4-2 量子场论习题五

4-2-1

> 由表象变换 $\psi' = S\psi, \hat{F}' = S\hat{F}S^\dagger, SS^\dagger = I$，证明力学量 $\hat{F}$ 的本征值、平均值和态矢量的内积均为不变量。

本征值不变

设 $\hat{F}$ 的本征方程为：

$$\hat{F}\psi_f = f\psi_f$$

由定义，有：

$$\hat{F}' \equiv S\hat{F}S^\dagger, \quad \psi'_f \equiv S\psi_f$$

对本征方程作如下的变形：

$$S\hat{F}S^\dagger S\psi_f = fS\psi_f$$

即：

$$\hat{F}'\psi'_f = f\psi'_f$$

因此本征值不变。

平均值不变

$\hat{F}$ 在 $|\psi\rangle$ 态下的平均值：

$$\begin{aligned}\left\langle\psi\middle|\hat{F}\middle|\psi\right\rangle &= \left\langle\psi\middle|S^\dagger S\hat{F}S^\dagger S\middle|\psi\right\rangle \\ &= \left\langle\psi'\middle|\hat{F}'\middle|\psi'\right\rangle\end{aligned}$$

因此平均值不变。

态矢量的内积不变

$$|\alpha'\rangle \equiv S|\alpha\rangle, \quad \langle\alpha'| = \langle\alpha|S^\dagger$$

$$|\beta'\rangle \equiv S|\beta\rangle$$

态矢量内积：

$$\begin{aligned}\langle\alpha\,|\,\beta\rangle &= \left\langle\alpha\,\middle|\,S^\dagger S\,\middle|\,\beta\right\rangle \\ &= \langle\alpha'\,|\,\beta'\rangle\end{aligned}$$

因此态矢量的内积不变。

4-2-2

> 证明
>
> $$\theta_-(-x) = \theta_+(x)$$
>
> $$\theta_+(-x) = \theta_-(x)$$

> $$\varepsilon(x)=\theta_+(x)-\theta_-(x)$$

由定义：

$$\theta_+(x)\equiv\begin{cases}1,x>0\\0,x<0\end{cases},\quad\theta_-(x)\equiv\begin{cases}0,x>0\\1,x<0\end{cases},\quad\varepsilon(x)\equiv\begin{cases}1,x>0\\-1,x<0\end{cases}$$

因此：

$$\theta_-(-x)\equiv\begin{cases}0,-x>0\\1,-x<0\end{cases}=\begin{cases}1,x>0\\0,x<0\end{cases}=\theta_+(x)$$

$$\theta_+(-x)\equiv\begin{cases}1,-x>0\\0,-x<0\end{cases}=\begin{cases}0,x>0\\1,x<0\end{cases}=\theta_-(x)$$

$$\theta_+(x)-\theta_-(x)=\begin{cases}1-0,x>0\\0-1,x<0\end{cases}=\begin{cases}1,x>0\\-1,x<0\end{cases}=\varepsilon(x)$$

4-2-3

> 推导 $\hat{H}_{--}=0$

实标量场的能量密度：

$$W=\frac{1}{2}\left(\nabla\phi\cdot\nabla\phi+\partial_t\phi\partial_t\phi+m^2\phi^2\right)$$

实标量场的傅里叶分解：

$$\hat{\phi}(x)=\hat{\phi}^{(+)}(x)+\hat{\phi}^{(-)}(x)$$

$$\hat{\phi}^{(+)}(x)=\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{-\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(+)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}$$

$$\hat{\phi}^{(-)}(x)=\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(-)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}$$

哈密顿算符：

$$\begin{aligned}\hat{H}&=\frac{1}{2}\int\left[\left(\nabla\hat{\phi}\right)^2+\left(\frac{\partial\hat{\phi}}{\partial t}\right)^2+m^2\hat{\phi}^2\right]\mathrm{d}V\\&=\frac{1}{2}\int\left[\left(\nabla\hat{\phi}^{(+)}(x)+\nabla\hat{\phi}^{(-)}(x)\right)^2+\left(\partial_t\hat{\phi}^{(+)}(x)+\partial_t\hat{\phi}^{(-)}(x)\right)^2+m^2\left(\hat{\phi}^{(+)}(x)+\hat{\phi}^{(-)}(x)\right)^2\right]\mathrm{d}V\end{aligned}$$

注意到，展开式中

$$
\begin{aligned}
&\int\left(\nabla\hat{\phi}^{(\pm)}(x)\cdot\nabla\hat{\phi}^{(\pm)}(x)+\partial_t\hat{\phi}^{(\pm)}(x)\partial_t\hat{\phi}^{(\pm)}(x)+m^2\hat{\phi}^{(\pm)}(x)\hat{\phi}^{(\pm)}(x)\right)\mathrm{d}^3\vec{x}\\
=&\left(\frac{1}{2\pi}\right)^3\int\mathrm{d}^3\vec{x}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{d}^3\vec{k}\int\limits_{k_0'=\varepsilon_{\vec{k}'}}\mathrm{d}^3\vec{k}'\left(-\vec{k}\cdot\vec{k}'-\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}+m^2\right)\mathrm{e}^{\mp\mathrm{i}\left(\vec{k}+\vec{k}'\right)\cdot\vec{x}}\mathrm{e}^{\pm\mathrm{i}\left(\varepsilon_{\vec{k}}+\varepsilon_{\vec{k}'}\right)t}\frac{\hat{a}^{(\pm)}\left(\vec{k}\right)\hat{a}^{(\pm)}\left(\vec{k}'\right)}{2\sqrt{\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}}}\\
=&\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{d}^3\vec{k}\int\limits_{k_0'=\varepsilon_{\vec{k}'}}\mathrm{d}^3\vec{k}'\left(-\vec{k}\cdot\vec{k}'-\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}+m^2\right)\delta\left(\vec{k}+\vec{k}'\right)\mathrm{e}^{\pm\mathrm{i}\left(\varepsilon_{\vec{k}}+\varepsilon_{\vec{k}'}\right)t}\frac{\hat{a}^{(\pm)}\left(\vec{k}\right)\hat{a}^{(\pm)}\left(\vec{k}'\right)}{2\sqrt{\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}}}\\
=&\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{d}^3\vec{k}\left(\vec{k}^2-\varepsilon_{\vec{k}}^2+m^2\right)\mathrm{e}^{\pm\mathrm{i}\left(\varepsilon_{\vec{k}}+\varepsilon_{\vec{k}'}\right)t}\frac{\hat{a}^{(\pm)}\left(\vec{k}\right)\hat{a}^{(\pm)}\left(\vec{-k}\right)}{2\varepsilon_{\vec{k}}}\\
=&0
\end{aligned}
$$

其中包括:

$$
\begin{aligned}
\hat{H}_{--}&=\frac{1}{2}\int\left(\nabla\hat{\phi}^{(-)}(x)\cdot\nabla\hat{\phi}^{(-)}(x)+\partial_t\hat{\phi}^{(-)}(x)\partial_t\hat{\phi}^{(-)}(x)+m^2\hat{\phi}^{(-)}(x)\hat{\phi}^{(-)}(x)\right)\mathrm{d}^3\vec{x}\\
&=0
\end{aligned}
$$

4-2-4

> 对标量场，由
>
> $$\vec{p}=-\int\nabla\phi\partial_t\phi\mathrm{d}^3\vec{x}\Longrightarrow\vec{p}=\frac{1}{2}\sum_{\vec{k}}\vec{k}\left\{a_{\vec{k}}^{(+)},a_{\vec{k}}^{(-)}\right\}$$

实标量场的傅里叶分解:

$$\hat{\phi}(x)=\hat{\phi}^{(+)}(x)+\hat{\phi}^{(-)}(x)$$

$$\hat{\phi}^{(+)}(x)=\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{-\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(+)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}$$

$$\hat{\phi}^{(-)}(x)=\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(-)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}$$

$$\hat{\vec{p}}=-\int\left(\nabla\hat{\phi}^{(+)}(x)+\nabla\hat{\phi}^{(-)}(x)\right)\left(\partial_t\hat{\phi}^{(+)}(x)+\partial_t\hat{\phi}^{(-)}(x)\right)\mathrm{d}^3\vec{x}$$

注意到:

$$
\begin{aligned}
\int\nabla\hat{\phi}^{(\pm)}(x)\partial_t\hat{\phi}^{(\pm)}(x)\mathrm{d}^3\vec{x}&=\int\mathrm{d}^3\vec{k}\int\mathrm{d}^3\vec{k}'\varepsilon_{\vec{k}}\vec{k}'\mathrm{e}^{\pm\mathrm{i}\left(\varepsilon_{\vec{k}}+\varepsilon_{\vec{k}'}\right)t}\delta\left(\vec{k}+\vec{k}'\right)\frac{\hat{a}^{(\pm)}\left(\vec{k}\right)\hat{a}^{(\pm)}\left(\vec{k}\right)}{2\sqrt{\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}}}\\
&=-\frac{1}{2}\int\mathrm{d}^3\vec{k}\vec{k}\mathrm{e}^{\pm2\mathrm{i}\varepsilon_{\vec{k}}t}\hat{a}^{(\pm)}\left(\vec{k}\right)\hat{a}^{(\pm)}\left(-\vec{k}\right)\\
&=0
\end{aligned}
$$

有贡献的仅为交叉项:

$$
\begin{aligned}
\hat{\vec{p}} &= -\int \mathrm{d}^3\vec{x}\left(\nabla\hat{\phi}^{(+)}(x)\partial_t\hat{\phi}^{(-)}(x)+\nabla\hat{\phi}^{(-)}(x)\partial_t\hat{\phi}^{(+)}(x)\right)\\
&= \left(\frac{1}{2\pi}\right)^3\int \mathrm{d}^3\vec{x}\int \mathrm{d}\vec{k}\int \mathrm{d}\vec{k}'\left[\vec{k}\varepsilon_{\vec{k}'}\mathrm{e}^{\mathrm{i}\left(\varepsilon_{\vec{k}}-\varepsilon_{\vec{k}'}\right)t}\mathrm{e}^{-\mathrm{i}\left(\vec{k}-\vec{k}'\right)\cdot\vec{x}}\frac{\hat{a}^{(+)}\left(\vec{k}\right)\hat{a}^{(-)}\left(\vec{k}\right)}{2\sqrt{\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}}}+\vec{k}\varepsilon_{\vec{k}'}\mathrm{e}^{-\mathrm{i}\left(\varepsilon_{\vec{k}}-\varepsilon_{\vec{k}'}\right)t}\mathrm{e}^{\mathrm{i}\left(\vec{k}-\vec{k}'\right)\cdot\vec{x}}\frac{\hat{a}^{(-)}\left(\vec{k}\right)\hat{a}^{(+)}\left(\vec{k}\right)}{2\sqrt{\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}}}\right]\\
&= \int \mathrm{d}^3\vec{k}\int \mathrm{d}^3\vec{k}'\delta\left(\vec{k}-\vec{k}'\right)\vec{k}\varepsilon_{\vec{k}'}\frac{1}{2\sqrt{\varepsilon_{\vec{k}}\varepsilon_{\vec{k}'}}}\left[\mathrm{e}^{\mathrm{i}\left(\varepsilon_{\vec{k}}-\varepsilon_{\vec{k}'}\right)t}\hat{a}^{(+)}\left(\vec{k}\right)\hat{a}^{(-)}\left(\vec{k}\right)+\mathrm{e}^{-\mathrm{i}\left(\varepsilon_{\vec{k}}-\varepsilon_{\vec{k}'}\right)t}\hat{a}^{(-)}\left(\vec{k}\right)\hat{a}^{(+)}\left(\vec{k}\right)\right]\\
&= \frac{1}{2}\int \vec{k}\left\{\hat{a}^{(+)}\left(\vec{k}\right),\hat{a}^{(-)}\left(\vec{k}\right)\right\}\mathrm{d}^3\vec{k}
\end{aligned}
$$

利用:

$$
\int \mathrm{d}^3\vec{k}\cdots=\frac{(2\pi)^3}{V}\sum_{\vec{k}}\cdots
$$

$$
\hat{a}^{(\pm)}_{\vec{k}}\equiv\frac{(2\pi)^{3/2}}{\sqrt{V}}\hat{a}^{(\pm)}\left(\vec{k}\right),\quad \hat{a}^{(\pm)}\left(\vec{k}\right)=\frac{\sqrt{V}}{(2\pi)^{3/2}}\hat{a}^{(\pm)}_{\vec{k}}
$$

有:

$$
\begin{aligned}
\hat{\vec{p}} &= \frac{1}{2}\int \vec{k}\left\{\hat{a}^{(+)}\left(\vec{k}\right),\hat{a}^{(-)}\left(\vec{k}\right)\right\}\mathrm{d}^3\vec{k}\\
&= \frac{1}{2}\frac{(2\pi)^3}{V}\sum_{\vec{k}}\left\{\frac{\sqrt{V}}{(2\pi)^{3/2}}\hat{a}^{(+)}_{\vec{k}},\frac{\sqrt{V}}{(2\pi)^{3/2}}\hat{a}^{(-)}_{\vec{k}}\right\}\\
&= \frac{1}{2}\sum_{\vec{k}}\vec{k}\left\{\hat{a}^{(+)}_{\vec{k}},\hat{a}^{(-)}_{\vec{k}}\right\}
\end{aligned}
$$

4-2-5

> 证明
>
> $$\Delta^{(\pm)}(x-x')=-\Delta^{(\mp)}(x'-x)$$
>
> $$\Delta^{(\pm)}\left(\vec{x},t\right)=-\Delta^{(\pm)}\left(-\vec{x},t\right)$$
>
> $$\Delta^{(+)}(\vec{x},t)=\Delta^{(-)}(\vec{x},-t)$$

$$
\Delta^{(+)}(x-x')\equiv\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4k\mathrm{e}^{-\mathrm{i}k\left(x-x'\right)}\delta\left(k^2+m^2\right)\theta_+(k_0)
$$

$$
\Delta^{(-)}(x-x')\equiv-\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4k\mathrm{e}^{\mathrm{i}k\left(x-x'\right)}\delta\left(k^2+m^2\right)\theta_+(k_0)
$$

合写为:

$$
\Delta^{(\pm)}(x-x')\equiv\frac{\pm\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4k\mathrm{e}^{\mp\mathrm{i}k\left(x-x'\right)}\delta\left(k^2+m^2\right)\theta_+(k_0)
$$

$$
\begin{aligned}
\Delta^{(\pm)}(x-x') &\equiv \frac{\pm\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4k\mathrm{e}^{\mp\mathrm{i}k\left(x-x'\right)}\delta\left(k^2+m^2\right)\theta_+(k_0)\\
&= -\frac{\mp\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4k\mathrm{e}^{\pm\mathrm{i}k\left(x'-x\right)}\delta\left(k^2+m^2\right)\theta_+(k_0)\\
&= -\Delta^{(\mp)}\left(x'-x\right)
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{(\pm)}(\vec{x},t) &\equiv \frac{\pm\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4k\mathrm{e}^{\mp\mathrm{i}kx}\delta\left(k^2+m^2\right)\theta_+(k_0)\\
&\equiv \frac{\pm\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\int \mathrm{d}k_0\mathrm{e}^{\mp\mathrm{i}\left(\vec{k}\cdot\vec{x}-k_0t\right)}\delta\left(\vec{k}^2+m^2-k_0^2\right)\theta_+(k_0)\\
&\equiv \frac{\pm\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\int \mathrm{d}k_0\mathrm{e}^{\mp\mathrm{i}\left(\vec{k}\cdot\vec{x}-k_0t\right)}\frac{1}{2\varepsilon_{\vec{k}}}\left(\delta\left(k_0+\varepsilon_{\vec{k}}\right)+\delta\left(k_0-\varepsilon_{\vec{k}}\right)\right)\theta_+(k_0)\\
&= \frac{\pm\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\mathrm{e}^{\mp\mathrm{i}\left(\vec{k}\cdot\vec{x}-\varepsilon_{\vec{k}}t\right)}\frac{1}{2\varepsilon_{\vec{k}}}\\
&= \frac{\pm\mathrm{i}}{(2\pi)^3}\int -\mathrm{d}^3\vec{k}'\mathrm{e}^{\mp\mathrm{i}\left(\vec{k}'\cdot(-\vec{x})-\varepsilon_{\vec{k}'}t\right)}\frac{1}{2\varepsilon_{\vec{k}'}}\\
&= -\Delta^{(\pm)}(-\vec{x},t)
\end{aligned}
$$

$$
\Delta^{(\pm)}(\vec{x},t) = \frac{\pm\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\mathrm{e}^{\mp\mathrm{i}\left(\vec{k}\cdot\vec{x}-\varepsilon_{\vec{k}}t\right)}\frac{1}{2\varepsilon_{\vec{k}}}
$$

$$
\begin{aligned}
\Delta^{(+)}(\vec{x},t) &= \frac{+\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\mathrm{e}^{-\mathrm{i}\left(\vec{k}\cdot\vec{x}-\varepsilon_{\vec{k}}t\right)}\frac{1}{2\varepsilon_{\vec{k}}}\\
&= \frac{+\mathrm{i}}{(2\pi)^3}\int -\mathrm{d}^3\vec{k}'\mathrm{e}^{-\mathrm{i}\left(-\vec{k}'\cdot\vec{x}-\varepsilon_{\vec{k}'}t\right)}\frac{1}{2\varepsilon_{\vec{k}'}}\\
&= \frac{-\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^3\vec{k}'\mathrm{e}^{\mathrm{i}\left(\vec{k}'\cdot\vec{x}-\varepsilon_{\vec{k}'}(-t)\right)}\frac{1}{2\varepsilon_{\vec{k}'}}\\
&= \Delta^{(-)}(\vec{x},-t)
\end{aligned}
$$

4-2-6

> 利用标量场算符满足 Heisenberg 方程和对易关系，证明它仍满足标量场方程，即
>
> $$\left(\Box-m^2\right)\hat{\phi}=0$$

$$\hat{\phi}(x)=\hat{\phi}^{(+)}(x)+\hat{\phi}^{(-)}(x)$$

$$\hat{\phi}^{(\pm)}(x)=\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(\pm)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}$$

$$\left[\hat{H},\hat{a}^{(-)}\left(\vec{k}\right)\right]=-\varepsilon_k\hat{a}^{(-)}\left(\vec{k}\right)$$

$$\left[\hat{H},\hat{a}^{(+)}\left(\vec{k}\right)\right]=\varepsilon_k\hat{a}^{(+)}\left(\vec{k}\right)$$

海森堡方程：

$$
\begin{aligned}
\frac{\partial\hat{\phi}^{(\pm)}(x)}{\partial t} &= \mathrm{i}\left[\hat{H},\hat{\phi}^{(\pm)}(x)\right]\\
&= \mathrm{i}\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\left[\hat{H},\hat{a}^{(\pm)}\left(\vec{k}\right)\right]\mathrm{d}^3\vec{k}\\
&= \mathrm{i}\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\left(\pm\varepsilon_{\vec{k}}\hat{a}^{(\pm)}\left(\vec{k}\right)\right)\mathrm{d}^3\vec{k}\\
&= \pm\mathrm{i}\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\varepsilon_{\vec{k}}\hat{a}^{(\pm)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
\partial_t^2\hat{\phi}^{(\pm)}(x) &= \mathrm{i}\left[\hat{H},\partial_t\hat{\phi}^{(\pm)}(x)\right] \\
&= \mp\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\varepsilon_{\vec{k}}\left[\hat{H},\hat{a}^{(\pm)}\left(\vec{k}\right)\right]\mathrm{d}^3\vec{k} \\
&= \mp\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\varepsilon_{\vec{k}}\left(\pm\varepsilon_{\vec{k}}\hat{a}^{(\pm)}\left(\vec{k}\right)\right)\mathrm{d}^3\vec{k} \\
&= -\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\varepsilon_{\vec{k}}^2\hat{a}^{(\pm)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
\partial_i\partial_i\hat{\phi}^{(\pm)}(x) &= \left(\frac{1}{2\pi}\right)^{3/2}\partial_i\partial_i\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(\pm)}\left(\vec{k}\right)\mathrm{d}^3\vec{k} \\
&= \left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}-k_ik_i\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(\pm)}\left(\vec{k}\right)\mathrm{d}^3\vec{k} \\
&= -\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\vec{k}^2\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(\pm)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
\left(\Box-m^2\right)\hat{\phi}^{(\pm)}(x) &= \left(\partial_i\partial_i-\partial_t^2-m^2\right)\hat{\phi}^{(\pm)}(x) \\
&= \left(\frac{1}{2\pi}\right)^{3/2}\left\{\int\limits_{k_0=\varepsilon_{\vec{k}}}\left(-\vec{k}^2+\varepsilon_{\vec{k}}^2-m^2\right)\mathrm{e}^{\mp\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}\hat{a}^{(\pm)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}\right\} \\
&= 0
\end{aligned}
$$

因此:

$$
\begin{aligned}
\left(\Box-m^2\right)\hat{\phi}(x) &= \left(\Box-m^2\right)\left(\hat{\phi}^{(+)}(x)+\hat{\phi}^{(-)}(x)\right) \\
&= 0
\end{aligned}
$$

4-2-7

> 讨论复标量场的量子化。

Fourier 表示

$$
\hat{\phi}^{(+)}(x) = \left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{-\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}b^{(+)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}
$$

$$
\hat{\phi}^{(-)}(x) = \left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}a^{(-)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}
$$

$$
\hat{\phi}^{*(+)}(x) = \left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{-\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}a^{(+)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}
$$

$$
\hat{\phi}^{*(-)}(x) = \left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{k_0=\varepsilon_{\vec{k}}}\mathrm{e}^{\mathrm{i}kx}\frac{1}{\sqrt{2\varepsilon_{\vec{k}}}}b^{(-)}\left(\vec{k}\right)\mathrm{d}^3\vec{k}
$$

物理量算符表示

$$\hat{H}=\int\varepsilon_{\vec{k}}\left[\hat{a}^{(+)}\left(\vec{k}\right)\hat{a}^{(-)}\left(\vec{k}\right)+b^{(-)}\left(\vec{k}\right)b^{(+)}\left(\vec{k}\right)\right]\mathrm{d}^3\vec{k}$$

$$\hat{\vec{p}}=\int\vec{k}\left[\hat{a}^{(+)}\left(\vec{k}\right)\hat{a}^{(-)}\left(\vec{k}\right)+b^{(-)}\left(\vec{k}\right)b^{(+)}\left(\vec{k}\right)\right]\mathrm{d}^3\vec{k}$$

$$\hat{Q}=\int e\left[\hat{a}^{(+)}\left(\vec{k}\right)\hat{a}^{(-)}\left(\vec{k}\right)-b^{(-)}\left(\vec{k}\right)b^{(+)}\left(\vec{k}\right)\right]\mathrm{d}^3\vec{k}$$

基本对易关系

$$\left[\hat{H},\hat{a}^{(-)}\left(\vec{k}\right)\right]=-\varepsilon_{\vec{k}}\hat{a}^{(-)}\left(\vec{k}\right)$$

$$\left[\hat{H},\hat{a}^{(+)}\left(\vec{k}\right)\right]=\varepsilon_{\vec{k}}\hat{a}^{(+)}\left(\vec{k}\right)$$

$$\left[\hat{H},\hat{b}^{(-)}\left(\vec{k}\right)\right]=-\varepsilon_{\vec{k}}\hat{b}^{(-)}\left(\vec{k}\right)$$

$$\left[\hat{H},\hat{b}^{(+)}\left(\vec{k}\right)\right]=\varepsilon_{\vec{k}}\hat{b}^{(+)}\left(\vec{k}\right)$$

$$\left[\hat{a}^{(-)}\left(\vec{k}\right),\hat{a}^{(+)}\left(\vec{k}'\right)\right]=\delta\left(\vec{k}-\vec{k}'\right)$$

$$\left[\hat{b}^{(-)}\left(\vec{k}\right),\hat{b}^{(+)}\left(\vec{k}'\right)\right]=\delta\left(\vec{k}-\vec{k}'\right)$$

复标量场对易关系

$$\left[\hat{\phi}^{*(+)}(x),\hat{\phi}^{(-)}(x')\right]=\mathrm{i}\Delta^{(+)}(x-x')$$

$$\left[\hat{\phi}^{*(-)}(x),\hat{\phi}^{(+)}(x')\right]=\mathrm{i}\Delta^{(+)}(x-x')$$

$$\left[\hat{\phi}^{*}(x),\hat{\phi}(x')\right]=\mathrm{i}\Delta(x-x')$$

4-2-8

> 证明
>
> $$\Delta^F(x-x')=\Delta^F(x'-x)$$
>
> $$D^F(x-x')=D^F(x'-x)$$

由于

$$\Delta^F(x)=2\mathrm{i}\left[\Delta^{(-)}(x)\theta_+(t)-\Delta^+(x)\theta_-(t)\right]$$

$$D^F(x)=2\mathrm{i}\left[D^{(-)}(x)\theta_+(t)-D^+(x)\theta_-(t)\right]$$

利用

$$\Delta^{(\pm)}(x-x')=-\Delta^{(\mp)}(x'-x)$$

$$D^{(\pm)}(x-x')=-D^{(\mp)}(x'-x)$$

$$\theta_\pm(t)=\theta_\mp(-t)$$

有：

$$\begin{aligned}\Delta^F(x-x') &= 2\mathrm{i}\left[\Delta^{(-)}(x-x')\theta_+(t-t') - \Delta^{(+)}(x-x')\theta_-(t-t')\right] \\ &= 2\mathrm{i}\left[-\Delta^{(+)}(x'-x)\theta_+(t-t') + \Delta^{(-)}(x'-x)\theta_-(t-t')\right] \\ &= 2\mathrm{i}\left[-\Delta^{(+)}(x'-x)\theta_-(t'-t) + \Delta^{(-)}(x'-x)\theta_+(t'-t)\right] \\ &= \Delta^F(x'-x)\end{aligned}$$

同理有：

$$D^F(x-x') = D^F(x'-x)$$

4-2-9

> 讨论 $\overline{A_\mu(x_1)A_\nu(x_2)}, \overline{\psi(x_1)\psi(x_2)}$ 的物理意义。

$\overline{A_\mu(x_1)A_\nu(x_2)} \equiv \langle 0 \,|\, P\left[A_\mu(x_1)A_\nu(x_2)\right] \,|\, 0\rangle$ 是矢量场 $A_\mu(x)$ 两点的关联函数，代表光子的传播振幅，即从时空点 x_2 发出一个光子，在不确定的中间过程中经过演化，在时空点 x_1 被探测到的概率振幅。

$\overline{\psi(x_1)\psi(x_2)}$ 是旋量场的两点关联函数，描述电子（或正电子）从 x_2 传播到 x_1 的概率幅。

4-2-10

> 利用旋量场算符满足的 Heisenberg 方程，证明
>
> $$\left[H, a_i^{(+)}(\vec{p})\right] = E_{\vec{p}} a_i^{(+)}(\vec{p})$$
>
> $$\left[H, b_i^{(-)}(\vec{p})\right] = -E_{\vec{p}} b_i^{(-)}(\vec{p})$$

旋量场 Fourier 表示：

$$\hat{\psi}^{(+)}(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{-\mathrm{i}px} \hat{b}_i^{(+)}(\vec{p})\, v_i(\vec{p})\, \mathrm{d}^3\vec{p}$$

$$\hat{\psi}^{(-)}(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{a}_i^{(-)}(\vec{p})\, u_i(\vec{p})\, \mathrm{d}^3\vec{p}$$

$$\hat{\bar{\psi}}^{(+)}(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{-\mathrm{i}px} \hat{a}_i^{(+)}(\vec{p})\, \bar{u}_i(\vec{p})\, \mathrm{d}^3\vec{p}$$

$$\hat{\bar{\psi}}^{(-)}(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}(\vec{p})\, \bar{v}_i(\vec{p})\, \mathrm{d}^3\vec{p}$$

海森堡方程：

$$\partial_t \hat{\bar{\psi}}^{(+)}(x) = \mathrm{i}\left[\hat{H}, \hat{\bar{\psi}}(x)\right]$$

左边：

$$\begin{aligned}\partial_t \hat{\bar{\psi}}^{(+)}(x) &= \partial_t \left\{ \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{-\mathrm{i}px} \hat{a}_i^{(+)}(\vec{p})\, \bar{u}_i(\vec{p})\, \mathrm{d}^3\vec{p} \right\} \\ &= \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{i}E_{\vec{p}} \mathrm{e}^{-\mathrm{i}px} \hat{a}_i^{(+)}(\vec{p})\, \bar{u}_i(\vec{p})\, \mathrm{d}^3\vec{p}\end{aligned}$$

右边：

$$\begin{aligned}\mathrm{i}\left[\hat{H}, \hat{\bar{\psi}}^{(+)}(x)\right] &= \mathrm{i}\left[\hat{H}, \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{-\mathrm{i}px} \hat{a}_i^{(+)}(\vec{p})\, \bar{u}_i(\vec{p})\, \mathrm{d}^3\vec{p}\right] \\ &= \mathrm{i}\left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{-\mathrm{i}px} \left[\hat{H}, \hat{a}_i^{(+)}(\vec{p})\right] \bar{u}_i(\vec{p})\, \mathrm{d}^3\vec{p}\end{aligned}$$

对比可得：

$$\boxed{\left[\hat{H}, \hat{a}_i^{(+)}(\vec{p})\right] = E_{\vec{p}} a_i^{(+)}(\vec{p})}$$

海森堡方程：

$$\partial_t \hat{\bar{\psi}}^{(-)}(x) = \mathrm{i}\left[\hat{H}, \hat{\bar{\psi}}^{(-)}(x)\right]$$

左边：

$$\begin{aligned}\partial_t \hat{\bar{\psi}}^{(-)}(x) &= \partial_t \left\{ \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}(\vec{p})\, \bar{v}_i(\vec{p})\, \mathrm{d}^3\vec{p} \right\} \\ &= \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} -\mathrm{i}E_{\vec{p}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}(\vec{p})\, \bar{v}_i(\vec{p})\, \mathrm{d}^3\vec{p}\end{aligned}$$

右边：

$$\begin{aligned}\mathrm{i}\left[\hat{H}, \hat{\bar{\psi}}^{(-)}(x)\right] &= \mathrm{i}\left[\hat{H}, \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}(\vec{p})\, \bar{v}_i(\vec{p})\, \mathrm{d}^3\vec{p}\right] \\ &= \mathrm{i}\left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{p_0=E_{\vec{p}}} \mathrm{e}^{\mathrm{i}px} \left[\hat{H}, \hat{b}_i^{(-)}(\vec{p})\right] \bar{v}_i(\vec{p})\, \mathrm{d}^3\vec{p}\end{aligned}$$

对比可得：

$$\boxed{\left[\hat{H}, \hat{b}_i^{(-)}(\vec{p})\right] = -E_{\vec{p}} \hat{b}_i^{(-)}(\vec{p})}$$