1

计算牛顿近似中的所有非零联络 $\Gamma^{\lambda}_{\mu\nu}$.

牛顿近似线元:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2\left(1-h_{00}\right)+\left(\mathrm{d}x^1\right)^2+\left(\mathrm{d}x^2\right)^2+\left(\mathrm{d}x^3\right)^2 \tag{1}$$

度规:

$$g_{00}=-\left(1-h_{00}\right), \quad g_{0i}=g^{0i}=0, \quad g_{ij}=\delta_{ij} \tag{2}$$

利用小量近似 $1/(1-x)\approx 1+x, |x|\ll 1$ 有逆度规:

$$g^{00}=-(1+h_{00}), \quad g^{0i}=g^{i0}=0, \quad g^{ij}=\delta^{ij} \tag{3}$$

知道了度规和逆度规，就可以计算黎曼联络:

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu}\right) \tag{4}$$

- $\lambda=i$

$$\begin{aligned}\Gamma^{i}_{\mu\nu}&=\frac{1}{2}g^{i\sigma}\left(\partial_\mu g_{\sigma\nu}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu}\right)\\&=\frac{1}{2}g^{ij}\left(\partial_\mu g_{j\nu}+\partial_\nu g_{j\mu}-\partial_j g_{\mu\nu}\right)\\&=\frac{1}{2}\delta^{ij}\left(\partial_\mu g_{j\nu}+\partial_\nu g_{j\mu}-\partial_j g_{\mu\nu}\right)\end{aligned} \tag{5}$$

- ◦ $(\mu,\nu)=(l,m)$

$$\begin{aligned}\Gamma^{i}_{lm}&=\frac{1}{2}\delta^{ij}\left(\partial_l g_{jm}+\partial_m g_{jl}-\partial_j g_{lm}\right)\\&=0\end{aligned} \tag{6}$$

- ◦ $(\mu,\nu)=(0,m)$

$$\begin{aligned}\Gamma^{i}_{0m}&=\frac{1}{2}\delta^{ij}\left(\partial_0 g_{jm}+\partial_m g_{j0}-\partial_j g_{0m}\right)\\&=0\end{aligned} \tag{7}$$

- ◦ $(\mu,\nu)=(l,0)$

$$\Gamma^{i}_{l0}=0 \tag{8}$$

- ◦ $(\mu,\nu)=(0,0)$

$$\begin{aligned}\Gamma^{i}_{00}&=\frac{1}{2}\delta^{ij}\left(\partial_0 g_{j0}+\partial_0 g_{j0}-\partial_j g_{00}\right)\\&=-\frac{1}{2}\delta^{ij}\partial_j\left[-(1-h_{00})\right]\\&=-\frac{1}{2}\delta^{ij}\partial_j h_{00}\\&=-\frac{1}{2}\partial_i h_{00}\end{aligned} \tag{9}$$

- $\lambda=0$

$$\begin{aligned}\Gamma^{0}_{\mu\nu}&=\frac{1}{2}g^{0\sigma}\left(\partial_\mu g_{\sigma\nu}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu}\right)\\&=\frac{1}{2}g^{00}\left(\partial_\mu g_{0\nu}+\partial_\nu g_{0\mu}-\partial_0 g_{\mu\nu}\right)\end{aligned} \tag{10}$$

- ◦ $(\mu,\nu)=(l,m)$

$$\begin{aligned}\Gamma^0_{lm}&=\frac{1}{2}g^{00}\left(\partial_l g_{0m}+\partial_m g_{0l}-\partial_0 g_{lm}\right)\\&=0\end{aligned}\tag{11}$$

- ◦ $(\mu,\nu)=(0,m)$

$$\begin{aligned}\Gamma^0_{0m}&=\frac{1}{2}g^{00}\left(\partial_0 g_{0m}+\partial_m g_{00}-\partial_0 g_{0m}\right)\\&=\frac{1}{2}g^{00}\partial_m g_{00}\\&=\frac{1}{2}\left[-(1+h_{00})\right]\partial_m\left[-(1-h_{00})\right]\\&\approx-\frac{1}{2}\partial_m h_{00}\end{aligned}\tag{12}$$

> 舍去二阶小量

- ◦ $(\mu,\nu)=(l,0)$

$$\Gamma^0_{l0}=-\frac{1}{2}\partial_l h_{00}\tag{13}$$

- ◦ $(\mu,\nu)=(0,0)$

$$\begin{aligned}\Gamma^0_{00}&=\frac{1}{2}g^{00}\left(\partial_0 g_{00}+\partial_0 g_{00}-\partial_0 g_{00}\right)\\&=0\end{aligned}\tag{14}$$

2

写出逆变、协变与二阶张量的坐标变换式，已知 $A^\mu_\nu=\partial\bar{x}^\mu/\partial x^\nu,\bar{A}^\mu_\nu=\partial x^\mu/\partial\bar{x}^\nu$ 且 $\det\left[A^\mu_\nu\right]\neq 0$

$$\phi'^\mu(x')=A^\mu_\alpha\phi^\alpha(x)\tag{15}$$

$$\phi'_\mu(x')=\bar{A}^\alpha_\mu\phi_\alpha(x)\tag{16}$$

$$\phi'^{\mu\nu}(x')=A^\mu_\alpha A^\nu_\beta\phi^{\alpha\beta}(x)\tag{17}$$

$$\phi'^\mu_\nu(x')=A^\mu_\alpha\bar{A}^\beta_\nu\phi^\alpha_\beta(x)\tag{18}$$

$$\phi'_{\mu\nu}(x')=\bar{A}^\alpha_\mu\bar{A}^\beta_\nu\phi_{\alpha\beta}(x)\tag{19}$$

$$\nabla_\mu\phi^\nu=\partial_\mu\phi^\nu+\Gamma^\nu_{\mu\lambda}\phi^\lambda\tag{20}$$

$$\nabla_\mu\phi_\nu=\partial_\mu\phi_\nu-\Gamma^\lambda_{\mu\nu}\phi_\lambda\tag{21}$$

$$\nabla_\mu\phi^{\nu\lambda}=\partial_\mu\phi^{\nu\lambda}+\Gamma^\nu_{\mu\rho}\phi^{\rho\lambda}+\Gamma^\lambda_{\mu\rho}\phi^{\nu\rho}\tag{22}$$

$$\nabla_\mu\phi^\nu_\lambda=\partial_\mu\phi^\nu_\lambda+\Gamma^\nu_{\mu\rho}\phi^\rho_\lambda-\Gamma^\rho_{\mu\lambda}\phi^\nu_\rho\tag{23}$$

$$\nabla_\mu\phi_{\nu\lambda}=\partial_\mu\phi_{\nu\lambda}-\Gamma^\rho_{\mu\nu}\phi_{\rho\lambda}-\Gamma^\rho_{\mu\lambda}\phi_{\nu\rho}\tag{24}$$

3

由比安基恒等式证明 $\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)_{;\nu}=0$

比安基恒等式：

$$\nabla_\lambda R^\rho{}_{\sigma\mu\nu}+\nabla_\mu R^\rho{}_{\sigma\nu\lambda}+\nabla_\nu R^\rho{}_{\sigma\lambda\mu}=0\tag{25}$$

令 $\rho=\mu$ 得到

$$\nabla_\lambda R^\mu{}_{\sigma\mu\nu} + \nabla_\mu R^\mu{}_{\sigma\nu\lambda} + \nabla_\nu R^\mu{}_{\sigma\lambda\mu} = 0 \tag{26}$$

由 $R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu}$ 可知第一项化为

$$\nabla_\lambda R^\mu{}_{\sigma\mu\nu} = \nabla_\lambda R_{\sigma\nu} \tag{27}$$

由 $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$ 知第三项化为

$$\nabla_\nu R^\mu{}_{\sigma\lambda\mu} = -\nabla_\nu R^\mu{}_{\sigma\mu\lambda} = -\nabla_\nu R_{\sigma\lambda} \tag{28}$$

于是有

$$\nabla_\nu R_{\sigma\lambda} = \nabla_\lambda R_{\sigma\nu} + \nabla_\mu R^\mu{}_{\sigma\nu\lambda} \tag{29}$$

两边同乘 $g^{\sigma\lambda}$，利用 $\nabla_\nu g^{\sigma\lambda} = 0$ 、协变微分的莱布尼兹律以及 $R \equiv g^{\mu\nu} R_{\mu\nu}, R^\mu{}_\nu \equiv g^{\mu\rho} R_{\rho\nu}$ 有

$$\nabla_\nu R = \nabla_\lambda R^\lambda{}_\nu + \nabla_\mu \left(g^{\sigma\lambda} R^\mu{}_{\sigma\nu\lambda}\right) \tag{30}$$

上式右边第二项，利用 $R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho}, R^\lambda{}_{\sigma\mu\nu} = -R^\lambda{}_{\sigma\nu\mu}$ 有

$$\begin{aligned} g^{\sigma\lambda} R^\mu{}_{\sigma\nu\lambda} &= g^{\mu\rho} g^{\sigma\lambda} R_{\rho\sigma\nu\lambda} = g^{\mu\rho} g^{\sigma\lambda} R_{\nu\lambda\rho\sigma} \\ &= -g^{\mu\rho} g^{\sigma\lambda} R_{\lambda\nu\rho\sigma} = -g^{\mu\rho} R^\sigma{}_{\nu\rho\sigma} = g^{\mu\rho} R^\sigma{}_{\nu\sigma\rho} \\ &= g^{\mu\rho} R_{\nu\rho} = R^\mu{}_\nu \end{aligned} \tag{31}$$

于是得到

$$\nabla_\nu R = \nabla_\lambda R^\lambda{}_\nu + \nabla_\mu R^\mu{}_\nu \tag{32}$$

也即

$$\nabla_\nu R = 2\nabla_\mu R^\mu{}_\nu, \quad \nabla_\mu R^\mu{}_\nu = \frac{1}{2}\nabla_\nu R = \frac{1}{2}\delta^\mu{}_\nu \nabla_\mu R \tag{33}$$

即

$$\nabla_\mu \left(R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R\right) = 0 \tag{34}$$

两边同乘 $g^{\rho\nu}$ 得

$$\nabla_\mu \left(R^{\mu\rho} - \frac{1}{2} g^{\rho\mu} R\right) = 0 \tag{35}$$

又 $R^{\mu\nu} = R^{\nu\mu}$ 于是

$$\nabla_\mu \left(R^{\rho\mu} - \frac{1}{2} g^{\rho\mu} R\right) = 0 \tag{36}$$

等价于

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\nu} = 0 \tag{37}$$

4

写出爱因斯坦作用量，并变分得到场方程。已知Palatini公式

$$\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \partial_\mu \left(\sqrt{-g} \phi^\mu\right) \tag{38}$$

且

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu} \tag{39}$$

总作用量

$$I=I_g+I_m \tag{40}$$

其中 I_g 是引力场作用量，I_m 是引力源物质作用量。

$$I_g=\int\limits_M L_g\sqrt{-g}\mathrm{d}^4x=\frac{c^3}{16\pi G}\int\limits_M R\sqrt{-g}\mathrm{d}^4x,\quad L_g=\frac{c^3}{16\pi G}R \tag{41}$$

想算 I_g 对度规的变分，要先算 $R\sqrt{-g}$ 对度规的变分。

$$\begin{aligned}\delta(R\sqrt{-g})&=\delta\left(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\right)\\&=\left(\delta g^{\mu\nu}\right)R_{\mu\nu}\sqrt{-g}+g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\sqrt{-g}+g^{\mu\nu}R_{\mu\nu}\delta\left(\sqrt{-g}\right)\end{aligned} \tag{42}$$

利用Palatini.II公式

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}=\partial_\mu\left(\sqrt{-g}\phi^\mu\right),\quad \phi^\mu\equiv g^{\lambda\nu}\delta\Gamma^\mu_{\lambda\nu}-g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\nu} \tag{43}$$

以及

$$\delta g=gg^{\mu\nu}\delta g_{\mu\nu}=-gg_{\mu\nu}\delta g^{\mu\nu} \tag{44}$$

$$\delta\left(\sqrt{-g}\right)=-\frac{1}{2}\frac{\delta g}{\sqrt{-g}}=-\frac{1}{2}\frac{-gg_{\mu\nu}\delta g^{\mu\nu}}{\sqrt{-g}}=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{45}$$

于是有

$$\begin{aligned}\delta(R\sqrt{-g})&=\left(\delta g^{\mu\nu}\right)R_{\mu\nu}\sqrt{-g}+g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\sqrt{-g}+g^{\mu\nu}R_{\mu\nu}\delta\left(\sqrt{-g}\right)\\&=R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}+\partial_\mu\left(\sqrt{-g}\phi^\mu\right)-\frac{1}{2}\sqrt{-g}Rg_{\mu\nu}\delta g^{\mu\nu}\end{aligned} \tag{46}$$

因此引力场作用量对度规的变分为

$$\begin{aligned}\delta I_g&=\frac{c^3}{16\pi G}\int\limits_M\delta\left(R\sqrt{-g}\right)\mathrm{d}^4x\\&=\frac{c^3}{16\pi G}\int\limits_M\left(R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}+\partial_\mu\left(\sqrt{-g}\phi^\mu\right)-\frac{1}{2}\sqrt{-g}Rg_{\mu\nu}\delta g^{\mu\nu}\right)\mathrm{d}^4x\\&=\frac{c^3}{16\pi G}\int\limits_M\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x\end{aligned} \tag{47}$$

接下来还需要引力源物质作用量对度规的变分。

$$I_m=\frac{1}{c}\int\limits_M L_m\sqrt{-g}\mathrm{d}^4x \tag{48}$$

假定拉式密度 L_m 只含 $g^{\mu\nu}$，而不含 $\partial_\lambda g^{\mu\nu}$，则

$$\begin{aligned}\delta I_m&=\frac{1}{c}\int\limits_M\delta\left(L_m\sqrt{-g}\right)\mathrm{d}^4x\\&=\frac{1}{c}\int\limits_M\frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\mathrm{d}^4x\end{aligned} \tag{49}$$

定义

$$T_{\mu\nu}\equiv-\frac{2}{\sqrt{-g}}\frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}},\quad \frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}=-\frac{1}{2}\sqrt{-g}T_{\mu\nu} \tag{50}$$

则

$$
\begin{aligned}
\delta I_m &= \frac{1}{c}\int\limits_M \frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\mathrm{d}^4x \\
&= -\frac{1}{2c}\int\limits_M T_{\mu\nu}\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x
\end{aligned}
\tag{51}
$$

最小作用量原理说，$\delta I = 0$ 给出体系的运动方程，也即

$$
\begin{aligned}
0 = \delta I = \delta I_g + \delta I_m &= \frac{c^3}{16\pi G}\int\limits_M \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x - \frac{1}{2c}\int\limits_M T_{\mu\nu}\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x \\
&= \frac{c^3}{16\pi G}\int\limits_M \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{8\pi G}{c^4}T_{\mu\nu}\right)\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x
\end{aligned}
\tag{52}
$$

最终得到爱因斯坦引力场方程：

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{53}
$$

5

求弱场线性近似的里奇张量，并导出线性化的场方程。

由于

$$
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \tag{54}
$$

线性近似理论只保留小量 $h_{\mu\nu}$ 的一阶项，于是

$$
\begin{aligned}
\Gamma^\mu_{\alpha\beta} &\equiv \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) \\
&\equiv \frac{1}{2}\left(\eta^{\mu\nu} + h^{\mu\nu}\right)\left[\partial_\alpha\left(\eta_{\beta\nu} + h_{\beta\nu}\right) + \partial_\beta\left(\eta_{\alpha\nu+h_{\alpha\nu}}\right) - \partial_\nu\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right)\right] \\
&= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\beta h_{\alpha\nu} + \partial_\alpha h_{\beta\nu} - \partial_\nu h_{\alpha\beta}\right) \\
&= \frac{1}{2}\left[\partial_\beta\left(\eta^{\mu\nu}h_{\alpha\nu}\right) + \partial_\alpha\left(\eta^{\mu\nu}h_{\beta\nu}\right) - \eta^{\mu\nu}\partial_\nu\left(h_{\alpha\beta}\right)\right] \\
&= \frac{1}{2}\left(\partial_\beta h^\mu_\alpha + \partial_\alpha h^\mu_\beta - \partial^\mu h_{\alpha\beta}\right) \\
&= \frac{1}{2}\left(h^\mu_{\alpha,\beta} + h^\mu_{\beta,\alpha} - h^{;\mu}_{\alpha\beta}\right)
\end{aligned}
\tag{55}
$$

克氏符 Γ 是 $h_{\mu\nu}$ 一阶小量的叠加，因此 Riemann 曲率张量中的 $\Gamma\Gamma$ 项可以舍去，此时 Riemann 曲率张量为：

$$
\begin{aligned}
R^\lambda_{\mu\alpha\nu} &\equiv \partial_\alpha\Gamma^\lambda_{\nu\mu} - \partial_\nu\Gamma^\lambda_{\alpha\mu} + \Gamma^\lambda_{\alpha\beta}\Gamma^\beta_{\nu\mu} - \Gamma^\lambda_{\nu\beta}\Gamma^\beta_{\alpha\mu} \\
&= \partial_\alpha\Gamma^\lambda_{\nu\mu} - \partial_\nu\Gamma^\lambda_{\alpha\mu} \\
&= \frac{1}{2}\left[\partial_\alpha\left(h^\lambda_{\nu,\mu} + h^\lambda_{\mu,\nu} - h^{;\lambda}_{\nu\mu}\right) - \partial_\nu\left(h^\lambda_{\alpha,\mu} + h^\lambda_{\mu,\alpha} - h^{;\lambda}_{\alpha\mu}\right)\right]
\end{aligned}
\tag{56}
$$

Ricci张量为

$$
\begin{aligned}
R_{\mu\nu} &= R^\alpha_{\mu\alpha\nu} \\
&= \frac{1}{2}\left[\partial_\alpha\left(h^\alpha_{\nu,\mu} + h^\alpha_{\mu,\nu} - h^{;\alpha}_{\nu\mu}\right) - \partial_\nu\left(h^\alpha_{\alpha,\mu} + h^\alpha_{\mu,\alpha} - h^{;\alpha}_{\alpha\mu}\right)\right] \\
&= \frac{1}{2}\left(\partial_\alpha h^\alpha_{\nu,\mu} + \partial_\nu h^{;\alpha}_{\alpha\mu} - \partial_\alpha h^{;\alpha}_{\nu\mu} - \partial_\nu h^\alpha_{\alpha,\mu}\right) \\
&= -\frac{1}{2}\left(h^{;\alpha}_{\nu\mu,\alpha} + h^\alpha_{\alpha,\mu,\nu} - h^\alpha_{\nu,\mu,\alpha} - h^{;\alpha}_{\alpha\mu,\nu}\right) \\
&= -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h^\alpha_{\alpha,\mu,\nu} - h^\alpha_{\nu,\mu,\alpha} - h^{;\alpha}_{\alpha\mu,\nu}\right)
\end{aligned}
\tag{57}
$$

注意到

$$h^{\alpha}_{\alpha,\mu,\nu} = \eta^{\lambda\alpha} h_{\lambda\alpha,\mu,\nu} = \left(\eta^{\lambda\alpha} h_{\lambda\alpha}\right)_{,\mu,\nu} = h_{,\mu,\nu} \tag{58}$$

$$h^{;\alpha}_{\alpha\mu,\nu} = \eta^{\alpha\beta} h_{\alpha\mu,\nu,\beta} = h^{\beta}_{\mu,\nu,\beta} = h^{\alpha}_{\mu,\nu,\alpha} \tag{59}$$

因此Ricci张量可化为

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h^{\alpha}_{\alpha,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{;\alpha}_{\alpha\mu,\nu}\right) \\ &= -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{\alpha}_{\mu,\nu,\alpha}\right) \end{aligned} \tag{60}$$

先算Ricci标量：

$$\begin{aligned} R &= \eta^{\mu\nu} R_{\mu\nu} \\ &= -\frac{1}{2}\eta^{\mu\nu}\left(h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{\alpha}_{\mu,\nu,\alpha}\right) \\ &= -\frac{1}{2}\left(h^{;\alpha}_{,\alpha} + h^{;\nu}_{,\nu} - h^{\alpha,\nu}_{\nu,\alpha} - h^{\alpha,\mu}_{\mu,\alpha}\right) \\ &= -\frac{1}{2}\left(2h^{;\alpha}_{,\alpha} - 2h^{\alpha,\beta}_{\beta,\alpha}\right) \\ &= -h^{;\alpha}_{,\alpha} + h^{\alpha,\beta}_{\beta,\alpha} \end{aligned} \tag{61}$$

利用

$$\bar{h} = -h, \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h = h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{h} \Longrightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{62}$$

$$h^{\alpha}_{\nu} = \bar{h}^{\alpha}_{\nu} - \frac{1}{2}\delta^{\alpha}_{\nu}\bar{h} \tag{63}$$

有

$$\begin{aligned} \bar{R}_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \\ &= -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{\alpha}_{\mu,\nu,\alpha}\right) - \frac{1}{2}\eta_{\mu\nu}\left(-h^{;\alpha}_{,\alpha} + h^{\alpha,\beta}_{\beta,\alpha}\right) \\ &= -\frac{1}{2}\left[h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{\alpha}_{\mu,\nu,\alpha} - \eta_{\mu\nu} h^{;\alpha}_{,\alpha} + \eta_{\mu\nu} h^{\alpha,\beta}_{\beta,\alpha}\right] \\ &= -\frac{1}{2}\left[\left(\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{;\alpha}_{,\alpha}\right) + \left(-\bar{h}_{,\mu,\nu}\right) - \left(\bar{h}^{\alpha}_{\nu,\mu,\alpha} - \frac{1}{2}\delta^{\alpha}_{\nu}\bar{h}_{,\mu,\alpha}\right) - \left(\bar{h}^{\alpha}_{\mu,\nu,\alpha} - \frac{1}{2}\delta^{\alpha}_{\mu}\bar{h}_{,\nu,\alpha}\right) - \left(-\eta_{\mu\nu}\bar{h}^{;\alpha}_{,\alpha}\right) + \left(\eta_{\mu\nu}\left(\bar{h}^{\alpha,\beta}_{\beta,\alpha} - \frac{1}{2}\delta^{\alpha}_{\beta}\bar{h}^{;\beta}_{,\alpha}\right)\right)\right] \\ &= -\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{\alpha,\beta}_{\beta,\alpha}\right] \\ &= -\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha} + \eta_{\mu\nu}\eta^{\alpha\rho}\bar{h}^{;\beta}_{\rho\beta,\alpha}\right] \\ &= -\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{;\rho,\beta}_{\rho\beta}\right] \\ &= -\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{;\alpha,\beta}_{\alpha\beta} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha}\right] \end{aligned} \tag{64}$$

最后，Einstein引力场方程

$$\bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{65}$$

就化为：

$$-\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{;\alpha,\beta}_{\alpha\beta} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha}\right] = 8\pi G T_{\mu\nu} \tag{66}$$

也即：

$$\bar{h}^{;\alpha}_{\mu\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{;\alpha,\beta}_{\alpha\beta} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha} = -16\pi G T_{\mu\nu} \tag{67}$$

6

推导光线在行星附近的轨道方程，并讨论其偏折，这证明了GR的什么效应?

考虑史瓦西时空，以线长 s 为参数，在 $\theta=\pi/2$ 平面内

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s}=0,\quad \text{or}\quad \mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{68}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{69}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{70}$$

注意到上面第三条方程

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0,\quad r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+2r\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0,\quad r^2\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}+\frac{\mathrm{d}\left(r^2\right)}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{71}$$

也即

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)=0 \tag{72}$$

因此

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}=\mathrm{const}\equiv\frac{h}{c} \tag{73}$$

这是 GR 中的角动量守恒。

我们要找轨道方程，因此需要找到 r,ϕ 的微分方程。

由于 $\theta=\pi/2$，则线元

$$\mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right),\quad \mathrm{e}^{\nu}=1-\frac{2GM}{c^2r} \tag{74}$$

可化简为

$$\mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{75}$$

利用线元 $\mathrm{d}s^2$ 与线长 $\mathrm{d}s$ 的关系

$$\mathrm{d}s=\sqrt{-\mathrm{d}s^2},\quad (\mathrm{d}s)^2=-\mathrm{d}s^2=c^2\mathrm{e}^{\nu}\mathrm{d}t^2-\mathrm{e}^{-\nu}\mathrm{d}r^2-r^2\mathrm{d}\phi^2 \tag{76}$$

两边同除 $(\mathrm{d}s)^2$，再同乘 e^{ν}，得到

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2=c^2\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-r^2\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2-\mathrm{e}^{\nu} \tag{77}$$

上式代入 r 关于 s 的二阶偏微分方程，就消去 t:

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{\nu'}{2}\mathrm{e}^{\nu}r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2+\frac{1}{2}\nu'\mathrm{e}^{\nu}=0 \tag{78}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\left(\frac{1}{2}r^2\nu'\mathrm{e}^{\nu}-r\mathrm{e}^{\nu}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2+\frac{1}{2}\nu'\mathrm{e}^{\nu}=0 \tag{79}$$

这就是GR中参数形式的行星轨道方程。

利用 GR 中的角动量守恒

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{c},\quad \frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{cr^2} \tag{80}$$

进一步化简为

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2}+\left(\frac{\nu'}{2}r^2-r\right)\mathrm{e}^{\nu}\left(\frac{h}{cr^2}\right)^2+\frac{1}{2}\nu'\mathrm{e}^{\nu}=0 \tag{81}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2}+\frac{1}{2}\nu'\mathrm{e}^{\nu}\left(\frac{h^2}{c^2r^2}+1\right)-r\mathrm{e}^{\nu}\left(\frac{h}{cr^2}\right)^2=0 \tag{82}$$

令 $u=\frac{1}{r}$，注意到

$$\mathrm{e}^{\nu}=1-\frac{2GM}{c^2r}=1-\frac{2GM}{c^2}u \tag{83}$$

$$\nu'\mathrm{e}^{\nu}=(\mathrm{e}^{\nu})'=\frac{2GM}{c^2r^2}=\frac{2GM}{c^2}u^2 \tag{84}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{cr^2}=\frac{h}{c}u^2 \tag{85}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}=\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\phi}=\frac{h}{c}u^2\frac{\mathrm{d}}{\mathrm{d}\phi} \tag{86}$$

$$\begin{aligned}\frac{\mathrm{d}^2 r}{\mathrm{d}s}&=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}(1/u)}{\mathrm{d}s}\right)=\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}s}\right)=\frac{\mathrm{d}}{\mathrm{d}s}\left[-\frac{1}{u^2}\cdot\left(\frac{h}{c}u^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\right]\\&=-\frac{h}{c}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)=-\frac{h}{c}\cdot\frac{h}{c}u^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\\&=-\frac{h^2}{c^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}\end{aligned} \tag{87}$$

于是轨道的参数方程就化为如下的轨道方程：

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2}+u=\frac{3GM}{c^2}u^2+\frac{GM}{h^2} \tag{88}$$

上式就是GR中的Binet方程。

设 u_0 满足牛顿力学中的Binet方程

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2}+u_0=\frac{GM}{h^2} \tag{89}$$

其解为

$$u_0=\frac{1}{p}\left(1+e\cos\phi\right),\quad p=\frac{h^2}{GM} \tag{90}$$

令 $\alpha=3GM/c^2$，则

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2}+u=\alpha u^2+\frac{GM}{h^2} \tag{91}$$

设 $u=u_0+\alpha u_1$，则

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2}+u_0+\alpha\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2}+\alpha u_1=\alpha\left(u_0+\alpha u_1\right)^2+\frac{GM}{h^2} \tag{92}$$

利用 u_0 满足的方程就得到

$$\alpha\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2}+\alpha u_1=\alpha\left(u_0+\alpha u_1\right)^2 \tag{93}$$

由于 α 是个小量，约去右边括号内的高阶小量，再利用 u_0 的表达式，就得到

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2}+u_1=u_0^2=\frac{(1+e\cos\phi)^2}{p^2}=\frac{1}{p^2}\left[\left(1+\frac{e^2}{2}\right)+2e\cos\phi+\frac{e^2}{2}\cos 2\phi\right] \tag{94}$$

设 u_1 的形式解为

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi \tag{95}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}\phi} = B\left(\sin\phi + \phi\cos\phi\right) - 2C\sin 2\phi \tag{96}$$

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\phi^2} = B\left(2\cos\phi - \phi\sin\phi\right) - 4C\cos 2\phi \tag{97}$$

代回方程得到

$$B\left(2\cos\phi - \phi\sin\phi\right) - 4C\cos 2\phi + A + B\phi\sin\phi + C\cos 2\phi = \frac{1}{p^2}\left[\left(1+\frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right] \tag{98}$$

对比各项前的系数就得到

$$A = \frac{1}{p^2}\left(1+\frac{e^2}{2}\right), \quad B = \frac{e}{p^2}, \quad C = -\frac{e^2}{6p^2} \tag{99}$$

因此

$$u_1 = \frac{1}{p^2}\left(1+\frac{e^2}{2}\right) + \frac{e}{p^2}\phi\sin\phi - \frac{e^2}{6p^2}\cos 2\phi \tag{100}$$

$$u = u_0 + \alpha u_1 = \frac{1}{p}\left[\left(1+e\cos\phi\right) + \frac{\alpha}{p}\left(1+\frac{e^2}{2}\right) + \frac{\alpha e}{p}\phi\sin\phi - \frac{1}{6}\frac{\alpha e^2}{p^2}\cos 2\phi\right] \tag{101}$$

上面只有 $\phi\sin\phi$ 项是累加的，只保留对轨道有长期影响的项：

$$u = \frac{1}{p}\left[\left(1+e\cos\phi\right) + \frac{\alpha e}{p}\phi\sin\phi\right] \tag{102}$$

由于 α 是小量，则

$$1 \approx \cos\left(\frac{\alpha}{p}\phi\right), \quad \frac{\alpha}{p}\phi \approx \sin\left(\frac{\alpha}{p}\phi\right) \tag{103}$$

于是

$$\begin{aligned} u &= \frac{1}{p}\left[\left(1+e\cos\phi\right) + \frac{\alpha e}{p}\phi\sin\phi\right] \\ &= \frac{1}{p}\left[1 + e\left(1\cdot\cos\phi + \frac{\alpha}{p}\phi\sin\phi\right)\right] \\ &\approx \frac{1}{p}\left[1 + e\left(\cos\left(\frac{\alpha}{p}\phi\right)\cos\phi + \sin\left(\frac{\alpha}{p}\phi\right)\sin\phi\right)\right] \\ &= \frac{1}{p}\left[1 + \cos\left(\left(1-\frac{\alpha}{p}\right)\phi\right)\right] \\ &\equiv \frac{1}{p}\left(1+e\cos\Phi\right) \end{aligned} \tag{104}$$

$$\Phi \equiv \left(1-\frac{\alpha}{p}\right)\phi \tag{105}$$

当 $\Phi = \Phi_n = (2n+1)\pi$ 时，$\cos\Phi = -1$，此时 u 最小，r 最大，相应 ϕ_n 为

$$\phi_n = \frac{\Phi_n}{1-\alpha/p} \approx \Phi_n\left(1+\frac{\alpha}{p}\right) \tag{106}$$

$$\phi_{n+1} \approx \Phi_{n+1}\left(1+\frac{\alpha}{p}\right) \tag{107}$$

于是

$$\phi_{n+1} - \phi_n = 2\pi\left(1+\frac{\alpha}{p}\right) \tag{108}$$

一周期进动角为

$$\Delta=\phi_{n+1}-\phi_n-2\pi=2\pi\frac{\alpha}{p}=\frac{6\pi GM}{c^2p}\tag{109}$$

由

$$T^2=\frac{4\pi^2}{GM}a^3,\quad GM=4\pi^2\frac{a^3}{T^2},\quad p=a\left(1-e^2\right)\tag{110}$$

可得

$$\Delta=\frac{24\pi^3a^2}{c^2T^2\left(1-e^2\right)}\tag{111}$$

7

推导史瓦西时空与闵可夫斯基时空的类光测地线，简述并画出Kruskal延拓图与彭罗斯图。

闵氏时空线元

$$\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\tag{112}$$

类光测地线满足

$$\mathrm{d}s^2=0\tag{113}$$

如果只考虑径向，则

$$-c^2\mathrm{d}t^2+\mathrm{d}r^2=0\tag{114}$$

取 $c=1$，则

$$\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2=1,\quad\frac{\mathrm{d}t}{\mathrm{d}r}=\pm1,\tag{115}$$

径向类光测地线为

$$t=\pm r+\mathrm{const}\tag{116}$$

史瓦西时空线元

$$\mathrm{d}s^2=-c^2\left(1-\frac{2GM}{c^2r}\right)\mathrm{d}t^2+\left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\tag{117}$$

如果只考虑径向，并取 $c=G=1$，则

$$\mathrm{d}s^2=-\left(1-\frac{2M}{r}\right)\mathrm{d}t^2+\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2\tag{118}$$

类光测地线满足

$$\mathrm{d}s^2=0\tag{119}$$

即

$$\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2=\left(1-\frac{2M}{r}\right)^{-2},\quad\frac{\mathrm{d}t}{\mathrm{d}r}=\pm\left(1-\frac{2M}{r}\right)^{-1}\tag{120}$$

$$\mathrm{d}t=\pm\frac{1}{1-2M/r}\mathrm{d}r=\pm\frac{r}{r-2M}\mathrm{d}r=\pm\frac{r-2M+2M}{r-2M}\mathrm{d}r=\pm\left(1+\frac{2M}{r-2M}\right)\mathrm{d}(r-2M)\tag{121}$$

两边积分得

$$t=\pm\left(r+2M\ln|r-2M|\right)+\mathrm{const}\tag{122}$$

史瓦西时空线元：

$$\mathrm{d}s^2 = -(1-2M/r)\,\mathrm{d}t^2 + (1-2M/r)^{-1}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{123}$$

考虑坐标变换

$$\begin{cases} T = 4M\left(\frac{r}{2M}-1\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\sinh\left(\frac{t}{4M}\right) \\ R = 4M\left(\frac{r}{2M}-1\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\cosh\left(\frac{t}{4M}\right) \end{cases}, r > 2M, \mathrm{I}$$

$$\begin{cases} T = -4M\left(\frac{r}{2M}-1\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\sinh\left(\frac{t}{4M}\right) \\ R = -4M\left(\frac{r}{2M}-1\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\cosh\left(\frac{t}{4M}\right) \end{cases}, r > 2M, \mathrm{II}$$

$$\begin{cases} T = 4M\left(1-\frac{r}{2M}\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\cosh\left(\frac{t}{4M}\right) \\ R = 4M\left(1-\frac{r}{2M}\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\sinh\left(\frac{t}{4M}\right) \end{cases}, r < 2M, F$$

$$\begin{cases} T = -4M\left(1-\frac{r}{2M}\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\cosh\left(\frac{t}{4M}\right) \\ R = -4M\left(1-\frac{r}{2M}\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\sinh\left(\frac{t}{4M}\right) \end{cases}, r < 2M, P$$

Kruskal坐标下的线元：

$$\mathrm{d}s^2 = \frac{2M}{r}\mathrm{e}^{-r/2M}\left(-\mathrm{d}T^2 + \mathrm{d}R^2\right) + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{124}$$

r 与 R, T 的关系：

$$R^2 - T^2 = 16M^2\left(\frac{r}{2M}-1\right)\mathrm{e}^{r/2M} \tag{125}$$

> 上式对四个区域都成立。

Kruskal坐标消除了度规分量在引力半径处的奇异性，但无法消除 $r=0$ 的奇点。

Kruskal度规具有最大解析区和最高完备性。

- 类时未来无穷远 I^+ : r 有限，$t \to +\infty$
- 类时过去无穷远 I^- : r 有限，$t \to -\infty$
- 类空无穷远 I^0 : t 有限，$r \to +\infty$
- 类光未来无穷远 J^+ : $(t-r)$ 有限，$(t+r) \to +\infty$
- 类光过去无穷远 J^- : $(t+r)$ 有限，$(t-r) \to +\infty$

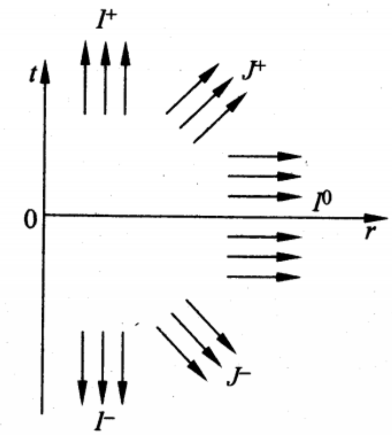

图 6.2.4　闵可夫斯基时空图

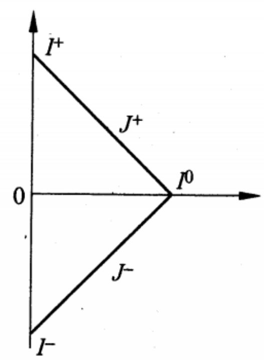

图 6.2.5　闵氏时空彭若斯图

克鲁斯卡坐标系可以统一描述整个史瓦西时空，它覆盖了黑洞内、外及视界。而且，从克鲁斯卡时空图(图 6.2.3)可知，它扩大了史瓦西时空。两条对角线是视界。Ⅰ区即通常的黑洞外部宇宙，F 区为黑洞区，P 区为白洞区，Ⅱ区是另一个洞外宇宙，它和我们的宇宙没有因果连通，没有任何信息交流。奇点 $r=0$ 分别出现在白洞区和黑洞区，以双曲线形式呈现。Ⅰ区和Ⅱ区中“$r=$常数”的双曲线，就是史瓦西时空中静止粒子的世界线。F 区和 P 区中“$r=$常数”的双曲线为等时线。应当注意，此图中的任何一点，都代表一个二维球面。光锥如图 6.2.3 所示，

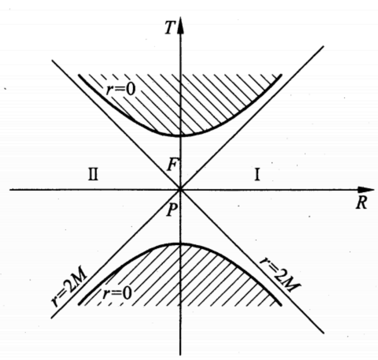

图 6.2.3　克鲁斯卡时空图

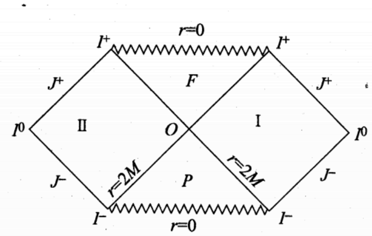

图 6.2.6　克鲁斯卡时空彭若斯图