Basic Foundations of Mathematical Logic

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Basic Logic

Our goal in this chapter is to build mathematical logic without any reference to set theory, or any other mathematical theory. We want to start from scratch, and we want to avoid any presumptions. So, part of our work is to make our assumptions clear, and to state them precisely. We should mention that in this chapter, any use of the words "set, number, sequence, ..." refers to the ordinary meaning of them, not to their meaning as mathematical objects.

We communicate through speaking and writing. We cannot build any theory of logic without using these tools, since we need to somehow communicate our ideas with each other, even with ourselves. So when we talk about starting from scratch, we do not mean that we will not use anything at all; although that would have been preferable, if it was possible. The **meta-language** is the language in which we are communicating, which in our case is English. It can be any other language too, like French, Persian, or Japanese. But as we said, we cannot create our theory without a meta-language, or some other way of communication.

However, we do not want to use the meta-language for stating results in our theory of logic, or later in our development of mathematics. There are several reasons for this. For example, our meta-language, which is an ordinary language spoken by people, is not precise enough. Another problem with ordinary languages is that they allow self-referencing. A famous example of this phenomena is the following sentence

"This sentence is false."

If the above sentence is true then it must be false, and vice versa!

Therefore we need to create a language for stating our logical and mathematical results, which has absolute precision, and does not allow self-referencing. This is our first step in this chapter. When we consider this language in contrast to the meta-language, we just call it the **language**. The next step after the construction of the language, is to develop our rules of logic. First we have to accept some basic rules as **axioms**, i.e. we have to accept them without any justification. This is the common approach in mathematics; and it is essentially unavoidable. Because if we want to deduce every statement from other statements, then we have to continue this process indefinitely; and this is not feasible. But it should be noted that although we do not logically justify the axioms from other valid statements, we have strong intuitive reasons regarding our choice of axioms.

A similar situation occurs when we define new notions. We cannot define every notion in terms of simpler notions; because then we have to continue this defining process indefinitely. So we have to work with some notions which do not have a definition. These notions are called **primitive notions**. But we still need to have some intuition about these undefined primitive notions, in order to be able to study them and other notions defined in terms of them. The axioms provide such intuitions. So in a sense, we can consider the axioms as the defining properties of the primitive notions.

After we constructed the language, and postulated our logical axioms, we need to deduce some properties from the axioms. But there are two problems here. Either we have not fully developed our logic yet; or we simply cannot apply it, because we want to avoid self-referencing, i.e. applying the logic to itself. However, we still need to be able to argue, so that we can transmit the fact that our choices are sound; and may be more importantly, to persuade ourselves that our choices are sound! In order to do that, we use argumentation in the meta-language. Note that we are actually doing this in this very paragraph! When we use reasoning in the meta-language, we are in fact using some rules of logic, or at least some elementary forms of them. We refer to these rules as **meta-logic**.

Note that the rules of the meta-logic are not different than the rules of the logic that we are going to construct. The difference lies at the level that we are applying them. So in some sense, we are using logic to construct logic! Although this is true, the situation is not as bad as it seems. Firstly, because we will only use very basic rules at the level of meta-logic (for more on this topic, see the discussion at the beginning of Section 1.7). Secondly, we can accept everything that we will prove about logic using meta-logic as true axioms. And we can consider those reasonings in meta-logic merely as convincing rationale for our choice of axioms. This is in fact one of the ways that mathematicians chose and continue to choose a set of axioms for a theory. They use tools outside the theory to convince themselves, and others, that those sets of axioms are appropriate. They often rely on their intuitions in this process; and as we will see, the reasonings in the meta-logic provide valuable intuitions for us. However, after we developed our logic, we must stop using the meta-logic completely.

1.1 Formulas

Let us start by constructing the language. This language is called the **language of set theory**. It is powerful enough to express all of mathematics. We will use it first to construct set theory. The language of set theory is an example of the so-called **formal languages**. There are many other formal languages; and their study is a major part of mathematical logic. But their proper study requires tools of set theory and other parts of mathematics. This is also true about the language of set theory itself. So we need to develop set theory before studying formal languages. However, we do not need to understand every aspect of the language of set theory, when we use it to construct set theory. Hence, we will not prove many results about the language of set theory at this point. We mainly develop it enough so that we can express our rules of logic, and then be able to construct our theory of sets.

Later, when we study formal languages, we can treat the language that we are going to construct now as a *mid-level meta-language*, i.e. a language between the ordinary language that we speak, and the formal language that we want to study. This way, we also avoid any circular reasoning, when we further study the formal language of set theory. Thus we will have two distinct copies of the language of set theory; one that we construct in this chapter, and another one which is a specific instance of formal languages. Although, we informally know that the two copies of the language of set theory are essentially the same.

Now, any language has an alphabet, i.e. a collection of letters. These are the symbols that we write on paper in order to communicate through that language. We do not define the alphabet though; we simply treat it as a primitive notion.

Primitive Notion 1.1. The **letters** are abstract notions that we represent by symbols on paper. We assume that we are able to recognize the letters from their symbols, and we can distinguish between them through their symbols. We refer to the collection of all letters as the **alphabet**. In the language of set theory, the letters are of the following types:

(i) Variables: $a, b, c, \dots, x, y, z, a_0, b_0, \dots, y_0, z_0, a_1, \dots, z_1, a_2, \dots$
(ii) Logical symbols: $\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \perp, \forall, \exists$
(iii) Special symbols: $=, \in$
(iv) Parentheses: $($, $)$

Remark. Note that the numbers used as an index in the variables are just symbols, and do not have any specific mathematical meaning at this point.

Remark. We tacitly assume that we have infinitely many distinct variables. Although in practice we will only use finitely many of them.

Primitive Notion 1.2. A **formula** is a primitive notion, which intuitively, is a finite sequence of letters that have some specific structure. These structures will be discussed in the next axiom. We represent formulas by writing the symbols of their letters successively. We assume that we are able to recognize formulas from their representations, and we can distinguish between them through their representations.

Before proceeding any further, let us mention an important process in meta-language, and also in mathematics. We are talking about the process of **naming** objects, and assigning **notations** to them. For example the symbol "x" is a name, or a notation, for the variable x in our alphabet. Later we will see that the variable x itself can be a name for a set in our universe of sets! We will also use names for formulas. For example we will see that $\forall x(x = x)$ is a formula in our language of set theory. We can call and denote this formula by ϕ. This process of naming formulas, and other objects, has many advantages. For example it makes it easier to talk about and refer to complicated formulas. Another important benefit is that we can use a name as a placeholder for many, or all, formulas; and state a general property about them. We will see many instances of this in the rest of the chapter.

Sometimes we assign a new notation to an object that already has a notation, in order to make it easier to read and comprehend the text. For example in arithmetic we write n^4 to denote the number $n \cdot n \cdot n \cdot n$. This type of notation is called **abbreviated notation**. There are many different ways of abbreviation in mathematics, and it is not feasible to try to formalize them. However, they usually do not create any confusion. Whenever the need arises, we will use abbreviated notations in these notes too.

Axiom 1.1.

(i) $\perp$ *is a formula.*

(ii) *For every variables like x, y the following are formulas*

$$x = y, \qquad x \in y.$$

Note that x, y can also be the same variable.

(iii) *If ϕ is a formula then*

$$\neg(\phi)$$

is also a formula.

(iv) *Let ϕ, ψ be two formulas, that are not necessarily distinct. Then the following are also formulas*

$$(\phi) \wedge (\psi), \quad (\phi) \vee (\psi), \quad (\phi) \to (\psi), \quad (\phi) \leftrightarrow (\psi).$$

(v) *If ϕ is a formula and x is a variable, then the following are also formulas*

$$\forall x(\phi), \qquad \exists x(\phi).$$

Remark. We assume that every formula is constructed after several applications of the above rules, and there is no other way to construct a formula. But in order to make this precise we need some basic mathematical tools; so we do not do this here. However, we will not state any other axiom about construction of formulas; so if something is not constructed in the above ways, it cannot be shown that it is a formula in the language of set theory.

Hence, every formula is built in the above ways by starting from $\perp$, or formulas of the form $x = y$ and $x \in y$, for some variables x, y. For this reason, $\perp$ and formulas of the form $x = y$ and $x \in y$ are called *atomic formulas*.

Remark. The above axiom is actually an **axiom schema**. This means that it is actually an infinite collection of axioms. For example in part (i), x, y can be any variables; so for example $a = b$, $u = c_1$, $z_2 \in z_2$, ... are all formulas. Thus we actually have an axiom for every pair of variables. Also in the other parts of the axiom, ϕ, ψ can be any formulas. So we actually have an axiom for every pair of formulas.

Notation. When there is no risk of confusion, we will usually omit some of the parentheses in the notation introduced in the above axiom. For example we may write

$$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \phi \to \psi, \quad \phi \leftrightarrow \psi, \quad \forall x\phi, \quad \exists x\phi.$$

When we drop some of the parentheses, some different formulas may look like the same. For example if we drop the parentheses in $(\phi)\wedge(\psi \to \tau)$ and $(\phi\wedge\psi) \to (\tau)$, we will get $\phi \wedge \psi \to \tau$. In order to avoid the confusions that arise in this way we will follow the following *order of precedence*. We assume that $=, \in$ bind stronger than $\neg$, and $\neg$ binds stronger than $\forall, \exists$, and $\forall, \exists$ bind stronger than $\wedge, \vee$, and $\wedge, \vee$ bind stronger than $\to, \leftrightarrow$. So by our convention, $\phi \wedge \psi \to \tau$ means $(\phi \wedge \psi) \to (\tau)$. As another example, consider

$$\forall x \neg\phi \wedge x \in z \leftrightarrow (\psi \vee \tau \to \exists y\sigma),$$

which is an abbreviation of the following formula

$$\big((\forall x(\neg\phi)) \wedge (x \in z)\big) \leftrightarrow \big((\psi \vee \tau) \to (\exists y(\sigma))\big).$$

In practice, we usually keep some of the parentheses to make reading the expression easier.

Let us mention that at this point, we merely consider formulas as sequences of symbols, and we do not assign any meaning to them. In other words, we are only concerned with the **syntax** of the language, i.e. the formal rules of constructing well-formed expressions, aka formulas. Later when we develop set theory, we can assign meanings and provide interpretations for the formulas, i.e. we can study the **semantics** of the language. However, it is illuminating to informally introduce some of the notions related to the semantics earlier. An important notion related to semantics is the notion of **truth**. We will not discuss this notion at length now; instead, we will use our intuitive understanding that a statement can be **true** or **false**, depending on whether its interpretation really happens or not. This also applies to the statements in the meta-language.

Next consider the logical symbols. The symbols $\bot, \neg, \wedge, \vee, \to, \leftrightarrow$ are called **logical connectives**. The symbol $\bot$ is called **falsum** or **absurdum**, and represents a false formula. It is included in the language mainly because it is more convenient to have a special notation for a formula which is always false. The symbol $\neg$ is called **negation**. Let ϕ be a formula. Then $\neg\phi$ means "not ϕ". The formula $\neg\phi$ is true when ϕ is false, and it is false when ϕ is true. The symbol $\wedge$ is called **conjunction**. For two formulas ϕ, ψ, the formula $\phi \wedge \psi$ means "ϕ and ψ". It is only true when both ϕ, ψ are true. The symbol $\vee$ is called **disjunction**. For two formulas ϕ, ψ, the formula $\phi \vee \psi$ means "ϕ or ψ". It is only true when at least one of ϕ, ψ is true; and it is false when both ϕ, ψ are false. Note that unlike some uses of the word "or" in ordinary language, the "or" in mathematical logic is *inclusive*. In other words, the truth of $\phi \vee \psi$ does not mean that exactly one of ϕ, ψ is true.

The symbol $\to$ is called **implication** or **conditional**. For two formulas ϕ, ψ, the formula $\phi \to \psi$ means "if ϕ then ψ". In the conditional formula $\phi \to \psi$, the formula ϕ is called the **antecedent**, and the formula ψ is called the **consequent**. The formula $\phi \to \psi$ is only false when ϕ is true and ψ is false. In particular, when both ϕ, ψ are false, then $\phi \to \psi$ is true. Note that unlike the usual use of "if ... then ..." in ordinary language, the truth of $\phi \to \psi$ does not mean that there is a causal relationship between ϕ, ψ. It merely means that if ϕ is true then ψ is true. Hence when ϕ is false, we cannot deduce anything about the truth of ψ, i.e. ψ can be true or false. To distinguish between the two concepts of implication and causal relationship, $\to$ is also called *material implication*. The symbol $\leftrightarrow$ is called **biconditional**. For two formulas ϕ, ψ, the formula $\phi \leftrightarrow \psi$ means "ϕ if and only if ψ" i.e. "if ϕ then ψ, and if ψ then ϕ". It is only true when ϕ, ψ are both true, or both false. Intuitively, $\phi \leftrightarrow \psi$ is true when both $\phi \to \psi$ and $\psi \to \phi$ are true. We will prove this fact later.

The following table summarizes the above information. It is called the *truth table*. Here T and F are abbreviations for true and false respectively. Note that at this moment, the truth table is just a tool to represent the informal meanings of logical connectives.

ϕ	ψ	$\neg\phi$	$\neg\psi$	$\phi \wedge \psi$	$\phi \vee \psi$	$\phi \to \psi$	$\phi \leftrightarrow \psi$	$\bot$
T	T	F	F	T	T	T	T	F
T	F	F	T	F	T	F	F	F
F	T	T	F	F	T	T	F	F
F	F	T	T	F	F	T	T	F

The symbols $\forall, \exists$ are called **quantifiers**. The symbol $\forall$ is called **universal quantifier**. Let ϕ be a formula. Suppose ϕ states some property about the variable x. To emphasize this we will write $\phi(x)$. Then $\forall x \phi(x)$ means that "for every x, $\phi(x)$ holds". Instead of "for every" we can also use "for all" or "for each". The formula $\forall x \phi(x)$ is only true when $\phi(x)$ is true for every choice of x. The symbol $\exists$ is called **existential quantifier**. The formula $\exists x \phi(x)$ means that "there exists x such that $\phi(x)$ holds". It is only true if $\phi(x)$ is true for at least one choice of x. An important concept regarding the quantifiers is their **domain of discourse**, which is also referred to as the **universe**. This is the collection of all objects x that we have to consider in order to examine the truth of $\forall x \phi$ or $\exists x \phi$. In the language of set theory, the domain of discourse is always the universe of all sets. But in meta-language, the domain of discourse can vary for different sentences. For example, we can express properties about all formulas; or we can have statements about all rules of logic.

Finally, consider the special symbols $=, \in$. The symbol $\in$ denotes the **set membership** relation. So $x \in y$ means that "x is an element of y", or equivalently "y contains x". The symbol $=$ denotes **equality**. So $x = y$ means that "x is equal to y". There is an important point here that should be noted. We have two notions of equality: one at the level of language, and one at the level of meta-language. When we say "x and y are equal" in the meta-language, we mean that x, y denote the same letter. But when we say "$x = y$" in the language, we mean that x, y denote the same set.

Notation. Instead of $\neg(x \in y)$ we usually write $x \notin y$. This means that "x does not belong to y". Also, instead of $\neg(x = y)$ we usually write $x \neq y$; which means that "x, y are not equal". Finally, instead of $\neg\bot$ we usually write $\top$; which represents a formula that is always true.

Remark. Let us emphasize again that the above semantic interpretations for the formulas of the language are all informal at this point. However, when we use connectives or quantifiers in meta-language, we assume that they have the above interpretations. In particular, our use of "or" is always inclusive; and our use of "if ... then ..." always indicates a material implication.

Remark. Another point that we wish to emphasize again is that a formula which is syntactically well-formed does not need to be true from a semantic viewpoint. In other words, there are formulas which are false. For example there might not be a set x such that $x \in x$, nevertheless, the formula $x \in x$ is syntactically well-formed.

Remark. The **converse** of the conditional formula $\phi \to \psi$ is the formula $\psi \to \phi$, and the **inverse** of $\phi \to \psi$ is the formula $\neg\phi \to \neg\psi$. If $\phi \to \psi$ is true, then its converse and inverse are not necessarily true, nor necessarily false. The **contrapositive** of $\phi \to \psi$ is the formula $\neg\psi \to \neg\phi$. We will see that the contrapositive of a conditional formula is equivalent to it, i.e. the contrapositive is true if and only if the original formula is true.

***Example* 1.1.** Let ϕ be a formula. Consider the formula $\phi \wedge \neg\phi$. Note that semantically, this formula is always false. Because if ϕ is true then $\neg\phi$ is false, and hence $\phi \wedge \neg\phi$ is false. And if ϕ is false then $\phi \wedge \neg\phi$ is false. For any formula like ϕ, we call $\phi \wedge \neg\phi$ a **contradiction**. We also consider $\perp$ as a contradiction.

As another example consider $\phi \vee \neg\phi$. We can similarly see that from a semantic viewpoint, this formula is always true.

1.2 Rules of Inference

Primitive Notion 1.3. The primitive notion of **entailment** is a relation between several formulas $\phi_1, \phi_2, \dots, \phi_n$ and another formula ψ. This relation is denoted by

$$\phi_1, \phi_2, \dots, \phi_n \vdash \psi.$$

The symbol $\vdash$ is called **turnstile**. The formulas $\phi_1, \phi_2, \dots, \phi_n$ are called **premises**, and the formula ψ is called **conclusion**. Intuitively, the above relation means that ϕ_1 and ϕ_2 and ... and ϕ_n together logically imply ψ. An entailment is allowed to have no premises; so we can have

$$\vdash \psi.$$

In this case we say that ψ is a **theorem**.

As we have done above, we represent an entailment by writing the representations of its formulas successively, and we separate the conclusion by a turnstile from the premises, and we separate the premises by " , ". We assume that we are able to recognize entailments from their representations, and we can distinguish between them through their representations. We also assume that we can recognize the formulas in an entailment from the representation of that entailment, and we can also figure out whether a given formula is a premise or the conclusion.

Remark. Let us emphasize again that the numbers which appear as indices in the names of formulas are just symbols, and do not have any specific mathematical meaning at this point.

Let us provide an informal semantic interpretation for the entailment relation. The entailment $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ means that if $\phi_1, \phi_2, \dots, \phi_n$ are all true, then ψ must also be true. It does not mean that ψ is necessarily true, because some of the $\phi_1, \phi_2, \dots, \phi_n$ might not be true. But if we have $\vdash \psi$, then ψ must be true. In other words, theorems are true statements.

Notation. To simplify the notation, we will use capital Greek letters to denote several formulas. For example if we denote $\phi_1, \phi_2, \ldots, \phi_n$ by Γ, then we can write

$$\Gamma \vdash \psi$$

to denote $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$. In this notation, we allow Γ to be empty too; so $\Gamma \vdash \psi$ may also denote $\vdash \psi$.

The process of showing that an entailment such as $\Gamma \vdash \psi$ exists, is called **deduction**. In order to perform deductions, we need some rules to know when an entailment exists. These rules are called **rules of inference**. They are the rules of logic that we use to prove theorems. We will express them in the next few axioms.

Axiom 1.2. *Suppose $\Gamma, \Delta, \Lambda_0, \Lambda_1, \Lambda_2$ denote collections of several formulas, which can be empty too. Let ψ be a formula. Then we have*

(i)
$$\psi \vdash \psi.$$

(ii) *If $\Gamma \vdash \psi$ then*
$$\Gamma, \Delta \vdash \psi.$$

(iii) *If $\Gamma, \Delta, \Delta \vdash \psi$ then*
$$\Gamma, \Delta \vdash \psi.$$

(iv) *If $\Lambda_0, \Gamma, \Lambda_1, \Delta, \Lambda_2 \vdash \psi$ then*
$$\Lambda_0, \Delta, \Lambda_1, \Gamma, \Lambda_2 \vdash \psi.$$

(v) *If $\Gamma \vdash \phi$ and $\Delta, \phi \vdash \psi$ then*
$$\Gamma, \Delta \vdash \psi.$$

The above rules are called **structural rules**. They do not refer to any logical connective or quantifier; rather, they depend on the structure of the entailments themselves. The first rule is self-evident; it states that any formula implies itself. The second rule is called **weakening**. It means that if ψ can be deduced from a collection of formulas Γ, then ψ can also be deduced from a larger collection of formulas Γ, Δ. In other words, the entailment can be weakened by adding extra formulas to the premises. The third rule is called **contraction**. It means that repetition of formulas in the premises is superfluous, and extra occurrences of repeated formulas can be eliminated. The forth rule is called **exchange**. It means that the order of formulas in the premises is irrelevant.

The last rule is called the **cut rule**. It states that entailment is a transitive relation. In other words, it formalizes the intuitive fact that if several formulas imply several other formulas, and those other formulas imply another formula, then the initial formulas also imply the last formula. Although we have stated the axiom in a way that there is only one formula in the middle, we can deduce the more general versions from this axiom. For example if $\Gamma \vdash \phi_1$ and $\Gamma \vdash \phi_2$, then $\Delta, \phi_1, \phi_2 \vdash \psi$ implies that $\Gamma, \Delta \vdash \psi$. To see this note that by using the cut rule we get $\Gamma, \Delta, \phi_1 \vdash \psi$. Notice that here we considered Δ, ϕ_1 in place of Δ. Now if we use the cut rule again we obtain $\Gamma, \Gamma, \Delta \vdash \psi$. Notice that here we considered Γ, Δ in place of Δ. Finally by using the contraction and exchange rules we get $\Gamma, \Delta \vdash \psi$, as desired. We can similarly extend the cut rule to the case of three or more formulas in the middle. But if we want to prove a general cut rule with an arbitrary number of formulas in the middle, we need mathematical induction. However, we usually do not have more than a few formulas in the middle; so we do not need the general version here.

Remark. A special case of the cut rule is when Δ is empty. In this case we have: If $\Gamma \vdash \phi$ and $\phi \vdash \psi$, then $\Gamma \vdash \psi$. This special case makes the transitivity of entailment more apparent.

Axiom 1.3 (Rules of inference for logical connectives)**.** *Suppose Γ denotes a collection of several formulas, which can be empty too. Let ϕ, ψ, τ be formulas. Then we have*

(i) *Introduction of $\wedge$:*
$$\phi, \psi \vdash \phi \wedge \psi.$$

(ii) *Elimination of $\wedge$:*
$$\phi \wedge \psi \vdash \phi, \qquad \textit{and} \qquad \phi \wedge \psi \vdash \psi.$$

(iii) *Introduction of $\rightarrow$:*
$$\textit{If} \quad \Gamma, \phi \vdash \psi \qquad \textit{then} \qquad \Gamma \vdash \phi \rightarrow \psi.$$

(iv) *Elimination of $\rightarrow$, or Modus ponens :*
$$\phi \rightarrow \psi,\ \phi \vdash \psi.$$

(v) *Introduction of $\vee$:*
$$\phi \vdash \phi \vee \psi, \qquad \textit{and} \qquad \psi \vdash \phi \vee \psi.$$

(vi) *Elimination of $\vee$, or Proof by cases :*
$$\textit{If} \quad \Gamma, \phi \vdash \tau \quad \textit{and} \quad \Gamma, \psi \vdash \tau \qquad \textit{then} \qquad \Gamma,\ \phi \vee \psi \vdash \tau.$$

(vii) *Introduction of $\leftrightarrow$:*
$$\textit{If} \quad \Gamma, \phi \vdash \psi \quad \textit{and} \quad \Gamma, \psi \vdash \phi \qquad \textit{then} \qquad \Gamma \vdash \phi \leftrightarrow \psi.$$

(viii) *Elimination of* $\leftrightarrow$:

$$\phi \leftrightarrow \psi,\ \phi \vdash \psi, \qquad \textit{and} \qquad \phi \leftrightarrow \psi,\ \psi \vdash \phi.$$

(ix) *Introduction of* $\neg$:

$$\textit{If} \quad \Gamma, \phi \vdash \bot \qquad \textit{then} \qquad \Gamma \vdash \neg\phi.$$

(x) *Elimination of* $\neg$:

$$\phi, \neg\phi \vdash \bot.$$

(xi) *Reductio ad absurdum* (RAA) :

$$\textit{If} \quad \Gamma, \neg\phi \vdash \bot \qquad \textit{then} \qquad \Gamma \vdash \phi.$$

Remark. Note that the axioms that we state about entailment are all axiom schemas. For example in the above axiom, ϕ, ψ can be any formulas. Similarly, Γ can be any collection of formulas. In other words, as we said before, for every formulas ϕ, ψ, and every collection of formulas Γ, we have an axiom as above.

The process of deduction is purely syntactic, i.e. it is a set of rules which tell us how to derive a formula from several other formulas, by looking only at their structure as sequences of symbols. However, when we consider the informal semantic interpretations for the formulas, we will see that the rules of inference are compatible with our intuitive understanding of the notion of reasoning. This is actually the reason that we choose them as rules of inference. For example we know that the truth of $\phi \wedge \psi$ is equivalent to the truth of both ϕ and ψ. Hence, the truth of ϕ and ψ implies the truth of $\phi \wedge \psi$, i.e. $\phi \wedge \psi$ is derivable from ϕ and ψ. This is exactly the rule of introduction of $\wedge$. Similarly, the truth of ϕ and ψ follows from the truth of $\phi \wedge \psi$, i.e. $\phi \wedge \psi$ implies both ϕ and ψ. And this is the rule of elimination of $\wedge$. The other rules also mirror our intuitive understanding of the logical connectives, and they can be justified similarly. Note that justification here means a heuristic argument which motivates the choice of some rule as an axiom, not a rigorous proof of the axiom!

Let us inspect the rules more closely. First consider the introduction of $\rightarrow$. It says that if we can deduce ψ from ϕ and some other hypotheses Γ, then we can infer from Γ alone that $\phi \rightarrow \psi$. In other words, Γ implies that "ϕ implies ψ". This is in agreement with our intuitive understandings of implication "$\rightarrow$" and entailment "$\vdash$". Next consider elimination of $\rightarrow$. It is also called **modus ponens**, which literally means "mood that affirms". It says that if $\phi \rightarrow \psi$ is true, and ϕ is true, then ψ must also be true. Note that this is in agreement with our intuition regarding material implication; we can even say that this is exactly the meaning of a conditional statement. The introduction and elimination of $\leftrightarrow$ can be interpreted similarly. Note that they both reflect the fact that $\phi \leftrightarrow \psi$ is regarded as "$\phi \to \psi$ and $\psi \to \phi$".

The introduction of $\vee$ has a simple meaning. It says that if one of the ϕ or ψ is true then $\phi \vee \psi$ is true. Note that this rule implies that $\vee$ is the inclusive "or". The elimination of $\vee$ says that if we can deduce τ by assuming ϕ, and we can deduce τ by assuming ψ, then we can also deduce τ by assuming "either ϕ or ψ", i.e. we can deduce τ by assuming $\phi \vee \psi$. This rule is also called **proof by cases**, because it says that if ϕ or ψ is true, and we can show that τ is true in the case that ϕ is true, and we can also show that τ is true in the case that ψ is true, then we have showed that τ is true.

Finally consider the rules concerning $\neg$. The introduction of $\neg$ says that if the assumption of ϕ leads to a contradiction, then ϕ must be false, which means $\neg\phi$ must be true. And, the elimination of $\neg$ says that $\phi, \neg\phi$ lead to a contradiction. The **reductio ad absurdum** (RAA), which literally means "reduction to absurdity", is a special rule among the above rules. It says that if the assumption of "not ϕ" leads to a contradiction, then ϕ must be true. Intuitively, this rule presuppose that every formula is either true or false. So if "not ϕ" is false then ϕ must be true. However, this fact does not follow from the other axioms. In fact we will prove it using the RAA.

In addition, we should mention that although RAA and introduction of $\neg$ look similar, they are different rules. To see this note that if we apply the introduction of $\neg$ to the premises of RAA, then we obtain $\neg\neg\phi$. And from a syntactic viewpoint, there is no reason that the formula $\neg\neg\phi$ must imply ϕ. In fact it can be shown that the other axioms do not imply that ϕ follows from $\neg\neg\phi$. Thus we have to prove this fact using RAA.

Remark. The application of the rules I$\neg$ and RAA in a deduction is also known as **proof by contradiction**.

***Example* 1.2.** Let us demonstrate a simple application of the rules of inference. We know that $\psi, \phi \vdash \psi \wedge \phi$, due to the introduction of $\wedge$. Now we can use the exchange rule to switch the order of ψ, ϕ in the premises, and conclude that

$$\phi, \psi \vdash \psi \wedge \phi.$$

A question that arises is that why do we have two notions of implication denoted by $\to$ and $\vdash$? To answer this question, first note that $\vdash$ is a symbol in meta-language that we introduced; and it is not part of the language of set theory. Whereas $\to$ is a symbol in the language of set theory. So the two notions lie at different levels.

Another distinction is that $\vdash$ denotes a process of deduction, but $\to$ is just a syntactic symbol which we use to construct formulas, and a priori it does not have a meaning. In other words, $\phi \to \psi$ is a formula that says "if ϕ then ψ", which might be true or false. But $\phi \vdash \psi$ means that we can deduce ψ from ϕ in a process of deduction. However, as we will see below, the two notions are closely related for conditional formulas which are true.

We know that if $\Gamma, \phi \vdash \psi$ then $\Gamma \vdash \phi \to \psi$, due to the introduction of $\to$. In other words, we can say that "entailment" implies "implication". Let us show that the converse also holds, i.e. if $\Gamma \vdash \phi \to \psi$ then $\Gamma, \phi \vdash \psi$. The reason is that by elimination of $\to$ we know that $\phi \to \psi, \phi \vdash \psi$. Hence by the exchange rule we get $\phi, \phi \to \psi \vdash \psi$. Now we get the desired by the cut rule. Therefore we have shown that

$$\Gamma, \phi \vdash \psi \qquad \text{if and only if} \qquad \Gamma \vdash \phi \to \psi.$$

In particular we have

$$\phi \vdash \psi \qquad \text{if and only if} \qquad \vdash \phi \to \psi.$$

Informally, this equivalence means that in order to show that $\phi \to \psi$ is true, it suffices to deduce ψ by assuming ϕ, i.e. to show that $\phi \vdash \psi$. In mathematics, conditional statements are usually proved in this way.

Remark. When $\vdash \phi \to \psi$, we say that ϕ is a **sufficient condition** for ψ, because in order for ψ to hold it suffices that ϕ holds. We also say that ψ is a **necessary condition** for ϕ, because if ϕ holds then ψ must necessarily hold too. In addition, when $\vdash \phi \leftrightarrow \psi$, we say that ϕ is a necessary and sufficient condition for ψ, and vice versa.

Remark. Suppose we know that $\vdash \phi$, i.e. ϕ is a theorem. Also, suppose we want to show that $\Gamma \vdash \psi$. Then it suffices to show that $\Gamma, \phi \vdash \psi$. Because by the cut rule, from $\vdash \phi$ and $\Gamma, \phi \vdash \psi$ we can conclude that $\Gamma \vdash \psi$. This argument shows that we may use theorems in the premises of entailments to deduce other formulas, and then we can discard those theorems.

Notation. For simplicity, we will denote the introduction and elimination rules by the letters "I" and "E" followed by the respective connectives. For example we will denote the introduction of $\to$ by I$\to$, and the elimination of $\vee$ by E$\vee$.

***Example* 1.3.** For every formula like ϕ we have $\phi, \neg\phi \vdash \bot$ by E$\neg$. Hence by RAA we get

$$\phi \vdash \phi.$$

Thus we can prove the above structural rule from the inference rules for connectives. However, it seems more natural to treat the above rule as an axiom. As another example note that we also have

$$\vdash \phi \to \phi.$$

Because we know that $\phi \vdash \phi$. Thus by I$\to$ we get $\vdash \phi \to \phi$, as desired.

***Example* 1.4.** By E$\wedge$ we know that $\phi \wedge \neg\phi \vdash \phi$ and $\phi \wedge \neg\phi \vdash \neg\phi$. On the other hand, by E$\neg$ we have $\phi, \neg\phi \vdash \bot$. Hence by the cut rule we get

$$\phi \wedge \neg\phi \vdash \bot.$$

More generally, suppose $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. Then by the cut rule and E$\neg$ we obtain $\Gamma \vdash \bot$.

In our first theorem, we present a few additional rules of inference. Note that here the meaning of "theorem" is different than its meaning in the Primitive Notion 1.3. Here, "theorem" is a statement in the meta-language which states a valid fact about logic; in contrast, "theorem" in Primitive Notion 1.3 is a formula in the language which states a true fact about sets. A better term for theorems in the meta-language could be "meta-theorem", but for simplicity we will keep calling them theorems.

Another important point is that the theorems that we state about entailment are also schemas, similar to the axioms. For example in the following theorem, ϕ, ψ, τ, σ can be any formulas. Similarly, Γ can be any collection of formulas. In other words, as we said before, for every formulas ϕ, ψ, τ, σ, and every collection of formulas Γ, we have a theorem as below.

Remark. We will usually try to cite every rule that we use inside a proof, but for simplicity, sometimes we will not mention our uses of structural rules, especially the exchange rule.

Theorem 1.1. *Suppose Γ denotes a collection of several formulas, which can be empty too. Let ϕ, ψ, τ, σ be formulas. Then we have*

(i) *Ex falso quodlibet (EFQ) :*

$$\phi \wedge \neg\phi \vdash \psi, \qquad \textit{and} \qquad \bot \vdash \psi.$$

(ii) *Law of excluded middle :*

$$\vdash \phi \vee \neg\phi.$$

(iii) *Law of non-contradiction :*

$$\vdash \neg(\phi \wedge \neg\phi), \qquad \textit{and} \qquad \vdash \top.$$

(iv) *Modus tollens :*

$$\phi \to \psi,\ \neg\psi \vdash \neg\phi.$$

(v) *Hypothetical syllogism :*

$$\phi \to \psi,\ \psi \to \tau \vdash \phi \to \tau.$$

(vi) *Disjunctive syllogism, or Modus tollendo ponens* :

$$\phi \vee \psi,\ \neg\phi \vdash \psi, \qquad \textit{and} \qquad \phi \vee \psi,\ \neg\psi \vdash \phi.$$

(vii) *Modus ponendo tollens* :

$$\neg(\phi \wedge \psi),\ \phi \vdash \neg\psi, \qquad \textit{and} \qquad \neg(\phi \wedge \psi),\ \psi \vdash \neg\phi.$$

(viii) *Constructive dilemma* :

$$\textit{If} \quad \Gamma, \phi \vdash \tau \quad \textit{and} \quad \Gamma, \psi \vdash \sigma \qquad \textit{then} \qquad \Gamma,\ \phi \vee \psi \vdash \tau \vee \sigma.$$

(ix) *Destructive dilemma* :

$$\textit{If} \quad \Gamma, \phi \vdash \tau \quad \textit{and} \quad \Gamma, \psi \vdash \sigma \qquad \textit{then} \qquad \Gamma,\ \neg\tau \vee \neg\sigma \vdash \neg\phi \vee \neg\psi.$$

(x)

$$\textit{If} \quad \Gamma, \phi \vdash \neg\phi \qquad \textit{then} \qquad \Gamma \vdash \neg\phi.$$

Proof. **(i)** We know that $\bot \vdash \bot$. Hence by the weakening rule we get $\bot, \neg\psi \vdash \bot$. Now by RAA we obtain $\bot \vdash \psi$, as desired. In addition we know that $\phi \wedge \neg\phi \vdash \bot$. Therefore by the cut rule we also get $\phi \wedge \neg\phi \vdash \psi$.

(ii) By I$\vee$ we have $\phi \vdash \phi \vee \neg\phi$. We also know that $\neg(\phi \vee \neg\phi) \vdash \neg(\phi \vee \neg\phi)$. So by the weakening rule we get $\neg(\phi \vee \neg\phi), \phi \vdash \phi \vee \neg\phi$, and $\neg(\phi \vee \neg\phi), \phi \vdash \neg(\phi \vee \neg\phi)$. Hence by the cut rule and E$\neg$ we obtain

$$\neg(\phi \vee \neg\phi), \phi \vdash \bot.$$

Thus by I$\neg$ we get $\neg(\phi \vee \neg\phi) \vdash \neg\phi$. Now by I$\vee$ and the cut rule we obtain $\neg(\phi \vee \neg\phi) \vdash \phi \vee \neg\phi$. Therefore by the cut rule and E$\neg$ we get

$$\neg(\phi \vee \neg\phi) \vdash \bot.$$

Hence by RAA we get $\vdash \phi \vee \neg\phi$, as desired.

(iii) We know that $\phi \wedge \neg\phi \vdash \bot$. Hence by I$\neg$ we get $\vdash \neg(\phi \wedge \neg\phi)$, as desired. Similarly, we know that $\bot \vdash \bot$. So by I$\neg$ we get $\vdash \neg\bot$, which is the same as saying $\vdash \top$.

(iv) By E$\to$ and the weakening rule we have $\phi \to \psi, \phi, \neg\psi \vdash \psi$. We also know that $\phi \to \psi, \phi, \neg\psi \vdash \neg\psi$. Thus by the exchange and cut rules, and E$\neg$, we get $\phi \to \psi, \neg\psi, \phi \vdash \bot$. Hence by I$\neg$ we obtain $\phi \to \psi, \neg\psi \vdash \neg\phi$, as desired.

(v) By E$\to$ we have $\phi \to \psi, \phi \vdash \psi$, and $\psi \to \tau, \psi \vdash \tau$. Hence by the cut and exchange rules we have $\phi \to \psi, \psi \to \tau, \phi \vdash \tau$. Thus by I$\to$ we get $\phi \to \psi, \psi \to \tau \vdash \phi \to \tau$, as desired.

(vi) By E$\neg$ we have $\phi, \neg\phi \vdash \bot$, and by EFQ rule we have $\bot \vdash \psi$. Hence by the cut rule we get $\phi, \neg\phi \vdash \psi$. Also, by the weakening rule we have $\psi, \neg\phi \vdash \psi$.

Therefore by E$\vee$ and exchange rule we get $\phi \vee \psi, \neg\phi \vdash \psi$, as desired. The other case can be proved similarly.

(vii) By the weakening rule and I$\wedge$ we have $\neg(\phi \wedge \psi), \phi, \psi \vdash \phi \wedge \psi$. We also know that $\neg(\phi \wedge \psi), \phi, \psi \vdash \neg(\phi \wedge \psi)$. Thus by the cut rule and E$\neg$ we get

$$\neg(\phi \wedge \psi), \phi, \psi \vdash \bot.$$

Hence by I$\neg$ we obtain $\neg(\phi \wedge \psi), \phi \vdash \neg\psi$; and by exchange rule and I$\neg$ we obtain $\neg(\phi \wedge \psi), \psi \vdash \neg\phi$, as desired.

(viii) By I$\vee$ we have $\Gamma, \phi \vdash \tau \vdash \tau \vee \sigma$, and $\Gamma, \psi \vdash \sigma \vdash \tau \vee \sigma$. Hence by the cut rule and E$\vee$ we get $\Gamma, \phi \vee \psi \vdash \tau \vee \sigma$.

(ix) By the weakening and exchange rules we have $\Gamma, \neg\tau, \phi \vdash \tau$. We also know that $\Gamma, \neg\tau, \phi \vdash \neg\tau$. Hence by the cut rule and E$\neg$ we get $\Gamma, \neg\tau, \phi \vdash \bot$. Thus by I$\neg$ we obtain $\Gamma, \neg\tau \vdash \neg\phi$. Therefore by I$\vee$ and the cut rule we get $\Gamma, \neg\tau \vdash \neg\phi \vee \neg\psi$. Similarly we can show that $\Gamma, \neg\sigma \vdash \neg\phi \vee \neg\psi$. Thus by E$\vee$ we get $\Gamma, \neg\tau \vee \neg\sigma \vdash \neg\phi \vee \neg\psi$, as desired.

(x) The assumption is that $\Gamma, \phi \vdash \neg\phi$. We also know that $\Gamma, \phi \vdash \phi$. Thus by the cut rule and E$\neg$ we get $\Gamma, \phi \vdash \bot$. Hence by I$\neg$ we obtain $\Gamma \vdash \neg\phi$, as desired. ■

The rule ex falso quodlibet (EFQ), which literally means “from falsehood anything (follows)”, states that a contradiction like $\phi \wedge \neg\phi$, or $\bot$, can imply any formula like ψ. The law of excluded middle states that for every formula like ϕ, either ϕ or $\neg\phi$ must be true. Hence, a formula is either true or false. And the law of non-contradiction states that a contradiction like $\phi \wedge \neg\phi$, or $\bot$, cannot be true. In other words, a formula cannot be both true and false.

The rule modus tollens, which literally means “mood that denies”, informally says that if ϕ implies ψ, and ψ is false, then ϕ must be false too. It is closely related to the law of contraposition, which is stated in Theorem 1.6. The hypothetical syllogism says that implication is transitive. The word *syllogism* is the name of inference rules in Aristotelean logic. Finally, let us mention that modus tollendo ponens literally means “mood that affirms by denying”, and modus ponendo tollens literally means “mood that denies by affirming”.

The last part of the above theorem says that if a formula implies its negation, then that formula must be false, i.e. its negation must be true. We can similarly show that if the negation of a formula implies the formula, then the negation must be false, i.e. the formula must be true. In other words:

$$\text{If} \quad \Gamma, \neg\phi \vdash \phi \qquad \text{then} \qquad \Gamma \vdash \phi.$$

This rule is known as *consequentia mirabilis*, which literally means “admirable consequence”. The proof of this fact is similar to the above, but we have to use RAA instead of I$\neg$. We can also prove it by using the last part of the above theorem and the double negation law.

***Example* 1.5.** Let ϕ, ψ be formulas. Then we have

$$\text{If} \quad \vdash \neg\phi \qquad \text{then} \qquad \vdash \phi \to \psi.$$

This confirms our intuitive understanding of material implication, namely, the fact that $\phi \to \psi$ is true when ϕ is false. To prove it, note that by weakening rule we have $\phi \vdash \neg\phi$. We also know that $\phi \vdash \phi$. Hence by E$\neg$ and the cut rule we get $\phi \vdash \bot$. Thus by EFQ rule we obtain $\phi \wedge \neg\phi \vdash \psi$. Therefore by the cut rule we get $\phi \vdash \psi$. Hence by I$\to$ we have $\vdash \phi \to \psi$, as desired. It is also easy to show that

$$\text{If} \quad \vdash \psi \qquad \text{then} \qquad \vdash \phi \to \psi.$$

In other words, $\phi \to \psi$ is true when ψ is true. Because by the weakening rule we have $\phi \vdash \psi$. Hence by I$\to$ we obtain $\vdash \phi \to \psi$, as desired.

1.3 Equivalent Formulas

Definition 1.1. We say two formulas ϕ, ψ are **equivalent** if $\phi \vdash \psi$ and $\psi \vdash \phi$. In this case we write

$$\phi \equiv \psi.$$

Remark. An important point to keep in mind is that the "if" in definitions actually means "if and only if". Thus the above definition actually says "ϕ, ψ are equivalent if and only if $\phi \vdash \psi$ and $\psi \vdash \phi$". But the tradition in mathematics is to use "if" in definitions instead, and we will adhere to this convention in these notes.

Remark. Note that by the introduction of $\leftrightarrow$, if $\phi \equiv \psi$ then we have

$$\vdash \phi \leftrightarrow \psi.$$

Thus from a semantic viewpoint, equivalent formulas are either both true, or both false.

Theorem 1.2. *Suppose ϕ, ψ, τ are formulas, and $\phi \equiv \psi$. Let Γ denote a collection of several formulas, which can be empty too. Then the entailment $\Gamma, \phi \vdash \tau$ holds if and only if the entailment $\Gamma, \psi \vdash \tau$ holds.*

Proof. To see this, suppose $\Gamma, \psi \vdash \tau$ holds. Then since $\phi \vdash \psi$, the cut rule implies that $\phi, \Gamma \vdash \tau$ holds too. Hence by the exchange rule we get $\Gamma, \phi \vdash \tau$, as desired. The converse can be proved similarly. ■

Remark. The above theorem means that we can replace a formula in the premises of an entailment by an equivalent formula, and the new entailment is equivalent to the original one.

Notation. Sometimes we may write $\Gamma \vdash \phi \vdash \psi$ to denote "$\Gamma \vdash \phi$ and $\phi \vdash \psi$". Note that due to the cut rule if $\Gamma \vdash \phi \vdash \psi$ then $\Gamma \vdash \psi$ too. In addition, we may write $\phi \equiv \psi \equiv \tau$ to denote "$\phi \equiv \psi$ and $\psi \equiv \tau$". Note that by the next theorem we can also conclude that $\phi \equiv \tau$. Both of these abbreviated notations can be used for more than three formulas.

Theorem 1.3. *Suppose ϕ, ψ, τ are formulas. Then we have*
(i) *Reflexivity* : $\phi \equiv \phi$.
(ii) *Symmetry* : *If $\phi \equiv \psi$ then $\psi \equiv \phi$.*
(iii) *Transitivity* : *If $\phi \equiv \psi$ and $\psi \equiv \tau$, then $\phi \equiv \tau$.*

Proof. **(i)** This is a trivial consequence of the fact that $\phi \vdash \phi$.

(ii) If $\phi \equiv \psi$ then by definition we have "$\phi \vdash \psi$ and $\psi \vdash \phi$". However this the same as saying "$\psi \vdash \phi$ and $\phi \vdash \psi$". Therefore we get $\psi \equiv \phi$ as desired. Note that here we are using the fact that the connective "and" in meta-language is commutative, i.e. the order of the phrases which are connected by "and" does not affect the truth of the compound statement.

(iii) If $\phi \equiv \psi$ and $\psi \equiv \tau$, then by definition we have $\phi \vdash \psi$ and $\psi \vdash \phi$, and $\psi \vdash \tau$ and $\tau \vdash \psi$. In other words we have $\phi \vdash \psi \vdash \tau$ and $\tau \vdash \psi \vdash \phi$. Hence by the cut rule we obtain $\phi \vdash \tau$ and $\tau \vdash \phi$. Thus $\phi \equiv \tau$ as desired. Note that we feel free to use our hypotheses in any order we want. In other words, we are using the exchange rule at the level of meta-logic. ■

Theorem 1.4. *Suppose ϕ, ψ, τ, σ are formulas. Also suppose $\phi \equiv \psi$ and $\tau \equiv \sigma$. Then we have*
(i) $\neg\phi \equiv \neg\psi$.
(ii) $\phi \wedge \tau \equiv \psi \wedge \sigma$.
(iii) $\phi \vee \tau \equiv \psi \vee \sigma$.
(iv) $\phi \to \tau \equiv \psi \to \sigma$.
(v) $\phi \leftrightarrow \tau \equiv \psi \leftrightarrow \sigma$.

Proof. **(i)** We know that $\phi \vdash \psi$ and $\psi \vdash \phi$. We have to show that $\neg\phi \vdash \neg\psi$ and $\neg\psi \vdash \neg\phi$. Note that $\neg\phi \vdash \neg\phi$. Thus by the weakening and exchange rules we have $\neg\phi, \psi \vdash \phi$ and $\neg\phi, \psi \vdash \neg\phi$. Therefore by E$\neg$ and cut rule we have $\neg\phi, \psi \vdash \perp$. Hence by I$\neg$ we get $\neg\phi \vdash \neg\psi$. Similarly we can show that $\neg\psi \vdash \neg\phi$.

(ii) By E$\wedge$ we have $\phi \wedge \tau \vdash \phi \vdash \psi$ and $\phi \wedge \tau \vdash \tau \vdash \sigma$. On the other hand, by I$\wedge$ we have $\psi, \sigma \vdash \psi \wedge \sigma$. Hence by the cut rule we have $\phi \wedge \tau \vdash \psi \wedge \sigma$. Similarly we can show that $\psi \wedge \sigma \vdash \phi \wedge \tau$. Thus $\phi \wedge \tau \equiv \psi \wedge \sigma$, as desired.

(iii) By I$\vee$ we have $\phi \vdash \psi \vdash \psi \vee \sigma$ and $\tau \vdash \sigma \vdash \psi \vee \sigma$. Hence by E$\vee$ we have $\phi \vee \tau \vdash \psi \vee \sigma$. Similarly we can show $\psi \vee \sigma \vdash \phi \vee \tau$, and conclude the desired result.

(iv) By E$\to$ we have $\phi \to \tau, \phi \vdash \tau \vdash \sigma$. Hence by Theorem 1.2 we also have $\phi \to \tau, \psi \vdash \sigma$, since $\phi \equiv \psi$. Thus by I$\to$ we get $\phi \to \tau \vdash \psi \to \sigma$. Similarly we can show that $\psi \to \sigma \vdash \phi \to \tau$.

(v) By E$\leftrightarrow$ we have $\phi \leftrightarrow \tau, \phi \vdash \tau \vdash \sigma$ and $\phi \leftrightarrow \tau, \tau \vdash \phi \vdash \psi$. Hence by Theorem 1.2 we also have $\phi \leftrightarrow \tau, \psi \vdash \sigma$ and $\phi \leftrightarrow \tau, \sigma \vdash \psi$, because $\phi \equiv \psi$ and $\tau \equiv \sigma$. Thus by I$\leftrightarrow$ we get $\phi \leftrightarrow \tau \vdash \psi \leftrightarrow \sigma$. Similarly we can show that $\psi \leftrightarrow \sigma \vdash \phi \leftrightarrow \tau$. ■

Suppose a formula is composed of several other formulas. Then the above theorem enables us to replace some of the components by equivalent formulas to obtain a formula equivalent to the original compound formula. We will not prove the general case of this fact here, but let us demonstrate it by an example. Suppose $\phi \equiv \psi$ and $\tau \equiv \sigma$. Consider the following formula

$$(\phi \wedge \phi_1) \to (\phi_2 \to \tau \vee \phi_3).$$

We claim that it is equivalent to $(\psi \wedge \phi_1) \to (\phi_2 \to \sigma \vee \phi_3)$. To see this note that any formula is equivalent to itself. Therefore by the above theorem we have

$$\phi \wedge \phi_1 \equiv \psi \wedge \phi_1, \qquad \tau \vee \phi_3 \equiv \sigma \vee \phi_3.$$

Now if we apply the theorem again we obtain $\phi_2 \to \tau \vee \phi_3 \equiv \phi_2 \to \sigma \vee \phi_3$, and therefore we get

$$(\phi \wedge \phi_1) \to (\phi_2 \to \tau \vee \phi_3) \equiv (\psi \wedge \phi_1) \to (\phi_2 \to \sigma \vee \phi_3),$$

as desired.

Theorem 1.5. *Suppose ϕ, ψ, τ are formulas. If $\phi \vdash \psi \vdash \tau \vdash \phi$ then we have $\phi \equiv \psi \equiv \tau$.*

Remark. This theorem provides a shortcut for proving that three formulas are equivalent with each other. It says that instead of proving that each formula is deduced from each of the other formulas, we can just show that we have a cycle of entailments. This is how such equivalences are usually proved in mathematics. Similarly we can show that the analogous statements are true for more than three formulas.

Proof. We know that $\psi \vdash \tau \vdash \phi$, so by the cut rule we have $\psi \vdash \phi$. On the other hand we know that $\phi \vdash \psi$. Hence we get $\phi \equiv \psi$. Similarly we know that $\psi \vdash \tau$. We also know that $\tau \vdash \phi \vdash \psi$. Thus by the cut rule we get $\tau \vdash \psi$. Therefore we have $\psi \equiv \tau$ too. Note that the equivalence $\phi \equiv \tau$ is not formally part of the statement $\phi \equiv \psi \equiv \tau$, but as we said before, it follows from the transitivity of $\equiv$. ■

Theorem 1.6. *Suppose ϕ, ψ, τ are formulas. Then we have*

(i) *Commutativity* :

$$\phi \wedge \psi \equiv \psi \wedge \phi, \qquad \textit{and} \qquad \phi \vee \psi \equiv \psi \vee \phi.$$

(ii) *Associativity* :

$$(\phi \wedge \psi) \wedge \tau \equiv \phi \wedge (\psi \wedge \tau), \qquad \textit{and} \qquad (\phi \vee \psi) \vee \tau \equiv \phi \vee (\psi \vee \tau).$$

(iii) *Distributivity* :

$$\tau \vee (\phi \wedge \psi) \equiv (\tau \vee \phi) \wedge (\tau \vee \psi), \qquad \textit{and} \qquad \tau \wedge (\phi \vee \psi) \equiv (\tau \wedge \phi) \vee (\tau \wedge \psi).$$

(iv) *Idempotency* :

$$\phi \wedge \phi \equiv \phi, \qquad \textit{and} \qquad \phi \vee \phi \equiv \phi.$$

(v) *Absorption* :

$$\phi \wedge (\phi \vee \psi) \equiv \phi, \qquad \textit{and} \qquad \phi \vee (\phi \wedge \psi) \equiv \phi.$$

(vi) *De Morgan's laws* :

$$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi, \qquad \textit{and} \qquad \neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi.$$

(vii) *Double negation law* :

$$\neg\neg\phi \equiv \phi.$$

(viii) *Law of material implication* :

$$\phi \to \psi \equiv \neg\phi \vee \psi.$$

(ix) *Law of contraposition* :

$$\phi \to \psi \equiv \neg\psi \to \neg\phi.$$

(x) *Negation of a conditional* :

$$\neg(\phi \to \psi) \equiv \phi \wedge \neg\psi.$$

(xi)

$$\phi \leftrightarrow \psi \equiv (\phi \to \psi) \wedge (\psi \to \phi),$$
$$\psi \leftrightarrow \phi \equiv \phi \leftrightarrow \psi \equiv \neg\phi \leftrightarrow \neg\psi.$$

(xii)

$$\phi \wedge \neg\phi \equiv \bot, \qquad \textit{and} \qquad \phi \vee \neg\phi \equiv \top.$$

(xiii)

$$\phi \wedge \top \equiv \phi, \qquad \textit{and} \qquad \phi \vee \bot \equiv \phi,$$
$$\phi \vee \top \equiv \top, \qquad \textit{and} \qquad \phi \wedge \bot \equiv \bot.$$

(xiv)

$$\neg\phi \equiv \phi \to \bot.$$

$\boxed{\textbf{Proof.}}$ **(i)** By E$\wedge$ we have $\phi \wedge \psi \vdash \psi$ and $\phi \wedge \psi \vdash \phi$. On the other hand by I$\wedge$ we have $\psi, \phi \vdash \psi \wedge \phi$. Hence by the cut rule we get $\phi \wedge \psi \vdash \psi \wedge \phi$. Similarly we can show that $\psi \wedge \phi \vdash \phi \wedge \psi$.

Next, by I$\vee$ we have $\phi \vdash \psi \vee \phi$ and $\psi \vdash \psi \vee \phi$. Hence by E$\vee$ we get $\phi \vee \psi \vdash \psi \vee \phi$. Similarly we can show that $\psi \vee \phi \vdash \phi \vee \psi$.

(ii) By E$\wedge$ we have

$$(\phi \wedge \psi) \wedge \tau \vdash \phi \wedge \psi \vdash \phi, \qquad (\phi \wedge \psi) \wedge \tau \vdash \phi \wedge \psi \vdash \psi,$$

and $(\phi \wedge \psi) \wedge \tau \vdash \tau$. Thus by the cut rule and I$\wedge$ we get $(\phi \wedge \psi) \wedge \tau \vdash \psi \wedge \tau$, and therefore $(\phi \wedge \psi) \wedge \tau \vdash \phi \wedge (\psi \wedge \tau)$. Similarly we can show that $\phi \wedge (\psi \wedge \tau) \vdash (\phi \wedge \psi) \wedge \tau$.

Next, by I$\vee$ we have $\phi \vdash \phi \vee (\psi \vee \tau)$, and

$$\psi \vdash \psi \vee \tau \vdash \phi \vee (\psi \vee \tau), \qquad \tau \vdash \psi \vee \tau \vdash \phi \vee (\psi \vee \tau).$$

Thus by E$\vee$ we get $\phi \vee \psi \vdash \phi \vee (\psi \vee \tau)$, and therefore $(\phi \vee \psi) \vee \tau \vdash \phi \vee (\psi \vee \tau)$. Similarly we can show that $\phi \vee (\psi \vee \tau) \vdash (\phi \vee \psi) \vee \tau$.

(iii) By E$\wedge$ and I$\vee$ we have $\phi \wedge \psi \vdash \phi \vdash \tau \vee \phi$, and $\phi \wedge \psi \vdash \psi \vdash \tau \vee \psi$. Hence by the cut rule and I$\wedge$ we get $\phi \wedge \psi \vdash (\tau \vee \phi) \wedge (\tau \vee \psi)$. Now by I$\vee$ we have $\tau \vdash \tau \vee \phi$, and $\tau \vdash \tau \vee \psi$. Hence by the cut rule and I$\wedge$ we get $\tau \vdash (\tau \vee \phi) \wedge (\tau \vee \psi)$. Therefore by E$\vee$ we get

$$\tau \vee (\phi \wedge \psi) \vdash (\tau \vee \phi) \wedge (\tau \vee \psi).$$

On the other hand, by I$\wedge$ and I$\vee$ we have $\phi, \psi \vdash \phi \wedge \psi \vdash \tau \vee (\phi \wedge \psi)$. Also, by the weakening rule and I$\vee$ we have $\phi, \tau \vdash \tau \vee (\phi \wedge \psi)$. Hence by E$\vee$ we get $\phi, \tau \vee \psi \vdash \tau \vee (\phi \wedge \psi)$. If we use the exchange rule we get $\tau \vee \psi, \phi \vdash \tau \vee (\phi \wedge \psi)$. Now by the weakening rule and I$\vee$ we also have $\tau \vee \psi, \tau \vdash \tau \vee (\phi \wedge \psi)$. Therefore by E$\vee$ we obtain $\tau \vee \psi, \tau \vee \phi \vdash \tau \vee (\phi \wedge \psi)$. Thus by E$\wedge$ and the cut rule we get

$$(\tau \vee \phi) \wedge (\tau \vee \psi) \vdash \tau \vee (\phi \wedge \psi).$$

Next, by E$\wedge$ and I$\vee$ we have $\tau \wedge \phi \vdash \phi \vdash \phi \vee \psi$, and $\tau \wedge \psi \vdash \psi \vdash \phi \vee \psi$. Also, by E$\wedge$ we have $\tau \wedge \phi \vdash \tau$, and $\tau \wedge \psi \vdash \tau$. Hence by the cut rule and I$\wedge$ we get $\tau \wedge \phi \vdash \tau \wedge (\phi \vee \psi)$, and $\tau \wedge \psi \vdash \tau \wedge (\phi \vee \psi)$. Thus by E$\vee$ we obtain

$$(\tau \wedge \phi) \vee (\tau \wedge \psi) \vdash \tau \wedge (\phi \vee \psi).$$

Conversely, by I$\wedge$ and I$\vee$ we have $\tau, \phi \vdash \tau \wedge \phi \vdash (\tau \wedge \phi) \vee (\tau \wedge \psi)$, and $\tau, \psi \vdash \tau \wedge \psi \vdash (\tau \wedge \phi) \vee (\tau \wedge \psi)$. Hence by E$\vee$ we get $\tau, \phi \vee \psi \vdash (\tau \wedge \phi) \vee (\tau \wedge \psi)$. Therefore by E$\wedge$ and the cut rule we obtain

$$\tau \wedge (\phi \vee \psi) \vdash (\tau \wedge \phi) \vee (\tau \wedge \psi).$$

(iv) By I$\wedge$ we know that $\phi, \phi \vdash \phi \wedge \phi$. Thus by the contraction rule we get $\phi \vdash \phi \wedge \phi$. On the other hand, by E$\wedge$ we have $\phi \wedge \phi \vdash \phi$.

Next, by I$\vee$ we have $\phi \vdash \phi \vee \phi$. Conversely, we know that $\phi \vdash \phi$. If we use this entailment twice, and apply E$\vee$, we get $\phi \vee \phi \vdash \phi$.

(v) We know that $\phi \vdash \phi$. Also by I$\vee$ we have $\phi \vdash \phi \vee \psi$. Thus by the cut rule and I$\wedge$ we get $\phi \vdash \phi \wedge (\phi \vee \psi)$. Conversely, by E$\wedge$ we have $\phi \wedge (\phi \vee \psi) \vdash \phi$.

Next, by I$\vee$ we have $\phi \vdash \phi \vee (\phi \wedge \psi)$. On the other hand, we know that $\phi \vdash \phi$. Also by E$\wedge$ we have $\phi \wedge \psi \vdash \phi$. Hence by E$\vee$ we get $\phi \vee (\phi \wedge \psi) \vdash \phi$.

(vi) By the weakening rule and E$\wedge$ we have $\neg\phi, \phi \wedge \psi \vdash \phi$. We also have $\neg\phi, \phi \wedge \psi \vdash \neg\phi$. Hence by the cut rule and E$\neg$ we get $\neg\phi, \phi \wedge \psi \vdash \bot$. Thus by I$\neg$ we obtain $\neg\phi \vdash \neg(\phi \wedge \psi)$. Similarly we have $\neg\psi \vdash \neg(\phi \wedge \psi)$. Therefore by E$\vee$ we get

$$\neg\phi \vee \neg\psi \vdash \neg(\phi \wedge \psi).$$

On the other hand, by the weakening rule and I$\vee$ we have $\neg(\neg\phi \vee \neg\psi), \neg\phi \vdash \neg\phi \vee \neg\psi$. Thus by the cut rule and E$\neg$ we get

$$\neg(\neg\phi \vee \neg\psi), \neg\phi \vdash \bot.$$

Hence by RAA we get $\neg(\neg\phi \vee \neg\psi) \vdash \phi$. Similarly we obtain $\neg(\neg\phi \vee \neg\psi) \vdash \psi$. Thus by the cut rule and I$\wedge$ we get $\neg(\neg\phi \vee \neg\psi) \vdash \phi \wedge \psi$. Therefore by the weakening and cut rules, and E$\neg$, we obtain

$$\neg(\phi \wedge \psi), \neg(\neg\phi \vee \neg\psi) \vdash \bot.$$

Hence by RAA we get $\neg(\phi \wedge \psi) \vdash \neg\phi \vee \neg\psi$, as desired.

Next, by the weakening rule and I$\vee$ we have $\neg(\phi \vee \psi), \phi \vdash \phi \vee \psi$. Hence by the cut rule and E$\neg$ we get $\neg(\phi \vee \psi), \phi \vdash \bot$. Thus by I$\neg$ we get $\neg(\phi \vee \psi) \vdash \neg\phi$. Similarly we obtain $\neg(\phi \vee \psi) \vdash \neg\psi$. Therefore by the cut rule and I$\wedge$ we get

$$\neg(\phi \vee \psi) \vdash \neg\phi \wedge \neg\psi.$$

Conversely, by the weakening rule and E$\wedge$ we have $\neg\phi \wedge \neg\psi, \phi \vdash \neg\phi$. Hence by the cut rule and E$\neg$ we get $\neg\phi \wedge \neg\psi, \phi \vdash \bot$. Thus by I$\neg$ we get

$$\phi \vdash \neg(\neg\phi \wedge \neg\psi).$$

Similarly we obtain $\psi \vdash \neg(\neg\phi \wedge \neg\psi)$. Therefore by E$\vee$ we get $\phi \vee \psi \vdash \neg(\neg\phi \wedge \neg\psi)$. Hence by the weakening and cut rules, and E$\neg$, we obtain

$$\neg\phi \wedge \neg\psi, \phi \vee \psi \vdash \bot.$$

Thus by I$\neg$ we get $\neg\phi \wedge \neg\psi \vdash \neg(\phi \vee \psi)$, as desired.

(vii) By E$\neg$ we have $\phi, \neg\phi \vdash \bot$. Hence by I$\neg$ we get $\phi \vdash \neg\neg\phi$. On the other hand, by the exchange rule and E$\neg$ we have $\neg\neg\phi, \neg\phi \vdash \bot$. Thus by RAA we get $\neg\neg\phi \vdash \phi$, as desired.

(viii) By the weakening rule we have $\phi, \psi \vdash \psi$. Also, by E$\neg$ and the EFQ rule (stated in Theorem 1.1) we have $\phi, \neg\phi \vdash \bot \vdash \psi$. Hence by E$\vee$ we obtain $\phi, \neg\phi \vee \psi \vdash \psi$. Thus by the exchange rule and I$\to$ we get $\neg\phi \vee \psi \vdash \phi \to \psi$.

Conversely, by E$\to$ and I$\vee$ we have $\phi \to \psi, \phi \vdash \psi \vdash \neg\phi \vee \psi$. Also, by the weakening rule and I$\vee$ we have $\phi \to \psi, \neg\phi \vdash \neg\phi \vdash \neg\phi \vee \psi$. Hence by E$\vee$ we get $\phi \to \psi, \phi \vee \neg\phi \vdash \neg\phi \vee \psi$. But we know that $\vdash \phi \vee \neg\phi$; so in particular we have $\phi \to \psi \vdash \phi \vee \neg\phi$, due to the weakening rule. Thus by the cut and contraction rules we get $\phi \to \psi \vdash \neg\phi \vee \psi$, as desired.

(ix) By modus tollens (stated in Theorem 1.1) we have $\phi \to \psi, \neg\psi \vdash \neg\phi$. Thus by I$\to$ we get

$$\phi \to \psi \vdash \neg\psi \to \neg\phi.$$

Conversely, note that if we repeat the above argument, and replace ϕ, ψ by $\neg\phi, \neg\psi$ respectively, we obtain $\neg\phi \to \neg\psi \vdash \neg\neg\psi \to \neg\neg\phi$. However, we have shown that $\neg\neg\phi \equiv \phi$ and $\neg\neg\psi \equiv \psi$. Hence by Theorem 1.4 we have $\neg\neg\psi \to \neg\neg\phi \equiv \psi \to \phi$. In particular we have $\neg\neg\psi \to \neg\neg\phi \vdash \psi \to \phi$. Thus by the cut rule we get $\neg\phi \to \neg\psi \vdash \psi \to \phi$, as desired.

(x) We have shown that $\phi \to \psi \equiv \neg\phi \vee \psi$. Thus by Theorem 1.4 and De Morgan's law we have $\neg(\phi \to \psi) \equiv \neg(\neg\phi \vee \psi) \equiv \neg\neg\phi \wedge \neg\psi$. However, we also know that $\neg\neg\phi \equiv \phi$. Therefore by using Theorem 1.4 again we obtain $\neg\neg\phi \wedge \neg\psi \equiv \phi \wedge \neg\psi$. Hence by the transitivity of $\equiv$ we get $\neg(\phi \to \psi) \equiv \phi \wedge \neg\psi$, as desired.

(xi) By the weakening rule and E$\to$ we have $\phi \to \psi, \psi \to \phi, \phi \vdash \psi$, and $\phi \to \psi, \psi \to \phi, \psi \vdash \phi$. Thus by I$\leftrightarrow$ we get $\phi \to \psi, \psi \to \phi \vdash \phi \leftrightarrow \psi$. Hence by E$\wedge$ and the cut rule we obtain

$$(\phi \to \psi) \wedge (\psi \to \phi) \vdash \phi \leftrightarrow \psi.$$

Conversely, by E$\leftrightarrow$ we have $\phi \leftrightarrow \psi, \phi \vdash \psi$, and $\phi \leftrightarrow \psi, \psi \vdash \phi$. Thus by I$\to$ we get $\phi \leftrightarrow \psi \vdash \phi \to \psi$, and $\phi \leftrightarrow \psi \vdash \psi \to \phi$. Therefore by the cut rule and I$\wedge$ we obtain

$$\phi \leftrightarrow \psi \vdash (\phi \to \psi) \wedge (\psi \to \phi).$$

Next, consider $\psi \leftrightarrow \phi$. By the above argument and the commutativity of $\wedge$ we have

$$\psi \leftrightarrow \phi \equiv (\psi \to \phi) \wedge (\phi \to \psi) \equiv (\phi \to \psi) \wedge (\psi \to \phi) \equiv \phi \leftrightarrow \psi.$$

The desired result follows from transitivity of $\equiv$. Finally consider $\neg\phi \leftrightarrow \neg\psi$. By the above argument, law of contraposition, and Theorem 1.4 we have

$$\begin{aligned} \neg\phi \leftrightarrow \neg\psi &\equiv (\neg\phi \to \neg\psi) \wedge (\neg\psi \to \neg\phi) \\ &\equiv (\psi \to \phi) \wedge (\phi \to \psi) \equiv \psi \leftrightarrow \phi \equiv \phi \leftrightarrow \psi. \end{aligned}$$

And again the desired result follows from transitivity of $\equiv$.

(xii) By the EFQ rule (stated in Theorem 1.1) we know that a contradiction implies any formula. So we have $\phi \wedge \neg\phi \vdash \bot$ and $\bot \vdash \phi \wedge \neg\phi$.

Next, note that by the law of excluded middle and the law of non-contradiction we respectively have $\vdash \phi \vee \neg\phi$ and $\vdash \top$. Hence by the weakening rule we get $\top \vdash \phi \vee \neg\phi$ and $\phi \vee \neg\phi \vdash \top$, as desired.

(xiii) By E$\wedge$ and I$\vee$ we respectively obtain

$$\phi \wedge \top \vdash \phi, \qquad \top \vdash \phi \vee \top.$$

Conversely, by the law of non-contradiction we have $\vdash \top$. Thus by the weakening rule we get

$$\phi \vee \top \vdash \top,$$

and $\phi \vdash \top$. We also know that $\phi \vdash \phi$. Therefore by the cut rule and I$\wedge$ we get $\phi \vdash \phi \wedge \top$.

Next, note that by I$\vee$ and E$\wedge$ we respectively obtain

$$\phi \vdash \phi \vee \bot, \qquad \phi \wedge \bot \vdash \bot.$$

On the other hand, by the EFQ rule (stated in Theorem 1.1) we know that

$$\bot \vdash \phi \wedge \bot,$$

and $\bot \vdash \phi$. We also know that $\phi \vdash \phi$. Therefore by E$\vee$ we get $\phi \vee \bot \vdash \phi$ as desired.

(xiv) We have $\phi \to \bot \equiv \neg\phi \vee \bot \equiv \neg\phi$. ■

Remark. Consider the formula $(\phi \wedge \psi) \wedge \tau$. We know that it is equivalent to $\phi \wedge (\psi \wedge \tau)$. Sometimes we abuse the notation, and denote these equivalent formulas simply by $\phi \wedge \psi \wedge \tau$. Similarly, we may write $\phi \vee \psi \vee \tau$ to denote the equivalent formulas $(\phi \vee \psi) \vee \tau$ and $\phi \vee (\psi \vee \tau)$. The associativity also implies that if several formulas are all connected by conjunction or disjunction, then the arrangement of parentheses between them does not alter the truth of the compound formula. In other words, all the possible arrangements of parentheses result in equivalent formulas. So for example, $\phi \vee ((\psi \vee \tau) \vee \sigma)$ is equivalent to $(\phi \vee \psi) \vee (\tau \vee \sigma)$. We may denote these equivalent formulas by $\phi \vee \psi \vee \tau \vee \sigma$. Similar abbreviated notations can be used when we have more formulas. However, we do not have the tools to state the general version of this fact precisely, and to prove it rigorously.

Remark. The equivalence $\phi \to \psi \equiv \neg\phi \vee \psi$ affirms the fact that $\phi \to \psi$ is only false if ϕ is true and ψ is false. Note that although the truth value of the material implication is not completely evident, we deduced the above equivalence from the inference rules which are intuitively more obvious. So we can consider this as an informal justification of the truth value of the material implication.

Also as we saw, the above equivalence implies that the negation of the conditional formula $\phi \to \psi$ is equivalent to $\phi \wedge \neg\psi$. Informally, this means that if ϕ does not imply ψ, then ϕ must be true while ψ is false.

Remark. The application of the law of contraposition in a deduction is also known as **proof by contraposition**. Namely, in order to show that $\phi \to \psi$ is true, sometimes it is easier to show that $\neg\psi \to \neg\phi$ is true. Then we get the desired by the law of contraposition. Informally, in this method, instead of showing that if ϕ is true then ψ must be true too, we show that if ψ is false then ϕ must be false too. Thus we can conclude that if ϕ is true then ψ cannot be false, so ψ must be true.

Remark. The equivalence $\phi \leftrightarrow \psi \equiv (\phi \to \psi) \wedge (\psi \to \phi)$ confirms our initial intuition about biconditional statements. It also implies that if we have $\vdash \phi \leftrightarrow \psi$ then by the cut rule and E$\wedge$ we also have $\vdash \phi \to \psi$ and $\vdash \psi \to \phi$. But as we saw before, this means that $\phi \vdash \psi$ and $\psi \vdash \phi$. Hence $\phi \equiv \psi$. We have seen that the converse of this fact holds too. Therefore we get

$$\phi \equiv \psi \qquad \text{if and only if} \qquad \vdash \phi \leftrightarrow \psi.$$

Thus, we could have also used the above as the definition of the equivalence of two formulas.

Remark. In the law of material implication, we have seen that we can express $\to$ in terms of $\vee, \neg$. More generally, it is possible to express each connective in terms of the other connectives. To see this, we just need to replace ϕ, ψ by $\neg\phi, \neg\psi$ in some parts of the above theorem, and use the double negation law and Theorem 1.4, to conclude

$$\begin{aligned} \phi \wedge \psi &\equiv \neg(\neg\phi \vee \neg\psi) &&\equiv \neg(\phi \to \neg\psi), \\ \phi \vee \psi &\equiv \neg(\neg\phi \wedge \neg\psi) &&\equiv \neg\phi \to \psi \equiv \neg\psi \to \phi, \\ \phi \to \psi &\equiv \neg(\phi \wedge \neg\psi) &&\equiv \neg\phi \vee \psi. \end{aligned}$$

We have also seen that we can express $\neg$ in terms of $\to, \perp$.

Also note that the above equivalences for $\phi \vee \psi$ give us useful tools for proving it. Namely, as explained in the following example, in order to prove $\phi \vee \psi$ we assume that one of ϕ, ψ is false, and then we show that the other one must be true.

***Example* 1.6.** Suppose Γ denotes a collection of several formulas, which can be empty too. Let ϕ, ψ be formulas. Then we have

$$\text{If} \quad \Gamma, \neg\phi \vdash \psi \qquad \text{then} \qquad \Gamma \vdash \phi \vee \psi.$$

Because by I$\to$ we have $\Gamma \vdash \neg\phi \to \psi$. Hence by the law of material implication we get $\Gamma \vdash \neg\neg\phi \vee \psi$. Finally, by the double negation law and Theorem 1.4 we obtain $\Gamma \vdash \phi \vee \psi$, as desired.

Theorem 1.7. *Suppose ϕ, ψ, τ are formulas. Then we have*

(i)
$$(\phi \wedge \psi) \to \tau \equiv \phi \to (\psi \to \tau) \equiv \psi \to (\phi \to \tau).$$

(ii)
$$\tau \to (\phi \wedge \psi) \equiv (\tau \to \phi) \wedge (\tau \to \psi).$$

(iii)
$$(\phi \vee \psi) \to \tau \equiv (\phi \to \tau) \wedge (\psi \to \tau).$$

(iv)
$$\tau \to (\phi \vee \psi) \equiv (\tau \to \phi) \vee (\tau \to \psi).$$

Proof. **(i)** By I$\wedge$ and E$\to$ we know that $\phi, \psi \vdash \phi \wedge \psi$ and $\phi \wedge \psi \to \tau, \phi \wedge \psi \vdash \tau$. Thus by the cut and exchange rules we get $\phi \wedge \psi \to \tau, \phi, \psi \vdash \tau$. Now by applying I$\to$ twice we obtain $\phi \wedge \psi \to \tau, \phi \vdash \psi \to \tau$, and therefore

$$\phi \wedge \psi \to \tau \vdash \phi \to (\psi \to \tau).$$

Conversely, by applying E$\to$ twice we get $\phi \to (\psi \to \tau), \phi \vdash \psi \to \tau$, and $\psi \to \tau, \psi \vdash \tau$. Thus by the cut and exchange rules we get $\phi \to (\psi \to \tau), \phi, \psi \vdash \tau$. Now by the cut rule and E$\wedge$ we obtain $\phi \to (\psi \to \tau), \phi \wedge \psi \vdash \tau$. Hence by I$\to$ we get

$$\phi \to (\psi \to \tau) \vdash \phi \wedge \psi \to \tau.$$

Finally note that by the above argument, commutativity of $\wedge$, and Theorem 1.4 we have
$$\psi \to (\phi \to \tau) \equiv \psi \wedge \phi \to \tau \equiv \phi \wedge \psi \to \tau \equiv \phi \to (\psi \to \tau).$$
The desired result follows from transitivity of $\equiv$.

(ii) By the law of material implication, distributivity of $\vee$ over $\wedge$, and Theorem 1.4 we have

$$\tau \to (\phi \wedge \psi) \equiv \neg\tau \vee (\phi \wedge \psi) \equiv (\neg\tau \vee \phi) \wedge (\neg\tau \vee \psi) \equiv (\tau \to \phi) \wedge (\tau \to \psi).$$

(iii) By the law of material implication, De Morgan's law, distributivity of $\vee$ over $\wedge$, and Theorem 1.4 we have

$$\begin{aligned}(\phi \vee \psi) \to \tau &\equiv \neg(\phi \vee \psi) \vee \tau \equiv (\neg\phi \wedge \neg\psi) \vee \tau \\ &\equiv (\neg\phi \vee \tau) \wedge (\neg\psi \vee \tau) \equiv (\phi \to \tau) \wedge (\psi \to \tau).\end{aligned}$$

(iv) By the law of material implication; idempotency, commutativity, and associativity of $\vee$; and Theorem 1.4 we have

$$\begin{aligned}\tau \to (\phi \vee \psi) &\equiv \neg\tau \vee (\phi \vee \psi) \equiv (\neg\tau \vee \neg\tau) \vee (\phi \vee \psi) \\ &\equiv \neg\tau \vee (\neg\tau \vee (\phi \vee \psi)) \equiv \neg\tau \vee ((\neg\tau \vee \phi) \vee \psi) \\ &\equiv (\neg\tau \vee (\neg\tau \vee \phi)) \vee \psi \equiv ((\neg\tau \vee \phi) \vee \neg\tau) \vee \psi \\ &\equiv (\neg\tau \vee \phi) \vee (\neg\tau \vee \psi) \equiv (\tau \to \phi) \wedge (\tau \to \psi).\end{aligned}$$

■

1.4 Variables and their Substitution

Primitive Notion 1.4. A variable can be a **variable in a formula**. This relation between variables and formulas is a primitive notion. But informally it means that the variable has appeared in the formula. There are two kinds of variables in a formula, namely **bound variables** and **free variables**. These are also primitive notions. We will provide their intuitive meanings after the next axiom.

Notation. Suppose ϕ is a formula, and $x_1, \ldots, x_n$ are its free variables. Then to denote this, we write

$$\phi(x_1, \ldots, x_n).$$

Note that $x_1, \ldots, x_n$ are not necessarily all the free variables of ϕ. We just want to emphasize that they are among the free variables of ϕ. For example we can just write $\phi(x_1)$ to state that x_1 is a free variable in ϕ.

Remark. Note that in the above notation, ϕ can be any formula. Also, $x_1, \ldots, x_n$ are just a notation for several variables, and any other collection of variables can be used in their place.

Axiom 1.4.

(i) *$\perp$ has no free or bound variables.*

(ii) *For every variable like x, the free variable of the formulas*

$$x = x, \qquad x \in x,$$

is x. Also for every distinct variables like x, y, the free variables of the formulas

$$x = y, \qquad x \in y,$$

are x, y. These formulas do not have any bound variables.

(iii) *Let ϕ be a formula. Then the free and bound variables of $\neg(\phi)$ are the free and bound variables of ϕ respectively.*

(iv) *Let ϕ, ψ be two formulas, that are not necessarily distinct. Consider the following formulas*

$$(\phi) \wedge (\psi), \quad (\phi) \vee (\psi), \quad (\phi) \to (\psi), \quad (\phi) \leftrightarrow (\psi).$$

A variable is a free variable of any of the above formulas if and only if it is a free variable of ϕ or a free variable of ψ. In other words, the collection of free variables of any of the above formulas is the union of the collection of free variables of ϕ and the collection of free variables of ψ. Similarly, a variable is a bound variable of any of the above formulas if and only if it is a bound variable of ϕ or a bound variable of ψ.

(v) *Let ϕ be a formula, and let x be a variable. Consider the following formulas*

$$\forall x(\phi), \qquad \exists x(\phi).$$

A variable is a free variable of any of the above formulas if and only if it is a free variable of ϕ, and it is not x. Also, a variable is a bound variable of any of the above formulas if and only if it is x, or it is a bound variable of ϕ.

Remark. The above axiom provides sufficient tools for finding the free and bound variables of every formula. We cannot prove this fact rigorously now, but informally it should be evident, since there is a rule for finding the free and bound variables corresponding to every rule for constructing a formula. Therefore in practice we can always find the free and bound variables of any formula we encounter.

Intuitively, a variable like x is a bound variable in a formula if and only if $\forall x$ and/or $\exists x$ appear somewhere in the formula. All other variables which appear in the formula are free. Note that a variable can be both free and bound in the same formula. For example, x is both free and bound in the following formulas:

$$(x \in y) \wedge (\forall x(x = x)), \qquad (\exists x(x \in y)) \to (\neg(x = y)).$$

This undesired phenomenon results from the fact that we do not impose any condition on two formulas like ϕ, ψ, when we combine them to construct other formulas like $\phi \vee \psi$. It is possible to exclude such expressions from being formulas, but it is easier to allow them to be formulas at this stage, and later develop some rules to deal with them.

Remark. We can rewrite the above formulas as follows:

$$(x \in y) \wedge (\forall z(z = z)), \qquad (\exists z(z \in y)) \to (\neg(x = y)).$$

Intuitively, these formulas have the same meaning as the previous ones. Also, no variable is both free and bound in them. We can always change the bound occurrences of a variable in a formula to produce another formula, as we did above, so that no variable is both free and bound in the new formula. We will not define this process rigorously, since we do not use it. Also, we do not have the mathematical tools to prove the above fact in general here. But we can easily check that it is true in every formula we encounter.

Another point to mention is that a variable can be the bound variable of several quantifiers in a formula. For example y in the formula

$$\forall x \exists y(x \in y) \vee \forall z \forall y(z \in y \to z \neq y)$$

is of this type. A related notion to this phenomenon is the notion of the **scope** of a quantifier in a formula. We will not try to define this notion precisely here, and we do not need its precise definition now. But informally it indicates the part of the formula over which the quantifier has effect. For example in the above formula, the scope of $\forall z$ is $\forall y(z \in y \to z \neq y)$, and the scope of $\exists y$ is $x \in y$. More explicitly, when we use a formula like ϕ to construct another formula like $\forall x\phi$, then ϕ is the scope of $\forall x$. But if we use the formula $\forall x\phi$ to construct another formula like $\forall x\phi \to \psi \wedge \exists x\tau$. Then the scope of $\forall x$ does not change, and is still ϕ.

***Example* 1.7.** Consider the formula

$$\exists x\forall y(x = y \to y \in z) \wedge \big((x = a \leftrightarrow x \in v) \vee \forall z\exists x(x \neq z \to z \neq x)\big).$$

The free variables of this formula are z, x, a, v, and its bound variables are x, y, z. The scope of $\forall y$ is $x = y \to y \in z$, and the scope of $\forall z$ is $\exists x(x \neq z \to z \neq x)$. There are two occurrences of $\exists x$ in the formula. The scope of the first $\exists x$ is $\forall y(x = y \to y \in z)$, and the scope of the second $\exists x$ is $x \neq z \to z \neq x$.

Remark. In the above formula x occurs as a bound variable once in the scope of the first $\exists x$, and again in the scope of the second $\exists x$. In addition, x occurs as a free variable in $x = a$ and $x \in v$. We will not try to make the notions of *free and bound occurrences* precise, but the above example should be sufficient to convey their meanings. Similarly, we have several occurrences of $\exists x$. We will not try to give an exact meaning to this either. Mainly because we do not need, and will not use, the exact meaning of these notions. An informal and intuitive understanding of them is sufficient for our purposes.

Finally let us mention some other anomalies that can happen when we add quantifiers to a formula. We do not require x to be a free variable in ϕ, or even a variable in ϕ, when we construct $\forall x\phi$ or $\exists x\phi$. Thus for example $\exists a(x = y)$ is a well-formed formula, even though $\exists a$ is superfluous here. Even worse is the case that a variable is bounded successively by two quantifiers, like $\forall x(\exists x(x = y))$.

Remark. As another example consider

$$\forall x(x \neq z \wedge \exists x(x = y)).$$

Here x is bounded by two quantifiers successively, but unlike the previous example the outer quantifier is not superfluous here. The usual interpretation of these kinds of formulas is to change the inner x to another bound variable. So for example, we can rewrite the above formula as follows:

$$\forall x(x \neq z \wedge \exists u(u = y)).$$

We will not need the apparatus to deal with these kinds of formulas, so we will not introduce it rigorously here.

Although the above formulas are problematic, and sometimes meaningless, they do not create much problem for us, and we can deal with them by introducing some simple rules later. So, similarly to most texts, we will not exclude them from being formulas.

Primitive Notion 1.5. The notion of **substitution** of free occurrences of a variable in a formula (if any) by another variable is a primitive notion. Intuitively, it means that if x is a free variable in a formula ϕ, and y is a variable which is not bound in ϕ, then we replace every free occurrence of x in the written representation of ϕ by y, and we obtain a new formula which we denote by

$$\phi[y/x].$$

Note that x need not be a free variable in ϕ; and in this case we intuitively know that $\phi[y/x]$ is the same formula as ϕ. Also note that x, y can be the same variable.

Remark. Note that y can be a free variable in ϕ before substitution. But y cannot be a bound variable in ϕ. Also note that if x is a variable which is both free and bound in ϕ, then when we substitute it with y, we only substitute the occurrences of x as a free variable, and we do not change the bound occurrences of x.

Remark. The reason that we do not allow y to be a bound variable in ϕ is that otherwise the truth of ϕ can change after substitution. For example consider

$$\exists y(x \neq y).$$

This formula says that there is a set y different than x, which is a true statement in the standard universe of sets. However if we replace x by y we get $\exists y(y \neq y)$. Now this new formula says that there is a set y that is not equal to itself, which is obviously a false statement.

Axiom 1.5. *Suppose ϕ is a formula, and x is a variable which is not a free variable in ϕ. Let y be a variable which is not bound in ϕ. Then $\phi[y/x]$ is the same formula as ϕ.*

Remark. As a consequence, $\perp$ does not change under substitution of variables, since $\perp$ has no free variables.

Axiom 1.6. *Suppose ϕ is a formula, x is a variable, and y is a variable which is not bound in ϕ. Then $\phi[y/x]$ is a formula whose bound variables are exactly the bound variables of ϕ. Also, if x is a free variable of ϕ, then a variable is a free variable of $\phi[y/x]$ if and only if it is a free variable of ϕ other than x, or it is y.*

Remark. Note that if x is not a free variable of ϕ, then $\phi[y/x]$ is the same formula as ϕ. Hence, in this case, a variable is a free variable of $\phi[y/x]$ if and only if it is a free variable of ϕ. Thus in particular, x is not a free variable of $\phi[y/x]$ either. This observation, and the above axiom, imply that x is never a free variable in $\phi[y/x]$, regardless of whether x is a free variable in ϕ or not.

Notation. Suppose $\phi(x)$ is a formula that has x as a free variable, and y is a variable which is not bound in ϕ. Then we sometimes denote $\phi[y/x]$ simply by $\phi(y)$. Note that this usage of the notation $\phi(\cdot)$ is compatible with the previous one, since in this case, y is a free variable in $\phi[y/x]$.

Axiom 1.7. *Suppose ϕ is a formula, and x is a variable which is not a bound variable of ϕ. Also suppose that y is a variable which is not a free or bound variable of ϕ. Let us denote $\phi[y/x]$ by ψ. Then we have*
(i) *$\phi[x/x]$ is the same formula as ϕ.*
(ii) *$\psi[x/y]$ is the same formula as ϕ.*

Remark. Note that we are allowed to substitute x for y in ψ, since x is not bound in ψ. Because the bound variables of ψ are the same as ϕ, and we assumed that x is not bound in ϕ. For the same reason, we are allowed to substitute x for x in ϕ.

Remark. The above axiom says that if we substitute a variable, and then substitute that variable back, we arrive at the original formula. It also says that if we substitute a variable with itself, the formula does not change. These facts are intuitively obvious, but we do not have the mathematical tools to prove them, so we accept them as axioms.

Remark. Note that if y is allowed to be a free variable in ϕ, then $\psi[x/y]$ and ϕ will be different; because y is not a free variable in $\psi[x/y]$, while it is a free variable in ϕ.

Axiom 1.8. *Suppose ϕ, ψ are formulas, x, z are variables, and y is a variable which is not bound in ϕ, ψ.*

(i) *If ϕ is one of the formulas $x = x$ or $x \in x$, then $\phi[y/x]$ is respectively*

$$y = y, \qquad \textit{or} \qquad y \in y.$$

Also if ϕ is one of the formulas $x = z$ or $x \in z$, where x, z are distinct, then $\phi[y/x]$ is respectively

$$y = z, \qquad \textit{or} \qquad y \in z.$$

Note that in this case y, z can also be the same variable. Similarly, if ϕ is one of the formulas $z = x$ or $z \in x$, where x, z are distinct, then $\phi[y/x]$ is respectively

$$z = y, \qquad \textit{or} \qquad z \in y.$$

Also, in this case y, z can be the same variable too.

(ii) *The formula $(\neg\phi)[y/x]$ is*

$$\neg(\phi[y/x]).$$

(iii) *The following formulas*

$$(\phi \wedge \psi)[y/x], \quad (\phi \vee \psi)[y/x], \quad (\phi \to \psi)[y/x], \quad (\phi \leftrightarrow \psi)[y/x],$$

are respectively

$$\begin{aligned} (\phi[y/x]) \wedge (\psi[y/x]), &\qquad (\phi[y/x]) \vee (\psi[y/x]), \\ (\phi[y/x]) \to (\psi[y/x]), &\qquad (\phi[y/x]) \leftrightarrow (\psi[y/x]). \end{aligned}$$

(iv) *If z is a variable different from x, y, then the formulas*

$$(\forall z\phi)[y/x], \qquad (\exists z\phi)[y/x],$$

are respectively

$$\forall z(\phi[y/x]), \qquad \exists z(\phi[y/x]).$$

Remark. Note that by the above axiom and Axiom 1.5 we can perform the substitution on any atomic formula. Because $\perp$ does not change under substitution as it does not have any variable. In addition, any atomic formula of the form $w = z$ or $w \in z$ is either considered in the above axiom (if one or both of w, z is the same as x), or it does not contain x as a free variable (if w, z are different from x), in which case the formula does not change under substitution by Axiom 1.5.

Remark. Note that in the last part of the above axiom, if we had $\forall y\phi$ or $\exists y\phi$, then we could not substitute y for x, because by Axiom 1.4, y is a bound variable in these formulas. In addition, if we had $\forall x\phi$ or $\exists x\phi$, then x is not a free variable in these formulas; therefore, by Axiom 1.5, these formulas do not change under the substitution for x.

Remark. Note that by Axiom 1.4, if y is not a bound variable in ϕ, ψ, then it is not a bound variable in the following formulas

$$\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \phi \to \psi, \quad \phi \leftrightarrow \psi, \quad \forall z\phi, \quad \exists z\phi.$$

Therefore, in the above axiom, the requirements for substitution are satisfied.

Remark. The above axiom and Axiom 1.5 provide sufficient tools to find $\phi[y/x]$ for every ϕ which has the necessary properties. We cannot prove this fact rigorously now, but informally it should be evident, since there is a rule for constructing $\phi[y/x]$ corresponding to every rule for constructing a formula. Therefore in practice we can always find $\phi[y/x]$ for any formula ϕ that we encounter.

***Example* 1.8.** Suppose ϕ is the formula

$$\forall z\forall u(z \in a \wedge u \in z \to u \in c) \wedge \forall y(y \in x \to \exists z(z \in a \wedge y \in z)) \leftrightarrow \forall a(a = a).$$

Then $\phi[c/a]$ is

$$\forall z \forall u(z \in c \wedge u \in z \to u \in c) \wedge \forall y(y \in x \to \exists z(z \in c \wedge y \in z)) \leftrightarrow \forall a(a = a).$$

Note that the bound occurrence of a in $\forall a(a = a)$ does not change. Also note that c is a free variable in ϕ before substitution, but this does not prevent us from substituting it for a.

1.5 Rules of Inference for Quantifiers

Let us recall that we write $\phi(x)$ to emphasize that x is a free variable in the formula ϕ. Also, if y is not a bound variable in ϕ, then we write $\phi(y)$ as a shorthand notation for $\phi[y/x]$. Note that this usage of the notation $\phi(\cdot)$ is compatible with the previous one, since in this case, y is a free variable in $\phi[y/x]$. In addition, remember that when x is not a free variable in ϕ, then $\phi[y/x]$ is the same formula as ϕ.

Axiom 1.9 (Rules of inference for quantifiers)**.** *Suppose* Γ *denotes a collection of several formulas, which can be empty too. Let* ϕ, ψ *be formulas, and let* x, y *be variables. Then we have*

(i) *Introduction of* $\forall$*, or Universal generalization* :

Suppose x *is not a free variable in any of the formulas in* Γ*.*
Then we have : *If* $\Gamma \vdash \phi$ *then* $\Gamma \vdash \forall x\phi$*.*

(ii) *Elimination of* $\forall$*, or Universal instantiation* :

$$\forall x\phi \vdash \phi,$$
and if y *is not a bound variable in* ϕ*,*
then we also have : $\forall x\phi \vdash \phi[y/x]$*.*

(iii) *Introduction of* $\exists$*, or Existential generalization* :

$$\phi \vdash \exists x\phi,$$
and if y *is not a bound variable in* ϕ*,*
then we also have : $\phi[y/x] \vdash \exists x\phi$*.*

(iv) *Elimination of* $\exists$:

Suppose x *is not a free variable in* ψ*, nor in any of the formulas in* Γ*.*
Then we have : *If* $\Gamma, \phi \vdash \psi$ *then* $\Gamma, \exists x\phi \vdash \psi$*.*

Remark. Note that we do not require x to be a free variable in ϕ, although this is the case that we are actually interested in. The reason is that we need the more general version of the axiom in order to be able to show that $\forall x, \exists x$ are redundant in $\forall x\phi, \exists x\phi$, when x is not a free variable in ϕ. See Theorem 1.9.

Notation. We will use shorthand notations for the inference rules, as we did in the last section. For example, the introduction of $\forall$ will be denoted by I$\forall$, and the elimination of $\exists$ will be denoted by E$\exists$.

Let us inspect the above rules more closely. Let us only consider the meaningful case where x is a free variable in ϕ. The I$\forall$, or *universal generalization*, says that if we can deduce $\phi(x)$, then we can also deduce $\forall x\phi(x)$. In other words, if $\phi(x)$ is true for an arbitrary x, then $\phi(x)$ is true for every x, i.e. $\forall x\phi(x)$ is true. This is how universal statements are usually proved in mathematics; we prove the statement for an arbitrary object, and then conclude that the statement must hold for every object. Intuitively, the reason is that when we prove the statement without assuming anything specific about the object, then the same reasoning can work for any other object, hence the statement is true for every object.

Note that the important part of the above reasoning is that we do not assume anything specific about the object. To incorporate this into the rule I$\forall$, we supposed that x is not a free variable in any of the formulas in Γ. In other words, we supposed that we are not stating anything specific about x in the premises. Furthermore, from a formal syntactic viewpoint, if we had allowed x to be a free variable in some of the premises, then we could come to conclusions which are obviously false. For example from the formula $x \in y$, which says that the set y contains some set x, we could deduce $\forall x(x \in y)$, which says that y contains every set!

Remark. Note that in particular, we cannot deduce $\forall x\phi(x)$ from $\phi(x)$, when x is a free variable in ϕ.

The E$\forall$, or *universal instantiation*, says that if $\forall x\phi(x)$ is true, i.e. if $\phi(x)$ is true for every x, then in particular $\phi(y)$ is true. Similarly, the I$\exists$, or *existential generalization*, says that if $\phi(y)$ is true for some y, then $\exists x\phi(x)$ is true. The only restriction is that y cannot be bound in ϕ, so that we can substitute y for x in ϕ. A question that arises is that why did we also include $\phi[y/x]$ in these rules, in addition to $\phi(x)$? The above intuitive explanations of the rules suggest that allowing $\phi[y/x]$ is semantically preferable, but there are also some technical reasons behind this choice. First, these rules allow us to prove results about substitution of variables in formulas.

The second reason is that sometimes we need to only quantify over some occurrences of a variable, not all of them. For example from $y = y$ we can deduce $\exists x(x = y)$. Because we can consider $y = y$ as $x = y$, in which we substituted y for x. But if we stated the rules only for $\phi(x)$, then from $y = y$ we could only deduce that $\exists y(y = y)$. Note that although both conclusions are true, they are stating different facts. Another point that should be noted is that we can consider ϕ as a special case of $\phi[y/x]$, namely we can consider ϕ as $\phi[x/x]$. Intuitively, it is obvious that if we substitute x with x, the formula ϕ does not change. However, if we want to treat this fact rigorously using the Axiom 1.7, we have to assume that x is not bound in ϕ, which is an unnecessary restriction. Although we can avoid this restriction by stating more axioms about substitution of variables, we prefer to avoid these complications altogether. Therefore we separated the case of ϕ in the rules E$\forall$ and I$\exists$.

Remark. An important assumption implicit in the rule I$\exists$ is that we are tacitly assuming that at least one set exists. Because otherwise we could not deduce that "there is a set x such that $\phi(x)$ holds", i.e. we could not deduce $\exists x\phi(x)$.

Finally, let us consider E$\exists$. It says that if we can deduce ψ from $\phi(x)$, then we can also deduce ψ from $\exists x\phi(x)$. In other words, if we can deduce that ψ holds by knowing that a particular set like x satisfies ϕ, then we can also deduce that ψ holds simply by knowing that there is a set that satisfies ϕ, i.e. by knowing that $\exists x\phi(x)$. Intuitively, in order for this argument to be valid, we cannot assume anything specific about x. Also, we cannot deduce anything specific about x. Therefore we require that x does not appear as a free variable in the conclusion ψ, nor in any of the formulas in the premises Γ. Hence in deducing ψ, the only property of x that we used is that it satisfies ϕ. Thus, just knowing that some set satisfies ϕ must be sufficient to deduce ψ.

In addition, from a formal syntactic viewpoint, if we had allowed x to be a free variable in some of the premises, or in the conclusion, then we could obtain false results. For example we have $x \in y \vdash x \in y$. Hence we could deduce that $\exists x(x \in y) \vdash x \in y$. Thus by I$\forall$ we could get $\exists x(x \in y) \vdash \forall x(x \in y)$, since x is not a free variable in $\exists x(x \in y)$. Therefore from the fact that the set y contains some set x, we could deduce that y contains every set! As another example consider the entailment $y = x, x = z \vdash y = z$, which reflects the transitivity of equality. If we were allowed to apply E$\exists$ here, we could deduce that $y = x, \exists x(x = z) \vdash y = z$. Then by exchange rule, and a valid application of E$\exists$, we could deduce that $\exists x(y = x), \exists x(x = z) \vdash y = z$. But this entailment is obviously false, because just by knowing that there is a set equal to y, and there is a set equal to z, we certainly cannot deduce that y, z are the same set!

Remark. An inference rule related to E$\exists$, which we do not accept as a valid rule, is *existential instantiation*. It says that $\exists x\phi \vdash \phi[y/x]$, provided that y is not a bound variable in ϕ. In other words, it says that if there is a set for which ϕ holds, then ϕ holds for some set y. This rule is closely related to the process of naming; a name, y, is given to the set for which ϕ holds. For this reason, existential instantiation seems to be a natural rule of inference. However, if we want to accept it as a valid rule of inference, we have to impose complicated constraints on when we can apply the other rules. Otherwise it leads us to wrong conclusions. For example, existential instantiation implies that $\exists x(x \in z) \vdash y \in z$. Hence by I$\forall$ we get $\exists x(x \in z) \vdash \forall y(y \in z)$, which cannot be a valid entailment as we saw before.

Therefore, in order to avoid the complications caused by existential instantiation, we do not include it in our valid rules of inference, and instead we use the elimination of $\exists$. Note that this choice does not limit us in proving theorems. Because if we want to deduce a formula ψ by assuming $\exists x\phi$, we can deduce ψ by assuming $\phi(x)$, and then conclude our desired entailment by applying E$\exists$. In other words, we can give the element which satisfies ϕ the name x, and then use x to deduce ψ. Finally we can use E$\exists$ to conclude that ψ can be deduced from $\exists x\phi$. This is how such proofs are usually carried out in mathematics. When in our assumptions we have a formula which says that an object with certain properties exists, i.e. we have a formula of the form $\exists x\phi$, we assume that some object satisfies that property, and we use that object to prove our desired results. The part of these proofs in which an object is picked is usually phrased like "Let x be an object such that $\phi(x)$ holds.".

Thus in practice there is not much difference between E$\exists$ and existential instantiation. The only difference is that in E$\exists$ we do not say that $\exists x\phi$ implies $\phi(x)$. Rather, we say that if we can deduce anything from $\phi(x)$, then we can also deduce it from $\exists x\phi$. Note that if we accept existential instantiation as a valid rule, then the last sentence can be obtained from it and the cut rule.

***Example* 1.9.** It is easy to see that for any formula ϕ we have

$$\forall x\phi \vdash \exists x\phi.$$

Because by E$\forall$ and I$\exists$ we have $\forall x\phi \vdash \phi \vdash \exists x\phi$. Hence we get the desired by the cut rule.

Theorem 1.8. *Suppose ϕ, ψ are formulas, and $\phi \equiv \psi$. Let x be a variable. Then we have*

(i) $\forall x\phi \equiv \forall x\psi$.

(ii) $\exists x\phi \equiv \exists x\psi$.

$\boxed{\textbf{Proof.}}$ **(i)** By E$\forall$ and equivalence of ϕ, ψ we have $\forall x\phi \vdash \phi \vdash \psi$. Thus by I$\forall$ we get $\forall x\phi \vdash \forall x\psi$, since x is not a free variable in the premises, i.e. in $\forall x\phi$. Similarly we can show that $\forall x\psi \vdash \forall x\phi$. Hence we get the desired.

(ii) By I$\exists$ and equivalence of ϕ, ψ we have $\phi \vdash \psi \vdash \exists x\psi$. Thus by E$\exists$ we get $\exists x\phi \vdash \exists x\psi$, since x is not a free variable in the conclusion, i.e. in $\exists x\psi$. Similarly we can show that $\exists x\psi \vdash \exists x\phi$. Hence we get the desired. ■

As we have seen after Theorem 1.4, if a formula is composed of several other formulas, then we can replace some of the components by equivalent formulas to obtain a formula equivalent to the original compound formula. The above theorem enables us to do this when the original compound formula also contains quantifiers. As before, we will not prove the general case of this fact here, rather, we demonstrate it by an example. Suppose $\phi \equiv \psi$ and $\tau \equiv \sigma$. Consider the following formula

$$\exists y(\phi \vee \phi_1) \leftrightarrow (\phi_2 \to \exists u \forall z \tau \wedge \phi_3).$$

We claim that it is equivalent to $\exists y(\psi \vee \phi_1) \leftrightarrow (\phi_2 \to \exists u \forall z \sigma \wedge \phi_3)$. To see this note that by the above theorem $\forall z \tau \equiv \forall z \sigma$, and therefore $\exists u \forall z \tau \equiv \exists u \forall z \sigma$. Also note that any formula is equivalent to itself. Therefore by Theorem 1.4 we have

$$\phi \vee \phi_1 \equiv \psi \vee \phi_1, \qquad \exists u \forall z \tau \wedge \phi_3 \equiv \exists u \forall z \sigma \wedge \phi_3.$$

Now if we apply the above theorem again we obtain $\exists y(\phi \vee \phi_1) \equiv \exists y(\psi \vee \phi_1)$. Finally, by Theorem 1.4 we obtain that $\phi_2 \to \exists u \forall z \tau \wedge \phi_3 \equiv \phi_2 \to \exists u \forall z \sigma \wedge \phi_3$, and therefore we get

$$\exists y(\phi \vee \phi_1) \leftrightarrow (\phi_2 \to \exists u \forall z \tau \wedge \phi_3) \equiv \exists y(\psi \vee \phi_1) \leftrightarrow (\phi_2 \to \exists u \forall z \sigma \wedge \phi_3),$$

as desired.

Theorem 1.9. *Suppose ϕ is a formula, and x is a variable which is not a free variable in ϕ. Then we have*

$$\forall x \phi \equiv \phi \equiv \exists x \phi.$$

Remark. This theorem shows that quantifying over variables which do not occur free is redundant.

Proof. We know that $\phi \vdash \phi$. Thus by I$\forall$ we have $\phi \vdash \forall x \phi$, since x is not a free variable in the premises, i.e. in ϕ. On the other hand, by E$\forall$ we have $\forall x \phi \vdash \phi$. Hence we have $\forall x \phi \equiv \phi$.

Similarly, by I$\exists$ we have $\phi \vdash \exists x \phi$. On the other hand since $\phi \vdash \phi$, and x is not a free variable in the conclusion ϕ, we can use E$\exists$ to obtain $\exists x \phi \vdash \phi$. Hence we also have $\phi \equiv \exists x \phi$. ■

Theorem 1.10. *Suppose ϕ, ψ are formulas, and x, y are variables. Then we have*

(i)
$$\neg(\exists x \phi) \equiv \forall x(\neg \phi).$$

(ii)
$$\neg(\forall x \phi) \equiv \exists x(\neg \phi).$$

(iii)
$$\exists x \exists y \phi \equiv \exists y \exists x \phi.$$

(iv)
$$\forall x \forall y \phi \equiv \forall y \forall x \phi.$$

(v)
$$\exists x \forall y \phi \vdash \forall y \exists x \phi.$$

(vi)
$$\exists x(\phi \vee \psi) \equiv \exists x \phi \vee \exists x \psi.$$

(vii)
$$\forall x(\phi \wedge \psi) \equiv \forall x \phi \wedge \forall x \psi.$$

(viii)
$$\exists x(\phi \wedge \psi) \vdash \exists x \phi \wedge \exists x \psi.$$

(ix)
$$\forall x \phi \vee \forall x \psi \vdash \forall x(\phi \vee \psi).$$

(x) *Suppose x is not a free variable in ϕ, then*
$$\exists x(\phi \wedge \psi) \equiv \phi \wedge \exists x \psi.$$

(xi) *Suppose x is not a free variable in ϕ, then*
$$\forall x(\phi \vee \psi) \equiv \phi \vee \forall x \psi.$$

Proof. **(i)** By the weakening rule and E$\forall$ we have $\forall x(\neg\phi), \phi \vdash \neg\phi$. Hence by E$\neg$ and the cut rule we have $\forall x(\neg\phi), \phi \vdash \perp$. Note that x is not a free variable in $\forall x(\neg\phi), \perp$ due to the Axiom 1.4. Now we can apply the E$\exists$ to conclude that $\forall x(\neg\phi), \exists x \phi \vdash \perp$. Finally by I$\neg$ we get
$$\forall x(\neg\phi) \vdash \neg(\exists x \phi).$$

Conversely, by the weakening rule and I$\exists$ we have $\neg(\exists x \phi), \phi \vdash \exists x \phi$. Hence by E$\neg$ and the cut rule we have $\neg(\exists x \phi), \phi \vdash \perp$. Thus by I$\neg$ we get $\neg(\exists x \phi) \vdash \neg\phi$. Now note that x is not a free variable in $\neg(\exists x \phi)$. Therefore by I$\forall$ we have
$$\neg(\exists x \phi) \vdash \forall x(\neg\phi),$$

as desired.

(ii) By the weakening rule and E$\forall$ we have $\forall x \phi, \neg\phi \vdash \phi$. Hence by E$\neg$ and the cut rule we have $\forall x \phi, \neg\phi \vdash \perp$. Note that $\forall x \phi, \perp$ do not have x as a free variable. Hence we can apply the E$\exists$ to conclude that $\forall x \phi, \exists x(\neg\phi) \vdash \perp$. Finally by exchange rule and I$\neg$ we get
$$\exists x(\neg\phi) \vdash \neg(\forall x \phi).$$

Conversely, note that by the previous part and E$\forall$ we have $\neg\exists x(\neg\phi) \vdash \forall x(\neg\neg\phi) \vdash \neg\neg\phi$. But by Theorem 1.6 we have $\neg\neg\phi \vdash \phi$. So by the cut rule we get $\neg\exists x(\neg\phi) \vdash \phi$. Now note that x is not a free variable in $\neg\exists x(\neg\phi)$. Therefore by I$\forall$ we have $\neg\exists x(\neg\phi) \vdash \forall x\phi$. Hence by E$\neg$ we obtain $\neg(\forall x\phi), \neg\exists x(\neg\phi) \vdash \perp$. Thus by RAA we get

$$\neg(\forall x\phi) \vdash \exists x(\neg\phi),$$

as desired.

(iii) By I$\exists$ we have $\phi \vdash \exists x\phi \vdash \exists y\exists x\phi$. Now note that x, y are not free variables in $\exists y\exists x\phi$ due to the Axiom 1.4. Hence by applying E$\exists$ twice we get $\exists y\phi \vdash \exists y\exists x\phi$, and $\exists x\exists y\phi \vdash \exists y\exists x\phi$. Similarly we can show that $\exists y\exists x\phi \vdash \exists x\exists y\phi$.

(iv) By E$\forall$ we have $\forall x\forall y\phi \vdash \forall y\phi \vdash \phi$. Now note that x, y are not free variables in $\forall x\forall y\phi$ due to the Axiom 1.4. Hence by applying I$\forall$ twice we get $\forall x\forall y\phi \vdash \forall x\phi$, and $\forall x\forall y\phi \vdash \forall y\forall x\phi$. Similarly we can show that $\forall y\forall x\phi \vdash \forall x\forall y\phi$.

(v) By E$\forall$ and I$\exists$ we have $\forall y\phi \vdash \phi \vdash \exists x\phi$. Now note that x is not a free variable in $\exists x\phi$. Hence by E$\exists$ we get $\exists x\forall y\phi \vdash \exists x\phi$. In addition, note that y is not a free variable in $\exists x\forall y\phi$. Thus by I$\forall$ we obtain $\exists x\forall y\phi \vdash \forall y\exists x\phi$, as desired.

(vi) By I$\exists$ and I$\vee$ we have $\phi \vdash \exists x\phi \vdash \exists x\phi \vee \exists x\psi$. Similarly we have $\psi \vdash \exists x\psi \vdash \exists x\phi \vee \exists x\psi$. Therefore by E$\vee$ we get $\phi \vee \psi \vdash \exists x\phi \vee \exists x\psi$. Now note that x is not a free variable in $\exists x\phi \vee \exists x\psi$. Thus by E$\exists$ we obtain

$$\exists x(\phi \vee \psi) \vdash \exists x\phi \vee \exists x\psi.$$

Conversely, by I$\vee$ and I$\exists$ we have $\phi \vdash \phi \vee \psi \vdash \exists x(\phi \vee \psi)$. Similarly we have $\psi \vdash \phi \vee \psi \vdash \exists x(\phi \vee \psi)$. Now note that x is not a free variable in $\exists x(\phi \vee \psi)$. Thus by E$\exists$ we obtain $\exists x\phi \vdash \exists x(\phi \vee \psi)$, and $\exists x\psi \vdash \exists x(\phi \vee \psi)$. Hence by E$\vee$ we get

$$\exists x\phi \vee \exists x\psi \vdash \exists x(\phi \vee \psi).$$

(vii) By E$\forall$ and E$\wedge$ we have $\forall x(\phi \wedge \psi) \vdash \phi \wedge \psi \vdash \phi$. Similarly we have $\forall x(\phi \wedge \psi) \vdash \phi \wedge \psi \vdash \psi$. Now note that x is not a free variable in $\forall x(\phi \wedge \psi)$. Thus by I$\forall$ we get $\forall x(\phi \wedge \psi) \vdash \forall x\phi$, and $\forall x(\phi \wedge \psi) \vdash \forall x\psi$. Hence by the cut rule and I$\wedge$ we obtain

$$\forall x(\phi \wedge \psi) \vdash \forall x\phi \wedge \forall x\psi.$$

Conversely, by E$\wedge$ and E$\forall$ we have $\forall x\phi \wedge \forall x\psi \vdash \forall x\phi \vdash \phi$. Similarly we have $\forall x\phi \wedge \forall x\psi \vdash \forall x\psi \vdash \psi$. Thus by the cut rule and I$\wedge$ we get $\forall x\phi \wedge \forall x\psi \vdash \phi \wedge \psi$. Now note that x is not a free variable in $\forall x\phi \wedge \forall x\psi$. Therefore by I$\forall$ we get

$$\forall x\phi \wedge \forall x\psi \vdash \forall x(\phi \wedge \psi),$$

as desired.

(viii) By E$\wedge$ and I$\exists$ we have $\phi\wedge\psi \vdash \psi \vdash \exists x\psi$. Similarly we have $\phi\wedge\psi \vdash \phi \vdash \exists x\phi$. Thus by the cut rule and I$\wedge$ we get $\phi \wedge \psi \vdash \exists x\phi \wedge \exists x\psi$. Now note that x is not a free variable in $\exists x\phi \wedge \exists x\psi$. Therefore by E$\exists$ we obtain

$$\exists x(\phi \wedge \psi) \vdash \exists x\phi \wedge \exists x\psi.$$

(ix) By E$\forall$ and I$\vee$ we have $\forall x\psi \vdash \psi \vdash \phi\vee\psi$. Similarly we have $\forall x\phi \vdash \phi \vdash \phi\vee\psi$. Thus by E$\vee$ we get $\forall x\phi \vee \forall x\psi \vdash \phi \vee \psi$. Now note that x is not a free variable in $\forall x\phi \vee \forall x\psi$. Hence by I$\forall$ we obtain

$$\forall x\phi \vee \forall x\psi \vdash \forall x(\phi \vee \psi).$$

(x) By E$\wedge$ and I$\exists$ we have $\phi\wedge\psi \vdash \psi \vdash \exists x\psi$. Similarly, by E$\wedge$ we have $\phi\wedge\psi \vdash \phi$. Thus by the cut rule and I$\wedge$ we get $\phi \wedge \psi \vdash \phi \wedge \exists x\psi$. Now note that x is not a free variable in $\phi \wedge \exists x\psi$. Therefore by E$\exists$ we obtain

$$\exists x(\phi \wedge \psi) \vdash \phi \wedge \exists x\psi.$$

Conversely, by I$\wedge$ and I$\exists$ we have $\phi, \psi \vdash \phi \wedge \psi \vdash \exists x(\phi \wedge \psi)$. Now note that x is not a free variable in ϕ and $\exists x(\phi \wedge \psi)$. Thus by E$\exists$ we get $\phi, \exists x\psi \vdash \exists x(\phi \wedge \psi)$. Hence by E$\wedge$ and the cut rule we obtain

$$\phi \wedge \exists x\psi \vdash \exists x(\phi \wedge \psi).$$

(xi) By the previous part we have

$$\neg\phi \wedge \exists x(\neg\psi) \equiv \exists x(\neg\phi \wedge \neg\psi);$$

because x is not a free variable in $\neg\phi$. Now by Theorem 1.4 we have

$$\neg(\neg\phi \wedge \exists x(\neg\psi)) \equiv \neg(\exists x(\neg\phi \wedge \neg\psi)).$$

Hence by De Morgan's law and part (i) of this theorem we get

$$\neg\neg\phi \vee \neg\exists x(\neg\psi) \equiv \forall x(\neg(\neg\phi \wedge \neg\psi)).$$

If we apply the De Morgan's law and part (i) of this theorem again, and use Theorems 1.4, 1.8, we obtain

$$\neg\neg\phi \vee \forall x(\neg\neg\psi) \equiv \forall x(\neg\neg\phi \vee \neg\neg\psi).$$

Finally, by double negation law, and Theorems 1.4, 1.8 we get

$$\phi \vee \forall x\psi \equiv \forall x(\phi \vee \psi),$$

as desired. ■

Remark. The equivalence $\neg(\exists x\phi) \equiv \forall x(\neg\phi)$ says that if there does not exists an x such that ϕ holds, then for every x, $\neg\phi$ must hold. Similarly, the equivalence $\neg(\forall x\phi) \equiv \exists x(\neg\phi)$ says that if it is not the case that for every x, ϕ holds, then there must exist an x such that $\neg\phi$ holds. In other words, if a universal statement is not valid, then a **counterexample** to it must exist.

Remark. As shown in the above theorem, the order of consecutive quantifiers can be changed, as long as they are of the same type. However, we cannot change the order of quantifiers of different types; because in general, $\forall x\exists y\phi$ does not imply $\exists y\forall x\phi$. For example, the formula $\forall x\exists y(y = x)$ says that for every set x there is a set y which is equal to it. Intuitively, this formula is true, since we can take y to be the same as x. But if we change the order of quantifiers we get the formula $\exists y\forall x(y = x)$, which says that there is a set y which is equal to every set x! And this formula is obviously false in the standard theory of sets.

Remark. In the first two parts of the above axiom, if we replace ϕ by $\neg\phi$, and use the fact that $\neg\neg\phi \equiv \phi$, then by Theorem 1.8 we obtain

$$\neg\exists x(\neg\phi) \equiv \forall x\phi, \qquad \neg\forall x(\neg\phi) \equiv \exists x\phi.$$

Therefore we can express each quantifier in terms of the other one.

Remark. In general, $\exists x\phi \wedge \exists x\psi$ does not imply $\exists x(\phi \wedge \psi)$; and $\forall x(\phi \vee \psi)$ does not imply $\forall x\phi \vee \forall x\psi$. Intuitively, these can be seen as follows. Suppose ϕ says that x has no element, and ψ says that x has some element. Then it is easy to check that both $\exists x\phi \wedge \exists x\psi$ and $\forall x(\phi \vee \psi)$ are true, while both $\exists x(\phi \wedge \psi)$ and $\forall x\phi \vee \forall x\psi$ are false.

Theorem 1.11. *Suppose ϕ is a formula, and x is a free variable in ϕ, which is not bound in ϕ. Let y be a variable which is not a free variable nor a bound variable in ϕ. Then we have*

$$\forall x\phi(x) \equiv \forall y\phi(y), \qquad \exists x\phi(x) \equiv \exists y\phi(y).$$

Remark. This theorem provides a tool for substituting bound variables in some cases. Note that we assume that x is not bound in ϕ, hence x is only bounded by one quantifier in the above formulas. However, this special case is sufficient for most of our purposes. Also note that we can apply this theorem repeatedly to change several bound variables in a formula.

Proof. By E$\forall$ we have $\forall x\phi(x) \vdash \phi(y)$. Hence by I$\forall$ we get $\forall x\phi(x) \vdash \forall y\phi(y)$, since y is not a free variable in $\forall x\phi(x)$, because we assumed that y is not a free variable in ϕ. On the other hand, by E$\forall$ we have $\forall y\phi(y) \vdash \phi(y)[x/y]$. Here $\phi(y)[x/y]$ is the result of substituting x back in ϕ, after we had substituted it with y. But by Axiom 1.7, and our assumptions about x, y, we can conclude that $\phi(y)[x/y]$ is the same formula as $\phi(x)$. Also, note that x is neither a free variable nor a bound variable in $\phi(y)$ and $\forall y\phi(y)$, due to the Axioms 1.4, 1.6, and the fact that x is not bound in ϕ. Therefore by I$\forall$ we have $\forall y\phi(y) \vdash \forall x\phi(x)$. Hence we get $\forall x\phi(x) \equiv \forall y\phi(y)$, as desired.

Now consider the second equivalence. By I$\exists$ we have $\phi(y) \vdash \exists x\phi(x)$. Hence by E$\exists$ we get $\exists y\phi(y) \vdash \exists x\phi(x)$, since y is not a free variable in $\exists x\phi(x)$, because we assumed that it is not a free variable in ϕ. On the other hand, by I$\exists$ we have $\phi(y)[x/y] \vdash \exists y\phi(y)$. Again, we know that $\phi(y)[x/y]$ is the same formula as $\phi(x)$. Therefore by E$\exists$ we have $\exists x\phi(x) \vdash \exists y\phi(y)$, since similarly to the last paragraph, we can see that x is not a free variable in $\exists y\phi(y)$. Hence we get $\exists x\phi(x) \equiv \exists y\phi(y)$, as desired. ■

Finally, let us introduce a helpful shorthand notation for quantification.

Notation. Let ϕ be a formula that has z as a free variable. Then

$\forall z \in x\ \phi(z)$	is a shorthand notation for	$\forall z(z \in x \to \phi(z))$,
$\exists z \in x\ \phi(z)$	is a shorthand notation for	$\exists z(z \in x \land \phi(z))$.

Remark. Let ϕ be a formula that has z as a free variable. If x does not have any element, i.e. if $\vdash \forall z(z \notin x)$, then we have

$$\vdash \forall z \in x\ \phi(z).$$

In other words, if x has no element, then $\phi(z)$ is true for every z in x. In this case we say that $\forall z \in x\ \phi(z)$ is **vacuously true**. To show this note that by E$\forall$ we have $\vdash z \notin x$. Hence by I$\land$ and the cut rule we get $z \in x \vdash (z \in x) \land (z \notin x)$. Therefore by EFQ rule (Theorem 1.1) we obtain

$$z \in x \vdash (z \in x) \land (z \notin x) \vdash \phi(z).$$

Thus by I$\to$, and then by I$\forall$ we get $\vdash \forall z(z \in x \to \phi(z))$, as desired.

The following technical result will be needed in later chapters.

Theorem 1.12. *Suppose ϕ, ψ are formulas, and x, z are variables. Suppose z is not a free variable in ψ, but it is a free variable in ϕ. Then we have*

(i)
$$\big(\forall z \in x\ \phi(z)\big) \lor \psi \equiv \forall z \in x\ \big(\phi(z) \lor \psi\big),$$

(ii)
$$\big(\exists z \in x\ \phi(z)\big) \land \psi \equiv \exists z \in x\ \big(\phi(z) \land \psi\big).$$

If in addition we assume that x is nonempty, i.e. if $\vdash \exists z(z \in x)$, then we have

(iii)
$$\big(\forall z \in x\ \phi(z)\big) \land \psi \equiv \forall z \in x\ \big(\phi(z) \land \psi\big),$$

(iv)
$$\big(\exists z \in x\ \phi(z)\big) \vee \psi \equiv \exists z \in x\ \big(\phi(z) \vee \psi\big).$$

Proof. We will use Theorems 1.4, 1.8 repeatedly in the following parts without explicit citation.

(i) We have

$$\begin{aligned}
\big(\forall z \in x\ \phi(z)\big) \vee \psi &\equiv \forall z(z \in x \to \phi(z)) \vee \psi && \text{(definition)}\\
&\equiv \forall z\big((z \in x \to \phi(z)) \vee \psi\big) && \text{(Theorem 1.10)}\\
&\equiv \forall z\big((\neg(z \in x) \vee \phi(z)) \vee \psi\big) && \text{(law of material implication)}\\
&\equiv \forall z\big(\neg(z \in x) \vee (\phi(z) \vee \psi)\big) && \text{(associativity of } \vee)\\
&\equiv \forall z\big(z \in x \to (\phi(z) \vee \psi)\big) && \text{(law of material implication)}\\
&\equiv \forall z \in x\ \big(\phi(z) \vee \psi\big). && \text{(definition)}
\end{aligned}$$

(ii) We have

$$\begin{aligned}
\big(\exists z \in x\ \phi(z)\big) \wedge \psi &\equiv \exists z(z \in x \wedge \phi(z)) \wedge \psi && \text{(definition)}\\
&\equiv \exists z\big((z \in x \wedge \phi(z)) \wedge \psi\big) && \text{(Theorem 1.10)}\\
&\equiv \exists z\big(z \in x \wedge (\phi(z) \wedge \psi)\big) && \text{(associativity of } \wedge)\\
&\equiv \exists z \in x\ \big(\phi(z) \wedge \psi\big). && \text{(definition)}
\end{aligned}$$

(iii) We have

$$\begin{aligned}
\big(\forall z \in x\ \phi(z)\big) \wedge \psi &\equiv \forall z(z \in x \to \phi(z)) \wedge \psi && \text{(definition)}\\
&\equiv \forall z\big((z \in x \to \phi(z)) \wedge \psi\big) && \text{(Theorem 1.10)}\\
&\equiv \forall z\big((\neg(z \in x) \vee \phi(z)) \wedge \psi\big) && \text{(law of material implication)}\\
&\equiv \forall z\big((z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi)\big). && \text{(distributivity of } \wedge \text{ over } \vee)
\end{aligned}$$

Now by E$\wedge$ and I$\vee$ we have

$$z \notin x \wedge \psi \vdash z \notin x \vdash z \notin x \vee (\phi(z) \wedge \psi).$$

By I$\vee$ we also have $\phi(z) \wedge \psi \vdash z \notin x \vee (\phi(z) \wedge \psi)$. Thus by E$\forall$ and E$\vee$ we get

$$\begin{aligned}
\forall z\big((z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi)\big) &\vdash (z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi)\\
&\vdash z \notin x \vee (\phi(z) \wedge \psi).
\end{aligned}$$

Therefore by I$\forall$ we obtain

$$\forall z\big((z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi)\big) \vdash \forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big).$$

On the other hand, by I$\wedge$ and EFQ rule (Theorem 1.1) we have

$$z \notin x, z \in x \vdash z \notin x \wedge z \in x \vdash \psi.$$

Also, by E$\wedge$ and the weakening rule we have $\phi(z) \wedge \psi, z \in x \vdash \psi$. Hence by E$\vee$ we get $z \notin x \vee (\phi(z) \wedge \psi), z \in x \vdash \psi$. Now by E$\forall$ and the cut rule we obtain

$$\forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big), z \in x \vdash \psi.$$

Thus by E$\exists$ we get $\forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big), \exists z(z \in x) \vdash \psi$, since z is not a free variable in ψ. However, we assumed that $\vdash \exists z(z \in x)$. Therefore by the cut rule we obtain $\forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big) \vdash \psi$. Now by I$\wedge$, the cut rule, and I$\vee$ we have

$$\forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big), z \notin x \vdash z \notin x \wedge \psi \vdash (z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi).$$

Also, by I$\vee$ and the weakening rule we have

$$\forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big), \phi(z) \wedge \psi \vdash (z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi).$$

Hence by E$\vee$ we get

$$\forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big), z \notin x \vee (\phi(z) \wedge \psi) \vdash (z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi).$$

Thus by E$\forall$ and the cut and contraction rules we obtain

$$\forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big) \vdash (z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi).$$

Finally, by I$\forall$ we get

$$\forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big) \vdash \forall z\big((z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi)\big).$$

So we have shown that $\forall z\big((z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi)\big) \equiv \forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big)$. Using this equivalence and the equivalences at the beginning of the proof of part (ii), we obtain

$$\begin{aligned}\big(\forall z \in x\ \phi(z)\big) \wedge \psi &\equiv \forall z\big((z \notin x \wedge \psi) \vee (\phi(z) \wedge \psi)\big) \\ &\equiv \forall z\big(z \notin x \vee (\phi(z) \wedge \psi)\big) \\ &\equiv \forall z\big(z \in x \to (\phi(z) \wedge \psi)\big) && \text{(law of material implication)} \\ &\equiv \forall z \in x\ \big(\phi(z) \wedge \psi\big), && \text{(definition)}\end{aligned}$$

as desired.

(iv) By definition we have $\big(\exists z \in x\ \phi(z)\big) \vee \psi \equiv \exists z(z \in x \wedge \phi(z)) \vee \psi$. Now by I$\vee$ we have $z \in x \vdash z \in x \vee \psi$. Hence by E$\wedge$, the cut rule, and I$\wedge$ we have

$$z \in x \wedge (\phi(z) \vee \psi) \vdash (z \in x \vee \psi) \wedge (\phi(z) \vee \psi).$$

Thus by I$\exists$ we get $z \in x \wedge (\phi(z) \vee \psi) \vdash \exists z\big((z \in x \vee \psi) \wedge (\phi(z) \vee \psi)\big)$. Therefore

$$\begin{aligned}
\exists z\big(z \in x \wedge (\phi(z) \vee \psi)\big) &\vdash \exists z\big((z \in x \vee \psi) \wedge (\phi(z) \vee \psi)\big) && \text{(by E}\exists\text{)}\\
&\vdash \exists z\big((z \in x \wedge \phi(z)) \vee \psi\big) && \text{(distributivity of } \vee \text{ over } \wedge\text{)}\\
&\vdash \exists z(z \in x \wedge \phi(z)) \vee \psi. && \text{(Theorem 1.10)}
\end{aligned}$$

On the other hand, by I$\vee$ we have $\psi \vdash \phi(z) \vee \psi$. Hence by I$\wedge$ and the cut rule we get

$$z \in x, \psi \vdash z \in x \wedge (\phi(z) \vee \psi).$$

Thus by I$\exists$ we obtain $z \in x, \psi \vdash \exists z\big(z \in x \wedge (\phi(z) \vee \psi)\big)$. Therefore by E$\exists$ we have

$$\exists z(z \in x), \psi \vdash \exists z\big(z \in x \wedge (\phi(z) \vee \psi)\big),$$

since z is not a free variable in ψ. However, we assumed that $\vdash \exists z(z \in x)$. Therefore by the cut rule we get

$$\psi \vdash \exists z\big(z \in x \wedge (\phi(z) \vee \psi)\big). \tag{$*$}$$

In addition, by I$\vee$ we have $\phi(z) \vdash \phi(z) \vee \psi$. Hence by E$\wedge$, the cut rule, and I$\wedge$ we have

$$z \in x \wedge \phi(z) \vdash z \in x \wedge (\phi(z) \vee \psi).$$

Thus by I$\exists$ we get $z \in x \wedge \phi(z) \vdash \exists z\big(z \in x \wedge (\phi(z) \vee \psi)\big)$. Therefore by E$\exists$ we obtain

$$\exists z\big(z \in x \wedge \phi(z)\big) \vdash \exists z\big(z \in x \wedge (\phi(z) \vee \psi)\big).$$

So by applying E$\vee$ to the above entailment and $(*)$ we get

$$\exists z\big(z \in x \wedge \phi(z)\big) \vee \psi \vdash \exists z\big(z \in x \wedge (\phi(z) \vee \psi)\big).$$

Hence we have shown that $\exists z\big(z \in x \wedge \phi(z)\big) \vee \psi \equiv \exists z\big(z \in x \wedge (\phi(z) \vee \psi)\big)$, which in shorthand notation can be written as

$$\big(\exists z \in x\ \phi(z)\big) \vee \psi \equiv \exists z \in x\ \big(\phi(z) \vee \psi\big),$$

as desired. ■

1.6 Rules of Inference for Equality

Our last set of inference rules are the rules related to the equality relation.

Axiom 1.10 (Rules of inference for equality)**.** *For every variables like x, y, z we have*

(i) *Reflexivity* :

$$\vdash x = x.$$

(ii) *Symmetry* :
$$x = y \vdash y = x.$$

(iii) *Transitivity* :
$$x = y,\ y = z \vdash x = z.$$

(iv)
$$x = y,\ y \in z \vdash x \in z.$$

(v)
$$x = y,\ z \in y \vdash z \in x.$$

Remark. The above axiom is an axiom schema, so x, y, z can be any variables. Note that in particular, either two of the x, y, z, or all three of them, can be the same variable. Hence for example we also have

$$x = y,\ y \in y \vdash x \in x.$$

Because by part (iv) we have $x = y,\ y \in y \vdash x \in y$. Then by part (v) we have $x = y,\ x \in y \vdash x \in x$. Therefore, by the cut rule we have $x = y,\ y \in y,\ x = y \vdash x \in x$. Finally we get the desired by the exchange and contraction rules.

The first three parts of the above axiom state the elementary properties of equality. The reflexivity of $=$ means that any set is equal to itself. The transitivity of $=$ means that if two sets are equal to a third set, then they are equal. The symmetry of $=$ means that equality is a reciprocal relation, namely, if x equals y then y equals x too. Note that since we can change the variables in the above axiom by any other variables, we also have

$$y = x \vdash x = y.$$

Hence $x = y$ is actually equivalent to $y = x$, i.e. $x = y \equiv y = x$.

Remark. We can also state the above axiom by using quantifiers. For example by I$\forall$ we have $\vdash \forall x(x = x)$. Note that the premises do not contain x as a free variable, so the application of I$\forall$ is justified. Similarly, by E$\wedge$, the cut rule, I$\to$ and I$\forall$ we get

$$\begin{aligned}
&\vdash \forall x \forall y \forall z(x = y \wedge y = z \to x = z),\\
&\vdash \forall x \forall y \forall z(x = y \wedge y \in z \to x \in z),\\
&\vdash \forall x \forall y \forall z(x = y \wedge z \in y \to z \in x).
\end{aligned}$$

Also, by I$\to$ and I$\forall$ we have $\vdash \forall x \forall y(x = y \to y = x)$. In fact, since $x = y$ and $y = x$ are equivalent, we can show that $\vdash \forall x \forall y(x = y \leftrightarrow y = x)$.

The last two parts of the above axiom say that if $x = y$ then we can replace y by x in the atomic formulas $y \in z$ and $z \in y$. Note that we can also interpret the transitivity of equality as above, namely if $x = y$ then we can replace y by x in the atomic formula $y = z$. Furthermore, by using the symmetry of equality, we can also show that

$$x = y,\ z = y \vdash z = x.$$

Because by symmetry of equality we have $z = y \vdash y = z$. Now if we apply the cut rule to this entailment and the entailment given in the transitivity of equality, we obtain $z = y,\ x = y \vdash z = x$. And we get the desired by the exchange rule.

Hence if $x = y$ then we can replace y by x in the atomic formula $z = y$. In addition, similarly to the above remark, we can show that if $x = y$ then we can replace y by x in the atomic formula $y = y$. Therefore if $x = y$ then we can replace y by x in any atomic formula that has y as a free variable. If we combine this fact with Axiom 1.1, and use the rules of inference, we get the following: Suppose ϕ is a formula, and y is a free variable in ϕ. Let x be a variable which is not a bound variable in ϕ. Then we have

$$x = y,\ \phi(y) \vdash \phi(x).$$

In other words, if $x = y$ then we can replace the free occurrences of y by x in any formula that has y as a free variable.

Intuitively, this fact is obvious. Because if $x = y$ then x, y are denoting the same object. In other words, x, y are names for the same object. Thus, every formula which states a property about the object whose name is y, can be rephrased to state that property about the object whose name is x. And these different phrasings are equivalent, since they state the same property about the same object. However, we do not have the tools to prove this fact here; and if we need it, we have to accept it as an axiom. But since in practice we can show that it holds for any particular formula ϕ, and we will not use its general form, we do not state this fact as an axiom. The following example illustrates how we can check this fact for a particular formula.

Remark. There is a more general version of the rule $x = y,\ \phi(y) \vdash \phi(x)$, namely

$$x = y,\ \phi[y/z] \vdash \phi[x/z],$$

provided that x, y are not bound variables in ϕ. This more general version can be used when we need to only substitute some occurrences of y with x, not all of them.

Remark. Note that since $x = y$ and $y = x$ are equivalent, we can replace $x = y$ by $y = x$ in the premises of any entailment, due to Theorem 1.2.

***Example* 1.10.** Let ϕ be the formula $\forall z(z = y \to \neg(z \in y))$. Then we have $x = y, z = y \vdash z = x$. On the other hand, we have $y = x, z \in x \vdash z \in y$. Hence by the weakening rule and E$\neg$ we have

$$x = y, \neg(z \in y), z \in x \vdash \perp.$$

Note that in the premises, we replaced $y = x$ by $x = y$, by using Theorem 1.2. Therefore by I$\neg$ we get

$$x = y, \neg(z \in y) \vdash \neg(z \in x). \tag{$*$}$$

Now note that $x = y, z = x \vdash z = y$. So by the cut rule and E$\to$ we have

$$x = y, z = y \to \neg(z \in y), z = x \vdash \neg(z \in y). \tag{$**$}$$

Thus if we apply the cut rule to $(*), (**)$ we get

$$x = y, z = y \to \neg(z \in y), z = x \vdash \neg(z \in x).$$

Then by I$\to$ we obtain $x = y, z = y \to \neg(z \in y) \vdash z = x \to \neg(z \in x)$. Now by E$\forall$ and the cut rule we get

$$x = y, \forall z(z = y \to \neg(z \in y)) \vdash z = x \to \neg(z \in x).$$

Finally, since z is not free in the premises of the above entailment, we can apply I$\forall$ to obtain

$$x = y, \forall z(z = y \to \neg(z \in y)) \vdash \forall z(z = x \to \neg(z \in x)).$$

Hence we have shown that $x = y$, $\phi(y) \vdash \phi(x)$, as desired.

Remark. Note that a crucial reason that we can carry out the proof in the above example is the symmetry of equality, i.e. the fact that we can replace $x = y$ by $y = x$ whenever we need.

1.7 The Notion of Proof

We have informally used the notion of proof so far. We considered it as a convincing argument toward establishing a theorem, in which we used the axioms, and some elementary inference rules of meta-logic. For example in the proof of Theorem 1.11, when we showed that $\forall x\phi(x) \vdash \forall y\phi(y)$ due to I$\forall$, we had to first check that $\forall x\phi(x) \vdash \phi(y)$, and y is not a bound variable in $\forall x\phi(x)$. Then we could use I$\forall$ and conclude the desired result. But I$\forall$ is a conditional statement in meta-language. So we concluded the consequent of the conditional statement I$\forall$ by checking that its antecedent holds. Thus we have actually used the modus ponens rule in meta-logic.

This is also the case when we have used the other rules which are expressed as conditional statements in meta-language, like I$\rightarrow$ or E$\vee$.

Also note that the antecedent of the rule I$\forall$ consists of two parts. Thus in the above example we had to both check that $\forall x \phi(x) \vdash \phi(y)$, and that y is not a bound variable in $\forall x \phi(x)$. Of course we did this by separately checking each statement. Therefore when we concluded that the antecedent of I$\forall$ holds, we were actually using the rule of introduction of "and" in meta-logic. We can continue this exploration and collect all types of meta-logical arguments we presented in this chapter, but that will take us far from our main subject. So we do not pursue this any further. However, let us recall that as we discussed in the introduction to this chapter, we can accept all the results of this chapter as axioms; and we can consider those reasonings in meta-logic as mere convincing rationale for our choice of axioms.

The proofs mentioned in the above paragraphs are proofs in meta-logic. Let us now turn to the notion of proof inside logic. Conceptually, the proofs inside logic and meta-logic are not different. But we can make the notion of proof precise, when we work in logic.

Primitive Notion 1.6. A **proof** is a primitive notion, which intuitively, is a finite sequence of entailments. We represent a proof by writing the representations of its entailments successively, and we separate the entailments by " ; ". We assume that we are able to recognize proofs from their representations, and we can distinguish between them through their representations. We also assume that we can recognize the entailments in a proof from the representation of the proof.

Remark. A proof is also called a **derivation** or a **formal proof**.

Axiom 1.11. *Suppose $\Gamma, \Delta, \Lambda_0, \Lambda_1, \Lambda_2$ denote collections of several formulas, which can be empty too. Let ϕ, ψ, τ be formulas. Then we have*

(i) *Let $\Gamma \vdash \psi$ be an entailment. Then $\Gamma \vdash \psi$ is also a proof.*
(ii) *If $\mathscr{D}$ is a proof, and $\Gamma \vdash \psi$ is an entailment, then $\mathscr{D}; \Gamma \vdash \psi$ is also a proof.*
(iii) *If $\mathscr{D}$ is a proof that contains the entailment $\Gamma \vdash \psi$, then $\mathscr{D}; \Gamma, \Delta \vdash \psi$ is also a proof.*
(iv) *If $\mathscr{D}$ is a proof that contains the entailment $\Gamma, \Delta, \Delta \vdash \psi$, then $\mathscr{D}; \Gamma, \Delta \vdash \psi$ is also a proof.*
(v) *If $\mathscr{D}$ is a proof that contains the entailment $\Lambda_0, \Gamma, \Lambda_1, \Delta, \Lambda_2 \vdash \psi$, then $\mathscr{D}; \Lambda_0, \Delta, \Lambda_1, \Gamma, \Lambda_2 \vdash \psi$ is also a proof.*
(vi) *If $\mathscr{D}$ is a proof that contains the entailments $\Gamma \vdash \phi$ and $\Delta, \phi \vdash \psi$, then $\mathscr{D}; \Gamma, \Delta \vdash \psi$ is also a proof.*
(vii) *If $\mathscr{D}$ is a proof that contains the entailment $\Gamma, \phi \vdash \psi$, then $\mathscr{D}; \Gamma \vdash \phi \rightarrow \psi$ is also a proof.*

(viii) *If $\mathscr{D}$ is a proof that contains the entailments $\Gamma, \phi \vdash \tau$ and $\Gamma, \psi \vdash \tau$, then $\mathscr{D}; \Gamma, \phi \vee \psi \vdash \tau$ is also a proof.*

(ix) *If $\mathscr{D}$ is a proof that contains the entailments $\Gamma, \phi \vdash \psi$ and $\Gamma, \psi \vdash \phi$, then $\mathscr{D}; \Gamma \vdash \phi \leftrightarrow \psi$ is also a proof.*

(x) *If $\mathscr{D}$ is a proof that contains the entailment $\Gamma, \phi \vdash \bot$, then $\mathscr{D}; \Gamma \vdash \neg\phi$ is also a proof.*

(xi) *If $\mathscr{D}$ is a proof that contains the entailment $\Gamma, \neg\phi \vdash \bot$, then $\mathscr{D}; \Gamma \vdash \phi$ is also a proof.*

(xii) *If $\mathscr{D}$ is a proof that contains the entailment $\Gamma \vdash \phi$, and x is not a free variable in any of the formulas in Γ, then $\mathscr{D}; \Gamma \vdash \forall x \phi$ is also a proof.*

(xiii) *If $\mathscr{D}$ is a proof that contains the entailment $\Gamma, \phi \vdash \psi$, and x is not a free variable in ψ, nor in any of the formulas in Γ, then $\mathscr{D}; \Gamma, \exists x \phi \vdash \psi$ is also a proof.*

Remark. We assume that every proof is constructed after several applications of the above rules, and there is no other way to construct a proof.

As it is evident from the above axiom, the proofs are constructed by using the rules of inference. Each part of the axiom, except the first two, corresponds to a rule of inference. Note that the entailment added in each part to the proof $\mathscr{D}$ is a valid entailment, due to the corresponding rule of inference. The first part of the axiom allows us to start a proof. The second part allows us to add a separately proven entailment to a proof. These two parts also incorporate the rules of inference which are not explicitly stated in the axiom, like I$\exists$ or E$\rightarrow$; because those rules only assert that certain entailments hold. More importantly, the first two parts allow us to use the axioms of equality, or the axioms of set theory (which will be added later) inside a proof.

Remark. When $\Gamma \vdash \psi$ is the last entailment that appears in a proof $\mathscr{D}$, we say that $\mathscr{D}$ is a proof of the entailment $\Gamma \vdash \psi$.

The virtue of a proof of an entailment, is that it encapsulates all the data needed to ensure that the entailment holds. However, it is not easy to understand a formal proof if there are no explanations around it expressed in the meta-language. So, mathematicians usually provide the proofs in the informal style expressed in the meta-language, similar to the proofs that we presented in previous sections. We will also adhere to this convention.

Remark. There is another approach to proofs which considers them as finite sequences of formulas. In this approach, we extend a proof by adding formulas that can be deduced from the previous formulas in the proof. So instead of recording the entailments in an argument, we record the formulas used in those entailments. Thus the two approaches to proofs are essentially the same. However, considering proofs as sequences of entailments seems more natural, and is easier to deal with axiomatically. Hence we chose this approach here.

***Example* 1.11.** Let ϕ, ψ be two formulas. The following is a formal proof of the entailment $\phi \to \psi, \neg\psi \vdash \neg\phi$, which is an instance of modus tollens.

$$\phi \to \psi, \phi \vdash \psi;\ \psi, \neg\psi \vdash \bot;\ \neg\psi, \psi \vdash \bot;\ \phi \to \psi, \phi, \neg\psi \vdash \bot;$$
$$\phi \to \psi, \neg\psi, \phi \vdash \bot;\ \phi \to \psi, \neg\psi \vdash \neg\phi.$$

As you can see, it is rather hard to comprehend a formal proof. To overcome this, we will usually write each entailment of a formal proof in a separate line, and we will mark the inference rule which implies that entailment. We will also number each line. With these conventions, the above formal proof can be rewritten as follows:

1	$\phi \to \psi, \phi \vdash \psi;$	(by E$\to$)
2	$\psi, \neg\psi \vdash \bot;$	(by E$\neg$)
3	$\neg\psi, \psi \vdash \bot;$	(by exchange rule)
4	$\phi \to \psi, \phi, \neg\psi \vdash \bot;$	(by cut rule applied to lines 1,3)
5	$\phi \to \psi, \neg\psi, \phi \vdash \bot;$	(by exchange rule)
6	$\phi \to \psi, \neg\psi \vdash \neg\phi.$	(by I$\neg$)

Remark. Note that the above proof is only a proof of the entailment $\phi \to \psi, \neg\psi \vdash \neg\phi$ for the particular instance of the formulas ϕ, ψ which we started with. And although the same proof works if we replace ϕ, ψ with any other particular pair of formulas, the above proof does not imply that the entailment $\phi \to \psi, \neg\psi \vdash \neg\phi$ holds for every pair of formulas ϕ, ψ. Because the logic that we have constructed only applies to the language of set theory, and that language can only talk about sets. So, statements about arbitrary pairs of formulas in the language of set theory lie outside of the language of set theory itself! Thus the logic that we have constructed cannot deal with such statements, and if we want to prove them we have to enter the realm of meta-logic.

As a side note, let us mention that we can repeat the above proof with some particular formulas ϕ, ψ, whenever we need to use the rule modus tollens. Therefore we do not need to accept modus tollens as a general rule of inference, if we prefer to avoid the theorems proved by meta-logical reasonings. The same remark applies to the other theorems proved in this chapter.

***Example* 1.12.** Let ϕ, ψ be two formulas. The following is a formal proof of the entailment $\vdash \exists x(\phi \vee \psi) \leftrightarrow \exists x\phi \vee \exists x\psi$. Informally, we can also say that the following is a formal proof of the equivalence $\exists x(\phi \vee \psi) \equiv \exists x\phi \vee \exists x\psi$.

1	$\phi \vdash \exists x\phi$;	(by I$\exists$)
2	$\exists x\phi \vdash \exists x\phi \vee \exists x\psi$;	(by I$\vee$)
3	$\phi \vdash \exists x\phi \vee \exists x\psi$;	(by cut rule applied to lines 1,2)
4	$\psi \vdash \exists x\psi$;	(by I$\exists$)
5	$\exists x\psi \vdash \exists x\phi \vee \exists x\psi$;	(by I$\vee$)
6	$\psi \vdash \exists x\phi \vee \exists x\psi$;	(by cut rule applied to lines 4,5)
7	$\phi \vee \psi \vdash \exists x\phi \vee \exists x\psi$;	(by E$\vee$ applied to lines 3,6)
8	$\exists x(\phi \vee \psi) \vdash \exists x\phi \vee \exists x\psi$;	(by E$\exists$)
9	$\phi \vdash \phi \vee \psi$;	(by I$\vee$)
10	$\phi \vee \psi \vdash \exists x(\phi \vee \psi)$;	(by I$\exists$)
11	$\phi \vdash \exists x(\phi \vee \psi)$;	(by cut rule applied to lines 9,10)
12	$\exists x\phi \vdash \exists x(\phi \vee \psi)$;	(by E$\exists$)
13	$\psi \vdash \phi \vee \psi$;	(by I$\vee$)
14	$\psi \vdash \exists x(\phi \vee \psi)$;	(by cut rule applied to lines 13,10)
15	$\exists x\psi \vdash \exists x(\phi \vee \psi)$;	(by E$\exists$)
16	$\exists x\phi \vee \exists x\psi \vdash \exists x(\phi \vee \psi)$;	(by E$\vee$ applied to lines 12,15)
17	$\vdash \exists x(\phi \vee \psi) \leftrightarrow \exists x\phi \vee \exists x\psi$.	(by I$\leftrightarrow$ applied to lines 8,16)