer (1995) and Øksendal (2003) Section 12.1., respectively.

$$
\begin{aligned}
&\mathrm{d}(\xi(\mathbf{x}_t,t,T)P(\mathbf{x}_t,t,T)) \\
&\qquad = \xi(\mathbf{x}_t,t,T)\,\mathrm{d}P(\mathbf{x}_t,t,T) + P(\mathbf{x}_t,t,T)\;\mathrm{d}\xi(\mathbf{x}_t,t,T) \\
&\qquad\quad + \mathrm{d}P(\mathbf{x}_t,t,T)\;\mathrm{d}\xi(\mathbf{x}_t,t,T) \\
&\qquad = \xi(\mathbf{x}_t,t,T)\mu_P\,\mathrm{d}t + \xi(\mathbf{x}_t,t,T)\boldsymbol{\sigma}'_P\,\mathrm{d}\mathbf{W}^{\mathbb{P}} \\
&\qquad\quad + \xi(\mathbf{x}_t,t,T)\mathbf{J}_P\mathbf{N}(\lambda^{\mathbb{P}}) \qquad (2.17)\\
&\qquad\quad - P(\mathbf{x}_t,t,T)\,\xi(\mathbf{x}_t,t,T)r(\mathbf{x}_t)\;\mathrm{d}t \\
&\qquad\quad - P(\mathbf{x}_t,t,T)\,\xi(\mathbf{x}_t,t,T)\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)'\;\mathrm{d}\mathbf{W}^{\mathbb{P}} \\
&\qquad\quad - P(\mathbf{x}_t,t,T)\,\xi(\mathbf{x}_t,t,T)\boldsymbol{\Lambda}'_{\boldsymbol{\lambda}}\left(\mathrm{d}\mathbf{N}(\boldsymbol{\lambda}^{\mathbb{P}}t) - \boldsymbol{\lambda}^{\mathbb{P}}\,\mathrm{d}t\right) \\
&\qquad\quad - \xi(\mathbf{x}_t,t,T)\boldsymbol{\sigma}'_P\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}(\mathbf{x}_t)\;\mathrm{d}t - \xi(\mathbf{x}_t,t,T)\mathbf{J}_P\mathbf{I}_N^{\boldsymbol{\lambda}^{\mathbb{P}}}\boldsymbol{\Lambda}_{\boldsymbol{\lambda}}\,\mathrm{d}t.
\end{aligned}
$$

In the last equation, we used for the infinitesimal time increments the relation

$$\mathrm{d}t\,\mathrm{d}t = 0,$$

and for the vector of uncorrelated Brownian motions

$$\mathrm{d}\mathbf{W}^{\mathbb{P}}\,\mathrm{d}\mathbf{W}^{\mathbb{P}'} = \mathbf{I}_M\,\mathrm{d}t.$$

Similarly, the corresponding expression for the vector of independent Poisson processes is

$$\mathrm{d}\mathbf{N}(\boldsymbol{\lambda}^{\mathbb{P}}t)\,\mathrm{d}\mathbf{N}(\boldsymbol{\lambda}^{\mathbb{P}}t)' = \mathbf{I}_N^{\boldsymbol{\lambda}^{\mathbb{P}}}\;\mathrm{d}t,$$

where $\mathbf{I}_N^{\boldsymbol{\lambda}^{\mathbb{P}}}$ represents a matrix consisting of the diagonal elements

$$\operatorname{diag}\left[\mathbf{I}_N^{\boldsymbol{\lambda}^{\mathbb{P}}}\right] = \boldsymbol{\lambda}^{\mathbb{P}},$$

and zeros otherwise. In the next step, we divide for notational ease all coefficients of the zero-bond SDE (2.12) by $P(\mathbf{x}_t,t,T)$. Hence, we use hereafter the normalized coefficients,

$$
\begin{aligned}
\tilde{\mu}_P &= \frac{\mu_P}{P(\mathbf{x}_t,t,T)}, \\
\tilde{\boldsymbol{\sigma}}_P &= \frac{\boldsymbol{\sigma}_P}{P(\mathbf{x}_t,t,T)}, \\
\tilde{\mathbf{J}}_P &= \frac{\mathbf{J}_P}{P(\mathbf{x}_t,t,T)}.
\end{aligned}
$$

Combining condition (2.16) and equation (2.17), and keeping in mind that under $\mathbb{P}$-dynamics, the Brownian motions and the compensated Poisson processes in equation (2.11) are martingales, we get for the expectation

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\left(\xi(\mathbf{x}_t,t,T)P\left(\mathbf{x}_t,t,T\right)\right)}{\xi(\mathbf{x}_t,t,T)P\left(\mathbf{x}_t,t,T\right)}\right] &= \tilde{\mu}_P\,\mathrm{d}t + \mathbb{E}_{\mathbf{J}}\left[\tilde{\mathbf{J}}_P\right]\boldsymbol{\lambda}^{\mathbb{P}}\,\mathrm{d}t \\ &\quad - r\left(\mathbf{x}_t\right)\,\mathrm{d}t - \tilde{\boldsymbol{\sigma}}'_P\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}\left(\mathbf{x}_t\right)\,\mathrm{d}t \\ &\quad - \mathbb{E}_{\mathbf{J}}\left[\tilde{\mathbf{J}}_P\right]\mathbf{I}_N^{\boldsymbol{\lambda}^{\mathbb{P}}}\boldsymbol{\Lambda}_{\boldsymbol{\lambda}}\,\mathrm{d}t \equiv 0.\end{aligned} \tag{2.18}$$

If we now solve equation (2.18) for the modified drift coefficient $\tilde{\mu}_P$, subsequently eliminating all $\mathrm{d}t$ terms, we eventually end up with the relation

$$\tilde{\mu}_P = r\left(\mathbf{x}_t\right) + \tilde{\boldsymbol{\sigma}}'_P\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}\left(\mathbf{x}_t\right) + \mathbb{E}_{\mathbf{J}}\left[\tilde{\mathbf{J}}_P\right]\left(\mathbf{I}_N^{\boldsymbol{\lambda}^{\mathbb{P}}}\boldsymbol{\Lambda}_{\boldsymbol{\lambda}} - \boldsymbol{\lambda}^{\mathbb{P}}\right), \tag{2.19}$$

which means that the rate of return of a zero bond must be equal to the risk free short rate plus some terms reflecting the particular risk premiums of the different sources of uncertainty.

We are now ready to identify the corresponding formal expressions under $\mathbb{Q}$-dynamics of the coefficient parameters $\boldsymbol{\mu}^{\mathbb{P}}$ and $\boldsymbol{\lambda}^{\mathbb{P}}$. Comparing equation (2.13) with (2.19) lead us to the fundamental partial differential equation for zero-bond prices[34]

$$\begin{aligned}&\frac{\partial P\left(\mathbf{x}_t,t,T\right)}{\partial t} + \frac{\partial P\left(\mathbf{x}_t,t,T\right)}{\partial \mathbf{x}'_t}\left(\boldsymbol{\mu}^{\mathbb{P}} - \boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}\left(\mathbf{x}_t\right)\right) \\ &\quad + \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Sigma}(\mathbf{x}_t)'\frac{\partial^2 P\left(\mathbf{x}_t,t,T\right)}{\partial \mathbf{x}_t\partial \mathbf{x}'_t}\right] \\ &\quad + \mathbb{E}_{\mathbf{J}}\left[\mathbf{J}_P\right]\left(\boldsymbol{\lambda}^{\mathbb{P}} - \mathbf{I}_N^{\boldsymbol{\lambda}^{\mathbb{P}}}\boldsymbol{\Lambda}_{\boldsymbol{\lambda}}\right) = r\left(\mathbf{x}_t\right)P\left(\mathbf{x}_t,t,T\right).\end{aligned} \tag{2.20}$$

According to equation (2.20), together with Itô's Lemma, and the Feynman-Kac representation, we are able to express the risk-neutral parameters as

$$\boldsymbol{\mu}^{\mathbb{Q}} = \boldsymbol{\mu}^{\mathbb{P}} - \boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Lambda}_{\boldsymbol{\Sigma}}\left(\mathbf{x}_t\right) = \boldsymbol{\mu}_0^{\mathbb{Q}} + \boldsymbol{\mu}_1^{\mathbb{Q}}\mathbf{x}_t, \tag{2.21}$$

$$\boldsymbol{\lambda}^{\mathbb{Q}} = \boldsymbol{\lambda}^{\mathbb{P}} - \mathbf{I}_N^{\boldsymbol{\lambda}^{\mathbb{P}}}\boldsymbol{\Lambda}_{\boldsymbol{\lambda}}. \tag{2.22}$$

Since the jump intensities $\boldsymbol{\lambda}^{\mathbb{Q}}$ have to be positive, we need $\boldsymbol{\Lambda}_{\boldsymbol{\lambda}}$ small enough to ensure the positiveness of the jump intensities under the risk-neutral measure $\mathbb{Q}$ given the intensity vector $\boldsymbol{\lambda}^{\mathbb{P}}$. The constant coefficients in the variance-covariance matrix (2.3) remain unchanged under the new measure $\mathbb{Q}$. This

[34] Once the risk-neutral coefficients for the interest-rate process are determined, equation (2.20) can be used to price any European contingent claim by exchanging the terminal condition and replacing $P\left(\mathbf{x}_t,t,T\right)$ with the particular function representing the price of the derivative security to be calculated.

phenomenon is often referred to as the diffusion invariance principle, although this terminology is not completely correct. We want to emphasize that the variations of the Brownian motions only coincide under both measures $\mathbb{P}$ and $\mathbb{Q}$, if the variance-covariance matrix exclusively exhibits constant coefficients[35]. Otherwise, we are implicitly dealing with a different time-dependent variance-covariance matrix, since the vector $\mathbf{x}_t$ experiences a drift correction and therefore affects the relation given in equation (2.3). Consequently, the probability transformation law of the process $\mathbf{x}_t$ from $\mathbb{P}$ to $\mathbb{Q}$ does not only contain a drift compensation. Moreover, besides the jump intensity correction, the very shape of the probability density itself can be changed, due to the implicitly altered variations of the diffusion terms.

Hence, calibrating the theoretical term-structure model to zero-bond yields, whether estimating the parameters of the left or the right sides of equations 2.21 and 2.22, results in the following SDE governing the particular factors under risk-neutral dynamics

$$\mathrm{d}\mathbf{x}_t = \boldsymbol{\mu}^{\mathbb{Q}}(\mathbf{x}_t)\,\mathrm{d}t + \boldsymbol{\Sigma}(\mathbf{x}_t)\,\mathrm{d}\mathbf{W}_t^{\mathbb{Q}} + \mathbf{J}\,\mathrm{d}\mathbf{N}\left(\boldsymbol{\lambda}^{\mathbb{Q}}t\right), \tag{2.23}$$

which we use in the subsequent sections as starting point for our calculations.

## 2.4 The Characteristic Function

In this section, we first give a brief overview of the abilities of characteristic functions and show afterwards how the characteristic function of an exponential-affine process, as given in equation (2.1), can be derived. We generalize the principle of building characteristic functions for some scalar process $g(\mathbf{x}_t)$, which is essential for our derivatives pricing technique. Since characteristic functions play a major part in our derivation of semi closed-form solutions for interest-rate derivatives, we discuss also some of their fundamental properties.

Before we introduce the characteristic function itself, we first need to state a definition of Fourier Transformations of some deterministic variable $x$[36].

[35] In this case, we would deal with the matrix $\boldsymbol{\Sigma}(\mathbf{x})\boldsymbol{\Sigma}(\mathbf{x})' = \boldsymbol{\Sigma}_0$.

[36] In the literature, there seems to exist various definitions for this type of transformation. Thus, we want to clarify the issue by giving a straightforward definition

This concept belongs to the field of integral transformations[37] and is a widely used tool in engineering disciplines, especially in signal processing.

**Definition 2.4.1 (General one-dimensional Fourier Transformation and its Inversion).** *We define the Fourier Transformation $\mathcal{F}^x[\,\cdot\,]$ of some function $f(x)$ with respect to the independent variable $x$ as*

$$\mathcal{F}^x[f(x)] = \int_{-\infty}^{\infty} e^{\imath z x} f(x)\,\mathrm{d}x = \hat{f}(z), \tag{2.24}$$

*where $z \in \mathbb{C}$ denotes the transform variable in Fourier space, satisfying the restriction $\mathrm{Im}(z) \in (\underline{\chi}, \overline{\chi})$ with $\underline{\chi}$ and $\overline{\chi}$ denoting some lower and upper boundaries guaranteeing the existence of the Fourier Transformation, $\imath = \sqrt{-1}$ as the standard imaginary unit, and $\hat{f}(z)$ as the shorthand notation for the Fourier Transformation of $f(x)$ with respect to its argument $x$.*

*Accordingly, the inverse transformation operator $\mathcal{F}^{-1}[\cdot]$ is then defined by*

$$\mathcal{F}^{-1}[\hat{f}(z)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\imath z x} \hat{f}(z)\,\mathrm{d}z = f(x). \tag{2.25}$$

Due to the exponential character of the Fourier Transformation, we need to establish in equation (2.25) a normalization factor of $2\pi$. The terminology *general* one-dimensional Fourier Transformation, in contrast to an *ordinary* one-dimensional Fourier Transformation, is used because we do not limit the transformation variable $z$ to be on the real line[38]. Thus, we allow $z$ to be complex-valued, which makes equation (2.24) and (2.25) a line integral, performed parallel to the real line. Note that both the transform and its inverse

in this section. In financial studies our definition according to equation (2.24) of a Fourier Transformation seems to be commonly accepted. See, for example, Carr and Madan (1999), Bakshi and Madan (2000) and Raible (2000). On the other hand in engineering sciences, the opposite definition of a Fourier Transformation and its inverse operation does exist. See, for example, Duffy (2004).

[37] Other popular integral transformations are e.g. the Laplace transformation or the z-transformation. A comprehensive discussion of the Laplace Transformation is given in Doetsch (1967).

[38] Hence, the equivalent expression *complex* Fourier Transformation is sometimes used in the literature.

operation have to take place on the same strip going through $\mathrm{Im}(z)$, in order to reconstruct the original function $f(x)$.

The advantage in performing this *general* Fourier Transformation is the possibility to derive image functions in cases where the *ordinary* transform approach would fail, e.g. for functions which are unbounded[39]. However, in these cases, the *general* approach enables us to derive solutions for their Fourier Transformations. For example, if we want to compute the Fourier Transformation of a function[40]

$$G(x) = \max(e^x - K, 0),$$

the *ordinary* transformation approach appears to be useless, since

$$\mathcal{F}^x[G(x)] \to \infty.$$

Performing a *general* transformation, in this case within the strip $\mathrm{Im}(z) \in (1, \infty)$, we get[41]

$$\mathcal{F}^x[G(x)] = \frac{K^{1+\imath z}}{\imath z(1+\imath z)}, \tag{2.26}$$

where $\mathrm{Im}(z)$ can be fixed at every value within the above mentioned strip to derive the original function by applying the inverse Fourier Transformation. The different contours in Fourier space of the transformed payoff function given in equation (2.26) are depicted in Figure 2.1. Having derived the fundamental technique to compute Fourier Transformations, which is an essential part in this thesis, we go further and have a look at Fourier Transformations of density functions of stochastic variables, which are commonly known as characteristic functions.

**Definition 2.4.2 (Scalar Characteristic Functions).** *We define the scalar characteristic function* $\psi^{x^{(m)}}(\mathbf{x}_t, z, w_0, \mathbf{w}, t, T)$ *as the expected value of the terminal condition* $G(\mathbf{x}_T) = e^{\imath z x_T^{(m)}}$*, given the state* $\mathbf{x}_t$ *at time* $t \leq T$*. This can be expressed more formally as*

[39] This is the case for most payoff structures of option contracts, e.g. plain vanilla call or put options.

[40] This function represents, for instance, the payoff function of a plain vanilla call option in an asset pricing environment, where $x$ is the natural logarithm of the underlying asset price.

[41] In Section 5.3, Fourier Transformations are derived in detail for different types of payoff functions.

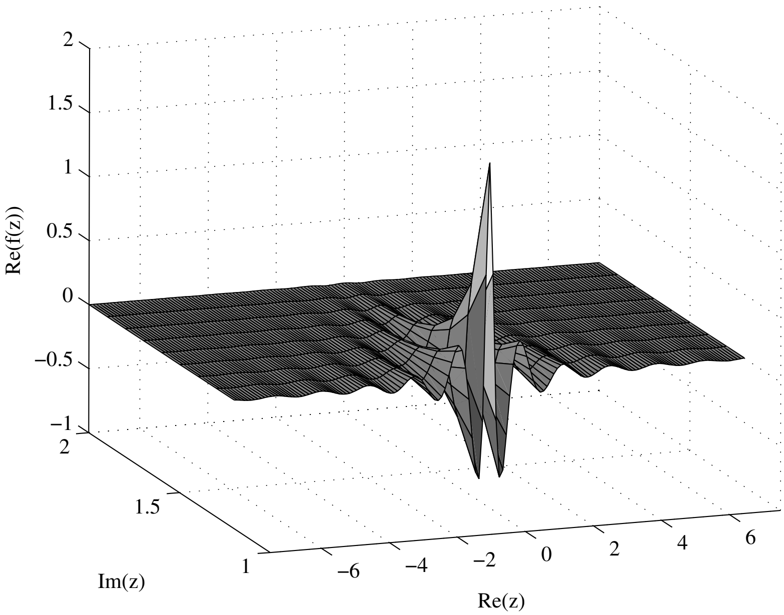


**Fig. 2.1.** Different contours of the Fourier transform in equation (2.26) for a strike of 90 units.

$$
\begin{aligned}
\psi^{x^{(m)}}(\mathbf{x}_t, z, w_0, \mathbf{w}, t, T) =& \mathbb{E}\left[ e^{-\int\limits_t^T r(\mathbf{x}_s)\,\mathrm{d}s + \imath z x_T^{(m)}} \right] \\
=& \int\limits_{\mathbb{R}^M} e^{\imath z x_T^{(m)}} p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)\,\mathrm{d}\mathbf{x}_T,
\end{aligned}
\tag{2.27}
$$

*for all* $m = 1, \ldots, M$. *In the last equality of equation (2.27), the function* $p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)$ *represents the (discounted) transition probability density, starting with an initial state* $\mathbf{x}_t$ *and ending up in time* $T$ *at* $\mathbf{x}_T$*. The continuous discounting is conducted with respect to* $r(\mathbf{x}_{t^*})$ *for* $t > t^* \geq T$*.*

Obviously, if the stochastic process consists only of one variable $x_t$, the characteristic function $\psi^x(x_t, z, 0, 0, t, T)$ is then just the Fourier Transformation of the particular transition density function $p(x_t, x_T, 0, 0, t, T)$. Although the transform operation in equation (2.27) is performed with respect to the terminal state of one single random variable $x_T^{(m)}$, we have to consider the state of the vector $\mathbf{x}_t$ as an argument of the characteristic function. In fact, since we are looking at the overall expectation, equation (2.27) is generally built as the $M$-dimensional integral over the entire state vector $\mathbf{x}_T$[42]. Therefore, we are also able to apply the definition presented above of building a characteristic function for the more general case

$$g(\mathbf{x}_T) = g_0 + \mathbf{g}'\mathbf{x}_T \tag{2.28}$$

with $g_0 \in \mathbb{R}$ and $\mathbf{g} \in \mathbb{R}^M$. This implies, as long as $g(\mathbf{x}_T)$ is a linear combination of the elements in $\mathbf{x}_T$ that only one single transformation variable $z$ necessary. Hence, if we are able to build the characteristic function for the scalar $g(\mathbf{x}_T)$[43], there is only a one-dimensional integral for the inverse operation to be performed, independent of the number of state variables included in $g(\mathbf{x}_T)$. Note, this powerful result will be used in our multi-factor framework. Equipped with these definitions we state next some general and important properties of Fourier Transformations on which we rely in our thesis.

**Proposition 2.4.3 (Important Properties of Characteristic Functions and Fourier Transformations).** *Let $\alpha, \beta, x, y \in \mathbb{R}$, and $f(x), g(y)$ some real-valued functions with Fourier transforms $\hat{f}(z), \hat{g}(z)$ and Fourier Transformation variable $z \in \mathbb{C}$. Then the following relations hold:*

1. *Linearity:*
$$\mathcal{F}^x[\alpha f(x) + \beta g(x)] = \alpha \hat{f}(z) + \beta \hat{g}(z).$$
2. *Differentiation:*
$$\mathcal{F}^x\left[\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\right] = (\imath z)^\alpha \hat{f}(z).$$
3. *Convolution:*
$$\mathcal{F}^x[f(x) * g(x)] = \hat{f}(z)\hat{g}(z).$$
4. *Symmetry:*
$$\pi f(x) = \int_0^\infty e^{-\imath z x} \hat{f}(z)\,\mathrm{d}z = \int_{-\infty}^0 e^{-\imath z x} \hat{f}(z)\,\mathrm{d}z.$$

[42] If $x_t^{(m)}$ would be no subordinated process and independent from all other state variables, equation (2.27) could still utilize the joint density function $p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)$ due to the possible discount factor including $r(\mathbf{x}_t)$.

[43] For example, calculating the general characteristic function for the short rate $r(\mathbf{x}_t)$ itself, we set $g(\mathbf{x}_T) = r(\mathbf{x}_T)$.

5. *Relation of the Moment-Generating and the Characteristic Function:*

$$\mathbb{E}\left[x^{\alpha}\right] = (-\imath)^{\alpha} \left.\frac{\mathrm{d}^{\alpha}\psi^{x}(x_t, z, 0, \mathbf{0}_M, t, T)}{\mathrm{d}z^{\alpha}}\right|_{z=0}.$$

Taking a second glance at Figure 2.1, we are able to justify the symmetry of the Fourier Transformation (2.26) of a real-valued function, mentioned in Proposition 2.4.3. Furthermore, one can clearly identify the dampening property of the characteristic function which is essential in developing a numerical algorithm to compute derivative prices. In the following, we show how the characteristic function for a scalar function $g\left(\mathbf{x}_T\right)$ is derived within the exponential-affine framework. Following Bakshi and Madan (2000), we interpret the characteristic function as a hypothetical contingent claim. Taking more elaborated payoff structures into account, we have to extend the list of permissible arguments for the characteristic function. The more general representation of the characteristic function, which we use hereafter is $\psi^{g(\mathbf{x})}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, t, T)$ with the complex-valued payoff representation at maturity $T$,

$$\psi^{g(\mathbf{x})}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, T, T) = e^{\imath z g(\mathbf{x}_T)}. \tag{2.29}$$

As discussed in the last section, we have to consider that all contingent claims need to be priced under the risk-neutral probability measure $\mathbb{Q}$. Hence, all prices are derived as discounted expectations. Consequently, the discounted expectation of the general form of the terminal condition can be represented as

$$\psi^{g(\mathbf{x})}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, t, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s + \imath z g(\mathbf{x}_T)}\right]. \tag{2.30}$$

However, we need to compute discounted expectations, e.g. for vanilla zero-bond calls, or undiscounted expectations, e.g. in the case of futures instruments. Hence, for futures-style contracts, $w_0$ equals zero and $\mathbf{w}$ is a zero valued vector[44].

In calculating European derivative prices, we rather need the general characteristic function $\psi^{g(\mathbf{x})}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, t, T)$ than the special case of the

[44] The characteristic marking to market for standardized futures-style contracts results in the non-existence of a discount factor in the pricing formula and the relevant PDE, respectively, of such a contract under the risk-neutral measure $\mathbb{Q}$.

characteristic function without considering any discount factor, which is just $\psi^{g(\mathbf{x})}(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, t, T)$, where $\mathbf{0}_M$ represents a $M \times 1$ vector containing exclusively zeros. Applying Theorem 2.2.3 to our hypothetical claim with a solution according to equation (2.30), we take advantage of the Feynman-Kac representation to derive the partial differential equation. Simplifying and suppressing unnecessary notation, we write henceforth $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \equiv \psi^{g(\mathbf{x})}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, t, T)$ and then get the partial differential equation

$$\begin{aligned}
&\frac{\partial \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\partial t} + \boldsymbol{\mu}^{\mathbb{Q}}(\mathbf{x}_t)' \frac{\partial \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\partial \mathbf{x}_t} \\
&+ \frac{1}{2} \operatorname{tr}\left[\boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Sigma}(\mathbf{x}_t)' \frac{\partial^2 \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\partial \mathbf{x}_t \partial \mathbf{x}_t'}\right] \\
&+ \mathbb{E}_{\mathbf{J}}\left[\boldsymbol{\psi}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau)' - \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\right] \boldsymbol{\lambda}^{\mathbb{Q}} \\
&\qquad = \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) r\left(\mathbf{x}_t\right),
\end{aligned} \tag{2.31}$$

where the complex-valued vector $\boldsymbol{\psi}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau)$ contains all jump components with particular elements $(\boldsymbol{\psi}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau))_n = \psi(\mathbf{x}_t + \mathbf{j}_n, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$. The vector $\mathbf{j}_n \in \mathbb{R}^M$ contains as $m$th element the random variable $J_{mn}$ of the amplitude matrix $\mathbf{J}$. Every contingent claim or function dependent on $\mathbf{x}_t$, an arbitrage-free environment presupposed, has to satisfy the same Partial differential equation structure as given in equation (2.31). For example, the corresponding risk-neutral transition density for the characteristic function $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, w_0, \mathbf{w}, \tau)$, with $g\left(\mathbf{x}_T\right) = r\left(\mathbf{x}_T\right)$, which is actually $p(r(\mathbf{x}_t), r(\mathbf{x}_T), w_0, \mathbf{w}, t, T)$ need to satisfy the same partial differential equation as the characteristic function itself[45]. The only difference between them would be the particular terminal payoff condition. Hence, solving the above partial differential equation for $p(r(\mathbf{x}_t), r(\mathbf{x}_T), w_0, \mathbf{w}, t, T)$, we would impose the Dirac delta function as the relevant terminal condition, having its density mass exclusively concentrated in an infinite spike for $r(\mathbf{x}_T)$ at time $T$. Solving equation (2.31) together with this type of boundary condition can be quite challenging and is in many cases just impossible[46]. Thus, it is feasible to first solve equation (2.31) for the general characteristic function, with its smooth and continuous boundary function at $T$, and afterwards do some sort

[45] See Heston (1993), p. 331.

[46] A prominent example is given with the stochastic volatility model of Heston (1993), for which no closed-form representation of the transition density of the underlying equity log-price variable exists.

of normalized integration, the inverse Fourier Transformation, probably in a numerical manner, to get the desired result. Proceeding like this is a very elegant way to find some semi-analytic solution. In contrast, if we want to interpret the terminal payoff function in equation (2.29) as a hypothetical futures-style contract, with solution

$$\psi(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \tau) = \mathbb{E}^{\mathbb{Q}}\left[e^{\imath z g(\mathbf{x}_T)}\right], \tag{2.32}$$

we have a slightly different partial differential equation. In this case the dynamic behavior of $\psi(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \tau)$ is described by the slightly altered PDE

$$\begin{aligned}
&\frac{\partial\psi(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \tau)}{\partial t} + \boldsymbol{\mu}^{\mathbb{Q}}(\mathbf{x}_t)'\frac{\partial\psi(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \tau)}{\partial \mathbf{x}_t} \\
&+ \frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Sigma}(\mathbf{x}_t)'\frac{\partial^2\psi(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \tau)}{\partial\mathbf{x}_t\partial\mathbf{x}_t'}\right] \\
&+ \mathbb{E}_{\mathbf{J}}\left[\boldsymbol{\psi}(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \mathbf{J}, \tau)' - \psi(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \tau)\right]\boldsymbol{\lambda}^{\mathbb{Q}} \\
&= 0,
\end{aligned} \tag{2.33}$$

Hence, the only difference to PDE (2.31) is that the right hand side is now equal to zero to contribute the missing discount rate. Moreover, we can use this futures-style characteristic function $\psi(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \tau)$ to obtain the particular values of the undiscounted transition density function. Thus, to compute the probability density function of the short rate $r(\mathbf{x}_t)$, we use this futures-style solution of the characteristic function together with the identity $g(\mathbf{x}_t) = r(\mathbf{x}_t)$.

Consequently, using a *separation of variables* approach, the partial differential equations in (2.31) and (2.33) can be decoupled into a system of ordinary differential equations. Therefore, we assume for $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ the exponential-affine structure

$$\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) = e^{a(z,\tau)+\mathbf{b}(z,\tau)'\mathbf{x}_t+\imath z g_0}, \tag{2.34}$$

with the scalar and complex-valued coefficient function $a(z, \tau)$ and

$$\mathbf{b}(z,\tau) = \begin{pmatrix} \tilde{b}^{(1)}(z,\tau) \\ \tilde{b}^{(2)}(z,\tau) \\ \vdots \\ \tilde{b}^{(M)}(z,\tau) \end{pmatrix} + \imath z \begin{pmatrix} g^{(1)} \\ g^{(2)} \\ \vdots \\ g^{(M)} \end{pmatrix} = \tilde{\mathbf{b}}(z,\tau) + \imath z \mathbf{g},$$

denotes some complex-valued coefficient vector. In the next step we plug the required expressions of the candidate function (2.34) into equation (2.31). Starting with the time derivative, we get

$$\begin{aligned} &\frac{\partial \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\partial t} \\ &\qquad\qquad = -\left(a(z,\tau)_\tau + \mathbf{b}(z,\tau)'_\tau \mathbf{x}_t\right) \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau), \end{aligned} \tag{2.35}$$

where $a(z,\tau)_\tau$ and $\mathbf{b}(z,\tau)_\tau$ are the first derivatives with respect to the time to maturity variable $\tau$. The gradient vector with respect to the state variables $\mathbf{x}_t$ is given by

$$\frac{\partial \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\partial \mathbf{x}_t} = \mathbf{b}(z,\tau)\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau), \tag{2.36}$$

the Hesse matrix is

$$\frac{\partial^2 \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\partial \mathbf{x}_t \partial \mathbf{x}_t'} = \mathbf{b}(z,\tau)\mathbf{b}(z,\tau)'\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau), \tag{2.37}$$

and the jump component in equation (2.31) can be derived as

$$\begin{aligned} &\mathbb{E}_{\mathbf{J}}\left[\boldsymbol{\psi}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau)' - \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\right] = \\ &\quad \mathbb{E}_{\mathbf{J}}\left[\boldsymbol{\psi}^*(z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau)' - 1\right] \psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau), \end{aligned} \tag{2.38}$$

with the normalized vector

$$\begin{aligned} \boldsymbol{\psi}^*(z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau) &= \frac{\boldsymbol{\psi}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau)}{\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)} \\ &= \begin{pmatrix} e^{\mathbf{b}(z,\tau)'\mathbf{J}_1} \\ e^{\mathbf{b}(z,\tau)'\mathbf{J}_2} \\ \vdots \\ e^{\mathbf{b}(z,\tau)'\mathbf{J}_N} \end{pmatrix}. \end{aligned} \tag{2.39}$$

In this affine framework, it can be easily checked that the normalized amplitude vector $\boldsymbol{\psi}^*(z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau)$ is independent of the actual state of $\mathbf{x}_t$, which results in the special form given by equation (2.39). Therefore, we are able to express the system of ODEs resulting from equations (2.31) and (2.33), respectively, and the affine form proposed in (2.34) in terms of the risk-neutral coefficients derived in Section 2.3.2. According to Theorem 2.2.3, the ODE which has to be solved for the scalar coefficient $a(z,\tau)$ is then

$$
\begin{aligned}
a(z,\tau)_\tau = -w_0 + \boldsymbol{\mu}_0^{\mathbb{Q}\prime}\mathbf{b}(z,\tau) + \frac{1}{2}\mathbf{b}(z,\tau)'\boldsymbol{\Sigma}_0\mathbf{b}(z,\tau) \\
+ \mathbb{E}_{\mathbf{J}}\left[\boldsymbol{\psi}^*(z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau)' - 1\right]\boldsymbol{\lambda}^{\mathbb{Q}},
\end{aligned}
\tag{2.40}
$$

whereas for the vector coefficient $\mathbf{b}(z,\tau)$ we have to solve

$$
\mathbf{b}(z,\tau)_\tau = -\mathbf{w} + \boldsymbol{\mu}_1^{\mathbb{Q}\prime}\mathbf{b}(z,\tau) + \frac{1}{2}\mathbf{b}(z,\tau)'\boldsymbol{\Sigma}_1\mathbf{b}(z,\tau), \tag{2.41}
$$

with boundary conditions $a(z,0) = 0$ and $\mathbf{b}(z,0) = \imath z\mathbf{g}$, respectively. The parameters $w_0$ and $\mathbf{w}$, determine whether we consider a discount rate or not for the characteristic function. The $m$th element of $\mathbf{b}(z,\tau)'\boldsymbol{\Sigma}_1\mathbf{b}(z,\tau)$ can be computed as $\sum_{i,j} b(z,\tau)_i \, (\boldsymbol{\Sigma}_1)_{ijm} \, b(z,\tau)_j$[47]. Moreover, we want to emphasize that the trace operator is circular, meaning the equality

$$
\mathrm{tr}\left[\boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Sigma}(\mathbf{x}_t)'\mathbf{b}(z,\tau)\mathbf{b}(z,\tau)'\right] = \mathrm{tr}\left[\mathbf{b}(z,\tau)'\boldsymbol{\Sigma}(\mathbf{x}_t)\boldsymbol{\Sigma}(\mathbf{x}_t)'\mathbf{b}(z,\tau)\right] \tag{2.42}
$$

holds. Obviously, the right hand side of this last equation represents a scalar and therefore we are able to neglect the trace operator in equation (2.40) and equation (2.41), respectively.

In order to calculate derivatives prices, the coefficients $a(z,\tau)$ and $\mathbf{b}(z,\tau)$ need not exhibit closed-form solutions in any case. There are several scenarios conceivable, e.g. the time integrated expectations of the jump amplitudes have no closed-form representations, or the processes themselves have such complicated structures that there simply does not exist a closed-form solution of the coefficients $a(z,\tau)$ or $\mathbf{b}(z,\tau)$ of the characteristic function. However, if we are able to represent $a(z,\tau)$ and $\mathbf{b}(z,\tau)$ in terms of their ordinary differential equations (2.40) and (2.41), solutions can be efficiently obtained via a Runge-Kutta solver and appropriately integrated within our numerical pricing procedure, such that time consuming Monte-Carlo studies for the pricing of European interest-rate derivatives can be avoided.

[47] See Duffie, Pan and Singleton (2000), p. 1351.

# 3

# Theoretical Prices of European Interest-Rate Derivatives

## 3.1 Overview

In this section, we want to give a representative selection of different interest-rate contracts for which the pricing framework used in this thesis is able to produce semi closed-form solutions[48]. In doing this we distinguish, for didactical purposes, between contracts based on the short rate $r(\mathbf{x}_t)$ and contracts based on a simple yield $Y(\mathbf{x}_t, t, T)$ over a specified time period $\tau$. These yields to maturity are often referred to as simple compound rates, e.g. LIBOR rates, and denote the constant compounding of wealth over a fixed period of time $\tau$, which is related to a zero bond with corresponding time to maturity.

**Definition 3.1.1 (Simply-Compounded Yield to Maturity).** *The simple yield to maturity $Y(\mathbf{x}_t, t, T)$ of a zero bond $P(\mathbf{x}_t, t, T)$, maturing after the time period $\tau$, is defined through the equality*

$$\frac{1}{1 + \tau Y(\mathbf{x}_t, t, T)} = P(\mathbf{x}_t, t, T). \tag{3.1}$$

*Therefore the simple yield to maturity can be derived as*

$$Y(\mathbf{x}_t, t, T) = \frac{P(\mathbf{x}_t, t, T)^{-1} - 1}{\tau} = \frac{1 - P(\mathbf{x}_t, t, T)}{\tau P(\mathbf{x}_t, t, T)}. \tag{3.2}$$

In the following sections, we generally distinguish in the derivation of theoretical prices of contingent claims between contracts based on the instantaneous interest rate $r(\mathbf{x}_t)$ and contracts depending on the simple yield

[48] A comprehensive summary of different valuation formulae of fixed-income securities is given, e.g. Brigo and Mercurio (2001) and Musiela and Rutkowski (2005).

$Y(\mathbf{x}_t, t, T)$. Moreover, we differentiate between contracts with unconditional and conditional exercise rights. This distinction is introduced because of the different mathematical derivation of the particular model prices. For contracts with unconditional exercise, we obtain pricing formulae, which bear strong resemblance to moment-generating functions of the particular underlying state process whereas contracts with conditional exercise rights, i.e. option contracts, need an explicit integration due to the natural exercise boundary. All derivative prices for which we derive the corresponding pricing formulae are European-style derivatives, meaning that the exercise can only be performed at maturity $T$.

## 3.2 Derivatives with Unconditional Payoff Functions

This derivatives class is characterized by the trivial exercise of the contract at maturity. This means that the contract is always exercised, no matter if the holder suffers a loss or make a profit as consequence of the exercise. Although trivially exercised, a zero-coupon bond is a special case of this class since it pays at maturity a predefined *riskless* quantity of monetary units.

**Definition 3.2.1 (Zero-Coupon Bond).** *A zero-coupon bond maturing at time $T$ guarantees its holder the payment of one monetary unit at maturity. The value of this contract at $t < T$ is then denoted as $P(\mathbf{x}_t, t, T)$, which is the expected value of the discounted terminal condition $G(x_T) = 1$. This can be formally expressed as,*

$$P(\mathbf{x}_t, t, T) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \right] \tag{3.3}$$

It is easily seen that the payoff function $G(\mathbf{x}_T)$ used in equation (3.3) is independent both of the time variable and the state variables in $\mathbf{x}_T$. Using the formal definition in equation (3.3), a zero-coupon bond, or as shorthand a zero bond, is just the present value of one monetary unit paid at time $T$. Hence, we are able to interpret $P(\mathbf{x}_t, t, T)$ as the expected discount factor relevant for the time period $t$ up to $T$. Due to this intuitive interpretation, these contracts are often used in calibrating interest-rate models to empirical data sets.

A slightly more elaborated contract is given by the combination of *certain* payments at different times. We denote this contract then as a coupon-bearing bond.

**Definition 3.2.2 (Coupon-Bearing Bond).** *A coupon-bearing bond guarantees its holder a number of $A$ deterministic payments $c_a \in \mathbf{c}$ at specific coupon dates $T_a \in \mathbf{T}$ for $a = 1, \ldots, A$. Typically, at maturity $T_A$, a nominal face value $C$ is included in $c_A$ in addition to the ordinary coupon. The present value of a coupon bond $CB(\mathbf{x}_t, \mathbf{c}, t, \mathbf{T})$ is then given as*

$$CB(\mathbf{x}_t, \mathbf{c}, t, \mathbf{T}) = \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} c_a \right] = \sum_{a=1}^{A} P\left(\mathbf{x}_t, t, T_a\right) c_a. \tag{3.4}$$

Obviously, a coupon-bearing bond, or as shorthand a coupon bond, is just the cumulation of payments $c_a$ discounted with the particular zero-bond prices $P\left(\mathbf{x}_t, t, T_a\right)$.

If a firm is requiring a hedge position for a risk exposure in the form of a future payment of interest, due to an uncertain floating interest rate, we are able to conclude a forward-rate agreement.

**Definition 3.2.3 (Forward-Rate Agreement).** *A forward-rate agreement concluded in time $t$ guarantees its holder the right to exchange his variable interest payments to a fixed rate $K$, scaled upon a notional principal $Nom$. The contract is sold in $t$. The interest payments exchanged relate then to the time period, say $[T, \hat{T}]$ with $t < T < \hat{T}$. We distinguish the cases, where the forward-rate agreement refers to the short rate $r\left(\mathbf{x}_t\right)$ and to the yield $Y\left(\mathbf{x}_t, t, T\right)$. Hence, for a contract based on the short rate, the relevant time interval is then $[T, \hat{T}] = [T, T + dT]$. The price of this contract is given as*

$$\begin{aligned} FRA_r(\mathbf{x}_t, K, Nom, t, T) &= \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r_s\,\mathrm{d}s} \left(K - r\left(\mathbf{x}_T\right)\right) \right] Nom \\ &= \left( K\, P\left(\mathbf{x}_t, t, T\right) - \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r_s\,\mathrm{d}s} r\left(\mathbf{x}_T\right) \right] \right) Nom. \end{aligned} \tag{3.5}$$

*The price for a forward-rate agreement over a discrete time period of length* $\hat{\tau} = \hat{T} - T$*, written on a yield* $Y\left(\mathbf{x}_T, T, \hat{T}\right)$ *and paid in arrears, can be represented as*[49]

$$
\begin{aligned}
FRA_Y(\mathbf{x}_t, &K, Nom, t, T, \hat{T}) \\
&= \hat{\tau}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - Y\left(\mathbf{x}_T, T, \hat{T}\right)\right)\right] Nom \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\left(\hat{\tau}K - P\left(\mathbf{x}_T, T, \hat{T}\right)^{-1} + 1\right)\right] Nom \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T} r(\mathbf{x}_s)\,\mathrm{d}s}\left(P\left(\mathbf{x}_T, T, \hat{T}\right)(\hat{\tau}K + 1) - 1\right)\right] Nom \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T} r(\mathbf{x}_s)\,\mathrm{d}s}\left(P\left(\mathbf{x}_T, T, \hat{T}\right) - \tilde{K}\right)\right] \frac{Nom}{\tilde{K}} \\
&= \left(P\left(\mathbf{x}_t, t, \hat{T}\right) - \tilde{K}P\left(\mathbf{x}_t, t, T\right)\right) \frac{Nom}{\tilde{K}},
\end{aligned}
\tag{3.6}
$$

*with* $\tilde{K} = \frac{1}{\hat{\tau}K+1}$.

To give a more illustrative example, we consider a firm, which has to make a future payment subject to an uncertain, floating rate of interest. Reducing the immanent interest-rate risk exposure, this firm wants to transform this payment into a certain cash-flow, locked at a fixed rate $K$. This can be achieved by contracting a forward-rate agreement, therefore exchanging the floating interest rate to the fixed rate $K$. Thus, the firm is, in its future calculation, independent of the evolution of the term structure.

[49] Here we use the fact that the exponential-affine model exhibits the Markov ability. Thus, the expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\right] = P\left(\mathbf{x}_t, t, \hat{T}\right)$ can be represented as the iterated expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T} r(\mathbf{x}_s)\,\mathrm{d}s}\mathbb{E}^{\mathbb{Q}_T}\left[e^{-\int_T^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\right]\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T} r(\mathbf{x}_s)\,\mathrm{d}s}P\left(\mathbf{x}_T, T, \hat{T}\right)\right]$, where the inner expectation is made with respect to time $T$.

Another point, we want to mention is the special strike value $K = K_{FRA}$ for which the yield-based forward-rate agreement becomes a fair zero value at time $t$. This value is commonly referred to as the forward rate and corresponds then to the simply-compounded rate

$$\begin{aligned} K_{FRA} &= \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\left(P\left(\mathbf{x}_T, T, \hat{T}\right)^{-1} - 1\right)\right]}{\hat{\tau} P\left(\mathbf{x}_t, t, \hat{T}\right)} \\ &= \frac{P\left(\mathbf{x}_t, t, T\right) - P\left(\mathbf{x}_t, t, \hat{T}\right)}{\hat{\tau} P\left(\mathbf{x}_t, t, \hat{T}\right)} \\ &= \frac{1}{\hat{\tau}}\left(\frac{P\left(\mathbf{x}_t, t, T\right)}{P\left(\mathbf{x}_t, t, \hat{T}\right)} - 1\right). \end{aligned} \tag{3.7}$$

Most of the time a firm does not want to insure itself against a floating interest payment for only one time period. For example, the firm has to serve a debt contract, which is linked to a LIBOR interest rate. In this case, the firm possibly wants to reduce its risk exposure due to the floating interest accrues over time and it is desired to make an exchange of interest payments for several successive time periods, where in each period the payment for the relevant floating rate is exchanged with a fixed rate $K$. This task can be achieved buying a receiver swap contract.

**Definition 3.2.4 (Swap).** *A forward-starting interest-rate receiver swap is defined as a portfolio of forward-rate agreements for different time periods $T_{a+1} - T_a$ with $T_a \in \mathbf{T}$ and $t < T_a$ for $a = 1, \ldots, A$ on the same strike rate $K$. The payments of the contract are made at dates $T_2, \ldots, T_A$, whereas the contract is said to reset the floating rate at dates $T_1, \ldots, T_{A-1}$.*

*Due to the instantaneous character of the floating rate based swap contract, the payment and reset dates coincide. Hence, the swap contract in this case, with nominal principal $Nom$ and $A$ payment dates contained in the vector $\mathbf{T}$, can be represented as*

$$
\begin{aligned}
SWA_r(\mathbf{x}_t, &K, Nom, t, \mathbf{T}) \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{a=1}^{A} e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - r\left(\mathbf{x}_{T_a}\right)\right)\right] Nom \\
&= Nom \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - r\left(\mathbf{x}_{T_a}\right)\right)\right] \\
&= Nom \left(K \sum_{a=1}^{A} P\left(\mathbf{x}_t, t, T_a\right)\right. \\
&\qquad \left. - \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} r\left(\mathbf{x}_{T_a}\right)\right]\right).
\end{aligned}
\tag{3.8}
$$

*The equivalent representation for a swap contract, exchanging a yield-based floating rate at $A-1$ payment dates paid in-arrears is then*

$$
\begin{aligned}
&SWA_Y(\mathbf{x}_t, K, Nom, t, \mathbf{T}) \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{a=1}^{A-1} e^{-\int_t^{T_{a+1}} r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - Y\left(\mathbf{x}_{T_a}, T_a, T_{a+1}\right)\right) \hat{\tau}_{a+1}\right] Nom \\
&= Nom \times \\
&\quad \sum_{a=1}^{A-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left((K\hat{\tau}_{a+1} + 1) P\left(\mathbf{x}_{T_a}, T_a, T_{a+1}\right) - 1\right)\right] \\
&= Nom \sum_{a=1}^{A-1} \Big(\left(K\hat{\tau}_{a+1} + 1\right) P\left(\mathbf{x}_t, t, T_{a+1}\right) - P\left(\mathbf{x}_t, t, T_a\right)\Big) \\
&= Nom \left(P\left(\mathbf{x}_t, t, T_A\right) - P\left(\mathbf{x}_t, t, T_1\right)\right. \\
&\qquad \left. + K \sum_{a=1}^{A-1} \hat{\tau}_{a+1} P\left(\mathbf{x}_t, t, T_{a+1}\right)\right),
\end{aligned}
\tag{3.9}
$$

*with* $\hat{\tau}_{a+1} = T_{a+1} - T_a$.

In contrast to the total number of $A$ swap payments in equation (3.8), where these payments refer merely to specific time dates, for the yield-based swap contracts we have to consider $A-1$ time periods, which explains the resulting summation term in equation (3.9). Subsequently, a swap contract can be interpreted as the sum of successive forward-rate agreements.

Similar to forward-rate agreements we are able to introduce the terminology of a special strike $K_S$, which makes the yield-based swap contract a fair zero valued contract. This special strike is then denoted as the swap rate and can be represented in the case of a yield-based swap as

$$
\begin{aligned}
K_S &= \frac{\sum_{a=1}^{A-1}\left(P\left(\mathbf{x}_t,t,T_a\right)-P\left(\mathbf{x}_t,t,T_{a+1}\right)\right)}{\sum_{a=1}^{A-1}\hat{\tau}_{a+1}P\left(\mathbf{x}_t,t,T_{a+1}\right)} \\
&= \frac{P\left(\mathbf{x}_t,t,T_1\right)-P\left(\mathbf{x}_t,t,T_A\right)}{\sum_{a=1}^{A-1}\hat{\tau}_{a+1}P\left(\mathbf{x}_t,t,T_{a+1}\right)} .
\end{aligned}
\tag{3.10}
$$

The last contract with unconditional exercise right which we include in the pricing methodology used is an Asian-type average-rate contract based on the floating rate $r\left(\mathbf{x}_t\right)$. These contracts do not belong to the class of traded derivatives in any exchange. However, this type of interest-rate derivative seems to be quite popular in over-the-counter markets[50]. Asian contracts belong to the field of path-dependent derivatives. Thus, the payoff consists not only of the terminal value of the underlying rate at maturity but of the complete sample path over the averaging period.

**Definition 3.2.5 (Unconditional Average-Rate Contract).** *An unconditional average-rate agreement concluded in time $t$ guarantees its holder the right at maturity $T$ to exchange the continuously measured average of the floating rate $r\left(\mathbf{x}_t\right)$ over the period $T-t$ against a fixed strike rate $K$. The value of this difference is then scaled by a nominal principal $Nom$. Hence, the price of this contract is given as*

$$
\begin{aligned}
&UARC_r\left(\mathbf{x}_t,K,Nom,t,T\right) \\
&\quad=\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K-\frac{1}{T-t}\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s\right)\right]Nom \\
&\quad=Nom\left(P\left(\mathbf{x}_t,t,T\right)K-\frac{1}{\tau}\,\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s\right]\right).
\end{aligned}
\tag{3.11}
$$

Consequently, in contrast to the forward-rate agreement according to equation (3.5), where the sole expectation of $r(\mathbf{x}_T)$ played the major part, we are

[50] See Ju (1997).

interested in the discounted expectation of the integral of $r(\mathbf{x}_t)$ over the time to maturity at this point.

## 3.3 Derivatives with Conditional Payoff Functions

In the last subsection, we considered the pricing formulae for contracts with unconditional exercise at maturity under the risk-neutral measure $\mathbb{Q}$. Obviously, these contracts can be expressed e.g. in terms of zero bonds and some constants. In this section we want to derive general pricing formulae for contracts with conditional or optional exercise rights at maturity. These derivatives contracts are therefore often referred to as option contracts. Basically, we are interested in calculating the particular option prices with underlying contracts of the form (3.5), (3.6), and (3.9) with optional exercise rights. Basically, the particular pricing formulae can be separated into zero bond and coupon-bond options, respectively, can be seen as a portfolio of several zero-bond options in case of a yield-based swap contract. Hence, we begin the introduction with option contracts written on a zero bond.

**Definition 3.3.1 (Zero Bond Option).** *We define a zero-bond call (put) option as a contract giving its holder the right, not the obligation, to buy (sell) a zero bond $P\left(\mathbf{x}_t, t, \hat{T}\right)$ for a strike price $K$ at time $T$. The remaining time to maturity of this zero bond at the exercise date of the option is then given as $\hat{\tau}$. Formally, the price of a zero-bond call can be obtained as*

$$\begin{aligned} &ZBC\left(\mathbf{x}_t, K, t, T, \hat{T}\right) \\ &\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int\limits_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \max\left(P\left(\mathbf{x}_T, T, \hat{T}\right) - K, 0\right)\right] \\ &\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int\limits_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \left(P\left(\mathbf{x}_T, T, \hat{T}\right) - K\right)^+\right], \end{aligned} \tag{3.12}$$

*whereas a zero-bond put option can be calculated as*

$$ZBP\left(\mathbf{x}_t, K, t, T, \hat{T}\right) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int\limits_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - P\left(\mathbf{x}_T, T, \hat{T}\right)\right)^+\right]. \tag{3.13}$$

Zero bond options can be used to price two contracts commonly used to hedge interest-rate risk. Namely, we want to introduce cap and floor contracts. In this terminology, a cap contract is meant to hedge upside interest-rate risk exposure. This is often required for a firm which holds some debt position with interest payments on a floating rate base and fears that future interest rates are rising. So it wants the interest rate capped at some fixed level, in order to limit its risk position due to this fixed rate. In contrast to the above introduced forward-rate agreement or swap, a firm can now both participate on advantageously low interest rates and simultaneously cap its interest payments against high rates. The opposite effect can be observed, if an institution or firm has outstanding loans based on a floating rate. In this case the firm is interested in limiting the downside risk, since low floating rates correspond to low interest payments. The contract with the desired properties is then a floor, where interest payments are exchanged under an agreed fixed rate.

**Definition 3.3.2 (Cap and Floor Contract).** *A cap (floor) contract is defined as a portfolio of caplets (floorlets) for different time periods $T_{a+1} - T_a$ with $T_a \in \mathbf{T}$ and $t < T_a$ for $a = 1, \ldots, A$ on the same strike rate $K$. The payments of the contract are made at dates $T_2, \ldots, T_A$, whereas the contract is said to reset the floating rate at dates $T_1, \ldots, T_{A-1}$.*

*Due to the short rate, the character of the floating rate based swap contract, the payment and reset dates coincide. Hence, the model price of a caplet with nominal principal $Nom$ and $A$ payment dates contained within the vector $\mathbf{T}$, is then given by*

$$CPL_r\left(\mathbf{x}_t, K, Nom, t, T_a\right) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s}\left(r\left(\mathbf{x}_{T_a}\right) - K\right)^+\right] Nom. \quad (3.14)$$

*The price of a cap contract, as a simple summation of caplets for different times $T_a \in \mathbf{T}$, can then be represented as*

$$\begin{aligned} &CAP_r\left(\mathbf{x}_t, K, Nom, t, \mathbf{T}\right) \\ &\quad= \sum_{a=1}^{A} CPL_r(\mathbf{x}_t, K, Nom, t, T_a) \\ &\quad= Nom \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s}\left(r\left(\mathbf{x}_{T_a}\right) - K\right)^+\right]. \end{aligned} \quad (3.15)$$

*Subsequently, we have for a floor the pricing formula*

$$FLR_r\left(\mathbf{x}_t, K, Nom, t, \mathbf{T}\right) = Nom \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - r\left(\mathbf{x}_{T_a}\right)\right)^+\right]. \tag{3.16}$$

*The particular yield-based cap and floor options, exchanging, if exercised, arbitrary yields with a fixed rate $K$ at $A-1$ payment dates, are given by*

$$\begin{aligned} CAP_Y\left(\mathbf{x}_t, K, Nom, t, \mathbf{T}\right) &= \sum_{a=1}^{A-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s}\left(\tilde{K}_a - P\left(\mathbf{x}_{T_a}, T_a, T_{a+1}\right)\right)^+\right]\frac{Nom}{\tilde{K}_a} \\ &= \sum_{a=1}^{A-1} ZBP\left(\mathbf{x}_t, \tilde{K}_a, t, T_a, T_{a+1}\right)\frac{Nom}{\tilde{K}_a}, \end{aligned} \tag{3.17}$$

*and*

$$\begin{aligned} FLR_Y\left(\mathbf{x}_t, K, Nom, t, \mathbf{T}\right) &= \sum_{a=1}^{A-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s}\left(P\left(\mathbf{x}_{T_a}, T_a, T_{a+1}\right) - \tilde{K}_a\right)^+\right]\frac{Nom}{\tilde{K}_a} \\ &= \sum_{a=1}^{A-1} ZBC\left(\mathbf{x}_t, \tilde{K}_a, t, T_a, T_{a+1}\right)\frac{Nom}{\tilde{K}_a}, \end{aligned} \tag{3.18}$$

*with* $\tilde{K}_a = \frac{1}{\hat{\tau}_{a+1}K+1}$.

Definition 3.3.2 shows that a cap or floor contract is just the summation of their legs, the caplets and floorlets, respectively. Especially for the more realistic case of yield-based contracts, we can identify the similarity to zero-bond options, since contract prices can be obtained as the summation of these options.

The yield-based options are said to be at the money if the modified strike rate $\bar{K}_a$ is equal to equation (3.10). A cap is therefore in the money if the modified strike rate is less than $K_S$, and for $\bar{K}_a > K_S$ it is out of the money. The opposite results hold for a floor contract. Furthermore, we can conclude that holding a cap contract long and a floor contract short, both with the same contract specifications, we are able to replicate a swap contract. This can be easily justified comparing the payoff of such a portfolio given for a yield $Y(\hat{\tau}_{a+1})$, which is then

$$\begin{aligned}(Y(\mathbf{x}_{T_a}, T_a, T_{a+1}) - K)^+ - (K - Y(\mathbf{x}_{T_a}, T_a, T_{a+1}))^+ \\ = Y(\mathbf{x}_{T_a}, T_a, T_{a+1}) - K,\end{aligned} \tag{3.19}$$

and the corresponding swap payment. Taking the discounted expectation of the sum of terms in equation (3.19) for all periods, we have the equivalent swap contract.

A more challenging contract in calculating model prices is a coupon-bond option. This option is only exercised if the coupon-bond price at maturity exceeds the strike $K$. Hence, we have to apply the maximum operator to the discounted sum of all outstanding coupon payments and the strike price. This is in contrast to the other option contracts mentioned above, where we applied the maximum operator to each term of the sum separately.

**Definition 3.3.3 (Coupon-Bond Option).** *A coupon-bond call (put) option is defined as the right but not the obligation to buy (sell) a coupon bond $CB(\mathbf{x}_T, \mathbf{c}, t, \mathbf{T})$ with payment dates $T_a \in \mathbf{T}$, with $T_a > T$ for $a = 1, \ldots, A$ and strike price $K$. The price of a coupon-bond call option is given by*

$$\begin{aligned}CBC(\mathbf{x}_t, \mathbf{c}, K, t, T, \mathbf{T}) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(CB(\mathbf{x}_T, \mathbf{c}, T, \mathbf{T}) - K\right)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(\sum_{a=1}^{A} P(\mathbf{x}_T, T, T_a)\, c_a - K\right)^+\right],\end{aligned} \tag{3.20}$$

*and the corresponding coupon-bond put option is given by*

$$\begin{aligned}CBP(\mathbf{x}_t, \mathbf{c}, K, t, T, \mathbf{T}) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - CB(\mathbf{x}_T, \mathbf{c}, T, \mathbf{T})\right)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - \sum_{a=1}^{A} P(\mathbf{x}_T, T, T_a) c_a\right)^+\right].\end{aligned} \tag{3.21}$$

Since the maximum operator is not distributive with respect to sums, the term inside the maximum operator in equation (3.20) and (3.21) cannot be decomposed easily without making further assumptions. Another popular option we want to discuss is an option on a swap contract or as shorthand often referred to as a swaption. With a swaption one can choose at the maturity of the option if it is advantageous to enter the underlying swap contract or otherwise leave the option unexercised.

**Definition 3.3.4 (Swaption).** *We define a forward-starting swaption as a contract conferring the right, but not the obligation to enter a forward starting receiver swap at maturity $T$. The particular underlying receiver swap contract is defined according to definition 3.2.4, with $T_1 \geq T$. Formally, the yield-based forward-starting receiver swaption for an underlying swap with $A-1$ payment periods is given as*

$$\begin{aligned} &SWP_Y\left(\mathbf{x}_t, K, Nom, t, T, \mathbf{T}\right) \\ &\quad= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(SWA_Y(\mathbf{x}_T, K, Nom, T, \mathbf{T})\right)^+\right] \\ &\quad= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K\left(\sum_{a=1}^{A-1} P(\mathbf{x}_T, T, T_{a+1})\hat{\tau}_{a+1}\right)\right.\right. \\ &\qquad\left.\left. + P(\mathbf{x}_T, T, T_A) - P(\mathbf{x}_T, T, T_1)\right)^+\right] Nom. \end{aligned} \tag{3.22}$$

Typically, the swaption maturity coincides with the first reset date of the underlying swap contract. Thus, a yield-based receiver swaption with $T_1 = T$, can be equivalently represented as a coupon-bond call option

$$SWP_Y\left(\mathbf{x}_t, K, Nom, t, T_1, \mathbf{T}^*\right) = CBC\left(\mathbf{x}_t, \mathbf{c}_{SWP}, 1, t, T_1, \mathbf{T}^*\right), \tag{3.23}$$

with

$$\mathbf{c}_{SWP} = \begin{pmatrix} K\hat{\tau}_2 \\ K\hat{\tau}_3 \\ \vdots \\ 1 + K\hat{\tau}_A \end{pmatrix} \times Nom,$$

and new time dates

$$\mathbf{T}^* = \begin{pmatrix} T_2 \\ T_3 \\ \vdots \\ T_A \end{pmatrix}.$$

Subsequently, we reduce the valuation problem of a swaption to the calculation of an equivalent coupon-bond option with strike one, a coupon vector $\mathbf{c}_{SWP}$ and a vector with payment dates $\mathbf{T}^*$.

According to the unconditional contract defined in equation (3.11), we are also able to price an average-rate option contract. The definition of the model price of an average-rate option is given below.

**Definition 3.3.5 (Average-Rate Option).** *An average-rate cap option gives its holder the right, but not the obligation to exchange at expiration a fixed strike rate $K$, over the period $T-t$, against the continuously measured average of the short rate $r(\mathbf{x}_t)$. Formally, the price of an average-rate cap option can be obtained as*

$$\begin{aligned} &ARC_r(\mathbf{x}_t, K, Nom, t, T) \\ =&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(\frac{1}{\tau}\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s - K\right)^+\right] Nom. \end{aligned} \tag{3.24}$$

*Consequently, we have for an average-rate floor the pricing formula*

$$\begin{aligned} &ARF_r(\mathbf{x}_t, K, Nom, t, T) \\ =&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - \frac{1}{\tau}\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s\right)^+\right] Nom. \end{aligned} \tag{3.25}$$

Asian options show the advantageous ability to exhibit reduced risk positions in comparison to ordinary options because of the time-averaging of the underlying price process. Moreover, asian option contracts are more robust against price manipulations since the option payoff includes the sample path over a finite time period. These options are not standard instruments traded on exchanges. However, they are popular over-the-counter contracts used by banks and corporations to hedge their interest-rate risk over a time period[51].

For all theoretical option prices presented in this section, we give in Section 5.3 the corresponding pricing formulae which have to be used in a numerical

[51] See, for example, Ju (1997).

scheme. Thus, we distinguish between the calculation of a portfolio of options, e.g. used for the pricing of cap and floor contracts and as a special case for zero-bond options, respectively, and the computation of options on a portfolio which is the case for coupon-bond options and swaption contracts. This is done because only in case of a one-factor interest-rate process semi closed-form solutions for swaptions and coupon bonds can be calculated.

# 4

# Three Fourier Transform-Based Pricing Approaches

## 4.1 Overview

Interest-rate derivatives are widely used instruments to cover possible interest-rate risk exposures. However, to model the term structure more realistically, sophisticated models are required. One way to enhance the capability of the term-structure model is to incorporate more stochastic factors, by, for instance, incorporating a stochastic mean and/or a stochastic volatility, or modeling the term structure with help of an additive interest-rate process. Another way, which would especially enrich the model with the ability to reflect price shocks, lies in implementing jump components in the shape of different Poisson processes with arbitrary stochastic jump amplitudes. Unfortunately, in most cases the pricing of derivatives securities, while incorporating for the underlying interest-rate process both features mentioned above, can only be accomplished with inefficient Monte-Carlo simulations. Hence, more efficient methods are needed to circumvent these time-consuming calculations. As shown in the prominent work of Heston (1993), a way out of this dilemma is achieved by using Fourier Transformation techniques. Doing this, we only need to solve one standardized inversion integral to evaluate the distribution function and then compute the desired derivative prices. The astonishing fact of the approach applied by Heston (1993) is that this Fourier-based valuation technique is independent of the underlying stochastic dynamics of the short-rate process and can be applied as long as the particular characteristic function exists[52]. Bakshi and Madan (2000) generalized this method to interpret the characteristic function itself as a derivative contract with a trigonometric payoff[53]. Zhu (2000) derived various pricing formulae for options with underlying stock prices, where stochastic interest rates, volatilities and jumps were included in a modularized manner. There, the stochastic factors are integrated by parts and the author ends up with a system of ordinary differential equations, which then has to be solved. In this thesis, we go a step further and, by using the transform methods of Lewis (2001), are able to generalize the modular aspect of Fourier-based derivatives pricing into parts of the underlying stochastic behavior and the contract type. This enables us to present valuation techniques, which can be adapted to every desired European-style contract without greater effort, assuming that the generalized Fourier Transformation of the payoff function exists in closed form.

We consider the general exponential-affine model introduced in Section 2.1 for the short rate $r(\mathbf{x}_t)$ and derive a flexible valuation procedure according to the approach given in Lewis (2001). Although we focus in our thesis on the exponential-affine setup, we are also able to extend the framework to incorporate non-affine term-structure models[54], such as the Longstaff (1989) model or the class of quadratic Gaussian models as discussed in Beaglehole and Tenney (1992)[55] and Filipovic (2001), respectively. All we need in the underlying model specification is the exponential separability of the coefficients in the general characteristic function. However, in applying these non-affine model specifications, we have to ignore the possibility of jumps for non-affine factors in order to avoid mixture terms in the fundamental partial differential equation, which would subsequently render the pricing procedure unattainable[56].

[52] Due to our pricing framework we can relax this restriction to the existence of a system of ordinary differential equations.
[53] This methodology is covered in Section 4.2.
[54] See Chapter 10.
[55] In fact, the model of Longstaff (1989) can be represented as a quadratic Gaussian model as shown in Beaglehole and Tenney (1992).
[56] The same holds for the term-structure model in Cheng and Scaillet (2004) where the terminology of a linear-quadratic jump-diffusion model is introduced. Despite the name, jump parts are only valid for linear factors, whereas the quadratic part is not allowed to bear jump parts. This issue is discussed in Section 9.3.

The outline of this chapter is as follows. We start with the comparison of three state of the art Fourier Transformation methodologies used in derivatives research. The Fourier-transformed Arrow-Debreu securities pricing approach is based on the work of Heston (1993)[57]. Afterwards, we present the transform methodology as proposed by Carr and Madan (1999) and then discuss the generalized derivatives pricing setup of Lewis (2001), which display similarities in the derivation of the model price of a contingent claim. Both approaches focus on the Fourier Transformation of the payoff function, whereas Carr and Madan (1999) apply the transform for the strike value, Lewis (2001) does a Fourier Transformation with respect to the state variable. Nevertheless, we provide an extension of the work in Lewis (2001), since we consider a multi-factor environment. One important difference between the pricing approach utilizing Fourier-transformed Arrow-Debreu securities, according to Heston (1993), the Carr and Madan (1999) methodology, and the method of Lewis (2001) is that the latter two approaches do not need to invoke Fourier Transformations for every single term in the pricing formula. Therefore, the transformation is applied on the entire contingent claim, which in a numerical sense is more efficient. Additionally, the these two approaches provide a more stable solution due to the freedom of choosing a contour path for the integration parallel to the real axis in the inversion formulae[58].

Generally, the derivatives we want to price are written on some functional of the underlying stochastic vector process $\mathbf{x}_t$, say $g(\mathbf{x}_t)$. Contingent claims on the short rate and on the yield are European-style derivatives and therefore pay only at maturity $T$ a payoff $G(\mathbf{x}_T)$. The solution of the pricing problems we seek then takes the following form.

**Definition 4.1.1 (General Valuation Problem for European-Style Derivatives).** *We define the general valuation problem of a contract $V(\mathbf{x}_t, t, T)$ as the time $T$ expectation of some (discounted) payoff function $G(\mathbf{x}_T)$ under the risk-neutral probability measure $\mathbb{Q}$, formally defined as*

[57] Recent work with further development and unification was made in Duffie, Pan and Singleton (2000), Bakshi and Madan (2000) and especially on the field of interest-rate derivatives in Chacko and Das (2002).

[58] See Carr and Madan (1999) and Lewis (2001).

$$V(\mathbf{x}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int\limits_t^T r(\mathbf{x}_s)\,\mathrm{d}s} G(\mathbf{x}_T)\right] \\ = \int\limits_{\mathbb{R}^M} G(\mathbf{x}_T)\, p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)\,\mathrm{d}\mathbf{x}_T. \tag{4.1}$$

*The contract can only be exercised at maturity $T$.*

Apart from the underlying stochastic dynamics, the solution to equation (4.1) depends on how $\mathbf{x}_T$ is incorporated within the payoff function $G(\mathbf{x}_T)$. Thus, we follow Chacko and Das (2002) and distinguish for didactical purposes between payoff functions which can be either linear, exponential-linear or integro-linear in $\mathbf{x}_t$. These idealized payoff types are illustrated in Table 4.1 below[59].

**Table 4.1.** Idealized call option payoff functions

| Payoff type | $G(\mathbf{x}_T)$ |
|---|---|
| Linear | $G(\mathbf{x}_T) = (g(\mathbf{x}_T) - K)^+$ |
| Exponential-linear | $G(\mathbf{x}_T) = \left(e^{g(\mathbf{x}_T)} - K\right)^+$ |
| Integro-linear | $G(\mathbf{x}_T) = \left(\int_t^T g(\mathbf{x}_s)\,\mathrm{d}s - K\right)^+$ |

In contrast to option-pricing models written on equities, where constant interest rates are often assumed, in calculating equation (4.1), we are confronted with a more difficult situation. Since both the discount factor and the payoff function $G(\mathbf{x}_T)$ depend on the same stochastic process, we are not able to evaluate these expectations separately and multiply them afterwards[60]. We have

[59] In case of unconditional payoff functions, we use the same classification.

[60] This is a direct consequence of the choice of numeraire made in Section 2.3.

to consider that both expressions are obviously not independent and therefore have to derive the solution of equation (4.1) under their joint stochastic dynamics. However, thanks to the fact that the discount factor itself has an exponential-affine representation[61], we are still able to use the general characteristic function $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ in derivatives pricing. Consequently, equation (4.1) is the starting point for all of the following derivatives pricing approaches.

## 4.2 Heston Approach

Pricing derivatives, using Fourier-transformed Arrow-Debreu securities and state prices, respectively, was introduced in Heston (1993). Since then, several articles utilizing Fourier Transformations in derivatives pricing have been published. Among others we want to mention, because of their relevance, Duffie, Pan and Singleton (2000) and Bakshi and Madan (2000). In the article of Duffie, Pan and Singleton (2000), a comprehensive survey is provided as to how this Fourier inversion methodology can be used to solve derivative prices for general stochastic dynamics. On the other hand, Bakshi and Madan (2000) offer a rigorous survey, of how Fourier-transformed Arrow-Debreu securities can be used to span the underlying market and to price derivative prices. In principle, both articles use the same pricing mechanism, shown below[62].

The basic principle behind the pricing approach with transformed Arrow-Debreu securities is that all derivatives based on the interest rate $r(\mathbf{x}_t)$ described by equation (4.1) have to solve the same partial differential equations (2.31) and (2.33) for futures-style contracts, respectively. The only difference between them is that they need to satisfy different terminal conditions. This statement holds also for the discounted probability density and the characteristic function of the interest-rate process. Therefore, they can be interpreted as hypothetical contingent claims solving the above-mentioned partial differential equations. Whereas derivative prices and probability densities are often

[61] One can easily validate this statement by solving equation (2.30) and (2.34) and setting $z$ equal to zero.

[62] In the context of interest-rate derivatives, Chacko and Das (2002) used this methodology to price the different payoff structures as given in Table 4.1.

hard to obtain, due to their discontinuous terminal conditions[63], the solution for the particular general characteristic function can be recovered, even if jump components are encountered in the stochastic vector process $\mathbf{x}_t$. This is due to a special ability of characteristic functions; their terminal condition is infinitely differentiable and smooth, which make them, from a mathematical point of view, more tractable.

**Definition 4.2.1 (Arrow-Debreu Security).** *We define an Arrow-Debreu security as a contingent claim paying one unit of money at maturity $T$ if and only if a specified state $\mathcal{A}$ occurs. The value $AD(\mathbf{x}_t, t, T)$ of an Arrow-Debreu security under probability measure $\mathbb{Q}_*$ at time $t$ is then given by*

$$AD(\mathbf{x}_t, t, T) = \mathbb{E}^{\mathbb{Q}_*}\left[\mathbb{1}_{\mathcal{A}}\right]. \tag{4.2}$$

*The expression $\mathbb{1}_{\mathcal{A}}$ denotes the indicator function for the event $\mathcal{A}$ in time $T$, which is unity if the state $\mathcal{A}$ occurs and zero otherwise.*

To demonstrate the pricing methodology, we consider the following example of a European call option with a linear payoff function $G(\mathbf{x}_T) = (g(\mathbf{x}_T) - K)^+$ and $g(\mathbf{x}_T)$ is given in equation (2.28)[64]. The solution for this option can then be represented as

$$\begin{aligned} V(\mathbf{x}_t, t, T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(g(\mathbf{x}_T) - K\right)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g(\mathbf{x}_T)\,\mathbb{1}_{g(\mathbf{x}_T)\geq K}\right] \\ &\quad - K\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\mathbb{1}_{g(\mathbf{x}_T)\geq K}\right], \end{aligned} \tag{4.3}$$

[63] For many underlying stochastic dynamics, the solutions cannot be calculated in closed form.

[64] The derivation of option-pricing formulae for exponential-linear and integro-linear payoff structures differs slightly from the derivation of the theoretical option price formula of a linear payoff function as given in this section. The derivation of the particular solutions for these payoff functions can be looked up in Chacko and Das (2002), Sections 2 and 3.

where the expectation is separated into parts. However, the expectations in equation (4.3) are not yet Arrow-Debreu securities in the sense of definition 4.2.1. These expressions still lack some sort of standardization to guarantee the outcome of one monetary unit. Thus, we need to apply the unconditional expectations[65]

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g\left(\mathbf{x}_T\right)\right] \qquad \text{and} \qquad \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\right] = P(\mathbf{x}_t, t, T).$$

Expanding the terms in equation (4.3) with their particular unconditional counterparts, we get

$$\begin{aligned} V(\mathbf{x}_t, t, T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g\left(\mathbf{x}_T\right)\right] \times \\ &\quad \mathbb{E}^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g\left(\mathbf{x}_T\right) \mathbb{1}_{g(\mathbf{x}_T)\geq K}}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g\left(\mathbf{x}_T\right)\right]}\right] \\ &\quad - KP(\mathbf{x}_t, t, T)\mathbb{E}^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \mathbb{1}_{g(\mathbf{x}_T)\geq K}}{P(\mathbf{x}_t, t, T)}\right] \\ &= \Pi_0(\mathbf{x}_t, t, T)\Pi_1(\mathbf{x}_t, t, T) - KP(\mathbf{x}_t, t, T)\Pi_2(\mathbf{x}_t, t, T). \end{aligned} \tag{4.4}$$

Obviously, the normalized functions $\Pi_1(\mathbf{x}_t, t, T)$ and $\Pi_2(\mathbf{x}_t, t, T)$ are two contingent claims and can be interpreted as Arrow-Debreu securities[66]. On the other hand, $\Pi_1(\mathbf{x}_t, t, T)$ can be interpreted as the discounted forward price of the underlying contract. Introducing two artificial changes of measure defined through the Radon-Nikodym derivatives, we get

$$\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{Q}} = \frac{e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g(\mathbf{x}_T)}{\Pi_0(\mathbf{x}_t, t, T)} \qquad \text{and} \qquad \frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{Q}} = \frac{e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}}{P(\mathbf{x}_t, t, T)}.$$

Consequently, we express the above call option price in terms of the particular Arrow-Debreu prices, which is

[65] See Chacko and Das (2002), p. 205.

[66] In the last equation of (4.4), we adopted the notation given in Chacko and Das (2002).

$$
\begin{aligned}
V(\mathbf{x}_t, t, T) = & \Pi_0(\mathbf{x}_t, t, T)\mathbb{E}^{\mathbb{Q}_1}\left[\mathbb{1}_{g(\mathbf{x}_T)\geq K}\right] \\
& - KP(\mathbf{x}_t, t, T)\mathbb{E}^{\mathbb{Q}_2}\left[\mathbb{1}_{g(\mathbf{x}_T)\geq K}\right].
\end{aligned}
\tag{4.5}
$$

Obviously, in calculating the option price in equation (4.5), we need only the general characteristic function with terminal condition $e^{\imath z g(\mathbf{x}_T)}$ and its derivative with respect to $z$, respectively. However, calculations within this pricing framework for the particular functions $\Pi_i(\mathbf{x}_t, t, T)$ are quite different for linear, exponential-linear and integro-linear payoff versions of $G(\mathbf{x}_T)$[67]. Thus, only $P(\mathbf{x}_t, t, T)$ remains unchanged, since this quantity is completely independent of the characteristic payoff part $g(\mathbf{x}_T)$.

Recalling the formal structure of the general characteristic function in (2.30) and the connection between the moment-generating and characteristic function[68], we are able to express $\Pi_0(\mathbf{x}_t, t, T)$ with the help of the derivative of the general characteristic function with respect to the frequency parameter $z$, evaluated at $z = 0$, which is given by[69]

$$
\begin{aligned}
\Pi_0(\mathbf{x}_t, t, T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g(\mathbf{x}_T)\right] \\
&= \frac{1}{\imath}\left(\frac{\mathrm{d}}{\mathrm{d}z}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} e^{\imath z g(\mathbf{x}_T)}\right]\right)\Bigg|_{z=0} \\
&= \frac{\psi_z(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\imath}.
\end{aligned}
\tag{4.6}
$$

Here, the subscript denotes partial differentiation with respect to $z$[70]. Taking into account the exponential-affine structure of the general characteristic function in (2.34), we are able to write equation (4.6) alternatively as

[67] See Chacko and Das (2002).

[68] See Proposition 2.4.3.

[69] Compare with Theorem 1 (c) in Bakshi and Madan (2000).

[70] The result in equation (4.6) is always real, see e.g. Bakshi and Madan (2000). Therefore, the operator Re[...] in this calculation is not necessary at all, which can be justified by checking that all imaginary parts in this equation cancel out except in the term $\imath g(\mathbf{x}_T)$.

$$
\begin{aligned}
\frac{\psi_z(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau)}{\imath} =& \frac{\psi(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau)}{\imath} \times \\
& \left(\frac{\mathrm{d}}{\mathrm{d}z} \ln\left[\psi(\mathbf{x}_t,z,w_0,\mathbf{w},g_0,\mathbf{g},\tau)\right]\right)\Bigg|_{z=0} \\
=& \frac{\psi(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau)}{\imath}\,\phi_z(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau).
\end{aligned}
$$

In the last equation, we used the function $\phi(\mathbf{x}_t,z,w_0,\mathbf{w},g_0,\mathbf{g},\tau)$, which is just the natural logarithm of $\psi(\mathbf{x}_t,z,w_0,\mathbf{w},g_0,\mathbf{g},\tau)$ in our exponential-affine model setup. Thus, the derivative with respect to $z$ of the exponent of the characteristic function is then

$$
\phi_z(\mathbf{x}_t,z,w_0,\mathbf{w},g_0,\mathbf{g},\tau) = a_z(z,\tau) + \tilde{\mathbf{b}}_z(z,\tau)'\mathbf{x}_t + \imath g\left(\mathbf{x}_t\right).
$$

Using the same technique as before, we obtain the value of an ordinary zero bond as

$$
\begin{aligned}
P\left(\mathbf{x}_t,t,T\right) =& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\right] = \mathbb{E}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} e^{\imath z g(\mathbf{x}_T)}\right]\Bigg|_{z=0} \\
=& \psi(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau).
\end{aligned}
\tag{4.7}
$$

Finally, we are left with the calculation of the Arrow-Debreu prices. As mentioned before, these functions $\Pi_1(\mathbf{x}_t,t,T)$ and $\Pi_2(\mathbf{x}_t,t,T)$ can also be interpreted as probabilities. Hence, we apply a tool to determine probabilities from characteristic functions. This can be done with a Fourier inverse transform as proposed in Gil-Pelaez (1951).

**Theorem 4.2.2 (Inversion Theorem of Gil-Pelaez).** *If $\psi^{x_T}(x_t,z,t,T)$ is the characteristic function of a one-dimensional stochastic variable $x_t$ then the probability $\Pr(x_T \geq K)$, given some state $x_t$ and some constant $K$, can be calculated as*

$$
\Pr(x_T \geq K) = \frac{1}{2} + \frac{1}{\pi}\int_{0^+}^{\infty} \mathrm{Re}\left[\frac{\psi^{x_T}(x_t,z,t,T)e^{-\imath zK}}{\imath z}\right]\mathrm{d}z, \tag{4.8}
$$

*with $z \in \mathbb{R}$.*

The expression $0^+$ in equation (4.8) denotes the right-sided limit to the origin. Obviously, the integrand is not defined for a zero-valued transformation variable $z$[71]. Note that the inversion theorem in 4.2.2 is not limited to recover only probabilities for the case of symmetric probability density functions, which might be implicated due to the term $\frac{1}{2}$. Equation (4.9) holds for general probability distributions. The only condition to be satisfied is the existence of the characteristic function or its system of ODEs. Moreover, we are also able to use Theorem 4.2.2 for the linear combination $g(\mathbf{x}_t)$, as long as the outcome is a scalar random variable.

As long as we are able to obtain the general characteristic functions $\psi_1(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ and $\psi_2(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ corresponding to the particular measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$, we are able to compute the values of $\Pi_1(\mathbf{x}_t, t, T)$ and $\Pi_2(\mathbf{x}_t, t, T)$. In analogy to equations (4.6) and (4.7), and keeping the normalization made in (4.4) in mind, we therefore have

$$\psi_1(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) = \frac{\psi_z(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\imath \Pi_0(\mathbf{x}_t, t, T)},$$

and

$$\psi_2(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) = \frac{\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{P\left(\mathbf{x}_t, t, T\right)}.$$

Subsequently, the values of the required Arrow-Debreu securities can be calculated as[72]

$$\Pi_{1,2}(\mathbf{x}_t, t, T) = \frac{1}{2} + \frac{1}{\pi} \int_{0^+}^{\infty} \mathrm{Re}\left[\frac{\psi_{1,2}(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) e^{-\imath z K}}{\imath z}\right] \mathrm{d}z. \quad (4.9)$$

Although the derivation of option prices within this methodology is comprehensible, this technique does entail some drawbacks. Firstly, a general advantage which holds for all pricing methodologies based on Fourier Transformation techniques is that we are not restricted to simple stochastic dynamics of the underlying short-rate process, where the probability density function $p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)$ is explicitly known in closed form[73]. With the continuum of characteristic functions at hand, we are able to calculate option prices for a much broader class of stochastic dynamics. Despite the apparent elegance of this approach, there are also some issues to discuss. Since we expressed the

[71] More on this topic and residue calculus is discussed in Section 4.3.
[72] Compare with the general result in Bakshi and Madan (2000), Theorem 1.
[73] However, there exist density functions for which no characteristic function exists, e.g. a log-normal distributed random variable.

option price as a decomposition of probabilities multiplied with their normalization factors, we have to calculate for a sum of $N$ terms in $G(\mathbf{x}_T)$ the same number of separate Fourier inversions and therefore to perform $N$ numerical integrations. Especially in one-factor interest-rate models, this fact can be avoided using a Fourier transform with respect to $r_T$[74]. From a computational point of view, this can be very time consuming and therefore inefficient compared to the pricing approaches of Carr and Madan (1999) and Lewis (2001). Additionally, the denominator in the integrand of equation (4.9) decays only linearly for the idealized payoff functions, compared to the payoff-transform approaches discussed in the subsequent sections[75]. Another matter we want to address is the integration procedure itself. In equation (4.8), we need to consider carefully the pole at the origin. Sometimes, this can lead to rather unstable results. Another point to mention is that the structure of the option contract dictates the calculation procedure of the particular function $\Pi_j(\mathbf{x}_t, t, T)$. Hence, it first has to be determined whether the payoff function $G(\mathbf{x}_T)$ exhibits linear, exponential-linear or integro-linear terms of $g(\mathbf{x}_T)$[76], which result in different valuation formulae for the option price. This can complicate unnecessarily the computation of option prices in contrast to the approaches discussed in the following sections, where Fourier Transformations of the payoff function are used.[77].

## 4.3 Carr-Madan Approach

Carr and Madan (1999) develop a different method for retrieving option prices using characteristic functions. Instead of applying general characteristic functions to obtain the exercise probabilities and the Arrow-Debreu security prices

[74] See, for example, the pricing of coupon bonds in Section 5.3.3.

[75] The denominator in the payoff transforms of the interest-rate option contracts in table 4.1 are quadratic and therefore have a higher rate of convergence. Compare with the particular transformations given in Section 5.3.

[76] See Chacko and Das (2002) for a comprehensive discussion and classification of payoff functions and derivation of the particular option prices in this transformed Arrow-Debreu security framework.

[77] See Bakshi and Madan (2000), pp. 218-220, cases 1-3, on how to derive the particular $\psi_j(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ for general payment structures. Chacko and Das (2002) also derive the respective valuation algorithms for these payoff structures.

under the particular probability measures $\mathbb{Q}_{1,2}$ as done in the last section, they propose an alternative approach. The intention behind this framework is to formulate a valuation procedure, that can incorporate the FFT, a very efficient tool in deriving Fourier Transformations for different values of the underlying random variable. However, they first perform a Fourier Transformation on the payoff function with respect to the strike variable $K$. Afterwards, interchanging the order of integration, they are able to compute the desired fair price of the option as an inverse Fourier Transformation, thus applying the relevant characteristic function, an example is given below. Obviously, a first advantage of this strategy is that, since we deal with only one transform operation on the option price, in order to compute model price we need only one inverse transformation. As the authors mention, a closed-form solution of the option price in Fourier space is presupposed[78]. Since option prices commonly have at least two terms in the payoff function $G(\mathbf{x}_T)$, numerical calculations with this method are approximately twice as fast. A problem in this approach mostly arises if a Fourier transform on the payoff function with a real-valued frequency variable $z \in \mathbb{R}$ is applied. As mentioned in Bakshi and Madan (2000), the transformed payoff function would not exist at all, due to the unbounded option payoff functions[79]. To circumvent this issue, Carr and Madan (1999) introduce an artificial *dampening* parameter $\alpha$ and derive a modified transformed option price, upon which they apply the inverse transformation procedure. In the following presentation of this methodology we do not refer to an artificial *dampening* parameter $\alpha$; rather we want to introduce a general Fourier Transformation as defined in definition 2.4.1 with $z \in \mathbb{C}$. Moreover, we show that the *dampening* parameter coincides with the negative fixed imaginary part $z_i$ of the frequency variable $z = z_r + \imath z_i$. Following this trail, we get a more intuitive concept of the nature of the *dampening* factor $\alpha$ used by Carr and Madan (1999).

Demonstrating the pricing technique, we rely on the same contract type as in (4.3) with $G(\mathbf{x}_T) = (g(\mathbf{x}_T) - K)^+$ to maintain the comparability to

[78] See Carr and Madan (1999), p. 61. We extend this methodology to allow for characteristic functions with no closed-form representations. This topic is discussed in Chapter 6.

[79] See Bakshi and Madan (2000), p. 215. An exception would be a contract which is bounded on two sides, e.g. a butterfly contract.

previously obtained solutions of our example in equation (4.5). Starting with a Fourier Transformation on the payoff function with respect to $K$, we have

$$\begin{aligned}\mathcal{F}^K\left[G\left(\mathbf{x}_T\right)\right] &= \int_{-\infty}^{\infty} e^{\imath zK} G\left(\mathbf{x}_T\right) \mathrm{d}K = \int_{-\infty}^{\infty} e^{\imath zK}\left(g\left(\mathbf{x}_T\right)-K\right)^{+} \mathrm{d}K \\ &= \int_{-\infty}^{g(\mathbf{x}_T)} e^{\imath zK}\left(g\left(\mathbf{x}_T\right)-K\right) \mathrm{d}K \\ &= \left[-e^{\imath zK}\frac{1+\left(g\left(\mathbf{x}_T\right)-K\right)\imath z}{z^2}\right]_{-\infty}^{g(\mathbf{x}_T)} \\ &= -\frac{e^{\imath z g(\mathbf{x}_T)}}{z^2} \qquad \text{with} \quad \operatorname{Im}(z)<0.\end{aligned} \tag{4.10}$$

The restriction in equation (4.10) upon the imaginary part of $z$ guarantees the finiteness of the transformed payoff function. Thus, we are able to interpret (4.10) as a line integral, which is evaluated parallel to the real axis going through $\imath z_i$. Apart from considerations about the regularity of the payoff transform, the value of $z_i$ can also be used to optimize numerical accuracy of the valuation algorithm[80]. Exploiting the symmetry of real-valued Fourier transforms, the payoff function $G\left(\mathbf{x}_T\right)$ for our specific example, can be expressed by the following inverse transformation problem

$$G\left(\mathbf{x}_T\right) = -\frac{1}{\pi}\int_{0}^{\infty} e^{-\imath zK}\frac{e^{\imath z g(\mathbf{x}_T)}}{z^2}\,\mathrm{d}z. \tag{4.11}$$

Carrying out this inverse operation, we need $z_i$ to be fixed on the same strip used for the transformation. Otherwise, the original function and its image function in dual space would not correspond to each other[81].

The essential part, in expressing the valuation formula as an inverse Fourier-style problem, is the interchanging of the integration order. Furthermore, we have in equation (4.11) an exponential term for both the underlying

[80] See Lee (2004), for a comprehensive analysis of the effect of $z_i$ on the accuracy of the computational result. Note, the derived error bounds in this article are only valid for one particular strike. These results have to be treated carefully for algorithms, where option prices for different strike rates, such as ITM, ATM, and OTM options, are computed simultaneously.

[81] This fact is discussed in Section 2.4.

stochastic variable and the strike rate enabling the application of the characteristic function methodology and afterwards to calculate prices with the FFT. Denoting our exemplary valuation problem of (4.3) in terms of (4.11), we get the following integral representation

$$\begin{aligned} V(\mathbf{x}_t, t, T) &= \int_{\mathbb{R}^M} \left(g\left(\mathbf{x}_T\right) - K\right)^+ p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)\, \mathrm{d}\mathbf{x}_T \\ &= -\frac{1}{\pi} \int_{\mathbb{R}^M} \left( \int_0^\infty e^{-\imath z K} \frac{e^{\imath z g(\mathbf{x}_T)}}{z^2}\, \mathrm{d}z \right) p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)\, \mathrm{d}\mathbf{x}_T. \end{aligned} \tag{4.12}$$

Due to Fubini's theorem, the order of integration can be interchanged[82]. Therefore, we are able to use the alternative representation

$$\begin{aligned} V(\mathbf{x}_t, t, T) &= -\frac{1}{\pi} \int_0^\infty \frac{e^{-\imath z K}}{z^2} \underbrace{\int_{\mathbb{R}^M} e^{\imath z g(\mathbf{x}_T)} p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)\, \mathrm{d}\mathbf{x}_T\ \mathrm{d}z}_{\text{eqn. (2.30)}} \\ &= -\frac{1}{\pi} \int_0^\infty e^{-\imath z K} \frac{\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{z^2}\, \mathrm{d}z. \end{aligned} \tag{4.13}$$

Eventually, we get the Fourier-style valuation formula for the price at time $t$ of a European call option, based on the payoff function $G\left(\mathbf{x}_T\right) = (g(\mathbf{x}_T) - K)^+$. The relationship between the artificial dampening factor $\alpha$ in Carr and Madan (1999) and $z_i$ becomes apparent if we substitute $z = z_r + \imath z_i$ in equation (4.13), which gives

$$\begin{aligned} V(\mathbf{x}_t, t, T) &= -\frac{1}{\pi} \int_0^\infty e^{-\imath (z_r + \imath z_i) K} \frac{\psi(\mathbf{x}_t, z_r + \imath z_i, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{(z_r + \imath z_i)^2}\, \mathrm{d}z_r \\ &= -\frac{e^{z_i K}}{\pi} \int_0^\infty e^{-\imath z_r K} \frac{\psi(\mathbf{x}_t, z_r + \imath z_i, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{z_r^2 + 2\imath z_r z_i - z_i^2}\, \mathrm{d}z_r. \end{aligned} \tag{4.14}$$

Obviously, compared to the corresponding option price formula in Lee (2004), it can easily be verified that the identity $z_i \equiv -\alpha$ holds[83].

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[82] Since all parts of the integral are real-valued, we are able to change the order of integration without any problems.

[83] The modified transformed option price for our example is also given in Lee (2004) Theorem 4.2 as $\hat{c}_{\alpha, G_2}(u)$, where u matches $z_r$. Also compare this result with the general Fourier-style valuation formula in Carr and Wu (2004), p. 136.

In contrast to the Heston pricing approach, the Carr-Madan methodology provides an additional degree of freedom, since we are no longer limited to the case of a real-valued transformation variable $z$. This is of major importance in a numerical scheme for computing derivative prices[84]. Furthermore, we are able to shift the integration contour around any existing pole. However, in these cases the residue of the particular pole must be taken into account[85]. Proceeding like this, the accuracy of the valuation algorithm can be drastically increased[86]. Nevertheless, we are also free to choose the imaginary part in (4.14), such that the contour integrals have to be performed right through a pole. Doing this we first consider the residuals of the poles and then evaluate the integral due to Cauchy's theorem[87].

Generally, the advantage in this approach lies in the availability of a fast numerical integration routine, the FFT algorithm. A properly set procedure, based e.g. on our example in (4.14), can calculate a vast number of derivative prices for alternative strike rates in fractions of a second. On the other hand, Fourier-style solutions in this framework cannot be properly decomposed into parts of the general characteristic function and the transformed payoff function[88]. Thus, we needed a specific payoff function in the derivation of the transformed option price. It would be more convenient and from a numerical perspective more desirable if the integral in (4.14) could be clearly separated into a part of the general characteristic function, which depends on the underlying stochastic dynamics, and a part determined by the contract we want to price. Moreover, there seems to exist a problem for particular models with specific parameter constellations[89]. Finally, we do not prefer this methodology in the first place because it cannot be properly applied for coupon-bond

[84] The choice of the optimal value of $z_i$ is discussed in Section 6.3.3.

[85] See Lee (2004) equations (6) and (7).

[86] This can be validated by Tables 2 and 3 in Lee (2004). The error bounds presented there are up to a thousand times lower, if the integrals are evaluated on contours with no existing poles.

[87] In the next section, we derive valuation formulae using different values of $z_i$.

[88] For example, the transformed option price in equation (4.13) is $-\frac{\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{z^2}$.

[89] Itkin (2005) analyzed the FFT method of Carr and Madan (1999) for the case of an underlying Variance-Gamma process and reports some numerical issues for different lengths of time to maturity $\tau$.

options and swaptions, respectively, with an underlying one-factor interest-rate process. The reason for this is that we need the exercise boundary to be explicit in $r_T$ in order to present the valuation formula in terms of the characteristic function. If this is not the case, we lose characteristics of the stochastic process, which are relevant in the valuation formula and therefore have to be considered within the integration. For example, in the case of coupon-bond options, we encounter the problem of determining numerically a critical value $r_T^*$[90], thus making it impossible to compute the particular option prices. These problems can be circumvented with the approach discussed in the following section.

## 4.4 Lewis Approach

Lewis (2001) presented in his work an alternative way to retrieve not only option prices, but general derivatives prices[91]. The approach is similar to the previously discussed methodology of Carr and Madan (1999), but can be applied to a wider area of pricing problems. Thus, we are able to calculate all derivatives prices presented in Chapter 3 with a single general valuation formula. Fortunately, within this framework, it is also possible to use an efficient numerical tool to compute derivative prices with comparable speed to the FFT algorithm, namely the IFFT algorithm. In contrast to the approach in Carr and Madan (1999), Lewis (2001) introduced a derivatives pricing framework starting with a Fourier Transformation of the payoff function, but this time with respect to the underlying stochastic variable, where the frequency parameter $z \in \mathbb{C}$ is also supposed to be complex-valued. Thus, the advantages discussed in the last section still hold.

As before, our starting point is the payoff function $G(\mathbf{x}_T)$ of a derivatives contract. As in the previous section, the Fourier Transformation is performed on the payoff function, in this case with respect to the scalar $g(\mathbf{x}_T)$. Accordingly, the transformed payoff function is

[90] See Jamshidian (1989).

[91] As mentioned before, the methodology was firstly used in Lewis (2000). However, we refer to Lewis (2001) because of the more detailed derivation and comprehensive discussion of this pricing framework.

$$\mathcal{F}^{g(\mathbf{x}_T)}\left[G(\mathbf{x}_T)\right] = \int_{-\infty}^{\infty} e^{\imath z g(\mathbf{x}_T)} G(\mathbf{x}_T)\,\mathrm{d}g\left(\mathbf{x}_T\right). \tag{4.15}$$

To guarantee the finiteness of the integral in equation (4.15) and the existence of $\mathcal{F}^{g(\mathbf{x}_T)}\left[G\left(\mathbf{x}_T\right)\right]$, respectively, the imaginary part of $z$ has to be restricted, where its domain depends on the specific contract.

Continuing with our example in pricing an interest-rate cap of the form $G\left(\mathbf{x}_T\right) = \left(g\left(\mathbf{x}_T\right) - K\right)^+$, we first calculate the transformed payoff function with respect to $g\left(\mathbf{x}_T\right)$ as

$$\begin{aligned}\mathcal{F}^{g(\mathbf{x}_T)}\left[G(\mathbf{x}_T)\right] &= \int_{-\infty}^{\infty} e^{\imath z g(\mathbf{x}_T)}\left(g\left(\mathbf{x}_T\right) - K\right)^+ \mathrm{d}g\left(\mathbf{x}_T\right) \\ &= -\frac{e^{\imath z K}}{z^2}\end{aligned} \tag{4.16}$$

with

$$\operatorname{Im}(z) > 0.$$

Although this formula bears a strong resemblance to equation (4.10), one remarkable difference between them is the interval of $z_i$, for which the Fourier transform of the particular payoff function exists[92]. Another point we would like to mention is that the transformed payoff function displays the strike rate $K$ in the exponential function instead of $g\left(\mathbf{x}_T\right)$, according to the methodology of Carr and Madan (1999).

Representing the time $t$ option price with the help of the transformed payoff function, we have at the general valuation formula[93]

$$\begin{aligned}V(\mathbf{x}_t, t, T) =& \frac{1}{\pi}\int_{\mathbb{R}^M}\left(\int_0^{\infty} e^{-\imath z g(\mathbf{x}_T)}\mathcal{F}^{g(\mathbf{x}_T)}\left[G\left(\mathbf{x}_T\right)\right]\,\mathrm{d}z\right)\times \\ & p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T)\,\mathrm{d}\mathbf{x}_T,\end{aligned} \tag{4.17}$$

which is for our specific example of an interest-rate cap

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[92] In comparison to equation (4.10), $z_i$ has to be negative.

[93] Again, we take advantage of the symmetry of Fourier Transformations for real-valued functions.

$$= -\frac{1}{\pi} \int\limits_{\mathbb{R}^M} \left( \int\limits_0^\infty e^{-\imath z g(\mathbf{x}_T)} \frac{e^{\imath z K}}{z^2} \,\mathrm{d}z \right) p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T) \,\mathrm{d}\mathbf{x}_T. \tag{4.18}$$

Again, we apply Fubini's theorem, implicating the possibility of interchanging the order of integration in (4.17). Thus, for general payoff functions we obtain

$$\begin{aligned} V(\mathbf{x}_t, t, T) = \frac{1}{\pi} \int\limits_0^\infty & \mathcal{F}^{g(\mathbf{x}_T)} \left[ G\left(\mathbf{x}_T\right) \right] \times \\ & \left( \int\limits_{\mathbb{R}^M} e^{-\imath z g(\mathbf{x}_T)} p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T) \,\mathrm{d}\mathbf{x}_T \right) \mathrm{d}z. \end{aligned} \tag{4.19}$$

Firstly, we focus on the inner integral. In line with the formal definition of the characteristic function, according to equation (2.27), we are able to establish the relation

$$\begin{aligned} \int\limits_{\mathbb{R}^M} & e^{\imath(-z) g(\mathbf{x}_T)} p(\mathbf{x}_t, \mathbf{x}_T, w_0, \mathbf{w}, t, T) \,\mathrm{d}\mathbf{x}_T \\ & = \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau). \end{aligned} \tag{4.20}$$

Inserting this result into equation (4.19), we eventually get the general version of the Fourier-style valuation formula

$$V(\mathbf{x}_t, t, T) = \frac{1}{\pi} \int\limits_0^\infty \mathcal{F}^{g(\mathbf{x}_T)} \left[ G\left(\mathbf{x}_T\right) \right] \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \,\mathrm{d}z, \tag{4.21}$$

which is for our example of a call contract with underlying variable $g\left(\mathbf{x}_T\right)$,

$$-\frac{1}{\pi} \int\limits_0^\infty \frac{e^{\imath z K}}{z^2} \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \,\mathrm{d}z$$

with

$$\mathrm{Im}(z) > 0.$$

In contrast to the pricing procedure introduced by Carr and Madan (1999), we have a strict separation of functionals, which depend either on the contract type or on the underlying stochastic dynamics. The respective part for the contract type is therefore represented by the transformed payoff function, whereas the stochastic dynamics of the underlying process is implemented in terms of the characteristic function. Hence, we have a real modular pricing framework, in which each part in (4.21) can be exchanged without greater effort. Moreover, we can apply this methodology consistently to contracts, whether they are unconditionally exercised or bear an optional exercise right[94]. In particular, for one-factor models with multiple jump components, we are able to take advantage of the fact that for most contracts the domains of $z_i$ are overlapping. This means that $z_i$ can be chosen arbitrarily, subject to compliance with numerical accuracy[95]. Thus, we usually have to evaluate $\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ only once for different values of $z_r$. Afterwards, these precomputed values can be used for all relevant contract types needed. This drastically improves the efficiency of the numerical valuation scheme.

The payoff-transform approach according to Lewis (2001) is extremely versatile. For example, with this pricing technique, we can also derive the quantities $\Pi_1(\mathbf{x}_t, t, T)$ and $\Pi_2(\mathbf{x}_t, t, T)$, without need of any derivative function $\psi_z(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$, as done in formula (4.9). Although the numerical integration on a line integral (partly) including a pole exhibits the undesirable numerical properties discussed earlier, we want to show the derivation of the Gil-Pelaez style valuation formulae for $\Pi_2(\mathbf{x}_t, t, T)$, as given in Theorem 4.2.2 within the Lewis methodology[96], for demonstration purposes. Recalling that the payoff of an Arrow-Debreu security can be formally represented by the indicator function, we apply a Fourier Transformation on this special function in order to calculate $\Pi_2(\mathbf{x}_t, t, T)$. Under the probability measure $\mathbb{Q}_2$, the simple payoff representation is then given by the incomplete Fourier Trans-

[94] This is demonstrated in the next chapter.

[95] In addition to the restrictions for $z_i$, due to the validity for the transformed payoff function, in some cases we need to restrict the domain for the imaginary part of the transformation variable further to ensure the regularity of the characteristic function. One example, where $z_i$ has an additional constraint due to this issue is the characteristic function for the variance gamma process which is discussed in Itkin (2005).

[96] In contrast to equation (4.9), we would get an alternative representation for $\Pi_1(\mathbf{x}_t, t, T)$, without needing any derivative of $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ and $\phi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$, respectively.

formation[97]

$$\mathcal{F}^{g(\mathbf{x}_T)}\left[\mathbb{1}_{g(\mathbf{x}_T)>K}\right] = -\frac{e^{\imath zK}}{\imath z} \tag{4.22}$$

with

$$\operatorname{Im}(z) > 0.$$

Using this formula, together with $z_i$ in the appropriate domain, we are almost ready to calculate $\Pi_2(\mathbf{x}_t, t, T)$. In fact, we consider the residue theorem and apply a suitable closed-contour integral to recover the exact formula according to equation (4.9). Hence, evaluating the integral including the pole at $z_i = 0$ gives the desired result, which is demonstrated below.

We start with a slightly modified function $\tilde{\Pi}_2(t,T)$ to compensate for the influence of the probability law $\mathbb{Q}_2$[98], which is defined as

$$\tilde{\Pi}_2(\mathbf{x}_t, t, T) = \Pi_2(\mathbf{x}_t, t, T)P(\mathbf{x}_t, t, T). \tag{4.23}$$

Inserting the transformed payoff function (4.22) into our general valuation formula (4.21) gives

$$\tilde{\Pi}_2(\mathbf{x}_t, t, T) = -\frac{1}{\pi}\int_0^\infty \frac{e^{\imath zK}}{\imath z}\,\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\,\mathrm{d}z, \tag{4.24}$$

with

$$\operatorname{Im}(z) > 0.$$

Equation (4.24) can already be used for valuation purposes. Since we want to show the similarity of this formula to the transformed Arrow-Debreu security pricing approach, we encounter the problem of integrating through a pole, and therefore must apply Cauchy's residue theorem for *analytic* functions[99].

**Theorem 4.4.1 (Cauchy's Residue Theorem).** *Assume the function $f(z)$ is analytic within a closed, counter-clockwise performed integration contour $C$,*

[97] One-sided Fourier Transformations are commonly referred to as incomplete Fourier Transformations.

[98] This has to be done, since we use the general characteristic function $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$.

[99] This means, the function has to satisfy the Cauchy-Riemann equations. See Duffy (2004), p. 16.

*except at points $z_d \in \mathbb{C}$, where $f(z_d)$ encounters singularities. Then the value of the closed contour integral for this function can be calculated as*

$$\oint_C f(z)\,\mathrm{d}z = 2\imath\pi \sum_d \mathrm{Res}\left[f(z)|z = z_d\right]. \tag{4.25}$$

*The residues at the singularities corresponding to points $z_d$ can be derived as*

$$\mathrm{Res}\left[f(z)|z = z_d\right] = \lim_{z\to z_d} \frac{1}{(n-1)!}\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[(z - z_d)^n f(z)\right]. \tag{4.26}$$

*The parameter $n$ represents the order of the pole.*

Hence, if we want to evaluate the integral in (4.24) for $\mathrm{Im}(z) = 0$, we have to deal with a simple pole of order $n = 1$. To facilitate the calculations, we first introduce the original, two-sided integral representation for $\tilde{\Pi}_2(\mathbf{x}_t, t, T)$ in the manner of equation (2.25), which is simply

$$\tilde{\Pi}_2(\mathbf{x}_t, t, T) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\imath zK}}{\imath z}\,\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\,\mathrm{d}z. \tag{4.27}$$

Proceeding like this, we add to the former line integral, which has to be evaluated parallel to the real axis with distance $\mathrm{Im}(z)$, several additional integral paths to build a rectangular shape on the upper imaginary half-plane[100]. This gives us a contour $C$, which is performed, as illustrated in Figure 4.1.

Setting

$$\tilde{\Pi}_2(\mathbf{x}_t, t, T) = \int_{-\infty}^{\infty} f(z)\,\mathrm{d}z, \tag{4.28}$$

with

$$f(z) = -\frac{e^{\imath zK}\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{2\pi\imath z},$$

we are able to express the contour integral as

$$\oint_C f(z)\,\mathrm{d}z = \sum_{j=1}^{6}\int_{C_j} f(z)\,\mathrm{d}z = \sum_{j=1}^{6} I_j. \tag{4.29}$$

[100] In manipulating equation (4.27), we could also have chosen the lower half-plane. Subsequently, we would then have to be careful about the direction, how the pole is encircled, making its contribution to the integration either in a positive or negative sense.

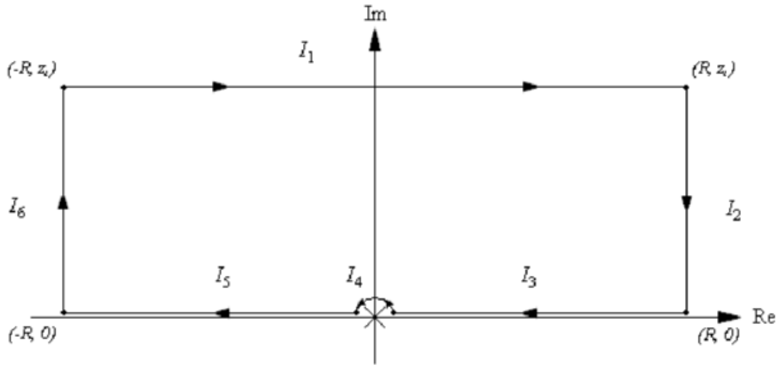


**Fig. 4.1.** Clockwise performed integral path for the derivation of $\tilde{\Pi}_2(\mathbf{x}_t, t, T)$ in equation (4.27) on the real line. The cross represents the pole.

Referring to Figure 4.1, the integral part $I_4$ forms a half arc around the pole of the meromorphic function[101] with radius $\epsilon$. Thus, excluding the pole, we can state, due to Cauchy's integral theorem,

$$\oint_C f(z)\,\mathrm{d}z = 0. \tag{4.30}$$

In the next step, we need to determine the values of the specific integrals $I_j$. Starting with $I_1$, we have just the value of $\tilde{\Pi}_2(\mathbf{x}_t, t, T)$ given in equation (4.27). Recognizing that for $0 \leq \mathrm{Im}(z) < \infty$, we have $\lim\limits_{R\to\pm\infty} f(R + \imath z_i) = 0$, we immediately obtain

$$I_2 + I_6 = 0. \tag{4.31}$$

Subsequently, we are left with the computation of the remaining integral parts $I_3$, $I_4$ and $I_5$. According to Theorem 4.4.1, if we consider an arc performed in a counter-clockwise fashion around a pole, we would have to take into account the entire contribution of the pole. Therefore, by assuming the radius $\epsilon$ of the half arc $I_4$ to be infinitesimally small, we eventually obtain half of the particular contribution. Thus, we have to consider the residue

[101] A function $f(z)$ is said to be meromorphic, if it only has some isolated singularities. This means that such a function is analytic everywhere, except at these poles. See Duffy (2004), p. 16.

$$\begin{aligned} I_4 &= \imath\pi \mathrm{Res}\left[f(z)|z=0\right] = \imath\pi \lim_{z\to 0} z f(z) \\ &= -\frac{\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{2} = -\frac{P(\mathbf{x}_t, t, T)}{2}. \end{aligned} \tag{4.32}$$

Likewise, assuming the distance to the origin for integrals $I_3$ and $I_5$ to be infinitesimally small, we are able to represent them in the limit as[102]

$$I_3 + I_5 = \int_{\infty}^{0^+} f(z)\,\mathrm{d}z + \int_{0^-}^{-\infty} f(z)\,\mathrm{d}z. \tag{4.33}$$

Having derived the required expressions for all integral parts $I_j$ in equations (4.31), (4.32) and (4.33), additionally using equation (4.30), we eventually end up with an alternative representation for $\tilde{\Pi}_2(\mathbf{x}_t, t, T)$, which is given by

$$\begin{aligned} \tilde{\Pi}_2(\mathbf{x}_t, t, T) &= -\left(I_3 + I_4 + I_5\right) \\ &= \frac{P(\mathbf{x}_t, t, T)}{2} - \left( \int_{\infty}^{0^+} f(z)\,\mathrm{d}z + \int_{0^-}^{-\infty} f(z)\,\mathrm{d}z \right) \\ &= \frac{P(\mathbf{x}_t, t, T)}{2} - 2\int_{0^-}^{-\infty} f(z)\,\mathrm{d}z \end{aligned} \tag{4.34}$$

In equation (4.34), the symmetry of characteristic functions for real-valued functions is exploited, due to Proposition 2.4.3. Therefore, the two integrals in the above equation can be aggregated. In a last step, we reinsert the detailed expression of $f(z)$ and substitute $z^* = -z$. This results in the relation

$$\begin{aligned} \tilde{\Pi}_2(\mathbf{x}_t, t, T) =& \frac{P(\mathbf{x}_t, t, T)}{2} \\ &+ \frac{1}{\pi}\int_{0^+}^{\infty} \mathrm{Re}\left[\frac{e^{-\imath z^* K}\psi(\mathbf{x}_t, z^*, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\imath z^*}\right] \mathrm{d}z^*, \end{aligned} \tag{4.35}$$

with

$$\mathrm{Im}(z^*) = 0.$$

Dividing equation (4.35) by $P(\mathbf{x}_t, t, T)$ and considering only the relevant real part of the solution, we obtain the Heston-style solution of equation (4.9),

[102] Here, we use again the convention $0^{\pm}$ denoting the right- and left-hand sided limit towards zero.

which we intentionally wanted to reproduce with the payoff-transformation approach of Lewis (2001).

In contrast to the Fourier-transform approach introduced in Carr and Madan (1999), the methodology discussed above is not that popular. One reason might be that the FFT algorithm cannot be applied to the valuation formula. Albeit, simply using an IFFT algorithm provides equivalent functionality and efficiency in solving derivatives prices. On the other hand, we prefer the method of Lewis (2001) because of the clear separation of different valuation components in the pricing formula. Additionally, this framework enables us to consistently use the valuation formula presented in equation (4.21) for both unconditional and conditional derivatives contracts by using residue calculus. Moreover, with this methodology even swaptions and options on coupon bonds can be priced in case of one-factor interest-rate models.

# 5

# Payoff Transformations and the Pricing of European Interest-Rate Derivatives

## 5.1 Overview

In this chapter we derive semi closed-form solutions of European interest-rate derivatives in terms of their transformed payoff functions, for all contracts given in Chapter 3. Equipped with this frequency representation of the payoff function, the contract can be priced with the general valuation formula according to equation (4.21). This procedure, combined with a standardized numerical integration routine, can then be used to compute the desired quantities. Apart from the generality of this method, we observe that all call and put option contracts exhibit identical payoff representations in Fourier space. The difference between them are the different strips in the imaginary plane, parallel to the real axis, on which the transform operation is valid for the particular contract.

As before, we distinguish between contracts with unconditional and conditional exercise rights. The reason for this separation of the payoff-transformed formulae is that contracts with unconditional exercise rights can be calculated as simple unconditional expectations. Using the residue theorem, solutions for the underlying contracts can be computed in terms of the general characteristic function, without evaluating numerically any integral at all. However, if the characteristic function is not known in closed form but can be represented as a system of ODEs, theoretical prices have to be numerically obtained via a Runge-Kutta algorithm. On the other hand, contracts with optional exercise rights are computed by numerical integration in every case.

## 5.2 Unconditional Payoff Functions

This section is organized as follows. First we compute some fundamental Fourier Transformations for functionals containing $g(\mathbf{x}_T)$[103], henceforth referred to as building blocks. These blocks, combined with the particular characteristic function, can then be used to compute the contract prices of Section 3.2 in the form of Fourier-style valuation formulae via equation (4.21). In calculating the payoff transform, we do not have to pay attention to the question of whether the derivative to be priced is a normal or futures-style contract. This is captured by the choice of the relevant characteristic function, which can be either $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ or $\psi(\mathbf{x}_t, z, 0, \mathbf{0}_M, g_0, \mathbf{g}, \tau)$.

At first sight, a problem arises in pricing unconditional interest-rate derivatives, due to the unbounded integration range of the expectation. As shown in the option-pricing example in Sections 4.3 and 4.4, the imaginary part of $z$ can be used to ensure the existence of the payoff transform by sufficiently dampening the integral on one side, which could be either the upper or lower. Unfortunately, the dampening effect cannot be accomplished simultaneously on both integration boundaries. Thus, we need additional considerations in order to derive an appropriate representation of the valuation formula in frequency space. Nevertheless, after some manipulation of the transformed payoff, we derive in the upcoming section the particular valuation formulae.

### 5.2.1 General Results

We begin with two basic interest-rate derivatives, the zero-bond contract as defined in equation (4.7) and the expectation of $g(\mathbf{x}_T)$ as given by equation (4.6). According to Section 4.2, the value of a zero bond equals $\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ whereas the latter quantity can be obtained via the calculation of its first derivative. These general results hold for arbitrary linear combinations $g(\mathbf{x}_t)$. In contrast, the payoff-transformation technique as presented in Section 4.4 seems at first sight to have difficulties in recovering these particular expectations, due to the unbounded integration domain. Hence, the first step in this subsection is to prove the former results obtained

[103] Although not explicitly displaying the variable $g(\mathbf{x}_T)$ in the payoff function, we also interpret in the following the Fourier Transformation of a constant as encountered in zero-bond contracts as a building block.

in equations (4.6) and (4.7) and therefore show that the payoff-transform methodology can be applied without exceptions.

If we set $G(\mathbf{x}_T) = 1$, which represents the riskless return of one unit money at maturity, it seems at first that the ordinary payoff transform is no longer finite. Unfortunately, with help of the imaginary part of the transformation variable $z$, we are only able to dampen the integrand on one side, which can be either in the direction of the positive or the negative real half-plane. Thus, we cannot dampen the underlying payoff function for both sides simultaneously, and consequently cannot perform the inverse Fourier Transformation on the same strip in the imaginary plane. However, performing the integration on different strips in the imaginary plane, we are again able to use the payoff-transform methodology. Dividing the integration domain $(-\infty, \infty)$ into two separate subdomains $(-\infty, \varepsilon)$ and $(\varepsilon, \infty)$ with arbitrary $\varepsilon \in \mathbb{R}$, we end up with two frequency functions defined on different strips in the imaginary plane. At first glance, this seems to complicate the situation. In fact, with the help of Cauchy's residue theorem, the calculations are rather simplified.

The payoff transform of an ordinary zero bond can be calculated as[104],

$$\mathcal{F}^{g(\mathbf{x}_T)}[1] = \int_{-\infty}^{\varepsilon} e^{\imath z g(\mathbf{x}_T)}\, \mathrm{d}g(\mathbf{x}_T) + \int_{\varepsilon}^{\infty} e^{\imath \overline{z} g(\mathbf{x}_T)}\, \mathrm{d}g(\mathbf{x}_T) = \frac{e^{\imath z \varepsilon}}{\imath z} - \frac{e^{\imath \overline{z} \varepsilon}}{\imath \overline{z}}, \tag{5.1}$$

with

$$\mathrm{Im}(z) < 0,$$

and $\overline{z}$ representing the complex conjugate of $z$[105]. Working with this transformed payoff function, we are already able to recover the zero-bond price due to the integral representation

$$\begin{aligned} P(\mathbf{x}_t, t, T) =& \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\imath z \varepsilon}}{\imath z}\, \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\, \mathrm{d}z \\ & - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\imath \overline{z} \varepsilon}}{\imath \overline{z}}\, \psi(\mathbf{x}_t, -\overline{z}, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\, \mathrm{d}\overline{z}. \end{aligned} \tag{5.2}$$

[104] Obviously, in pricing a zero bond, the choice of $g(\mathbf{x}_T)$ is irrelevant. In fact, $g(\mathbf{x}_T)$ can be set to any value, since the payoff function itself is independent of $g(\mathbf{x}_T)$.

[105] We make this assumption for convenience. Generally, the imaginary part of the transform variable used in the latter integral can be independently chosen on the positive half-axis.

Interchanging the integration boundaries of the latter integral in equation (5.2) and closing the contour with two additional paths from points $(R, \imath z_i)$ to $(R, -\imath z_i)$ for $R \to \pm\infty$, thus forming a closed contour integral with the resulting four integrals, we are able to use Cauchy's residue theorem again. The rectangular contour including the singularity is shown in Figure 5.1.

Due to the direction of the path, we have to consider a counter-clockwise encircled simple pole at $z = 0$, which is completely inside the contour. Consequently, the contour integral equals $2\pi\imath \mathrm{Res}\left[f(z)|z=0\right]$ with

$$f(z) = \frac{e^{\imath z\varepsilon}}{2\pi\imath z}\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau),$$

and the value of a zero bond is[106]

$$\begin{aligned} P(\mathbf{x}_t, t, T) = & \int_{-\infty}^{\infty} f(z)\,\mathrm{d}z + \int_{\infty}^{-\infty} f(\overline{z})\,\mathrm{d}\overline{z} = 2\pi\imath \mathrm{Res}\left[f(z)|\, z=0\right] \\ =&\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau). \end{aligned} \tag{5.3}$$

Here, the calculations for the residue are analogous to the ones made in equation (4.32), but this time considering the entire residue.

The same result would have been obtained using the Dirac Delta function $\delta(z)$ in the transformed payoff function. It is a well-known result that

$$\mathcal{F}^{g(\mathbf{x}_T)}\left[1\right] = \int_{-\infty}^{\infty} e^{\imath z g(\mathbf{x}_T)}\,\mathrm{d}g\left(\mathbf{x}_T\right) = 2\pi\delta(z), \tag{5.4}$$

with

$$\mathrm{Im}(z) = 0.$$

Hence, the fair value of a zero bond can be alternatively calculated as[107]

$$\begin{aligned} P(\mathbf{x}_t, t, T) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(z)\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\,\mathrm{d}z \\ =&\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau), \end{aligned} \tag{5.5}$$

[106] Starting from here, all zero-valued integrals are ignored.

[107] Obviously, for arbitrary real-valued $w$, the relation $\int_{-\infty}^{\infty} \delta(z-w)f(z)\,\mathrm{d}z = f(w)$ holds.

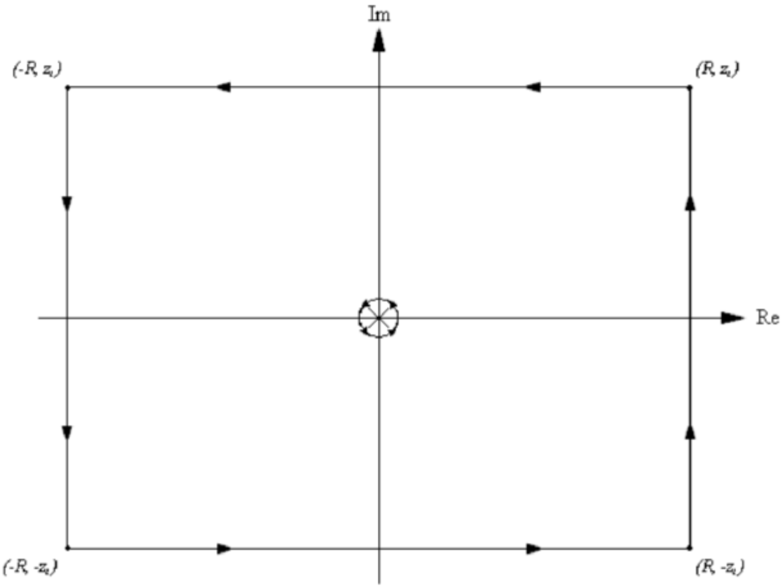


**Fig. 5.1.** Closed contour integral path for the derivation of $P(\mathbf{x}_t, t, T)$ in equation (5.2). The pole is completely encircled in a counter-clockwise manner.

which justifies the above statement.

So far, we have shown the result of one important building block, the model price of a zero bond, with the help of the payoff-transform methodology. In order to price interest-rate contracts bearing unconditional exercise rights, we also need the expected value of the payoff function $G(\mathbf{x}_T) = g(\mathbf{x}_T)$ as given by equation (4.6). In the following, we want to prove this general result within the payoff-transform methodology.

Starting our calculations, we assume a linear payoff function based on $g(\mathbf{x}_T)$ and then apply two incomplete Fourier Transformations, this time with an artificial integration boundary $\varepsilon$ for the particular integrals. Hence, the transformed payoff function of $G(\mathbf{x}_T) = g(\mathbf{x}_T)$ can be calculated as

$$
\begin{aligned}
\mathcal{F}^{g(\mathbf{x}_T)}\left[g\left(\mathbf{x}_T\right)\right] = & \int_{-\infty}^{\varepsilon} e^{\imath z g(\mathbf{x}_T)} g\left(\mathbf{x}_T\right)\, \mathrm{d}g\left(\mathbf{x}_T\right) \\
& + \int_{\varepsilon}^{\infty} e^{\imath \overline{z} g(\mathbf{x}_T)} g\left(\mathbf{x}_T\right)\, \mathrm{d}g\left(\mathbf{x}_T\right) \\
= & \frac{e^{\imath z \varepsilon}(1 - \imath z \varepsilon)}{z^2} - \frac{e^{\imath \overline{z} \varepsilon}(1 - \imath \overline{z} \varepsilon)}{\overline{z}^2},
\end{aligned} \tag{5.6}
$$

with

$$\mathrm{Im}(z) < 0.$$

This time, we build a rectangular integration path, performed in a clockwise manner which is depicted in Figure 5.2. Hence, we get for the discounted expectation[108]

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g\left(\mathbf{x}_T\right)\right] \\
& = -\frac{1}{2\pi} \int_{\infty}^{-\infty} \frac{e^{\imath z \varepsilon}(1 - \imath z \varepsilon)}{z^2}\, \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\, \mathrm{d}z \\
& \quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\imath \overline{z} \varepsilon}(1 - \imath \overline{z} \varepsilon)}{\overline{z}^2}\, \psi(\mathbf{x}_t, -\overline{z}, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\, \mathrm{d}\overline{z} \\
& = -2\pi\imath \mathrm{Res}\left[ -\frac{e^{\imath z \varepsilon}(1 - \imath z \varepsilon)}{2\pi z^2}\, \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \middle| z = 0 \right].
\end{aligned} \tag{5.7}
$$

Using again Cauchy's residue theorem, the contribution of the pole at the origin[109] can be derived as

[108] According to the clockwise performed integration path, the contribution of the pole in this case is $-2\pi\imath$ times the residue.

[109] According to a removable singularity, we have in fact at $z = 0$ two different poles, a simple and a second order pole.

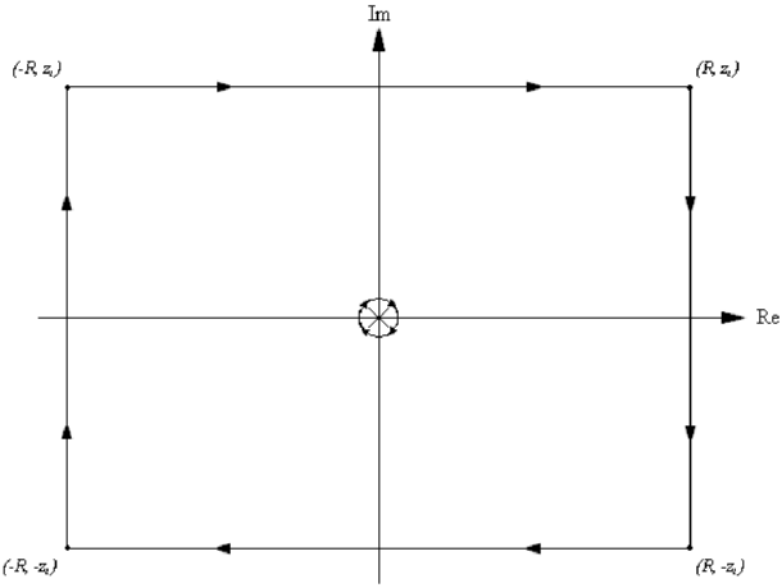


**Fig. 5.2.** Closed contour integral path for the discounted expectation of $g(\mathbf{x}_T)$. The pole is completely encircled in a clockwise manner.

$$
\begin{aligned}
&\mathrm{Res}\left[-\frac{e^{\imath z\varepsilon}(1-\imath z\varepsilon)\psi(\mathbf{x}_t,-z,w_0,\mathbf{w},g_0,\mathbf{g},\tau)}{2\pi z^2}\middle|\, z=0\right] \\
&=\mathrm{Res}\left[-\frac{e^{\imath z\varepsilon}\psi(\mathbf{x}_t,-z,w_0,\mathbf{w},g_0,\mathbf{g},\tau)}{2\pi z^2}\middle|\, z=0\right] \\
&\quad+\mathrm{Res}\left[-\frac{e^{\imath z\varepsilon}\psi(\mathbf{x}_t,-z,w_0,\mathbf{w},g_0,\mathbf{g},\tau)\,\varepsilon}{2\pi\imath z}\middle|\, z=0\right] \\
&=\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{e^{\imath z\varepsilon}\psi(\mathbf{x}_t,-z,w_0,\mathbf{w},g_0,\mathbf{g},\tau)}{2\pi}\right) \\
&\quad+\lim_{z\to 0}\left(-\frac{e^{\imath z\varepsilon}\psi(\mathbf{x}_t,-z,w_0,\mathbf{w},g_0,\mathbf{g},\tau)\,\varepsilon}{2\pi\imath}\right) \\
&=\frac{\psi_z(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau)-\imath\psi(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau)\,\varepsilon}{2\pi} \\
&\quad-\frac{\psi(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau)\,\varepsilon}{2\pi\imath} \\
&=\frac{\psi_z(\mathbf{x}_t,0,w_0,\mathbf{w},g_0,\mathbf{g},\tau)}{2\pi}.
\end{aligned}
\tag{5.8}
$$

Inserting this result in equation (5.7), we eventually obtain the general expression for the expected value of $g(\mathbf{x}_T)$, which is

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g(\mathbf{x}_T)\right] &= -\imath\psi_z(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \\ &= \frac{\psi_z(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\imath}.\end{aligned} \tag{5.9}$$

Thus, we have also derived the result in equation (4.6) within the payoff-transform methodology.

The remaining building block represents the unconditional expectation under the risk-neutral measure of an integro-linear variable where the payoff function satisfies $G(\mathbf{x}_T) = \int_t^T g(\mathbf{x}_s)\,\mathrm{d}s$. Because of the integrated expression in the payoff function, this quantity has to be treated differently. Pricing an unconditional contract, including such an integrated term, we are interested in the expected value

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \int_t^T g(\mathbf{x}_s)\,\mathrm{d}s\right]. \tag{5.10}$$

In the following, we first want to show how equation (5.10) can be recovered manipulating the expectation itself, as done in equations (4.6) and (4.7). Obviously, the calculations are very similar compared to equation (4.6). Afterwards, we show that the payoff-transform methodology replicates the same result without any problems.

Making the same considerations as for the derivation of the expected value of $g(\mathbf{x}_T)$, we compute (5.10) as the derivative with respect to the transform variable, evaluated at $z = 0$. Note that the characteristic function itself consists only of one sole exponential discounting term, since we have

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} e^{\imath z\int_t^T g(\mathbf{x}_s)\,\mathrm{d}s}\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(\mathbf{x}_s) - \imath z g(\mathbf{x}_s))\,\mathrm{d}s}\right]. \tag{5.11}$$

Obviously, this particular characteristic function is equivalent to the value of a zero-bond contract, but with a hypothetical complex-valued short rate of

$$r^A(\mathbf{x}_t, z) = r(\mathbf{x}_t) - \imath z g(\mathbf{x}_t) = (w_0 - \imath z g_0) + (\mathbf{w}' - \imath z \mathbf{g}')\mathbf{x}_t. \tag{5.12}$$

In the last equation we considered that both the instantaneous interest rate $r(\mathbf{x}_t)$ and the payoff-characterizing function $g(\mathbf{x}_t)$ are linear combinations of $\mathbf{x}_t$. Since we deal with a zero bond like contract, the solution for this model price also exhibits an exponential-affine form. Thus, in analogy to the considerations made for zero bonds, we are able to represent the solution as an exponential-affine function. Introducing new parameters characterizing the modified short rate, we have

$$w_0^A(z) = w_0 - \imath z g_0 \quad \text{and} \quad \mathbf{w}^A(z) = \mathbf{w} - \imath z \mathbf{g}.$$

The resulting characteristic function for pricing average-rate derivatives is then

$$\psi\left(\mathbf{x}_t, z, w_0^A(z), \mathbf{w}^A(z), 0, \mathbf{0}_M, \tau\right),$$

and the relevant payoff function for this modified characteristic function is $G(\mathbf{x}_T) = 1$.

As mentioned above, this characteristic function exhibits a strong resemblance compared to the Fourier-style zero-bond representation in equation (5.3), where the original characteristic function was evaluated at some point $z = 0$. This can be traced back to the fact that both payoff functions are independent of the Fourier Transformation variable. The difference between them is that the function $\psi\left(\mathbf{x}_t, z, w_0^A(z), \mathbf{w}^A(z), 0, \mathbf{0}_M, \tau\right)$ generates zero-bond prices with respect to the modified short rate $r^A(\mathbf{x}_t, z)$, independently of the value of the transformation variable $z$. Thus, the coefficient functions in this particular case, $a(z,\tau)$ and $\mathbf{b}(z,\tau)$ solve again the system of ordinary differential equations (2.40) and (2.41), with terminal conditions $a(z,0) = 0$, $\mathbf{b}(z,0) = \mathbf{0}_M$. The hypothetical discount rate is defined by $w_0^A(z)$ and $\mathbf{w}^A(z)$, respectively, whereas the terminal value is given by

$$\psi\left(\mathbf{x}_t, z, w_0^A(z), \mathbf{w}^A(z), 0, \mathbf{0}_M, 0\right) = 1.$$

Having found the characteristic function for this special case, the same considerations can be applied as for the expected value of $g(\mathbf{x}_T)$. Using the technique of Fourier-transformed prices, we eventually express equation (5.11) as[110]

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\int_t^T g\left(\mathbf{x}_s\right)\,\mathrm{d}s\right] &= \frac{\mathrm{d}}{\mathrm{d}z}\,\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(\mathbf{x}_s)-\imath z g(\mathbf{x}_s))\,\mathrm{d}s}\right]\Bigg|_{z=0} \\ &= \frac{\psi_z\left(\mathbf{x}_t, 0, w_0^A(0), \mathbf{w}^A(0), 0, \mathbf{0}_M, \tau\right)}{\imath}.\end{aligned} \tag{5.13}$$

Alternatively, we are also able to obtain this result using the payoff-transform methodology together with the contour integration technique. For convenience, we first set up the substitution

$$\gamma(T) = \int_t^T g\left(\mathbf{x}_s\right)\,\mathrm{d}s,$$

and afterwards perform the Fourier Transformation with respect to this new variable $\gamma(T)$. Thus, the transformation of the particular payoff function is the same as the one used in deriving equation (5.6). Therefore, we can immediately adopt the result of equation (5.9) by exchanging the general characteristic function $\psi\left(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau\right)$ with its modified pendant $\psi\left(\mathbf{x}_t, z, w_0^A(z), \mathbf{w}^A(z), 0, \mathbf{0}_M, \tau\right)$[111]. Afterwards, we get the desired result according to equation (5.13).

In this section we proved the general results of unconditional expectations for zero bonds, and linear and integro-linear payoff functions, respectively, obtained within the payoff-transform framework[112]. Moreover, apart from the traditional formulae, where the desired value is derived by manipulation of the

[110] Obviously, the values of the functions $\psi_z\left(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau\right)$ and $\psi\left(\mathbf{x}_t, z, w_0^A(z), \mathbf{w}^A(z), 0, \mathbf{0}_M, \tau\right)$ are equal for $z = 0$. However, the derivatives with respect to $z$ evaluated at this point, do not share this similarity. This is the reason why we make the dependence of $z$ in the modified short rate explicit, although $w_0^A(0) = w_0$ and $\mathbf{w}^A(0) = \mathbf{w}$.

[111] The path of the contour integral and the location of the pole is given in Figure 5.2.

[112] The particular derivation for the exponential-linear case was not derived in this section since it is not needed in this work. However, the calculations are straightforward using the *integration-by-parts* methodology, where the relevant pole is at $z = \imath$.

expectation itself, as shown in Section 4.2, we have with the payoff-transform approach the freedom to choose among a set of infinite solution formulae due to the contour integration in the complex plane. This fact becomes especially important in computing the expectation $\mathbb{E}^{\mathbb{Q}_1}[1]$ and the expectation for the unconditional average-rate contract where the derivative of the characteristic function with respect to the transformation variable $z$ has to be used. In these cases we are provided with the alternative to use the simple payoff transform and apply equation (4.21) on the appropriate strip in the imaginary plane.

Hence, using the building blocks above, we are able to price all interest-rate derivatives introduced in Section 3.2 with Fourier-style formulae. According to the results in equations (5.5), (5.9) and (5.13) we arrive at completely closed-form pricing formulae, which are illustrated in the next subsection[113].

### 5.2.2 Pricing Unconditional Interest-Rate Contracts

So far, the three building blocks for general unconditional payoff functions have been derived. In this section, these blocks are translated into the valuation formulae for the particular yield-based and level-based interest-rate contracts discussed in Section 3.2.

Starting with yield-based contracts, we need first a translation of yields into Fourier-style solutions. This is easily done as follows

$$Y(\mathbf{x}_t, t, T) = \frac{\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)^{-1} - 1}{\tau}. \tag{5.14}$$

The model price for zero bonds can then be obtained by using equation (5.5), whereas prices of coupon bonds can be calculated as

$$CB(\mathbf{x}_t, \mathbf{c}, t, \mathbf{T}) = \sum_{a=1}^{A} \psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau_a) c_a. \tag{5.15}$$

The price of a forward-rate agreement is given as

[113] This statement is valid if the characteristic function or its derivative with respect to $z$ can be displayed in closed form. In cases where the characteristic function cannot be explicitly expressed, but its coefficient functions $a(z,\tau)$ and $\mathbf{b}(z,\tau)$ are solutions to the system of ordinary differential equations according to (2.40) and (2.41), a Runge-Kutta algorithm can be used to obtain the relevant values.

$$\begin{aligned} &FRA_Y(\mathbf{x}_t, K, Nom, t, T, \hat{T}) \\ &\quad = Nom \left( \frac{\psi\left(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \hat{\tau}\right)}{\tilde{K}} - \psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \right), \end{aligned} \tag{5.16}$$

and a yield-based swap can be similarly computed in terms of the general characteristic functions as

$$\begin{aligned} &SWA_Y(\mathbf{x}_t, K, Nom, t, \mathbf{T}) \\ &\quad = Nom \left( \sum_{a=1}^{A-1} \frac{\psi\left(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau_{a+1}\right)}{\tilde{K}_a} \right. \\ &\qquad \left. - \sum_{a=1}^{A-1} \psi\left(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau_a\right) \right). \end{aligned} \tag{5.17}$$

On the other hand, pricing contracts linearly based on the function $g\left(\mathbf{x}_T\right)$, we foremost need the derivative of the general characteristic function $\psi\left(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau\right)$ with respect to $z$. Hence, a level-based forward-rate agreement defined in equation (3.5) is represented by

$$\begin{aligned} &FRA_r(\mathbf{x}_t, K, Nom, t, T) \\ &\quad = Nom \left( K\, \psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) - \frac{\psi_z(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\imath} \right) \\ &\quad = Nom \left( K - \frac{\phi_z(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\imath} \right) \psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau). \end{aligned} \tag{5.18}$$

Accordingly, the corresponding swap contract in this framework can be obtained as

$$\begin{aligned} &SWA_r(\mathbf{x}_t, K, Nom, t, \mathbf{T}) \\ &\quad = Nom \left( K \sum_{a=1}^{A} \psi\left(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau_a\right) \right. \\ &\qquad \left. - \sum_{a=1}^{A} \frac{\psi_z\left(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau_a\right)}{\imath} \right) \\ &\quad = Nom \sum_{a=1}^{A} \left( K - \frac{\phi_z\left(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau_a\right)}{\imath} \right) \times \\ &\qquad \psi\left(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau_a\right). \end{aligned} \tag{5.19}$$

The last unconditional contract to be priced is the average-rate contract. Here, the integro-linear payoff function can be interpreted as an interest-rate contract based on the short rate itself. According to equation (3.11) and (5.13), the price of this contract can be calculated as

$$
\begin{aligned}
&UARC_r(\mathbf{x}_t, K, Nom, t, T) \\
&\quad = Nom \left( K\,\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \right. \\
&\qquad \left. - \frac{\psi_z(\mathbf{x}_t, 0, w_0^A(0), \mathbf{w}^A(0), 0, \mathbf{0}_M, \tau)}{\imath} \right) .
\end{aligned} \tag{5.20}
$$

For the special case $g(\mathbf{x}_T) = r(\mathbf{x}_T)$, we use the simplified versions $w_0^A(z) = (1 - \imath z)w_0$ $\mathbf{w}^A(z) = (1 - \imath z)\mathbf{w}$, respectively.

## 5.3 Conditional Payoff Functions

So far, we derived closed-form solutions for contracts with unconditional exercise rights. In contrast to the calculations in the last section, where contracts merely depended on the simple evaluation of the terms $\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$, $\psi_z(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ and $\psi_z(\mathbf{x}_t, 0, w_0^A(0), \mathbf{w}^A(0), 0, \mathbf{0}_M, \tau)$, respectively, the option-pricing problem confronts us with a different situation. The integration by parts method is not of use anymore due to a natural integration boundary, characterized by some strike value $K$. Including this optional exercise right within the payoff-transform methodology, we end up with some semi closed-form solutions, which means we have to solve a standardized Fourier integral in order to compute the desired model prices of interest-rate options. Although the payoff-transform methodology enables us to price consistently the option prices with payoff functions according to Table 4.1, without adapting the valuation formula (4.21) to the different cases, we distinguish for convenience between linear, exponential-linear and integro-linear payoff functions. As before, we first derive some basic payoff transforms for general $g(\mathbf{x}_T)$ and afterwards take into account the interest-rate options discussed in Chapter 3. Eventually, we develop as a special case the Fourier-transformed payoff function of a coupon-bond option for the case of a one-factor interest-rate model[114] with $x_t = r_t$.

[114] The term *one-factor model* refers to the fact that only one Brownian motion is incorporated in the model.

### 5.3.1 General Results

Besides the elementary payoff functions, we also differentiate between call and put options, because of the conditional exercise property of the contracts. The transformed payoff functions for call and put contracts display a strong resemblance, which is demonstrated in this section, allowing a more general implementation of the valuation algorithms. Due to the exercise boundary and the different ways of incorporating $g(\mathbf{x}_T)$ and its integro-linear counterpart, respectively, in the payoff function $G(\mathbf{x}_T)$, we introduce the critical value

$$\alpha(K) = \begin{cases} \ln[K] & \text{Exponential-linear Case.} \\ K & \text{Linear and Integro-linear Case,} \end{cases} \tag{5.21}$$

for which the option payoff is exactly *at the money*. The Fourier Transformation for different call payoff structures can be generally represented as

$$\begin{aligned} \mathcal{F}^{g(\mathbf{x}_T)}\left[G(\mathbf{x}_T)\right] &= \int_{-\infty}^{\infty} e^{\imath z g(\mathbf{x}_T)} G(\mathbf{x}_T) \mathbb{1}_{g(\mathbf{x}_T) \geq \alpha(K)} \, \mathrm{d}g\left(\mathbf{x}_T\right) \\ &= \int_{\alpha(K)}^{\infty} e^{\imath z g(\mathbf{x}_T)} G(\mathbf{x}_T) \, \mathrm{d}g\left(\mathbf{x}_T\right), \end{aligned} \tag{5.22}$$

whereas the particular put payoff transform in its general form is given by

$$\begin{aligned} \mathcal{F}^{g(\mathbf{x}_T)}\left[G(\mathbf{x}_T)\right] &= \int_{-\infty}^{\infty} e^{\imath z g(\mathbf{x}_T)} G(\mathbf{x}_T) \mathbb{1}_{g(\mathbf{x}_T) \leq \alpha(K)} \, \mathrm{d}g\left(\mathbf{x}_T\right) \\ &= \int_{-\infty}^{\alpha(K)} e^{\imath z g(\mathbf{x}_T)} G(\mathbf{x}_T) \, \mathrm{d}g\left(\mathbf{x}_T\right). \end{aligned} \tag{5.23}$$

In deriving the solution for the exponential-linear case, we have to use the transform

$$\begin{aligned} \mathcal{F}^{g(\mathbf{x}_T)}\left[\left(e^{g(\mathbf{x}_T)} - K\right)^+\right] &= \int_{\alpha(K)}^{\infty} e^{\imath z g(\mathbf{x}_T)} \left(e^{g(\mathbf{x}_T)} - K\right) \mathrm{d}g\left(\mathbf{x}_T\right) \\ &= \left[\frac{e^{(1+\imath z) g(\mathbf{x}_T)}}{1+\imath z} - \frac{K e^{\imath z g(\mathbf{x}_T)}}{\imath z}\right]_{\alpha(K)}^{\infty} \\ &= \frac{e^{(1+\imath z)\alpha(K)}}{\imath z(1+\imath z)} = \frac{K^{1+\imath z}}{\imath z(1+\imath z)}. \end{aligned} \tag{5.24}$$

Due to the exponential-linear dependence of the payoff-characterizing variable we set $\alpha(K) = \ln[K]$ and obtain the equivalent transformation as given in equation (2.26). Since the frequency representation of a call option payoff only exists on a strip with

$$\mathrm{Im}(z) > 1,$$

a general Fourier Transformation is needed. Although exhibiting different payoff structures the corresponding payoff transform of a put option has the identical formal structure as given in equation (5.24). This can be easily proved by

$$\begin{aligned}\mathcal{F}^{g(\mathbf{x}_T)}\left[\left(K - e^{g(\mathbf{x}_T)}\right)^+\right] &= \int_{-\infty}^{\alpha(K)} e^{\imath z g(\mathbf{x}_T)}\left(K - e^{g(\mathbf{x}_T)}\right) \mathrm{d}g\left(\mathbf{x}_T\right) \\ &= \frac{K^{1+\imath z}}{\imath z(1+\imath z)},\end{aligned} \tag{5.25}$$

but with

$$\mathrm{Im}(z) < 0.$$

Based on this result, both call and put option prices can be recovered using the same payoff transform and as a direct consequence, only one single program code is needed for evaluating values for both interest-rate option contracts. The only difference are the different sets and strips on which $\mathrm{Im}(z)$ is valid for the inverse operation. Whereas the condition for the call contract assured the dampening of the integrand on the positive half-axis, we need for the put option the condition to guarantee the same on the negative equivalent.

An interesting feature of the payoff-transform methodology is, due to the equivalent transformed payoff functions of calls and puts, the applicability of a closed contour integral to obtain in a very elegant way the particular put-call parity[115]. Without loss of generality, we set

$$f(z) = \frac{K^{1+\imath z}\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{2\pi\imath z(1+\imath z)}. \tag{5.26}$$

Thus, we have

[115] The relevant integration path is depicted in Figure 5.3.

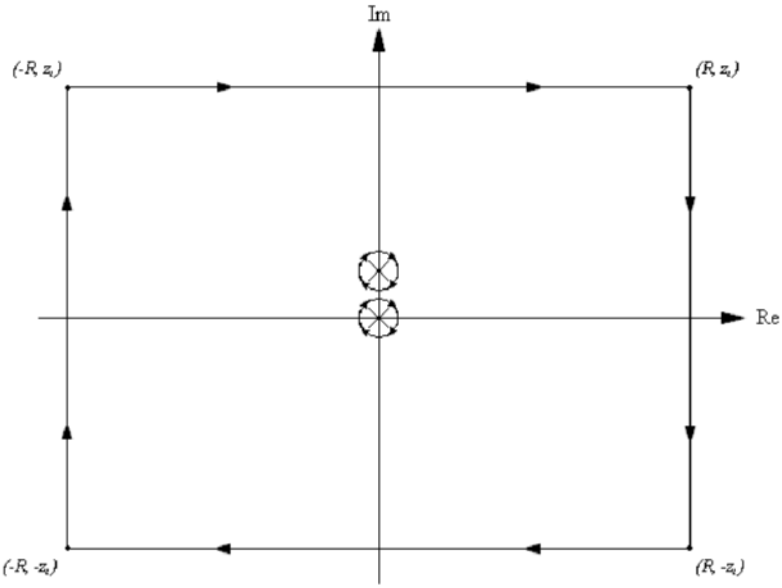


**Fig. 5.3.** Closed contour integral path for the derivation of the put-call parity in equation (5.27). The poles at $z = 0$ and $z = \imath$ are completely encircled in a clockwise manner.

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(e^{g(\mathbf{x}_T)} - K\right)^+\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - e^{g(\mathbf{x}_T)}\right)^+\right]& \\ = \int_{-\infty}^{\infty} f(z)\,\mathrm{d}z + \int_{\infty}^{-\infty} f(\overline{z})\,\mathrm{d}\overline{z}& \\ = -2\pi\imath\left(\mathrm{Res}\left[f(z)|z=0\right] + \mathrm{Res}\left[f(z)|z=\imath\right]\right),&\end{aligned} \tag{5.27}$$

with

$$\mathrm{Im}(z) > 1.$$

The imaginary part of the Fourier variable $z$ in equation (5.27) can be chosen arbitrarily as long as the existence of the payoff transformations is guaranteed[116]. Obviously, this clockwise performed contour integral now encircles two simple poles of the function $f(z)$, one at the origin and the other one located at $z = \imath$. Due to the closed contour, we only have to calculate the residues of all included poles in order to obtain the desired put-call parity. Comparing equation (5.26) with (5.5), the residue of $f(z)$ at the origin is just

$$\mathrm{Res}\left[f(z)|z=0\right] = K\frac{\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{2\pi\imath},$$

whereas the residue at $z = \imath$ is

$$\mathrm{Res}\left[f(z)|z=\imath\right] = -\frac{\psi(\mathbf{x}_t, -\imath, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{2\pi\imath}.$$

Hence, equation (5.27) equals

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(e^{g(\mathbf{x}_T)} - K\right)^+\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - e^{g(\mathbf{x}_T)}\right)^+\right] \\ &\qquad = \psi(\mathbf{x}_t, -\imath, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) - K\,\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau).\end{aligned} \tag{5.28}$$

According to the result in equation (4.7), the term $\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ simply represents the price of a zero bond with maturity $\tau$. The other term, the quantity $\psi(\mathbf{x}_t, -\imath, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ equals the discounted forward price of the exponential of the variable $g(\mathbf{x}_t)$[117]. Therefore, setting $z = -\imath$, we get

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} e^{g(\mathbf{x}_T)}\right].$$

For a call option, linearly based on $g(\mathbf{x}_T)$, we get

$$\begin{aligned}\mathcal{F}^{g(\mathbf{x}_T)}\left[(g(\mathbf{x}_T) - K)^+\right] &= \left[e^{\imath z g(\mathbf{x}_T)}\frac{1 + \imath z(K - g(\mathbf{x}_T))}{z^2}\right]_{\alpha(K)}^{\infty} \\ &= -\frac{e^{\imath z \alpha(K)}}{z^2} = -\frac{e^{\imath z K}}{z^2},\end{aligned} \tag{5.29}$$

with

[116] For convenience, we work with the complex conjugate for the latter integral. In fact, due to the exponential-linear payoff function the restriction for the put option transform can be independently chosen according to equation (5.25).

[117] See, for example, Bakshi and Madan (2000), p. 212. There, this quantity is alternatively denoted as the scaled-forward price.

$$\operatorname{Im}(z) > 0.$$

Similar to the call representation in Fourier space, the put option transform is

$$\mathcal{F}^{g(\mathbf{x}_T)}\left[(K - g(\mathbf{x}_T))^+\right] = -\frac{e^{\imath z K}}{z^2}. \tag{5.30}$$

The only difference between the call and put option transform is that equation (5.30) is defined on the opposite imaginary half-plane. Consequently, we use the complex conjugate of the Fourier variable in equation (5.29). The put-call parity for the linear case can be derived as[118]

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\,(g(\mathbf{x}_T) - K)^+\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\,(K - g(\mathbf{x}_T))^+\right] \\
&\qquad = \imath \operatorname{Res}\left[\left.\frac{e^{\imath z K}\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{z^2}\right| z = 0\right] \\
&\qquad = \frac{\psi_z(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\imath} - K\,\psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau).
\end{aligned} \tag{5.31}$$

Due to the payoff similarities of the linear and integro-linear case, the payoff transformations are equivalent for both cases in Fourier space. Hence, to compute the average-rate option prices (3.24) and (3.25), equations (5.29), (5.30) and (5.31) can be used together with the modified characteristic function.

Although not directly applicable for tradable option contracts, but nevertheless important for theoretical issues is the Fourier-transformed payoff function of a hypothetical contingent claim according to the Dirac delta function, which is $\delta(g(\mathbf{x}_T) - \alpha(K))$. As mentioned before, the Dirac delta function has an infinite spike for $g(\mathbf{x}_T) = \alpha(K)$. The Fourier Transformation of the Dirac delta function can be simply expressed as

$$\mathcal{F}^{g(\mathbf{x}_T)}\left[\delta(g(\mathbf{x}_T) - \alpha(K))\right] = e^{\imath z \alpha(K)}, \tag{5.32}$$

with no need to set up any restriction on the imaginary part of the transform variable $z$. Since the Dirac delta function states the terminal condition of a probability density function, equation (5.32) may be used to recover the relevant transition density function. Especially for illustrating the behavior of a particular stochastic process $g(\mathbf{x}_t)$, the transition density function is useful to explain its characteristics. The other special function we want to derive, is

[118] The relevant integration path is depicted in Figure 5.2.

the Fourier Transformation of the cumulative probability function $\Pr(g(\mathbf{x}_T) < \alpha(K))$. The payoff corresponding to this terminal condition is given by the indicator function of the event $g(\mathbf{x}_T) < \alpha(K)$. Thus, the transformed payoff can be expressed as[119]

$$\mathcal{F}^{g(\mathbf{x}_T)}\left[\mathbb{1}_{g(\mathbf{x}_T)<\alpha(K)}\right] = \frac{e^{\imath z \alpha(K)}}{\imath z}, \tag{5.33}$$

with

$$\mathrm{Im}(z) < 0.$$

Accordingly, we plug (5.33) into our general valuation formula (4.21) or alternatively use a slightly modified version of the Gil-Pelaez formula[120] which is

$$\begin{aligned}\Pr(g(\mathbf{x}_T) < K) \\ = \frac{1}{2} - \frac{1}{\pi}\int_{0+}^{\infty} \mathrm{Re}\left[\frac{\psi(x_t, z^*, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) e^{-\imath z^* \alpha(K)}}{\imath z^*}\right] \mathrm{d}z^*,\end{aligned} \tag{5.34}$$

with

$$\mathrm{Im}(z^*) = 0.$$

### 5.3.2 Pricing of Zero-Bond Options and Interest-Rate Caps and Floors

In this section valuation formulae for the specific interest-rate contracts in Section 3.3 are derived. Since the transformed payoff functions are independent of the variable $g(\mathbf{x}_T)$, most of the contracts share a similar payoff transform. According to the previous section, differences between the various pricing formulae lay in the particular characteristic function to be used. Therefore, we focus on the general forms of the characteristic function and refer only to the relevant payoff transformations, constructed in the previous section. Like contracts with unconditional exercise rights, we start with yield-based option contracts and discuss afterwards the particular level-based contracts. We exclude in this section the Fourier-style pricing formulae for coupon-bond options and swaptions, respectively, because these special contracts can only

119 See equation (5.1).

120 Since $\Pr(g(\mathbf{x}_T) < K) + \Pr(g(\mathbf{x}_T) \geq K) \equiv 1$, equation (5.34) can be immediately derived from equation (4.8).

be priced in a one-factor environment, due to the more complicated exercise boundary[121]. Thus, we give for these contracts the specific valuation formulae in the next section.

Beginning with zero-bond options, we use for the transformed payoff function of a call option the equation (5.24) and equation (5.25) for a put option. Taking into account the terminal condition at expiration of the option contract of a zero bond with remaining time to maturity $\hat{\tau} = \hat{T} - T$, we set the relation

$$g_0 = a(0, \hat{\tau}) \qquad \text{and} \qquad \mathbf{g} = \mathbf{b}(0, \hat{\tau}).$$

The relevant characteristic function is then

$$\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, a(0, \hat{\tau}), \mathbf{b}(0, \hat{\tau}), \tau),$$

and option prices can be calculated by plugging the relevant payoff transform and the characteristic function into the general valuation formula (4.21), which gives for a call option

$$\begin{aligned} &ZBC\left(\mathbf{x}_t, K, t, T, \hat{T}\right) \\ &\quad = \frac{1}{\pi} \int_0^\infty \frac{K^{1+\imath z}}{\imath z(1+\imath z)} \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, a(0, \hat{\tau}), \mathbf{b}(0, \hat{\tau}), \tau) \,\mathrm{d}z, \end{aligned} \tag{5.35}$$

with

$$\operatorname{Im}(z) > 1.$$

In contrast, a zero-bond put option price can be derived via equation (5.35) but with the restriction

$$\operatorname{Im}(z) < 0.$$

According to equation (3.17) and (3.18), a yield-based cap and floor contract can be immediately expressed as the summation over the particular zero-bond options, scaled with some quantity $\frac{Nom}{K_a}$. Hence, the model price of a yield-based cap contract is

[121] However, there exist some articles which derive approximated values for these contracts in a multi-factor framework, see e.g. Singleton and Umantsev (2002) or Collin-Dufresne and Goldstein (2002).

$$CAP_Y(\mathbf{x}_t, K, Nom, t, \mathbf{T}) = \frac{Nom}{\pi} \sum_{a=1}^{A-1} \int_0^{\infty} \frac{\overline{K}_a^{\imath z}}{\imath z(1+\imath z)} \times \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, a(0,\hat{\tau}_a), \mathbf{b}(0,\hat{\tau}_a), \tau_a)\,\mathrm{d}z, \tag{5.36}$$

with

$$\mathrm{Im}(z) > 0,$$

$\hat{\tau}_a = T_{a+1} - T_a$, and $\tau_a = T_a - t$. Subsequently, a yield-based floor contract can be priced using equation (5.36) with

$$\mathrm{Im}(z) < 0.$$

Next, we derive the particular pricing formulae of level-based interest-rate contracts and interest-rate options written on the short rate $r(\mathbf{x}_t)$ itself. Starting with a cap contract according to equation (3.15), we use the payoff transform (5.29) with

$$g_0 = w_0 \qquad \text{and} \qquad \mathbf{g} = \mathbf{w},$$

and therefore apply the characteristic function

$$\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, w_0, \mathbf{w}, \tau_a).$$

Thus, the cap contract can be priced as

$$CAP_r(\mathbf{x}_t, K, Nom, t, \mathbf{T}) = -\frac{Nom}{\pi} \sum_{a=1}^{A} \int_0^{\infty} \frac{e^{\imath z K}}{z^2}\, \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, w_0, \mathbf{w}, \tau_a)\,\mathrm{d}z, \tag{5.37}$$

with

$$\mathrm{Im}(z) > 0.$$

Hence, the model price of a floor contract with equivalent input parameters can be recovered using equation (5.37) again but evaluating the integrals on the negative imaginary half-plane with

$$\mathrm{Im}(z) < 0.$$

The last option contracts for which we want to give a payoff-transformed solution are the average-rate options due to equation (3.24) and (3.25). Thus, the payoff of the average-rate cap option contract at expiration can be expressed as

$$\frac{Nom}{\tau}\left(K^* - \int_t^T r(\mathbf{x}_s)\,\mathrm{d}s\right)^+,$$

with $K^* = \tau K$. Taking the same considerations into account as done for the unconditional average-rate contract, the relevant characteristic function for

$$\gamma(T) = \int_t^T r(\mathbf{x}_s)\,\mathrm{d}s,$$

is given by

$$\psi(\mathbf{x}_t, z, w_0^A(z), \mathbf{w}^A(z), 0, \mathbf{0}_M, \tau).$$

Together with the payoff transform in equation (5.29), we are able to postulate the model price of an average-rate cap as

$$\begin{aligned} &ARC_r\left(\mathbf{x}_t, K, Nom, t, T\right) \\ &\quad = -\frac{Nom}{\tau\pi}\int_0^\infty \frac{e^{\imath z K^*}}{z^2}\,\psi(\mathbf{x}_t, -z, w_0^A(-z), \mathbf{w}^A(-z), 0, \mathbf{0}_M, \tau)\,\mathrm{d}z, \end{aligned} \tag{5.38}$$

with

$$\mathrm{Im}(z) > 0.$$

The respective average-rate floor contract can be priced, using equation (5.38) with

$$\mathrm{Im}(z) < 0.$$

### 5.3.3 Pricing of Coupon-Bond Options and Yield-Based Swaptions

So far, we have excluded the valuation formulae for coupon-bond options and yield-based swaptions, respectively. In contrast to the option contracts discussed in the last section, where we computed only a single option price and a portfolio of different option prices, respectively, we deal here with a option on a portfolio of future cash flows. Consequently, the determination of a unique critical exercise value $\alpha(K)$ in a multi-factor setting is not possible anymore[122]. However, dealing with a one-factor interest-rate model setup with $r(x_t) = r_t$[123], we are able to circumvent this issue. Hence, we follow the technique proposed in Jamshidian (1989) to derive the theoretical price of a coupon-bond option using the payoff-transform methodology in pricing this derivative contract, which is shown below.

Setting $x_t = r_t$, we are able to exploit the coefficient structure of the affine term-structure model. The special form of the characteristic function is of the form

$$\psi(r_t, z, 0, 1, 0, 1, \tau) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s \, \mathrm{d}s + \imath z r_T}\right] = e^{a(z,\tau)+b(z,\tau)r_t}.$$

Because a yield-based swaption can be interpreted as an option on a coupon bond[124], we focus on the valuation of the particular coupon-bond option.

In a one-factor setup the coupon-bond call option payoff is given by

$$\begin{aligned}(CB(r_T, \mathbf{c}, T, \mathbf{T}) - K)^+ &= \left(\sum_{a=1}^{A} P(r_T, T, T_a)c_a - K\right)^+ \\ &= \left(\sum_{a=1}^{A} e^{a(0,\tau_a)+b(0,\tau_a)r_T} c_a - K\right)^+ .\end{aligned}$$

In the last equation, we inserted the particular Fourier-style zero-bond prices generated by the exponential-affine model. The above presented payoff function is then a continuous and strictly decreasing function in $r_T$[125]. In these models we have[126]

$$\frac{\partial P(r_t, t, T)}{\partial r_t} = b(0, \tau) < 0 \quad \forall \quad T > t.$$

Consequently, the payoff function exhibits a unique zero value for the critical short rate $r_T^*$ for which the coupon-bond call is exercised. However, dealing

[122] See, for example, Singleton and Umantsev (2002).

[123] Without loss of generality, we set in the following $w_0 = 0$ and $w_1 = 1$.

[124] See the alternative presentation of a swaption payoff in Section 3.3.

[125] The particular characteristic functions are derived in Chapter 8.

[126] See e.g. Duffie and Kan (1996) for the properties of $b(0, \tau)$ in common one-factor interest-rate models.

with a single-factor environment, we cannot explicitly express this critical value $r_T^*$ in closed form, which is due to the sum of exponentials in the payoff function. Thus, the critical exercise value has to be computed numerically. Having determined the value of $r_T^*$, the Fourier Transformation of a coupon-bond call payoff can be calculated as[127]

$$\begin{aligned}
&\mathcal{F}^{r_T}\left[(CB(r_T,\mathbf{c},t,T,\mathbf{T})-K)^+\right] \\
&\quad = \int_{-\infty}^{r_T^*} e^{\imath z r_T}\left(\sum_{a=1}^{A} e^{a(0,\tau_a)+b(0,\tau_a)r_T} c_a - K\right) \mathrm{d}r_T \\
&\quad = e^{\imath z r_T^*}\left(\sum_{a=1}^{A} \frac{e^{a(0,\tau_a)+b(0,\tau_a)r_T^*}}{b(0,\tau_a)+\imath z} c_a - \frac{K}{\imath z}\right),
\end{aligned} \tag{5.39}$$

with

$$\operatorname{Im}(z) < \min_a \left[b(0,\tau_a)\right].$$

Note that in contrast to the valuation formula a zero-bond call option, where the Fourier Transformation of the payoff function was made with respect to $g(\mathbf{x}_T)$, we now perform the transform operation with respect to $r_T$. Therefore, we need a different restriction for the imaginary part of the transform variable $z$. Because the coefficient $b(0,\tau_a)$ is generally negative, we take the smallest value of $b(0,\tau_a)$ as an upper bound for the domain of valid values for $\operatorname{Im}(z)$, which is due to the monotonicity simply $b(0,\tau_A)$. Eventually, using the general valuation formula (4.21), we are able to compute the price of a coupon-bond call option as

$$\begin{aligned}
&CBC\left(r_t,\mathbf{c},K,t,T,\mathbf{T}\right) \\
&\quad = \frac{1}{\pi}\int_0^{\infty} e^{\imath z r_T^*}\left(\sum_{a=1}^{A} \frac{e^{a(0,\tau_a)+b(0,\tau_a)r_T^*}}{b(0,\tau_a)+\imath z} c_a - \frac{K}{\imath z}\right) \times \\
&\qquad \psi(r_t,-z,0,1,0,1,\tau)\,\mathrm{d}z.
\end{aligned} \tag{5.40}$$

As before, the payoff transform of the particular put option is also given by equation (5.40), but with the slightly modified restriction

$$\operatorname{Im}(z) > 0.$$

[127] Since the integration variable is no longer $g\left(\mathbf{x}_T\right)$, we have to switch the integration boundaries, due to the negativeness of $b(0,\tau)$.

Having derived the proper Fourier Transformation of a coupon-bond option payoff, the equivalent expression for a yield-based swaption contract is given by the alternative representation of a swaption contract according to equation (3.23), with coupon payment vector $\mathbf{c}_{SWP}$ and payment dates contained in $\mathbf{T}^*$. On the other hand, the particular forward-start payer swaption can be interpreted as a coupon-bond put option with strike one and the same coupon payment vector and the same payment dates as used before. Hence, for the transformed payoff function to be existent, we have to ensure that the inequality $\mathrm{Im}(z) > 0$ holds.

# 6

# Numerical Computation of Model Prices

## 6.1 Overview

In this chapter we develop a new pricing algorithm to compute model prices for the derivatives contracts previously discussed. Here, we distinguish, as before, between contracts with unconditional and conditional exercise rights. The distinction is made because of the separate fundamental calculation procedure for these prices. Whereas derivatives with unconditional exercise rights can be calculated in terms of the general characteristic function $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ and in terms of the relevant moment-generating function[128], respectively, without evaluating any integral at all if the characteristic function is known in closed form, we need for option-type contracts to apply a numerical integration scheme in order to calculate their model prices. Carr and Madan (1999) showed in their prominent article a very convenient method to compute option prices for a given strike range, using the FFT. The advantage in applying the FFT to option-pricing problems, is its considerable computational speed improvement compared to other numerical integration schemes. Due to the payoff transform methodology, we use another pricing algorithm, which shares the same desirable, numerical properties of the FFT. Unfortunately, implementing the pricing approach according to Lewis (2001), it is necessary to impose the transform with respect to the strike. Therefore, one cannot use the FFT any longer to obtain option prices in one pass for a strike range[129].

[128] See Section 5.2.

[129] See Lee (2004), p. 61. However, comparing the structure in equation (4.21) it is possible to obtain model prices with the help of a FFT procedure for different levels of $g(\mathbf{x}_t)$.

In order to circumvent this problem within the payoff-transform pricing approach, we need an another numerical algorithm. Therefore, we incorporate in our pricing algorithm the IFFT, to compute model prices for different strike values[130]. Furthermore, to enhance the quality of results[131], the fractional Fourier Transform of Bailey and Swarztrauber (1994) is used. This refinement was introduced by Chourdakis (2005) in pricing equity option prices with the transformed option price methodology of Carr and Madan (1999).

However, we sometimes encounter the problem that $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ cannot be calculated in closed form[132]. For these cases, we implement a Runge-Kutta solver in our IFFT pricing algorithm. This algorithm is then used to compute the relevant values for different $z$ in $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$ by solving the ODEs (2.40) and (2.41) numerically and providing the procedure with the needed values.

## 6.2 Contracts with Unconditional Exercise Rights

As explained in Section 5.2.2 all contracts with unconditional exercise rights can be calculated as mere function evaluations of the general characteristic function $\psi(\mathbf{x}_t, z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)$, its first order derivative with respect to $z$, and for integro-linear payoff functions with the help of the first order derivative $\psi_z\left(\mathbf{x}_t, z, w_0^A(z), \mathbf{w}^A(z), 0, \mathbf{0}_M, \tau\right)$. As shown, these unconditional expectations can be obtained by contour integration in closed form. Thus, we do not need to develop a numerical integration routine at all in order to calculate the relevant model prices. The calculations reduce in these cases to

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\right] = \psi(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau),$$

[130] We find it natural to use the FFT and the IFFT algorithm to obtain the desired Fourier Transformation. Other numerical integration schemes are also possible, like for example the numerical integration via Laguerre polynomials as used in Tahani (2004).

[131] The ordinary IFFT pricing algorithm suffers, like the particular FFT algorithm, from the fixed scale of increments of strike values and transformation variable, which is discussed in Section 6.3.1.

[132] This could be the case e.g. for some subordinated processes $r_t$ or for jump components where $\mathbb{E}_{\mathbf{J}}\left[\boldsymbol{\psi}^*(z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau)\right]$ cannot be solved explicitly.

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int\limits_t^T r(\mathbf{x}_s)\,\mathrm{d}s} g\left(\mathbf{x}_T\right)\right] = \frac{\psi_z(\mathbf{x}_t, 0, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)}{\imath},$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int\limits_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \gamma(T)\right] = \frac{\psi_z(\mathbf{x}_t, 0, w_0^A(0), \mathbf{w}^A(0), 0, \mathbf{0}_M, \tau)}{\imath},$$

for arbitrary times to maturity $\tau$. For normal contracts, the discount rate used in the characteristic function is based on the short rate $r\left(\mathbf{x}_t\right)$ and is zero for futures-style contracts. In case of an average-rate contract where the underlying is the geometric average of the short rate, we have to use the characteristic function with a modified discount rate $r^A\left(\mathbf{x}_t\right)$.

If the general characteristic function cannot be expressed in closed form although defined by a system of ODEs, we apply a numerical algorithm to evaluate the needed values. In this case we implement a Runge-Kutta solver for the system of ODEs (2.40) and (2.41).

## 6.3 Contracts with Conditional Exercise Rights

### 6.3.1 Calculating Option Prices with the IFFT

We start with the integral representation of the general option valuation formula (4.21). Since we are interested in calculating option prices in one pass for a given strike range simultaneously with the IFFT, we have to reduce the presence of $K$ in the integral to the expression $e^{\imath z\alpha(K)}$ for both exponential-linear, linear, and integro-linear type payoff functions. In the case of coupon-bond options and swaptions we have to divide the payoff function up into $A$ separate parts. The alternative representation of the valuation formula is

$$V(\mathbf{x}_t, t, T) = \frac{e^{\alpha(K)d}}{\pi} \int\limits_0^\infty e^{\imath z\alpha(K)} \hat{g}(z)\psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau)\,\mathrm{d}z, \qquad (6.1)$$

with

$$\mathcal{F}^{g(\mathbf{x}_T)}\left[G\left(\mathbf{x}_T\right)\right] = e^{(d+\imath z)\alpha(K)}\hat{g}(z),$$

and $\alpha(K) = K$ for the case of a floating-rate based contract and an asian-type contract, respectively, and $\alpha(K) = \ln[K]$ for a yield-based contract[133]. The

[133] See equation (5.21).

parameter $d$ is chosen in a way to eliminate all dependency of $\alpha(K)$ in $\hat{g}(z)$, which is crucial for the IFFT algorithm to work properly[134]. A first problem might arise using multi-valued functions, e.g. the complex-valued logarithm, square-root, and the confluent hypergeometric function $\mathrm{KU}(a;b;y)$. Thus, we have to carefully keep track of the integration path to avoid any discontinuities[135]. However, using a numerical algorithm to compute the particular values of the characteristic function such as a Runge-Kutta algorithm we do not encounter these problems[136].

The first step in deriving the IFFT pricing algorithm is to truncate the integration domain as

$$f(\alpha(K)) \approx \int_0^{\omega} e^{\imath z \alpha(K)} \hat{g}(z) \psi(\mathbf{x}_t, -z, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \, \mathrm{d}z. \tag{6.2}$$

Applying an $U$-point approximation with increment $\Delta = \frac{\omega}{U}$, we discretize the domain of the transform variable into

$$z_u = \left(u - \frac{1}{2}\right) \Delta + \imath z_i$$

with $u = 1, \ldots, U$ and $z_i$ corresponding to a fixed value for which the Fourier-transformed payoff function exists. The integration interval $[0, \infty]$ is then replaced with a discrete, truncated region such that the integrand of $f(\alpha(K))$ is negligible for $z_U$. Hence, the discrete approximation to equation (6.2) is

$$\begin{aligned} f(\alpha(K)) &\approx \sum_{u=1}^{U} e^{\imath z_u \alpha(K)} \hat{g}(z_u) \psi(\mathbf{x}_t, -z_u, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau) \, \Delta \\ &= \Delta e^{-z_i \alpha(K)} \sum_{u=1}^{U} e^{\imath (u-1) \, \Delta \alpha(K)} e^{\frac{\imath \Delta}{2} \, \alpha(K)} \hat{g}_u \psi_u, \end{aligned} \tag{6.3}$$

[134] Otherwise, the IFFT algorithm is not applicable to the valuation problem at hand. Fortunately, we are able to reduce the dependency of $K$ in the particular integrals to the specific term $e^{\imath z \alpha(K)}$, for all contracts discussed in Chapter 3.

[135] This topic is covered comprehensively in Nagel (2001), Appendix 4.

[136] In case of the Fong and Vasicek (1991a) model, we made the same experience as mentioned in Tahani (2004), Footnote 4, and compute values of the characteristic function with help of an explicit Runge-Kutta algorithm in the first place. Thus, besides the prevention of discontinuities, the Runge-Kutta algorithm can be more efficient than the explicit computation of the confluent hypergeometric function.

with

$$\hat{g}_u = \hat{g}(z_u) \qquad \text{and} \qquad \psi_u = \psi(\mathbf{x}_t, -z_u, w_0, \mathbf{w}, g_0, \mathbf{g}, \tau).$$

The sum above is commonly referred to as a discrete inverse Fourier Transformation[137] of the function $e^{\frac{\imath \Delta}{2}\alpha(K)}\hat{g}_u\psi_u$. We also want to mention that in computing this sum we eventually obtain the option price for only one particular strike value $K$. Since we are interested in calculating option prices for a strike range we also have to discretize $\alpha(K)$, which yields

$$\alpha_v = \alpha(K_1) + (v-1)\eta,$$

with step size $\eta$ and $v = 1, \ldots, U$[138]. Thus, inserting the explicit expression for $\alpha_v$ inside the brackets of equation (6.3) gives

$$\begin{aligned} f(\alpha_v) &= \Delta\, e^{-z_i\alpha_v} \sum_{u=1}^{U} e^{\imath(u-1)\,\Delta(\alpha_1+(v-1)\eta)} e^{\frac{\imath\Delta}{2}(\alpha_1+(v-1)\eta)} \hat{g}_u\psi_u \\ &= \Delta\, e^{-z_i\alpha_v} e^{\frac{\imath\Delta\eta}{2}(v-1)} \sum_{u=1}^{U} e^{\imath(u-1)(v-1)\,\Delta\eta} e^{\imath\,\Delta\alpha_1(u-\frac{1}{2})} \hat{g}_u\psi_u. \end{aligned} \tag{6.4}$$

The form of $f(\alpha_v)$ is almost ready to be inserted into the IFFT algorithm.

The IFFT algorithm is developed to calculate simultaneously the discrete inverse Fourier Transformation for a range of values $\alpha_v$. The main advantage is that it reduces the number of calculations from an order of $U^2$ to the order of $U\log_2[U]$, which makes a significant difference in computational speed[139]. It efficiently computes the sum

$$f(v, \mathbf{h}) = \frac{1}{U}\sum_{u=1}^{U} e^{\imath(u-1)(v-1)\frac{2\pi}{U}} h_u \qquad \text{for } v = 1, \ldots, U. \tag{6.5}$$

---

[137] Although we defined the transform operations in Section 2.4 vice versa, in this chapter we rely on the term discrete *inverse* transform, which belongs to engineering disciplines and is in line with the expression used afterwards for the IFFT.

[138] We use the same discretization scheme for $\alpha(K)$ as used in Lee (2004). The advantage, in contrast to the discretization schemes applied in Carr and Madan (1999) and Raible (2000), is the possibility to adjust the numerical scheme for the lower bound of the strike rates. Thus, one does not necessarily have to compute option prices for negligible strike rates, which is a more efficient procedure.

[139] See Cooley and Tukey (1965).

Introducing the vectors

$$\mathbf{u} = \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ U \end{pmatrix},$$

equation (6.5) can be displayed in a more compact form, which is

$$\mathbf{f}(\mathbf{h}) = \mathrm{IFFT}[\mathbf{h}], \tag{6.6}$$

with $\mathbf{h} \in \mathbb{C}^U$.

By comparing equation (6.5) with (6.4), we obviously need the relation

$$\Delta\eta = \frac{2\pi}{U},$$

in order to apply the IFFT algorithm properly to equation (6.4). Because $\frac{2\pi}{U}$ remains constant for a fixed number of points $U$, we have only the freedom to choose either $\Delta$ or $\eta$ independently. Thus, there is a tradeoff between the accuracy of the calculated results and the coarseness of the strike-value grid. According to these considerations, more accurate results of option prices corresponding to specific strike rates have to be paid with more points in the integration scheme due to the rule $U \times 2^n$. This rule ensures that the algorithm computes option prices for specific strike values and illustrates the exponential cost for more accurate results. Calculating the same number of option prices, most of them outside a desired strike range, entails a substantial waste of computational time[140].

To give a more compact writing, we use henceforth the vectors $\boldsymbol{\alpha} = (\alpha_v)_{v=1}^U$, $\hat{\mathbf{g}} = (\hat{g}_u)_{u=1}^U$ and $\boldsymbol{\psi} = (\psi_u)_{u=1}^U$. Eventually, the vector $\mathbf{V}(\mathbf{x}_t, t, T)$ containing the option values for different strikes, can be computed as

$$\begin{aligned} \mathbf{V}(\mathbf{x}_t, t, T) =& \frac{U\,\Delta\, e^{(d-z_i)\boldsymbol{\alpha}}}{\pi} \\ &\odot \mathrm{Re}\left[ e^{\frac{\pi \imath}{U}(\mathbf{v}-1)} \odot \mathrm{IFFT}[e^{\imath\,\Delta\alpha_1(\mathbf{u}-\frac{1}{2})} \odot \hat{\mathbf{g}} \odot \boldsymbol{\psi}] \right], \end{aligned} \tag{6.7}$$

where the operator $\odot$ denotes the vector-dot product of two arbitrary vectors of the same length. This pricing algorithm is already capable of calculating

[140] This particular problem is addressed in the next section.

option prices. However, as stated before, equation (6.7) displays the problem of computing option prices for many irrelevant strike rates, given a desired level of accuracy.

### 6.3.2 Refinement of the IFFT Pricing Algorithm

The purpose of this subsection is to solve the problem of the inverse relationship of $\Delta$ and $\eta$ mentioned in the last section. The numerical efficiency can be enhanced by using a modified version of the ordinary IFFT algorithm to ensure that all calculated option prices are at least within an interval of relevant strike values. Bailey and Swarztrauber (1994) developed a method based on the FFT to choose $\Delta$ and $\eta$ independently. Their method, called the fractional Fourier Transformation, henceforth denoted as the FRFT, incorporates a new auxiliary parameter $\zeta$[141], which successfully dissects the otherwise fixed relation $\Delta\eta \equiv \frac{2\pi}{U}$. Chourdakis (2005) used this refined algorithm in pricing European options on equities based on the Carr and Madan (1999) pricing framework.

The FRFT was developed to efficiently compute the sum

$$f(v, \mathbf{h}, \zeta) = \sum_{u=1}^{U} e^{-2\pi\imath(u-1)(v-1)\zeta} h_u \qquad \text{for } \; v = 1, \ldots, U. \tag{6.8}$$

Thus, introducing the FRFT operator, we define the compact expression

$$\mathbf{f}(\mathbf{h}, \zeta) = \mathrm{FRFT}\left[\mathbf{h}; \zeta\right].$$

Although, the parameter $\zeta$ is usually real-valued, it is not restricted to the set of $\mathbb{R}$. Obviously, the FRFT is strongly connected to the FFT and the IFFT. For example, by comparing equation (6.5) with (6.8), we have the equivalence

$$\mathrm{IFFT}\left[\mathbf{h}\right] \equiv \frac{1}{U}\,\mathrm{FRFT}\left[\mathbf{h}; -\frac{1}{U}\right].$$

The key insight to compute the FRFT in terms of the FFT and the IFFT algorithm, respectively, is to recognize that the product $2(u-1)(v-1)$ can be expressed as

[141] The fractional Fourier Transformation parameter $\zeta$ in this thesis corresponds to $\alpha$ in the original article of Bailey and Swarztrauber (1994).

$$(u-1)^2 + (v-1)^2 - (v-u)^2.$$

Inserting this relation into equation (6.8), subsequently doing some algebraic transformations and using the discrete version of the convolution theorem of Fourier Transformations[142], we are able to efficiently compute equation (6.8) with the help of both the FFT and the IFFT algorithm as follows[143]. Defining the vectors $\mathbf{p}$ and $\mathbf{q}$ with elements

$$p_u = \begin{cases} \frac{h_u}{a_u} & \text{for} \quad 1 \leq u \leq U \\ 0 & \text{for} \quad U < u \leq 2U, \end{cases}$$

and

$$q_u = \begin{cases} a_u & \text{for} \quad 1 \leq u \leq U \\ a_{(2U+2-u)} & \text{for} \quad U < u \leq 2U, \end{cases}$$

with

$$a_u = e^{\imath\pi\zeta(u-1)^2},$$

we compute first the raw transformation as

$$\hat{\mathbf{f}}(\mathbf{h},\zeta) = \text{IFFT}\left[\text{FFT}\left[\mathbf{p}\right] \odot \text{FFT}\left[\mathbf{q}\right]\right].$$

The last $U$ elements in $\hat{\mathbf{f}}(\mathbf{h},\zeta)$ can be discarded due to the zero padding made in the vector $\mathbf{p}$. Thus, we store the first half of the vector $\hat{\mathbf{f}}(\mathbf{h},\zeta)$ in a new vector $\hat{\mathbf{f}}^-(\mathbf{h},\zeta)$. The FRFT is then

$$\mathbf{f}(\mathbf{h},\zeta) = \hat{\mathbf{f}}^-(\mathbf{h},\zeta) \odot \mathbf{a}_{-\mathbf{u}}. \tag{6.9}$$

Obviously, by comparing the term inside the sum operator in equation (6.4) with the corresponding term inside the sum in equation (6.8) we have to establish the relation

$$\zeta = -\frac{\Delta\eta}{2\pi},$$

where both $\Delta$ and $\eta$ can be chosen arbitrarily[144]. Thus, our general option-pricing formula (6.7), can be rewritten in terms of the FRFT as

[142] See Proposition 2.4.3.

[143] The detailed derivation of the FRFT algorithm is given in Bailey and Swarztrauber (1994).

[144] Note that the factor $\frac{1}{U}$ used in equation (6.4) is already included in $\Delta$.

$$\begin{aligned}\mathbf{V}(\mathbf{x}_t, t, T) \\ &= \frac{\Delta\, e^{(d-z_i)\boldsymbol{\alpha}}}{\pi} \odot \\ &\quad \mathrm{Re}\left[e^{-\pi\imath(\mathbf{v}-1)\zeta} \odot \mathrm{FRFT}\left[e^{\imath\,\Delta\alpha_1(\mathbf{u}-\frac{1}{2})} \odot \hat{\mathbf{g}} \odot \boldsymbol{\psi}; -\frac{\Delta\eta}{2\pi}\right]\right].\end{aligned} \tag{6.10}$$

Although we have to compute two FFTs and one IFFT in order to obtain one FRFT, there is a substantial improvement due to the now independent choice of strike interval and integration domain, which saves in the end computer time. This fact becomes more important for the computation of characteristic functions for which no closed-form solutions exist and therefore the system of ODEs (2.40) and (2.41) must be solved numerically for each sampling point $z_u$.

### 6.3.3 Determination of the Optimal Parameters for the Numerical Scheme

As discussed in Lee (2004) and Lord and Kahl (2007), the choice of $z_i$, determining the specific contour in the complex plane used for the numerical integration routine is crucial in computing option prices. Lee (2004) finds that for different option payoff functions, for different strike values and driving processes, respectively, the optimal value of $z_i$, thus minimizing the numerical error, varies substantially[145]. Furthermore, the parameter $\omega$ concerning the truncation error is also of the utmost importance in a numerical option-pricing scheme. Thus, both parameters influence the accuracy of numerical solutions. This is illustrated in Figure 6.1 for zero-bond call options and the jump-enhanced models of Vasicek (1977) and Cox, Ingersoll and Ross (1985b)[146]. Obviously, setting $\omega$ too small results in a highly oscillating solution vector. On the other hand choosing $\omega$ too high, the absolute error of the numerical so-

[145] See Lee (2004) Table 2 and 3. The same observation is made in Lord and Kahl (2007), Figure 1.

[146] Both interest-rate models are enhanced with an exponentially distributed jump component. The coefficients for the characteristic function of the jump-enhanced Vasicek model are given in equations (8.6), (8.7), and (8.8). The particular coefficients in case of the jump-enhanced CIR model are given in equations (8.11), (8.12), and (8.13). A discussion of these models is given in Chapter 8.

lutions increase exponentially. The opposite statement holds for $z_i$. Therefore, these parameters should be chosen to avoid minimize both effects.

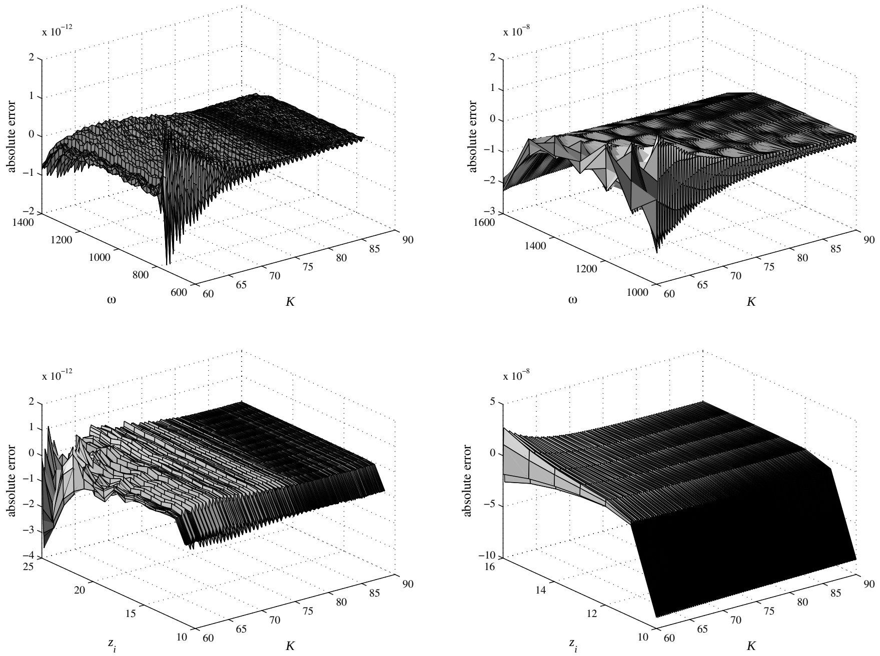


**Fig. 6.1.** Graphs in the first row depict absolute errors of 512 zero-bond call prices for alternating values of $\omega$. In the second row, the particular errors are depicted for varying values of $z_i$. An exponential-jump version of the Vasicek (CIR) model is used in the left (right) column. The parameters are: $r_t = 0.05(0.03), \kappa = 0.4(0.3), \theta = 0.05(0.03), \sigma = 0.01(0.1), \eta = 0.005(0.005), \lambda = 2(2), \tau = 0.5(0.5), \hat{\tau} = 2(2)$.

Since we want to price a vector of option prices with the computation of one FRFT operation, thus considering one specific parameter setting for the entire strike range, we are interested in finding the optimal parameter setting for the pricing algorithm, $(\omega^*, z_i^*)$, which minimizes the overall numerical error in equation (6.10). Hence, we need a criterion which measures the cumulative error of both positive and negative deviations from the theoretical solutions. Consequently, we apply in the following analysis the root mean-squared error (RMSE), which is

$$\text{RMSE} = \sqrt{\frac{(\mathbf{V}^{Num} - \mathbf{V}^{True})'(\mathbf{V}^{Num} - \mathbf{V}^{True})}{U}}, \tag{6.11}$$

where $\mathbf{V}^{Num}$ denotes some numerical solution vector and $\mathbf{V}^{True}$ represents the corresponding vector of closed-form solutions. To give an idea of the error behavior of the FRFT pricing algorithm, we first compare quasi closed-form solutions computed with the QUADL integration routine in MATLAB[147] according to equation (6.1) and the corresponding values due to the FRFT algorithm as defined in equation (6.10) for a fixed number of 512 different strike rates. The particular natural logarithms of the RMSE for zero-bond call option prices are depicted in Figure 6.2. We make two remarkable observations. Firstly, for differing values of $\omega$ and $z_i$ both models have a global minimum of the RMSE of computed option prices. Secondly, the logarithmic presentation of the RMSE implies a rapid and monotonic descent towards this minimum, starting with small values of $\omega$ and $z_i$[148]. In case of the jump-enhanced CIR model, the specific error-minimizing parameter couple is clearly evident according to the contour plot of the logarithmic RMSE given in the lower right graph of Figure 6.2. On the other hand, the particular contour plot of the logarithmic RMSE for zero-bond call options under the jump-enhanced Vasicek model also clearly indicates a region of parameter couples exhibiting approximately the same RMSE magnitude.

Consequently, we exploit this monotonic decrease of the RMSE to develop an algorithm, which is capable of finding an optimal parameter setting $(\omega^*, z_i^*)$ and simultaneously giving an estimate of the magnitude of errors of numerical solutions even when the closed-form solutions are not known. The technique we use for the approximation of the numerical error is based on the exponential decreasing of the mean-squared error between two successive parameter values in the numerical scheme. Thus, we define the approximate RMSE as

$$\text{RMSE}^a = \sqrt{\frac{(\mathbf{V}^{Num} - \mathbf{V}^{Num(+)})'(\mathbf{V}^{Num} - \mathbf{V}^{Num(+)})}{U}}, \tag{6.12}$$

where $\mathbf{V}^{Num}$ and $\mathbf{V}^{Num^{(+)}}$ denote numerical solutions of two successive parameter values, whether in $\omega$ or in $z_i$ direction.

[147] This integration routine uses an adaptive Lobatto quadrature scheme. In the calculation of quasi closed-form solutions, we set its error tolerance to $10^{-15}$.

[148] This phenomenon shows up for all interest-rate model/payoff combinations mentioned in this thesis.

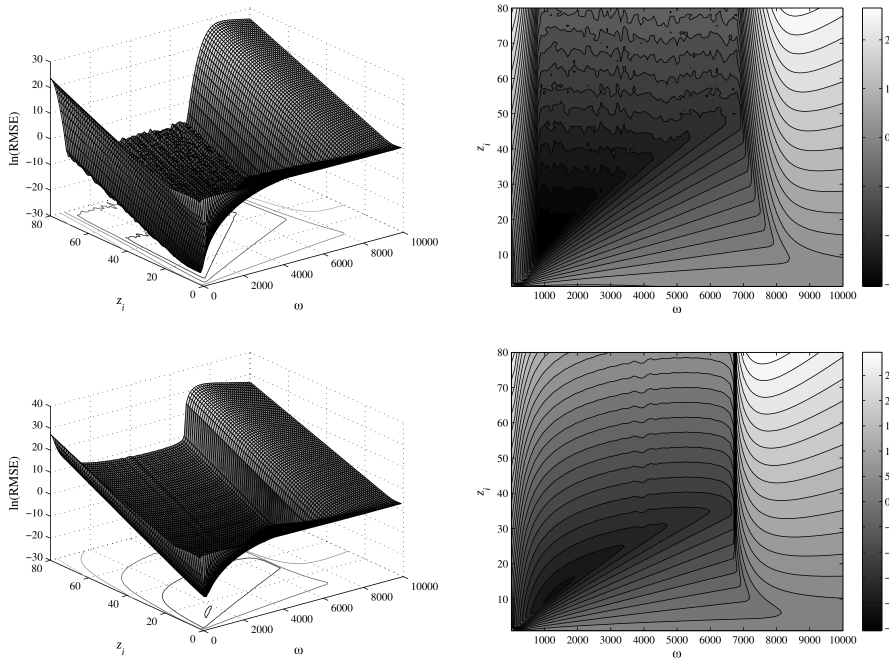


**Fig. 6.2.** Logarithmic RMSEs of 512 zero-bond call option prices. In the upper (lower) row the underlying interest rate is modeled by a jump-enhanced Vasicek (CIR) model. The parameters are: $r_t = 0.05(0.03), \kappa = 0.4(0.3), \theta = 0.05(0.03), \sigma = 0.01(0.1), \eta = 0.005(0.005), \lambda = 2(2), \tau = 0.5(0.5), \hat{\tau} = 2(2)$ and a strike range of $K \in [60, 90]$.

In Figure 6.3, differences of the logarithmic RMSE$^a$, for two successive parameter values of $z_i$, and the logarithmic RMSE according to equation (6.11) are depicted for zero-bond call prices for varying $z_i$ values. Obviously, the approximate and exact RMSEs show nearly the same magnitude until the minimum RMSE is reached. Afterwards, the difference, still very small, becomes oscillating in case of the Vasicek model and experiences a decrease of its level in case of the CIR model, respectively. This characteristic behavior of the RMSE$^{\mathrm{a}}$ is used in our algorithm to find the optimal parameter couple $(\omega^*, z_i^*)$ and enables the formulation of an approximate error bound for the numerical solution vector.

As mentioned above, our algorithm to find the optimal parameter couple $(\omega^*, z_i^*)$ utilizes a steepest descent technique on the logarithm of the RMSE$^{\mathrm{a}}$.

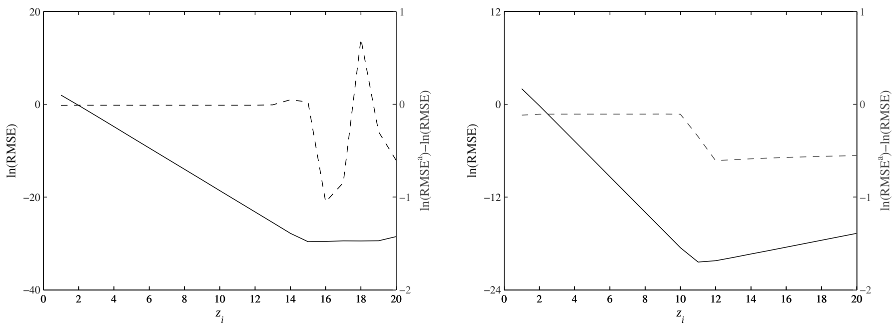


**Fig. 6.3.** The dashed line represents the difference of the logarithmic RMSE$^a$ and the exact RMSE of 512 zero-bond call option prices and increasing values of $z_i$. Both graphs are drawn for $\omega = 1400$. The straight line depicts the logarithmic RMSE in dependence of $z_i$. The underlying model in the left (right) graph is a jump-enhanced Vasicek (CIR) model with parameters: $r_t = 0.05(0.03), \kappa = 0.4(0.3), \theta = 0.05(0.03), \sigma = 0.01(0.1), \eta = 0.005(0.005), \lambda = 2(2), \tau = 0.5(0.5), \hat{\tau} = 2(2)$ and a strike range of $K \in [60, 90]$.

Thus, initializing the algorithm, we first evaluate the numerical solution $\mathbf{V}^{Num}$ for some parameter values $(\omega^o, z_i^0)$[149]. Subsequently, we compute two additional solution vectors for ascending parameter values in the direction of both $\omega$ and $z_i$ which are then used to derive the particular first order finite differences. Afterwards, if the slope in $\omega$ direction is smaller than the one in $z_i$ direction, thus more negative, the next numerical solution is computed with an exalted $\omega$ and vice versa. The next step in the numerical scheme is then again to obtain the necessary numerical solution vectors in order to derive the particular finite differences and so on. The algorithm aborts if the smallest value of ln(RMSE$^a$) is reached over some interval where the curve experienced its reversal point. In Figure 6.4, the paths with an initial value of $z_i^0 = 2$ and $\omega^0 = 10$ for the jump-enhanced Vasicek and CIR model are shown. Obviously, the algorithm finds for both interest-rate models the optimal parameter setting, which can be justified by the graphs in the left column of Figure 6.4. In case of the optimal parameter couple using the jump-enhanced Vasicek (CIR) model, we get a difference of exact and approximate RMSEs of $9.02924 \times 10^{-14}$

[149] Since we observe the steepest descent starting at the origin the initial value for $z_i^0$ and $\omega^0$ has to be near the origin subject to the particular regularity conditions of the Fourier-transformed payoff function.

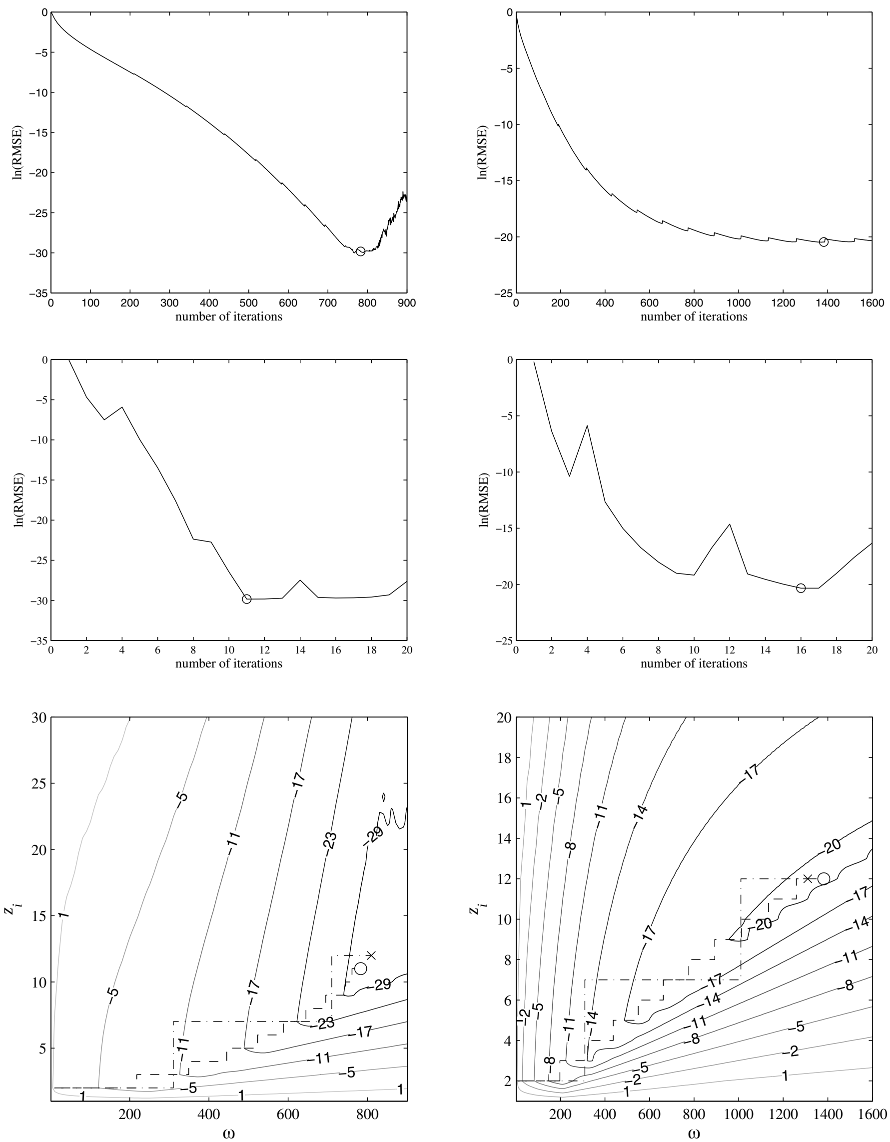


**Fig. 6.4.** Search for the optimal parameter couple $(\omega^*, z_i^*)$. In the first (second) column particular graphs are shown for the Vasicek (CIR) model with the data used in Figure 6.3. In the first row, the particular ln(RMSE) is depicted for the search algorithm with increments $(\Delta\omega, \Delta z_i) = (1, 1)$. In the second row the same search is made with increments (100, 5). In the third row the dashed (dash-dotted) line denotes the particular search path for small (high) increment values, where the optimal choice is marked by a circle and cross, respectively.

($8.27740 \times 10^{-10}$), whereas the exact RMSE is $1.11766 \times 10^{-13}$ ($1.30453 \times 10^{-9}$). Thus, we have in both models a difference which is of smaller order than the effective error according to the RMSE. Consequently, the RMSE$^a$ gives a good prediction for the corresponding exact value, which justifies the application of the approximate RMSE. In the first row of Figure 6.4, we used very small increments for the search of the optimal parameter couple to give a detailed impression of the search path and the particular logarithmic RMSE. According to the graphs in the second row of Figure 6.4, a comparable result is achieved by running the algorithm with higher increments[150]. However, due to the reduced number of iterations, the search algorithm with high increments is in case of the jump-enhanced Vasicek (CIR) model up to 71 (86) times faster. Dealing with a characteristic function known in closed form together with a FRFT-based pricing algorithm, the search takes only a second at all even for small increments. Thus, if the general characteristic function is known in closed form, the step-size does not matter. However, if values of the general characteristic need to be determined numerically via a Runge-Kutta algorithm, we usually set the increments high enough to keep the overall number of iterations small.

Finally, we use the RMSE$^a$ to derive an upper error bound for the numerical solutions contained in $\mathbf{V}^{Num}$. The first step in deriving this particular error bound is to consider a hypothetical solution vector $\mathbf{V}^{Num}$, where all elements equal their true solutions except the result given in the first position of the solution vector, namely $V_1^{Num}$. Without loss of generality, we assume the numerical error of this particular option price to be of magnitude $|a|$. Therefore, solving equation (6.11) in this special case gives

$$a = \mathrm{RMSE}\sqrt{U}. \tag{6.13}$$

Additionally, we are also able to state the inequality

$$\sqrt{(\mathbf{V}^{Num} - \mathbf{V}^{True})'(\mathbf{V}^{Num} - \mathbf{V}^{True})} \geq |V_v^{Num} - V_v^{True}|, \tag{6.14}$$

to hold for every element of the numerical solution vector $\mathbf{V}^{Num}$. According to equation (6.13), the RMSE scaled by some constant $\sqrt{U}$ states the value

[150] The second run of the algorithm, with higher increments, gives an absolute error for the optimal parameter couple $(\omega^*, z_i^*)$ for zero-bond calls under the jump-enhanced Vasicek (CIR) model of $1.13911 \times 10^{-13}$ ($1.46601 \times 10^{-9}$).

of the maximum attainable error. Furthermore, this result together with the inequality in (6.14) generally implies that the absolute error of one particular numerical solution $V_v^{Num}$ cannot exceed the absolute value $|a|$. Therefore, the RMSE can be used in formulating a boundary for the highest possible error. Consequently, we use the quantity $\mathrm{RMSE}^{\mathrm{a}}$, scaled by some constant $\sqrt{U}$, as a conservative upper error bound for the results generated by the pricing algorithm.

# 7

# Jump Specifications for Affine Term-Structure Models

## 7.1 Overview

In this thesis, we discuss jump-diffusion interest-rate models. Thus, both diffusion and jump components are included in order to model more realistic term-structure models. The jump sizes considered are governed either by exponential, normal or gamma distributions. The exponential jump distribution is a very popular approach in modeling term structure and equity models[151], since it yields closed-form formulae for most derivatives contracts. Das and Foresi (1996) and Chacko and Das (2002) have conducted recent studies with a double-sided version of this jump type using a Vasicek model for the diffusion part[152]. Our second candidate, the normal jump-size distribution is used in Baz and Das (1996) and Das (2002)[153]. The last jump-size distribution candidate for the interest-rate process is a gamma distribution, which is used in Kispert (2005) to support the stochastic dynamics of the volatility in electricity derivative contracts. This jump type is used for the first time in a jump-diffusion interest-rate model. As a special case, the gamma distribution covers the exponential distribution. Hence, we can build a more flexible

[151] Das and Foresi (1996) used this jump specification in modeling short rates whereas Kou (2002) uses this type of jump-size distribution modeling equities.

[152] Jumps in the instantaneous interest rate are governed by an exponential jump-size distribution. The direction of the jump itself is modeled either by a Bernoulli distribution or via two different Poisson processes. See Section 8.2.

[153] The articles consider a discrete version of the Vasicek interest-rate model with normally distributed jump shocks.

jump shock component in contrast to the exponential case, by extending the repertoire of jump-size distributions to the gamma distribution case.

In the following, we do not restrict ourselves solely to one jump component for each factor. Due to the independence of the jump distributions from the state of $\mathbf{x}_t$[154], we are able to add an unlimited amount of different jump components. However, we need to consider possible nonnegativity constraints of the particular diffusion process. Thus, we do not combine normally distributed jump parts, or negatively directed exponentially and gamma distributed jump parts, with a Square-Root diffusion model. This is in fact no real drawback, that is to say we can think of a *bad news* effect rather as a discontinuous *increase* in interest rates than the opposite effect[155]. All possible combinations for diffusion and jump components are illustrated in Figure 7.1.

According to equation (2.40), the coefficient function $a(z,\tau)$ can be split into a part containing the characteristics of the diffusion process and a part containing the additional jump characteristics[156]. This results in a modular representation of the ODE for the coefficient function $a(z,\tau)$, which is

$$a(z,\tau)_\tau = a^0(z,\tau)_\tau + a^1(z,\tau)_\tau, \tag{7.1}$$

with

$$a^0(z,\tau)_\tau = -w_0 + \boldsymbol{\mu}_0^{\mathbb{Q}\prime}\mathbf{b}(z,\tau) + \frac{1}{2}\,\mathbf{b}(z,\tau)'\boldsymbol{\Sigma}_0\mathbf{b}(z,\tau),$$

and

$$a^1(z,\tau)_\tau = \mathbb{E}_{\mathbf{J}}\left[\boldsymbol{\psi}^*(z,w_0,\mathbf{w},g_0,\mathbf{g},\mathbf{J},\tau)' - 1\right]\boldsymbol{\lambda}^{\mathbb{Q}}.$$

Unless otherwise stated, the coefficient function $a^0(z,\tau)$ denotes the diffusion part, whereas $a^1(z,\tau)$ represents the solution for the jump part, which is frequently called the jump transform[157].

As mentioned above, each diffusion process can be augmented with an infinite number of jump processes. Thus, taking the expectation in (2.39)

[154] This statement also holds for different jumps triggered by the same poisson process. See equation (7.2).

[155] See, for example, Schöbel and Zhu (2000), p. 5.

[156] Note that the jump part affects the coefficient $a(z,\tau)$, whereas the coefficient vector $\mathbf{b}(z,\tau)$ is independent from the jump amplitude and intensity.

[157] See Duffie, Pan and Singleton (2000).

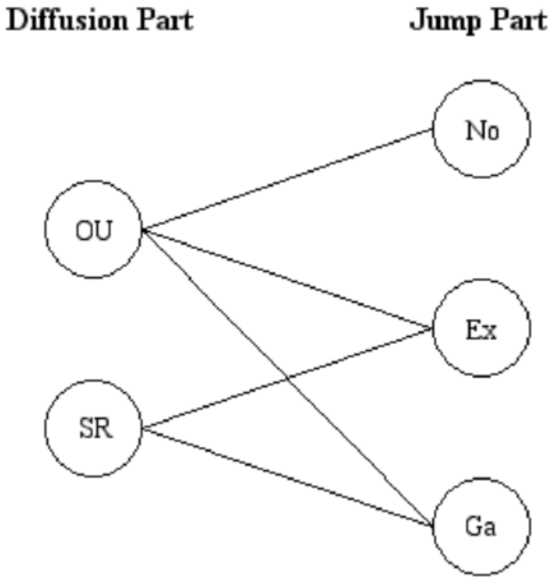


**Fig. 7.1.** Possible combinations of the Ornstein-Uhlenbeck (OU) process and the Square-Root (SR) process with the exponential (Ex), normal (No), and gamma (Ga) jump distributions.

with respect to the jump amplitudes $\mathbf{J}$, we obviously are able to check that every element of the resulting vector is expressible as the product of different expectations. Formally, we have

$$\mathbb{E}_{\mathbf{J}}\left[\begin{pmatrix} e^{\mathbf{b}(z,\tau)'\mathbf{j}_1} \\ e^{\mathbf{b}(z,\tau)'\mathbf{j}_2} \\ \vdots \\ e^{\mathbf{b}(z,\tau)'\mathbf{j}_N} \end{pmatrix}\right] = \begin{pmatrix} \mathbb{E}_{\mathbf{j}_1}\left[e^{\mathbf{b}(z,\tau)'\mathbf{j}_1}\right] \\ \mathbb{E}_{\mathbf{j}_2}\left[e^{\mathbf{b}(z,\tau)'\mathbf{j}_2}\right] \\ \vdots \\ \mathbb{E}_{\mathbf{j}_N}\left[e^{\mathbf{b}(z,\tau)'\mathbf{j}_N}\right] \end{pmatrix}.$$

Selecting one element of this vector as an example, say $\mathbb{E}_{\mathbf{j}_n}\left[e^{\mathbf{b}(z,\tau)'\mathbf{j}_n}\right]$, and manipulating the expectation operator, we get

$$\begin{aligned}
\mathbb{E}_{\mathbf{j}_n}\left[e^{\mathbf{b}(z,\tau)'\mathbf{j}_n}\right] &= \int\limits_{\mathbb{R}^M}\left(e^{\mathbf{b}(z,\tau)'\mathbf{j}_n}\prod_{m=1}^{M}\nu(J_{mn})\right)\mathrm{d}\mathbf{j}_n \\
&= \prod_{m=1}^{M}\int\limits_{\mathbb{R}} e^{b^{(m)}(z,\tau)J_{mn}}\nu(J_{mn})\,\mathrm{d}J_{mn} \qquad (7.2)\\
&= \prod_{m=1}^{M}\mathbb{E}_{J_{mn}}\left[e^{b^{(m)}(z,\tau)J_{mn}}\right].
\end{aligned}$$

The function $\nu(J_{mn})$ represents the probability density of the particular jump amplitude $J_{mn}$. As demonstrated in equation (7.2) the joint density function can be expressed as the product of different density $\nu(J_{mn})$ on account of the independence of the jump amplitudes. Consequently, issues are simplified in equation (7.2) by successively evaluating all integrals one by one, which yields the cumulative product of different expectations. Thus, we express the solution of the jump part as

$$a^1(z,\tau) = -\tau\boldsymbol{\iota}_N'\boldsymbol{\lambda}^{\mathbb{Q}} + \sum_{n=1}^{N}\lambda^{\mathbb{Q}^{(n)}}\left(\int\limits_0^{\tau}\prod_{m=1}^{M}\mathbb{E}_{J_{mn}}\left[e^{b^{(m)}(z,l)J_{mn}}\right]\mathrm{d}l\right), \qquad (7.3)$$

with $n = 1,\ldots,N$ and each element of the vector $\boldsymbol{\iota}_N \in \mathbb{R}^N$ equals one.

Since we need to calculate the integral over the time variable there is the possibility of ending up without any closed-form solution[158]. In contrast, the Bates (1996) model, which is a jump-diffusion model in an equity context, in which a normally distributed jump component is used for the log-asset price process, the coefficient for the jump size yields a nice closed-form expression in Fourier space[159]. In term-structure models, normal and gamma size distributions allow only the formulation of the coefficient $a^1(z,\tau)$ in terms of its underlying differential equation. Thus, a possible reason why normal and gamma jump distributions are not as popular in interest-rate option pricing might be tracked back to the unavailability of appropriate valuation formulae for interest-rate contracts. Nevertheless, these jump size candidates provide a

[158] Both gamma and normally distributed jump amplitudes have no closed-form jump transform for all models discussed in this thesis.

[159] In one-factor *equity* models the computation can be simplified to $\mathbb{E}_{J_n}\left[e^{\imath z J_n}\right]$, which is obviously easier to handle, since the exponential function inside the expectation operator is independent of the time to maturity variable $\tau$. See Cont and Tankov (2004), p. 477.

valuable contribution in generating a realistic overall probability distribution of short rates. However, the algorithm presented in Chapter 6 can compute derivative prices under these interest-rate dynamics. The only condition that needs to be met is the availability of separable ODEs of the coefficient functions $a(z,\tau)$ and $\mathbf{b}(z,\tau)$, which lets us apply a Runge-Kutta algorithm[160].

## 7.2 Exponentially Distributed Jumps

The exponential distribution is a widely used shock specification in jump-diffusion models. Thus, it can be found in both equity and interest-rate models[161]. The probability density function $p_{Ex}(J,\eta)$ of an exponentially distributed variable $J \sim Ex(\eta)$ is defined as

$$p_{Ex}(J,\eta) = \begin{cases} 0 & \text{if} \quad J < 0 \\ \dfrac{1}{\eta} e^{-\frac{J}{\eta}} & \text{if} \quad J \geq 0; \eta > 0. \end{cases}$$

Hence, the expected value for $J$ and its variance is

$$\mathbb{E}_J[J] = \eta,$$

and

$$\mathbb{VAR}_J[J] = \eta^2.$$

The shape of the density function $p_{Ex}(J,\eta)$ for different values of $\eta$ is shown in Figure 7.2.

For a positively directed jump, with distribution parameter $\eta_+$, we get

[160] An interest-rate model which clearly opposes this separability ability of the coefficient functions of the general characteristic function is given in Ahn and Gao (1999) and Aït-Sahalia (1999), Example 3. However, closed-form solutions of zero-bond prices under these short-rate dynamics can be derived. See Ahn and Gao (1999), Proposition 1.

[161] Kou (2002), Kou and Wang (2004) implemented this jump specification for equity models, whereas Das and Foresi (1996) integrated this jump type in an Ornstein-Uhlenbeck short-rate model.

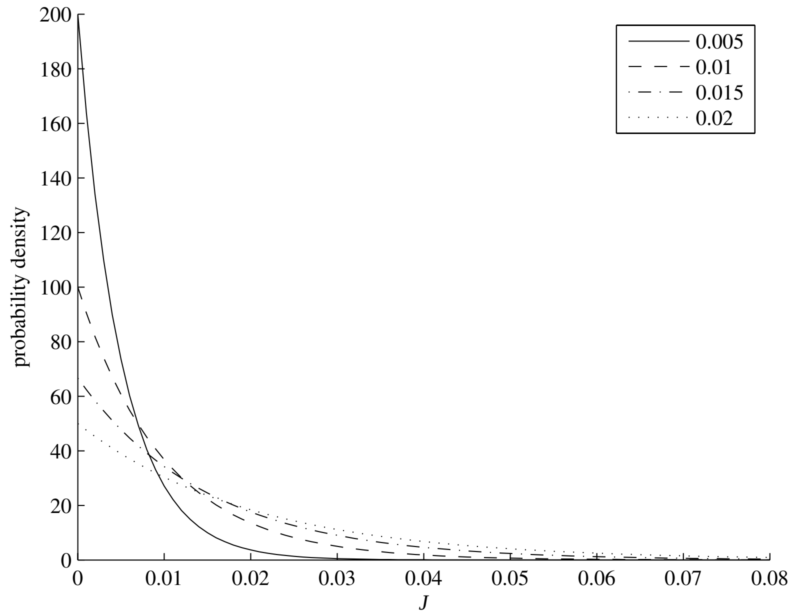


**Fig. 7.2.** The density function $p_{Ex}(J, \eta)$ for varying $\eta$ of an exponentially distributed random variable.

$$
\begin{aligned}
\mathbb{E}_J\left[e^{b^{(m)}(z,\tau)J} - 1\right] &= \frac{1}{\eta_+}\left[\frac{e^{\left(b^{(m)}(z,\tau)-\frac{1}{\eta_+}\right)J}}{b^{(m)}(z,\tau)-\frac{1}{\eta_+}}\right]_0^\infty - 1 \\
&= \frac{1}{1-b^{(m)}(z,\tau)\eta_+} - 1 \\
&= \frac{b^{(m)}(z,\tau)\eta_+}{1-b^{(m)}(z,\tau)\eta_+}.
\end{aligned}
$$

Accordingly, a negatively directed jump with parameter $\eta_-$, has an expected value of

$$
\begin{aligned}
\mathbb{E}_J\left[e^{-b^{(m)}(z,\tau)J} - 1\right] &= -\frac{1}{\eta_-}\left[\frac{e^{-\left(b^{(m)}(z,\tau)+\frac{1}{\eta_-}\right)J}}{b^{(m)}(z,\tau)+\frac{1}{\eta_-}}\right]_0^\infty - 1 \\
&= \frac{1}{1+b^{(m)}(z,\tau)\eta_-} - 1 \\
&= -\frac{b^{(m)}(z,\tau)\eta_-}{1+b^{(m)}(z,\tau)\eta_-}.
\end{aligned}
$$

In order to guarantee the existence of the jump transform, we need the real part $\mathrm{Re}\left[b^{(m)}(z,\tau)\right] \leq \frac{1}{\eta_+}$ for the positively sized jump and $\mathrm{Re}\left[b^{(m)}(z,\tau)\right] \geq -\frac{1}{\eta_-}$ for the negatively directed jump[162], respectively. Thus, multiplying the recently derived expectations with the jump intensity of the particular Poisson jump, the jump part of the coefficient function $a(z,\tau)$ can be generally represented as,

$$a^1_{Ex^\pm}(z,\tau) = \pm \int_0^\tau \frac{\lambda^{\mathbb{Q}^{(n)}} b^{(m)}(z,l)\eta_\pm}{1 \mp b^{(m)}(z,l)\eta_\pm} \, \mathrm{d}l. \tag{7.4}$$

The transform for this jump candidate is the only one that can be expressed in closed form for the interest-rate models discussed in the next chapter.

## 7.3 Normally Distributed Jumps

The second candidate we consider for the jump-size distribution is the normal distribution. As mentioned before, this specification is not as popular in interest-rate pricing frameworks compared to the exponentially distributed case. One reason might be that the jump transform in an interest-rate jump-diffusion framework cannot be expressed in closed form. In this setup, the jump amplitude $J \sim N\left(\mu_J, \sigma_J^2\right)$ is distributed according to a probability density function:

$$p_{No}\left(J, \mu_J, \sigma_J\right) = \frac{e^{-\frac{(J-\mu_J)^2}{2\sigma_J^2}}}{\sqrt{2\pi}\sigma_J} \qquad \forall \qquad J \in \mathbb{R},$$

with mean

$$\mathbb{E}_J\left[J\right] = \mu_J,$$

and variance

$$\mathbb{VAR}_J\left[J\right] = \sigma_J^2.$$

The shape of the density function $p_{No}\left(J, \mu_J, \sigma_J\right)$ for different values of $\sigma_J$ is shown in Figure 7.3.

The few articles which mention this particular jump type can be quickly summarized. Baz and Das (1996), Durham (2005) and Durham (2006) implemented the Gaussian jump within a Vasicek base model. Since this type

[162] Since $b^{(m)}(z,\tau) < 0$ and $\frac{1}{\eta_-}$ is usually very large, we assume both conditions to be fulfilled.

of jump might violate a non-negativity constraint of the underlying diffusion process, it is only meaningful in a context of a real-valued process. Therefore, we do not consider the normally distributed jump candidate in case of a Square-Root diffusion process.

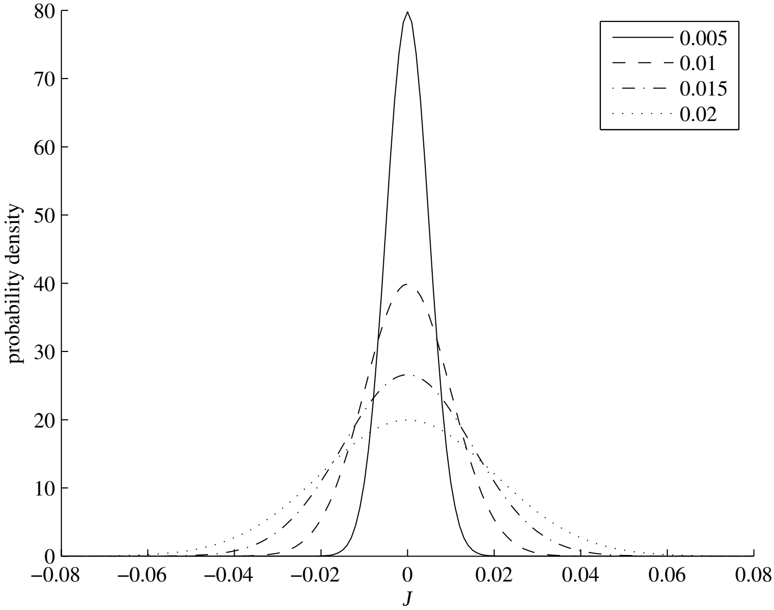


**Fig. 7.3.** The density function $p_{No}\left(J, \mu_J, \sigma_J\right)$ for fixed $\mu_J = 0$ and varying $\sigma_J$ of a normally distributed random variable.

Baz and Das (1996) approximate the expectation in equation (7.3) via a Taylor series approximation[163]. The series-approximation approach mentioned there considers two terms. Consequently, they first approximate the expression inside the expectation operator, and then take the expectation of the resulting terms. Hence, the approximation is given as

[163] The Taylor series approximation of the exponential function $f(x) = e^x$ is given by $\sum_{i=0}^{\infty} \frac{x^i}{i!}$.

$$\mathbb{E}_J\left[e^{b^{(m)}(z,\tau)J}-1\right]\approx\mathbb{E}_J\left[b^{(m)}(z,\tau)J+\frac{b^{(m)}(z,\tau)^2}{2}J^2\right]$$
$$=b^{(m)}(z,\tau)\mu_J+\frac{b^{(m)}(z,\tau)^2}{2}(\mu_J^2+\sigma_J^2).$$

In the last equation, the particular parameters of the normal distribution $\mu_J$ and $\sigma_J^2$ are used. Of course, it is possible to use a Taylor series considering more terms in order to enhance the accuracy of the calculations.

Another slightly different approximation technique is presented in Durham (2005) and Durham (2006), respectively. Here, the author applies a Taylor series approximation *after* taking the expectation in equation (7.3). This incorporates the distributional parameters in a more explicit fashion. Applying a two-term Taylor expansion gives[164]

$$\mathbb{E}_J\left[e^{b^{(m)}(z,\tau)J}-1\right]\approx b^{(m)}(z,\tau)\mu_J+\frac{b^{(m)}(z,\tau)^2}{2}\left(\mu_J^2+\sigma_J^2\right)$$
$$+\frac{b^{(m)}(z,\tau)^3}{2}\mu_J\sigma_J^2+\frac{b^{(m)}(z,\tau)^4}{8}\sigma_J^4.$$

Obviously, there is an advantage in applying either one of these *analytic approximations* for the jump transform. Using these simplifications, we are able to solve the ODE (7.1) in a consistent manner, meaning that no numerical integration of the jump transform is needed anymore, since only terms of $b^{(m)}(z,\tau)$ are left, yielding an *approximate* closed-form solution for the characteristic function[165]. As an additional benefit of the analytical approximations, Baz and Das (1996) mention the computational speed enhancement, facilitating the calibration to empirical data. However, the major drawback of both approximation techniques results from the application of the Taylor series. In order to produce accurate results, the term $b^{(m)}(z,\tau)$ and the difference inside the expectation operator, respectively, must be very small. Hence, with an increasing mean of the jump component and increasing variance, the results

[164] In Durham (2006) the negative sign of the coefficient $b^{(m)}(z,\tau)$ is extracted which explains the slightly different representation.

[165] For example, taking the Vasicek one-factor base model of equation (8.5) and the linear approximation due to Baz and Das (1996), we encounter the simple problem of solving equation (7.1) with extended parameters $\hat{\mu}_0^{\mathbb{Q}}=\mu_0^{\mathbb{Q}}+\lambda^{\mathbb{Q}}\mu_J$ and $\hat{\sigma}_0=\sqrt{\sigma_0^2+\sigma_J^2+\mu_J^2}$. Subsequently, the coefficient function $a_\tau(z,\tau)$ with these modified parameters has to be solved.

get more and more inaccurate. Consequently, the *analytical* approximation procedures should be applied only to scenarios where the jump component exhibits a small mean and variance.

Since our numerical procedure is designed to handle implicitly the ODE part of the jump transform, we need only the expected value of $e^{b^{(m)}(z,\tau)J}$ to be explicit. Under a normally distributed jump size regime, this is

$$\mathbb{E}_J\left[e^{b^{(m)}(z,\tau)J}-1\right]=e^{b^{(m)}(z,\tau)\mu_J+\frac{\left(b^{(m)}(z,\tau)\sigma_J\right)^2}{2}}-1, \tag{7.5}$$

which leads to the particular coefficient function

$$a^1_{No}(z,\tau)=\lambda^{\mathbb{Q}^{(n)}}\left(-\tau+\int\limits_0^\tau e^{b^{(m)}(z,l)\mu_J+\frac{\left(b^{(m)}(z,\tau)\sigma_J\right)^2}{2}}\,\mathrm{d}l\right). \tag{7.6}$$

The value of the integral can then be numerically approximated via a Runge-Kutta algorithm. Despite the numerical integration, the computational effort is very small due to our implemented FRFT procedure. But in contrast to the Taylor-series approach mentioned above, our results do not suffer from inaccuracies due to high mean and volatility parameters of the jump component. Hence, with our valuation procedure we gain superior accuracy. Furthermore, we are able to compute model prices for this jump specification for the first time, not only for ordinary zero bonds, but for all derivatives contracts, presented in Sections 3.2 and 3.3.

## 7.4 Gamma Distributed Jumps

The last jump-size distribution we want to implement in an interest-rate model is the gamma distribution. The probability density function $p_{Ga}\left(J,\eta,p\right)$ of the random variable $J\sim Ga(\eta,p)$ is given as

$$p_{Ga}\left(J,\eta,p\right)=\begin{cases}0 & \text{if} \quad J<0\\ \dfrac{1}{\eta^p\Gamma(p)}J^{p-1}e^{-\frac{J}{\eta}} & \text{if} \quad J\geq 0; p,\eta>0.\end{cases}$$

Thus, the expected value and variance for $J$ is

$$\mathbb{E}_J\left[J\right] = \eta p,$$

and

$$\mathbb{VAR}_J\left[J\right] = \eta^2 p.$$

The function $\Gamma(p)$ denotes the gamma function. Setting the parameter $p = 1$, the gamma distribution replicates the exponential distribution, since we have then the relation $p_{Ga}\left(J,\eta,1\right) = p_{Ex}\left(J,\eta\right)$. Additionally, we can use the gamma distribution to generate a chi-squared distribution. In this case we set $\eta = 2$ and $p = \frac{q}{2}$, where $q$ is a positively valued integer. The resulting chi-squared distribution has then $2p$ and $q$ degrees of freedom[166]. Another special case of the gamma distribution is the Erlang distribution. Here, we only need $p$ to be a positive integer value[167]. Thus, the Erlang distribution can be interpreted as the sum of $p$ independent exponentially distributed random variables with equal parameter $\eta$. The graph in Figure 7.4 shows the probability density function for different values of $p$. Comparing the different curves in Figure 7.4, it is obvious that this jump-size distribution is able to substantially enhance the short-rate model.

In Heston (1995) a pure jump interest-rate model is proposed. Accordingly, instead of a diffusion component, the innovations of the process in this model are governed solely by gamma distributed jumps. To our knowledge, Kispert (2005) was the first to use gamma distributed jump sizes within a jump-diffusion model. However, in pricing European options he needs inefficient Monte-Carlo routines, using the gamma and normal jump amplitude specification. These numerical problems can be circumvented by applying the FRFT-based algorithm together with a Runge-Kutta algorithm for the ODEs.

Next, we want to derive the particular jump transform. The expectation for a positively directed jump can be computed as

$$\mathbb{E}_J\left[e^{b^{(m)}(z,\tau)J} - 1\right] = \frac{1}{\eta_+^p \Gamma(p)} \int_0^\infty e^{-J\left(\frac{1}{\eta_+} - b^{(m)}(z,\tau)\right)} J^{p-1}\, \mathrm{d}J - 1.$$

[166] This is easily checked by comparing the particular moment-generating functions. See Stuart and Ord (1994), p. 541.

[167] See, for example, Balakrishnan, Johnson and Kotz (1994), p. 337.

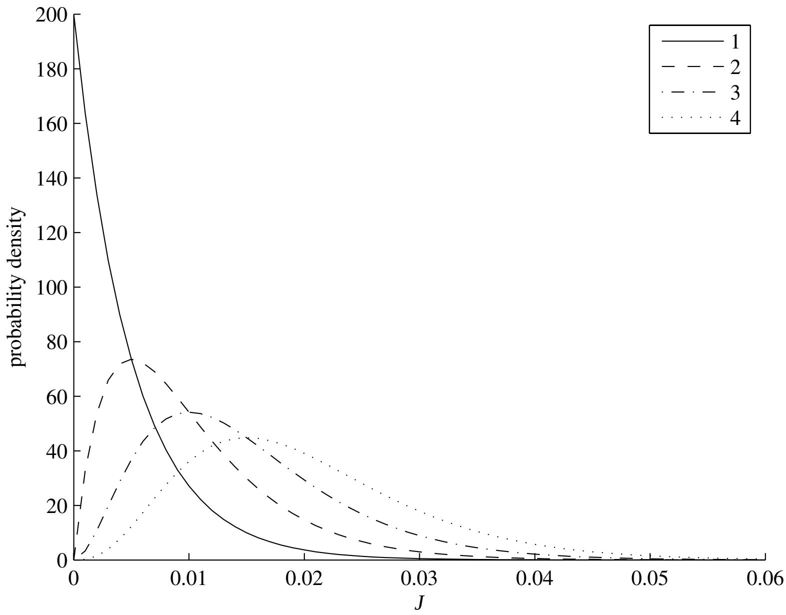


**Fig. 7.4.** The density function $p_{Ga}(J, \eta, p)$ for fixed $\eta = 0.005$ and varying $p$ of a gamma distributed random variable.

Introducing the substitution $m = J\left(\frac{1}{\eta_+} - b^{(m)}(z,\tau)\right)$, we arrive at the simpler representation[168]

$$\mathbb{E}_J\left[e^{b^{(m)}(z,\tau)J} - 1\right] = \frac{1}{\Gamma(p)\left(\frac{1}{\eta_+} - b^{(m)}(z,\tau)\right)^p \eta_+^p}\left(\underbrace{\int_0^\infty e^{-m} m^{p-1}\,\mathrm{d}m}_{\Gamma(p)}\right) - 1$$

$$= \frac{1}{\left(1 - b^{(m)}(z,\tau)\eta_+\right)^p} - 1.$$

Hence, the corresponding expression for a negatively sized jump is

$$\mathbb{E}_J\left[e^{-b^{(m)}(z,\tau)J} - 1\right] = \frac{1}{\left(1 + b^{(m)}(z,\tau)\eta_-\right)^p} - 1.$$

168 To ensure the existence of the jump transform, we have the same inequalities for $\mathrm{Re}\left[b^{(m)}(z,\tau)\right]$ to be satisfied as in the case of exponentially distributed jumps.

Having derived the relevant expectations, we immediately are able to formulate the respective jump transforms $a^1_{Ga^\pm}(z,\tau)$. Thus, integrating over the time axis and multiplying the result by the relevant jump intensity $\lambda^{\mathbb{Q}^{(n)}}$ we obtain,

$$a^1_{Ga^\pm}(z,\tau) = \lambda^{\mathbb{Q}^{(n)}} \left( -\tau + \int\limits_0^\tau \frac{1}{(1 \mp b^{(m)}(z,l)\eta_\pm)^p} \, \mathrm{d}l \right) . \tag{7.7}$$

Since it is not possible to derive closed-form solutions for general values of $p$, we apply a Runge-Kutta algorithm implemented in our FRFT procedure in order to efficiently calculate option prices. Again, for a strictly positive interest-rate model, we use only a positively directed version of the jump candidate.

# 8

# Jump-Enhanced One-Factor Interest-Rate Models

## 8.1 Overview

In order to implement the previously proposed pricing procedure, we need in addition to the payoff transformations derived in Chapter 5, the particular characteristic functions. The goal of this chapter is to provide these necessary functions for the case of an underlying one-factor interest-rate model and to examine the behavior of the particular density functions and prices of selected contingent claims, according to Table 4.1, influenced by jump components. Thus, we focus our efforts exclusively on the exponential-affine term-structure models generated by the one-factor version of equation (2.23). Since a one-factor model implicates the incorporation of one Brownian motion, this statement does not entail the restriction of including one sole jump component. Therefore, we apply different jump components in our examples. The general version of the one-factor instantaneous interest rate is then given by

$$r_t = w_0 + w_1 x_t,$$

and the factor $x_t$ is defined by the one-dimensional stochastic differential equation

$$\mathrm{d}x_t = \mu^{\mathbb{Q}}(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}W_t^{\mathbb{Q}} + \mathbf{j}\,\mathrm{d}\mathbf{N}\left(\boldsymbol{\lambda}^{\mathbb{Q}}t\right), \tag{8.1}$$

where $\mathbf{j} \in \mathbb{R}^N$, $\mu^{\mathbb{Q}}(x_t)$ and $\sigma(x_t)$ are the one-factor counterparts of the original parameters $\mathbf{J}$, $\boldsymbol{\mu}^{\mathbb{Q}}(\mathbf{x}_t)$ and $\boldsymbol{\Sigma}(\mathbf{x}_t)$ used in equation (2.23). All parameters are postulated under the risk-neutral probability measure $\mathbb{Q}$. Therefore, the solution of the general characteristic function $\psi(x_t, z, w_0, w_1, g_0, g_1, \tau)$ for these models is given by the simplified versions of the ODEs (2.40) and (2.41), which are

$$a^0(z,\tau)_\tau = \mu_0^{\mathbb{Q}} b(z,\tau) + \frac{\sigma_0^2}{2} b(z,\tau)^2 - w_0, \tag{8.2}$$

$$a^1(z,\tau)_\tau = \mathbb{E}_{\mathbf{j}}\left[\left(e^{b(z,\tau)J_1}, e^{b(z,\tau)J_2}, \ldots, e^{b(z,\tau)J_N}\right) - 1\right] \boldsymbol{\lambda}^{\mathbb{Q}}, \tag{8.3}$$

and

$$b(z,\tau)_\tau = \mu_1^{\mathbb{Q}} b(z,\tau) + \frac{\sigma_1^2}{2} b(z,\tau)^2 - w_1, \tag{8.4}$$

with terminal conditions $a(z,0) = 0$ and $b(z,0) = \imath z g_1$.

In the upcoming sections, we discuss jump-enhanced short-rate models where the diffusion part is either modeled as a Ornstein-Uhlenbeck or a Square-Root process and the jump components are governed by the distributions presented in the previous chapter.

## 8.2 The Ornstein-Uhlenbeck Model

### 8.2.1 Derivation of the Characteristic Function

Modeling the factor $x_t$ as a stochastic process according to Ornstein and Uhlenbeck (1930) exhibits a strong resemblance to the well-known model given in Vasicek (1977)[169]. The so-called Vasicek model has become very popular in interest-rate modeling. The instantaneous interest rate is modeled as an Ornstein-Uhlenbeck process, with a mean-reverting component and a Brownian motion, which yields a time-homogenous Markov process. The approach used in Vasicek (1977) to derive prices for contingent claims under the risk-neutral probability measure, is similar to the methodology used in the article of Black and Scholes (1973), based on a hedging argument[170]. Due to its popularity, many authors have made attempts to extend this diffusion model with jumps. Das and Foresi (1996) introduced a jump-enhanced Vasicek model, where the jump size is governed by an exponential distribution and the jump direction is modeled as a Bernoulli random variable which results in a double-sided jump component. However, given a jump intensity $\lambda$ for

[169] The process used in Vasicek (1977) and the process discussed in this section coincide for the case of $r_t = x_t$, thus setting the discount parameters to $w_0 = 0$ and $w_1 = 1$.

[170] In contrast to the Black-Scholes model, an appropriate market price of risk has to be additionally considered, since the short rate $r_t$ is no traded quantity. See Section 2.3.

the Poisson process, this model can be easily subsumed by applying a single exponentially distributed jump size and setting the modified intensity for the upward jump trigger to $\psi\lambda$ and the downward jump intensity to $(1-\psi)\lambda$[171], respectively. Another model, where the Vasicek model is extended with a normally distributed jump size, is given in Baz and Das (1996), Das (2002), Durham (2005). In Baz and Das (1996) and Durham (2005), approximation techniques are presented for pricing option contracts under these interest-rate dynamics, as explained in Section 7.3. In Das (2002), the author utilizes this jump-diffusion model for the estimation of the term structure and subsequent calibration of the particular parameters according to Fed Funds data.

The the risk-neutral coefficients are

$$\mu_0^{\mathbb{Q}} = \kappa\theta, \quad \mu_1^{\mathbb{Q}} = -\kappa, \quad \sigma_0 = \sigma, \quad \sigma_1 = 0,$$

where the mean-reverting feature of the instantaneous interest rate $r_t$ is guaranteed for $\kappa > 0$. Thus, the diffusion part of the SDE (8.1) is

$$\mathrm{d}x_t = \kappa(\theta - x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}. \tag{8.5}$$

Under these dynamics the stochastic process starting with $x_t$ the Ornstein-Uhlenbeck process reflects a normal distribution with expectation

$$\mathbb{E}^{\mathbb{Q}}[x_T] = x_t e^{-\kappa\tau} + \theta(1 - e^{-\kappa\tau}),$$

and variance

$$\mathbb{VAR}^{\mathbb{Q}}[x_T] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right).$$

Modeling the term structure with this Ornstein-Uhlenbeck type process, has the attractive feature that solutions for many important contingent claims can be derived within closed-form formulae. Moreover, the model is likely to be used for its high tractability. Finally, one major drawback of the model is the ability to produce negative short rates with a positive probability.

According to equations (8.2) and (8.4), straightforward calculations show that the diffusion-related coefficients of the general characteristic function can

[171] In Das and Foresi (1996), the parameter $\psi$ denotes the probability that the sign of the jump is positive.

be derived as[172]

$$\tilde{b}(z,\tau) = \left(\frac{w_1}{\kappa} + \imath z g_1\right)\left(e^{-\kappa\tau} - 1\right), \tag{8.6}$$

and

$$\begin{aligned} a^0(z,\tau) = &- w_0\tau - \frac{\imath z g_1 \sigma^2}{2\kappa}\,\tilde{b}(z,\tau) \\ &- \left(\theta - \frac{w_1\sigma^2}{2\kappa^2}\right)\left(\tilde{b}(z,\tau) + w_1\tau\right) - \frac{\sigma^2}{4\kappa}\,\tilde{b}(z,\tau)^2. \end{aligned} \tag{8.7}$$

Equipped with these time-dependent coefficient functions corresponding to the diffusion parts of the short-rate model, we must determine in the next step the particular jump part $a^1(z,\tau)$. Since this function is independent of $a^0(z,\tau)$, we are able to derive it separately.

Unfortunately, a closed-form solution for the coefficient $a^1(z,\tau)$ exists only in case of an exponentially distributed jump size. Thus, according to equation (7.4) we obtain for the Ornstein-Uhlenbeck model, where the $n$th jump in $x_t$ is governed by an exponentially distributed jump size, the relevant coefficient function as

$$a^1_{Ex^\pm} = -\lambda^{\mathbb{Q}^{(n)}}\tau + \frac{\lambda^{\mathbb{Q}^{(n)}}}{\kappa \pm w_1\eta_\pm}\ln\left[\frac{1 \mp b(z,\tau)\eta_\pm}{(1 \mp \imath z g_1 \eta_\pm)\, e^{-\kappa\tau}}\right]. \tag{8.8}$$

In equation (8.8), the signs in the index of $a^1_{Ex^\pm}$ denotes an upward and a downward jump, respectively. Considering normally and/or gamma distributed jumps, we have to apply a Runge-Kutta algorithm to solve equations (7.6) and (7.7).

### 8.2.2 Numerical Results

Next, we want to examine and demonstrate the impact of the particular jump specifications for the case of a jump-enhanced Ornstein-Uhlenbeck process. Thus, we first compare the probability density for different jump amplitude specifications, and afterwards look briefly at values of option prices for interest-rate derivatives corresponding to the payoff structures given in Table 4.1.

[172] Here, the coefficient $\tilde{b}(z,\tau)$ denotes the scalar version of $\tilde{\mathbf{b}}(z,\tau)$ and therefore complies with the relation $b(z,\tau) = \tilde{b}(z,\tau) + \imath z g_1$.

Figures 8.1 - 8.3 depict probability density functions of short rates under different jump regimes with diffusion parameters $r_t = 0.05, \kappa = 0.4, \theta = 0.05, \sigma = 0.01$ and $T = 1$. In each figure, we focus exclusively on one particular jump candidate, while ignoring other jump specifications. The probability density functions are then examined for different arrival rates and jump amplitudes[173], respectively. Additionally, in case of a normally distributed jump component, we also examine the influence of the jump amplitude volatility, whereas in case of a gamma distributed jump distribution, the impact for different values of $p$ is displayed.

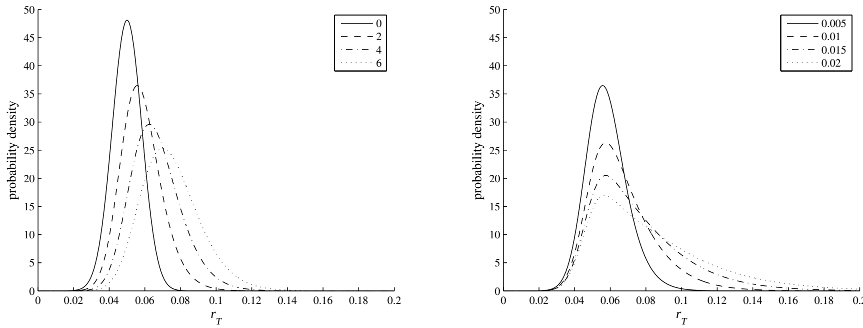


**Fig. 8.1.** Probability densities for a short rate governed by a Vasicek diffusion model enhanced with an exponentially distributed jump component. In the left (right) graph the density function for varying jump intensities (means) are depicted. The base parameters are: $r_t = 0.05, \kappa = 0.4, \theta = 0.05, \sigma = 0.01, \lambda = 2, \eta = 0.005, T = 1$.

The first impression from Figures 8.1 - 8.3, is that increased jump intensity results in all three cases in a positively skewed density function with a slightly right-shifted mode[174]. The asymmetric shape is in line with empirical findings[175]. Increasing the mean of the jump amplitude, the density functions

[173] In case of exponentially and gamma distributed jumps, the arrival rates belong only to positively directed jumps, thus leaving downward jumps with zero jump intensities.

[174] This effect becomes more apparent for higher values of jump amplitudes $\eta$ and $\mu_J$, respectively.

[175] See Arapis and Gao (2006), Figure 3. The authors apply alternatively a nonparametric estimator for the short-rate probability density of three-month Treasury bill rates and seven-day Eurodollar deposit rates.

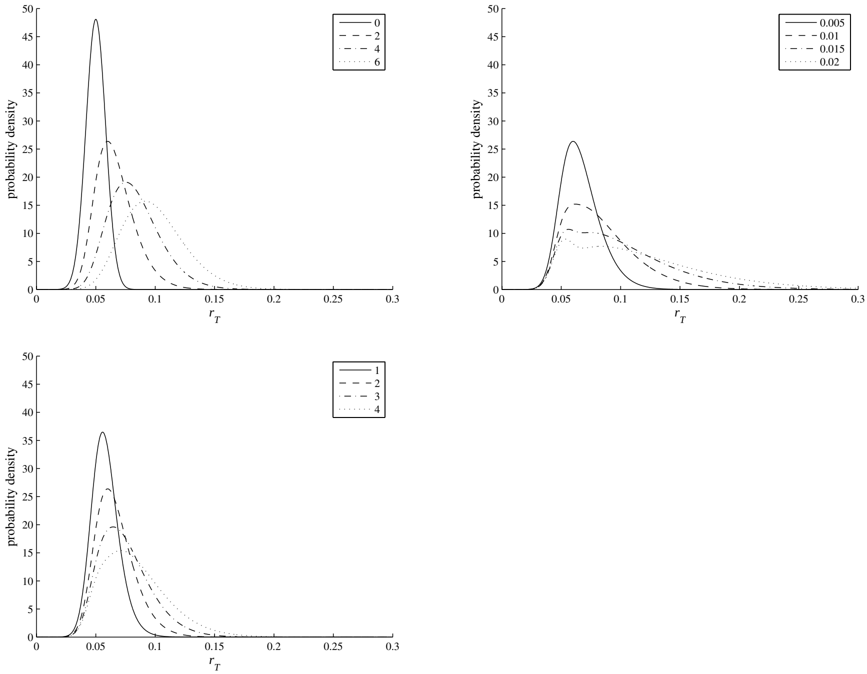


**Fig. 8.2.** Probability densities for a short rate governed by a Vasicek diffusion model enhanced with a gamma distributed jump component. In the upper left (right) graph the density functions for varying jump intensities (means) are depicted. The lower graph shows the density behavior for alternating values of $p$. The base parameters are: $r_t = 0.05, \kappa = 0.4, \theta = 0.05, \sigma = 0.01, \lambda = 2, \eta = 0.005, p = 2, T = 1$.

of all jump candidates show positive skewness while concurrently maintaining the mode of the density function. Comparing the particular density functions of an exponentially and gamma distributed jump-enhanced short-rate model, we encounter, in case of a gamma and normal distribution, a bi-modal density function. The impact of the volatility parameter $\sigma_J$ for a normally distributed jump is more complex. For high values of the jump volatility, the density function displays a leptokurtic behavior compared. In addition, we observe, due to the possibility of negative jump sizes, raised tails on both sides of the particular density function as well. This effect is rather visible to the right tail, since we have a positive mean of the jump-size distribution. Due to the possibility to produce negative short rates in the Ornstein-Uhlenbeck case, we have the undesirable ability to obtain a density function with non-negligible

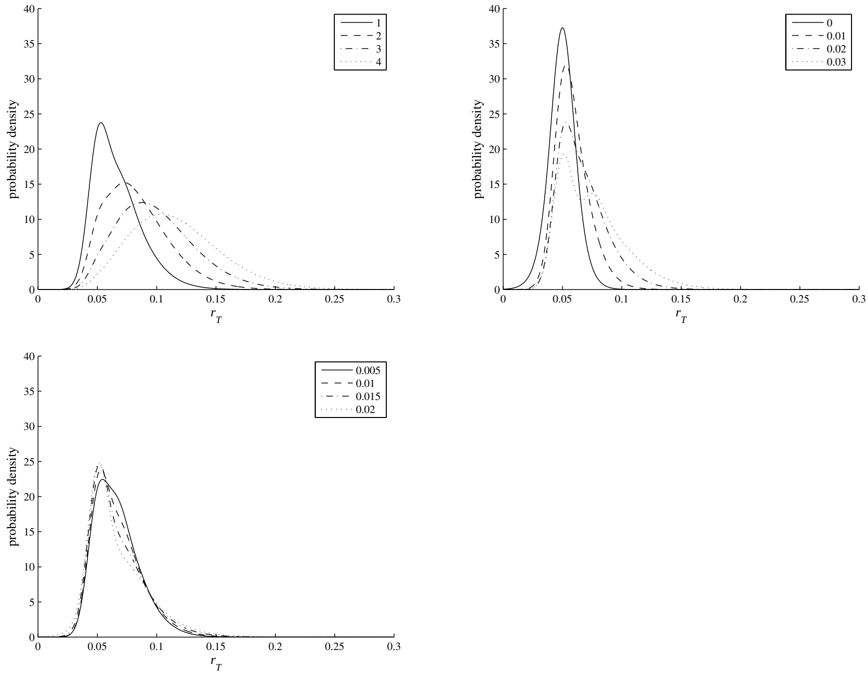


**Fig. 8.3.** Probability densities for a short rate governed by a Vasicek diffusion model enhanced with a normally distributed jump component. In the upper left (right) graph the density functions for varying jump intensities (means) are depicted. The lower graph shows the density behavior for alternating values of $\sigma_J$. The base parameters are: $r_t = 0.05, \kappa = 0.4, \theta = 0.05, \sigma = 0.01, \lambda = 1, \mu_J = 0.02, \sigma_J = 0.01, T = 1$.

probabilities for negative rates, which becomes more severe depending on the absolute height of the volatility. Obviously, besides the asymmetric shape, all density functions are skewed, in contrast to the plain Ornstein-Uhlenbeck model, which is another advantage in including jump components.

Theoretical prices of interest-rate derivatives are computed with the following base parameters: $r_t = 0.05$, $\kappa = 0.4$, $\theta = 0.05$, $\sigma = 0.01$, $\lambda_\Gamma = 2$, $\eta = 0.005$, $p = 2$, $\lambda_N = 2$, $\mu_J = 0.015$, $\sigma_J = 0.01$ and $\tau = 0.5$. Table 8.1 reports values of zero-bond calls, according to equation (3.12), for a strike range from 60 to 90 units computed with the FRFT pricing algorithm. We choose this particular strike range to cover either ITM, ATM and OTM option prices[176]. Here, we only considered normally and gamma distributed jumps, thus excluding exponentially distributed jumps, because of the similarity to the gamma jump specification. Solutions are given for different jump intensities, amplitudes, and volatilities in case of normally distributed jumps and for different values of $p$ in case of the gamma jump size specification, respectively. Examining values of zero-bond calls for different values of the parameters $\eta$, $\lambda_\Gamma$ and $p$ one-by-one, we observe that the jump intensity has the greatest influence upon call values followed by the parameter $p$. The jump mean has the smallest effect upon option prices although it significantly alters the particular density function of the short rate. However, price differences for ITM options are relatively diminutive, whereas for OTM options the above mentioned impact is quite considerable. Applying a normally distributed jump component in the short-rate model, we observe for increased jump volatilities higher option prices, which can be explained based on the above mentioned two-sided enlargement of the probability density function compared to the cases where the jump mean or jump intensity is increased. Theoretical prices for cap contracts and average-rate caps on the short rate are presented in Tables 8.2 and 8.3, respectively, for a strike range from 2 to 8 units. The cap contracts have both only one payment date, which is paid at the maturity of the contract. Here, we observe the opposite effect due to the direct influence of the short rate on the payoff function. Since the contract is based on $r_T$, positively directed jumps increase the contract value. Remarkably, the effect of the jump volatility $\sigma_J$ is twofold. Firstly, it lowers the value of the option contract for ITM options. On the other hand cap values are being raised, if the option contract is OTM. Obviously, the geometric average is less sensitive to discontinuous jumps. Since the interest rate is influenced by positively sized jumps, we compute higher values for the ordinary cap contract compared to the corresponding average-rate contract due to the averaging process itself. Thus, we are able to validate the statement that average-rate options are more robust to price manipulations, thus reducing risk exposures[177].

[176] The value of a zero bond with remaining time to maturity of two years priced with the base parameters is 83.768 units.

[177] Compare with the comments made on p. 41.

**Table 8.1.** Values of zero-bond call options for the jump-enhanced OU model, where the underlying zero-bond contract has a nominal value of 100 units.

| **K** | 60 | 65 | 70 | 75 | 80 | 85 | 90 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta}$ | | | | | | | |
| 0.005 | 20.595 | 15.747 | 10.899 | 6.067 | 1.666 | 0.004 | 0 |
| 0.01 | 17.277 | 12.442 | 7.631 | 3.117 | 0.273 | 0 | 0 |
| 0.015 | 14.168 | 9.383 | 4.827 | 1.246 | 0.003 | 0 | 0 |
| 0.02 | 11.296 | 6.711 | 2.745 | 0.314 | 0 | 0 | 0 |
| $\boldsymbol{\lambda_\Gamma}$ | | | | | | | |
| 0 | 24.134 | 19.274 | 14.415 | 9.556 | 4.727 | 0.686 | 0 |
| 2 | 20.595 | 15.747 | 10.899 | 6.067 | 1.666 | 0.004 | 0 |
| 4 | 17.220 | 12.383 | 7.552 | 2.928 | 0.170 | 0 | 0 |
| 6 | 14.001 | 9.177 | 4.436 | 0.774 | 0 | 0 | 0 |
| $\boldsymbol{p}$ | | | | | | | |
| 1 | 22.338 | 17.485 | 12.631 | 7.779 | 3.054 | 0.094 | 0 |
| 2 | 20.595 | 15.747 | 10.899 | 6.067 | 1.666 | 0.004 | 0 |
| 3 | 18.902 | 14.060 | 9.219 | 4.459 | 0.738 | 0 | 0 |
| 4 | 17.258 | 12.422 | 7.601 | 3.051 | 0.253 | 0 | 0 |
| $\boldsymbol{\mu_J}$ | | | | | | | |
| 0.005 | 24.123 | 19.264 | 14.404 | 9.545 | 4.703 | 0.627 | 0 |
| 0.01 | 22.333 | 17.480 | 12.626 | 7.773 | 3.032 | 0.096 | 0 |
| 0.015 | 20.595 | 15.747 | 10.899 | 6.067 | 1.666 | 0.004 | 0 |
| 0.02 | 18.907 | 14.065 | 9.225 | 4.472 | 0.757 | 0 | 0 |
| $\boldsymbol{\lambda_N}$ | | | | | | | |
| 0 | 25.945 | 21.080 | 16.215 | 11.350 | 6.486 | 1.772 | 0 |
| 2 | 20.595 | 15.747 | 10.899 | 6.067 | 1.666 | 0.004 | 0 |
| 4 | 15.610 | 10.780 | 5.982 | 1.747 | 0.027 | 0 | 0 |
| 6 | 10.967 | 6.194 | 1.990 | 0.093 | 0 | 0 | 0 |
| $\boldsymbol{\sigma_J}$ | | | | | | | |
| 0.005 | 20.580 | 15.732 | 10.884 | 6.044 | 1.593 | 0.002 | 0 |
| 0.01 | 20.595 | 15.747 | 10.899 | 6.067 | 1.666 | 0.004 | 0 |
| 0.015 | 20.620 | 15.772 | 10.925 | 6.107 | 1.774 | 0.014 | 0 |
| 0.02 | 20.656 | 15.807 | 10.962 | 6.167 | 1.907 | 0.043 | 0 |

**Table 8.2.** Values of short-rate caps for the jump-enhanced OU model, with a nominal value of 100 units.

| **K** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta}$ | | | | | | | |
| 0.005 | 5.096 | 4.126 | 3.160 | 2.241 | 1.476 | 0.909 | 0.525 |
| 0.01 | 5.952 | 4.984 | 4.020 | 3.097 | 2.296 | 1.642 | 1.136 |
| 0.015 | 6.798 | 5.833 | 4.871 | 3.948 | 3.135 | 2.446 | 1.878 |
| 0.02 | 7.635 | 6.672 | 5.712 | 4.790 | 3.974 | 3.267 | 2.665 |
| $\boldsymbol{\lambda_\Gamma}$ | | | | | | | |
| 0 | 4.230 | 3.258 | 2.295 | 1.430 | 0.826 | 0.445 | 0.223 |
| 2 | 5.096 | 4.126 | 3.160 | 2.241 | 1.476 | 0.909 | 0.525 |
| 4 | 5.957 | 4.990 | 4.024 | 3.080 | 2.223 | 1.512 | 0.972 |
| 6 | 6.815 | 5.850 | 4.886 | 3.931 | 3.024 | 2.215 | 1.543 |
| $\boldsymbol{p}$ | | | | | | | |
| 1 | 4.664 | 3.693 | 2.727 | 1.822 | 1.115 | 0.636 | 0.338 |
| 2 | 5.096 | 4.126 | 3.160 | 2.241 | 1.476 | 0.909 | 0.525 |
| 3 | 5.526 | 4.557 | 3.592 | 2.668 | 1.872 | 1.243 | 0.782 |
| 4 | 5.954 | 4.986 | 4.022 | 3.096 | 2.284 | 1.613 | 1.092 |
| $\boldsymbol{\mu_J}$ | | | | | | | |
| 0.005 | 4.231 | 3.261 | 2.306 | 1.435 | 0.786 | 0.388 | 0.175 |
| 0.01 | 4.664 | 3.694 | 2.730 | 1.825 | 1.104 | 0.613 | 0.314 |
| 0.015 | 5.096 | 4.126 | 3.160 | 2.241 | 1.476 | 0.909 | 0.525 |
| 0.02 | 5.525 | 4.557 | 3.591 | 2.666 | 1.877 | 1.257 | 0.800 |
| $\boldsymbol{\lambda_N}$ | | | | | | | |
| 0 | 3.797 | 2.824 | 1.858 | 0.993 | 0.441 | 0.177 | 0.065 |
| 2 | 5.096 | 4.126 | 3.160 | 2.241 | 1.476 | 0.909 | 0.525 |
| 4 | 6.385 | 5.419 | 4.454 | 3.512 | 2.645 | 1.901 | 1.302 |
| 6 | 7.665 | 6.702 | 5.741 | 4.789 | 3.875 | 3.031 | 2.288 |
| $\boldsymbol{\sigma_J}$ | | | | | | | |
| 0.005 | 5.097 | 4.127 | 3.160 | 2.233 | 1.444 | 0.856 | 0.467 |
| 0.01 | 5.096 | 4.126 | 3.160 | 2.241 | 1.476 | 0.909 | 0.525 |
| 0.015 | 5.093 | 4.126 | 3.166 | 2.266 | 1.531 | 0.988 | 0.610 |
| 0.02 | 5.093 | 4.132 | 3.188 | 2.312 | 1.606 | 1.084 | 0.710 |

**Table 8.3.** Values of average-rate caps for the jump-enhanced OU model, with a nominal value of 100 units.

| **K** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta}$ | | | | | | | |
| 0.005 | 4.037 | 3.067 | 2.098 | 1.168 | 0.533 | 0.214 | 0.077 |
| 0.01 | 4.474 | 3.507 | 2.540 | 1.607 | 0.915 | 0.488 | 0.248 |
| 0.015 | 4.906 | 3.940 | 2.975 | 2.043 | 1.327 | 0.840 | 0.522 |
| 0.02 | 5.331 | 4.368 | 3.405 | 2.474 | 1.745 | 1.222 | 0.851 |
| $\boldsymbol{\lambda_\Gamma}$ | | | | | | | |
| 0 | 3.593 | 2.621 | 1.650 | 0.754 | 0.285 | 0.096 | 0.029 |
| 2 | 4.037 | 3.067 | 2.098 | 1.168 | 0.533 | 0.214 | 0.077 |
| 4 | 4.478 | 3.511 | 2.544 | 1.597 | 0.844 | 0.390 | 0.162 |
| 6 | 4.918 | 3.953 | 2.987 | 2.033 | 1.202 | 0.625 | 0.291 |
| $\boldsymbol{p}$ | | | | | | | |
| 1 | 3.816 | 2.845 | 1.874 | 0.953 | 0.386 | 0.139 | 0.045 |
| 2 | 4.037 | 3.067 | 2.098 | 1.168 | 0.533 | 0.214 | 0.077 |
| 3 | 4.257 | 3.288 | 2.320 | 1.387 | 0.709 | 0.323 | 0.134 |
| 4 | 4.476 | 3.508 | 2.541 | 1.607 | 0.902 | 0.462 | 0.219 |
| $\boldsymbol{\mu_J}$ | | | | | | | |
| 0.005 | 3.594 | 2.622 | 1.654 | 0.760 | 0.251 | 0.070 | 0.017 |
| 0.01 | 3.816 | 2.845 | 1.875 | 0.956 | 0.375 | 0.126 | 0.037 |
| 0.015 | 4.037 | 3.067 | 2.098 | 1.168 | 0.533 | 0.214 | 0.077 |
| 0.02 | 4.256 | 3.288 | 2.320 | 1.385 | 0.715 | 0.334 | 0.142 |
| $\boldsymbol{\lambda_N}$ | | | | | | | |
| 0 | 3.372 | 2.399 | 1.426 | 0.530 | 0.126 | 0.028 | 0.006 |
| 2 | 4.037 | 3.067 | 2.098 | 1.168 | 0.533 | 0.214 | 0.077 |
| 4 | 4.697 | 3.731 | 2.765 | 1.819 | 1.046 | 0.537 | 0.250 |
| 6 | 5.352 | 4.389 | 3.427 | 2.475 | 1.623 | 0.968 | 0.529 |
| $\boldsymbol{\sigma_J}$ | | | | | | | |
| 0.005 | 4.038 | 3.068 | 2.099 | 1.163 | 0.505 | 0.182 | 0.056 |
| 0.01 | 4.037 | 3.067 | 2.098 | 1.168 | 0.533 | 0.214 | 0.077 |
| 0.015 | 4.035 | 3.066 | 2.099 | 1.184 | 0.574 | 0.259 | 0.109 |
| 0.02 | 4.033 | 3.066 | 2.106 | 1.212 | 0.624 | 0.311 | 0.149 |

## 8.3 The Square-Root Model

### 8.3.1 Derivation of the Characteristic Function

Modeling the short rate as a Square-Root process was introduced in Cox, Ingersoll and Ross (1985b) to demonstrate the equilibrium approach described in Cox, Ingersoll and Ross (1985a). In contrast to the arbitrage-based approach used in Vasicek (1977), the relevant interest-rate dynamics of the CIR model was derived within an equilibrium-based approach. The main advantage in modeling the short rate as a Square-Root process lies in its nonnegativity property. Thus, interest rates governed by a Square-Root process always stay positive[178]. This ability, together with the maintained tractability, offers a very useful tool in modeling the term structure of interest rates. Ahn and Thompson (1988) extend the diffusion model with a constant jump size, which is triggered by a Poisson process[179]. Zhou (2001) uses a CIR model augmented with a uniformly distributed jump size for estimation purposes.

Similar to the Vasicek model, this short-rate process has a mean-reverting component, which is crucial in depicting the term structure faithfully. However, the coefficient governing the diffusion part has now a stochastic component governed by the factor $x_t$ itself. The the risk-neutral coefficients are

$$\mu_0^{\mathbb{Q}} = \kappa\theta, \quad \mu_1^{\mathbb{Q}} = -\kappa, \quad \sigma_0 = 0, \quad \sigma_1 = \sigma\sqrt{x_t}.$$

Thus, the diffusion part of the SDE (8.1) is

$$\mathrm{d}x_t = \kappa(\theta - x_t)\,\mathrm{d}t + \sigma\sqrt{x_t}\,\mathrm{d}W_t^{\mathbb{Q}}. \tag{8.9}$$

Modeling the short-rate process this way bears several advantages. Firstly, as mentioned above, the interest-rate model displays a stochastic volatility without incorporating an additional factor. Secondly, as long as the initial value suffices $x_t \geq 0$ together with the condition $2\kappa\theta \geq \sigma^2$, the model guarantees that the short rate never reaches the origin and therefore stays strictly positive[180]. In contrast to the normally distributed short-rate process in Vasicek (1977), the mean-reverting Square-Root process exhibits a non-central Chi-Square distribution with expectation

[178] Setting the discount parameters to $w_0 = 0$ and $w_1 = 1$, the general Square-Root model as used in this thesis and the CIR model coincides.

[179] See Ahn and Thompson (1988), p. 168.

[180] See Feller (1951), p. 173.

$$\mathbb{E}^{\mathbb{Q}}[x_T] = x_t e^{-\kappa\tau} + \theta(1 - e^{-\kappa\tau}),$$

and variance

$$\mathbb{VAR}^{\mathbb{Q}}[x_T] = x_t \frac{\sigma^2}{\kappa}\left(e^{-\kappa\tau} - e^{-2\kappa\tau}\right) + \theta\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right)^2.$$

Due to the stochastic volatility term $\sigma\sqrt{x_t}$, the derivation of the general characteristic function is more tedious, but also straightforward. In this case, the ordinary differential equation for the coefficient function $\tilde{b}(z,\tau)$ has the form of the well-known Riccati equation, for which several solution methods exist. In order to solve for $\tilde{b}(z,\tau)$, we prepare our differential equation by substituting the coefficient $b(z,\tau)$ in equation (8.4) with $\tilde{b}(z,\tau)$. This leads to the alternative representation

$$\tilde{b}(z,\tau)_\tau = -\left(w_1 + \imath z\kappa g_1 + \frac{\sigma^2 z^2 g_1^2}{2}\right) + \left(\imath z\sigma^2 g_1 - \kappa\right)\tilde{b}(z,\tau) + \frac{\sigma^2}{2}\tilde{b}(z,\tau)^2.$$

Thus, introducing the parameters

$$\begin{aligned}
c_0(z) &= -\left(w_1 + \imath z\kappa g_1 + \frac{\sigma^2 z^2 g_1^2}{2}\right),\\
c_1(z) &= \imath z\sigma^2 g_1 - \kappa,\\
c_2(z) &= \frac{\sigma^2}{2},
\end{aligned}$$

we are able to express this ODE simply as

$$\tilde{b}(z,\tau)_\tau = c_0(z) + c_1(z)\tilde{b}(z,\tau) + c_2(z)\tilde{b}(z,\tau)^2, \tag{8.10}$$

for which standardized solution techniques exist. Eventually, we obtain the functional form of the coefficient function $\tilde{b}(z,\tau)$ as[181]

$$\tilde{b}(z,\tau) = \frac{2c_0(z)\sinh\left[\frac{\vartheta(z)\tau}{2}\right]}{\epsilon(z,\tau)}, \tag{8.11}$$

with

$$\epsilon(z,\tau) = \vartheta(z)\cosh\left[\frac{\vartheta(z)\tau}{2}\right] - c_1(z)\sinh\left[\frac{\vartheta(z)\tau}{2}\right],$$

and

$$\vartheta(z) = \sqrt{c_1(z)^2 - 4c_0(z)c_2(z)}.$$

[181] The detailed derivation of the coefficient functions $\tilde{b}(z,\tau)$ and $a^0(z,\tau)$ is shown in Appendix A.

Given the coefficient function $\tilde{b}(z,\tau)$, we can proceed onward with the calculation of $a^0(z,\tau)$, which represents the antiderivative of $b(z,\tau) = \tilde{b}(z,\tau) + \imath z g_1$, scaled by some constant factor $\kappa\theta$. Applying a logarithmic integration approach, the solution is formally given by

$$a^0(z,\tau) = (\imath z\kappa\theta g_1 - w_0)\tau - \frac{\kappa\theta}{2c_2(z)}\left(\tau c_1(z) + 2\ln\left[\frac{\epsilon(z,\tau)}{\vartheta(z)}\right]\right). \tag{8.12}$$

Equipped with these two coefficient functions, we are already able to price interest-rate derivatives for ordinary diffusion specifications of the short rate without considering any jump components.

Implementing a jump component in the Square-Root model, one must be careful about the jump specifications. Due to the strict positiveness of the model, we have to limit ourselves to cases of positively sized exponentially and gamma distributed jump sizes, thus excluding the normal distribution for the jump size specifications[182]. Similar to the Ornstein-Uhlenbeck model, a closed-form formula of the general characteristic function exists only in case of an exponentially distributed jump component. Thus, calculating the jump transform for this specification, we obtain for a jump-enhanced Square-Root model, where the $n$th (positively directed) jump in $x_t$ is governed by an exponential distribution with mean $\eta$, the coefficient function

$$\begin{aligned} a^1_{Ex} = &- \lambda^{\mathbb{Q}^{(n)}}\tau + \lambda^{\mathbb{Q}^{(n)}} \times \\ &\frac{\left(c_2(z)c_3(z) + \frac{\eta c_1(z)}{2}\right)\tau - \eta\ln\left[\frac{\epsilon(z,\tau)}{\vartheta(z)}\left(1 - \frac{\eta}{c_3(z)}\tilde{b}(z,\tau)\right)\right]}{c_2(z)c_3^2(z) + \eta\left(c_0(z)\eta + c_1(z)c_3(z)\right)}, \end{aligned} \tag{8.13}$$

with

$$c_3(z) = 1 - \imath z\eta g_1.$$

For a gamma distributed jump size, we again use a Runge-Kutta solver to recover the relevant values for the coefficient function $a^1(z,\tau)$.

### 8.3.2 Numerical Results

Given the two different jump candidates, we want to demonstrate the impact on the density function as well as interest-rate derivative prices. Figures 8.4

[182] However, Ahn and Thompson (1988) implemented a constant, negatively sized jump component in a CIR short-rate model. Accordingly, they have to choose carefully the fixed jump amplitude to ensure that interest rates remain positive over the trading interval $\tau$.

and 8.5 depict density functions for short-rate models with diffusion parameters $r_t = 0.03, \kappa = 0.3, \theta = 0.03, \sigma = 0.1$ and $T = 1$. In each figure, we focus exclusively on one particular jump candidate, while ignoring other jump specifications.The probability density functions are examined for different jump intensities, means, and in case of a gamma jump size specification we also investigate the behavior of the density function for varying $p$. Thus, we have in each figure the diffusion base model exclusively combined with one jump specification. Subsequently, model prices of idealized interest-rate contracts are derived similar to the payoff functions in Table 4.1. Here, we only compute derivative prices for the gamma jump-enhanced diffusion model because the gamma distribution is able to generate the exponential distribution as a special case.

In contrast to the Vasicek model, the density function of the pure diffusion model innately shows an asymmetric shape, since the instantaneous interest rate $r_t$ features a non-central chi-square probability density function. The effect of jump components can be seen by comparing the density function of the ordinary CIR diffusion model, which is depicted in the particular (upper) left graphs of Figure 8.4 and 8.5 for $\lambda = 0$, with the behavior of the jump-enhanced density function. Particularly, empirical findings of right-skewed density functions[183] can be assembled within the jump-enhanced model. As mentioned earlier, we consider only positively sized, exponentially and gamma distributed jumps due to the positivity constraint of the Square-Root process, thus neglecting the normal distribution specification for jump candidates in the CIR model. For both jump specifications we notice a higher skewness of the density function compared to the pure diffusion case. However, the jump intensity and jump size mean parameters influence the density function differently. According to the (upper) left graphs in Figures 8.4 and 8.5, increased arrival times show the effect of distributing the probability mass over a broader range and shifting the mode of the density to the right, which is characteristic for the intensity parameter. Compared to the Vasicek model, this effect is not that pronounced, which might be due to the non-central chi-squared distribution of the short rate. On the other hand, increasing the parameter of the jump size mean results in fat tails to the right. Accordingly, the density functions display a lower kurtosis. Comparing the particular graphs for the exponential

[183] See, for example, Arapis and Gao (2006).

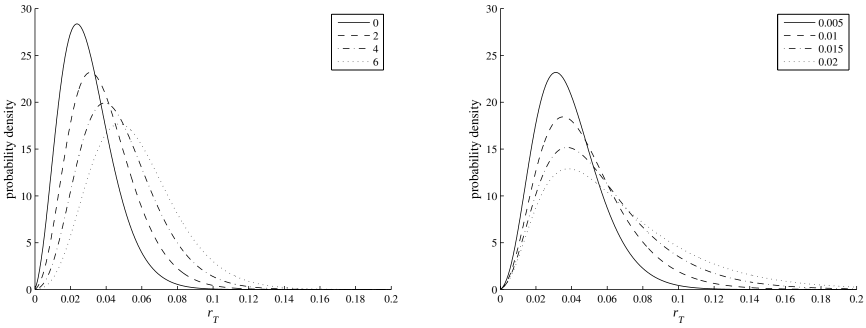


**Fig. 8.4.** Probability densities for a short rate governed by a CIR diffusion model enhanced with an exponentially distributed jump component. In the left (right) graph the density functions for varying jump intensities (means) are depicted. The base parameters are: $r_t = 0.03, \kappa = 0.3, \theta = 0.03, \sigma = 0.1, \lambda = 2, \eta = 0.005, T = 1$.

and gamma jump case and interpret Figure 8.4 as a special case of a gamma distributed jump size variable with $p = 1$, we clearly identify the multiplying effect of $p$ on the jump intensity. Especially in the upper right graph of Figure 8.5, we notice the extremely flat tail of the density function for $\eta = 0.02$ compared to the behavior of the particular graph in Figure 8.4.

Examining the effect of jump parameters on derivative prices, we assume for all contingent claims the following base parameters: $r_t = 0.03$, $\kappa = 0.3$, $\theta = 0.03$, $\sigma = 0.1$, $\lambda = 2$, $\eta = 0.005$, $p = 2$ and $\tau = 0.5$. Firstly, we have a look at Table 8.4, where values of zero-bond calls are computed according to a strike range of 60 to 90 units. The strike range is chosen in a way to include either ITM, ATM and OTM option prices[184]. As in the Vasicek framework, we observe in Table 8.4 the jump intensity $\lambda$ to have the greatest influence on zero-bond call values, followed by the jump mean parameter $\eta$ and the parameter $p$. Since varying the jump mean $\eta$ and the parameter $p$ keeps the mode of the density nearly unchanged, a smaller amount of the probability mass is moved out of the exercise region of the zero-bond call. Comparing zero-bond call prices of the particular parameter settings and strike prices, thus keeping the overall expected jump size, based on $\eta$, $\lambda_\Gamma$ and $p$, equal, we observe relatively low spreads between ITM option prices, while spreads for OTM option prices are high. Turning our attention to the cap contracts,

[184] The value of a zero bond with remaining maturity of $\hat{T} - T = 3$ is 85.525 units.

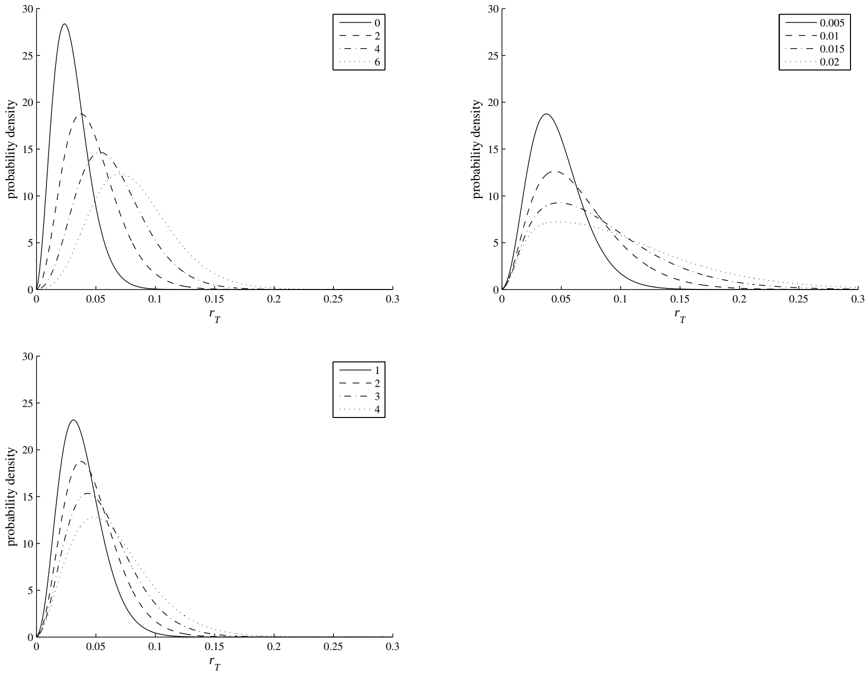


**Fig. 8.5.** Probability densities for a short rate governed by a CIR diffusion model enhanced with a gamma distributed jump component. In the upper left (right) graph the density functions for varying jump intensities (means) are depicted. The lower graph shows the density behavior for alternating values of $p$. The base parameters are: $r_t = 0.03, \kappa = 0.3, \theta = 0.03, \sigma = 0.1, \lambda = 2, \eta = 0.005, p = 2, T = 1$.

we only consider one payment date at the maturity of both the ordinary and the average-rate cap. In Table 8.5, it is first of all evident that the influence of jump parameters is reversed. Thus, the jump mean involves the greatest increase in cap prices, whereas the arrival rate results in a smaller increase in cap prices. Comparing contract values for alternating jump intensities, we have, in absence of any jump, relatively close values for ITM options of the ordinary and the average-rate cap. However, for ATM options we observe relatively large differences between both contracts. By neglecting positively sized jumps, the opposite effect could be observed for $r_t < \theta$, because of the averaging process.

**Table 8.4.** Values of zero-bond call options for the jump-enhanced SR model, where the underlying zero-bond contract has a nominal value of 100 units.

| **K** | 60 | 65 | 70 | 75 | 80 | 85 | 90 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta}$ | | | | | | | |
| 0.005 | 23.625 | 18.711 | 13.797 | 8.890 | 4.117 | 0.595 | 0 |
| 0.01 | 17.013 | 12.128 | 7.345 | 3.044 | 0.339 | 0 | 0 |
| 0.015 | 11.185 | 6.610 | 2.704 | 0.297 | 0 | 0 | 0 |
| 0.02 | 6.454 | 2.762 | 0.374 | 0 | 0 | 0 | 0 |
| $\boldsymbol{\lambda}$ | | | | | | | |
| 0 | 31.018 | 26.093 | 21.167 | 16.242 | 11.316 | 6.394 | 1.770 |
| 2 | 23.625 | 18.711 | 13.797 | 8.890 | 4.117 | 0.595 | 0 |
| 4 | 16.861 | 11.960 | 7.096 | 2.648 | 0.198 | 0 | 0 |
| 6 | 10.678 | 5.879 | 1.811 | 0.079 | 0 | 0 | 0 |
| $\boldsymbol{p}$ | | | | | | | |
| 1 | 27.225 | 22.305 | 17.385 | 12.466 | 7.551 | 2.816 | 0.121 |
| 2 | 23.625 | 18.711 | 13.797 | 8.890 | 4.117 | 0.595 | 0 |
| 3 | 20.207 | 15.300 | 10.400 | 5.603 | 1.567 | 0.023 | 0 |
| 4 | 16.963 | 12.068 | 7.252 | 2.921 | 0.310 | 0 | 0 |

**Table 8.5.** Values of short-rate caps (Panel A) and average-rate caps (Panel B), for the jump-enhanced SR model, with a nominal value of 100 units.

| **Panel A: Caps** | | | | | | | |
|---|---|---|---|---|---|---|---|
| **K** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\boldsymbol{\eta}$ | | | | | | | |
| 0.005 | 1.931 | 1.146 | 0.609 | 0.296 | 0.135 | 0.058 | 0.024 |
| 0.01 | 2.821 | 2.003 | 1.375 | 0.924 | 0.610 | 0.395 | 0.252 |
| 0.015 | 3.705 | 2.877 | 2.212 | 1.692 | 1.284 | 0.965 | 0.719 |
| 0.02 | 4.580 | 3.748 | 3.065 | 2.507 | 2.043 | 1.655 | 1.332 |
| $\boldsymbol{\lambda}$ | | | | | | | |
| 0 | 1.070 | 0.443 | 0.140 | 0.035 | 0.007 | 0.001 | 0 |
| 2 | 1.931 | 1.146 | 0.609 | 0.296 | 0.135 | 0.058 | 0.024 |
| 4 | 2.811 | 1.938 | 1.236 | 0.735 | 0.411 | 0.218 | 0.110 |
| 6 | 3.699 | 2.778 | 1.968 | 1.316 | 0.834 | 0.503 | 0.290 |
| $\boldsymbol{p}$ | | | | | | | |
| 1 | 1.490 | 0.760 | 0.323 | 0.118 | 0.039 | 0.012 | 0.003 |
| 2 | 1.931 | 1.146 | 0.609 | 0.296 | 0.135 | 0.058 | 0.024 |
| 3 | 2.376 | 1.563 | 0.959 | 0.557 | 0.308 | 0.164 | 0.085 |
| 4 | 2.821 | 1.994 | 1.347 | 0.876 | 0.550 | 0.335 | 0.199 |

| **Panel B: Average-Rate Caps** | | | | | | | |
|---|---|---|---|---|---|---|---|
| **K** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\boldsymbol{\eta}$ | | | | | | | |
| 0.005 | 1.451 | 0.618 | 0.189 | 0.050 | 0.012 | 0.003 | 0.001 |
| 0.01 | 1.907 | 1.052 | 0.528 | 0.262 | 0.129 | 0.063 | 0.030 |
| 0.015 | 2.357 | 1.495 | 0.923 | 0.578 | 0.361 | 0.224 | 0.137 |
| 0.02 | 2.801 | 1.935 | 1.338 | 0.942 | 0.662 | 0.463 | 0.322 |
| $\boldsymbol{\lambda}$ | | | | | | | |
| 0 | 0.994 | 0.261 | 0.028 | 0.001 | 0 | 0 | 0 |
| 2 | 1.451 | 0.618 | 0.189 | 0.050 | 0.012 | 0.003 | 0.001 |
| 4 | 1.909 | 1.016 | 0.428 | 0.154 | 0.050 | 0.015 | 0.004 |
| 6 | 2.367 | 1.439 | 0.730 | 0.319 | 0.125 | 0.045 | 0.015 |
| $\boldsymbol{p}$ | | | | | | | |
| 1 | 1.222 | 0.423 | 0.083 | 0.012 | 0.002 | 0 | 0 |
| 2 | 1.451 | 0.618 | 0.189 | 0.050 | 0.012 | 0.003 | 0.001 |
| 3 | 1.680 | 0.830 | 0.334 | 0.123 | 0.043 | 0.014 | 0.005 |
| 4 | 1.908 | 1.048 | 0.506 | 0.230 | 0.100 | 0.041 | 0.016 |

# 9

# Jump-Enhanced Two-Factor Interest-Rate Models

## 9.1 Overview

In this chapter, we derive the characteristic functions for one specific additive interest-rate model and one subordinated stochastic volatility interest-rate model. As in the one-factor case, we extend these pure diffusion models with additional jump components. The diffusion part of the additive model, which is discussed consists of both a factor governed by an Ornstein-Uhlenbeck process, and a factor modeled as a Square-root process. Other popular additive models are given by pure multi-factor versions of Ornstein-Uhlenbeck and Square-Root processes[185]. The additive interest-rate model was first introduced in Schöbel and Zhu (2000). Here the authors apply a Heston-like transformation methodology to price interest-rate derivatives, as demonstrated in Section 4.2. The subordinated model we choose for our analysis was presented in Fong and Vasicek (1991a). Here, both the short rate and its stochastic volatility are modeled as Square-Root processes with additional jump components. In this thesis, we extend both models to incorporate various jump components. In each model, the behavior of the particular density function and numerical values of idealized interest-rate options are examined.

[185] The interest rate is then modeled as the sum either of some Ornstein-Uhlenbeck processes defined by the SDE (8.5) or of some mean-reverting Square-Root processes according to (8.9). Modeling the short rate as an additive Square-Root model, all Brownian motions have to be uncorrelated in order to derive closed-form solutions for the general characteristic function.

## 9.2 The Additive OU-SR Model

### 9.2.1 Derivation of the Characteristic Function

Basically, additive multi-factor short-rate models consist only of either additive mean-reverting Ornstein-Uhlenbeck, or Square-Root processes. For example, an additive model for the short rate is used in Chen and Scott (1992), Longstaff and Schwartz (1992), and Chen and Scott (1995). There, the short rate is modeled as the sum of two independent Square-Root processes. A multivariate, additive Gaussian interest-rate model with correlated factors is given in e.g. Langetieg (1980). Collin-Dufresne and Goldstein (2002) also consider both additive multivariate Ornstein-Uhlenbeck and Square Root processes in pricing swaptions. In the case of Ornstein-Uhlenbeck processes, the different Brownian motions driving the particular factors can be correlated. On the other hand, if taking an additive Square-Root model, we have to impose the restriction that all Brownian motions be mutually uncorrelated. Otherwise, the separation approach is no longer valid and no closed-form solution for the general characteristic function would exist[186]. Exemplary for the set of additive model candidates we select a term-structure model where the short-rate process consists of two factors. The first factor $x_t^{OU}$ is modeled as an Ornstein-Uhlenbeck process, according to equation (8.5), whereas the second factor $x_t^{SR}$ is governed by a Square-Root process, as given in equation (8.9)[187]. Since we want to extend the model setup, we allow for both factors to include jump components subject to possible non-negativity constraints. Subsequently, the short rate is built as the weighted sum of those factors with a scaling factor $w \in [0, 1]$, which gives

$$r(\mathbf{x}_t) = w x_t^{OU} + (1 - w) x_t^{SR}. \tag{9.1}$$

Therefore, the coefficients characterizing the short rate are $\mathbf{w} = (w, 1 - w)'$ and $w_0 = 0$. Accordingly, we use a slightly modified version of the model setup introduced in Schöbel and Zhu (2000).

[186] In this case, even a Runge-Kutta solver cannot be applied to the valuation problem, due to the missing system of ODEs.

[187] We assume the parameters for the particular processes to be $\kappa^i$, $\theta^i$, $\sigma^i$ with $i \in \{OU, SR\}$. Furthermore, we use the payoff-characterizing coefficients $g_0^i$ and $g_1^i$.

Setting $w = 1$ we obtain the Vasicek model and for $w = 0$ we obtain the CIR model according to Section 8.3. Although the factors are linearly combined within the short rate, all derivative functions, e.g. the probability density function, are not just simple linear combinations of their particular one-factor counterparts, which is illustrated in Figure 9.1. Thus, the additive process allows more flexibility in modeling the term structure of interest rates compared to the one-factor models discussed in Chapter 8, while maintaining the simple structure of coefficients used in the general characteristic function.

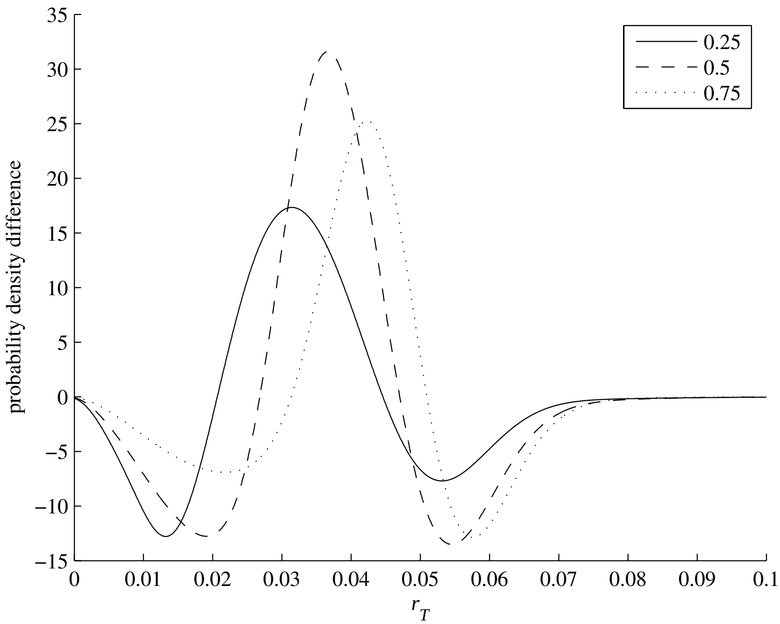


**Fig. 9.1.** Differences of the pure diffusion OU-SR model density function and the sum of the particular one-factor pendants for different weighting factors. The parameters used are: $\mathbf{x}_t = (0.05, 0.03)', \boldsymbol{\kappa} = (0.4, 0.3)', \boldsymbol{\theta} = (0.05, 0.03)', \boldsymbol{\sigma} = (0.01, 0.1)', T = 1$. In case of the one-factor models the first (last) elements correspond to the Vasicek (CIR) model.

Due to the independence of the two Brownian motions, the particular time-dependent coefficients exhibit the same formal structure as the ones derived in the one-factor Vasicek and CIR interest-rate models. Thus, the general characteristic function of this additive interest-rate model has the time-dependent vector function

$$\tilde{\mathbf{b}}(z,\tau) = \begin{pmatrix} \tilde{b}_{OU}(z,\tau) \\ \tilde{b}_{SR}(z,\tau) \end{pmatrix},$$

with $\tilde{b}_{OU}(z,\tau)$ and $\tilde{b}_{SR}(z,\tau)$ given by their one-factor representations in equation (8.6) and (8.11) with adapted parameters. Consequently, we obtain for the coefficient function $a(z,\tau)$ the relation

$$a^0(z,\tau) = a^0_{OU}(z,\tau) + a^0_{SR}(z,\tau),$$

where $a^0_{OU}(z,\tau)$ and $a^0_{SR}(z,\tau)$ correspond to equation (8.7) and (8.12). The jumps contained in the vectors $\mathbf{j}_{x^{OU}}$ and $\mathbf{j}_{x^{SR}}$ are both triggered by the same Poisson vector process $\mathbf{N}(\boldsymbol{\lambda}^{\mathrm{Q}})$[188].Due to the independence of the two factors governing the short rate, we are also able to adapt the jump transforms of the particular one-factor models without altering their formal structure.

### 9.2.2 Numerical Results

In this section, we show the behavior of the density function and compute values for some common interest-rate options under the additive jump-diffusion model. As base parameters for both the density function and the interest-rate contracts, we use the particular parameters according to their one-factor counterparts. The default value of the scaling parameter $w$ is set to $\frac{1}{2}$. The impact of jumps on the short-rate density is demonstrated in Figures 9.2 and 9.3. As before, we focus exclusively on one particular jump candidate, while ignoring other jump specifications. Thus, the graphs in the first row and the left graph in the second row in Figure 9.2, respectively, display only the impact of gamma jump component of the Ornstein-Uhlenbeck process. The other three graphs in this figure depict the influence of the normal jump component on the short-rate density. Consequently, in Figure 9.3, we only consider the gamma jump component of the Square-Root process. Option prices for varying parameters of normally and gamma distributed jump amplitudes are given in Tables 9.1 - 9.6. Again, we focus on the sensitivity of option prices to jump parameters and neglect the exponentially distributed jump size since the exponential distribution is a special case of the gamma distribution.

[188] However, setting elements in the jump vectors $\mathbf{j}_{x^{OU}}$ and $\mathbf{j}_{x^{SR}}$ to zero, it is possible to assign jump components to particular processes.

Comparing the figures of the densities functions depicted in 8.2, 8.3 and 8.5 with the corresponding graphs in Figures 9.2 and 9.3, we obviously notice more skewness in the densities of the two-factor model compared to the densities in the Vasicek model and a more leptokurtic behavior of the densities in the additive model compared to the ones in a CIR model. Firstly, taking a look at the influence of the gamma distributed jump size specification, we determine a similar influence of the gamma jump component belonging to the Vasicek and CIR model. However, the Vasicek part has a significantly weaker effect on the probability density function compared to the relevant one-factor model. On the other hand, the gamma distributed jump size component of the Square-Root process has a considerable impact on the probability density, which can be justified by the similar shape of the probability density functions in Figures 8.5 and 9.3. Examining the impact of a normally distributed jump component in this model, we observe only small changes in contrast to the one-factor equivalent Vasicek model. Thus, in the multi-factor setup we no longer encounter the characteristic strong curvature in the probability density function and the bimodal distribution displayed in the upper right graph of Figure 8.3. However, the effect of an increased volatility of the normally distributed jump size, which raises both tails of the probability density function, remains immanent.

Computing numerical values of interest-rate derivatives in this model, we assume for all contingent claims the following base diffusion parameters:

$$\mathbf{x}_t = \begin{pmatrix} 0.05 \\ 0.03 \end{pmatrix}, \quad \boldsymbol{\kappa} = \begin{pmatrix} 0.4 \\ 0.3 \end{pmatrix}, \quad \boldsymbol{\theta} = \begin{pmatrix} 0.05 \\ 0.03 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\sigma} = \begin{pmatrix} 0.01 \\ 0.1 \end{pmatrix}.$$

In each vector, the first element corresponds to the Ornstein-Uhlenbeck part, whereas the second element states the parameter value for the Square-Root component in this particular additive interest-rate model. The default jump parameters used for the valuation are in case of a gamma distributed jump size candidate $\lambda_{\Gamma}^{i} = 2$, $\eta^{i} = 0.005$ and $p^{i} = 2$ with $i \in \{OU, SR\}$. The normally distributed jump component of the Ornstein-Uhlenbeck process is governed by the parameters $\lambda_N^{OU} = 2$, $\mu_J^{OU} = 0.015$ and $\sigma_J^{OU} = 0.01$. All contracts have a remaining time to maturity of a half year. Let us discuss first Tables 9.1 and 9.2, where numerical values of zero-bond calls are reported according to equation (3.12) for a strike range from 60 to 90 units. The strike

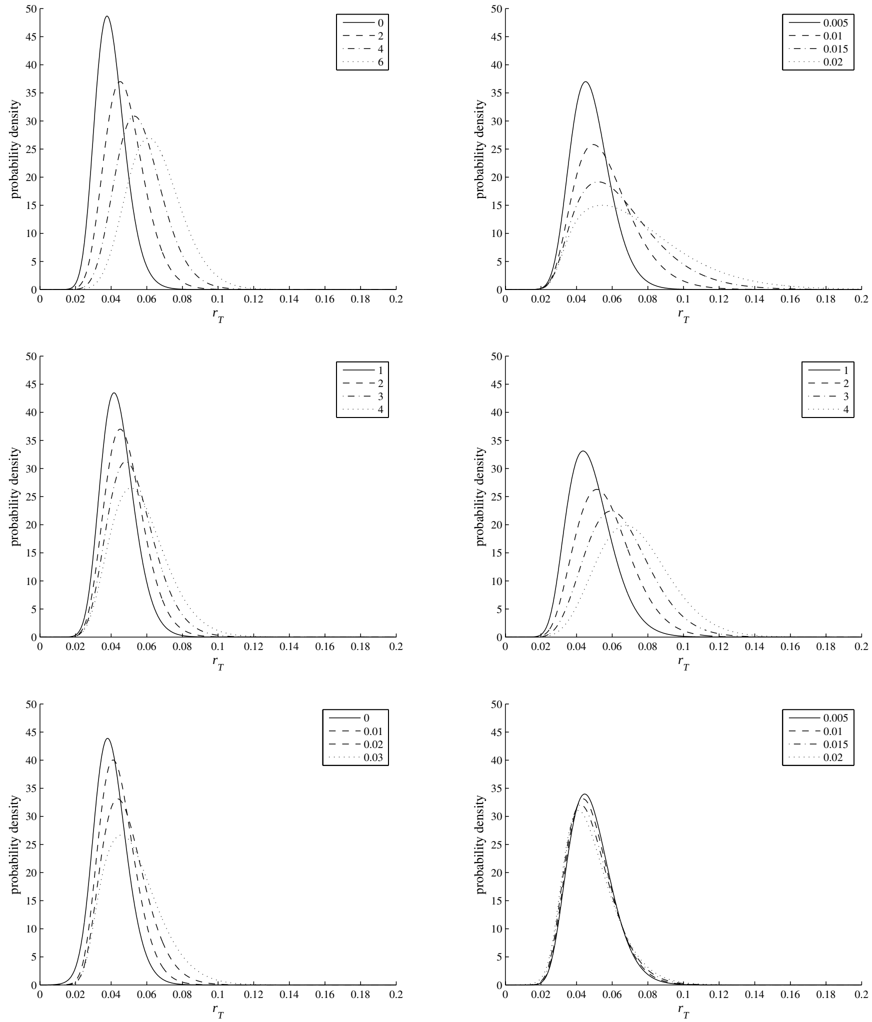


**Fig. 9.2.** Probability densities for a short rate governed by an OU-SR diffusion model enhanced with either a gamma or normally distributed jump component for the OU process. In the upper left (right) graph density functions for varying jump intensities (means) of the gamma distributed jump component are depicted. The graphs in the second row show the density behavior for alternating values of $p^{OU}$ and the jump intensity $\lambda_N^{OU}$ of the normally distributed jump component. In the last row, the left (right) graph shows density functions for different values of jump mean (volatility) of the normally distributed jump component. The base parameters are: $\mathbf{x}_t = (0.05, 0.03)', \boldsymbol{\kappa} = (0.4, 0.3)', \boldsymbol{\theta} = (0.05, 0.03)', \boldsymbol{\sigma} = (0.01, 0.1)', \lambda_\Gamma^{OU} = 2, \eta^{OU} = 0.005, p^{OU} = 2, \lambda_N^{OU} = 1, \mu_J^{OU} = 0.02, \sigma_J^{OU} = 0.01, T = 1$.

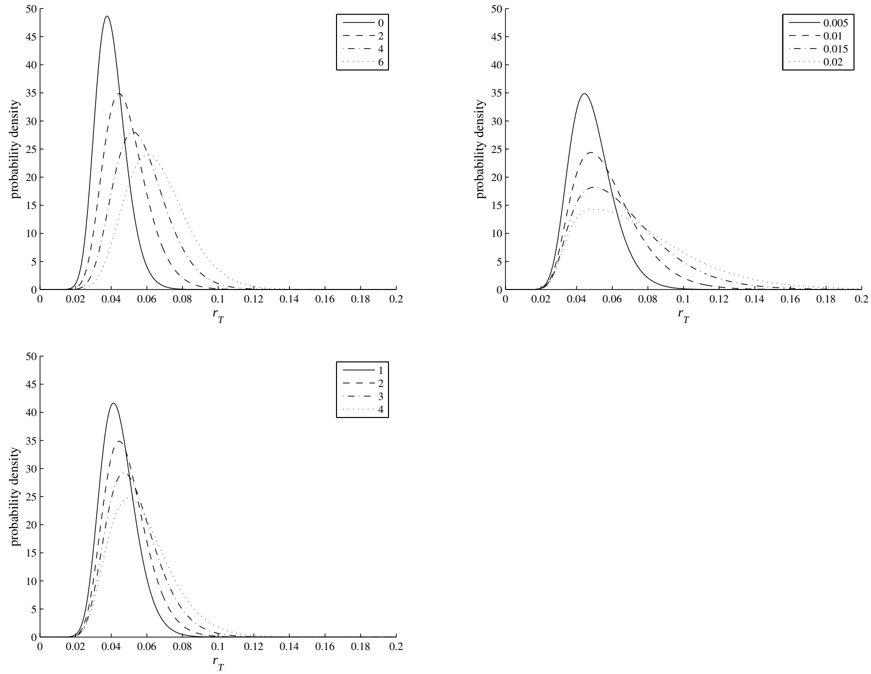


**Fig. 9.3.** Probability densities for a short rate governed by an OU-SR diffusion model enhanced with a gamma distributed jump component for the SR process. In the upper left (right) graph the density functions for varying jump intensities (means) are depicted. The lower graph shows the density behavior for alternating values of $p$. The base parameters are: $\mathbf{x}_t = (0.05, 0.03)', \boldsymbol{\kappa} = (0.4, 0.3)', \boldsymbol{\theta} = (0.05, 0.03)', \boldsymbol{\sigma} = (0.01, 0.1)', \lambda_\Gamma^{SR} = 2, \eta^{SR} = 0.005, p^{SR} = 2, T = 1$.

range is chosen to include either ITM, ATM and OTM option prices[189]. As encountered in the one-factor framework, we observe in case of the gamma jump-size distribution that the intensity $\lambda_\Gamma^i$ has the greatest influence on zero-bond call values, followed by the parameter $p^i$ and the jump mean $\eta^i$. This can be explained by the shifting of the density function to the right for increasing arrival rates. Contrary, varying jump mean $\eta^i$ and parameter $p^i$ keeps the mode of the density nearly unchanged, so that a smaller amount of the probability mass is moved out of the zero-bond call exercise region. Varying the parameters of the normally distributed jump component, we also

[189] The value of a zero bond with remaining maturity of two years, priced with base parameters, is 87.359 units.

observe a very similar behavior of option prices in comparison to the one-factor model. Comparing relative zero-bond call price differences of particular parameter settings and strike rates, thus keeping the overall expected jump size equal, we observe low spreads between ITM option prices, whereas spreads for OTM option prices are relative high. For the cap contracts, we only allow one payment date, which is at maturity. At first, we observe the same positive effect of the jump components as encountered in the one-factor pendants. Thus, the jump mean involves the greatest increase of cap prices, whereas the arrival rate results in a smaller increase of cap prices. However, due to the scaling factor $w$, the effect of different jump components is not as strong as we encountered in the particular one-factor models. Comparing the effect of a gamma distributed jump on the average-rate cap, we compute nearly the same contract values, whether we have the jumps in the Ornstein-Uhlenbeck or in the Square-Root part of the model.

**Table 9.1.** Values of zero-bond call options for the jump-enhanced OU-SR model, where the underlying zero-bond contract has a nominal value of 100 units.

| **K** | 60 | 65 | 70 | 75 | 80 | 85 | 90 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta^{OU}}$ | | | | | | | |
| 0.005 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 0.01 | 23.022 | 18.147 | 13.271 | 8.396 | 3.573 | 0.201 | 0 |
| 0.015 | 21.275 | 16.406 | 11.537 | 6.680 | 2.137 | 0.021 | 0 |
| 0.02 | 19.584 | 14.720 | 9.862 | 5.092 | 1.125 | 0 | 0 |
| $\boldsymbol{\lambda_{\Gamma}^{OU}}$ | | | | | | | |
| 0 | 26.688 | 21.801 | 16.915 | 12.028 | 7.142 | 2.326 | 0.003 |
| 2 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 4 | 23.006 | 18.131 | 13.256 | 8.380 | 3.531 | 0.152 | 0 |
| 6 | 21.230 | 16.360 | 11.491 | 6.622 | 1.926 | 0.006 | 0 |
| $\boldsymbol{p^{OU}}$ | | | | | | | |
| 1 | 25.750 | 20.866 | 15.982 | 11.098 | 6.215 | 1.521 | 0 |
| 2 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 3 | 23.914 | 19.036 | 14.158 | 9.280 | 4.412 | 0.438 | 0 |
| 4 | 23.017 | 18.141 | 13.266 | 8.391 | 3.556 | 0.190 | 0 |
| $\boldsymbol{\mu_{J}^{OU}}$ | | | | | | | |
| 0.005 | 26.685 | 21.799 | 16.912 | 12.025 | 7.139 | 2.312 | 0.005 |
| 0.01 | 25.748 | 20.865 | 15.981 | 11.097 | 6.214 | 1.511 | 0 |
| 0.015 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 0.02 | 23.916 | 19.037 | 14.159 | 9.281 | 4.415 | 0.445 | 0 |
| $\boldsymbol{\lambda_{N}^{OU}}$ | | | | | | | |
| 0 | 27.631 | 22.741 | 17.852 | 12.962 | 8.073 | 3.197 | 0.029 |
| 2 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 4 | 22.118 | 17.245 | 12.373 | 7.501 | 2.711 | 0.048 | 0 |
| 6 | 19.505 | 14.641 | 9.777 | 4.924 | 0.792 | 0 | 0 |
| $\boldsymbol{\sigma_{J}^{OU}}$ | | | | | | | |
| 0.005 | 24.821 | 19.940 | 15.059 | 10.178 | 5.299 | 0.847 | 0 |
| 0.01 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 0.015 | 24.832 | 19.951 | 15.070 | 10.189 | 5.312 | 0.918 | 0 |
| 0.02 | 24.841 | 19.960 | 15.080 | 10.199 | 5.325 | 0.973 | 0.001 |

**Table 9.2.** Values of zero-bond call options for the jump-enhanced OU-SR model, where the underlying zero-bond contract has a nominal value of 100 units.

| **K** | 60 | 65 | 70 | 75 | 80 | 85 | 90 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta^{SR}}$ | | | | | | | |
| 0.005 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 0.01 | 22.891 | 18.016 | 13.141 | 8.266 | 3.466 | 0.183 | 0 |
| 0.015 | 21.022 | 16.153 | 11.284 | 6.442 | 1.997 | 0.014 | 0 |
| 0.02 | 19.217 | 14.354 | 9.504 | 4.794 | 1.001 | 0 | 0 |
| $\boldsymbol{\lambda_{\Gamma}^{SR}}$ | | | | | | | |
| 0 | 26.828 | 21.941 | 17.055 | 12.168 | 7.281 | 2.444 | 0.004 |
| 2 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 4 | 22.873 | 17.997 | 13.122 | 8.247 | 3.407 | 0.136 | 0 |
| 6 | 20.969 | 16.100 | 11.230 | 6.362 | 1.737 | 0.004 | 0 |
| $\boldsymbol{p^{SR}}$ | | | | | | | |
| 1 | 25.819 | 20.935 | 16.051 | 11.167 | 6.284 | 1.566 | 0 |
| 2 | 24.825 | 19.944 | 15.063 | 10.183 | 5.304 | 0.875 | 0 |
| 3 | 23.847 | 18.969 | 14.091 | 9.213 | 4.349 | 0.422 | 0 |
| 4 | 22.885 | 18.010 | 13.135 | 8.260 | 3.443 | 0.173 | 0 |

**Table 9.3.** Values of short-rate caps for the jump-enhanced OU-SR model, with a nominal value of 100 units.

| **K** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta^{OU}}$ | | | | | | | |
| 0.005 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 0.01 | 3.943 | 2.969 | 2.024 | 1.220 | 0.651 | 0.312 | 0.136 |
| 0.015 | 4.376 | 3.403 | 2.457 | 1.635 | 1.012 | 0.592 | 0.330 |
| 0.02 | 4.806 | 3.835 | 2.889 | 2.058 | 1.404 | 0.928 | 0.597 |
| $\boldsymbol{\lambda_{\Gamma}^{OU}}$ | | | | | | | |
| 0 | 3.069 | 2.094 | 1.186 | 0.534 | 0.193 | 0.058 | 0.015 |
| 2 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 4 | 3.944 | 2.970 | 2.014 | 1.176 | 0.579 | 0.242 | 0.087 |
| 6 | 4.380 | 3.407 | 2.443 | 1.554 | 0.854 | 0.403 | 0.164 |
| $\boldsymbol{p^{OU}}$ | | | | | | | |
| 1 | 3.288 | 2.313 | 1.384 | 0.668 | 0.261 | 0.085 | 0.023 |
| 2 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 3 | 3.726 | 2.751 | 1.807 | 1.015 | 0.482 | 0.196 | 0.069 |
| 4 | 3.943 | 2.969 | 2.023 | 1.210 | 0.631 | 0.288 | 0.117 |
| $\boldsymbol{\mu_J^{OU}}$ | | | | | | | |
| 0.005 | 3.070 | 2.096 | 1.184 | 0.518 | 0.176 | 0.048 | 0.011 |
| 0.01 | 3.289 | 2.314 | 1.384 | 0.662 | 0.252 | 0.079 | 0.021 |
| 0.015 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 0.02 | 3.725 | 2.751 | 1.807 | 1.019 | 0.489 | 0.202 | 0.073 |
| $\boldsymbol{\lambda_N^{OU}}$ | | | | | | | |
| 0 | 2.850 | 1.875 | 0.972 | 0.370 | 0.107 | 0.025 | 0.005 |
| 2 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 4 | 4.162 | 3.188 | 2.232 | 1.378 | 0.736 | 0.340 | 0.137 |
| 6 | 4.814 | 3.842 | 2.878 | 1.971 | 1.210 | 0.660 | 0.320 |
| $\boldsymbol{\sigma_J^{OU}}$ | | | | | | | |
| 0.005 | 3.508 | 2.532 | 1.589 | 0.815 | 0.335 | 0.112 | 0.031 |
| 0.01 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 0.015 | 3.507 | 2.533 | 1.603 | 0.859 | 0.392 | 0.155 | 0.054 |
| 0.02 | 3.506 | 2.537 | 1.621 | 0.896 | 0.435 | 0.189 | 0.075 |

**Table 9.4.** Values of short-rate caps for the jump-enhanced OU-SR model, with a nominal value of 100 units.

| **K** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta^{SR}}$ | | | | | | | |
| 0.005 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 0.01 | 3.953 | 2.979 | 2.035 | 1.232 | 0.664 | 0.325 | 0.147 |
| 0.015 | 4.397 | 3.424 | 2.478 | 1.657 | 1.037 | 0.618 | 0.353 |
| 0.02 | 4.838 | 3.866 | 2.920 | 2.089 | 1.439 | 0.965 | 0.633 |
| $\boldsymbol{\lambda_{\Gamma}^{SR}}$ | | | | | | | |
| 0 | 3.059 | 2.083 | 1.173 | 0.518 | 0.181 | 0.052 | 0.013 |
| 2 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 4 | 3.955 | 2.980 | 2.026 | 1.190 | 0.593 | 0.252 | 0.093 |
| 6 | 4.402 | 3.428 | 2.465 | 1.579 | 0.881 | 0.427 | 0.181 |
| $\boldsymbol{p^{SR}}$ | | | | | | | |
| 1 | 3.283 | 2.308 | 1.378 | 0.661 | 0.254 | 0.080 | 0.021 |
| 2 | 3.507 | 2.532 | 1.593 | 0.832 | 0.358 | 0.129 | 0.040 |
| 3 | 3.731 | 2.756 | 1.813 | 1.021 | 0.490 | 0.202 | 0.073 |
| 4 | 3.954 | 2.980 | 2.034 | 1.222 | 0.645 | 0.301 | 0.127 |

**Table 9.5.** Values of average-rate caps for the jump-enhanced OU-SR model, with a nominal value of 100 units.

| **K** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta^{OU}}$ | | | | | | | |
| 0.005 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 0.01 | 2.977 | 2.002 | 1.053 | 0.389 | 0.112 | 0.027 | 0.006 |
| 0.015 | 3.199 | 2.225 | 1.275 | 0.579 | 0.232 | 0.087 | 0.032 |
| 0.02 | 3.419 | 2.446 | 1.496 | 0.781 | 0.385 | 0.184 | 0.087 |
| $\boldsymbol{\lambda_{\Gamma}^{OU}}$ | | | | | | | |
| 0 | 2.528 | 1.551 | 0.625 | 0.135 | 0.019 | 0.002 | 0 |
| 2 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 4 | 2.978 | 2.003 | 1.046 | 0.349 | 0.077 | 0.012 | 0.002 |
| 6 | 3.202 | 2.228 | 1.264 | 0.494 | 0.129 | 0.025 | 0.004 |
| $\boldsymbol{p^{OU}}$ | | | | | | | |
| 1 | 2.641 | 1.664 | 0.725 | 0.174 | 0.027 | 0.003 | 0 |
| 2 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 3 | 2.865 | 1.890 | 0.941 | 0.298 | 0.064 | 0.010 | 0.001 |
| 4 | 2.977 | 2.002 | 1.052 | 0.379 | 0.099 | 0.020 | 0.003 |
| $\boldsymbol{\mu_J^{OU}}$ | | | | | | | |
| 0.005 | 2.528 | 1.551 | 0.625 | 0.123 | 0.014 | 0.001 | 0 |
| 0.01 | 2.641 | 1.664 | 0.725 | 0.168 | 0.024 | 0.002 | 0 |
| 0.015 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 0.02 | 2.865 | 1.889 | 0.941 | 0.302 | 0.067 | 0.011 | 0.002 |
| $\boldsymbol{\lambda_N^{OU}}$ | | | | | | | |
| 0 | 2.416 | 1.438 | 0.516 | 0.079 | 0.007 | 0.001 | 0 |
| 2 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 4 | 3.089 | 2.115 | 1.158 | 0.434 | 0.112 | 0.022 | 0.003 |
| 6 | 3.424 | 2.452 | 1.488 | 0.680 | 0.227 | 0.057 | 0.012 |
| $\boldsymbol{\sigma_J^{OU}}$ | | | | | | | |
| 0.005 | 2.753 | 1.777 | 0.829 | 0.215 | 0.033 | 0.004 | 0 |
| 0.01 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 0.015 | 2.753 | 1.777 | 0.838 | 0.248 | 0.053 | 0.009 | 0.001 |
| 0.02 | 2.752 | 1.777 | 0.849 | 0.272 | 0.069 | 0.015 | 0.003 |

**Table 9.6.** Values of average-rate caps for the jump-enhanced OU-SR model, with a nominal value of 100 units.

| **K** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta^{SR}}$ | | | | | | | |
| 0.005 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 0.01 | 2.980 | 2.005 | 1.056 | 0.393 | 0.116 | 0.030 | 0.007 |
| 0.015 | 3.206 | 2.232 | 1.282 | 0.587 | 0.240 | 0.094 | 0.035 |
| 0.02 | 3.429 | 2.457 | 1.507 | 0.792 | 0.396 | 0.195 | 0.094 |
| $\boldsymbol{\lambda_{\Gamma}^{SR}}$ | | | | | | | |
| 0 | 2.524 | 1.547 | 0.621 | 0.130 | 0.017 | 0.002 | 0 |
| 2 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 4 | 2.981 | 2.006 | 1.050 | 0.354 | 0.079 | 0.013 | 0.002 |
| 6 | 3.209 | 2.235 | 1.272 | 0.502 | 0.136 | 0.027 | 0.004 |
| $\boldsymbol{p^{SR}}$ | | | | | | | |
| 1 | 2.639 | 1.662 | 0.723 | 0.171 | 0.025 | 0.003 | 0 |
| 2 | 2.753 | 1.777 | 0.832 | 0.229 | 0.041 | 0.005 | 0.001 |
| 3 | 2.867 | 1.891 | 0.943 | 0.300 | 0.066 | 0.011 | 0.002 |
| 4 | 2.980 | 2.005 | 1.055 | 0.383 | 0.103 | 0.022 | 0.004 |

## 9.3 The Fong-Vasicek Model

### 9.3.1 Derivation of the Characteristic Function

Apart from the additive modeling approach, the term structure within our exponential-affine framework can also be modeled with the help of subordinated factors. Hence, it is possible to explicitly incorporate the long term mean and/or the volatility as a stochastic factor itself[190]. For example, the assumption of a constant volatility in one-factor short-rate models is frequently criticized. In Fong and Vasicek (1991b), the authors argue that a model with a deterministic volatility parameter cannot produce a meaningful volatility exposure. Therefore, they propose to model the variance of a CIR-like short-rate model as a Square-Root process itself. In this section, we use a slightly modified version of the Fong and Vasicek (1991a) model. Thus, the SDEs for the base diffusion model are[191]

$$\mathrm{d}r_t = \kappa(\theta - r_t)\,\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}W_{1t}, \tag{9.2}$$

$$\mathrm{d}v_t = \alpha(\bar{v} - v_t)\,\mathrm{d}t + \beta\sqrt{v_t}\,\mathrm{d}W_{2t}. \tag{9.3}$$

Equivalently to the target rate $\theta$, often also referred to as the long term mean, of the short rate, $\bar{v}$ expresses the parameter for a long-term mean of the variance factor $v_t$. In addition to Fong and Vasicek (1991a), we extend this base diffusion model with additional jump components $\mathbf{j}_r$ and $\mathbf{j}_v$ for the short rate and its volatility factor[192], both triggered by the same vector of Poisson processes $\mathbf{N}(\boldsymbol{\lambda}^{\mathrm{Q}})$. In contrast to the additive OU-SR model discussed in the last section, the Brownian motions $W_{1t}$ and $W_{2t}$ can be correlated as follows:

[190] Beaglehole and Tenney (1991) model the long term mean $\theta$ of a mean-reverting, normally distributed short rate as a subordinated factor governed by an Ornstein-Uhlenbeck process. In Balduzzi, Das, Foresi and Sundaram (1996), the authors model a CIR like short rate with subordinated stochastic mean and volatility factor. However, in both cases, the authors do not give any option prices and derive only zero-bond prices and yields of zero bonds, respectively.

[191] For this interest-rate model, we assume $w_0 = 0$ and $\mathbf{w} = (1,0)'$. However, the derivation of the time-dependent coefficients of the characteristic function is shown in Appendix B for general discounting parameter values $w_0$ and $\mathbf{w}$.

[192] We only allow strictly positively sized jumps, thus restricting ourselves to positively directed exponentially and gamma distributed jump amplitudes.

$$\mathrm{d}W_{1t}\,\mathrm{d}W_{2t} = \rho\,\mathrm{d}t. \tag{9.4}$$

However, according to Section 2.1, all Brownian motions have to be uncorrelated within our modeling approach. Fortunately, the feature of correlated Brownian motions can be easily incorporated into our framework[193] by using the following diffusion-specific matrix

$$\mathbf{\Sigma}(v_t) = \beta\sqrt{v_t}\begin{pmatrix} \frac{\sigma}{\beta} & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}.$$

For convenience, we introduce for this model the following representation of time-dependent coefficient functions[194]

$$a^0(z,\tau) = A^0(z,\tau),$$

and

$$\mathbf{b}(z,\tau) = \begin{pmatrix} B(z,\tau) \\ C(z,\tau) \end{pmatrix} + \imath z\begin{pmatrix} \bar{B} \\ \bar{C} \end{pmatrix}.$$

Thus, in order to derive the general characteristic function of the state vector $\mathbf{x}_t = (r_t, v_t)'$, we explicitly have to solve the following system of ODEs

$$\begin{aligned} A^0(z,\tau)_\tau =& \kappa\theta(B(z,\tau) + \imath z\bar{B}) + \alpha\bar{v}(C(z,\tau) + \imath z\bar{C}) \\ =& A^{01}(z,\tau)_\tau + A^{02}(z,\tau)_\tau, \end{aligned} \tag{9.5}$$

$$B(z,\tau)_\tau = -\kappa(B(z,\tau) + \imath z\bar{B}) - w, \tag{9.6}$$

$$\begin{aligned} C(z,\tau)_\tau =& -\alpha(C(z,\tau) + \imath z\bar{C}) \\ & + \frac{\sigma^2}{2}(B(z,\tau) + \imath z\bar{B})^2 + \frac{\beta^2}{2}(C(z,\tau) + \imath z\bar{C})^2 \\ & + \sigma\beta\rho(B(z,\tau) + \imath z\bar{B})(C(z,\tau) + \imath z\bar{C}). \end{aligned} \tag{9.7}$$

Hence, we are dealing with a system of coupled ODEs. Fortunately, there are no two-sided interdependencies, enabling us to successively solve the differential equations one-by-one. Starting with equation (9.6), the solution to this differential equation is easy to obtain and coincides with equation (8.6) for $w_1 = w$ and $g_1 = \bar{B}$. Also straightforward, but more tedious, is the derivation

[193] A standard decomposition of two correlated Brownian motions is applied. Thus, two correlated Brownian motions as given in equation (9.4) allow for the alternative representation $\mathrm{d}W_{2t} = \rho\,\mathrm{d}W_{1t} + \sqrt{1-\rho^2}\,\mathrm{d}W^*_{2t}$, where the processes $W^*_{2t}$ and $W_{1t}$ are neither correlated.

[194] In this model setup, the constant parameter $g_0$ is represented by the term $\bar{A}$.

of the coefficient function solving the ODE (9.7). Performing some appropriate transformations on the particular ODE, the solution of the coefficient function corresponding to $v_t$ can be obtained as[195]

$$\begin{aligned} C(z,\tau) =& - M(z,\tau) \\ &+ J(z,\tau)\Big((1+Q(z)-S(z))\mathrm{KU}[Q(z)+1;S(z);Y(z,\tau)] \\ &\qquad + \Upsilon(z)\,\mathrm{KM}[Q(z)+1;S(z);Y(z,\tau)]\Big), \end{aligned} \tag{9.8}$$

with

$$\begin{aligned} M(z,\tau) &= \frac{\kappa}{\beta^2}\left(1+\frac{f_1(z)}{\kappa}+2Q(z)-S(z)+\left(1+\frac{\rho}{\sqrt{\rho^2-1}}\right)Y(z,\tau)\right), \\ J(z,\tau) &= \frac{2\kappa Q(z)}{\beta^2\left(\mathrm{KU}[Q(z);S(z);Y(z,\tau)]+\Upsilon(z)\,\mathrm{KM}[Q(z);S(z);Y(z,\tau)]\right)}, \\ Y(z,\tau) &= \frac{\sigma\beta\sqrt{\rho^2-1}}{\kappa}\left(\frac{w}{\kappa}+\imath z\bar{B}\right)e^{-\kappa\tau}, \end{aligned}$$

$$\begin{aligned} Q(z) &= \frac{S(z)}{2}+\frac{(f_3(z)+\kappa)\rho-\beta f_2(z)}{2\kappa\sqrt{\rho^2-1}}, \\ S(z) &= 1+\sqrt{\frac{f_3(z)^2-2\beta^2 f_1(z)}{\kappa^2}}, \\ \Upsilon(z) &= \frac{M(z,0)\mathrm{KU}[Q(z);S(z);Y(z,0)]}{\Xi(z)} \\ &\quad -\frac{(1+Q(z)-S(z))\,\mathrm{KU}[Q(z)+1;S(z);Y(z,0)]}{\frac{\beta^2}{2\kappa Q(z)}\,\Xi(z)}, \\ \Xi(z) &= 2\frac{\kappa}{\beta^2}Q(z)\mathrm{KM}[Q(z)+1;S(z);Y(z,0)] \\ &\quad - M(z,0)\mathrm{KM}[Q(z);S(z);Y(z,0)], \end{aligned}$$

and

$$\begin{aligned} f_1(z) &= -\imath z\bar{C}\left(\alpha-\frac{\imath z\beta^2\bar{C}}{2}+\frac{\sigma\beta\rho w}{\kappa}\right)+\frac{\sigma^2 w^2}{2\kappa^2}, \\ f_2(z) &= \imath z\beta\rho\bar{C}-\frac{\sigma w}{\kappa}, \\ f_3(z) &= \imath z\beta^2\bar{C}-\alpha-\frac{\sigma\beta\rho w}{\kappa}. \end{aligned}$$

In the equations above the function $\mathrm{KM}[a;b;y]$ is commonly referred to as the Kummer function (of the first kind) and $\mathrm{KU}[a;b;y]$ represents a confluent

[195] The detailed derivation of the coefficient function $C(z,\tau)$ is given in Appendix B.

hypergeometric function[196]. Both functions represent the two independent solutions of the Kummer equation[197].

In contrast to the last derivation, both parts of the coefficient function $A^0(z,\tau)$ are relatively easy to obtain. Due to the simple structure of the solution given in equation (9.6), we are able to state immediately

$$A^{01}(z,\tau) = -\theta\left(B(z,\tau) + w\tau\right). \tag{9.9}$$

For the second part of the diffusion component in $A^0(z,\tau)$, we exploit the formal structure and attempt a logarithmic integration, which is shown in Appendix B. Thus, the solution of the second part of the coefficient function $A^0(z,\tau)$ can be written as[198]

$$A^{02}(z,\tau) = -\frac{2\alpha\bar{v}}{\beta^2}\ln\left[\frac{L(z,\tau)}{L(z,0)}\,\frac{J(z,0)}{J(z,\tau)}\right] + \imath z\alpha\bar{v}\bar{C}\tau, \tag{9.10}$$

with

$$\begin{aligned}\ln\left[L(z,\tau)\right] = &\left(\frac{S(z) - \left(1+\frac{f_3(z)}{\kappa}\right)}{2}\right)\ln[Y(z,\tau)] \\ &-\left(1+\frac{\rho}{\sqrt{\rho^2-1}}\right)\frac{Y(z,\tau)}{2} \\ &+\left(\frac{1}{2}+\frac{f_3(z)}{2\kappa}\right)\ln\left[\frac{\beta\sqrt{\rho^2-1}}{\kappa}\right].\end{aligned} \tag{9.11}$$

Having calculated the diffusion-related coefficients of the general characteristic function, we are now ready for the corresponding jump parts. In case

[196] The confluent hypergeometric function is sometimes also denoted as the Kummer function of the second kind and is – like the complex-valued square-root and logarithm – a multi-valued function. Thus, one has to track carefully the path of integration by using this type of function to avoid discontinuities according to the principal branch used by standard mathematical programming environments.

[197] The functions $\mathrm{KM}[a;b;y]$ and $\mathrm{KU}[a;b;y]$ are two independent solutions of the differential equation $y\frac{\mathrm{d}^2w(y)}{\mathrm{d}y^2} + (b-y)\frac{\mathrm{d}w(y)}{\mathrm{d}y} = aw(y)$. More information on confluent hypergeometric functions, especially about the computation of $\mathrm{KM}[a;b;y]$ and $\mathrm{KU}[a;b;y]$ can be found in Abramowitz and Stegun (1972), p. 504.

[198] The detailed derivation of $A^{02}(z,\tau)$ is given in Appendix B.

of a exponential jump size specification on the short rate we are able to use the same jump transformation derived for the Vasicek model. In any other case – meaning gamma distributed jump sizes in the short rate, and for any jump components incorporated in the volatility factor[199] – we have to apply a Runge-Kutta algorithm.

### 9.3.2 Numerical Results

In this subsection, we want to demonstrate the impact of different jump components and the correlation parameter $\rho$ on the probability density function as well as on option prices with payoff functions similar to Table 4.1, respectively. To the best of our knowledge, option prices for this model, whether of the exponential-affine, linear, or integro-linear type, are presented for the first time in this thesis. Articles do exist, which cover the computation of numerical values under the base model. However, only prices of unconditional contracts, such as zero bonds and likewise yields of zero-bond prices, are computed[200]. These model prices are easy to obtain due to their similarity of the moment-generating function of the short rate[201].

The base diffusion parameters we use in computing the probability density function are $r_t = 0.08, v_t = 0.04, \kappa = 0.2, \theta = 0.08, \sigma = 0.1, \alpha = 0.4, \bar{v} = 0.04, \beta = 0.1, \rho = -0.5$ and $T = 1$. Firstly, we examine the behavior of the probability density for alternating correlation specifications. To avoid the results being biased by the influence of jump components, we examine first the density function of the pure diffusion model according to equations (9.2) and (9.3). Obviously, Figure 9.4 shows that the correlation between the short rate and its volatility has an effect on the probability density function, and therefore on the price of any contingent claim. Thus, for low interest rates

[199] Due to the complicated structure of the coefficient function $C(z, \tau)$, even for exponentially distributed jump amplitudes there exist no closed-form jump transforms.

[200] See Fong and Vasicek (1991b). Selby and Strickland (1995) also compute numerical values of zero-bond prices for the base model, but present a technique avoiding the application of hypergeometric functions. In Balduzzi, Das, Foresi and Sundaram (1996), zero-bond prices are computed for an extended version of the Fong-Vasicek model, where the mean of the short rate is also modeled as a stochastic factor.

[201] See Proposition 2.4.3.

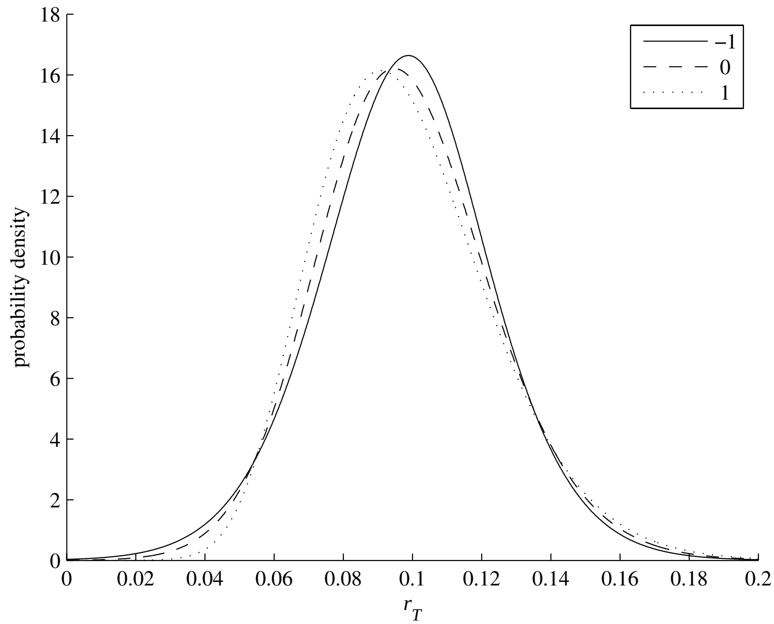


**Fig. 9.4.** Probability density functions of the Fong-Vasicek pure diffusion model for different values of the correlation parameter $\rho$. The parameters used are: $r_t = 0.08, v_t = 0.04, \kappa = 0.2, \theta = 0.08, \sigma = 0.1, \alpha = 0.4, \bar{v} = 0.04, \beta = 0.1, T = 1$.

and negatively correlated factors we observe substantially increased values of the probability density function, in contrast to the probability density function with positively correlated random variables. Since in this scenario we encounter a tendency toward higher volatilities in case of low interest rates, we are dealing with a more volatile process, which explains the behavior of the density. Finally, the correlation parameter can be used to adjust the skewness of the probability probability density function, which is advantageous for calibrating empirical term structures.

Next, we examine the influence of jump specifications on the density function of the short rate. Thus, we fix the correlation to $\rho = -0.5$. Comparing the graphs in Figure 9.5 with the particular graphs in Figure 8.5, we observe the same behavior of the density functions considering a gamma distributed jump component on the short rate. This fact is not surprising since the short rate in the Fong-Vasicek interest-rate model is governed by a Square-Root

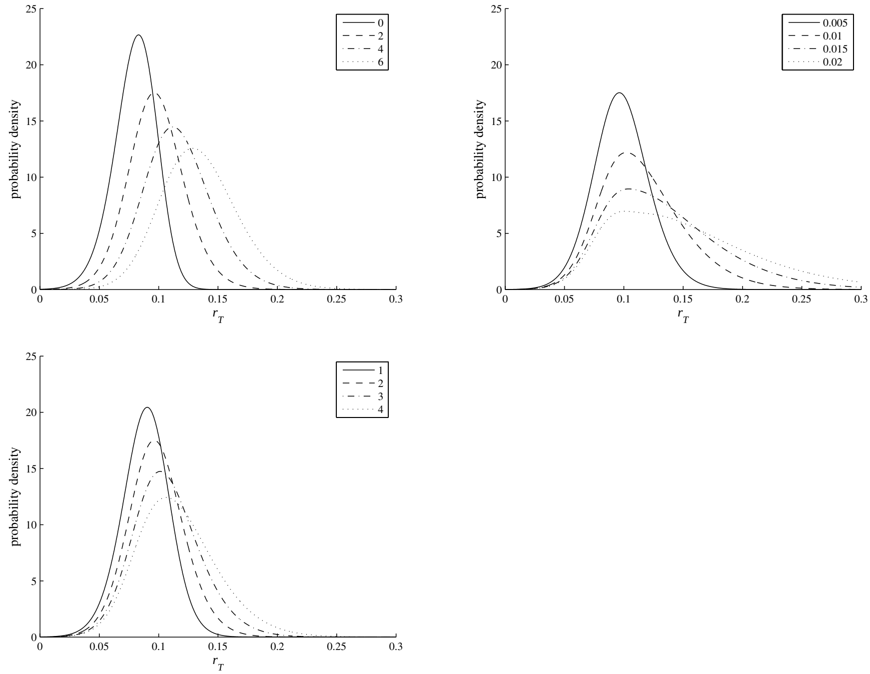


**Fig. 9.5.** Probability densities for a short rate governed by the Fong-Vasicek diffusion model enhanced with a gamma distributed jump component for the short-rate process. In the upper left (right) graph the density functions for varying jump intensities (means) are depicted. The lower graph shows the density behavior for alternating values of $p$. The base parameters are: $r_t = 0.08, v_t = 0.04, \kappa = 0.2, \theta = 0.08, \sigma = 0.1, \alpha = 0.4, \bar{v} = 0.04, \beta = 0.1, \rho = -0.5, \lambda_r = 2, \eta_r = 0.005, p_r = 2, T = 1$.

process. However, allowing for a gamma distributed jump in the volatility process, the density functions in Figure 9.6 show rather increased values on both tails while maintaining their mode. Distinguishing between the impact of jump parameters on the density, we clearly identify the jump intensity to have the strongest influence on the short-rate process.

The base diffusion parameters used for the computation of theoretical prices given in Tables 9.7 and 9.8 are the same as above. The default parameters for the different gamma distributed jump components are $\lambda = 2$, $\eta = 0.005$ and $p = 2$ for both the short-rate and volatility process. All contracts have a remaining time to maturity of half a year. At first, we turn our attention to Table 9.7, where zero-bond call option prices are given for

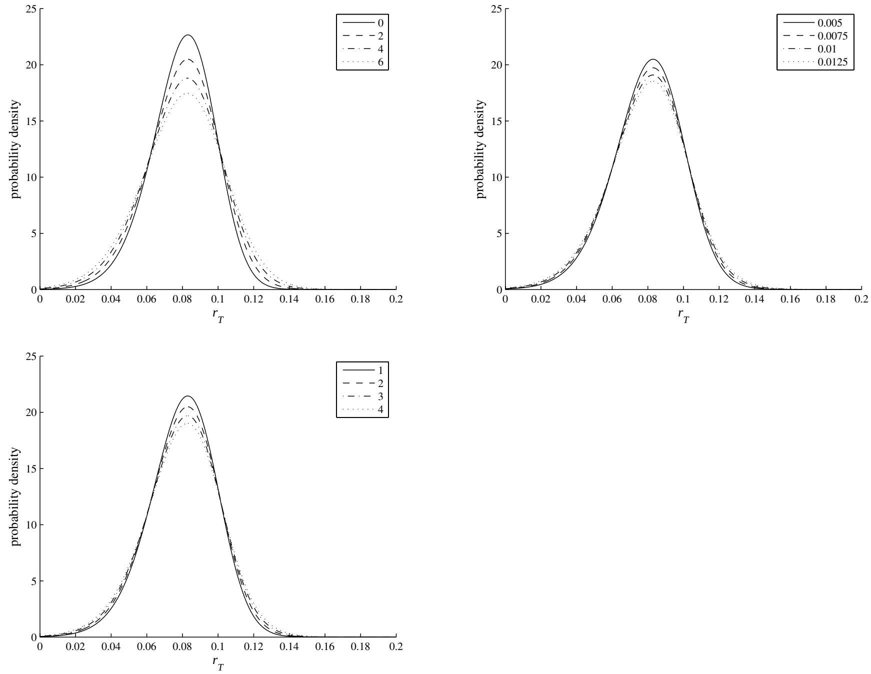


**Fig. 9.6.** Probability densities for a short rate governed by the Fong-Vasicek diffusion model enhanced with a gamma distributed jump component for the volatility process. In the upper left (right) graph the density functions for varying jump intensities (means) are depicted. The lower graph shows the density behavior for alternating values of $p$. The base parameters are: $r_t = 0.08, v_t = 0.04, \kappa = 0.2, \theta = 0.08, \sigma = 0.1, \alpha = 0.4, \bar{v} = 0.04, \beta = 0.1, \rho = -0.5, \lambda_v = 2, \eta_v = 0.005, p_v = 2, T = 1$.

a strike range from 60 to 90 units. The range is chosen to cover either ITM, ATM and OTM option prices[202]. Similar to findings for the probability density function, the jump behavior of gamma distributed jump component on the short rate under the Fong-Vasicek model and also in the Square-Root model shows a strong resemblance on account of the Square-Root process. Looking at the gamma distributed jump component in the volatility process, we observe higher values of the particular derivative price for increasing jump parameters. This effect is due to the more increased tails of the density in comparison to the ordinary diffusion case. Accordingly, the effect of volatility

[202] The value of a zero bond with remaining maturity of two years, priced with base parameters, is 82.335 units.

jumps on derivative contracts is small, which explains the nearly linear relation between jump parameters and contract values. Turning our attention to Tables 9.8 and 9.9, we computed prices for caps with one payment date made at the maturity of the particular contract. Thus, we examine cap prices similar to the idealized payoff function in Table 4.1. Since we restrict this model to positively sized jump components due to the Square-Root limitations, we have the ordinary cap dominating its average-rate counterpart. This is a direct consequence of the geometric average of an increasing function. Comparing the corresponding values of the one-factor CIR model with the cap values in this section, we indicate the same impact of jumps on the short rate generally. Thus, this model also has a less sensitive average-rate option in contrast to the ordinary cap contract. However, jumps on the volatility component have roughly speaking no effect at all on average-rate options, whereas ordinary caps encounter small changes for either ITM, ATM and OTM option values.

**Table 9.7.** Values of zero-bond call options for the jump-enhanced Fong-Vasicek model, where the underlying zero-bond contract has a nominal value of 100 units.

| **K** | 60 | 65 | 70 | 75 | 80 | 85 | 90 |
|---|---|---|---|---|---|---|---|
| $\eta_r$ | | | | | | | |
| 0.005 | 20.232 | 15.439 | 10.647 | 5.867 | 1.562 | 0.052 | 0 |
| 0.01 | 16.456 | 11.68 | 6.965 | 2.689 | 0.255 | 0.002 | 0 |
| 0.015 | 12.970 | 8.316 | 4.050 | 0.921 | 0.021 | 0 | 0 |
| 0.02 | 9.868 | 5.579 | 2.088 | 0.181 | 0.001 | 0 | 0 |
| $\lambda_r$ | | | | | | | |
| 0 | 24.302 | 19.498 | 14.694 | 9.890 | 5.087 | 0.910 | 0.015 |
| 2 | 20.232 | 15.439 | 10.647 | 5.867 | 1.562 | 0.052 | 0 |
| 4 | 16.378 | 11.597 | 6.828 | 2.393 | 0.169 | 0.001 | 0 |
| 6 | 12.728 | 7.967 | 3.409 | 0.469 | 0.006 | 0 | 0 |
| $p_r$ | | | | | | | |
| 1 | 22.232 | 17.434 | 12.636 | 7.838 | 3.103 | 0.237 | 0.002 |
| 2 | 20.232 | 15.439 | 10.647 | 5.867 | 1.562 | 0.052 | 0 |
| 3 | 18.299 | 13.512 | 8.730 | 4.067 | 0.660 | 0.010 | 0 |
| 4 | 16.430 | 11.651 | 6.910 | 2.596 | 0.238 | 0.002 | 0 |
| $\eta_v$ | | | | | | | |
| 0.005 | 20.232 | 15.439 | 10.647 | 5.867 | 1.562 | 0.052 | 0 |
| 0.01 | 20.247 | 15.455 | 10.663 | 5.885 | 1.597 | 0.064 | 0 |
| 0.015 | 20.263 | 15.471 | 10.678 | 5.902 | 1.631 | 0.077 | 0.001 |
| 0.02 | 20.279 | 15.486 | 10.694 | 5.920 | 1.664 | 0.090 | 0.002 |
| $\lambda_v$ | | | | | | | |
| 0 | 20.216 | 15.424 | 10.631 | 5.850 | 1.525 | 0.041 | 0 |
| 2 | 20.232 | 15.439 | 10.647 | 5.867 | 1.562 | 0.052 | 0 |
| 4 | 20.247 | 15.455 | 10.663 | 5.884 | 1.598 | 0.063 | 0 |
| 6 | 20.263 | 15.471 | 10.678 | 5.901 | 1.634 | 0.075 | 0.001 |
| $p_v$ | | | | | | | |
| 1 | 20.224 | 15.432 | 10.639 | 5.859 | 1.543 | 0.046 | 0 |
| 2 | 20.232 | 15.439 | 10.647 | 5.867 | 1.562 | 0.052 | 0 |
| 3 | 20.240 | 15.447 | 10.655 | 5.876 | 1.580 | 0.057 | 0 |
| 4 | 20.247 | 15.455 | 10.663 | 5.884 | 1.597 | 0.063 | 0 |

**Table 9.8.** Values of short-rate caps for the jump-enhanced Fong-Vasicek model, with a nominal value of 100 units.

| **K** | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\eta_r}$ | | | | | | | |
| 0.005 | 2.854 | 1.976 | 1.222 | 0.663 | 0.317 | 0.137 | 0.055 |
| 0.01 | 3.742 | 2.852 | 2.061 | 1.422 | 0.952 | 0.624 | 0.400 |
| 0.015 | 4.623 | 3.730 | 2.925 | 2.253 | 1.721 | 1.306 | 0.981 |
| 0.02 | 5.494 | 4.601 | 3.790 | 3.101 | 2.536 | 2.069 | 1.678 |
| $\boldsymbol{\lambda_r}$ | | | | | | | |
| 0 | 1.979 | 1.162 | 0.541 | 0.181 | 0.04 | 0.006 | 0.001 |
| 2 | 2.854 | 1.976 | 1.222 | 0.663 | 0.317 | 0.137 | 0.055 |
| 4 | 3.738 | 2.827 | 1.991 | 1.291 | 0.769 | 0.423 | 0.217 |
| 6 | 4.625 | 3.696 | 2.812 | 2.018 | 1.358 | 0.858 | 0.510 |
| $\boldsymbol{p_r}$ | | | | | | | |
| 1 | 2.412 | 1.554 | 0.850 | 0.375 | 0.131 | 0.038 | 0.009 |
| 2 | 2.854 | 1.976 | 1.222 | 0.663 | 0.317 | 0.137 | 0.055 |
| 3 | 3.298 | 2.410 | 1.627 | 1.011 | 0.583 | 0.317 | 0.163 |
| 4 | 3.743 | 2.849 | 2.049 | 1.394 | 0.906 | 0.565 | 0.339 |
| $\boldsymbol{\eta_v}$ | | | | | | | |
| 0.005 | 2.854 | 1.976 | 1.222 | 0.663 | 0.317 | 0.137 | 0.055 |
| 0.01 | 2.861 | 1.990 | 1.242 | 0.684 | 0.334 | 0.148 | 0.061 |
| 0.015 | 2.868 | 2.003 | 1.260 | 0.704 | 0.351 | 0.159 | 0.067 |
| 0.02 | 2.876 | 2.016 | 1.278 | 0.723 | 0.366 | 0.170 | 0.073 |
| $\boldsymbol{\lambda_v}$ | | | | | | | |
| 0 | 2.848 | 1.962 | 1.200 | 0.639 | 0.300 | 0.127 | 0.050 |
| 2 | 2.854 | 1.976 | 1.222 | 0.663 | 0.317 | 0.137 | 0.055 |
| 4 | 2.860 | 1.990 | 1.243 | 0.685 | 0.335 | 0.148 | 0.060 |
| 6 | 2.867 | 2.004 | 1.263 | 0.707 | 0.352 | 0.158 | 0.065 |
| $\boldsymbol{p_v}$ | | | | | | | |
| 1 | 2.851 | 1.969 | 1.211 | 0.651 | 0.308 | 0.132 | 0.052 |
| 2 | 2.854 | 1.976 | 1.222 | 0.663 | 0.317 | 0.137 | 0.055 |
| 3 | 2.857 | 1.983 | 1.232 | 0.674 | 0.326 | 0.142 | 0.058 |
| 4 | 2.861 | 1.990 | 1.242 | 0.684 | 0.334 | 0.148 | 0.060 |

**Table 9.9.** Values of average-rate caps for the jump-enhanced Fong-Vasicek model, with a nominal value of 100 units.

| **K** | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|
| $\eta_r$ | | | | | | | |
| 0.005 | 2.377 | 1.443 | 0.655 | 0.205 | 0.051 | 0.012 | 0.003 |
| 0.01 | 2.827 | 1.892 | 1.077 | 0.543 | 0.268 | 0.131 | 0.063 |
| 0.015 | 3.271 | 2.336 | 1.512 | 0.936 | 0.586 | 0.366 | 0.226 |
| 0.02 | 3.708 | 2.775 | 1.946 | 1.346 | 0.948 | 0.668 | 0.468 |
| $\lambda_r$ | | | | | | | |
| 0 | 1.922 | 1.004 | 0.308 | 0.036 | 0.001 | 0 | 0 |
| 2 | 2.377 | 1.443 | 0.655 | 0.205 | 0.051 | 0.012 | 0.003 |
| 4 | 2.831 | 1.889 | 1.040 | 0.449 | 0.159 | 0.050 | 0.014 |
| 6 | 3.283 | 2.337 | 1.450 | 0.751 | 0.329 | 0.127 | 0.044 |
| $p_r$ | | | | | | | |
| 1 | 2.150 | 1.221 | 0.466 | 0.096 | 0.012 | 0.001 | 0 |
| 2 | 2.377 | 1.443 | 0.655 | 0.205 | 0.051 | 0.012 | 0.003 |
| 3 | 2.603 | 1.668 | 0.859 | 0.352 | 0.127 | 0.043 | 0.013 |
| 4 | 2.828 | 1.892 | 1.072 | 0.523 | 0.237 | 0.101 | 0.041 |
| $\eta_v$ | | | | | | | |
| 0.005 | 2.377 | 1.443 | 0.655 | 0.205 | 0.051 | 0.012 | 0.003 |
| 0.01 | 2.377 | 1.446 | 0.661 | 0.211 | 0.053 | 0.012 | 0.003 |
| 0.015 | 2.378 | 1.449 | 0.667 | 0.216 | 0.056 | 0.013 | 0.003 |
| 0.02 | 2.378 | 1.452 | 0.672 | 0.221 | 0.058 | 0.014 | 0.003 |
| $\lambda_v$ | | | | | | | |
| 0 | 2.377 | 1.441 | 0.648 | 0.200 | 0.049 | 0.011 | 0.002 |
| 2 | 2.377 | 1.443 | 0.655 | 0.205 | 0.051 | 0.012 | 0.003 |
| 4 | 2.377 | 1.446 | 0.661 | 0.211 | 0.053 | 0.012 | 0.003 |
| 6 | 2.377 | 1.449 | 0.668 | 0.216 | 0.055 | 0.013 | 0.003 |
| $p_v$ | | | | | | | |
| 1 | 2.377 | 1.442 | 0.651 | 0.202 | 0.05 | 0.012 | 0.002 |
| 2 | 2.377 | 1.443 | 0.655 | 0.205 | 0.051 | 0.012 | 0.003 |
| 3 | 2.377 | 1.445 | 0.658 | 0.208 | 0.052 | 0.012 | 0.003 |
| 4 | 2.377 | 1.446 | 0.661 | 0.211 | 0.053 | 0.012 | 0.003 |

# 10

# Non-Affine Term-Structure Models and Short-Rate Models with Stochastic Jump Intensity

## 10.1 Overview

Although the model setup proposed in this thesis is of the exponential-affine class, we can also extend the framework to allow for certain non-affine models and models with state-dependent jump intensities $\boldsymbol{\lambda}^{\mathbb{Q}}(\mathbf{x}_t)$. Moreover, option prices under these more sophisticated model dynamics can be priced in our numerical scheme without greater effort, due to an exponential separable structure of the governing characteristic function. However, working with a non-affine model, we have to abandon jump components for those particular non-affine factors. A stochastic jump intensity in the general exponential-affine model framework is introduced in Duffie, Pan and Singleton (2000). Consequently, the jump transform is no longer independent of the coefficient function $a(z, \tau)$, and therefore a complicated system of ODEs has to be determined numerically anyway. Since both approaches need to establish further restrictions, they are only discussed as possibilities for extending and modifying the base model, respectively.

## 10.2 Quadratic Gaussian Models

Non-Affine exponential separable models are characterized by a non-affine structure of the factors in the relevant moment-generating function, as well as the general characteristic function, while preserving the separability of coefficient functions for different powers of the particular factors included in the model. Thus, the essential system of ODEs can be derived. Prominent representatives of this model class are in an equity context the stochastic volatility model of Schöbel and Zhu (1999), which is a generalized version of the Stein and Stein (1991) model. In case of interest rates, we have, e.g. the Double Square-Root model of Longstaff (1989), the quadratic Gaussian model approach of Beaglehole and Tenney (1991)[203], and the general linear-quadratic jump-diffusion model of Cheng and Scaillet (2004)[204].

Although the quadratic Gaussian and the Double Square-Root model seem quite attractive to implement, it is impossible to compute theoretical model prices within the Fourier-based pricing framework if jumps are incorporated, while Monte-Carlo pricing approaches might still work. This stems from the fact that in equation (2.39), for the $n$th jump $J_{mn}$ in the non-affine factor $x_t^{(m)}$, there would be a corresponding term $(x_t^{(m)} + J_{mn})^2$ resulting in a mixed expression. Hence, the exponential separation approach will no longer be available in deriving the general characteristic function. Since none of the non-affine interest-rate models are capable of exhibiting any jump component we completely ignored these models in our base setup according to Section 2.1.

The one-factor quadratic Gaussian approach models the short rate under the risk-neutral measure, as the square of some factor $x_t$ governed by an Ornstein-Uhlenbeck process according to equation (8.5). In order to price interest-rate derivatives for this particular process, we need to have the general characteristic function to consider both the state variable $x_t$ and its square $x_t^2$. Thus, for the squared Gaussian interest-rate model we use the following form of the general characteristic function

$$\psi\left(\mathbf{y}_t, z, 0, \mathbf{w}, g_0, \mathbf{g}, \tau\right) = e^{a(z,\tau)+\mathbf{b}(z,\tau)'\mathbf{y}_t+\imath z g_0},$$

with

$$\mathbf{y}_t = \begin{pmatrix} x_t \\ x_t^2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For convenience, we use again the time-dependent coefficient functions[205]

[203] Ahn, Dittmar and Gallant (2002) give a good overview of general multi-dimensional linear-quadratic Gaussian interest-rate models.

[204] Linear-quadratic in this context means all factors contained in the state vector $\mathbf{x}_t$ are allowed to enter the interest rate both in a linear and quadratic fashion.

[205] The constant parameter $g_0$ is represented by the term $\bar{A}$.

$$a(z,\tau) = A(z,\tau),$$

and

$$\mathbf{b}(z,\tau) = \begin{pmatrix} B(z,\tau) \\ C(z,\tau) \end{pmatrix} + \imath z \begin{pmatrix} \bar{B} \\ \bar{C} \end{pmatrix}.$$

Inserting the above characteristic function in equation (2.33) and applying the separation approach result again in a system of coupled ODEs[206]

$$\begin{aligned} A(z,\tau)_\tau =& \kappa\theta(B(z,\tau) + \imath z\bar{B}) + \sigma^2(C(z,\tau) + \imath z\bar{C}) \\ &+ 2\sigma^2(B(z,\tau) + \imath z\bar{B})^2, \\ B(z,\tau)_\tau =& (B(z,\tau) + \imath z\bar{B})(\sigma^2(C(z,\tau) + \imath z\bar{C}) - \kappa) \\ &+ 2\kappa\theta(C(z,\tau) + \imath z\bar{C}), \\ C(z,\tau)_\tau =& -2\kappa(C(z,\tau) + \imath z\bar{C}) + 2\sigma^2(C(z,\tau) + \imath z\bar{C})^2 - 1, \end{aligned}$$

which can be solved successively. The advantage of this modeling approach lies in its tractability while describing a more elaborated interest-rate behavior. Additionally, the short rate in this approach is always positive, compared to possible negative short rates using the Vasicek model. In the Double Square-Root model according to Longstaff (1989), we encounter a very similar situation, since we are able to transform the model into a quadratic Gaussian model and vice versa but with additional restrictions on the parameter set[207].

Cheng and Scaillet (2004) introduce a linear-quadratic jump-diffusion model. Here, the diffusion part of some random variable, for example the short rate $r(\mathbf{x}_t)$ or the payoff characteristic function $g(\mathbf{x}_t)$, is built similarly to the multivariate quadratic Gaussian model in Beaglehole and Tenney (1991), as the sum of linear and quadratic terms of the state vector $\mathbf{x}_t$ containing correlated Ornstein-Uhlenbeck processes. To gain a closed-form solution for the general characteristic function, additional jump parts only occur in the affine terms of $\mathbf{x}_t$. Therefore, we can think of this interest-rate model as a simple combination of an additive multivariate Ornstein-Uhlenbeck model augmented with jump components and an additive multivariate quadratic Gaussian model.

[206] Although the vector $\mathbf{y}_t$ occurs in the characteristic function, derivatives remain still to be taken with respect to the unique state variables which is in this one-dimensional model just the factor $x_t$.

[207] See Beaglehole and Tenney (1992), pp. 346-347.

## 10.3 Stochastic Jump Intensity

Another possibility for extending the base model setup stated in Section 2.1 is to implement stochastic jump intensities. Duffie, Pan and Singleton (2000) introduced, with their affine jump-diffusion model, a vector of stochastic jump intensities where the stochastic component is affine in the state variable $\mathbf{x}_t$. Thus, they implement stochastic intensities without overly aggravating their solution technique. Defining the vector of jump intensities as[208]

$$\boldsymbol{\lambda}^{\mathbb{Q}}(\mathbf{x}_t) = \boldsymbol{\lambda}_0^{\mathbb{Q}} + \boldsymbol{\lambda}_1^{\mathbb{Q}} \mathbf{x}_t,$$

with $(\boldsymbol{\lambda}_0^{\mathbb{Q}}, \boldsymbol{\lambda}_1^{\mathbb{Q}}) \in \mathbb{R}^M \times \mathbb{R}^{M \times M}$, we therefore get a slightly modified system of ODEs for the vector coefficient $\mathbf{b}(z,\tau)$ compared to equation (2.41), which is

$$\begin{aligned}\mathbf{b}(z,\tau)_\tau = & -\mathbf{w} + \boldsymbol{\mu}_1^{\mathbb{Q}\prime}\mathbf{b}(z,\tau) + \frac{1}{2}\,\mathbf{b}(z,\tau)'\boldsymbol{\Sigma}_1\mathbf{b}(z,\tau) \\ & + \boldsymbol{\lambda}_1^{\mathbb{Q}}\mathbb{E}_{\mathbf{J}}\left[\boldsymbol{\psi}^*(z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau) - 1\right].\end{aligned}$$

Obviously, in implementing this type of jump intensity, values of the coefficient vector $\mathbf{b}(z,\tau)$ must be determined numerically due to the complicated structure of the relevant ODE. Subsequently, the same statement holds also for the coefficient function $a(z,\tau)$, which depends on $\mathbf{b}(z,\tau)$. Although this type of jump specification enriches the modeling capabilities of the short-rate dynamics, it is infrequently implemented in interest-rate models because of the numerical difficulties mentioned above. However, our FRFT algorithm presented in Chapter 6 can be easily modified to handle this type of stochastic jump intensity.

[208] To stay conform with our base model setup in equation (2.1), we suggest to include $N$ Poisson processes with stochastic intensities.

# 11

# Conclusion

In this thesis, we have introduced a general jump-diffusion short-rate model. The model approach we proposed extends the interest-rate model of Duffie and Kan (1996) by considering $N$ different possible Poison processes in the underlying factors. Using the findings in Carr and Madan (1999) and Lewis (2001) to interchange the order of integration of an integral-transformed option price in an equity context, we derived a general pricing formula valid for various popular interest-rate contracts. However, we eventually preferred the approach of Lewis (2001) over the technique presented in Carr and Madan (1999). The pricing scheme used in this thesis exhibits a rigorous modular structure. Thus, we took one step towards successfully extending the spirit of a modular pricing framework as proposed in Zhu (2000) by modularizing not only the stochastic parts, but also modularizing the derivative price in terms of its payoff structure. Hence, all pricing formulae developed in this thesis can be split into parts of the Fourier-transformed payoff function and of the underlying process, characterized by its characteristic function, respectively. Hence, we were able to state one single valuation formula, equation (4.21), to price derivatives of the linear, exponential-linear, and integro-linear types. Especially for the integro-linear case, the payoff-transform approach offers an elegant alternative to the methods proposed, e.g. in Bakshi and Madan (2000), Chacko and Das (2002), and Ju (1997). In addition, we presented within the pricing framework of Lewis (2001) for the first time the consistent inclusion of both unconditional and conditional interest-rate derivatives. Hence, facilitating a integration by parts method, we successfully applied the residue theorem to recover prices of contracts with unconditional exercise rights and the particular put-call parities, respectively. As a special case in Section 5.3.3, we also derived a Fourier-style solution of coupon-bond options, where the interest rate is governed by a one-factor process, which can be compared to the two-sided Laplace-style solution presented in Kluge (2005), Section 2.4. However, we want to emphasize that the term *one-factor* does not correspond to the number of jump components incorporated in the short-rate process. Thus, theoretically speaking, within our pricing scheme we are able to price coupon-bond options and swaptions, respectively, as long as the underlying process exhibits only one single Brownian motion. This is in fact a powerful result, since the additional inclusion of jump processes can result in more realistic models.

In Chapter 6, we employed the IFFT algorithm for the first time to compute option prices within the pricing framework of Lewis (2001). The obtained pricing algorithm is then refined by translating the IFFT procedure into the FRFT algorithm. Subsequently, we dealt with the issue of finding the optimal and therefore error-minimizing parameter setting for the FRFT algorithm by utilizing a steepest descent technique. Doing this, we focused our efforts to minimize the overall error of the solution vector generated by the FRFT pricing algorithm, rather than the error of one single option price[209]. In our opinion, this procedure is a more powerful procedure, since the advantage of the FFT- and IFFT-based pricing algorithms is the simultaneous computation of option prices for a given strike range. Therefore, we used for the error measuring the RMSE of the numerical solution vector. Fortunately, it became apparent that the logarithmic RMSE of the numerical solution is a nearly linear descending function for increasing values of $z_i$ and $\omega$, starting with the smallest possible values not violating any regularity conditions. Thus, we used a steepest-descent technique to identify the optimal parameters for the numerical algorithm. Furthermore, exploiting this linearity we were also able to formulate an approximate error bound for the numerical solution vector.

After discussing the numerical algorithm, we analyzed a selection of both one-factor and two-factor jump-diffusion short-rate models. We first specified our jump size candidates, which were the exponential, gamma, and normal

[209] Lee (2004) and Lord and Kahl (2007) study the error behavior of Fourier transform-based algorithms for only a single strike value.

distribution, and derived their particular jump transforms. Subsequently, we derived the relevant general characteristic function of the jump-diffusion process, and then computed numerical values of the particular density functions and contract values. Widely used, the exponentially distributed jump size assumption presents no difficulties in derivatives pricing because of the closed-form jump transforms for both Ornstein-Uhlenbeck and Square-Root diffusion processes. However, we also applied normal and gamma distributions for the jump size. The normal distribution for the jump component within a Vasicek model is also used in the articles of Baz and Das (1996) and Durham (2005), where an approximation technique for zero-bond prices is described. Unfortunately, under some circumstances both approaches deliver inaccurate values for the respective derivatives contracts[210]. Our pricing algorithm is able to circumvent these issues and compute accurate numerical values of interest-rate derivatives in any case. Moreover, we introduced gamma distributed jumps within a jump-diffusion short-rate model framework for the first time. We then combined these jump candidates with the one-factor Ornstein-Uhlenbeck, the Square-Root processes, the two-factor version of a combined OU-SR, and the stochastic volatility model of Fong and Vasicek (1991a), and computed densities and option values. In particular, our contribution besides the implementation of the normal and gamma jump-size distribution in interest-rate option pricing, is to present an algorithm capable of computing option prices for the (jump-extended) Fong and Vasicek (1991a) interest-rate model. Up to now, only zero-bond prices have been computed for the jump-enhanced Vasicek and CIR model[211] and the (pure diffusion) Fong and Vasicek (1991a) model[212], but no option prices have been presented so far. Due to the general applicability of the solution formula (4.21), we were able to compute numerical solutions for all important interest-rate derivatives. Comparing the different results from the jump-diffusion term-structure models, it is obvious that jump components can enhance the stochastic dynamics. Accordingly, we were able to model probability density functions, which show bimodality and the important feature of fat tails.

[210] See the concluding remarks in Durham (2006) and the comments in Section 7.3.
[211] Compare with the comments in Sections 8.2 and 8.3.
[212] See, for example, Selby and Strickland (1995).

Although the model setup used in this thesis is of the exponential-affine type, the pricing technique can be extended to special non-affine processes, namely to the family of quadratic Gaussian processes due to their exponential-affine structure of the particular characteristic function. As discussed in Chapter 10, this model class cannot be enhanced with jump components since the resulting PDE would then no longer be separable. Another possible way of extending the base model specification, which we briefly discussed, is given by the inclusion of stochastic jump intensities, where the intensity is an affine function of the state vector $\mathbf{x}_t$. However, we have then to numerically determine the coefficient functions $a(z, \tau)$ and $\mathbf{b}(z, \tau)$, which can be a challenging task due to the elaborated jump transforms.

We presented a sophisticated alternative to time-consuming Monte-Carlo simulations, which have to be applied otherwise due to the complicated jump-diffusion dynamics. Combined with the highly efficient FRFT algorithm, this numerical pricing approach offers an accuracy and efficiency, which can be hardly achieved by other methods. However, the methodology is restricted, in this form, to price only European-type derivatives. Thus, possible research can focus on developing a pricing procedure based on the algorithm in this thesis, which is also capable of valuing American-type derivatives. The early-exercise feature of these American-type derivatives might then be implemented by using some sort of time-stepping scheme of the Fourier-transformed derivative value or by using backward induction as proposed in Lord, Fang, Bervoets and Oosterlee (2007). Although we discussed one- and two-factor interest-rate models, we can easily extend the pricing framework to include also jump-enhanced versions of higher factor models, such as e.g. the multi-factor models presented in Balduzzi, Das, Foresi and Sundaram (1996) and Collin-Dufresne and Goldstein (2002). Another possibility for further research might be an empirical validation of the family of gamma jump-enhanced diffusion models, as for example done in the studies by Lin and Yeh (1999) and Das (2002).

# A

# Derivation of the Complex-Valued Coefficients for the Characteristic Function in the Square-Root Model

Our starting point for deriving the time-dependent coefficient function $\tilde{b}(z,\tau)$ is equation (8.10). Thus, making the standard transformation for this type of differential equation, we assume

$$\tilde{b}(z,\tau) = -\frac{1}{c_2(z)} \frac{E(z,\tau)_\tau}{E(z,\tau)}. \tag{A.1}$$

Consequently, substituting the particular expressions in equation (8.10), function $E(z,\tau)$ satisfies the following homogeneous ODE

$$E(z,\tau)_{\tau\tau} = c_1(z)E(z,\tau)_\tau - c_0(z)c_2(z)E(z,\tau), \tag{A.2}$$

with $E(z,0)_\tau = 0$, due to the terminal condition $\tilde{b}(z,0) = 0$. Additionally, we assume for the moment an unspecified constant $E(z,0) = E_0$ and guess a solution of the form

$$E(z,\tau) = e^{\upsilon(z)\tau}.$$

Hence, plugging this function together with its particular derivatives into equation (A.2), we arrive at the so-called characteristic equation for this second order type ODE, which after some simplifications is

$$\upsilon^2(z) - c_1(z)\upsilon(z) + c_0(z)c_2(z) = 0.$$

The solution of this quadratic form is given by

$$\upsilon_\pm(z) = \frac{c_1(z) \pm \vartheta(z)}{2},$$

with $\vartheta(z)$ defined according to Section 8.3. Since the discriminant of the square-root function $\vartheta(z)$ is

$$\kappa^2 + 2\sigma^2 w_1 > 0,$$

the characteristic equation has two different real-valued solutions and therefore the general solution can be represented by the linear combination

$$E(z,\tau) = \Psi_1(z)e^{\upsilon(z)_+\tau} + \Psi_2(z)e^{\upsilon(z)_-\tau}.$$

Consequently, we get for $\tau = 0$ the following terminal conditions

$$\begin{aligned} E(z,0) &= \Psi_1(z) + \Psi_2(z), \\ E(z,0)_\tau &= \Psi_1(z)\upsilon_+(z) + \Psi_2(z)\upsilon_-(z). \end{aligned}$$

Keeping in mind that $E(z,0)_\tau \equiv 0$, we use the two equations above to determine the coefficient functions $\Psi_1(z)$ and $\Psi_2(z)$. Eventually, the solution of $E(z,\tau)$ can be obtained as

$$\begin{aligned} E(z,\tau) &= \frac{E_0 e^{\frac{c_1(z)}{2}\tau}}{\vartheta(z)} \left( \vartheta(z) \frac{e^{\frac{\vartheta(z)}{2}\tau} + e^{-\frac{\vartheta(z)}{2}\tau}}{2} - c_1(z) \frac{e^{\frac{\vartheta(z)}{2}\tau} - e^{-\frac{\vartheta(z)}{2}\tau}}{2} \right) \\ &= \frac{E_0 e^{\frac{c_1(z)}{2}\tau}}{\vartheta(z)} \left( \vartheta(z) \cosh\left[\frac{\vartheta(z)\tau}{2}\right] - c_1(z) \sinh\left[\frac{\vartheta(z)\tau}{2}\right] \right), \end{aligned} \tag{A.3}$$

and the particular derivative with respect to the time-to-maturity variable $\tau$ can be calculated as

$$E(z,\tau)_\tau = -2 \frac{E_0 e^{\frac{c_1(z)}{2}\tau}}{\vartheta(z)} c_0(z) c_2(z) \sinh\left[\frac{\vartheta(z)\tau}{2}\right].$$

Finally, inserting the functions $E(z,\tau)$, now up to a constant $E_0$ determined, and $E(z,\tau)_\tau$ into equation (A.1), we end up with

$$\tilde{b}(z,\tau) = \frac{2c_0(z)\sinh\left[\frac{\vartheta(z)\tau}{2}\right]}{\vartheta(z)\cosh\left[\frac{\vartheta(z)\tau}{2}\right] - c_1(z)\sinh\left[\frac{\vartheta(z)\tau}{2}\right]},$$

which coincides with the solution given in equation (8.11)[1].

Having obtained $\tilde{b}(z,\tau)$, it is a very simple task to derive the coefficient function $a^0(z,\tau)$ because of the approach taken in (A.1). Thus, using a logarithmic integration approach we immediately arrive at

[1] The terminal condition $\tilde{b}(z,0) = 0$ is satisfied, which can be easily justified due to the relation $\sinh[0] = 0$.

$$
\begin{aligned}
a^0(z,\tau) &= -w_0\tau + \kappa\theta \int_0^\tau \left(\tilde{b}(z,s) + \imath z g_0\right) \mathrm{d}s \\
&= (\imath z g_0 \kappa\theta - w_0)\tau - \frac{\kappa\theta}{c_2(z)} \int_0^\tau \frac{E(z,s)_\tau}{E(z,s)} \mathrm{d}s \\
&= (\imath z g_0 \kappa\theta - w_0)\tau - \frac{\kappa\theta}{c_2(z)} \left(\ln\left[E(z,\tau)\right] - \ln\left[E(z,0)\right]\right).
\end{aligned}
\tag{A.4}
$$

Because of the terminal condition $a^0(z,0) \equiv 0$, we must set the constant $E_0 = 1$ in equation (A.3). Eventually, after simplifying the resulting expression in equation (A.4) we are able to state the desired form given in (8.12).

# B

# Derivation of the Complex-Valued Coefficients for the Characteristic Function in the Fong-Vasicek Model

Starting with the time-dependent coefficient function $B(z,\tau)$, we adopt the solution according to equation (8.6). Thus, we exchange the parameter $g_1$ with $\bar{B}$. Subsequently, we show that the derivation of the time-dependent coefficient $A^{01}(z,\tau)$, the volatility-related part of $A^0(z,\tau)$, states no problem on account of logarithmic integration.

Thus, the next task is to recover the coefficient function $C(z,\tau)$. Therefore, plugging in the explicit solution of $B(z,\tau)$ into ODE (9.7) results in

$$\begin{aligned} C(z,\tau)_\tau =& f_1(z) + f_2(z)X(z,\tau) + \frac{1}{2}X(z,\tau)^2 + f_3(z)C(z,\tau) \\ &+ \beta\rho X(z,\tau)C(z,\tau) + \frac{\beta^2}{2}C(z,\tau)^2, \end{aligned} \tag{B.1}$$

with time-independent coefficients $f_i(z)$ according to Section 9.3 and

$$X(z,\tau) = \sigma\left(\frac{w}{\kappa} + \imath z\bar{B}\right)e^{-\kappa\tau}.$$

Similar to the derivation of the time-dependent coefficient function $\tilde{b}(z,\tau)$ in the SR model, we assume for $C(z,\tau)$ a solution of the form

$$C(z,\tau) = -\frac{2}{\beta^2}\frac{U(z,\tau)_\tau}{U(z,\tau)}. \tag{B.2}$$

Inserting this alternative representation into equation (B.1) and simplifying the resulting ODE for the new function $U(z,\tau)$ gives

$$\begin{aligned} U(z,\tau)_{\tau\tau} =& \left(f_3(z) + \beta\rho X(z,\tau)\right)U(z,\tau)_\tau \\ &- \frac{\beta^2}{2}\left(f_1(z) + f_2(z)X(z,\tau) + \frac{1}{2}X(z,\tau)^2\right)U(z,\tau). \end{aligned} \tag{B.3}$$

Subsequently, we apply another substitution

$$V(X(z,\tau)) = U(z,\tau), \tag{B.4}$$

and get a new ODE, with derivatives taken with respect to $X(z,\tau)$. For convenience, we express the particular derivatives with $V(X(z,\tau))_X$ and $V(X(z,\tau))_{XX}$, respectively. Thus, the resulting ODE has the formal structure

$$\begin{aligned} & X(z,\tau)^2 V(X(z,\tau))_{XX} \\ &+ \left( \left( 1 + \frac{f_3(z)}{\kappa} \right) X(z,\tau) + \frac{\beta\rho}{\kappa} X(z,\tau)^2 \right) V(X(z,\tau))_X \\ &+ \frac{\beta^2}{2\kappa^2} \left( f_1(z) + f_2(z) X(z,\tau) + \frac{1}{2} X(z,\tau)^2 \right) V(X(z,\tau)) = 0. \end{aligned} \tag{B.5}$$

Finally, the solution of this particular ODE can be obtained by applying a last substitution of the form

$$V(X(z,\tau)) = L(z,\tau)\, W(Y(z,\tau)),$$

with $L(z,\tau)$ and $Y(z,\tau)$ as defined in Section 9.3. Hence, inserting this substitution into equation (B.5) and simplifying the resulting ODE, we end up with

$$\begin{aligned} & - Q(z) W(Y(z,\tau)) \\ & + (S(z) - Y(z,\tau)) W(Y(z,\tau))_Y \\ & + Y(z,\tau) W(Y(z,\tau))_{YY} = 0. \end{aligned} \tag{B.6}$$

Again, the explicit expressions of $Q(z)$ and $S(z)$ are given in Section 9.3. Equation (B.6) is better known as the prominent Kummer equation, which has the general solution[2]

$$W(Y(z,\tau)) = \Psi_1(z) \mathrm{KM}[Q(z), S(z), Y(z,\tau)] + \Psi_2(z) \mathrm{KU}[Q(z), S(z), Y(z,\tau)].$$

Thus, in order to obtain the solution for the coefficient $C(z,\tau)$, we also need the first derivative with respect to $\tau$ of the function $U(z,\tau)$. Hence, according to the chain rule we have the relation

$$U(z,\tau)_\tau = -\kappa X(z,\tau) V(X(z,\tau))_X.$$

[2] See, for example, Abramowitz and Stegun (1972), p. 504. Our solution is customized to account for the parametric form due to the frequency representation.

The desired derivative of $V(X(z,\tau))$ with respect to $X(z,\tau)$ in the above equation can be represented as

$$\begin{aligned}
V(X(z,\tau))_X = \\
&\left(-\frac{1}{2X(z,\tau)}\right) L(z,\tau) \\
&\times \Big[-2Q(z) \\
&\quad \times \Big(\Psi_1(z)\mathrm{KM}[Q(z)+1, S(z), Y(z,\tau)] \\
&\qquad + (1+Q(z)-S(z))\Psi_2(z)\mathrm{KU}[Q(z)+1, S(z), Y(z,\tau)]\Big) \\
&\quad + \frac{\beta^2}{\kappa} M(z,\tau) \\
&\quad \times \Big(\Psi_1(z)\mathrm{KM}[Q(z), S(z), Y(z,\tau)] \\
&\qquad + \Psi_2(z)\mathrm{KU}[Q(z), S(z), Y(z,\tau)]\Big)\Big].
\end{aligned}$$

Thus, according to the approach taken in equation (B.2), the coefficient function $C(z,\tau)$ can be recovered as (9.8), which is in terms of $V(X(z,\tau))$

$$C(z,\tau) = \frac{2\kappa}{\beta^2} X(z,\tau) \frac{V(X(z,\tau))_X}{V(X(z,\tau))}.$$

Next, checking the validity of the terminal condition

$$C(z,0) = U(z,0)_\tau = V(X(z,0))_X \equiv 0,$$

we only need the explicit form of the time-independent function $\Upsilon(z)$, which is just the fraction

$$\Upsilon(z) = \frac{\Psi_1(z)}{\Psi_2(z)}.$$

Arranging terms for $\Psi_1(z)$ and $\Psi_2(z)$ in the first derivative of $V(X(z,\tau))$ evaluated at $X(z,0) = \sigma\left(\frac{w}{\kappa} + \imath z\bar{B}\right)$, we get

$$\Psi_1(z)\bigg(\frac{\beta^2}{\kappa} M(z,\tau)\mathrm{KM}[Q(z)+1,S(z),Y(z,0)] \\ - 2Q(z)\mathrm{KM}[Q(z)+1,S(z),Y(z,0)]\bigg) = \\ \Psi_2(z)\bigg(2Q(z)(1+Q(z)-S(z))\mathrm{KU}[Q(z),S(z),Y(z,0)] \\ - \frac{\beta^2}{\kappa} M(z,\tau)\mathrm{KU}[Q(z),S(z),Y(z,0)]\bigg).$$

Obviously, solving for the particular fraction, the specific form of $\Upsilon(z)$ can be validated by checking its definition given in Section 9.3. Thus, the coefficient function $C(z,\tau)$ with specified time-independent function $\Upsilon(z)$ coincides with the result given in equation (9.8).

For the calculation of $A^{02}(z,\tau)$, we exploit the functional form chosen in the derivation of the coefficient function $C(z,\tau)$. Thus, we apply a logarithmic integration approach and recover the antiderivative of $A^{02}(z,\tau)_\tau$ as

$$A^{02}(z,\tau) = -\frac{2\alpha\bar{v}}{\beta^2}\ln[U(z,\tau)] + \imath z\alpha\bar{v}\bar{C}\tau.$$

In order to guarantee the terminal condition of

$$A^{02}(z,0) = 0,$$

at the maturity of the contract, we have to ensure that

$$U(z,0) = 1.$$

Thus, rewriting $U(z,\tau)$ as

$$U(z,\tau) = L(z,\tau)\Psi_2(z) \\ \times \left(\Upsilon(z)\mathrm{KM}[Q(z),S(z),Y(z,\tau)] + \mathrm{KM}[Q(z),S(z),Y(z,\tau)]\right),$$

we immediately arrive at

$$\frac{1}{\Psi_2(z)} = L(z,0)\left(\Upsilon(z)\mathrm{KM}[Q(z),S(z),Y(z,0)] + \mathrm{KM}[Q(z),S(z),Y(z,0)]\right).$$

Therefore, the time-dependent function $A^{02}(z,\tau)$ can be written in terms of

$$J(z,\tau) = \frac{2Q(z)\kappa}{\beta^2\left(\mathrm{KU}[Q(z);S(z);Y(z,\tau)] + \Upsilon(z)\,\mathrm{KM}[Q(z);S(z);Y(z,\tau)]\right)}$$

and $L(z,\tau)$, given in (9.11), which concludes the derivation of the coefficient functions in Section 9.3.

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