4-4

由自由旋量场的拉格朗日密度

$${\cal L}_0 = -rac{1}{2}\left(ar{\psi}\gamma_lpha\partial_lpha\psi - \partial_lphaar{\psi}\gamma_lpha\psi
ight) - mar{\psi}\psi$$

求其能量动量张量、能量密度和动量密度

$$egin{aligned} T_{\mu
u} &= rac{1}{2} \left(ar{\psi} \gamma_{
u} \partial_{\mu} \psi - \partial_{\mu} ar{\psi} \gamma_{
u} \psi
ight) \ W &= rac{1}{2\mathrm{i}} \left(\partial_t \psi^\dagger \psi - \psi^\dagger \partial_t \psi
ight) \ G_i &= rac{1}{2\mathrm{i}} \left(\psi^\dagger \partial_i \psi - \partial_i \psi^\dagger \psi
ight) \end{aligned}$$

旋量场拉格朗日密度:

$$egin{align} L &= -rac{1}{2} \left(ar{\psi} \gamma_{\mu} \partial_{\mu} \psi - \partial_{\mu} ar{\psi} \gamma_{\mu} \psi
ight) - m ar{\psi} \psi \ & rac{\partial L}{\partial \left(\partial_{
u} \psi
ight)} = -rac{1}{2} ar{\psi} \gamma_{
u} \ & rac{\partial L}{\partial \left(\partial_{
u} ar{\psi}
ight)} = rac{1}{2} \gamma_{
u} \psi \ & \end{split}$$

能量动量张量:

$$\begin{split} T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\phi_{A}\right)}\partial_{\mu}\phi_{A} \\ &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\psi\right)}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\frac{\partial L}{\partial \left(\partial_{\nu}\bar{\psi}\right)} \\ &= \left[-\frac{1}{2}\left(\bar{\psi}\gamma_{\alpha}\partial_{\alpha}\psi - \partial_{\alpha}\bar{\psi}\gamma_{\alpha}\psi\right) - m\bar{\psi}\psi\right]\delta_{\mu\nu} - \left(-\frac{1}{2}\bar{\psi}\gamma_{\nu}\right)\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\left(\frac{1}{2}\gamma_{\nu}\psi\right) \\ &= \left[-\frac{1}{2}\left(\bar{\psi}\gamma_{\alpha}\partial_{\alpha}\psi - \partial_{\alpha}\bar{\psi}\gamma_{\alpha}\psi\right) - m\bar{\psi}\psi\right]\delta_{\mu\nu} + \frac{1}{2}\bar{\psi}\gamma_{\nu}\partial_{\mu}\psi - \frac{1}{2}\partial_{\mu}\bar{\psi}\gamma_{\nu}\psi \end{split}$$

利用旋量场及共轭旋量场运动方程

$$\gamma_{\mu}\partial_{\mu}\psi+m\psi=0$$

$$\partial_{\mu}ar{\psi}\gamma_{\mu}-mar{\psi}=0$$

进一步有:

$$\begin{split} T_{\mu\nu} &= \left[-\frac{1}{2} \left(\bar{\psi} \gamma_{\alpha} \partial_{\alpha} \psi - \partial_{\alpha} \bar{\psi} \gamma_{\alpha} \psi \right) - m \bar{\psi} \psi \right] \delta_{\mu\nu} + \frac{1}{2} \bar{\psi} \gamma_{\nu} \partial_{\mu} \psi - \frac{1}{2} \partial_{\mu} \bar{\psi} \gamma_{\nu} \psi \\ &= \left[\frac{m}{2} \bar{\psi} \psi + \frac{m}{2} \bar{\psi} \psi - m \bar{\psi} \psi \right] \delta_{\mu\nu} + \frac{1}{2} \left(\bar{\psi} \gamma_{\nu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma_{\nu} \psi \right) \\ &= \frac{1}{2} \left(\bar{\psi} \gamma_{\nu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma_{\nu} \psi \right) \end{split}$$

能量密度:

$$egin{aligned} W &= -T_{44} \ &= -rac{1}{2} \left(ar{\psi} \gamma_4 \partial_4 \psi - \partial_4 ar{\psi} \gamma_4 \psi
ight) \ &= -rac{1}{2} \left(-\mathrm{i} \psi^\dagger \gamma_4^2 \partial_t \psi + \mathrm{i} \partial_t \psi^\dagger \gamma_4^2 \psi
ight) \ &= rac{\mathrm{i}}{2} \left(\psi^\dagger \partial_t \psi - \partial_t \psi^\dagger \psi
ight) \ &= rac{1}{2\mathrm{i}} \left(\partial_t \psi^\dagger \psi - \psi^\dagger \partial_t \psi
ight) \end{aligned}$$

动量密度:

$$egin{aligned} G_i &= rac{1}{\mathrm{i}} T_{i4} \ &= rac{1}{\mathrm{i}} \cdot rac{1}{2} \left(ar{\psi} \gamma_4 \partial_i \psi - \partial_i ar{\psi} \gamma_4 \psi
ight) \ &= rac{1}{2\mathrm{i}} \left(\psi^\dagger \gamma_4^2 \partial_i \psi - \partial_i \psi^\dagger \gamma_4^2 \psi
ight) \ &= rac{1}{2\mathrm{i}} \left(\psi^\dagger \partial_i \psi - \partial_i \psi^\dagger \psi
ight) \end{aligned}$$