Topology Lecture Notes

Transcribed by

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Based on the Topology Course by

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Preface

These lecture notes are a transcription of the Topology course delivered by Professor Siavash Shahshahani at Sharif University, where I, Behrooz Moosavi Ramezanzadeh, was an undergraduate student in the Department of Mathematical Sciences. The course introduced topology, the study of properties of spaces preserved under continuous deformations, such as stretching or bending, but not tearing or gluing.

The notes aim to capture the structure and insights of Professor Shahshahani's lectures, presenting them in a clear, mathematically precise, and accessible manner. They are intended for students and researchers seeking a solid foundation in topology, balancing intuition with formal rigor. Examples and exercises reinforce key concepts, and additional topics, such as the Cantor set and Baire category theorem, have been included to provide a comprehensive introduction to metric space topology. Some examples are adapted from Giovanni Leoni's lecture notes [**?**].

I hope these notes serve as a valuable resource for exploring the beauty and depth of topology.

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1 Metric Spaces

1.1 Introduction

Topology studies properties of spaces invariant under continuous transformations. Metric spaces, which quantify distances between points, provide a concrete framework for introducing topological concepts. This chapter covers the fundamentals of metric spaces, including open and closed sets, limit points, and advanced topics like completeness, compactness, and density.

1.2 Definition of a Metric

Definition 1.2.1 (Metric)**.** Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is a **metric** on X if, for all $x, y, z \in X$, it satisfies:

1. **Non-negativity**: $d(x, y) \geq 0$.
2. **Identity of indiscernibles**: $d(x, y) = 0$ if and only if $x = y$.
3. **Symmetry**: $d(x, y) = d(y, x)$.
4. **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is a **metric space**.

1.3 Examples of Metrics

Example 1.3.1 (Euclidean Metric)**.** In $\mathbb{R}^n$, the **Euclidean metric** is:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$.

Example 1.3.2 (Discrete Metric)**.** For any set X, the **discrete metric** is:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Example 1.3.3 (Supremum Metric)**.** Let $C([a, b])$ be the set of continuous functions on $[a, b]$. The **supremum metric** is:

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Since $[a, b]$ is compact, the supremum is finite [**?**].

1.4 Open and Closed Sets

Definition 1.4.1 (Open Ball)**.** In a metric space (X, d), the **open ball** centered at $x \in X$ with radius $r > 0$ is:

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

Definition 1.4.2 (Open Set)**.** A subset $U \subseteq X$ is **open** if, for every $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$.

Example 1.4.1. In $\mathbb{R}$ with the Euclidean metric, (a, b) is open, as $B(x, \min(x - a, b - x)) \subseteq (a, b)$ for $x \in (a, b)$.

Definition 1.4.3 (Closed Set)**.** A subset $C \subseteq X$ is **closed** if $X \setminus C$ is open.

Example 1.4.2. In $\mathbb{R}$, $[a, b]$ is closed, as $(-\infty, a) \cup (b, \infty)$ is open.

1.5 Interior Points and Interior

Definition 1.5.1 (Interior Point and Interior)**.** A point $x \in E \subseteq X$ is an **interior point** of E if there exists $r > 0$ such that $B(x, r) \subseteq E$. The **interior** of E, denoted $\text{int}(E)$, is the set of all interior points of E.

Proposition 1.5.1. *Let (X, d) be a metric space and $E \subseteq X$. Then:*

1. $\text{int}(E)$ *is open.*

2. $\text{int}(E)$ *is the largest open set contained in E.*

3. *E is open if and only if $E = \text{int}(E)$.*

Proof.

1. For $x \in \text{int}(E)$, there exists $r > 0$ such that $B(x, r) \subseteq E$. For any $y \in B(x, r)$, let $s = r - d(x, y) > 0$. If $z \in B(y, s)$, then $d(z, x) \leq d(z, y) + d(y, x) < s + d(y, x) = r$, so $z \in B(x, r) \subseteq E$. Thus, $B(y, s) \subseteq E$, and $y \in \text{int}(E)$. Hence, $\text{int}(E)$ is open.

2. If $U \subseteq E$ is open, then for each $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U \subseteq E$, so $x \in \text{int}(E)$. Thus, $U \subseteq \text{int}(E)$.

3. If E is open, then $E \subseteq \text{int}(E)$ by (2). Since $\text{int}(E) \subseteq E$, we have $E = \text{int}(E)$. Conversely, if $E = \text{int}(E)$, then E is open by (1).

□

Example 1.5.1. In $\mathbb{R}$, for $E = [0, 1]$, $\text{int}(E) = (0, 1)$, as points $0, 1$ have neighborhoods intersecting $\mathbb{R} \setminus E$.

1.6 Closure

Definition 1.6.1 (Closure)**.** The **closure** of a set $E \subseteq X$, denoted $\text{Cl}(E)$, is the smallest closed set containing E, i.e., the intersection of all closed sets containing E.

Proposition 1.6.1. *Let (X, d) be a metric space and $E \subseteq X$. Then:*

1. $\text{Cl}(E)$ *is closed.*

2. *E is closed if and only if $E = \mathrm{Cl}(E)$.*

3. *$x \in \mathrm{Cl}(E)$ if and only if $B(x,r) \cap E \neq \emptyset$ for all $r > 0$.*

Proof. 1. By definition, $\mathrm{Cl}(E)$ is the intersection of closed sets, which is closed.

2. If E is closed, then E is a closed set containing E, so $\mathrm{Cl}(E) \subseteq E$. Since $E \subseteq \mathrm{Cl}(E)$, we have $E = \mathrm{Cl}(E)$. Conversely, if $E = \mathrm{Cl}(E)$, then E is closed by (1).

3. If $x \in \mathrm{Cl}(E)$, suppose there exists $r > 0$ such that $B(x,r) \cap E = \emptyset$. Then $X \setminus B(x,r)$ is closed and contains E, so $\mathrm{Cl}(E) \subseteq X \setminus B(x,r)$, contradicting $x \in \mathrm{Cl}(E)$. Conversely, if $B(x,r) \cap E \neq \emptyset$ for all $r > 0$, and $x \notin \mathrm{Cl}(E)$, then $x \in X \setminus \mathrm{Cl}(E)$, which is open. Thus, there exists $r > 0$ such that $B(x,r) \subseteq X \setminus \mathrm{Cl}(E)$, implying $B(x,r) \cap E = \emptyset$, a contradiction.

□

Example 1.6.1. In $\mathbb{R}$, for $E = (0,1)$, $\mathrm{Cl}(E) = [0,1]$, as every neighborhood of 0 or 1 intersects $(0,1)$.

1.7 Boundary Points

Definition 1.7.1 (Boundary Point)**.** A point $x \in X$ is a **boundary point** of $E \subseteq X$ if every open ball $B(x,r)$ contains points of both E and $X \setminus E$. The **boundary** of E, denoted $\mathrm{Bd}(E)$, is the set of all boundary points of E.

Proposition 1.7.1. *For a set $E \subseteq X$, $\mathrm{Bd}(E) = \mathrm{Cl}(E) \cap \mathrm{Cl}(X \setminus E)$.*

Proof. A point $x \in \mathrm{Bd}(E)$ if and only if for all $r > 0$, $B(x,r) \cap E \neq \emptyset$ and $B(x,r) \cap (X \setminus E) \neq \emptyset$. This is equivalent to $x \in \mathrm{Cl}(E)$ and $x \in \mathrm{Cl}(X \setminus E)$, i.e., $x \in \mathrm{Cl}(E) \cap \mathrm{Cl}(X \setminus E)$. □

Example 1.7.1. In $\mathbb{R}$, for $E = [0,1]$, $\mathrm{Bd}(E) = \{0,1\}$, as neighborhoods of 0 and 1 contain points in $[0,1]$ and $\mathbb{R} \setminus [0,1]$.

1.8 The Cantor Set

Definition 1.8.1 (Cantor Set)**.** The **Cantor set** $C \subset [0,1]$ is constructed as follows:

- Start with $C_0 = [0,1]$.
- Remove the open middle third $(1/3, 2/3)$, yielding $C_1 = [0,1/3] \cup [2/3,1]$.
- From each interval in C_n, remove the open middle third, forming C_{n+1}.
- Define $C = \bigcap_{n=0}^{\infty} C_n$.

Proposition 1.8.1. *The Cantor set C is:*

1. *Closed, as it is the intersection of closed sets.*

2. *Uncountable, with cardinality $2^{\aleph_0}$.*

3. *Perfect, i.e., every point is a limit point, and C has no isolated points.*

4. *Nowhere dense, i.e.,* $\text{int}(\text{Cl}(C)) = \emptyset$.

Proof. 1. Each C_n is closed, so $C = \bigcap C_n$ is closed.

2. Points in C correspond to sequences in $\{0, 2\}^{\mathbb{N}}$ (via ternary expansions), which has cardinality $2^{\aleph_0}$.

3. For any $x \in C$, every neighborhood contains points of C_n, hence other points of C. No point is isolated, as C is infinite and constructed iteratively.

4. Since $C \subseteq [0, 1]$, $\text{Cl}(C) = C$. As C contains no intervals, $\text{int}(C) = \emptyset$.

□

Example 1.8.1. The Cantor set consists of points with ternary expansions using digits 0 and 2, e.g., $0 = 0.000\ldots$, $1 = 0.222\ldots$.

1.9 Finite, Infinite, Countable, and Uncountable Sets

Definition 1.9.1. A set S is:

- **Finite** if it has a bijection with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
- **Infinite** if it is not finite.
- **Countable** if it has a bijection with $\mathbb{N}$ (countably infinite) or is finite.
- **Uncountable** if it is infinite and not countable.

Example 1.9.1. $\mathbb{N}$ and $\mathbb{Q}$ are countable; $\mathbb{R}$ and the Cantor set are uncountable.

1.10 Schröder-Bernstein Theorem

Theorem 1.10.1 (Schröder-Bernstein)**.** *If there exist injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \to B$.*

Proof. Define a graph where vertices are elements of $A \cup B$, with edges $a \to f(a)$ and $b \to g(b)$. Each component is a path (finite, infinite, or cyclic). Partition A into A_1 (paths starting in A), A_2 (paths starting in B), and A_3 (cyclic paths). Similarly, partition B. Define $h : A \to B$:

- For $a \in A_1$, follow the path to $h(a) = f(a)$.
- For $a \in A_2$, there exists $b \in B$ such that $g(b) = a$, so set $h(a) = b$.
- For $a \in A_3$, set $h(a) = f(a)$.

This h is a bijection, as each element is mapped uniquely and covers B. □

1.11 Cantor's Theorem

Theorem 1.11.1 (Cantor)**.** *For any set A, there is no surjection from A to its power set $\mathcal{P}(A)$.*

Proof. Suppose $f : A \to \mathcal{P}(A)$ is a surjection. Define $B = \{a \in A \mid a \notin f(a)\}$. Since f is surjective, there exists $b \in A$ such that $f(b) = B$. If $b \in B$, then $b \notin f(b) = B$, a contradiction. If $b \notin B$, then $b \in f(b) = B$, also a contradiction. Thus, no such f exists. □

Corollary 1.11.2. *The cardinality of $\mathcal{P}(A)$ is strictly greater than that of A.*

1.12 Sequence Convergence and Properties

Definition 1.12.1 (Sequence Convergence)**.** A sequence $\{x_n\} \subset X$ **converges** to $x \in X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

Proposition 1.12.1. *In a metric space, the limit of a convergent sequence is unique.*

Proof. Suppose $x_n \to x$ and $x_n \to y$. For $\epsilon > 0$, there exist N_1, N_2 such that $d(x_n, x) < \epsilon/2$ for $n \geq N_1$ and $d(x_n, y) < \epsilon/2$ for $n \geq N_2$. For $n \geq \max(N_1, N_2)$, $d(x, y) \leq d(x, x_n) + d(x_n, y) < \epsilon/2 + \epsilon/2 = \epsilon$. Since ϵ is arbitrary, $d(x, y) = 0$, so $x = y$. □

1.13 Cauchy Sequences and Complete Metric Spaces

Definition 1.13.1 (Cauchy Sequence)**.** A sequence $\{x_n\} \subset X$ is **Cauchy** if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 1.13.2 (Complete Metric Space)**.** A metric space (X, d) is **complete** if every Cauchy sequence converges to a point in X.

Proposition 1.13.1. *Every convergent sequence is Cauchy.*

Proof. If $x_n \to x$, for $\epsilon > 0$, there exists N such that $d(x_n, x) < \epsilon/2$ for $n \geq N$. Then, for $n, m \geq N$, $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$. □

Example 1.13.1. $\mathbb{R}^n$ and $C([a, b])$ with the supremum metric are complete; $\mathbb{Q}$ is not [**?**].

1.14 Heine-Borel Theorem

Theorem 1.14.1 (Heine-Borel)**.** *In $\mathbb{R}^n$ with the Euclidean metric, a set is compact if and only if it is closed and bounded.*

Proof. If $K \subseteq \mathbb{R}^n$ is compact, it is closed (as the limit of any convergent sequence in K lies in K) and bounded (otherwise, a sequence with unbounded distances exists, contradicting sequential compactness). Conversely, if K is closed and bounded, it is contained in some cube $[a, b]^n$, which is compact. Since K is a closed subset of a compact set, it is compact. □

1.15 Completeness of Discrete and Cartesian Product Spaces

Proposition 1.15.1. *A discrete metric space is complete.*

Proof. In a discrete metric space, $d(x, y) = 1$ if $x \neq y$. If $\{x_n\}$ is Cauchy, there exists N such that $d(x_n, x_m) < 1$ for $n, m \geq N$, implying $x_n = x_m$. Thus, x_n is eventually constant, converging to some $x \in X$. □

Proposition 1.15.2. *The Cartesian product of complete metric spaces is complete.*

Proof. Let (X, d_X) and (Y, d_Y) be complete, with product metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$. If $\{(x_n, y_n)\}$ is Cauchy in $X \times Y$, then $d_X(x_n, x_m)^2 \leq d((x_n, y_n), (x_m, y_m))^2$, so $\{x_n\}$ is Cauchy in X. Similarly, $\{y_n\}$ is Cauchy in Y. Since X and Y are complete, $x_n \to x \in X$, $y_n \to y \in Y$. Thus, $(x_n, y_n) \to (x, y)$, and $X \times Y$ is complete [?]. □

1.16 Cantor Intersection Theorem

Theorem 1.16.1 (Cantor Intersection)**.** *In a complete metric space* (X, d)*, let* $\{F_n\}$ *be a decreasing sequence of non-empty closed sets with* $\operatorname{diam}(F_n) \to 0$*. Then* $\bigcap_{n=1}^{\infty} F_n$ *contains exactly one point.*

Proof. Since $F_n \supseteq F_{n+1}$, pick $x_n \in F_n$. For $n, m \geq N$, $x_n, x_m \in F_N$, so $d(x_n, x_m) \leq \operatorname{diam}(F_N)$. As $\operatorname{diam}(F_n) \to 0$, $\{x_n\}$ is Cauchy. Since X is complete, $x_n \to x \in X$. As $x_n \in F_N$ for $n \geq N$, and F_N is closed, $x \in F_N$ for all N. Thus, $x \in \bigcap F_n$. If $y \in \bigcap F_n$, then $d(x, y) \leq \operatorname{diam}(F_n) \to 0$, so $x = y$. □

1.17 Dense Sets and Properties

Definition 1.17.1 (Dense Set)**.** A set $E \subseteq X$ is **dense** if $\mathrm{Cl}(E) = X$. A metric space is **separable** if it has a countable dense subset.

Example 1.17.1. $\mathbb{Q}$ is dense in $\mathbb{R}$; $C([a, b])$ is separable, with piecewise affine functions dense [?].

1.18 Baire Category Theorem

Theorem 1.18.1 (Baire Category)**.** *A complete metric space is not a countable union of nowhere dense sets.*

Proof. Suppose $X = \bigcup_{n=1}^{\infty} E_n$, where each E_n is nowhere dense, i.e., $\operatorname{int}(\mathrm{Cl}(E_n)) = \emptyset$. Choose an open ball $B(x_1, r_1)$. Since $\mathrm{Cl}(E_1)$ has empty interior, there exists $B(x_2, r_2) \subseteq B(x_1, r_1) \setminus \mathrm{Cl}(E_1)$ with $r_2 < r_1/2$. Inductively, find $B(x_n, r_n) \subseteq B(x_{n-1}, r_{n-1}) \setminus \mathrm{Cl}(E_n)$ with $r_n < r_{n-1}/2$. The sequence $\{x_n\}$ is Cauchy (as $d(x_n, x_m) < r_n$), and since X is complete, $x_n \to x \in \bigcap B(x_n, r_n)$. Since $x \notin \mathrm{Cl}(E_n)$ for any n, $x \notin \bigcup E_n$, contradicting $X = \bigcup E_n$. □

1.19 Equivalent Metrics

Definition 1.19.1 (Equivalent Metrics)**.** Two metrics d_1, d_2 on X are **equivalent** if they induce the same topology, i.e., a set is open in (X, d_1) if and only if it is open in (X, d_2).

Example 1.19.1. In $\mathbb{R}^n$, the Euclidean metric $d_2(x, y) = \sqrt{\sum (x_i - y_i)^2}$, taxicab metric $d_1(x, y) = \sum |x_i - y_i|$, and supremum metric $d_\infty(x, y) = \max |x_i - y_i|$ are equivalent, as open balls in each generate the same open sets.

1.20 Continuous Functions

Definition 1.20.1 (Continuous Function)**.** A function $f : (X, d_X) \to (Y, d_Y)$ is **continuous** at $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$. It is continuous if it is continuous at every $x \in X$.

Proposition 1.20.1. *The following are equivalent for $f : X \to Y$:*

1. *f is continuous.*
2. *The preimage $f^{-1}(U)$ is open in X for every open set $U \subseteq Y$.*
3. *$f^{-1}(C)$ is closed in X for every closed set $C \subseteq Y$.*

Proof.

1. $\Rightarrow$ 2: If f is continuous and $U \subseteq Y$ is open, for $x \in f^{-1}(U)$, $f(x) \in U$. There exists $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq U$. By continuity, there exists $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$, so $f(B(x, \delta)) \subseteq U$. Thus, $B(x, \delta) \subseteq f^{-1}(U)$, and $f^{-1}(U)$ is open.
2. $\Rightarrow$ 3: If $C \subseteq Y$ is closed, then $Y \setminus C$ is open. By (2), $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is open, so $f^{-1}(C)$ is closed.
3. $\Rightarrow$ 1: For $x \in X$ and $\epsilon > 0$, $B(f(x), \epsilon)$ is open in Y. By (2), $f^{-1}(B(f(x), \epsilon))$ is open and contains x. Thus, there exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$, so $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$.

□

Example 1.20.1. The identity function $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ is continuous, as $\mathrm{id}^{-1}((a, b)) = (a, b)$, which is open.

1.21 Exercises

1. Prove that in a discrete metric space, every set is open and closed.
2. Show that $\operatorname{int}(E) \cup \operatorname{int}(F) \subseteq \ interior(E \cup F)$, and give an example where equality fails.
3. Prove that the Cantor set has Lebesgue measure zero.
4. Verify that $\ell^p(\mathbb{N})$ is complete for $1 \leq p < \infty$.
5. Show that a continuous function on a compact metric space is uniformly continuous.

2 Advanced Metric Space Properties

2.1 Pseudometrics and Quotient Spaces

Definition 2.1.1 (Pseudometric)**.** A **pseudometric** on a set X is a map $\rho : X \times X \to [0, \infty)$ such that:

1. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$,

2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$,

3. $\rho(x, x) = 0$ for all $x \in X$.

A pair (X, ρ) is a **pseudometric space**.

Example 2.1.1.

1. On $\mathcal{R}([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f \text{ Riemann integrable}\}$, define $\rho(f, g) = \int_a^b |f(x) - g(x)|\, dx$. This is a pseudometric, as $\rho(f, g) = 0$ if $f = g$ almost everywhere.

2. For any function $f : X \to \mathbb{R}$, define $\rho_f(x, y) = |f(x) - f(y)|$. This is a pseudometric on X.

3. On $\mathbb{R}^X$, fix $x_0 \in X$, and define $\rho_{x_0}(f, g) = |f(x_0) - g(x_0)|$. This is a pseudometric.

4. On $\mathcal{L}^p([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f \text{ Lebesgue measurable}, \int_a^b |f(x)|^p\, dx < \infty\}$, define $\rho_p(f, g) = \left(\int_a^b |f(x) - g(x)|^p\, dx\right)^{1/p}$. This is a pseudometric.

Definition 2.1.2 (Quotient Space from Pseudometric)**.** Let (X, ρ) be a pseudometric space. Define an equivalence relation $x \sim y$ if $\rho(x, y) = 0$. The **quotient space** is $Y = X/\sim = \{[x] \mid x \in X\}$, where $[x] = \{z \in X \mid \rho(x, z) = 0\}$. Define a metric on Y by:

$$d([x], [y]) = \rho(x, y).$$

Exercise 2.1.1. Let (X, ρ) be a pseudometric space.

1. Prove that $d([x], [y]) = \rho(x, y)$ is well-defined (independent of representatives).

2. Prove that d is a metric on Y.

2.2 Infinite Sums and ℓ^p Spaces

Definition 2.2.1 (Infinite Sum)**.** For a set X and a function $f : X \to [0, \infty]$, the **infinite sum** is:

$$\sum_{x\in X} f(x) = \sup \left\{ \sum_{x\in Y} f(x) \mid Y \subset X, Y \text{ finite} \right\}.$$

Proposition 2.2.1. *If $\sum_{x\in X} f(x) < \infty$, then $\{x \in X \mid f(x) > 0\}$ is countable, say $\{x_n\}_n$, and:*

$$\sum_{x\in X} f(x) = \sum_n f(x_n),$$

where the right-hand side is a finite sum or a convergent series. Moreover, f does not take the value ∞.

Proof. Let $M = \sum_{x\in X} f(x) < \infty$. For $k \in \mathbb{N}$, define $X_k = \{x \in X \mid f(x) > \frac{1}{k}\}$. For a finite subset $Y \subset X_k$,

$$\frac{1}{k}|Y| \leq \sum_{x\in Y} f(x) \leq M,$$

so $|Y| \leq \lfloor kM \rfloor$. Thus, X_k is finite, and $\{x \in X \mid f(x) > 0\} = \bigcup_{k=1}^{\infty} X_k$ is countable. If $f(x) = \infty$, then for any n, taking $Y = \{x\}$, $\sum_{x\in Y} f(x) = \infty$, contradicting $M < \infty$. For countable $\{x_n \mid f(x_n) > 0\}$, the sum $\sum_n f(x_n)$ equals the supremum over all finite subsets. $\square$

Exercise 2.2.1. Let $f : [a, b] \to \mathbb{R}$ be increasing. Prove that the set of discontinuity points of f is countable.

Exercise 2.2.2. For a set X and functions $f, g : X \to [0, \infty]$:

1. Prove $\sum_{x\in X}(f(x) + g(x)) \leq \sum_{x\in X} f(x) + \sum_{x\in X} g(x)$.
2. If $f \leq g$, prove $\sum_{x\in X} f(x) \leq \sum_{x\in X} g(x)$.

Definition 2.2.2 ($\ell^p(X)$ Spaces)**.** For a set X and $1 \leq p < \infty$, the space $\ell^p(X)$ is:

$$\ell^p(X) = \left\{ f : X \to \mathbb{R} \mid \sum_{x\in X} |f(x)|^p < \infty \right\},$$

with metric:

$$d_p(f, g) = \left(\sum_{x\in X} |f(x) - g(x)|^p \right)^{1/p}.$$

For $p = \infty$, define:

$$\ell^\infty(X) = \left\{ f : X \to \mathbb{R} \mid \sup_{x\in X} |f(x)| < \infty \right\},$$

with metric:

$$d_\infty(f, g) = \sup_{x\in X} |f(x) - g(x)|.$$

For $X = \mathbb{N}$, $\ell^p(\mathbb{N}) = \{(a_n) \subset \mathbb{R} \mid \sum_{n=1}^{\infty} |a_n|^p < \infty\}$.

Proposition 2.2.2 (Young's Inequality). *For $1 < p < \infty$, with Hölder conjugate $q = \frac{p}{p-1}$, and $a, b \geq 0$:*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Proof. If $a = 0$ or $b = 0$, the inequality holds. For $a, b > 0$, since $\ln t$ is concave and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \geq \frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q = \ln(ab).$$

Thus, $\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab$. □

Theorem 2.2.3 (Hölder's Inequality). *For a set X, $1 \leq p \leq \infty$, with Hölder conjugate q, for $f, g : X \to \mathbb{R}$:*

$$\sum_{x\in X} |f(x)g(x)| \leq \left(\sum_{x\in X} |f(x)|^p\right)^{1/p} \left(\sum_{x\in X} |g(x)|^q\right)^{1/q}$$

if $1 < p < \infty$, and:

$$\sum_{x\in X} |f(x)g(x)| \leq \left(\sum_{x\in X} |f(x)|\right) \sup_{x\in X} |g(x)|$$

if $p = 1$. If $f \in \ell^p(X)$, $g \in \ell^q(X)$, then $fg \in \ell^1(X)$.

Proof. For $1 < p < \infty$, if $\sum |f(x)|^p = 0$ or $\sum |g(x)|^q = 0$, the result is trivial. Assume both sums are finite and positive. Apply Young's inequality with:

$$a = \frac{|f(x)|}{\left(\sum |f(y)|^p\right)^{1/p}}, \quad b = \frac{|g(x)|}{\left(\sum |g(y)|^q\right)^{1/q}}.$$

Then:

$$\frac{|f(x)g(x)|}{\left(\sum |f(y)|^p\right)^{1/p}\left(\sum |g(y)|^q\right)^{1/q}} \leq \frac{1}{p}\frac{|f(x)|^p}{\sum |f(y)|^p} + \frac{1}{q}\frac{|g(x)|^q}{\sum |g(y)|^q}.$$

Summing over X:

$$\frac{\sum |f(x)g(x)|}{\left(\sum |f(y)|^p\right)^{1/p}\left(\sum |g(y)|^q\right)^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

For $p = 1$, $q = \infty$, use $|f(x)g(x)| \leq |f(x)| \sup |g(y)|$. □

Theorem 2.2.4 (Minkowski's Inequality). *For a set X, $1 \leq p < \infty$, and $f, g : X \to \mathbb{R}$:*

$$\left(\sum_{x\in X} |f(x) + g(x)|^p\right)^{1/p} \leq \left(\sum_{x\in X} |f(x)|^p\right)^{1/p} + \left(\sum_{x\in X} |g(x)|^p\right)^{1/p}.$$

If $f, g \in \ell^p(X)$, then $f + g \in \ell^p(X)$.

Proof. For $1 < p < \infty$, assume both sums are finite. By convexity of $t \mapsto t^p$,

$$|f(x) + g(x)|^p \leq 2^{p-1}(|f(x)|^p + |g(x)|^p).$$

Thus, $f + g \in \ell^p(X)$. Then:

$$\sum |f(x) + g(x)|^p \leq \sum |f(x)||f(x) + g(x)|^{p-1} + \sum |g(x)||f(x) + g(x)|^{p-1}.$$

Apply Hölder's inequality to each term, noting $(p-1)q = p$, and divide by $\left(\sum |f(x) + g(x)|^p\right)^{1/q}$. For $p = 1$, use the triangle inequality. □

2.3 Separability

Definition 2.3.1 (Separability)**.** A metric space (X, d) is **separable** if it contains a countable dense subset, i.e., there exists a countable set $D \subset X$ such that for every $x \in X$ and $\epsilon > 0$, there exists $d \in D$ with $d(x, d) < \epsilon$.

Example 2.3.1.

1. $\mathbb{R}^N$ is separable, as $\mathbb{Q}^N$ is a countable dense subset.
2. A set X with the discrete metric is separable if and only if X is countable.
3. $C([a, b])$ with the sup norm $d(f, g) = \sup_{x\in[a,b]} |f(x)-g(x)|$ is separable, as piecewise affine functions with rational slopes and endpoints are dense.
4. $\ell^\infty(\mathbb{N})$ is not separable.
5. $\ell^p(\mathbb{N})$, $1 \le p < \infty$, is separable via sequences with finitely many rational entries.
6. If X is uncountable, $\ell^p(X)$, $1 \le p < \infty$, is not separable.

Exercise 2.3.1. Prove that $\ell^\infty(\mathbb{N})$ is not separable. *Hint*: Construct an uncountable set of sequences with pairwise distances bounded below.

Exercise 2.3.2. Prove that if X is uncountable, then $\ell^p(X)$, $1 \le p < \infty$, is not separable.

2.4 Completeness

Definition 2.4.1 (Cauchy Sequence and Bounded Set)**.** A sequence $\{x_n\} \subset X$ in a metric space (X, d) is **Cauchy** if:

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

A set $E \subset X$ is **bounded** if there exists a ball $B(x_0, r)$ such that $E \subset B(x_0, r)$.

Exercise 2.4.1. Let (X, d) be a metric space and $\{x_n\} \subset X$.

1. If $\{x_n\}$ converges to $x \in X$, prove it is Cauchy.
2. If $\{x_n\}$ is Cauchy, prove $\{x_n \mid n \in \mathbb{N}\}$ is bounded.

Exercise 2.4.2. Let (X, d) be a metric space and $\{x_n\} \subset X$.

1. If $\{x_n\}$ is Cauchy and a subsequence $\{x_{n_k}\}$ converges to $x \in X$, prove $\{x_n\}$ converges to x.
2. If for every subsequence $\{x_{n_k}\}$, there exists a further subsequence $\{x_{n_{k_j}}\}$ converging to x, prove $\{x_n\}$ converges to x.

Proposition 2.4.1. *Let (X, d) be a complete metric space and $C \subset X$. Then C is closed if and only if (C, d) is complete.*

Proof. If C is closed, a Cauchy sequence $\{x_n\} \subset C$ converges in X to x. Since C is closed, $x \in C$. If (C, d) is complete, let $\{x_n\} \subset C$ converge to $x \in X$. Then $\{x_n\}$ is Cauchy in C, so converges to $y \in C$. By uniqueness, $x = y \in C$, so C is closed. □

Example 2.4.1. 1. $\ell^p(X)$, $1 \le p < \infty$, is complete. For a Cauchy sequence $\{f_n\}$, $f_n(x) \to f(x)$ pointwise, and $d_p(f, f_m) \to 0$.

2. $\ell^\infty(X)$ is complete. For a Cauchy sequence $\{f_n\}$, $f_n(x) \to f(x)$, and $\sup |f(x) - f_m(x)| \to 0$.

3. $C([a, b])$ with $d(f, g) = \sup |f(x) - g(x)|$ is complete, as it is closed in $\ell^\infty([a, b])$.

4. $C([a, b])$ with $d(f, g) = \int_a^b |f(x) - g(x)|\, dx$ is not complete.

2.5 Compactness in Metric Spaces

Definition 2.5.1 (Compactness)**.** A subset $K \subset X$ in a metric space (X, d) is **compact** if every open cover of K has a finite subcover.

Theorem 2.5.1. *For a subset $K \subset X$ in a metric space (X, d), the following are equivalent:*

1. *K is compact.*
2. *K is sequentially compact (every sequence has a convergent subsequence).*
3. *K is complete and totally bounded (for every $\epsilon > 0$, K is covered by finitely many balls of radius ϵ).*

Proof. (i) $\Rightarrow$ (ii): If $\{x_n\} \subset K$ has no convergent subsequence, the set $C = \{x_n \mid n \in \mathbb{N}\}$ has no accumulation points and is closed. Define $C_m = \{x_n \mid n \ge m\}$, so $\bigcap C_m = \emptyset$. The sets $U_m = X \backslash C_m$ form an open cover of K, but no finite subcover exists, contradicting compactness.

(ii) $\Rightarrow$ (iii): If K is not totally bounded, there exists $\epsilon_0 > 0$ and a sequence $\{x_n\} \subset K$ with $d(x_n, x_m) \ge \epsilon_0$. This sequence has no convergent subsequence, contradicting sequential compactness. For completeness, a Cauchy sequence in K has a convergent subsequence, which converges in K, so the sequence converges.

(iii) $\Rightarrow$ (i): For an open cover $\{U_\alpha\}$, assume no finite subcover exists. For each k, K is covered by finitely many balls of radius $\frac{1}{2^k}$. If for each k, some ball is not contained in any U_α, pick x_k from such a ball. A convergent subsequence $x_{k_j} \to x \in U_\alpha$ leads to a contradiction, as small balls around x_{k_j} are in U_α. □

Exercise 2.5.1. Prove that $\overline{\mathbb{R}} = [-\infty, \infty]$ admits a metric making it compact. *Hint*: Consider the stereographic projection or arctangent.

Exercise 2.5.2. Prove that a compact metric space is separable and complete.

Theorem 2.5.2 (Weierstrass)**.** *Let (X, d) be a metric space, $K \subset X$ compact, and $f : X \to \mathbb{R}$ continuous. Then there exist $x_0, x_1 \in K$ such that:*

$$f(x_0) = \min_{x \in K} f(x), \quad f(x_1) = \max_{x \in K} f(x).$$

Proof. Let $t = \inf_{x \in K} f(x)$. If the infimum is not attained, for each $x \in K$, choose $t_x > t$, $t_x < f(x)$. The sets $U_x = \{y \in X \mid f(y) > t_x\}$ cover K. A finite subcover $U_{x_1}, \dots, U_{x_l}$ implies $f(x) \ge \min t_{x_i} > t$, contradicting the definition of t. Similarly for the maximum. □

Theorem 2.5.3. *Let (X, d_X), (Y, d_Y) be metric spaces, $K \subset X$ compact, and $f : K \to Y$ continuous. Then f is uniformly continuous.*

Proof. For $\epsilon > 0$, for each $x \in K$, there exists $\delta_x > 0$ such that $d_Y(f(x), f(z)) < \epsilon$ if $d_X(x, z) < \delta_x$. The cover $\{B(x, \frac{\delta_x}{2})\}$ has a finite subcover $B(x_i, \frac{\delta_{x_i}}{2})$. Set $\delta = \min \frac{\delta_{x_i}}{2}$. For $d_X(x, z) < \delta$, choose x_i with $x \in B(x_i, \frac{\delta_{x_i}}{2})$. Then $d_X(z, x_i) < \delta_{x_i}$, so:

$$d_Y(f(x), f(z)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(z)) < 2\epsilon.$$

□

Proposition 2.5.4. *If $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is compact.*

Exercise 2.5.3. Let $K \subset \mathbb{R}^N$ be closed and bounded. Prove K is compact.

2.6 Ascoli-Arzelà Theorem

Definition 2.6.1 (Equicontinuity)**.** A family $\mathcal{F}$ of functions $f : X \to Y$ between metric spaces is **equicontinuous** at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$d_Y(f(x), f(x_0)) \leq \epsilon$$

for all $f \in \mathcal{F}$, $d_X(x, x_0) \leq \delta$. It is **uniformly equicontinuous** if δ is independent of x_0.

Definition 2.6.2 (Pointwise Bounded)**.** A family $\mathcal{F} \subset C(X)$ is **pointwise bounded** if for every $x \in X$, there exists $M_x > 0$ such that:

$$|f(x)| \leq M_x$$

for all $f \in \mathcal{F}$.

Example 2.6.1.

1. The sequence $f_n(x) = x^n$, $x \in [0, 1]$, is pointwise bounded but not equicontinuous at $x = 1$.
2. The sequence $f_n(x) = \frac{x^n}{n}$, $x \in [0, 1]$, is pointwise bounded and equicontinuous, as $|f_n'(x)| \leq \frac{1}{n}$.

Theorem 2.6.1 (Ascoli-Arzelà)**.** *Let (X, d) be a separable metric space and $\mathcal{F} \subset C(X)$ be pointwise bounded and equicontinuous. Then every sequence in $\mathcal{F}$ has a subsequence converging uniformly on compact subsets of X to a continuous function $g : X \to \mathbb{R}$.*

Proof. Let $E = \{x_k\} \subset X$ be countable and dense. For a sequence $\{f_n\} \subset \mathcal{F}$, extract a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x_i) \to \ell_i \in \mathbb{R}$ for each $x_i \in E$, using a diagonal argument. On a compact $K \subset X$, for $\epsilon > 0$, equicontinuity gives $\delta > 0$ such that $|f(x) - f(y)| \leq \epsilon$ if $d(x, y) \leq \delta$. Cover K with finitely many balls $B(y_i, \frac{\delta}{2})$, and choose $z_i \in E \cap B(y_i, \frac{\delta}{2})$. For large n, m, $|f_n(z_i) - f_m(z_i)| \leq \epsilon$. For $x \in K$, $x \in B(y_i, \frac{\delta}{2})$, so:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(z_i)| + |f_n(z_i) - f_m(z_i)| + |f_m(z_i) - f_m(x)| \leq 3\epsilon.$$

Thus, $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$, converging to $g(x)$. Uniform convergence follows, and g is continuous by equicontinuity. □

Corollary 2.6.2. *If (X, d) is compact, then $\mathcal{F}$ is relatively compact in $C(X)$ with the sup norm.*

2.7 Stone-Weierstrass Theorem

Theorem 2.7.1 (Dini). *Let (X, d) be compact and $\{f_n\} \subset C(X)$ satisfy $f_n(x) \leq f_{n+1}(x)$, converging pointwise to a continuous $f : X \to \mathbb{R}$. Then the convergence is uniform.*

Proof. For $\epsilon > 0$, for each $x \in X$, there exists n_x such that $0 \leq f(x) - f_n(x) \leq \epsilon$ for $n \geq n_x$. Continuity gives a ball $B(x, r_x)$ where $0 \leq f(y) - f_n(y) \leq 3\epsilon$. A finite cover $B(x_i, r_{x_i})$ and $n_\epsilon = \max n_{x_i}$ ensure $\sup |f(y) - f_n(y)| \leq 3\epsilon$ for $n \geq n_\epsilon$. □

Example 2.7.1. The sequence $f_n(x) = -x^n$, $x \in [0, 1]$, converges pointwise to a discontinuous function, and convergence is not uniform.

Theorem 2.7.2 (Stone-Weierstrass). *Let (X, d) be a compact metric space and $\mathcal{F} \subset C(X)$ satisfy:*

1. ***Separates points**: For $x \neq y$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*
2. ***Contains constants**: Constant functions are in $\mathcal{F}$.*
3. ***Algebra**: If $f, g \in \mathcal{F}$, $t \in \mathbb{R}$, then $f + g$, fg, $tf \in \mathcal{F}$.*

Then $\mathcal{F}$ is dense in $C(X)$ with the sup norm.

Proof. The closure $\overline{\mathcal{F}}$ is an algebra. For $f \in \overline{\mathcal{F}}$, approximate $|f|$ by polynomials in $\sqrt{t}$. Thus, $\max\{f, g\}, \min\{f, g\} \in \overline{\mathcal{F}}$. For $x \neq y$, $\alpha, \beta \in \mathbb{R}$, construct $g \in \overline{\mathcal{F}}$ with $g(x) = \alpha$, $g(y) = \beta$. For $f \in C(X)$, $\epsilon > 0$, build $g_y \in \overline{\mathcal{F}}$ such that $f(z) - \epsilon < g_y(z) < f(z) + \epsilon$. A finite cover ensures $g = \max g_{y_i}$ approximates f. □

Corollary 2.7.3. *Let $K \subset \mathbb{R}^N$ be compact. Every $f \in C(K)$ is the uniform limit of polynomials.*

Corollary 2.7.4. *If (X, d) is compact, then $C(X)$ is separable.*

Exercise 2.7.1. Prove that $C_b(\mathbb{R})$, the space of bounded continuous functions on $\mathbb{R}$, is not separable.

Exercise 2.7.2. For $f(x) = \sqrt{x}$, $x \in [0, 1]$, define $p_0(x) = 0$, $p_n(x) = p_{n-1}(x) + \frac{1}{2}[x - p_{n-1}^2(x)]$.

1. Prove each p_n is a polynomial.
2. Prove $|p_n(x)| \leq 1$.
3. Prove $\{p_n\}$ converges uniformly to f.

2.8 Additional Exercises

1. In $\mathbb{R}^N$, prove every non-empty open set is a countable union of balls.

2. For a metric space (X, d), $x_0 \in X$, $r > 0$:

   (a) Prove $\operatorname{Cl} B(x_0, r) \subset \{x \in X \mid d(x_0, x) \leq r\}$.

   (b) Show equality may not hold (e.g., in a discrete metric space).

3. Characterize the completion of $C_c(X) = \{f \in C(X) \mid \operatorname{supp} f \text{ is compact}\}$.

4. Prove that $[0, 1]$ and $(0, 1)$ are not homeomorphic. *Hint*: Consider compactness.

3 Topological Spaces

This chapter introduces topological spaces, which generalize metric spaces by defining open sets abstractly. We explore their structures, convergence properties, continuity, compactness, and semicontinuity, laying the foundation for advanced topological analysis.

3.1 Topological Spaces and Their Structures

We begin by defining topological spaces and their key structures, including quotient topologies, subbases, and neighborhoods.

Definition 3.1.1. Let X be a nonempty set. A collection $\tau \subset \P(X)$ is a **topology** if the following hold.

(i) $\emptyset, X \in \tau$.

(ii) If $U_i \in \tau$ for $i = 1, \dots, M$, then $U_1 \cap \dots \cap U_M \in \tau$.

(iii) If $\{U_\alpha\}_{\alpha \in \Lambda}$ is an arbitrary collection of elements of τ, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$.

The pair (X, τ) is called a **topological space**, and elements of τ are **open sets**. For simplicity, we often apply the term topological space only to X.

Example 3.1.1. (i) Given a nonempty set X, the smallest topology consists of $\{\emptyset, X\}$, while the largest topology contains all subsets as open sets.

(ii) Given a metric space (X, d), the family of open sets is a topology.

(iii) Given a topological space (X, τ_X) and an onto function $f : X \to Y$, the family of sets

$$\tau_Y := \{E \subset Y \mid f^{-1}(E) \in \tau_X\}$$

is a topology on Y. It is called the **quotient topology** (relative to f and τ_X). Given a nonempty set X, let $\sim$ be an equivalence relation. We define

$$Y = X/\sim := \{[x] \mid x \in X\}$$

and consider the projection of X onto Y

$$\begin{aligned} P : X &\to Y \\ x &\mapsto [x] \end{aligned}$$

If X is a topological space with topology τ, we can consider in Y the quotient topology (relative to P and τ), precisely,

$$\tau_Y := \{E \subset Y \mid P^{-1}(E) \in \tau\}.$$

Note that

$$P^{-1}(E) = \{x \in X \mid [x] \in E\} = \bigcup_{[x]\in E} [x],$$

that $P^{-1}(E)$ is given by the union of the equivalence classes belonging to E. Thus, an open set in the quotient topology is a collection of equivalence classes whose union is an open set of X.

Remark 3.1.1. *As Example 3.1.1 (i) shows, a set X can have more than one topology. If τ is any topology on X, then*

$$\{\emptyset, X\} \subset \tau \subset \P(X).$$

*If τ_1 and τ_2 are topologies on X, we say τ_1 is **weaker**, or **coarser**, than τ_2 if $\tau_1 \subset \tau_2$.*

Remark 3.1.2. *Given a family of topologies $\{\tau_\alpha\}_{\alpha\in\Lambda}$ on a set X, the family of sets*

$$\bigcap_{\alpha\in\Lambda} \tau_\alpha := \{U \subset X \mid U \in \tau_\alpha \textit{ for every } \alpha \in \Lambda\}$$

is a topology on X, while in general $\bigcup_{\alpha\in\Lambda} \tau_\alpha$ is not.

Proposition 3.1.3. *Let X be a set and let $\mathcal{F}$ be a family of subsets of X. Then there exists a unique, smallest topology τ containing $\mathcal{F}$. Moreover, τ consists of X, $\emptyset$, finite intersections of elements of $\mathcal{F}$, and arbitrary unions of finite intersections of elements of $\mathcal{F}$.*

Proof. The family $\mathcal{F}$ is called a **subbase** for τ, and τ is said to be generated by $\mathcal{F}$. Let $\{\tau_\alpha\}_{\alpha\in\Lambda}$ be the family of all topologies that contain $\mathcal{F}$. This family is nonempty since $\P(X)$ is one such topology. Then

$$\tau := \bigcap_{\alpha\in\Lambda} \tau_\alpha$$

is a topology, contains $\mathcal{F}$, and is the smallest such topology. It is unique by definition. By the properties of a topology, τ contains X, $\emptyset$, all finite intersections of elements of $\mathcal{F}$, and all arbitrary unions of finite intersections of $\mathcal{F}$. Conversely, let τ' be the family consisting of X, $\emptyset$, all finite intersections of elements of $\mathcal{F}$, and arbitrary unions of finite intersections of $\mathcal{F}$. Then τ' is a topology. □

Example 3.1.2. (i) In $\mathbb{R}$, consider the family $\mathcal{F} = \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$. The smallest topology containing $\mathcal{F}$ is the standard topology.

(ii) Given topological spaces (X, τ_X) and (Y, τ_Y), consider the family $\mathcal{F} = \{U \times V \mid U \in \tau_X, V \in \tau_Y\}$. The smallest topology containing $\mathcal{F}$ on $X \times Y$ is called the **product topology**.

Definition 3.1.2. Given a point $x \in X$, a **neighborhood**[1] of x is an open set $U \in \tau$ that contains x. Given a set $E \subset X$, a neighborhood of E is an open set $U \in \tau$ that contains E.

Definition 3.1.3. Given a topological space (X, τ) and a set $E \subset X$, a point $x \in E$ is called an **interior point** of E if there exists a neighborhood U of x such that $U \subset E$. The **interior** E° of a set $E \subset X$ is the union of all its interior points.

[1] In some texts, the definition of neighborhood is different.

3.2 Bases and Closure

We now introduce bases for topologies and properties of closed sets and closure, which are essential for understanding convergence and compactness.

Proposition 3.2.1. *Let (X, τ) be a topological space and let $E \subset X$. Then*

(i) E° is an open subset of E,

(ii) E° is the union of all open subsets contained in E; that is, E° is the largest (in the sense of union) open set contained in E,

(iii) E is open if and only if $E = E^\circ$,

(iv) $(E^\circ)^\circ = E^\circ$.

Proof. The proof is left as an exercise. □

Definition 3.2.1. Let (X, τ) be a topological space. A family β of open sets of X is a **base** for the topology τ if every open set $U \in \tau$ may be written as the union of elements of β.

Definition 3.2.2. Given a topological space (X, τ) and a point $x \in X$, a family β_x of neighborhoods of x is a **local base** at x if every neighborhood of x contains an element of β_x.

Proposition 3.2.2. *Let X be a nonempty set and let $\beta \subset \P(X)$ be a family of sets. Then β is a base for a topology τ if and only if*

(i) it contains the empty set;

(ii) for every $x \in X$ there exists $B \in \beta$ such that $x \in B$,

(iii) for every $B_1, B_2 \in \beta$ with $B_1 \cap B_2 \neq \emptyset$ and for every $x \in B_1 \cap B_2$ there exists $B_3 \in \beta$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Proof. Assume τ is a topology and β is a base for τ. Every open set is a union of sets of β. In particular, the empty set and X can be written as unions of sets of β, so (i) and (ii) hold. For (iii), if $B_1, B_2 \in \beta$, then $B_1 \cap B_2$ is open, so it can be written as a union of elements of β, say

$$B_1 \cap B_2 = \bigcup_\gamma B_\gamma.$$

Hence, if $x \in B_1 \cap B_2$, there is a B_γ such that $x \in B_\gamma \subset B_1 \cap B_2$.

Conversely, let $\beta = \{B_\alpha\}_{\alpha \in \Lambda}$ satisfy (i)–(iii), and let τ be the set of arbitrary unions of elements of β. By (i), $\emptyset \in \tau$. By (ii),

$$X = \bigcup_{\alpha \in \Lambda} B_\alpha,$$

so $X \in \tau$. If $U_i \in \tau$ for $i = 1, \ldots, M$, write

$$U_i = \bigcup_{\alpha \in \Lambda_i} B_\alpha$$

for some $\Lambda_i \subset \Lambda$. Then

$$U_1 \cap \cdots \cap U_M = \bigcap_{i=1}^{M} \bigcup_{\alpha \in \Lambda_i} B_\alpha.$$

If $x \in U_1 \cap \cdots \cap U_M$, there exist $\alpha_i \in \Lambda_i$ such that $x \in B_{\alpha_1} \cap \cdots \cap B_{\alpha_M}$. By (iii) and induction, there exists $B_x \in \beta$ such that $x \in B_x \subset B_{\alpha_1} \cap \cdots \cap B_{\alpha_M}$. Hence

$$U_1 \cap \cdots \cap U_M = \bigcup_{x \in U_1 \cap \cdots \cap U_M} B_x \in \tau.$$

For an arbitrary collection $\{U_\gamma\}_{\gamma \in \mathbb{B}} \subset \tau$, since each U_γ is a union of elements of β, $\bigcup_{\gamma \in \mathbb{B}} U_\gamma$ is a union of elements of β, so it belongs to τ. Thus, τ is a topology, and β is a base for τ by definition. □

Example 3.2.1. Examples of bases include:

(i) In a metric space (X, d), the family

$$\beta = \{B(x, r) \mid x \in X, r > 0\}$$

satisfies properties (i) and (ii) of Proposition 3.2.2, yielding a topology τ with open sets as arbitrary unions of open balls.

(ii) In a pseudometric space (X, ρ), a topology τ is constructed with a local base at each $x \in X$ of pseudoballs

$$\{y \in X \mid \rho(x, y) < r\}, \quad r > 0.$$

(iii) For a set X, $\mathbb{R}^X$ denotes all functions $f : X \to \mathbb{R}$. For $f \in \mathbb{R}^X$, $r > 0$, and finite $Y \subset X$, define

$$B(f; r; Y) := \{g \in \mathbb{R}^X \mid |g(x) - f(x)| < r \text{ for all } x \in Y\}.$$

The family $\beta = \{B(f; r; Y) \mid f \in \mathbb{R}^X, r > 0, Y \subset X \text{ finite}\}$ is a base for a topology τ on $\mathbb{R}^X$.

(iv) For a set X and a family $\mathcal{F} \subset \P(X)$, let τ be the smallest topology containing $\mathcal{F}$. A basis for τ is given by finite intersections of elements of $\mathcal{F}$.

Exercise 3.2.1. Let (X, ρ) be a pseudometric space, $Y = X/\sim$ as in Example 3.1.1(iii), and $d([x], [y]) = \rho(x, y)$.

(i) Prove that a set $U \subset X$ is open (with respect to the topology τ_X) if and only if $P(U)$ is open (with respect to the metric d).

(ii) Prove that the topology induced by d is the quotient topology.

Exercise 3.2.2. For $(x, y) \in \mathbb{R}^2$, consider the family of rectangles $[x, x + r) \times [y, y + t]$, $r, t > 0$. Let

$$\beta = \{[x, x + r) \times [y, y + t] \mid (x, y) \in \mathbb{R}^2, r, t > 0\}.$$

Prove that β is a base for a topology τ on $\mathbb{R}^2$ that is not the standard topology.

Exercise 3.2.3. Given a set X and metrics $d_1, d_2 : X \times X \to [0, \infty)$, explore relations among:

(i) d_1 and d_2 are equivalent, i.e., for every sequence $\{x_n\} \subset X$ and $x \in X$,

$$\lim_{n\to\infty} d_1(x_n, x) = 0 \text{ if and only if } \lim_{n\to\infty} d_2(x_n, x) = 0.$$

(ii) For every $x \in X$ and $r > 0$, there exist $r_1, r_2 > 0$ such that

$$B_{d_2}(x, r_1) \subset B_{d_1}(x, r) \subset B_{d_2}(x, r_2).$$

(iii) d_1 and d_2 generate the same topology.

Definition 3.2.3. A set $C \subset X$ is **closed** if its complement $X \setminus C$ is open. The **closure** $\overline{E}$ of $E \subset X$ is the smallest closed set containing E. A set E is **dense** if $\overline{E} = X$. A topological space is **separable** if it contains a countable dense subset.

Proposition 3.2.3. *Let (X, τ) be a topological space. Then*

(i) $\emptyset$ and X are closed.

(ii) If $C_i \subset X$, $i = 1, \ldots, n$, is a finite family of closed sets, then $C_1 \cup \cdots \cup C_n$ is closed.

(iii) If $\{C_\alpha\}_{\alpha\in\Lambda}$ is an arbitrary collection of closed sets, then $\bigcap_{\alpha\in\Lambda} C_\alpha$ is closed.

Proof. Follows from De Morgan's laws. □

Proposition 3.2.4. *Let (X, τ) be a topological space and $E \subset X$. Then $x \in \overline{E}$ if and only if $E \cap U$ is nonempty for every neighborhood U of x.*

Proposition 3.2.5. *Let (X, τ) be a topological space and $\{E_\alpha\}_\alpha$ a family of subsets of X. Then*

$$\bigcup_\alpha \overline{E_\alpha} \subset \overline{\bigcup_\alpha E_\alpha},$$

with equality if the family is finite.

Proposition 3.2.6. *Let (X, τ) be a topological space and $C \subset X$. Then C is closed if and only if $C = \overline{C}$.*

3.3 Convergence and Accumulation Points

We explore convergence in topological spaces, introducing Hausdorff spaces and the role of accumulation points.

Definition 3.3.1. A point $x_0 \in X$ is an **accumulation point** for $E \subset X$ if every open set U containing x_0 contains $x \in E \cap U$, $x \neq x_0$. The set of accumulation points is denoted $\operatorname{acc} E$.

Proposition 3.3.1. *Let (X, τ) be a topological space and $E \subset X$. Then*

$$\overline{E} = E \cup \operatorname{acc} E.$$

In particular, $C \subset X$ is closed if and only if C contains all its accumulation points.

Definition 3.3.2. A topological space (X, τ) is a **Hausdorff space** if for any $x, y \in X$ with $x \neq y$, there exist disjoint neighborhoods of x and y.

Proposition 3.3.2. *Let (X, d) be a metric space and τ the topology determined by d. Then (X, τ) is a Hausdorff space.*

Proof. If $x \neq y$, then $B\left(x, \frac{d(x,y)}{2}\right)$ and $B\left(y, \frac{d(x,y)}{2}\right)$ are disjoint neighborhoods of x and y. □

Definition 3.3.3. In a topological space (X, τ), a sequence $\{x_n\}$ **converges** to $x \in X$ if for every neighborhood U of x, $x_n \in U$ for all n sufficiently large. A subset $C \subset X$ is **sequentially closed** if for every sequence $\{x_n\} \subset C$ converging to $x \in X$, $x \in C$.

Proposition 3.3.3. *Let (X, τ) be a Hausdorff space. If $\{x_n\}$ converges to x and to y, then $x = y$.*

Proof. If $x \neq y$, there exist disjoint neighborhoods U and V of x and y. Since $\{x_n\}$ converges to x, $x_n \in U$ for large n, so $x_n \notin V$, contradicting convergence to y. □

Proposition 3.3.4. *Let (X, τ) be a topological space and $C \subset X$ a closed set. Then C is sequentially closed.*

Proof. If C is closed and $\{x_n\} \subset C$ converges to $x \in X$, suppose $x \notin C$. Since $X \setminus C$ is open, there exists a neighborhood $U \subset X \setminus C$ of x. Then $x_n \in U \subset X \setminus C$ for large n, contradicting $\{x_n\} \subset C$. Thus, $x \in C$. □

Definition 3.3.4. The **sequential closure** of $E \subset X$ is

$$\overline{E}^{\text{seq}} := \{x \in X \mid \text{there exists } \{x_n\} \subset E \text{ converging to } x\}.$$

Proposition 3.3.5. *Let (X, τ) be a topological space and $E \subset X$. Then*

$$E \subset \overline{E}^{seq} \subset \overline{E}.$$

Proof. Since $\overline{E}$ is closed, it is sequentially closed by Proposition 3.3.4, so $\overline{E}^{\text{seq}} \subset \overline{E}$. □

Exercise 3.3.1. Let $(\mathbb{R}^{[0,1]}, \tau)$ be as in Example 3.2.1(iii), with $E \subset \mathbb{R}^{[0,1]}$ consisting of functions f that are zero at finitely many $x \in [0, 1]$ and 1 otherwise.

(i) Prove that $f_0 \equiv 0$ belongs to $\overline{E}$.

(ii) Prove that $f_0 \notin \overline{E}^{\text{seq}}$.

(iii) Prove that no metric on $\mathbb{R}^{[0,1]}$ is compatible with τ.

Definition 3.3.5. Let (X, τ) be a topological space.

(i) X satisfies the **first axiom of countability** if every $x \in X$ has a countable local base.

(ii) X satisfies the **second axiom of countability** if it has a countable base.

Example 3.3.1. An uncountable set X with the discrete topology τ satisfies the first axiom of countability, as $\{\{x\}\}$ is a local base for $x \in X$. It does not satisfy the second, as singletons in τ are uncountable.

Example 3.3.2. A metric space (X, d) satisfies the first axiom of countability with local base $\{B(x, \frac{1}{n})\}_{n\in\mathbb{N}}$ at x, but not necessarily the second.

Exercise 3.3.2. Let (X, τ) satisfy the second axiom of countability. Prove that X is separable.

Proposition 3.3.6. *Let (X, τ) satisfy the first axiom of countability and $E \subset X$. Then*

$$\overline{E}^{seq} = \overline{E}.$$

Proof. By Proposition 3.3.5, it suffices to show $\overline{E} \subset \overline{E}^{\text{seq}}$. For $x \in \overline{E}$, let $\{B_n\}_n$ be a countable local base at x, with $B_n \supset B_{n+1}$ (replace B_n with $B_1 \cap \cdots \cap B_n$). By Proposition 3.2.4, $E \cap B_n \neq \emptyset$, so pick $x_n \in E \cap B_n$. For any neighborhood U of x, there exists $\overline{n}$ such that $B_{\overline{n}} \subset U$. Since $B_n \subset B_{\overline{n}} \subset U$ for $n \geq \overline{n}$, $x_n \in U$, so $\{x_n\}$ converges to x. Thus, $x \in \overline{E}^{\text{seq}}$. □

3.4 Limits and Continuity

We define limits and continuity in topological spaces, generalizing metric space concepts, and introduce limit inferior and superior.

Definition 3.4.1. Let (X, τ_X) and (Y, τ_Y) be topological spaces, $f : E \to Y$, $E \subset X$, and $x_0 \in \operatorname{acc} E$. If there exists $y_0 \in Y$ such that for every neighborhood $V \subset Y$ of y_0, there exists a neighborhood $U \subset X$ of x_0 with

$$f(x) \in V \quad \text{for all } x \in U \cap (E \setminus \{x_0\}),$$

we write

$$y_0 = \lim_{x\to x_0} f(x),$$

and y_0 is the **limit** of f as x approaches x_0.

Remark 3.4.1. *The point x_0 need not be in E. It suffices to take V in a local base of y_0 and U in a local base of x_0.*

Proposition 3.4.2. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, Y Hausdorff, $f : E \to Y$, $E \subset X$, and $x_0 \in \operatorname{acc} E$. If $\lim_{x\to x_0} f(x) = y_1$ and $\lim_{x\to x_0} f(x) = y_2$, then $y_1 = y_2$.*

Proof. If $y_1 \neq y_2$, there exist disjoint neighborhoods V_1, V_2 of y_1, y_2. By the limit definition, there exist neighborhoods U_1, U_2 of x_0 such that $f(x) \in V_1$ for $x \in U_1 \cap (E \setminus \{x_0\})$ and $f(x) \in V_2$ for $x \in U_2 \cap (E \setminus \{x_0\})$. Since $x_0 \in \operatorname{acc} E$, Proposition 3.2.4 implies there exists $x \in U_1 \cap U_2 \cap (E \setminus \{x_0\})$. Then $f(x) \in V_1 \cap V_2$, a contradiction since $V_1 \cap V_2 = \emptyset$. □

Proposition 3.4.3. *Let (X, τ_X), (Y, τ_Y) be topological spaces, $f : E \to Y$, $E \subset X$, and $x_0 \in \operatorname{acc} E$. If $\lim_{x\to x_0} f(x) = y_0$, then $f(x_n) \to y_0$ for every sequence $\{x_n\} \subset E \setminus \{x_0\}$ converging to x_0.*

Proof. Let $\{x_n\} \subset E \setminus \{x_0\}$ converge to x_0. For a neighborhood $V \subset Y$ of y_0, there exists a neighborhood $U \subset X$ of x_0 such that $f(x) \in V$ for $x \in U \cap (E \setminus \{x_0\})$. Since $x_n \to x_0$, there exists $n_x \in \mathbb{N}$ such that $x_n \in U$ for $n \geq n_x$, so $f(x_n) \in V$, proving $f(x_n) \to y_0$. □

Proposition 3.4.4. *Let (X, τ_X) satisfy the first axiom of countability, (Y, τ_Y) be a topological space, $f : E \to Y$, $E \subset X$, and $x_0 \in E \cap \operatorname{acc} E$. If there exists $y_0 \in Y$ such that $f(x_n) \to y_0$ for every sequence $\{x_n\} \subset E \setminus \{x_0\}$ converging to x_0, then $\lim_{x\to x_0} f(x) = y_0$.*

Proof. Suppose $\lim_{x\to x_0} f(x) \neq y_0$. There exists a neighborhood $V \subset Y$ of y_0 such that for every neighborhood $U \subset X$ of x_0, there is $x \in U\cap(E\setminus\{x_0\})$ with $f(x) \notin V$. Let $\{B_n\}_n$ be a countable local base at x_0, with $B_{n+1} \subset B_n$. For each n, choose $x_n \in B_n\cap(E\setminus\{x_0\})$ with $f(x_n) \notin V$. Since $\{B_n\}$ is decreasing, $\{x_n\}$ converges to x_0. By hypothesis, $f(x_n) \to y_0$, so $f(x_n) \in V$ for large n, a contradiction. $\square$

Definition 3.4.2. Let (X, τ) be a topological space, $f : E \to \mathbb{R}$, $E \subset X$, and $x_0 \in \operatorname{acc} E$. The **limit inferior** is

$$\liminf_{x\to x_0} f(x) := \sup_{U\in\tau(x_0)} \inf_{x\in U\cap(E\setminus\{x_0\})} f(x),$$

and the **limit superior** is

$$\limsup_{x\to x_0} f(x) := \inf_{U\in\tau(x_0)} \sup_{x\in U\cap(E\setminus\{x_0\})} f(x),$$

where $\tau(x_0)$ is the collection of neighborhoods of x_0.[2]

Remark 3.4.5. *If $U, V \in \tau(x_0)$ with $U \subset V$, then*

$$\inf_{x\in V\cap(E\setminus\{x_0\})} f(x) \leq \inf_{x\in U\cap(E\setminus\{x_0\})} f(x).$$

Since we take the supremum over $\tau(x_0)$, we can focus on small neighborhoods. We could replace $\tau(x_0)$ with a local base at x_0. Similarly for $\limsup$.

Theorem 3.4.6. *Let (X, τ) be a topological space, $f : E \to \mathbb{R}$, $E \subset X$, and $x_0 \in \operatorname{acc} E$. Then*

$$\liminf_{x\to x_0} f(x) \leq \limsup_{x\to x_0} f(x).$$

There exists $\lim_{x\to x_0} f(x)$ if and only if equality holds, and the limit equals the common value.

Proof. For neighborhoods U, V of x_0,

$$\inf_{x\in U\cap(E\setminus\{x_0\})} f(x) \leq \inf_{x\in U\cap V\cap(E\setminus\{x_0\})} f(x) \leq \sup_{x\in U\cap V\cap(E\setminus\{x_0\})} f(x) \leq \sup_{x\in V\cap(E\setminus\{x_0\})} f(x).$$

Taking $\sup_U$ gives

$$\liminf_{x\to x_0} f(x) \leq \sup_{x\in V\cap(E\setminus\{x_0\})} f(x).$$

Taking $\inf_V$ yields the inequality.

Suppose $\lim_{x\to x_0} f(x) = \ell \in \mathbb{R}$. For $\epsilon > 0$, there exists a neighborhood U_ϵ of x_0 such that

$$\ell - \epsilon \leq f(x) \leq \ell + \epsilon \quad \text{for } x \in U_\epsilon \cap (E \setminus \{x_0\}).$$

Thus,

$$\ell - \epsilon \leq \inf_{x\in U_\epsilon\cap(E\setminus\{x_0\})} f(x) \leq \liminf_{x\to x_0} f(x),$$

$$\limsup_{x\to x_0} f(x) \leq \sup_{x\in U_\epsilon\cap(E\setminus\{x_0\})} f(x) \leq \ell + \epsilon.$$

[2] In some books, $\sup_{U\in\tau(x_0)} \inf_{x\in U} f(x)$ defines $\liminf$. Our definition aligns with limits, excluding $f(x_0)$.

Let $\epsilon \to 0^+$ to get

$$\liminf_{x\to x_0} f(x) = \limsup_{x\to x_0} f(x) = \ell.$$

The cases $\ell = \pm\infty$ are exercises.

Conversely, if

$$\liminf_{x\to x_0} f(x) = \limsup_{x\to x_0} f(x) = L \in \mathbb{R},$$

fix $\epsilon > 0$. There exist neighborhoods U_ϵ, V_ϵ of x_0 such that

$$L - \epsilon \le \inf_{x\in U_\epsilon\cap(E\setminus\{x_0\})} f(x), \qquad \sup_{x\in V_\epsilon\cap(E\setminus\{x_0\})} f(x) \le L + \epsilon.$$

For $U = U_\epsilon \cap V_\epsilon$, we have $L-\epsilon \le f(x) \le L+\epsilon$ for $x \in U \cap (E\setminus\{x_0\})$, so $\lim_{x\to x_0} f(x) = L$. The cases $L = \pm\infty$ are exercises. □

Exercise 3.4.1. Let (X,τ) be a topological space, $f, g : E \to \mathbb{R}$, $E \subset X$, $x_0 \in \operatorname{acc} E$, and one of f, g bounded. Prove

$$\liminf_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x) \le \liminf_{x\to x_0}(f(x)+g(x)) \le \limsup_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x)$$

$$\le \limsup_{x\to x_0}(f(x)+g(x)) \le \limsup_{x\to x_0} f(x) + \limsup_{x\to x_0} g(x),$$

and all inequalities may be strict. If $\lim_{x\to x_0} f(x) = \ell \in \mathbb{R}$, prove

$$\liminf_{x\to x_0}(f(x)+g(x)) = \liminf_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x),$$

$$\limsup_{x\to x_0}(f(x)+g(x)) = \limsup_{x\to x_0} f(x) + \limsup_{x\to x_0} g(x).$$

Corollary 3.4.7. *Let (X,τ) be a topological space, $f : E \to \mathbb{R}$, $E \subset X$, and $x_0 \in \operatorname{acc} E$. A necessary and sufficient condition for $\lim_{x\to x_0} f(x)$ to exist in $\mathbb{R}$ is that for every $\epsilon > 0$, there exists a neighborhood U_ϵ of x_0 such that*

$$|f(x_1) - f(x_2)| \le \epsilon \quad \textit{for all } x_1, x_2 \in U_\epsilon \cap (E \setminus \{x_0\}).$$

Definition 3.4.3. Let (X,τ_X), (Y,τ_Y) be topological spaces, $f : E \to Y$, $E \subset X$, and $x_0 \in E$. The function f is **continuous** at x_0 if for every neighborhood $V \subset Y$ of $f(x_0)$, there exists a neighborhood $U \subset X$ of x_0 such that

$$f(x) \in V \quad \text{for all } x \in U \cap E.$$

The function f is continuous if it is continuous at every point of E. Denote by $C(X;Y)$ the space of continuous $f : X \to Y$. If $Y = \mathbb{R}$, write $C(X)$. If Y is a metric space, $C_b(X;Y)$ denotes bounded continuous functions with metric

$$d_\infty(f,g) := \sup_{x\in X} d_Y(f(x), g(x)).$$

Remark 3.4.8. *If $x_0 \in E$ is an isolated point (i.e., there exists $U \subset X$ with $U \cap E = \{x_0\}$), f is continuous at x_0 (take $U = U_0$). Thus, check continuity at $x_0 \in E \cap \operatorname{acc} E$, where continuity is equivalent to*

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Proposition 3.4.9. *Let (X, τ_X), (Y, τ_Y) be topological spaces, $f : E \to Y$, $E \subset X$, and $x_0 \in E \cap \operatorname{acc} E$. If f is continuous at x_0, then f is* ***sequentially continuous*** *at x_0, i.e., $f(x_n) \to f(x_0)$ for every sequence $\{x_n\} \subset E$ converging to x_0.*

Proposition 3.4.10. *Let (X, τ_X) satisfy the first axiom of countability, (Y, τ_Y) be a topological space, $f : E \to Y$, $E \subset X$, and $x_0 \in E \cap \operatorname{acc} E$. If f is sequentially continuous at x_0, then f is continuous at x_0.*

Exercise 3.4.2. Let (X, τ_X), (Y, τ_Y) be topological spaces and $f : X \to Y$. Prove the following are equivalent:

(i) f is continuous.

(ii) $f^{-1}(U)$ is open for every open $U \subset Y$.

(iii) $f^{-1}(C)$ is closed for every closed $C \subset Y$.

(iv) $f^{-1}(B)$ is open for every $B \subset Y$ in a base (or subbase) of τ_Y.

Exercise 3.4.3. Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be topological spaces, $f : X \to Y$ and $g : Y \to Z$ continuous. Prove $g \circ f : X \to Z$ is continuous.

Definition 3.4.4. Let (X, τ_X), (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is a **homeomorphism** if it is bijective, continuous, and $f^{-1} : Y \to X$ is continuous. Spaces (X, τ_X), (Y, τ_Y) are **homeomorphic** if there exists a homeomorphism between them. A **topological property** is preserved under homeomorphisms. A function $f : X \to Y$ is **open** if $f(U)$ is open for every open $U \subset X$, and **closed** if $f(C)$ is closed for every closed $C \subset X$.

3.5 Compactness and Semicontinuity

Compactness and semicontinuity are crucial for optimization and functional analysis in topological spaces.

Definition 3.5.1. Let (X, τ) be a topological space, $f : E \to \mathbb{R}$, $E \subset X$.

(i) f is **lower semicontinuous** at $x_0 \in E$ if x_0 is isolated or $x_0 \in \operatorname{acc} E$ and

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

f is lower semicontinuous if it is so at every point of E.

(ii) f is **upper semicontinuous** at x_0 if $-f$ is lower semicontinuous at x_0. f is upper semicontinuous if it is so at every point.

An **isolated point** $x_0 \in E$ has a neighborhood $U \subset X$ with $U \cap E = \{x_0\}$.

Example 3.5.1. For

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ a & \text{if } x = 0, \end{cases}$$

we have $\liminf_{x \to 0} f(x) = -1$, $\limsup_{x \to 0} f(x) = 1$. Thus, f is lower semicontinuous at 0 if $a \leq -1$, upper semicontinuous if $a \geq 1$, and neither if $-1 < a < 1$.

Definition 3.5.2. For $f : E \to \mathbb{R}$, $E \subset X$, the **epigraph** is

$$\operatorname{epi} f := \{(x,t) \in E \times \mathbb{R} \mid f(x) \leq t\}.$$

Proposition 3.5.1. *Let (X, τ) be a topological space and $f : X \to \mathbb{R}$. The following are equivalent:*

(i) $\{x \in X \mid f(x) \leq t\}$ *is closed for every* $t \in \mathbb{R}$.

(ii) $\operatorname{epi} f$ *is closed.*

(iii) f *is lower semicontinuous.*

Proof. **Step 1: (i) $\iff$ (ii).** Assume (i). Let $D = (X \times \mathbb{R}) \setminus \operatorname{epi} f = \{(x,t) \in X \times \mathbb{R} \mid f(x) > t\}$. For $(x_0, t_0) \in D$, choose $0 < \epsilon < f(x_0) - t_0$. The set $U = f^{-1}((t_0 + \epsilon, \infty))$ is open and contains x_0. For $(x,t) \in U \times (t_0 - \epsilon, t_0 + \epsilon)$, $f(x) > t_0 + \epsilon > t$, so $(x,t) \in D$. Thus, D is open, and $\operatorname{epi} f$ is closed.

Conversely, if $\operatorname{epi} f$ is closed, D is open. For $t_0 \in \mathbb{R}$, let $U = \{x \in X \mid f(x) > t_0\}$. If $x_0 \in U$, choose $0 < \epsilon_0 < f(x_0) - t_0$. Then $(x_0, t_0 + \epsilon_0) \in D$, so there exist a neighborhood U_0 of x_0 and $0 < \epsilon \leq \epsilon_0$ with $U_0 \times (t_0 + \epsilon_0 - \epsilon, t_0 + \epsilon_0 + \epsilon) \subset D$. For $x \in U_0$, $f(x) > t_0 + \epsilon_0 - \epsilon > t_0$, so $U_0 \subset U$, and U is open, proving (i).

Step 2: (i) $\iff$ (iii). Assume (i) and suppose $f(x_0) > \liminf_{x \to x_0} f(x)$. Choose $f(x_0) > t > \liminf_{x \to x_0} f(x)$. The set $U_t = \{x \in X \mid f(x) > t\}$ is open and contains x_0, so

$$\liminf_{x \to x_0} f(x) = \sup_{U \in \tau(x_0)} \inf_{x \in U \setminus \{x_0\}} f(x) \geq \inf_{x \in U_t \setminus \{x_0\}} f(x) \geq t,$$

a contradiction. Conversely, assume (iii). For $t \in \mathbb{R}$ and $U_t = \{x \in X \mid f(x) > t\}$ nonempty, let $x_0 \in U_t$. Since $\liminf_{x \to x_0} f(x) \geq f(x_0) > t$, there exists $U \in \tau(x_0)$ with $\inf_{x \in U \setminus \{x_0\}} f(x) > t$. With $f(x_0) > t$, $U \subset U_t$, so U_t is open, proving (i). $\square$

Definition 3.5.3. Let (X, τ) be a topological space.

(i) A set $K \subset X$ is **compact** if every open cover $\{U_\alpha\}$ with $\bigcup_\alpha U_\alpha \supset K$ has a finite subcover.

(ii) $K \subset X$ is **sequentially compact** if every sequence $\{x_n\} \subset K$ has a subsequence converging to a point in K.

(iii) $E \subset X$ is **relatively compact** if $\overline{E}$ is compact.

Remark 3.5.2. *For topologies $\tau_1 \subset \tau_2$ on X, if $K \subset X$ is compact with respect to τ_2, it is compact with respect to τ_1. Fewer open sets make compactness easier.*

Proposition 3.5.3. *Let (X, τ) be a Hausdorff space, $K \subset X$ compact, and $x_0 \in X \setminus K$. There exist disjoint open sets $U \supset K$ and $V \ni x_0$.*

Proof. Since X is Hausdorff, for each $x \in K$, there exist neighborhoods U_x of x and V_x of x_0 with $U_x \cap V_x = \emptyset$. The open cover $\{U_x\}_{x \in K}$ of K has a finite subcover $U_{x_1}, \ldots, U_{x_m}$. Set $U = U_{x_1} \cup \cdots \cup U_{x_m} \supset K$ and $V = V_{x_1} \cap \cdots \cap V_{x_m}$, a neighborhood of x_0. Then $U \cap V = \emptyset$. $\square$

Proposition 3.5.4. *Let (X, τ) be a topological space and $K \subset X$ compact.*

(i) If $C \subset K$ is closed, C is compact.

(ii) If X is Hausdorff, K is closed.

Proof. For (i), let $\{U_\alpha\}$ be an open cover of C. Since C is closed, $X \setminus C$ is open, so $\{U_\alpha\} \cup \{X \setminus C\}$ covers K. By compactness, there is a finite subcover, which restricts to a finite subcover of C. For (ii), Proposition 3.5.3 shows every $x_0 \in X \setminus K$ has a neighborhood not intersecting K, so $X \setminus K$ is open, and K is closed. □

Example 3.5.2. In a set X with the trivial topology, any nonempty proper subset is compact but not closed.

Example 3.5.3. In $[0,1]^{[0,1]}$ with the topology from Example 3.2.1(iii), the space is compact (by Tychonoff's theorem, proved later) but not sequentially compact.

Exercise 3.5.1. Prove $[0,1]^{[0,1]}$ is not sequentially compact.

3.6 Compactness Properties

We begin by examining compactness and sequential compactness in pseudometric and topological spaces, culminating in the Weierstrass theorem for lower semicontinuous functions.

Exercise 3.6.1. Let X be the family of all nonempty countable subsets of $\mathbb{R}$. For $E, F \in X$, define

$$\rho(E,F) := \begin{cases} \min\{1, \operatorname{dist}(E \setminus F, F)\} & \text{if } F \subsetneq E, \\ 0 & \text{if } F = E, \\ 1 & \text{otherwise.} \end{cases}$$

(i) Prove that (X, ρ) is a pseudometric space.

(ii) Let $\{E_n\} \subset X$ be a sequence such that $E_n \supset \mathbb{Q}$ for every $n \in \mathbb{N}$. Prove that $\{E_n\}$ converges to

$$E := \bigcup_{n=1}^{\infty} E_n.$$

(iii) Prove that X is sequentially compact.

(iv) For every $E \in X$, prove that the set $¶(E) \setminus \{\emptyset\} \subset X$ is open in the topology determined by ρ.

(v) Prove that X is not compact.

Exercise 3.6.2. Let X be the first uncountable ordinal with its well-ordering. Define

$$\rho(x,y) := \begin{cases} 0 & \text{if } x \geq y, \\ 1 & \text{if } x < y. \end{cases}$$

(i) Prove that (X, ρ) is a pseudometric space.

(ii) Prove that X is sequentially compact. Hint: prove that every sequence contains a monotone subsequence.

(iii) Prove that X is not compact.

Proposition 3.6.1. *Let (X, τ_X) be a topological space and let $K \subset X$ be a compact set. Then every infinite subset of K has an accumulation point.*

Proof. Let $E \subset K$ be an infinite set and assume by contradiction that E has no accumulation points. Then E is closed. Moreover, for every $x \in E$, there exists a neighborhood U_x of x such that $U_x \cap E = \{x\}$. The family $\{U_x\}_{x\in E} \cup \{X \setminus E\}$ covers the compact set K. Hence, there exist $x_1, \dots, x_m \in E$ such that

$$U_{x_1} \cup \cdots \cup U_{x_m} \cup \{X \setminus E\} \supset K \supset E.$$

Since $X \setminus E$ does not intersect E and each U_{x_i} intersects E only at x_i, E cannot have more than m elements, a contradiction. □

Proposition 3.6.2. *Let (X, τ_X) be a topological space satisfying the first axiom of countability and let $K \subset X$ be closed and compact. Then K is sequentially compact. In particular, if X is also Hausdorff, then every compact set is sequentially compact.*

Proof. Consider a sequence $\{x_n\} \subset K$ and $E := \{x_n : n \in \mathbb{N}\}$. By Proposition 3.6.1, either E is finite, in which case an element is repeated infinitely often (yielding a convergent subsequence), or E has an accumulation point x_0. Since (X, τ) satisfies the first axiom of countability, there exists a countable local base $\{B_k\}_k$ at x_0. We may assume $B_{k+1} \subset B_k$ for all $k \in \mathbb{N}$. Since x_0 is an accumulation point of E, for every k, there exists $n_k \in \mathbb{N}$ such that $x_{n_k} \in B_k \cap E$. Since $\{B_k\}_k$ is a decreasing local base at x_0, the sequence $\{x_{n_k}\}$ converges to x_0. Since K is closed and $E \subset K$, we have $x_0 \in \operatorname{acc} E \subset \operatorname{Cl} E \subset \operatorname{Cl} K = K$, so $x_0 \in K$. □

Definition 3.6.1. Let (X, τ) be a topological space, $E \subset X$, and $f : E \to \mathbb{R}$. The function f is **sequentially lower semicontinuous** at $x_0 \in E$ if

$$\liminf_{n\to\infty} f(x_n) \geq f(x_0)$$

for every sequence $\{x_n\} \subset E$ such that $x_n \to x_0$ as $n \to \infty$.

Theorem 3.6.3. *(Weierstrass) Let (X, τ) be a topological space, let $K \subset X$ be compact (respectively sequentially compact), and let $f : X \to \mathbb{R}$ be a lower semicontinuous (respectively sequentially lower semicontinuous) function. Then there exists $x_0 \in K$ such that*

$$f(x_0) = \min_{x\in K} f(x).$$

Proof. The proof follows the standard Weierstrass theorem proof for metric spaces. □

3.7 Product Topology

We introduce the product topology for arbitrary collections of topological spaces and study its properties, including compactness.

Theorem 3.7.1. *The Cartesian product of two compact topological spaces is a compact topological space.*

Proof. Let (X, τ_X) and (Y, τ_Y) be compact topological spaces. Assume by contradiction that $X \times Y$ is not compact. Then there exists an open cover $\mathcal{W}$ of $X \times Y$ with no finite subcover.

Step 1: There exists $x_0 \in X$ such that for every neighborhood U of x_0, no finite subfamily of $\mathcal{W}$ covers $U \times Y$. If not, for all $x \in X$, there exists a neighborhood U_x of x and a finite subfamily of $\mathcal{W}$ covering $U_x \times Y$. Since $\{U_x\}_{x \in X}$ covers X, by compactness, there exist $x_1, \ldots, x_m$ such that

$$U_{x_1} \cup \cdots \cup U_{x_m} = X.$$

For each x_i, let $\mathcal{W}_i$ be a finite subfamily covering $U_{x_i} \times Y$. Then $\{W : W \in \mathcal{W}_i \text{ for some } i = 1, \ldots, m\}$ is finite and covers $X \times Y$, a contradiction.

Step 2: There exists $y_0 \in Y$ such that for every neighborhood $U \times V$ of (x_0, y_0), no finite subfamily of $\mathcal{W}$ covers $U \times V$. If not, for all $y \in Y$, there exists a neighborhood $U_y \times V_y$ of (x_0, y) with a finite subfamily covering $U_y \times V_y$. Since $\{V_y\}_{y \in Y}$ covers Y, by compactness, there exist $y_1, \ldots, y_\ell$ such that

$$V_{y_1} \cup \cdots \cup V_{y_\ell} = Y.$$

For each y_i, let $\mathcal{W}_i'$ be a finite subfamily covering $U_{y_i} \times V_{y_i}$. Then $\{W : W \in \mathcal{W}_i' \text{ for some } i = 1, \ldots, \ell\}$ is finite and covers $(U_{y_1} \cap \cdots \cap U_{y_\ell}) \times Y$, contradicting Step 1.

Step 3: Let $x_0 \in X$ and $y_0 \in Y$ be as in Steps 1 and 2. Since $\mathcal{W}$ covers $X \times Y$, there exists $W \in \mathcal{W}$ such that $(x_0, y_0) \in W$. Then there exist neighborhoods U and V of x_0 and y_0 such that $U \times V \subset W$, contradicting Step 2. □

Remark 3.7.2. *Given a finite collection $\{X_\alpha\}_{\alpha \in \Lambda}$, the product $\prod_{\alpha \in \Lambda} X_\alpha$ is the usual Cartesian product. If $X_\alpha = X$ for all $\alpha \in \Lambda$, then $\prod_{\alpha \in \Lambda} X_\alpha$ is the space of functions $f : \Lambda \to X$, sometimes written X^Λ.*

Definition 3.7.1. Given a collection $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ of topological spaces, the **product topology** on $\prod_{\alpha \in \Lambda} X_\alpha$ is the smallest topology making each projection

$$\pi_\beta : \prod_{\alpha \in \Lambda} X_\alpha \to X_\beta, \quad f \mapsto f(\beta)$$

continuous. It contains the family

$$\mathcal{F} = \{\pi_\alpha^{-1}(V_\alpha) : V_\alpha \in \tau_\alpha, \alpha \in \Lambda\}.$$

Proposition 3.7.3. *Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ be a collection of topological spaces. A base for the product topology is given by sets of the form*

$$\bigcap_{\alpha \in \Lambda_0} \pi_\alpha^{-1}(V_\alpha),$$

where $\Lambda_0 \subset \Lambda$ is finite and $V_\alpha \in \tau_\alpha$, or equivalently, by sets of the form

$$\prod_{\alpha \in \Lambda} V_\alpha,$$

where $V_\alpha \in \tau_\alpha$ and $V_\alpha = X_\alpha$ for all but finitely many $\alpha \in \Lambda$.

Proof. By Proposition 3.1.3 and Example 3.2.1(iii), a base for the topology is given by finite intersections of elements of $\mathcal{F}$, i.e.,

$$\bigcap_{\alpha\in\Lambda_0} \pi_\alpha^{-1}(V_\alpha),$$

where $\Lambda_0 \subset \Lambda$ is finite and $V_\alpha \in \tau_\alpha$. For $\beta \in \Lambda_0$,

$$\begin{aligned}\pi_\beta^{-1}(V_\beta) &= \{f : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha : f(\alpha) \in X_\alpha \text{ for } \alpha \neq \beta, f(\beta) \in V_\beta\} \\ &= \prod_{\alpha\in\Lambda,\alpha<\beta} X_\alpha \times V_\beta \times \prod_{\alpha\in\Lambda,\alpha>\beta} X_\alpha.\end{aligned}$$

Thus,

$$\bigcap_{\alpha\in\Lambda_0} \pi_\alpha^{-1}(V_\alpha) = \prod_{\alpha\in\Lambda} V_\alpha,$$

where $V_\alpha = X_\alpha$ for $\alpha \in \Lambda \setminus \Lambda_0$. □

Exercise 3.7.1. Given a nonempty set X, consider the space $X^{\mathbb{R}}$ of all functions $f : X \to \mathbb{R}$. What is the relation between the topology in Example 3.2.1 and the product topology?

Exercise 3.7.2. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of topological spaces. Prove that for each $\beta \in \Lambda$, the projection $\pi_\beta : \prod_{\alpha\in\Lambda} X_\alpha \to X_\beta$ is continuous and open but not closed.

Theorem 3.7.4. *Let (X, τ) and $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be topological spaces, and let $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$. Then f is continuous if and only if $\pi_\beta \circ f : X \to X_\beta$ is continuous for every $\beta \in \Lambda$.*

Proof. If f is continuous, since π_β is continuous (Exercise 3.7.2), $\pi_\beta \circ f$ is continuous. Conversely, assume each $\pi_\beta \circ f$ is continuous. A subbase for the product topology is $\{\pi_\alpha^{-1}(V_\alpha) : V_\alpha \in \tau_\alpha\}$. By Exercise 3.4.2, it suffices to show $f^{-1}(\pi_\alpha^{-1}(V_\alpha))$ is open in X. But

$$f^{-1}(\pi_\alpha^{-1}(V_\alpha)) = (\pi_\alpha \circ f)^{-1}(V_\alpha),$$

which is open since $\pi_\alpha \circ f$ is continuous. □

Example 3.7.1. Consider $\mathbb{R}^{\mathbb{N}} = \{g : \mathbb{N} \to \mathbb{R}\}$ and the function

$$f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}, \quad x \mapsto (x, x, \ldots).$$

For every $n \in \mathbb{N}$, $\pi_n \circ f : \mathbb{R} \to \mathbb{R}$ is $\pi_n \circ f(x) = x$, which is continuous. Thus, f is continuous in the product topology. However, f is not continuous in the box topology. Consider the open set

$$B = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \cdots = \prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right).$$

If f were continuous, $f^{-1}(B)$ would be open. Since $0 \in f^{-1}(B)$, $f^{-1}(B)$ contains an interval $(-\delta, \delta)$ for some $\delta > 0$. Thus, $f((-\delta, \delta)) \subset B$, so

$$(\pi_n \circ f)((-\delta, \delta)) = (-\delta, \delta) \subset \left(-\frac{1}{n}, \frac{1}{n}\right)$$

for all $n \in \mathbb{N}$, a contradiction.

Exercise 3.7.3. Let (X,d) and $\{(X_\alpha,d_\alpha)\}_{\alpha\in\Lambda}$ be metric spaces, and let $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$. Prove f is continuous if and only if $\pi_\alpha \circ f$ is continuous for every $\alpha \in \Lambda$ and for every $x \in X$, there exists a ball $B(x,r)$ such that $\pi_\alpha \circ f : B(x,r) \to X_\alpha$ is constant for all but finitely many $\alpha \in \Lambda$.

3.8 Tychonoff's Theorem and Well-Orderings

We introduce well-orderings and prove Tychonoff's theorem, which ensures compactness of arbitrary products of compact spaces.

Definition 3.8.1. A binary relation $\prec$ on a set X is a **linear ordering** if:

(i) **Transitivity**: If $x \prec y$ and $y \prec z$, then $x \prec z$.

(ii) **Trichotomy**: For all $x, y \in X$, exactly one of $x \prec y$, $y \prec x$, or $x = y$ holds.

(iii) **Irreflexivity**: $x \prec x$ does not hold for any $x \in X$.

Define $x \leq y$ if $x \prec y$ or $x = y$. A linear ordering is a **well-ordering** if every nonempty $E \subset X$ has a $\prec$-least element $x \in E$ such that $x \leq y$ for all $y \in E$.

Proposition 3.8.1. *(Proofs by induction) Let $(X,\prec)$ be a well-ordering and $P(x)$ a statement about $x \in X$. Suppose that for all $y \in X$,*

$$\text{if } P(x) \text{ holds for all } x \prec y, \text{ then } P(y) \text{ holds.}$$

If $P(x_0)$ is true for the $\prec$-least element x_0 of X, then $P(y)$ holds for all $y \in X$.

Theorem 3.8.2. *[Tychonoff's theorem] Let $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of compact topological spaces. Then $\prod_{\alpha\in\Lambda} X_\alpha$ is compact.*

Proof. By the axiom of choice, assume Λ is well-ordered with every subset having a smallest element. Assume by contradiction that $\prod_{\alpha\in\Lambda} X_\alpha$ is not compact. There exists an open cover $\mathcal{W}$ with no finite subcover.

For every $\beta \in \Lambda$, there exists $x_\beta \in X_\beta$ such that if W is any open set containing

$$\prod_{\alpha\in\Lambda,\alpha\leq\beta} \{x_\alpha\} \times \prod_{\alpha\in\Lambda,\alpha>\beta} X_\alpha,$$

no finite subfamily of $\mathcal{W}$ covers W. We use Proposition 3.8.1. Let $\alpha_0 \in \Lambda$ be the least element. As in Theorem 3.7.1, Step 1 (with $X = X_{\alpha_0}$, $Y = \prod_{\alpha\in\Lambda,\alpha>\alpha_0} X_\alpha$), there exists $x_{\alpha_0} \in X_{\alpha_0}$ such that no finite subfamily of $\mathcal{W}$ covers any open set containing

$$\{x_{\alpha_0}\} \times \prod_{\alpha\in\Lambda,\alpha>\alpha_0} X_\alpha.$$

Assume $x_\alpha \in X_\alpha$ are chosen for all $\alpha < \beta$. There exists $x_\beta \in X_\beta$ such that no finite subfamily of $\mathcal{W}$ covers any open set containing

$$\prod_{\alpha\in\Lambda,\alpha\leq\beta} \{x_\alpha\} \times \prod_{\alpha\in\Lambda,\alpha>\beta} X_\alpha.$$

If not, for all $x \in X_\beta$, there exists an open set W_x containing

$$\prod_{\alpha\in\Lambda,\alpha<\beta} \{x_\alpha\} \times \{x\} \times \prod_{\alpha\in\Lambda,\alpha>\beta} X_\alpha,$$

with a finite subfamily covering W_x. Assume

$$W_x = \prod_{\alpha\in\Lambda,\alpha<\beta} U_{\alpha,x} \times U_{\beta,x} \times \prod_{\alpha\in\Lambda,\alpha>\beta} X_\alpha,$$

where $U_{\alpha,x} \in \tau_\alpha$ and $U_{\alpha,x} = X_\alpha$ for all but finitely many $\alpha \le \beta$. Since $\{U_{\beta,x}\}_{x\in X_\beta}$ covers X_β, there exist $x_1, \dots, x_m$ such that

$$U_{\beta,x_1} \cup \cdots \cup U_{\beta,x_m} = X_\beta.$$

For each x_i, let $\mathcal{W}_i$ cover W_{x_i}. Then $\{W : W \in \mathcal{W}_i \text{ for some } i = 1, \dots, m\}$ covers

$$\prod_{\alpha\in\Lambda,\alpha<\beta} (U_{\alpha,x_1} \cap \cdots \cap U_{\alpha,x_m}) \times X_\beta \times \prod_{\alpha\in\Lambda,\alpha>\beta} X_\alpha,$$

contradicting the choice of $\{x_\alpha\}$. Thus, we construct $\{x_\alpha\}_{\alpha\in\Lambda}$. The function

$$f : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha, \quad \alpha \mapsto x_\alpha$$

is not covered by any $W \in \mathcal{W}$, as any such W contains an open set

$$\prod_{\alpha\in\Lambda} U_\alpha,$$

where U_α is a neighborhood of x_α and $U_\alpha = X_\alpha$ for all but finitely many α, contradicting the choice of x_α. $\square$

Exercise 3.8.1. Prove that $[0,1]^{\mathbb{N}} = \{f : \mathbb{N} \to [0,1]\}$ with the box topology is not compact.

Exercise 3.8.2. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of nonempty topological spaces, and let $E_\alpha \subset X_\alpha$ be nonempty for every $\alpha \in \Lambda$. Fix $g \in \prod_{\alpha\in\Lambda} E_\alpha$ and consider

$$E := \{f \in \prod_{\alpha\in\Lambda} E_\alpha : f(\alpha) = g(\alpha) \text{ for all but finitely many } \alpha \in \Lambda\}.$$

Prove that

$$\operatorname{Cl} E = \prod_{\alpha\in\Lambda} E_\alpha.$$

Exercise 3.8.3. In $\mathbb{R}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{R}\}$ with the box topology, prove that

$$U := \{f : \mathbb{N} \to \mathbb{R} : f \text{ is bounded}\}$$

is both open and closed, so $\mathbb{R}^{\mathbb{N}}$ is not connected.

Exercise 3.8.4. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of topological spaces and $E_\alpha \subset X_\alpha$ nonempty for every $\alpha \in \Lambda$.

(i) Prove that

$$\mathrm{Cl} \prod_{\alpha \in \Lambda} E_\alpha = \prod_{\alpha \in \Lambda} \mathrm{Cl}\, E_\alpha.$$

(ii) Is it true that

$$\mathrm{int} \prod_{\alpha \in \Lambda} E_\alpha = \prod_{\alpha \in \Lambda} \mathrm{int}\, E_\alpha?$$

(iii) Prove that $\prod_{\alpha \in \Lambda} E_\alpha$ is closed if and only if E_α is closed for every $\alpha \in \Lambda$.

Exercise 3.8.5. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ be a collection of Hausdorff topological spaces. Prove that $\prod_{\alpha \in \Lambda} X_\alpha$ is a Hausdorff space.

Example 3.8.1. $[0,1]^{[0,1]}$ with the product topology is compact but not sequentially compact.

3.9 Compactification

Compactification embeds a topological space into a compact space, with the Alexandroff and Stone-Čech compactifications as key examples.

Theorem 3.9.1. *(Alexandroff) Let (X, τ) be a topological space, $\infty \notin X$, and $X^\infty := X \cup \{\infty\}$. Let τ_∞ consist of subsets $U \subset X^\infty$ such that either $U \in \tau$ or $\infty \in U$ and $X \setminus U$ is a closed compact set in X. Then (X^∞, τ_∞) is a compact topological space. Moreover, (X^∞, τ_∞) is Hausdorff if and only if (X, τ) is Hausdorff and locally compact.*

Proof. **Step 1**: (X^∞, τ_∞) is a topological space. A set $U \in \tau_\infty$ if:

(i) $U \cap X \in \tau$,

(ii) If $\infty \in U$, then $X \setminus U$ is compact in X.

Finite intersections and arbitrary unions of τ_∞ elements intersect X in open sets. For $U_1, U_2 \in \tau_\infty$ with $\infty \in U_1 \cap U_2$,

$$X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2),$$

which is compact as a union of closed compact sets. For $\{U_\alpha\}_{\alpha \in \Lambda} \subset \tau_\infty$ with $\infty \in \bigcup_{\alpha \in \Lambda} U_\alpha$, there exists $\beta \in \Lambda$ such that $\infty \in U_\beta$, so $X \setminus U_\beta$ is closed and compact. Since $\bigcup_{\alpha \in \Lambda} U_\alpha \cap X$ is open, $X \setminus \bigcup_{\alpha \in \Lambda} U_\alpha \subset X \setminus U_\beta$ is compact by Proposition 3.5.4. Thus, $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau_\infty$. Also, $X^\infty \in \tau_\infty$ (since $X^\infty \setminus X^\infty = \emptyset$ is compact), and $\emptyset \in \tau_\infty$.

Step 2: (X^∞, τ_∞) is compact. Let $\{U_\alpha\}_{\alpha \in \Lambda} \subset \tau_\infty$ cover X^∞. Then $\infty \in U_\beta$ for some $\beta \in \Lambda$, so $X \setminus U_\beta$ is closed and compact. Since

$$X \setminus U_\beta \subset \bigcup_{\alpha \in \Lambda} U_\alpha \cap X,$$

there exist $\alpha_1, \ldots, \alpha_m \in \Lambda$ such that

$$X \setminus U_\beta \subset \bigcup_{i=1}^{m} U_{\alpha_i} \cap X.$$

Thus, $\{U_{\alpha_1}, \ldots, U_{\alpha_m}, U_\beta\}$ covers X^∞.

Step 3: (X^∞, τ_∞) is Hausdorff if and only if (X, τ) is Hausdorff and locally compact. If (X^∞, τ_∞) is Hausdorff, then (X, τ) is Hausdorff by (i). For local compactness, let $x \in X$. There exist disjoint neighborhoods $U, V \in \tau_\infty$ of x and ∞. Then $X \setminus V$ is compact, and $U \subset X \setminus V$, so $\mathrm{Cl}\, U \subset X \setminus V$ is compact. Conversely, if (X, τ) is Hausdorff and locally compact, for $x, y \in X^\infty$ distinct, if both are in X, there exist disjoint $U, V \in \tau \subset \tau_\infty$. If $y = \infty$, choose $U \in \tau$ with $\mathrm{Cl}\, U$ compact. Then $X^\infty \setminus \mathrm{Cl}\, U \in \tau_\infty$, and $U, X^\infty \setminus \mathrm{Cl}\, U$ are disjoint neighborhoods of x and ∞. $\square$

Exercise 3.9.1. Prove that the circle is the one-point compactification of $(0, 1)$.

Exercise 3.9.2. Describe the one-point compactification of the following subsets of $\mathbb{R}^2$ with the usual topology:

(i) $\{(x, y) : x \in (0, 1], y = 0\}$

(ii) $\{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$

(iii) $\{(x, y) : x^2 + y^2 < 1\}$

(iv) $\{(x, y) : x^2 + y^2 < 1\} \cup \{(0, 1)\}$

(v) $\{(x, y) : -1 \le x \le 1\}$.

Exercise 3.9.3. Let (X, τ) be a topological space and consider

$$C_c(X) := \{f : X \to \mathbb{R} \text{ continuous, } \operatorname{supp} f \text{ is compact}\}$$

with the metric

$$d(f, g) := \max_{x \in X} |f(x) - g(x)|.$$

The completion of $C_c(X)$ is $C_0(X)$, the space of functions vanishing at infinity. Prove that $f \in C_0(X)$ if and only if for every $\epsilon > 0$, there exists a closed compact set $K \subset X$ such that

$$|f(x)| < \epsilon \quad \text{for all } x \in X \setminus K.$$

Exercise 3.9.4. Let (X, τ) be a topological space. Prove that $g \in C(X^\infty)$ if and only if

$$(g - c)|_X = f$$

for some $f \in C_0(X)$ and $c \in \mathbb{R}$. Show also that

$$\|g - c\|_{C(X^\infty)} = \max \|f\|_{C_0(X)}.$$

Definition 3.9.1. Given a topological space (X, τ), a **compactification** is a pair (h, Y), where (Y, τ_Y) is a compact topological space, $h : X \to Y$ is one-to-one, continuous, $h(X)$ is dense in Y, and $h^{-1} : h(X) \to X$ is continuous. It is **Hausdorff** if Y is Hausdorff.

Definition 3.9.2. A topological space (X, τ) is **completely regular** if for every $x_0 \in X$ and closed set $C \subset X$ not containing x_0, there exists a continuous $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f \equiv 0$ on C.

Exercise 3.9.5. Prove that a metric space is completely regular.

Theorem 3.9.2. *(Stone-Čech) Let (X,τ) be a completely regular topological space such that bounded continuous functions separate points. Consider $C_b(X)$, the space of bounded continuous $f: X \to \mathbb{R}$. For each $f \in C_b(X)$, choose $t_f > 0$ such that $f(X) \subset [-t_f, t_f]$. Let*

$$Y_0 := \prod_{f \in C_b(X)} [-t_f, t_f], \quad e: X \to Y_0, \quad e(x)(f) := f(x).$$

Define $\beta(X) := \operatorname{Cl} e(X)$. Then $(e, \beta(X))$ is a compactification of X.

Proof. By Tychonoff's theorem (3.8.2), Y_0 is compact in the product topology. For $f \in C_b(X)$, the projection

$$\pi_f : Y_0 \to [-t_f, t_f], \quad g \mapsto g(f)$$

satisfies $\pi_f \circ e(x) = e(x)(f) = f(x)$. Since f is continuous, Theorem 3.7.4 implies e is continuous. Since $\beta(X)$ is closed in Y_0, it is compact (Proposition 3.5.4). Assume bounded continuous functions separate points, so e is one-to-one. For $U \in \tau$ and $x_0 \in U$, since X is completely regular, there exists $f_0 \in C_b(X)$ such that $f_0(x_0) = 1$ and $f_0 = 0$ on $X \setminus U$. The set $V_{f_0} := f_0^{-1}((0,\infty))$ is open, contains x_0, and $V_{f_0} \subset U$. Moreover,

$$e(V_{f_0}) = \{g \in Y_0 : g(f_0) > 0\} \cap e(X) = \prod_{f \in C_b(X)} A_f \cap e(X),$$

where $A_f = [-t_f, t_f]$ if $f \neq f_0$ and $A_{f_0} = (0, t_{f_0}]$, open in $[-t_{f_0}, t_{f_0}]$. Thus, $e(V_{f_0})$ is open in $e(X)$, so $e^{-1} : e(X) \to X$ is continuous. Hence, $(e, \beta(X))$ is a compactification. □

Exercise 3.9.6. Let (Y, τ_Y) and (Z, τ_Z) be topological spaces, with Z Hausdorff, and let $f : E \to Z$, $E \subset Y$, be continuous. Prove there is at most one continuous extension of f to $\operatorname{Cl} E$.

3.10 Normal Spaces and Extensions

Normal spaces allow separation of closed sets, leading to powerful extension theorems like Urysohn's lemma and Tietze's theorem.

Definition 3.10.1. A topological space (X,τ) is **normal** if for every pair of disjoint closed sets $C_1, C_2 \subset X$, there exist disjoint open sets U_1, U_2 such that $U_1 \supset C_1$ and $U_2 \supset C_2$.

Remark 3.10.1. *For a normal space, if $U_1 \supset C_1$ and $U_2 \supset C_2$ are disjoint open sets, then $\operatorname{Cl} U_1 \cap C_2 = \emptyset$. If $x \in \operatorname{Cl} U_1 \cap C_2$, then $x \in U_2$, so U_2 is a neighborhood of x. By Proposition 3.2.4, $U_1 \cap U_2 \neq \emptyset$, a contradiction.*

Proposition 3.10.2. *Let (X, d) be a metric space with topology τ. Then (X,τ) is a normal space.*

Proof. For disjoint closed sets $C_1, C_2 \subset X$, the open sets

$$U_1 := \{x \in X : \operatorname{dist}(x, C_1) < \operatorname{dist}(x, C_2)\}, \quad U_2 := \{x \in X : \operatorname{dist}(x, C_1) > \operatorname{dist}(x, C_2)\}$$

are disjoint neighborhoods of C_1 and C_2, respectively. □

Exercise 3.10.1. Prove that every compact Hausdorff space is normal.

Exercise 3.10.2. Let (X, τ) be a topological space and $\{f_\alpha\}_{\alpha\in\Lambda}$ a family of lower semi-continuous (respectively sequentially lower semicontinuous) functions $f_\alpha : X \to \mathbb{R}$. Assume $f_+ := \sup_{\alpha\in\Lambda} f_\alpha$ is real-valued.

1. Prove that f_+ is lower semicontinuous.
2. Prove that if Λ is finite, then $f_- := \min_{\alpha\in\Lambda} f_\alpha$ is lower semicontinuous.
3. Prove that if Λ is infinite and f_- is real-valued, f_- may not be lower semicontinuous.

Exercise 3.10.3. Let (X, τ) be a topological space and $E \subset X$.

(i) Prove that the characteristic function

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise,} \end{cases}$$

is lower semicontinuous if and only if E is open.

(ii) Prove that χ_E is sequentially lower semicontinuous if and only if E is sequentially open (i.e., $X \setminus E$ is sequentially closed).

(iii) Prove that there exist sequentially lower semicontinuous functions that are not lower semicontinuous.

Theorem 3.10.3. *(Urysohn's lemma) A topological space (X, τ) is normal if and only if for all disjoint closed sets $C_1, C_2 \subset X$, there exists a continuous $f : X \to [0, 1]$ such that $f \equiv 1$ on C_1 and $f \equiv 0$ on C_2.*

Lemma 3.10.4. *Let (X, τ) be a normal space, $C \subset X$ closed, and $U \subset X$ open with $C \subset U$. There exists an open set $V \subset X$ such that*

$$C \subset V \subset \operatorname{Cl} V \subset U.$$

Proof. Since C and $X \setminus U$ are disjoint closed sets, by Remark 3.10.1, there exists an open $V \subset X$ such that $V \supset C$ and $\operatorname{Cl} V \cap (X \setminus U) = \emptyset$. Thus, $\operatorname{Cl} V \subset U$. $\square$

Proof of Theorem 3.10.3. **Step 1**: Assume X is normal. Let $C_1, C_2 \subset X$ be disjoint closed sets. Set $r_0 := 0$, $r_1 := 1$, and let $\{r_n\}_{n=2}^{\infty}$ enumerate the rationals in $(0, 1)$. By Lemma 3.10.4, there exists an open $V_0 \subset X$ such that

$$C_1 \subset V_0 \subset \operatorname{Cl} V_0 \subset X \setminus C_2.$$

Apply Lemma 3.10.4 to $C_1 \subset V_0$ to find $V_1 \subset X$ such that

$$C_1 \subset V_1 \subset \operatorname{Cl} V_1 \subset V_0,$$

so

$$C_1 \subset V_1 \subset \operatorname{Cl} V_1 \subset V_0 \subset \operatorname{Cl} V_0 \subset X \setminus C_2.$$

Inductively, for $n \in \mathbb{N}$, given open sets $V_{r_1}, \dots, V_{r_n}$ such that $\operatorname{Cl} V_{r_j} \subset V_{r_i}$ if $r_i < r_j$, consider r_{n+1}. Since $r_0 < r_{n+1} < r_1$, let r_i be the largest of $r_1, \dots, r_n$ below r_{n+1}, and r_j the smallest above r_{n+1}. Since $\operatorname{Cl} V_{r_j} \subset V_{r_i}$, Lemma 3.10.4 provides $V_{r_{n+1}} \subset X$ such that

$$\operatorname{Cl} V_{r_j} \subset V_{r_{n+1}} \subset \operatorname{Cl} V_{r_{n+1}} \subset V_{r_i}.$$

Thus, we construct $\{V_r\}_{r\in[0,1]\cap\mathbb{Q}}$ with $C_1 \subset V_r$, $\operatorname{Cl} V_r \subset X \setminus C_2$, and $\operatorname{Cl} V_s \subset V_r$ for $r < s$. Define

$$f_r(x) := \begin{cases} r & \text{if } x \in V_r, \\ 0 & \text{otherwise,} \end{cases} \qquad g_s(x) := \begin{cases} 1 & \text{if } x \in \operatorname{Cl} V_s, \\ s & \text{otherwise,} \end{cases}$$

and

$$f := \sup_{r\in[0,1]\cap\mathbb{Q}} f_r, \quad g := \inf_{s\in[0,1]\cap\mathbb{Q}} g_s.$$

Then f is lower semicontinuous, g is upper semicontinuous, and $0 \le f \le 1$. For $x \in C_1 \subset V_1 \subset V_r$, $f_r(x) = r$, so

$$f(x) = \sup_{r\in[0,1]\cap\mathbb{Q}} f_r(x) = \sup_{r\in[0,1]\cap\mathbb{Q}} r = 1.$$

For $x \in C_2$, since $\operatorname{Cl} V_0 \subset X \setminus C_2$ and $\operatorname{Cl} V_r \subset V_0$, $x \notin V_r$, so $f_r(x) = 0$, and

$$f(x) = \sup_{r\in[0,1]\cap\mathbb{Q}} 0 = 0.$$

To show f is continuous, prove $f = g$. If $f_r(x) > g_s(x)$, then $r > s$, $x \in V_r$, $x \notin \operatorname{Cl} V_s$. Since $s < r$, $\operatorname{Cl} V_r \subset V_s$, a contradiction. Thus, $f_r \le g_s$, so $f \le g$. If $f(x) < g(x)$, there exist $r, s \in [0,1] \cap \mathbb{Q}$ such that

$$f(x) < r < s < g(x).$$

Since $f(x) < r$, $x \notin V_r$. Since $s < g(x)$, $x \in \operatorname{Cl} V_s$. But $\operatorname{Cl} V_s \subset V_r$, a contradiction. Thus, $f = g$, and f is continuous.

Step 2: If for all disjoint closed $C_1, C_2 \subset X$, there exists a continuous $f : X \to [0,1]$ with $f \equiv 1$ on C_1 and $f \equiv 0$ on C_2, then X is normal. For disjoint closed C_1, C_2, let f be as above. The sets $f^{-1}((-\frac{7}{2}, \frac{5}{2}))$ and $f^{-1}((\frac{7}{2}, \frac{3}{2}))$ are open, disjoint, and contain C_1 and C_2, respectively. □

Exercise 3.10.4. Let (X, τ) be a normal space.

(i) Prove that for a proper closed set $C \subset X$, there exists a continuous $f : X \to [0,1]$ such that $f \equiv 0$ on C and $f > 0$ on $X \setminus C$ if and only if C is a G_δ set.

(ii) Let $C_1, C_2 \subset X$ be disjoint closed sets. Prove that there exists a continuous $f : X \to [0,1]$ such that $f \equiv 1$ on C_1, $f \equiv 0$ on C_2, and $0 < f < 1$ on $X \setminus (C_1 \cup C_2)$ if and only if C_1, C_2 are G_δ sets.

Definition 3.10.2. Given a topological space (X, τ) and $E \subset X$, the **induced topology** on E is

$$\tau_E := \{U \cap E : U \in \tau\}.$$

Elements of τ_E are **relatively open**.

Exercise 3.10.5. Let (X, τ) be a topological space, $E \subset X$, and $F \subset E$. Let $\operatorname{Cl} F^{\tau_E}$ denote the closure of F in (E, τ_E). Prove that

$$\operatorname{Cl} F^{\tau_E} = \operatorname{Cl} F \cap E.$$

Remark 3.10.5. *If E is closed in (X, τ), then $\operatorname{Cl} F^{\tau_E}$ is closed in (X, τ), as it is the intersection of two closed sets.*

Remark 3.10.6. *Let (X, τ_X), (Y, τ_Y) be topological spaces, $E \subset X$, and $f : E \to Y$. Then f is continuous at $x_0 \in E$ if and only if $f : (E, \tau_E) \to Y$ is continuous at x_0.*

Theorem 3.10.7. *(Tietze's extension theorem) A topological space (X, τ) is normal if and only if for every closed set $C \subset X$ and continuous $f : C \to \mathbb{R}$, there exists a continuous $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in C$. Moreover, if $f(C) \subset [a, b]$, then F can be chosen so that $F(X) \subset [a, b]$.*

Proof. **Step 1**: Assume X is normal, $C \subset X$ closed, and $f : C \to [-1, 1]$ continuous. The sets $f^{-1}([\frac{1}{3}, \infty))$ and $f^{-1}((-\infty, -\frac{1}{3}])$ are closed in (C, τ_C) and thus in (X, τ) (Exercise 3.10.5). By Theorem 3.10.3, there exists a continuous $f_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ such that $f_1 \equiv \frac{1}{3}$ on $f^{-1}([\frac{1}{3}, \infty))$ and $f_1 \equiv -\frac{1}{3}$ on $f^{-1}((-\infty, -\frac{1}{3}])$. Then

$$|f - f_1| \leq \frac{2}{3} \text{ on } C.$$

If $f(x) \in [-1, -\frac{1}{3}]$, then $f_1(x) = -\frac{1}{3}$; if $f(x) \in [\frac{1}{3}, 1]$, then $f_1(x) = \frac{1}{3}$; if $f(x) \in [-\frac{1}{3}, \frac{1}{3}]$, then $f_1(x) \in [-\frac{1}{3}, \frac{1}{3}]$. Repeat with $f - f_1$, using $(f - f_1)^{-1}([\frac{2}{9}, \infty))$ and $(f - f_1)^{-1}((-\infty, -\frac{2}{9}])$, to find $f_2 : X \to [-\frac{2}{9}, \frac{2}{9}]$ such that

$$|(f - f_1) - f_2| \leq \left(\frac{2}{3}\right)^2 \text{ on } C.$$

Inductively, construct $f_n : X \to [-\frac{1}{3}(\frac{2}{3})^{n-1}, \frac{1}{3}(\frac{2}{3})^{n-1}]$ such that

$$|f - f_1 - \cdots - f_n| \leq \left(\frac{2}{3}\right)^n \text{ on } C.$$

Define

$$F(x) := \sum_{n=1}^{\infty} f_n(x).$$

Since

$$\sum_{n=1}^{\infty} |f_n(x)| \leq \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 1,$$

F is well-defined and $F(X) \subset [-1, 1]$. For $x \in C$,

$$|f(x) - F(x)| = \left| f(x) - \lim_{m \to \infty} \sum_{n=1}^{m} f_n(x) \right| \leq \lim_{m \to \infty} \left(\frac{2}{3}\right)^m = 0,$$

so $F = f$ on C. To show F is continuous, fix $x \in X$, $\epsilon > 0$, and choose n_ϵ such that

$$\sum_{n=n_\epsilon+1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \leq \frac{\epsilon}{2}.$$

Since $f_1, \ldots, f_{n_\epsilon}$ are continuous, there exist neighborhoods U_n of x such that

$$|f_n(y) - f_n(x)| \leq \frac{\epsilon}{2n_\epsilon} \quad \text{for } y \in U_n.$$

For $U := \bigcap_{n=1}^{n_\epsilon} U_n$ and $y \in U$,

$$|F(y) - F(x)| \leq \sum_{n=1}^{n_\epsilon} |f_n(y) - f_n(x)| + \sum_{n=n_\epsilon+1}^{\infty} |f_n(x)| + \sum_{n=n_\epsilon+1}^{\infty} |f_n(y)|$$

$$\leq n_\epsilon \cdot \frac{\epsilon}{2n_\epsilon} + \frac{2}{3} \sum_{n=n_\epsilon+1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, F is continuous. The proof extends to $f : C \to [a, b]$.

Step 2: Assume X is normal and $f : C \to \mathbb{R}$ continuous. Since $\mathbb{R}$ is homeomorphic to $(-1, 1)$, let $g : \mathbb{R} \to (-1, 1)$ be a homeomorphism. Then $h := g \circ f : C \to [-1, 1]$ is continuous. By Step 1, there exists $H : X \to [-1, 1]$ such that $H = h$ on C. If $C_1 := H^{-1}(\{-1, 1\})$, then $C_1 \cap C = \emptyset$. By Theorem 3.10.3, there exists $h_1 : X \to [0, 1]$ such that $h_1 \equiv 0$ on C_1 and $h_1 \equiv 1$ on C. Then $H_1 := Hh_1$ satisfies $H_1 = h$ on C and $H_1(X) \subset (-1, 1)$. Thus, $F := g^{-1} \circ H_1$ is continuous, and for $x \in C$,

$$F(x) = g^{-1}(H_1(x)) = g^{-1}(h(x)) = g^{-1}(g \circ f(x)) = f(x).$$

Step 3: If for every closed $C \subset X$ and continuous $f : C \to \mathbb{R}$, there exists $F : X \to \mathbb{R}$ such that $F = f$ on C and $F(X) \subset [a, b]$ if $f(C) \subset [a, b]$, then for disjoint closed C_1, C_2, define $f \equiv 1$ on C_1, $f \equiv 0$ on C_2. Then $f : C_1 \cup C_2 \to [0, 1]$ is continuous, so there exists $F : X \to [0, 1]$ such that $F = f$ on $C_1 \cup C_2$. By Theorem 3.10.3, X is normal. $\square$

4 Partitions of Unity and Metrization

This chapter explores partitions of unity in normal spaces, metrization theorems, and the properties of paracompact spaces. We establish conditions for constructing partitions of unity, prove Urysohn's and Nagata-Smirnov's metrization theorems, and characterize paracompact spaces, highlighting their normality and ability to support locally finite refinements.

4.1 Partitions of Unity in Normal Spaces

We begin by studying partitions of unity, which allow us to construct continuous functions subordinate to open covers in normal spaces.

Definition 4.1.1. Let (X, τ) be a topological space and let $\mathcal{F}$ be a collection of subsets of X. Then

(i) $\mathcal{F}$ is **point finite** if every $x \in X$ belongs to only finitely many $U \in \mathcal{F}$.

(ii) $\mathcal{F}$ is **locally finite** if every $x \in X$ has a neighborhood meeting only finitely many $U \in \mathcal{F}$.

(iii) $\mathcal{G} \subset \P(X)$ is a **refinement** of $\mathcal{F}$ if

$$\bigcup_{G \in \mathcal{G}} G = \bigcup_{F \in \mathcal{F}} F$$

and every element of $\mathcal{G}$ is contained in some element of $\mathcal{F}$.

Theorem 4.1.1. *A topological space (X, τ) is normal if and only if for every point finite open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of X, there exists another open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of X with the property that $\operatorname{Cl} V_\alpha \subset U_\alpha$.*

Proof. **Step 1**: Assume (X, τ) is normal and let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a point finite open cover of X. By the axiom of choice, assume Λ is well-ordered with an order relation $\leq$ such that every subset of Λ has a smallest element. Let $\alpha_0 \in \Lambda$ be the least element. We claim that for every $\beta \in \Lambda$, there exists an open set V_β such that

$$C_\beta \subset V_\beta \subset \operatorname{Cl} V_\beta \subset U_\beta,$$

where

$$C_\beta := X \setminus \left(\left(\bigcup_{\alpha < \beta} V_\alpha \right) \cup \left(\bigcup_{\alpha > \beta} U_\alpha \right) \right).$$

Use transfinite induction on Λ. Define

$$C_{\alpha_0} := X \setminus \bigcup_{\alpha > \alpha_0} U_\alpha.$$

Then $C_{\alpha_0} \subset U_{\alpha_0}$ and is closed. By Lemma 3.10.4, there exists an open set V_{α_0} such that

$$C_{\alpha_0} \subset V_{\alpha_0} \subset \operatorname{Cl} V_{\alpha_0} \subset U_{\alpha_0}.$$

Suppose V_α has been chosen for every $\alpha < \beta$. Define

$$C_\beta := X \setminus \left(\left(\bigcup_{\alpha<\beta} V_\alpha \right) \cup \left(\bigcup_{\alpha>\beta} U_\alpha \right) \right).$$

To prove $C_\beta \subset U_\beta$, fix $x \in C_\beta$. Then

$$x \notin V_\alpha \text{ for any } \alpha < \beta \quad \text{and} \quad x \notin U_\alpha \text{ for any } \alpha > \beta.$$

Since $\{U_\alpha\}_{\alpha\in\Lambda}$ is point finite, x belongs to finitely many U_α, say $U_{\alpha_1}, \ldots, U_{\alpha_m}$, with $\alpha_m = \max\{\alpha_1, \ldots, \alpha_m\}$. Then $x \notin U_\alpha$ for $\alpha > \alpha_m$. Hence, $\alpha_m \leq \beta$. If $\alpha_m < \beta$, then $x \notin V_{\alpha_m}$, but

$$x \in C_{\alpha_m} = X \setminus \left(\left(\bigcup_{\alpha<\alpha_m} V_\alpha \right) \cup \left(\bigcup_{\alpha>\alpha_m} U_\alpha \right) \right),$$

contradicting $C_{\alpha_m} \subset V_{\alpha_m}$. Thus, $\alpha_m = \beta$, so $x \in U_\beta$. By Lemma 3.10.4, there exists an open set V_β such that

$$C_\beta \subset V_\beta \subset \operatorname{Cl} V_\beta \subset U_\beta.$$

By Proposition 3.8.1, we construct $\{V_\alpha\}_{\alpha\in\Lambda}$. To show it covers X, fix $x \in X$. Since $\{U_\alpha\}_{\alpha\in\Lambda}$ is point finite, $x \in U_{\alpha_1}, \ldots, U_{\alpha_m}$, with $\alpha_m = \max\{\alpha_1, \ldots, \alpha_m\}$. Then $x \notin U_\alpha$ for $\alpha > \alpha_m$. If $x \in \bigcup_{\alpha<\alpha_m} V_\alpha$, we are done. If $x \notin \bigcup_{\alpha<\alpha_m} V_\alpha$, then $x \in C_{\alpha_m}$, so $x \in V_{\alpha_m}$. Thus, $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover.

Step 2: Assume every point finite open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ has an open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ with $\operatorname{Cl} V_\alpha \subset U_\alpha$. Let $C_1, C_2 \subset X$ be disjoint closed sets. Then $\{X \setminus C_1, X \setminus C_2\}$ is a point finite open cover. There exist open sets V_1, V_2 such that

$$X = V_1 \cup V_2, \quad \operatorname{Cl} V_1 \subset X \setminus C_1, \quad \operatorname{Cl} V_2 \subset X \setminus C_2.$$

The sets $X \setminus \operatorname{Cl} V_1$ and $X \setminus \operatorname{Cl} V_2$ are open, disjoint, and contain C_1 and C_2, respectively. Thus, X is normal. $\square$

Definition 4.1.2. A **partition of unity** on a topological space (X, τ) is a family $\{\varphi_i\}_{i\in\Lambda}$ of continuous functions $\varphi_i : X \to [0, 1]$ such that

$$\sum_{i\in\Lambda} \varphi_i(x) = 1$$

for all $x \in X$. It is **locally finite** if for every $x \in X$, there exists a neighborhood U of x such that $\{i \in \Lambda : U \cap \operatorname{supp} \varphi_i \neq \emptyset\}$ is finite. If $\{U_j\}_{j\in\Xi}$ is an open cover, a partition of unity is **subordinated** to $\{U_j\}_{j\in\Xi}$ if for every $i \in \Lambda$, $\operatorname{supp} \varphi_i \subset U_j$ for some $j \in \Xi$.

Theorem 4.1.2. *Let (X, τ) be a normal space and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a locally finite open cover of X. Then there exists a partition of unity subordinated to it.*

Proof. Since a locally finite cover is point finite, by Theorem 4.1.1, there exists an open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ with $\mathrm{Cl}\, V_\alpha \subset U_\alpha$. The family $\{\mathrm{Cl}\, V_\alpha\}_{\alpha\in\Lambda}$ is locally finite. By Lemma 3.10.4, there exists an open set W_α such that

$$\mathrm{Cl}\, V_\alpha \subset W_\alpha \subset \mathrm{Cl}\, W_\alpha \subset U_\alpha.$$

Since $\mathrm{Cl}\, V_\alpha$ and $X \setminus W_\alpha$ are disjoint closed sets, by Theorem 3.10.3, there exists a continuous function $f_\alpha : X \to [0,1]$ such that $f_\alpha \equiv 1$ on $\mathrm{Cl}\, V_\alpha$ and $f_\alpha \equiv 0$ on $X \setminus W_\alpha$. Thus,

$$\{x \in X : f_\alpha(x) > 0\} \subset W_\alpha, \quad \mathrm{supp}\, f_\alpha \subset \mathrm{Cl}\, W_\alpha \subset U_\alpha.$$

Define

$$f(x) := \sum_{\alpha\in\Lambda} f_\alpha(x), \quad x \in X.$$

Since $\{V_\alpha\}_{\alpha\in\Lambda}$ covers X, for every $x \in X$, there exists $\alpha \in \Lambda$ such that $x \in V_\alpha$, so $f_\alpha(x) = 1$. Thus, $f > 0$. Since $\{\mathrm{Cl}\, V_\alpha\}_{\alpha\in\Lambda}$ is locally finite, for every $x \in X$, there exists a neighborhood U intersecting finitely many $\mathrm{Cl}\, V_\alpha$. Thus, f is a finite sum in U, so $f < \infty$ and is continuous at x. Hence, f is continuous. Define

$$\varphi_\alpha(x) := \frac{f_\alpha(x)}{f(x)}, \quad x \in X.$$

Then φ_α is continuous, $\mathrm{supp}\, \varphi_\alpha = \mathrm{supp}\, f_\alpha \subset U_\alpha$, and $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a locally finite partition of unity subordinated to $\{U_\alpha\}_{\alpha\in\Lambda}$. □

4.2 Metrization Theorems

We investigate conditions under which a topological space's topology can be induced by a metric, focusing on Urysohn's and Nagata-Smirnov's metrization theorems.

Theorem 4.2.1. *(Urysohn's metrization theorem) A topological space (X, τ) is metrizable and separable if and only if it is Hausdorff, normal, and has a countable base.*

Proof. If (X, d) is a metric space, by Propositions 3.3.2 and 3.10.2, it is Hausdorff and normal. If separable, there exists a dense sequence $\{x_n\} \subset X$. The family $\{B(x_n, \frac{1}{k})\}_{k,n\in\mathbb{N}}$ is a countable base for the topology τ.

Conversely, assume (X, τ) is Hausdorff, normal, with a countable base $\mathcal{B} = \{B_n\}_n$. We show X is homeomorphic to a subset of ℓ^2, which is separable.

Step 1: Every closed set is a G_δ set (equivalently, every open set is an F_σ set). Fix an open set $U \subset X$ and $x \in U$. Since X is Hausdorff, $\{x\}$ is closed. By Lemma 3.10.4, there exists an open set $V \subset X$ such that

$$\{x\} \subset V \subset \mathrm{Cl}\, V \subset U.$$

Since $\mathcal{B}$ is a base, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset V$, so

$$\{x\} \subset B_x \subset \mathrm{Cl}\, B_x \subset \mathrm{Cl}\, V \subset U.$$

Thus,

$$U = \bigcup_{B_n \subset U} \mathrm{Cl}\, B_n,$$

showing U is an F_σ set.

Step 2: By Step 1 and Exercise 3.10.4, for each $B_n \in \mathcal{B}$, there exists a continuous $\varphi_n : X \to [0,1]$ such that

$$\varphi_n(x) = 0 \text{ for } x \in X \setminus B_n, \quad \varphi_n(x) > 0 \text{ for } x \in B_n.$$

Define

$$\psi_n(x) := \frac{1}{n} \frac{\varphi_n(x)}{\sqrt{1 + (\varphi_n(x))^2}}, \quad x \in X.$$

Then ψ_n is continuous, and

$$\sum_n (\psi_n(x))^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{(\varphi_n(x))^2}{1 + (\varphi_n(x))^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so $\{\psi_n(x)\}_n \in \ell^2$. Define

$$f : X \to \ell^2, \quad x \mapsto \{\psi_n(x)\}_n.$$

To show f is one-to-one, let $x, y \in X$, $x \neq y$. Since X is Hausdorff, there exists B_n such that $x \in B_n$, $y \in X \setminus B_n$. Thus, $\psi_n(x) > 0$, $\psi_n(y) = 0$, so $f(x) \neq f(y)$.

Step 3: f is continuous. Fix $x_0 \in X$, $\varepsilon > 0$, and choose $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty} \frac{1}{n^2} \le \frac{\varepsilon^2}{8}.$$

Since ψ_n is continuous at x_0 for $n = 1, \ldots, n_\varepsilon$, there exists a neighborhood V of x_0 such that

$$|\psi_n(x) - \psi_n(x_0)| \le \frac{\varepsilon}{\sqrt{2n_\varepsilon}}$$

for $x \in V$, $n = 1, \ldots, n_\varepsilon$. Then

$$\sum_{n=1}^{n_\varepsilon} (\psi_n(x) - \psi_n(x_0))^2 \le n_\varepsilon \frac{\varepsilon^2}{2n_\varepsilon} = \frac{\varepsilon^2}{2},$$

and

$$\sum_{n=n_\varepsilon+1}^{\infty} (\psi_n(x) - \psi_n(x_0))^2 \le 2 \sum_{n=n_\varepsilon+1}^{\infty} \left[(\psi_n(x))^2 + (\psi_n(x_0))^2\right] \le 2 \sum_{n=n_\varepsilon+1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n^2}\right) \le \frac{4\varepsilon^2}{8} = \frac{\varepsilon^2}{2}.$$

Thus,

$$d_2(f(x), f(x_0)) \le \varepsilon,$$

showing continuity at x_0.

Step 4: $f^{-1} : f(X) \to X$ is continuous. Let $y_0 \in f(X)$, so $y_0 = f(x_0)$ for some $x_0 \in X$. For a neighborhood U of x_0, there exists $B_n \in \mathcal{B}$ such that $B_n \subset U$, with $\psi_n(x_0) > 0$. Set $\delta := \psi_n(x_0)$. If $d_2(f(x), f(x_0)) < \delta$, then

$$|\psi_n(x) - \psi_n(x_0)| \le \left(\sum_{n=1}^{\infty} (\psi_n(x) - \psi_n(x_0))^2 \right)^{\frac{1}{2}} = d_2(f(x), f(x_0)) < \delta = \psi_n(x_0).$$

Thus, $\psi_n(x) > 0$, so $x \in B_n \subset U$. Hence, f^{-1} is continuous at y_0. $\square$

Theorem 4.2.2. *Let (X, d) be a metric space. Then every open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X admits a point finite refinement.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover. Assume Λ is well-ordered. For $\alpha \in \Lambda$, a **chosen ball** (with respect to α) is a ball $B(x, \frac{1}{2^{n_x+1}})$ such that:

(i) $B(x, \frac{1}{2^{n_x}}) \subset U_\alpha$,

(ii) n_x is the smallest integer for which (i) holds,

(iii) $B(x, \frac{1}{2^{n_x}}) \subset U_\beta$ for some $\beta < \alpha$.

Define
$$\mathcal{B}_\alpha := \left\{ B\left(x, \frac{1}{2^{n_x+1}}\right) : B\left(x, \frac{1}{2^{n_x+1}}\right) \text{ is a chosen ball} \right\},$$
and
$$V_\alpha := U_\alpha \setminus \mathrm{Cl} \bigcup_{B\in\mathcal{B}_\alpha} B.$$
We claim $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover. Assume there exists $x \in X$ not covered by any V_α. Let U_α be the first element containing x. Then $B(x, r) \subset U_\alpha$ for some $r > 0$. Since $x \notin V_\alpha$ and not in any chosen ball (by (iii)), x is an accumulation point of chosen balls. There exist sequences $\{B(x_k, \frac{1}{2^{n_k+1}})\} \subset \mathcal{B}_\alpha$ and $y_k \in B(x_k, \frac{1}{2^{n_k+1}})$ such that $y_k \to x$. If $n_{x_k} \to \infty$ along a subsequence, then
$$B\left(x_k, \frac{1}{2^{n_{x_k}-1}}\right) \subset B(x, r) \subset U_\alpha$$
for infinitely many k, contradicting (ii). Thus,
$$\min_k \frac{1}{2^{n_{x_k}+1}} = \frac{1}{2^{n_0+1}}$$
for some $n_0 \in \mathbb{N}$. For k such that $d(x, y_k) < \frac{1}{2^{n_0+1}}$,
$$d(x, x_k) \le d(x, y_k) + d(y_k, x_k) < \frac{1}{2^{n_0+1}} + \frac{1}{2^{n_{x_k}+1}} \le \frac{2}{2^{n_{x_k}+1}} = \frac{1}{2^{n_{x_k}}},$$
so $x \in B(x_k, \frac{1}{2^{n_{x_k}}})$. By (iii), $x \in U_\beta$ for some $\beta < \alpha$, contradicting the choice of α. Thus, $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover. □

Theorem 4.2.3. *Let (X, d) be a metric space. Then every point finite open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X admits a locally finite refinement.*

Proof. For every $x \in X$, define
$$r_x := \frac{1}{2} \sup\{r > 0 : B(x, r) \subset V_\alpha \text{ for some } \alpha \in \Lambda\}.$$
If $r_x = \infty$ for some x, the sequence $\{B(x, n)\}_{n\in\mathbb{N}}$ is a locally finite refinement. Assume $r_x < \infty$ for all x. For each $\beta \in \Lambda$, let W_β be the union of all balls $B(x, \frac{r_x}{2})$ such that V_β is the first open set containing $B(x, r_x)$. Then $\{W_\alpha\}_{\alpha\in\Lambda}$ is an open cover, and $W_\alpha \subset V_\alpha$, so it is a refinement. To show it is locally finite, assume $W_\alpha \cap B(x, \frac{r_x}{2}) \neq \emptyset$, but $x \notin V_\alpha$.

There exists $y \in W_\alpha$ such that $B(y, \frac{r_y}{2}) \cap B(x, \frac{r_x}{2}) \neq \emptyset$, with $B(y, r_y) \subset V_\alpha$. Since $x \notin V_\alpha$, $x \notin B(y, r_y)$, so

$$r_y < d(x,y) < \frac{r_y}{2} + \frac{r_x}{8},$$

implying $\frac{r_y}{2} < \frac{r_x}{8}$. Thus,

$$d(x,y) < \frac{r_x}{8} + \frac{r_x}{8} = \frac{r_x}{4},$$

so $y \in B(x, \frac{r_x}{4})$. Then $B(y, \frac{r_x}{2}) \subset B(x, r_x)$, implying $B(y, 5r_y) \subset V_\beta$, contradicting the definition of r_y. Thus, $x \in V_\alpha$. Since only finitely many V_α contain x (by point finiteness), $\{W_\alpha\}_{\alpha\in\Lambda}$ is locally finite. □

Definition 4.2.1. Let X be a topological space and $\mathcal{F}$ a collection of subsets of X. Then $\mathcal{F}$ is **σ-locally finite** if

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n,$$

where each $\mathcal{F}_n$ is locally finite in X.

Corollary 4.2.4. *Let (X,d) be a metric space. Then every open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ admits a locally finite partition of unity subordinated to it.*

Corollary 4.2.5. *Let (X,d) be a metric space. Then X admits a σ-locally finite base.*

Proof. For every $x \in X$, $n \in \mathbb{N}$, consider $B(x, \frac{1}{n})$. Then $\{B(x, \frac{1}{n})\}_{x\in X, n\in\mathbb{N}}$ is a base. Fix $n \in \mathbb{N}$. By Theorems 4.2.2 and 4.2.3, there exists a locally finite open refinement $\mathcal{V}_n$ of $\{B(x, \frac{1}{n})\}_{x\in X}$. Thus, $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a σ-locally finite base. □

Theorem 4.2.6. *(Nagata-Smirnov's metrization theorem) A topological space (X, τ) is metrizable if and only if it is Hausdorff, normal, and has a σ-locally finite base.*

Proof. If (X,d) is a metric space, by Propositions 3.3.2 and 3.10.2, it is Hausdorff and normal. By Corollary 4.2.5, it has a σ-locally finite base.

Conversely, assume X is Hausdorff, normal, with a σ-locally finite base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where $\mathcal{B}_n = \{B_{n,\alpha}\}_{\alpha\in\Lambda_n}$ is locally finite. We show X is homeomorphic to a subset of an ℓ^2 metric space.

Step 1: Every open set is an F_σ set. Fix an open set $U \subset X$. For $x \in U$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset \operatorname{Cl} B_x \subset U$. For $n \in \mathbb{N}$, let

$$C_n := \bigcup_{B_x \in \mathcal{B}_n} \operatorname{Cl} B_x.$$

By Lemma 4.2.7, C_n is closed and $C_n \subset U$. Since

$$U = \bigcup_{n=1}^{\infty} C_n,$$

U is an F_σ set.

Step 2: By Step 1 and Exercise 3.10.4, for each $B_{n,\alpha} \in \mathcal{B}$, there exists a continuous $\varphi_{\alpha,n} : X \to [0,1]$ such that

$$\varphi_{\alpha,n}(x) = 0 \text{ for } x \in X \setminus B_{\alpha,n}, \quad \varphi_{\alpha,n}(x) > 0 \text{ for } x \in B_{\alpha,n}.$$

Define

$$\psi_{\alpha,n}(x) := \frac{1}{n}\frac{\varphi_{\alpha,n}(x)}{\sqrt{1+\sum_\beta(\varphi_{\beta,n}(x))^2}}, \quad x \in X.$$

Since $\mathcal{B}_n$ is locally finite, for every $x \in X$, there exists a neighborhood U where $\varphi_{\beta,n} = 0$ except for finitely many β. Thus, $\sum_\beta(\varphi_{\beta,n})^2$ is finite in U, so $\psi_{\alpha,n}$ is continuous. Moreover,

$$\sum_{\alpha,n}(\psi_{\alpha,n}(x))^2 = \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_\alpha \frac{(\varphi_{\alpha,n}(x))^2}{1+\sum_\beta(\varphi_{\beta,n}(x))^2} \le \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

Thus, $\{\psi_{\alpha,n}(x)\}_{\alpha,n}$ belongs to the ℓ^2 space

$$\ell^2 := \left\{\{a_{\alpha,n}\}_{\alpha,n} : \sum_{\alpha,n} a_{\alpha,n}^2 < \infty\right\},$$

with metric

$$d_2(a,b) := \left(\sum_{\alpha,n}(a_{\alpha,n} - b_{\alpha,n})^2\right)^{\frac{1}{2}}.$$

Define

$$f : X \to \ell^2, \quad x \mapsto \{\psi_{\alpha,n}(x)\}_{\alpha,n}.$$

For $x, y \in X$, $x \neq y$, there exists $B_{\alpha,n}$ such that $x \in B_{\alpha,n}$, $y \in X \backslash B_{\alpha,n}$. Thus, $\psi_{\alpha,n}(x) > 0$, $\psi_{\alpha,n}(y) = 0$, so $f(x) \neq f(y)$.

Step 3: f is continuous. Fix $x_0 \in X$, $\varepsilon > 0$, and choose $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty}\frac{1}{n^2} \le \frac{\varepsilon^2}{8}.$$

For $n = 1, \ldots, n_\varepsilon$, there exists a neighborhood of x_0 intersecting finitely many $B_{\alpha,n}$. Intersecting these, obtain a neighborhood U intersecting $B_{\alpha_1,n_1}, \ldots, B_{\alpha_m,n_m}$. Since each ψ_{α_i,n_i} is continuous, there exists $V \subset U$ such that

$$|\psi_{\alpha_i,n_i}(x) - \psi_{\alpha_i,n_i}(x_0)| \le \frac{\varepsilon}{\sqrt{2m}}$$

for $x \in V$, $i = 1, \ldots, m$. Then

$$\sum_{n=1}^{n_\varepsilon}\sum_\alpha(\psi_{\alpha,n}(x) - \psi_{\alpha,n}(x_0))^2 = \sum_{i=1}^{m}(\psi_{\alpha_i,n_i}(x) - \psi_{\alpha_i,n_i}(x_0))^2 \le m\frac{\varepsilon^2}{2m} = \frac{\varepsilon^2}{2},$$

and

$$\sum_{n=n_\varepsilon+1}^{\infty}\sum_\alpha(\psi_{\alpha,n}(x)-\psi_{\alpha,n}(x_0))^2 \le 2\sum_{n=n_\varepsilon+1}^{\infty}\sum_\alpha\left[(\psi_{\alpha,n}(x))^2 + (\psi_{\alpha,n}(x_0))^2\right] \le 2\sum_{n=n_\varepsilon+1}^{\infty}\frac{2}{n^2} \le \frac{\varepsilon^2}{2}.$$

Thus,

$$d_2(f(x), f(x_0)) \le \varepsilon,$$

showing continuity.

Step 4: $f^{-1} : f(X) \to X$ is continuous. For $y_0 = f(x_0) \in f(X)$ and a neighborhood U of x_0, there exists $B_{\alpha,n} \in \mathcal{B}$ such that $B_{\alpha,n} \subset U$, with $\psi_{\alpha,n}(x_0) > 0$. Set $\delta := \psi_{\alpha,n}(x_0)$. If $d_2(f(x), f(x_0)) < \delta$, then

$$|\psi_{\alpha,n}(x) - \psi_{\alpha,n}(x_0)| \leq d_2(f(x), f(x_0)) < \delta = \psi_{\alpha,n}(x_0).$$

Thus, $\psi_{\alpha,n}(x) > 0$, so $x \in B_{\alpha,n} \subset U$. Hence, f^{-1} is continuous. □

Lemma 4.2.7. *Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be a locally finite family of sets. Then*

$$\operatorname{Cl} \bigcup_{\alpha\in\Lambda} E_\alpha = \bigcup_{\alpha\in\Lambda} \operatorname{Cl} E_\alpha.$$

In particular, the union of a locally finite family of closed sets is closed.

Proof. Define

$$E := \bigcup_{\alpha\in\Lambda} E_\alpha.$$

By Proposition 3.2.5, $\operatorname{Cl} E \supset \bigcup_{\alpha\in\Lambda} \operatorname{Cl} E_\alpha$. For the reverse, let $x \in \operatorname{Cl} E$. There exists a neighborhood U of x intersecting finitely many E_α, say $E_{\alpha_1}, \ldots, E_{\alpha_m}$. For any neighborhood V of x, since $x \in \operatorname{Cl} E$, $(V \cap U) \cap E \neq \emptyset$. Thus,

$$(V \cap U) \cap \bigcup_{i=1}^{m} E_{\alpha_i} \neq \emptyset.$$

Hence,

$$x \in \operatorname{Cl} \bigcup_{i=1}^{m} E_{\alpha_i} = \bigcup_{i=1}^{m} \operatorname{Cl} E_{\alpha_i},$$

completing the proof. □

4.3 Paracompact Spaces

Paracompact spaces generalize metric spaces by ensuring locally finite refinements for open covers, and they are closely tied to partitions of unity.

Definition 4.3.1. A topological space (X, τ) is **paracompact** if it is Hausdorff and every open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ admits a locally finite open refinement.

Proposition 4.3.1. *Let (X, τ) be a paracompact space. Then (X, τ) is normal.*

Proof. **Step 1**: Let $C \subset X$ be closed and $x \in X \setminus C$. For each $y \in C$, since X is Hausdorff, there exist disjoint neighborhoods V_x and V_y of x and y. Then $\{V_y\}_{y\in C} \cup \{X \setminus C\}$ is an open cover. Since X is paracompact, there exists a locally finite refinement $\{U_\alpha\}_{\alpha\in\Lambda}$. Define

$$U := \bigcup_{\alpha\in\Lambda: U_\alpha \cap C \neq \emptyset} U_\alpha.$$

Then U is open, contains C, and by Lemma 4.2.7,

$$\operatorname{Cl} U = \bigcup_{\alpha\in\Lambda: U_\alpha \cap C \neq \emptyset} \operatorname{Cl} U_\alpha.$$

Each U_α with $U_\alpha \cap C \neq \emptyset$ is contained in some V_y, so $\operatorname{Cl} U_\alpha \subset \operatorname{Cl} V_y$, and $x \notin \operatorname{Cl} V_y$. Thus, $x \notin \operatorname{Cl} U$. The sets $X \setminus \operatorname{Cl} U$ and U are open, disjoint, and contain x and C.

Step 2: For disjoint closed sets $C_1, C_2 \subset X$, apply Step 1 to C_2 and each $y \in C_2$ to find open V_y with $V_y \cap C_1 = \emptyset$. Define U as in Step 1 with $C = C_2$. Then $C_1 \cap \operatorname{Cl} U = \emptyset$, so $X \setminus \operatorname{Cl} U$ and U are open, disjoint, and contain C_1 and C_2. Thus, X is normal. □

Theorem 4.3.2. *(Michael) Let (X, τ) be a normal space. The following are equivalent:*

(i) (X, τ) is paracompact.

(ii) Every open cover has a locally finite refinement (not necessarily open).

(iii) Every open cover has a closed, locally finite refinement.

(iv) Every open cover has a σ-locally finite open refinement.

Proof. **(i) $\implies$ (ii)**: Trivial.

(ii) $\implies$ (iii): Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover. By Theorem 4.1.1, there exists an open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ with $\operatorname{Cl} V_\alpha \subset U_\alpha$. Apply (ii) to $\{V_\alpha\}_{\alpha \in \Lambda}$ to find a locally finite refinement $\mathcal{E}$. Set $\mathcal{C} := \{\operatorname{Cl} E : E \in \mathcal{E}\}$. Then $\mathcal{C}$ refines $\{U_\alpha\}_{\alpha \in \Lambda}$. For $x \in X$, since $\mathcal{E}$ is locally finite, there exists a neighborhood U intersecting $E_1, \ldots, E_m \in \mathcal{E}$. If $\operatorname{Cl} E \cap U \neq \emptyset$, then $E \cap U \neq \emptyset$, so $E \in \{E_1, \ldots, E_m\}$. Thus, $\mathcal{C}$ is locally finite.

(iii) $\implies$ (i): Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover. By (iii), there exists a closed, locally finite refinement $\mathcal{C}$. For $x \in X$, there exists a neighborhood V_x intersecting finitely many elements of $\mathcal{C}$. Since $\{V_x\}_{x \in X}$ is an open cover, apply (iii) to find a closed, locally finite refinement $\mathcal{K}$. For $C \in \mathcal{C}$, define $\mathcal{K}_C := \{K \in \mathcal{K} : K \cap C = \emptyset\}$ and

$$D_C := X \setminus \bigcup_{K \in \mathcal{K}_C} K.$$

Since $\mathcal{K}_C$ is locally finite, $\bigcup_{K \in \mathcal{K}_C} K$ is closed (Lemma 4.2.7), so D_C is open and contains C. Thus, $\{D_C\}_{C \in \mathcal{C}}$ is an open cover. If $K \in \mathcal{K}$, then $K \cap D_C \neq \emptyset$ if and only if $K \cap C \neq \emptyset$. For $x \in X$, since $\mathcal{K}$ is locally finite, there exists a neighborhood U intersecting $K_1, \ldots, K_m \in \mathcal{K}$. If $U \cap D_C \neq \emptyset$, then some $y \in U$ is not in $\bigcup_{K \in \mathcal{K}_C} K$, so $y \in K \notin \mathcal{K}_C$. Thus, $U \cap K \neq \emptyset$, and some K_i intersects C. Since each $K_i \subset V_{x_i}$ intersects finitely many $C \in \mathcal{C}$, $U \cap D_C \neq \emptyset$ for finitely many C. Define

$$\mathcal{V} := \{D_C \cap U_{\alpha_C} : C \in \mathcal{C}, C \subset U_{\alpha_C}\}.$$

Then $\mathcal{V}$ is a locally finite open refinement of $\{U_\alpha\}_{\alpha \in \Lambda}$.

(iv) $\implies$ (ii): Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover. By (iv), there exists a σ-locally finite open refinement $\mathcal{V} = \bigcup_{n=1}^\infty \mathcal{V}_n$, with each $\mathcal{V}_n$ locally finite. For $V \in \mathcal{V}_n$, define $\mathcal{V}_V := \{U \in \mathcal{V} : U \in \mathcal{V}_k, k < n\}$ and

$$E_V := V \setminus \bigcup_{U \in \mathcal{V}_V} U.$$

Then $\{E_V\}_{V \in \mathcal{V}}$ covers X. For $x \in X$, let n be the smallest integer such that $x \in V \in \mathcal{V}_n$. Then V does not intersect E_U for $U \in \mathcal{V}_k$, $k > n$. For $k \leq n$, there exists a neighborhood W_k intersecting finitely many $U \in \mathcal{V}_k$. The neighborhood $V \cap \bigcap_{k=1}^n W_k$ intersects finitely many E_U, so $\{E_V\}_{V \in \mathcal{V}}$ is locally finite. □

Theorem 4.3.3. *(Michael) Let (X, τ) be a normal space. The following are equivalent:*

(i) (X, τ) is paracompact.

(ii) Every open cover has a locally finite partition of unity subordinated to it.

(iii) Every open cover has a partition of unity subordinated to it.

Proof. **(i) $\implies$ (ii)**: Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover. Since X is paracompact, there exists a locally finite open refinement. By Proposition 4.3.1, X is normal. Apply Theorem 4.1.2 to find a locally finite partition of unity subordinated to the refinement, hence to $\{U_\alpha\}_{\alpha\in\Lambda}$.

(ii) $\implies$ (i): Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover with a locally finite partition of unity $\{\varphi_i\}_{i\in I}$ subordinated to it. For $n \in \mathbb{N}$, $i \in I$, define

$$V_{i,n} := \{x \in X : \varphi_i(x) > \frac{1}{n}\}.$$

Then $\{V_{i,n}\}_{i\in I, n\in\mathbb{N}}$ is an open cover, since for $x \in X$, $\sum_{i\in I} \varphi_i(x) = 1$ implies there exists i such that $\varphi_i(x) > \frac{1}{n}$ for large n. Since $V_{i,n} \subset \operatorname{supp}\varphi_i \subset U_\alpha$ for some α, it is a refinement. To show it is σ-locally finite, fix $x_0 \in X$, $n \in \mathbb{N}$. Write

$$1 = \sum_{i\in I} \varphi_i(x_0) = \sum_{i\in I_0} \varphi_i(x_0),$$

where $I_0 = \{i \in I : \varphi_i(x_0) > 0\}$ is countable. Choose a finite $I_1 \subset I_0$ such that

$$\sum_{i\in I_1} \varphi_i(x_0) > 1 - \frac{1}{2n}.$$

By continuity, there exists a neighborhood U of x_0 such that

$$\sum_{i\in I_1} \varphi_i(x) > 1 - \frac{1}{n}$$

for $x \in U$. If $i \notin I_1$, $U \cap V_{i,n} = \emptyset$, since otherwise $\sum_{i\in I} \varphi_i(x) > 1$. Thus, U intersects finitely many $V_{i,n}$. By Theorem 4.3.2, X is paracompact.

(ii) $\implies$ (iii): Trivial.

(iii) $\implies$ (ii): Any partition of unity can be modified to be locally finite by refining the cover, as shown in (i) $\implies$ (ii). $\square$