Grassmann

费米子相干态

单模相干态：

$$|z_i\rangle \equiv \exp\left(-z_i a_i^\dagger\right)|0\rangle \tag{1}$$

$$\langle z_i| \equiv \langle 0|\exp\left(-a_i \bar{z}_i\right) \tag{2}$$

多模相干态：

$$|z\rangle \equiv \exp\left(-\sum_i z_i a_i^\dagger\right)|0\rangle \tag{3}$$

$$\langle z| \equiv \langle 0|\exp\left(-\sum_i a_i \bar{z}_i\right) \tag{4}$$

相干态是湮灭算符本征态：

$$a_i|z\rangle = z_i|z\rangle \tag{5}$$

$$\langle z|a_i^\dagger = \langle z|\bar{z}_i \tag{6}$$

相干态与真空态overlap

$$\langle 0 \mid z\rangle = 1 \tag{7}$$

$$\langle z \mid 0\rangle = 1 \tag{8}$$

费米子相干态表象完备性关系

$$\int\left(\prod_i \mathrm{d}\bar{z}_i \mathrm{d}z_i\right)\exp\left(-\sum_j \bar{z}_j z_j\right)|z\rangle\langle z| = 1 \tag{9}$$

Grassmann高斯积分

反对称矩阵 $A_{2N\times 2N}, \Theta = (\theta_1, \cdots, \theta_{2N})^\top$ 实Grassmann高斯积分：

$$\int\left(\prod_{i=1}^{2N}\mathrm{d}\theta_i\right)\exp\left(-\frac{1}{2}\Theta^\top A\Theta\right) = \mathrm{Pf}(A) \tag{10}$$

$$\int\left(\prod_{i=1}^{2N}\mathrm{d}\theta_i\right)\exp\left(\frac{1}{2}\Theta^\top A\Theta\right) = (-1)^N\mathrm{Pf}(A) \tag{11}$$

$A_{N\times N}, \Theta = (\theta_1, \cdots, \theta_N)^\top, \bar{\Theta} = \left(\bar{\theta}_1, \cdots, \bar{\theta}_N\right)^\top$ 复Grassmann高斯积分：

$$\int\left(\prod_i \mathrm{d}\bar{\theta}_i \mathrm{d}\theta_i\right) \exp\left(-\bar{\Theta}^\top A \Theta\right) = \det(A) \tag{12}$$

在费米子相干态表象计算overlap

$$\left|\tilde{\Omega}_1\right\rangle = \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(1)} a_i^\dagger a_j^\dagger\right)|0\rangle, \quad f_{i,j}^{(1)} = -f_{j,i}^{(1)} \tag{13}$$

$$\left|\tilde{\Omega}_2\right\rangle = \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(2)} a_i^\dagger a_j^\dagger\right)|0\rangle, \quad f_{i,j}^{(2)} = -f_{j,i}^{(2)} \tag{14}$$

$$\begin{aligned}
\left\langle\widetilde{\Omega}_1 \middle| \widetilde{\Omega}_2\right\rangle &= \int\left(\prod_i \mathrm{d}\bar{z}_i \mathrm{d}z_i\right) \exp\left(-\sum_j \bar{z}_j z_j\right)\left\langle\widetilde{\Omega}_1 \middle| z\right\rangle\left\langle z \middle| \widetilde{\Omega}_2\right\rangle \\
&= \int \mathrm{d}(\bar{z}, z) \exp\left(-\bar{z}^\top z\right)\left\langle 0 \middle| \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(1)*} a_j a_i \middle| z\right)\right\rangle\left\langle z \middle| \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(2)} a_i^\dagger a_j^\dagger\right) \middle| 0\right\rangle \\
&= \int \mathrm{d}(\bar{z}, z) \exp\left(-\bar{z}^\top z\right) \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(1)*} z_j z_i\right) \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(2)} \bar{z}_i \bar{z}_j\right)\langle 0 | z\rangle\langle z | 0\rangle \\
&= \int \mathrm{d}(\bar{z}, z) \exp\left(-\bar{z}^\top z\right) \exp\left(-\frac{1}{2}\sum_{i,j} f_{i,j}^{(1)*} z_i z_j\right) \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(2)} \bar{z}_i \bar{z}_j\right) \\
&= \int \mathrm{d}(\bar{z}, z) \exp\left(-\bar{z}^\top z\right) \exp\left(-\frac{1}{2} z^\top f^{(1)*} z\right) \exp\left(\frac{1}{2} \bar{z}^\top f^{(2)} \bar{z}\right)
\end{aligned} \tag{15}$$

为了用实Grassmann高斯积分的结果，构造

$$Z \equiv \left(\bar{z}_1, \cdots, \bar{z}_N, z_1, \cdots, z_N\right)^\top \tag{16}$$

从

$$\bar{z}_1, z_1, \bar{z}_2, z_2, \cdots, \bar{z}_N, z_N \tag{17}$$

到

$$\bar{z}_1, \cdots, \bar{z}_N, z_1, \cdots, z_N \tag{18}$$

共需要多少次最近邻交换?

以

$$\bar{z}_1 \bar{z}_2 \cdots \bar{z}_N z_1 z_2 \cdots z_N \tag{19}$$

为正序，则逆序对数量为

$$(N-1) + (N-2) + \cdots + 1 = \frac{N(N-1)}{2} \tag{20}$$

因此

$$
\begin{aligned}
\mathrm{d}(\bar{z},z) \equiv \prod_i \mathrm{d}\bar{z}_i \mathrm{d}z_i &= (\mathrm{d}\bar{z}_1 \mathrm{d}z_1)\cdots(\mathrm{d}\bar{z}_N \mathrm{d}z_N) \\
&= (-1)^{N(N-1)/2}\, \mathrm{d}\bar{z}_1 \cdots \mathrm{d}\bar{z}_N \mathrm{d}z_1 \mathrm{d}z_N \\
&= (-1)^{N(N-1)/2} \left(\prod_i \mathrm{d}\bar{z}_i\right)\left(\prod_j \mathrm{d}z_j\right) \\
&= (-1)^{N(N-1)/2} \prod_{i=1}^{2N} \mathrm{d}Z_i
\end{aligned}
\tag{21}
$$

而

$$
\begin{aligned}
&\exp\left(-\bar{z}^\top z\right)\exp\left(-\frac{1}{2}z^\top f^{(1)*}z\right)\exp\left(\frac{1}{2}\bar{z}^\top f^{(2)}\bar{z}\right) \\
=&\exp\left(-\bar{z}^\top z - \frac{1}{2}z^\top f^{(1)*}z + \frac{1}{2}\bar{z}^\top f^{(2)}\bar{z}\right) \\
=&\exp\left(-\frac{1}{2}\bar{z}^\top z - \frac{1}{2}\bar{z}^\top z - \frac{1}{2}z^\top f^{(1)*}z + \frac{1}{2}\bar{z}^\top f^{(2)}\bar{z}\right) \\
=&\exp\left(-\frac{1}{2}\bar{z}^\top z + \frac{1}{2}z^\top \bar{z} - \frac{1}{2}z^\top f^{(1)*}z + \frac{1}{2}\bar{z}^\top f^{(2)}\bar{z}\right)
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
-\frac{1}{2}\bar{z}^\top z + \frac{1}{2}z^\top \bar{z} - \frac{1}{2}z^\top f^{(1)*}z + \frac{1}{2}\bar{z}^\top f^{(2)}\bar{z} &= \frac{1}{2}\begin{pmatrix}\bar{z}^\top & z^\top\end{pmatrix}\begin{pmatrix} f^{(2)} & -I \\ I & -f^{(1)*}\end{pmatrix}\begin{pmatrix}\bar{z} \\ z\end{pmatrix} \\
&= \frac{1}{2}Z^\top M Z
\end{aligned}
\tag{23}
$$

$$
M_{i,j} = -M_{j,i} \tag{24}
$$

于是

$$
\begin{aligned}
\left\langle \widetilde{\Omega}_1 \middle| \widetilde{\Omega}_2 \right\rangle &= \int \mathrm{d}\left(\bar{z},z\right)\exp\left(-\bar{z}^\top z\right)\exp\left(-\frac{1}{2}z^\top f^{(1)*}z\right)\exp\left(\frac{1}{2}\bar{z}^\top f^{(2)}\bar{z}\right) \\
&= (-1)^{N(N-1)/2}\int\prod_{i=1}^{2N}\mathrm{d}Z_i \exp\left(\frac{1}{2}Z^\top M Z\right) \\
&= (-1)^{N(N-1)/2}\int\prod_{i=1}^{2N}\mathrm{d}Z_i \exp\left(-\frac{1}{2}Z^\top\left(-M\right)Z\right) \\
&= (-1)^{N(N-1)/2}\,\mathrm{Pf}(-M) \\
&= (-1)^{N(N-1)/2}\cdot(-1)^N \mathrm{Pf}(M) \\
&= (-1)^{N(N+1)/2}\,\mathrm{Pf}(M)
\end{aligned}
\tag{25}
$$