1

1.1

> 证明弧元 $\mathrm{d}s^2 \equiv e g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ 是坐标变换下的不变量。

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu = A^\mu_\nu \mathrm{d}x^\nu \tag{1}$$

$$\begin{aligned}
\mathrm{d}s'^2 &\equiv e g'_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu \\
&= e\bar{A}^\alpha_\mu \bar{A}^\beta_\nu g_{\alpha\beta}\left(A^\mu_\lambda \mathrm{d}x^\lambda\right)\left(A^\nu_\rho \mathrm{d}x^\rho\right) \\
&= e g_{\alpha\beta}\left(\bar{A}^\alpha_\mu A^\mu_\lambda\right)\left(\bar{A}^\beta_\nu A^\nu_\rho\right)\mathrm{d}x^\lambda \mathrm{d}x^\rho \\
&= e g_{\alpha\beta}\delta^\alpha_\lambda \delta^\beta_\rho \mathrm{d}x^\lambda \mathrm{d}x^\rho \\
&= e g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta
\end{aligned} \tag{2}$$

1.2

> 由协变微商的协变性推导联络在坐标变换下的变换式。

由定义

$$\nabla_\mu \phi^\nu(x) \equiv \partial_\mu \phi^\nu(x) + \Gamma^\nu_{\mu\lambda}\phi^\lambda(x) \tag{3}$$

$$\nabla'_\mu \phi'^\nu(x') \equiv \partial'_\mu \phi'^\nu(x) + \Gamma'^\nu_{\mu\lambda}\phi'^\lambda(x') \tag{4}$$

我们知道 ∂_μ 是协变矢量，$\phi^\nu(x)$ 是逆变矢量，利用它们的变换规律

$$\partial'_\mu = \bar{A}^\alpha_\mu \partial_\alpha, \quad \phi'^\nu(x') = A^\nu_\beta \phi^\beta(x) \tag{5}$$

有

$$\begin{aligned}
\nabla'_\mu \phi'^\nu(x') &\equiv \partial'_\mu \phi'^\nu(x) + \Gamma'^\nu_{\mu\lambda}\phi'^\lambda(x') \\
&= \bar{A}^\alpha_\mu \partial_\alpha\left[A^\nu_\beta \phi^\beta(x)\right] + \Gamma'^\nu_{\mu\lambda}A^\lambda_\gamma \phi^\gamma(x) \\
&= \bar{A}^\alpha_\mu\left(\partial_\alpha A^\nu_\beta\right)\phi^\beta(x) + \bar{A}^\alpha_\mu A^\nu_\beta \partial_\alpha \phi^\beta(x) + \Gamma'^\nu_{\mu\lambda}A^\lambda_\gamma \phi^\gamma(x)
\end{aligned} \tag{6}$$

而我们希望逆变矢量的协变微商是一个张量，其满足张量的变换规律

$$\begin{aligned}
\nabla'_\mu \phi'^\nu(x') &= \bar{A}^\alpha_\mu A^\nu_\beta \nabla_\alpha \phi^\beta(x) \\
&= \bar{A}^\alpha_\mu A^\nu_\beta\left[\partial_\alpha \phi^\beta(x) + \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x)\right] \\
&= \bar{A}^\alpha_\mu A^\nu_\beta \partial_\alpha \phi^\beta(x) + \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x)
\end{aligned} \tag{7}$$

有

$$\bar{A}^\alpha_\mu\left(\partial_\alpha A^\nu_\beta\right)\phi^\beta(x) + \bar{A}^\alpha_\mu A^\nu_\beta \partial_\alpha \phi^\beta(x) + \Gamma'^\nu_{\mu\lambda}A^\lambda_\gamma \phi^\gamma(x) = \bar{A}^\alpha_\mu A^\nu_\beta \partial_\alpha \phi^\beta(x) + \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x) \tag{8}$$

化简为

$$\Gamma'^\nu_{\mu\lambda}A^\lambda_\gamma \phi^\gamma(x) = \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x) - \bar{A}^\alpha_\mu\left(\partial_\alpha A^\nu_\beta\right)\phi^\beta(x) \tag{9}$$

替换哑标得

$$\Gamma'^\nu_{\mu\lambda}A^\lambda_\gamma \phi^\gamma(x) = \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x) - \bar{A}^\alpha_\mu\left(\partial_\alpha A^\nu_\gamma\right)\phi^\gamma(x) \tag{10}$$

因此有

$$\Gamma'^\nu_{\mu\lambda}A^\lambda_\gamma = \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma} - \bar{A}^\alpha_\mu\left(\partial_\alpha A^\nu_\gamma\right) \tag{11}$$

两边同乘 $\bar{A}^{\gamma}_{\rho}$，并对 γ 求和，(11) 式左边

$$\begin{aligned}\Gamma'^{\nu}_{\mu\lambda}A^{\lambda}_{\gamma}\bar{A}^{\gamma}_{\rho} &= \Gamma'^{\nu}_{\mu\lambda}\delta^{\lambda}_{\rho} \\ &= \Gamma'^{\nu}_{\mu\rho}\end{aligned} \tag{12}$$

(11) 式右边

$$\begin{aligned}\bar{A}^{\gamma}_{\rho}\left[\bar{A}^{\alpha}_{\mu}A^{\nu}_{\beta}\Gamma^{\beta}_{\alpha\gamma} - \bar{A}^{\alpha}_{\mu}\left(\partial_{\alpha}A^{\nu}_{\gamma}\right)\right] &= \bar{A}^{\gamma}_{\rho}\bar{A}^{\alpha}_{\mu}A^{\nu}_{\beta}\Gamma^{\beta}_{\alpha\gamma} - \bar{A}^{\alpha}_{\mu}\bar{A}^{\gamma}_{\rho}\partial_{\alpha}A^{\nu}_{\gamma} \\ &= \bar{A}^{\gamma}_{\rho}\bar{A}^{\alpha}_{\mu}A^{\nu}_{\beta}\Gamma^{\beta}_{\alpha\gamma} - \bar{A}^{\alpha}_{\mu}\left[\partial_{\alpha}\left(\bar{A}^{\gamma}_{\rho}A^{\nu}_{\gamma}\right) - A^{\nu}_{\gamma}\partial_{\alpha}\bar{A}^{\gamma}_{\rho}\right] \\ &= \bar{A}^{\gamma}_{\rho}\bar{A}^{\alpha}_{\mu}A^{\nu}_{\beta}\Gamma^{\beta}_{\alpha\gamma} - \bar{A}^{\alpha}_{\mu}\left[\partial_{\alpha}\delta^{\nu}_{\rho} - A^{\nu}_{\gamma}\partial_{\alpha}\bar{A}^{\gamma}_{\rho}\right] \\ &= \bar{A}^{\gamma}_{\rho}\bar{A}^{\alpha}_{\mu}A^{\nu}_{\beta}\Gamma^{\beta}_{\alpha\gamma} + \bar{A}^{\alpha}_{\mu}A^{\nu}_{\gamma}\partial_{\alpha}\bar{A}^{\gamma}_{\rho} \\ &= A^{\nu}_{\beta}\bar{A}^{\alpha}_{\mu}\bar{A}^{\gamma}_{\rho}\Gamma^{\beta}_{\alpha\gamma} + A^{\nu}_{\gamma}\bar{A}^{\alpha}_{\mu}\partial_{\alpha}\bar{A}^{\gamma}_{\rho}\end{aligned} \tag{13}$$

于是得到联络的变换规律

$$\Gamma'^{\nu}_{\mu\rho} = A^{\nu}_{\beta}\bar{A}^{\alpha}_{\mu}\bar{A}^{\gamma}_{\rho}\Gamma^{\beta}_{\alpha\gamma} + A^{\nu}_{\gamma}\bar{A}^{\alpha}_{\mu}\partial_{\alpha}\bar{A}^{\gamma}_{\rho} \tag{14}$$

替换指标就得到

$$\Gamma'^{\mu}_{\nu\lambda} = A^{\mu}_{\alpha}\bar{A}^{\beta}_{\nu}\bar{A}^{\gamma}_{\lambda}\Gamma^{\alpha}_{\beta\gamma} + A^{\mu}_{\alpha}\bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\lambda} \tag{15}$$

1.3

> 证明挠率 $\Gamma^{\lambda}_{[\mu,\nu]}$ 是一个张量。

$$\Gamma'^{\lambda}_{\mu\nu} = A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} + A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\partial_{\beta}\bar{A}^{\alpha}_{\nu} \tag{16}$$

$$\Gamma'^{\lambda}_{\nu\mu} = A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\bar{A}^{\gamma}_{\mu}\Gamma^{\alpha}_{\beta\gamma} + A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\mu} \tag{17}$$

注意到

$$\begin{aligned}\bar{A}^{\beta}_{\mu}\partial_{\beta}\bar{A}^{\alpha}_{\nu} - \bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\mu} &\equiv \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}} - \frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \\ &= \frac{\partial}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}} - \frac{\partial}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \\ &= 0\end{aligned} \tag{18}$$

因此

$$\begin{aligned}\Gamma'^{\lambda}_{[\mu,\nu]} &\equiv \Gamma'^{\lambda}_{\mu\nu} - \Gamma'^{\lambda}_{\nu\mu} \\ &= \left(A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} + A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\partial_{\beta}\bar{A}^{\alpha}_{\nu}\right) - \left(A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\bar{A}^{\gamma}_{\mu}\Gamma^{\alpha}_{\beta\gamma} + A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\mu}\right) \\ &= \left(A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} - A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\bar{A}^{\gamma}_{\mu}\Gamma^{\alpha}_{\beta\gamma}\right) + A^{\lambda}_{\alpha}\left(\bar{A}^{\beta}_{\mu}\partial_{\beta}\bar{A}^{\alpha}_{\nu} - \bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\mu}\right) \\ &= A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} - A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\bar{A}^{\gamma}_{\mu}\Gamma^{\alpha}_{\beta\gamma} \\ &= A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} - A^{\lambda}_{\alpha}\bar{A}^{\gamma}_{\nu}\bar{A}^{\beta}_{\mu}\Gamma^{\alpha}_{\gamma\beta} \\ &= A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\left(\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}\right) \\ &= A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{[\beta,\gamma]}\end{aligned} \tag{19}$$

1.4

> 由 $\nabla_{\lambda}g_{\mu\nu} = 0$ 证明 $\nabla_{\lambda}g^{\mu\nu} = 0$.

$$g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu} \tag{20}$$

两边用协变微商作用

$$\nabla_{\lambda}\left(g^{\mu\alpha}g_{\alpha\nu}\right) = 0 \tag{21}$$

利用协变微商的莱布尼兹律

$$\begin{aligned}\nabla_\lambda\left(g^{\mu\alpha}g_{\alpha\nu}\right)&=\left(\nabla_\lambda g^{\mu\alpha}\right)g_{\alpha\nu}+g^{\mu\alpha}\left(\nabla_\lambda g_{\alpha\nu}\right)\\&=\left(\nabla_\lambda g^{\mu\alpha}\right)g_{\alpha\nu}\end{aligned}\tag{22}$$

得到

$$\left(\nabla_\lambda g^{\mu\alpha}\right)g_{\alpha\nu}=0\tag{23}$$

上式两边乘 $g^{\nu\beta}$ 并对 ν 求和

$$\begin{aligned}0&=\left(\nabla_\lambda g^{\mu\alpha}\right)g_{\alpha\nu}g^{\nu\beta}\\&=\left(\nabla_\lambda g^{\mu\alpha}\right)\delta_\alpha^\beta\\&=\nabla_\lambda g^{\mu\beta}\end{aligned}\tag{24}$$

1.5

> 假设有一对称张量 $f_{\mu\nu}$ 及其逆 $f^{\mu\nu}$ 可对张量指标进行升降
>
> $$f_{\mu\nu}\phi^\nu=\phi_\mu,\quad f^{\mu\nu}\phi_\nu=\phi^\mu\tag{25}$$
>
> 由协变微商定义及其性质，证明此张量的协变微商为零，即
>
> $$\nabla_\lambda f_{\mu\nu}=0\tag{26}$$

$$\phi_\mu=f_{\mu\nu}\phi^\nu\tag{27}$$

$$\begin{aligned}\nabla_\lambda\phi_\mu&=\nabla_\lambda\left(f_{\mu\nu}\phi^\nu\right)\\&=\left(\nabla_\lambda f_{\mu\nu}\right)\phi^\nu+f_{\mu\nu}\left(\nabla_\lambda\phi^\nu\right)\\&=\left(\nabla_\lambda f_{\mu\nu}\right)\phi^\nu+\nabla_\lambda\phi_\mu\end{aligned}\tag{28}$$

前后对比得

$$\left(\nabla_\lambda f_{\mu\nu}\right)\phi^\nu=0\tag{29}$$

由 ϕ^ν 的任意性有

$$\nabla_\lambda f_{\mu\nu}=0\tag{30}$$

2

2.1

> 已知 $\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=-\mathrm{d}\tau^2$，从变分原理 $\delta\int_A^B\mathrm{d}s=0$ 或 $\delta\int_A^B\left(\mathrm{d}\tau/\mathrm{d}\lambda\right)^2\mathrm{d}\lambda=0$ 求出“短”程线方程。

段先生书上已经有对线长变分求短程线的过程，也就是从

$$\delta\int_{\lambda_1}^{\lambda_2}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\mathrm{d}\lambda=0\tag{31}$$

出发找到 $x^\mu(\lambda)$ 要满足的方程。

这里

$$\mathrm{d}\tau=\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}=\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\mathrm{d}\lambda,\tag{32}$$

也就是从

$$\delta \int_{\lambda_1}^{\lambda_2} -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \mathrm{d}\lambda = 0 \tag{33}$$

出发推导短程线方程。也即

$$\int_{\lambda_1}^{\lambda_2} \delta(-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)\mathrm{d}\lambda = 0 \tag{34}$$

令:

$$F(x,\dot{x}) = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu, \tag{35}$$

$$\begin{aligned}
\delta F &= -\delta\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) \\
&= -\left[\dot{x}^\mu \dot{x}^\nu \delta g_{\mu\nu} + g_{\mu\nu}\dot{x}^\nu \delta\dot{x}^\mu + g_{\mu\nu}\dot{x}^\mu \delta \dot{x}^\nu\right] \\
&= -\left[\dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu}\delta x^\alpha + g_{\mu\nu}\dot{x}^\nu \frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\mu + g_{\mu\nu}\dot{x}^\mu \frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\nu\right] \\
&= -\left[\dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu}\delta x^\alpha + g_{\alpha\nu}\dot{x}^\nu \frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\alpha + g_{\mu\alpha}\dot{x}^\mu \frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\alpha\right] \\
&= -\left\{\dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu}\delta x^\alpha + \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\delta x^\alpha\right] - \delta x^\alpha \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\right\} \\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\delta x^\alpha\right] + \delta x^\alpha\left[\left(\dot{x}^\nu \partial_\beta g_{\alpha\nu}\dot{x}^\beta + g_{\alpha\nu}\ddot{x}^\nu + \dot{x}^\mu \partial_\beta g_{\mu\alpha}\dot{x}^\beta + g_{\mu\alpha}\ddot{x}^\mu\right) - \dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu}\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\delta x^\alpha\right] + \delta x^\alpha\left[\left(\dot{x}^\nu \partial_\mu g_{\alpha\nu}\dot{x}^\mu + g_{\alpha\mu}\ddot{x}^\mu + \dot{x}^\mu \partial_\nu g_{\mu\alpha}\dot{x}^\nu + g_{\alpha\mu}\ddot{x}^\mu\right) - \dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu}\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\delta x^\alpha\right] + \delta x^\alpha\left[\dot{x}^\mu \dot{x}^\nu\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) + 2g_{\alpha\mu}\ddot{x}^\mu\right]
\end{aligned} \tag{36}$$

由 $\int_{\lambda_1}^{\lambda}\delta F \mathrm{d}\lambda$ 可得

$$\dot{x}^\mu \dot{x}^\nu\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) + 2g_{\alpha\mu}\ddot{x}^\mu = 0 \tag{37}$$

上式两边同乘 $g_{\lambda\alpha}$ 可得

$$\Gamma^\lambda_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \ddot{x}^\lambda = 0 \tag{38}$$

也即短程线方程:

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\lambda^2} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0. \tag{39}$$

2.2

> 已知 $\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu}A_\lambda$，利用标量微分关系 $\nabla_\nu U = \partial_\nu U$ 以及莱布尼茨法则证明 $\nabla_\nu B^\mu = \partial_\nu B^\mu + \Gamma^\mu_{\nu\lambda}B^\lambda$.

一方面，协变微商满足莱布尼兹法则:

$$\begin{aligned}
\nabla_\nu\left(A_\mu B^\mu\right) &= B^\mu \nabla_\nu A_\mu + A_\mu \nabla_\nu B^\mu \\
&= B^\mu\left(\partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu}A_\lambda\right) + A_\mu \nabla_\nu B^\mu \\
&= B^\mu \partial_\nu A_\mu - B^\mu \Gamma^\lambda_{\nu\mu}A_\lambda + A_\mu \nabla_\nu B^\mu
\end{aligned} \tag{40}$$

另一方面，$A_\mu B^\mu$ 是标量，其协变微商等于普通偏微分:

$$\begin{aligned}
\nabla_\nu\left(A_\mu B^\mu\right) &= \partial_\nu\left(A_\mu B^\mu\right) \\
&= B^\mu \partial_\nu A_\mu + A_\mu \partial_\nu B^\mu
\end{aligned} \tag{41}$$

对比可得:

$$\begin{aligned}
A_\mu \nabla_\nu B^\mu &= A_\mu \partial_\nu B^\mu + B^\mu \Gamma^\lambda_{\nu\mu}A_\lambda \\
&= A_\mu \partial_\nu B^\mu + A_\lambda \Gamma^\lambda_{\nu\mu}B^\mu \\
&= A_\mu \partial_\nu B^\mu + A_\mu \Gamma^\mu_{\nu\lambda}B^\lambda
\end{aligned} \tag{42}$$

从而得到：

$$\nabla_\nu B^\mu = \partial_\nu B^\mu + \Gamma^\mu_{\nu\lambda} B^\lambda \tag{43}$$

2.3

> 一个嵌入三维欧氏空间的普通球面，选用球极坐标，其线元为 $\mathrm{d}s^2 = a^2\mathrm{d}\theta^2 + a^2\sin^2\theta\mathrm{d}\phi^2$，求：(1) $g^{\nu\mu}$；(2) 全部非零克氏符 $\Gamma^\lambda_{\mu\nu}$；(3) 全部非零 $R^\nu_{\mu\sigma\lambda}, R_{\mu\nu}, R$；(4) 写出该度规表示的球面空间的测地线方程。

度规

取 $\left(x^1, x^2\right) = (\theta, \phi)$，根据线元

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2\mathrm{d}\theta^2 + a^2\sin^2\theta\mathrm{d}\phi^2 \tag{44}$$

可得度规

$$g_{11} = a^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = a^2\sin^2\theta, \tag{45}$$

$$[g_{\mu\nu}] = \begin{bmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{bmatrix} \tag{46}$$

以及逆度规

$$[g^{\mu\nu}] = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2\sin^2\theta} \end{bmatrix} \tag{47}$$

$$g^{11} = \frac{1}{a^2}, \quad g^{12} = g^{21} = 0, \quad g^{22} = \frac{1}{a^2\sin^2\theta}, \tag{48}$$

克氏符

由度规 $g_{\mu\nu}$ 以及逆度规 $g^{\mu\nu}$ 可得克氏符：

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right), \tag{49}$$

$$\begin{aligned}
\Gamma^1_{\mu\nu} &= \frac{1}{2}g^{1\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \\
&= \frac{1}{2}g^{11}\left(\partial_\mu g_{1\nu} + \partial_\nu g_{1\mu} - \partial_1 g_{\mu\nu}\right) \\
&= \frac{1}{2a^2}\left(-\partial_1 g_{\mu\nu}\right) \\
&= -\frac{1}{2a^2}\partial_1 g_{\mu\nu}
\end{aligned} \tag{50}$$

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = 0, \tag{51}$$

$$\Gamma^1_{22} = -\frac{1}{2a^2}\partial_1 g_{22} = -\frac{1}{2a^2}\partial_\theta\left(a^2\sin^2\theta\right) = -\sin\theta\cos\theta, \tag{52}$$

$$\begin{aligned}
\Gamma^2_{\mu\nu} &= \frac{1}{2}g^{2\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \\
&= \frac{1}{2}g^{22}\left(\partial_\mu g_{2\nu} + \partial_\nu g_{2\mu} - \partial_2 g_{\mu\nu}\right) \\
&= \frac{1}{2}g^{22}\left(\partial_\mu g_{2\nu} + \partial_\nu g_{2\mu}\right) \\
&= \frac{1}{2a^2\sin^2\theta}\left(\partial_\mu g_{2\nu} + \partial_\nu g_{2\mu}\right)
\end{aligned} \tag{53}$$

$$\Gamma^2_{11} = \Gamma^2_{22} = 0, \tag{54}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2a^2\sin^2\theta}\left(\partial_1 g_{22} + \partial_2 g_{21}\right) = \frac{1}{2a^2\sin^2\theta}\partial_\theta\left(a^2\sin^2\theta\right) = \frac{\cos\theta}{\sin\theta} = \cot\theta \tag{55}$$

Riemann张量

$$R^{\lambda}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\nu\sigma} - \partial_\nu \Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\alpha}\Gamma^{\alpha}_{\mu\sigma} \tag{56}$$

$$R^{1}_{212} = \partial_1 \Gamma^{1}_{22} - \partial_2 \Gamma^{1}_{12} + \Gamma^{1}_{1\alpha}\Gamma^{\alpha}_{22} - \Gamma^{1}_{2\alpha}\Gamma^{\alpha}_{12} = \sin^2\theta \tag{57}$$

- $\lambda = 1$
  - $(\mu, \nu) = (1, 2)$

$$\begin{aligned} R^{1}_{\sigma 12} &= \partial_1 \Gamma^{1}_{2\sigma} - \partial_2 \Gamma^{1}_{1\sigma} + \Gamma^{1}_{1\alpha}\Gamma^{\alpha}_{2\sigma} - \Gamma^{1}_{2\alpha}\Gamma^{\alpha}_{1\sigma} \\ &= \partial_1 \Gamma^{1}_{2\sigma} - \Gamma^{1}_{22}\Gamma^{2}_{1\sigma} \end{aligned} \tag{58}$$

    - $\sigma = 1$

$$\begin{aligned} R^{1}_{112} &= \partial_1 \Gamma^{1}_{21} - \Gamma^{1}_{22}\Gamma^{2}_{11} \\ &= -\Gamma^{1}_{22}\Gamma^{2}_{11} \\ &= 0 \end{aligned} \tag{59}$$

    - $\sigma = 2$

$$\begin{aligned} R^{1}_{212} &= \partial_1 \Gamma^{1}_{22} - \Gamma^{1}_{22}\Gamma^{2}_{12} \\ &= \sin^2\theta \end{aligned} \tag{60}$$

$$R^{1}_{221} = -R^{1}_{212} = -\sin^2\theta \tag{61}$$

- $\lambda = 2$
  - $(\mu, \nu) = (1, 2)$

$$\begin{aligned} R^{2}_{\sigma 12} &= \partial_1 \Gamma^{2}_{2\sigma} - \partial_2 \Gamma^{2}_{1\sigma} + \Gamma^{2}_{1\alpha}\Gamma^{\alpha}_{2\sigma} - \Gamma^{2}_{2\alpha}\Gamma^{\alpha}_{1\sigma} \\ &= \partial_1 \Gamma^{2}_{2\sigma} + \Gamma^{2}_{12}\Gamma^{2}_{2\sigma} - \Gamma^{2}_{21}\Gamma^{1}_{1\sigma} \\ &= \partial_1 \Gamma^{2}_{2\sigma} + \cot\theta\,\Gamma^{2}_{2\sigma} - \cot\theta\,\Gamma^{1}_{1\sigma} \end{aligned} \tag{62}$$

    - $\sigma = 1$

$$\begin{aligned} R^{2}_{112} &= \partial_1 \Gamma^{2}_{21} + \cot\theta\,\Gamma^{2}_{21} - \cot\theta\,\Gamma^{1}_{11} \\ &= \partial_\theta \cot\theta + \cot\theta \cdot \cot\theta \\ &= -1 \end{aligned} \tag{63}$$

$$R^{2}_{121} = -R^{2}_{112} = 1 \tag{64}$$

    - $\sigma = 2$

$$\begin{aligned} R^{2}_{212} &= \partial_1 \Gamma^{2}_{22} + \cot\theta\,\Gamma^{2}_{22} - \cot\theta\,\Gamma^{1}_{12} \\ &= 0 \end{aligned} \tag{65}$$

Ricci张量

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = R^{1}_{\mu 1\nu} + R^{2}_{\mu 2\nu} \tag{66}$$

- $(\mu, \nu) = (1, 1)$

$$R_{11} = R^{1}_{111} + R^{2}_{121} = 1 \tag{67}$$

- $(\mu, \nu) = (1, 2)$

$$R_{12} = R^{1}_{112} + R^{2}_{122} = 0 \tag{68}$$

- $(\mu, \nu) = (2, 1)$

$$R_{21} = R^{1}_{211} + R^{2}_{221} = 0 \tag{69}$$

- $(\mu, \nu) = (2, 2)$

$$R_{22}=R^1_{212}+R^2_{222}=\sin^2\theta \tag{70}$$

曲率标量

$$R=g^{\mu\nu}R_{\mu\nu}=g^{11}R_{11}+g^{22}R_{22}=\frac{2}{a^2} \tag{71}$$

测地线方程

测地线方程：

$$\ddot{x}^\lambda+\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0 \tag{72}$$

- $\lambda=1$

$$\ddot{x}^1+\Gamma^1_{22}\dot{x}^2\dot{x}^2=0 \tag{73}$$

$$\ddot{\theta}-\sin\theta\cos\theta\dot{\phi}^2=0 \tag{74}$$

- $\lambda=2$

$$\ddot{x}^2+\Gamma^2_{12}\dot{x}^1\dot{x}^2+\Gamma^2_{21}\dot{x}^2\dot{x}^1=0 \tag{75}$$

$$\ddot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}=0 \tag{76}$$

2.4

> 由协变矢量双重协变微商非对称部分 $\phi_{\lambda;[\mu;\nu]}$ 推导曲率张量与挠率张量 $\nabla_\mu\nabla_\nu\phi_\lambda-\nabla_\nu\nabla_\mu\phi_\lambda=-R^\sigma_{\lambda\mu\nu}\phi_\sigma-T^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda$.

$$\begin{aligned}
\nabla_\mu\nabla_\nu\phi_\lambda-\mu\leftrightarrow\nu &= \nabla_\mu\left(\nabla_\nu\phi_\lambda\right)-\mu\leftrightarrow\nu\\
&=\partial_\mu\left(\nabla_\nu\phi_\lambda\right)-\Gamma^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda-\Gamma^\alpha_{\mu\lambda}\nabla_\nu\phi_\alpha-\mu\leftrightarrow\nu\\
&=\partial_\mu\left(\partial_\nu\phi_\lambda-\Gamma^\alpha_{\nu\lambda}\phi_\alpha\right)-\Gamma^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda-\Gamma^\alpha_{\mu\lambda}\left(\partial_\nu\phi_\alpha-\Gamma^\beta_{\nu\alpha}\phi_\beta\right)-\mu\leftrightarrow\nu\\
&=\partial_\mu\partial_\nu\phi_\lambda-\left(\partial_\mu\Gamma^\alpha_{\nu\lambda}\right)\phi_\alpha-\Gamma^\alpha_{\nu\lambda}\partial_\mu\phi_\alpha-\Gamma^\alpha_{\mu\lambda}\partial_\nu\phi_\alpha+\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\nu\alpha}\phi_\beta-\Gamma^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda-\mu\leftrightarrow\nu\\
&=-\left(\partial_\mu\Gamma^\alpha_{\nu\lambda}-\partial_\nu\Gamma^\alpha_{\mu\lambda}\right)\phi_\alpha+\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\nu\alpha}\phi_\beta-\Gamma^\alpha_{\nu\lambda}\Gamma^\beta_{\mu\alpha}\phi_\beta-\left(\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}\right)\nabla_\alpha\phi_\lambda\\
&=-\left(\partial_\mu\Gamma^\beta_{\nu\lambda}-\partial_\nu\Gamma^\beta_{\mu\lambda}\right)\phi_\beta+\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\nu\alpha}\phi_\beta-\Gamma^\alpha_{\nu\lambda}\Gamma^\beta_{\mu\alpha}\phi_\beta-\left(\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}\right)\nabla_\alpha\phi_\lambda\\
&=-\left(\partial_\mu\Gamma^\beta_{\nu\lambda}-\partial_\nu\Gamma^\beta_{\mu\lambda}+\Gamma^\alpha_{\nu\lambda}\Gamma^\beta_{\mu\alpha}-\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\nu\alpha}\right)\phi_\beta-\left(\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}\right)\nabla_\alpha\phi_\lambda\\
&=-R^\beta_{\lambda\mu\nu}\phi_\beta-T^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda
\end{aligned} \tag{77}$$

2.5

> 试求出适用任意时轴正交时空、任意观者的光速表达式，并由此验证广义相对论中的光速不变原理。提示：利用类光线元，并只考虑光线沿径向传播的简单情形。

$$\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{78}$$

时轴正交时空，$g_{0i}=0$，线元可写为

$$\mathrm{d}s^2=g_{tt}c^2\mathrm{d}t^2+g_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{79}$$

考虑光线沿径向传播，则类光线元满足

$$\mathrm{d}s^2=g_{tt}c^2\mathrm{d}t^2+g_{rr}\mathrm{d}r^2=0 \tag{80}$$

其中

$$g_{tt}<0,\quad g_{rr}>0 \tag{81}$$

考虑任意观者 $x^\mu(\tau)$，其四速度 $u^\mu(\tau)$ 满足

$$g_{\mu\nu}u^\mu u^\nu=-c^2 \tag{82}$$

观者观测的时间间隔为

$$\mathrm{d}t_{\mathrm{obs}}=-\frac{1}{c^2}u_\mu\mathrm{d}x^\mu=-\frac{1}{c^2}\left(u_0\mathrm{d}x^0+u_1\mathrm{d}x^1\right) \tag{83}$$

观者观测的空间距离为

$$\mathrm{d}l_{\mathrm{obs}}=\sqrt{h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}=\sqrt{\left(g_{\mu\nu}+\frac{u_\mu u_\nu}{c^2}\right)\mathrm{d}x^\mu\mathrm{d}x^\nu} \tag{84}$$

对于类光线元，

$$\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=0 \tag{85}$$

则

$$\mathrm{d}l_{\mathrm{obs}}=\sqrt{\frac{1}{c^2}\left(u_\mu\mathrm{d}x^\mu\right)^2}=-\frac{1}{c}u_\mu\mathrm{d}x^\mu \tag{86}$$

测量光速

$$c_{\mathrm{obs}}=\frac{\mathrm{d}l_{\mathrm{obs}}}{\mathrm{d}t_{\mathrm{obs}}}=c \tag{87}$$

3

3.1

弱引力场近似下，度规表示为

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu},\quad |h_{\mu\nu}|\ll 1 \tag{88}$$

线性近似理论中只保留 $h_{\mu\nu}$ 中的线性项（一阶小量）。定义

$$\bar{h}_{\mu\nu}\equiv h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h,\quad h\equiv\eta^{\mu\nu}h_{\mu\nu}, \tag{89}$$

试证明，它的逆变换是

$$\bar{\bar{h}}_{\mu\nu}\equiv\bar{h}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\bar{h}=h_{\mu\nu} \tag{90}$$

证明：

$$\begin{aligned}\bar{h}\equiv\eta^{\mu\nu}\bar{h}_{\mu\nu}&=\eta^{\mu\nu}\left(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\right)\\&=\eta^{\mu\nu}h_{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}h\\&=h-\frac{1}{2}\delta^\mu_\mu h\\&=-h\end{aligned} \tag{91}$$

于是

$$\begin{aligned}\bar{\bar{h}}_{\mu\nu}&\equiv\bar{h}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\bar{h}\\&=\left(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\right)-\frac{1}{2}\eta_{\mu\nu}\left(-h\right)\\&=h_{\mu\nu}\end{aligned} \tag{92}$$

3.2

线性近似理论中，证明克氏符

$$\Gamma_{\alpha\beta}^{\mu}=\frac{1}{2}\eta^{\mu\nu}\left(\partial_{\beta}h_{\alpha\nu}+\partial_{\alpha}h_{\beta\nu}-\partial_{\nu}h_{\alpha\beta}\right)=\frac{1}{2}\left(h_{\alpha,\beta}^{\mu}+h_{\beta,\alpha}^{\mu}-h_{\alpha\beta}^{,\mu}\right) \tag{93}$$

线性近似理论中张量指标的升降借助 $\eta_{\mu\nu}$ 和 $\eta^{\mu\nu}$ 进行。并求线性化后的Ricci张量

$$R_{\mu\nu}=\Gamma_{\mu\nu,\lambda}^{\lambda}-\Gamma_{\mu\lambda,\nu}^{\lambda}=-\frac{1}{2}\left(h_{\mu\nu,\alpha}^{,\alpha}+h_{,\mu,\nu}-h_{\mu,\nu,\alpha}^{\alpha}-h_{\nu,\mu,\alpha}^{\alpha}\right) \tag{94}$$

证明:

由于

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu},\quad |h_{\mu\nu}|\ll 1, \tag{95}$$

线性近似理论只保留小量 $h_{\mu\nu}$ 的一阶项，于是

$$\begin{aligned}\Gamma_{\alpha\beta}^{\mu}&\equiv\frac{1}{2}g^{\mu\nu}\left(\partial_{\alpha}g_{\beta\nu}+\partial_{\beta}g_{\alpha\nu}-\partial_{\nu}g_{\alpha\beta}\right)\\&\equiv\frac{1}{2}\left(\eta^{\mu\nu}+h^{\mu\nu}\right)\left[\partial_{\alpha}\left(\eta_{\beta\nu}+h_{\beta\nu}\right)+\partial_{\beta}\left(\eta_{\alpha\nu+h_{\alpha\nu}}\right)-\partial_{\nu}\left(\eta_{\alpha\beta}+h_{\alpha\beta}\right)\right]\\&=\frac{1}{2}\eta^{\mu\nu}\left(\partial_{\beta}h_{\alpha\nu}+\partial_{\alpha}h_{\beta\nu}-\partial_{\nu}h_{\alpha\beta}\right)\\&=\frac{1}{2}\left[\partial_{\beta}\left(\eta^{\mu\nu}h_{\alpha\nu}\right)+\partial_{\alpha}\left(\eta^{\mu\nu}h_{\beta\nu}\right)-\eta^{\mu\nu}\partial_{\nu}\left(h_{\alpha\beta}\right)\right]\\&=\frac{1}{2}\left(\partial_{\beta}h_{\alpha}^{\mu}+\partial_{\alpha}h_{\beta}^{\mu}-\partial^{\mu}h_{\alpha\beta}\right)\\&=\frac{1}{2}\left(h_{\alpha,\beta}^{\mu}+h_{\beta,\alpha}^{\mu}-h_{\alpha\beta}^{,\mu}\right)\end{aligned} \tag{96}$$

克氏符 Γ 是 $h_{\mu\nu}$ 一阶小量的叠加，因此 Riemann 曲率张量中的 $\Gamma\Gamma$ 项可以舍去，此时 Riemann 曲率张量为:

$$\begin{aligned}R_{\mu\alpha\nu}^{\lambda}&\equiv\partial_{\alpha}\Gamma_{\nu\mu}^{\lambda}-\partial_{\nu}\Gamma_{\alpha\mu}^{\lambda}+\Gamma_{\alpha\beta}^{\lambda}\Gamma_{\nu\mu}^{\beta}-\Gamma_{\nu\beta}^{\lambda}\Gamma_{\alpha\mu}^{\beta}\\&=\partial_{\alpha}\Gamma_{\nu\mu}^{\lambda}-\partial_{\nu}\Gamma_{\alpha\mu}^{\lambda}\\&=\frac{1}{2}\left[\partial_{\alpha}\left(h_{\nu,\mu}^{\lambda}+h_{\mu,\nu}^{\lambda}-h_{\nu\mu}^{,\lambda}\right)-\partial_{\nu}\left(h_{\alpha,\mu}^{\lambda}+h_{\mu,\alpha}^{\lambda}-h_{\alpha\mu}^{,\lambda}\right)\right]\end{aligned} \tag{97}$$

Ricci张量为

$$\begin{aligned}R_{\mu\nu}&=R_{\mu\alpha\nu}^{\alpha}\\&=\frac{1}{2}\left[\partial_{\alpha}\left(h_{\nu,\mu}^{\alpha}+h_{\mu,\nu}^{\alpha}-h_{\nu\mu}^{,\alpha}\right)-\partial_{\nu}\left(h_{\alpha,\mu}^{\alpha}+h_{\mu,\alpha}^{\alpha}-h_{\alpha\mu}^{,\alpha}\right)\right]\\&=\frac{1}{2}\left(\partial_{\alpha}h_{\nu,\mu}^{\alpha}+\partial_{\nu}h_{\alpha\mu}^{,\alpha}-\partial_{\alpha}h_{\nu\mu}^{,\alpha}-\partial_{\nu}h_{\alpha,\mu}^{\alpha}\right)\\&=-\frac{1}{2}\left(h_{\nu\mu,\alpha}^{,\alpha}+h_{\alpha,\mu,\nu}^{\alpha}-h_{\nu,\mu,\alpha}^{\alpha}-h_{\alpha\mu,\nu}^{,\alpha}\right)\\&=-\frac{1}{2}\left(h_{\mu\nu,\alpha}^{,\alpha}+h_{\alpha,\mu,\nu}^{\alpha}-h_{\nu,\mu,\alpha}^{\alpha}-h_{\alpha\mu,\nu}^{,\alpha}\right)\end{aligned} \tag{98}$$

注意到

$$h_{\alpha,\mu,\nu}^{\alpha}=\eta^{\lambda\alpha}h_{\lambda\alpha,\mu,\nu}=\left(\eta^{\lambda\alpha}h_{\lambda\alpha}\right)_{,\mu,\nu}=h_{,\mu,\nu} \tag{99}$$

$$h_{\alpha\mu,\nu}^{,\alpha}=\eta^{\alpha\beta}h_{\alpha\mu,\nu,\beta}=h_{\mu,\nu,\beta}^{\beta}=h_{\mu,\nu,\alpha}^{\alpha} \tag{100}$$

因此Ricci张量可化为

$$\begin{aligned}R_{\mu\nu}&=-\frac{1}{2}\left(h_{\mu\nu,\alpha}^{,\alpha}+h_{\alpha,\mu,\nu}^{\alpha}-h_{\nu,\mu,\alpha}^{\alpha}-h_{\alpha\mu,\nu}^{,\alpha}\right)\\&=-\frac{1}{2}\left(h_{\mu\nu,\alpha}^{,\alpha}+h_{,\mu,\nu}-h_{\nu,\mu,\alpha}^{\alpha}-h_{\mu,\nu,\alpha}^{\alpha}\right)\end{aligned} \tag{101}$$

3.3

由以上公式，证明线性化Einstein引力场方程

$$
\bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{102}
$$

具体化为

$$
\bar{h}_{\mu\nu,\alpha}^{,\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta}^{,\alpha,\beta} - \bar{h}_{\mu\alpha,\nu}^{,\alpha} - \bar{h}_{\nu\alpha,\mu}^{,\alpha} = -16\pi G T_{\mu\nu} \tag{103}
$$

证明:

先算Ricci标量:

$$
\begin{aligned}
R &= \eta^{\mu\nu}R_{\mu\nu} \\
&= -\frac{1}{2}\eta^{\mu\nu}\left(h_{\mu\nu,\alpha}^{,\alpha} + h_{,\mu,\nu} - h_{\nu,\mu,\alpha}^{\alpha} - h_{\mu,\nu,\alpha}^{\alpha}\right) \\
&= -\frac{1}{2}\left(h_{,\alpha}^{,\alpha} + h_{,\nu}^{,\nu} - h_{\nu,\alpha}^{\alpha,\nu} - h_{\mu,\alpha}^{\alpha,\mu}\right) \\
&= -\frac{1}{2}\left(2h_{,\alpha}^{,\alpha} - 2h_{\beta,\alpha}^{\alpha,\beta}\right) \\
&= -h_{,\alpha}^{,\alpha} + h_{\beta,\alpha}^{\alpha,\beta}
\end{aligned} \tag{104}
$$

利用

$$
\bar{h} = -h, \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h = h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{h} \Longrightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{105}
$$

$$
h_{\nu}^{\alpha} = \bar{h}_{\nu}^{\alpha} - \frac{1}{2}\delta_{\nu}^{\alpha}\bar{h} \tag{106}
$$

有

$$
\begin{aligned}
\bar{R}_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\
&= -\frac{1}{2}\left(h_{\mu\nu,\alpha}^{,\alpha} + h_{,\mu,\nu} - h_{\nu,\mu,\alpha}^{\alpha} - h_{\mu,\nu,\alpha}^{\alpha}\right) - \frac{1}{2}\eta_{\mu\nu}\left(-h_{,\alpha}^{,\alpha} + h_{\beta,\alpha}^{\alpha,\beta}\right) \\
&= -\frac{1}{2}\left[h_{\mu\nu,\alpha}^{,\alpha} + h_{,\mu,\nu} - h_{\nu,\mu,\alpha}^{\alpha} - h_{\mu,\nu,\alpha}^{\alpha} - \eta_{\mu\nu}h_{,\alpha}^{,\alpha} + \eta_{\mu\nu}h_{\beta,\alpha}^{\alpha,\beta}\right] \\
&= -\frac{1}{2}\left[\left(\bar{h}_{\mu\nu,\alpha}^{,\alpha} - \frac{1}{2}\eta_{\mu\nu}\bar{h}_{,\alpha}^{,\alpha}\right) + \left(-\bar{h}_{,\mu,\nu}\right) - \left(\bar{h}_{\nu,\mu,\alpha}^{\alpha} - \frac{1}{2}\delta_{\nu}^{\alpha}\bar{h}_{,\mu,\alpha}\right) - \left(\bar{h}_{\mu,\nu,\alpha}^{\alpha} - \frac{1}{2}\delta_{\mu}^{\alpha}\bar{h}_{,\nu,\alpha}\right) - \left(-\eta_{\mu\nu}\bar{h}_{,\alpha}^{,\alpha}\right) + \left(\eta_{\mu\nu}\left(\bar{h}_{\beta,\alpha}^{\alpha,\beta} - \frac{1}{2}\delta_{\beta}^{\alpha}\bar{h}_{,\alpha}^{,\beta}\right)\right)\right] \\
&= -\frac{1}{2}\left[\bar{h}_{\mu\nu,\alpha}^{,\alpha} - \bar{h}_{\nu,\mu,\alpha}^{\alpha} - \bar{h}_{\mu,\nu,\alpha}^{\alpha} + \eta_{\mu\nu}\bar{h}_{\beta,\alpha}^{\alpha,\beta}\right] \\
&= -\frac{1}{2}\left[\bar{h}_{\mu\nu,\alpha}^{,\alpha} - \bar{h}_{\nu,\mu,\alpha}^{\alpha} - \bar{h}_{\mu,\nu,\alpha}^{\alpha} + \eta_{\mu\nu}\eta^{\alpha\rho}\bar{h}_{\rho\beta,\alpha}^{,\beta}\right] \\
&= -\frac{1}{2}\left[\bar{h}_{\mu\nu,\alpha}^{,\alpha} - \bar{h}_{\nu,\mu,\alpha}^{\alpha} - \bar{h}_{\mu,\nu,\alpha}^{\alpha} + \eta_{\mu\nu}\bar{h}_{\rho\beta}^{,\rho,\beta}\right] \\
&= -\frac{1}{2}\left[\bar{h}_{\mu\nu,\alpha}^{,\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta}^{,\alpha,\beta} - \bar{h}_{\nu,\mu,\alpha}^{\alpha} - \bar{h}_{\mu,\nu,\alpha}^{\alpha}\right]
\end{aligned} \tag{107}
$$

最后，Einstein引力场方程

$$
\bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{108}
$$

就化为:

$$
-\frac{1}{2}\left[\bar{h}_{\mu\nu,\alpha}^{,\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta}^{,\alpha,\beta} - \bar{h}_{\nu,\mu,\alpha}^{\alpha} - \bar{h}_{\mu,\nu,\alpha}^{\alpha}\right] = 8\pi G T_{\mu\nu} \tag{109}
$$

也即:

$$
\bar{h}_{\mu\nu,\alpha}^{,\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta}^{,\alpha,\beta} - \bar{h}_{\nu,\mu,\alpha}^{\alpha} - \bar{h}_{\mu,\nu,\alpha}^{\alpha} = -16\pi G T_{\mu\nu} \tag{110}
$$

3.4

证明对于静态时空，线性化Ricci

$$R_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu,\lambda}-\Gamma^{\lambda}_{\mu\lambda,\nu}=-\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha}+h_{,\mu,\nu}-h^{\alpha}_{\mu,\nu,\alpha}-h^{\alpha}_{\nu,\mu,\alpha}\right) \tag{111}$$

可化为

$$R_{00}=-\frac{1}{2}h_{00,i,i},\quad R_{0i}=\frac{1}{2}\left(h_{k0,i,k}-h_{0i,k,k}\right) \tag{112}$$

$$R_{ij}=-\frac{1}{2}\left(-h_{00,i,j}+h_{kk,i,j}-h_{ki,j,k}-h_{kj,i,k}+h_{ij,k,k}\right) \tag{113}$$

证明:

静态时空满足

$$\partial_0 h_{\mu\nu}=0 \tag{114}$$

也即

$$h_{\mu\nu,0}=0 \tag{115}$$

$$h^{;0}_{\mu\nu}=\eta^{0\alpha}h_{\mu\nu,\alpha}=\eta^{00}h_{\mu\nu,0}=0 \tag{116}$$

$$h_{,0}=\eta^{\mu\nu}h_{\mu\nu,0}=0 \tag{117}$$

$$h^{;0}=\eta^{0\alpha}h_{,\alpha}=\eta^{00}h^{;0}=0 \tag{118}$$

$$h^{\mu}_{\nu,0}=\eta^{\mu\alpha}h_{\alpha\nu,0}=0 \tag{119}$$

于是可计算Ricci张量:

$$\begin{aligned}R_{00}&=-\frac{1}{2}\left(h^{;\alpha}_{00,\alpha}+h_{,0,0}-h^{\alpha}_{0,0,\alpha}-h^{\alpha}_{0,0,\alpha}\right)\\&=-\frac{1}{2}h^{;\alpha}_{00,\alpha}\\&=-\frac{1}{2}h^{;i}_{00,i}\end{aligned} \tag{120}$$

$$\begin{aligned}R_{0i}&=-\frac{1}{2}\left(h^{;\alpha}_{0i,\alpha}+h_{,0,i}-h^{\alpha}_{0,i,\alpha}-h^{\alpha}_{i,0,\alpha}\right)\\&=\frac{1}{2}\left(h^{\alpha}_{0,i,\alpha}-h^{;\alpha}_{0i,\alpha}\right)\\&=\frac{1}{2}\left(h^{k}_{0,i,k}-h^{;k}_{0i,k}\right)\end{aligned} \tag{121}$$

$$\begin{aligned}R_{ij}&=-\frac{1}{2}\left(h^{;\alpha}_{ij,\alpha}+h_{,i,j}-h^{\alpha}_{i,j,\alpha}-h^{\alpha}_{j,i,\alpha}\right)\\&=-\frac{1}{2}\left(h^{;k}_{ij,k}+h_{,i,j}-h^{k}_{i,j,k}-h^{k}_{j,i,k}\right)\\&=-\frac{1}{2}\left(h^{;k}_{ij,k}-h_{00,i,j}+h^{k}_{k,i,j}-h^{k}_{i,j,k}-h^{k}_{j,i,k}\right)\end{aligned} \tag{122}$$

由于

$$\eta_{\mu\nu}=\mathrm{diag}(-1,+1,+1,+1) \tag{123}$$

若认为一对重复的上/下指标也代表求和，则:

$$\begin{aligned}R_{00}&=-\frac{1}{2}h^{;i}_{00,i}\\&=-\frac{1}{2}\eta^{\alpha i}h_{00,i,\alpha}\\&=-\frac{1}{2}h_{00,i,i}\end{aligned} \tag{124}$$

$$
\begin{aligned}
R_{0i} &= \frac{1}{2}\left(h^{k}_{0,i,k} - h^{,k}_{0i,k}\right) \\
&= \frac{1}{2}\left(\eta^{k\alpha} h_{\alpha 0,i,k} - \eta^{k\alpha} h_{0i,k,\alpha}\right) \\
&= \frac{1}{2}\left(h_{k0,i,k} - h_{0i,k,k}\right)
\end{aligned} \tag{125}
$$

$$
\begin{aligned}
R_{ij} &= -\frac{1}{2}\left(h^{,k}_{ij,k} - h_{00,i,j} + h^{k}_{k,i,j} - h^{k}_{i,j,k} - h^{k}_{j,i,k}\right) \\
&= -\frac{1}{2}\left(\eta^{k\alpha} h_{ij,k,\alpha} - h_{00,i,j} + \eta^{k\alpha} h_{\alpha k,i,j} - \eta^{k\alpha} h_{\alpha i,j,k} - \eta^{k\alpha} h_{kj,i,k}\right) \\
&= -\frac{1}{2}\left(-h_{00,i,j} + h_{kk,i,j} - h_{ki,j,k} - h_{kj,i,k} + h_{ij,k,k}\right)
\end{aligned} \tag{126}
$$

4

4.1

由 $f(R)$ 引力作用量

$$
S = \frac{1}{16\pi G}\int \mathrm{d}^4 x \sqrt{-g} f(R) + S_m \tag{127}
$$

变分，推出 $f(R)$ 引力中的引力场方程

$$
f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^\lambda \nabla_\lambda\right) f'(R) = 8\pi G T_{\mu\nu} \tag{128}
$$

先看引力作用量

$$
S_g = \frac{1}{16\pi G}\int \mathrm{d}^4 x \sqrt{-g} f(R) \tag{129}
$$

对度规的变分。计算

$$
\delta\left[\sqrt{-g} f(R)\right] = \delta\left(\sqrt{-g}\right) f(R) + \sqrt{-g}\,\delta\left[f(R)\right] \tag{130}
$$

对于第一项，利用 $\delta g = -g g_{\mu\nu} \delta g^{\mu\nu}$ 有

$$
\begin{aligned}
\delta\left(\sqrt{-g}\right) f(R) &= -\frac{1}{2}\frac{\delta g}{\sqrt{-g}} f(R) \\
&= -\frac{1}{2}\frac{-g g_{\mu\nu} \delta g^{\mu\nu}}{\sqrt{-g}} f(R) \\
&= -\frac{1}{2}\sqrt{-g} g_{\mu\nu} f(R) \delta g^{\mu\nu}
\end{aligned} \tag{131}
$$

对于第二项，利用 $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu \phi^\mu$

$$
\begin{aligned}
\sqrt{-g}\,\delta\left[f(R)\right] &= \sqrt{-g} f'(R) \delta R \\
&= \sqrt{-g} f'(R) \delta\left(g^{\mu\nu} R_{\mu\nu}\right) \\
&= \sqrt{-g} f'(R) \left[\left(\delta g^{\mu\nu}\right) R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right] \\
&= \sqrt{-g} f'(R) R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} f'(R) \nabla_\mu \phi^\mu \\
&= \sqrt{-g} f'(R) R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g}\left\{\nabla_\mu\left[f'(R)\phi^\mu\right] - \phi^\mu \nabla_\mu f'(R)\right\} \\
&= \sqrt{-g} f'(R) R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g}\nabla_\mu\left[f'(R)\phi^\mu\right] - \sqrt{-g}\phi^\mu \nabla_\mu f'(R) \\
&= \sqrt{-g} f'(R) R_{\mu\nu} \delta g^{\mu\nu} - \sqrt{-g}\phi^\mu \nabla_\mu f'(R) + \left[\partial M \text{ term}\right]
\end{aligned} \tag{132}
$$

其中 ∂M term 在体积分中可化为边界面积分，为零。为了进一步计算，注意到

$$
\phi^\mu \equiv g^{\lambda\nu}\delta\Gamma^{\mu}_{\lambda\nu} - g^{\mu\nu}\delta\Gamma^{\lambda}_{\lambda\nu} \tag{133}
$$

利用联络对 $g_{\mu\nu}$ 的变分

$$\delta\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\lambda\sigma}\left(\nabla_{\mu}\delta g_{\nu\sigma}+\nabla_{\nu}\delta g_{\nu\sigma}-\nabla_{\sigma}\delta g_{\mu\nu}\right) \tag{134}$$

可以计算

$$\delta\Gamma^{\mu}_{\lambda\nu}=\frac{1}{2}g^{\mu\sigma}\left(\nabla_{\lambda}\delta g_{\nu\sigma}+\nabla_{\nu}\delta g_{\lambda\sigma}-\nabla_{\sigma}\delta g_{\lambda\nu}\right) \tag{135}$$

$$\delta\Gamma^{\lambda}_{\lambda\nu}=\frac{1}{2}g^{\lambda\sigma}\left(\nabla_{\lambda}\delta g_{\nu\sigma}+\nabla_{\nu}\delta g_{\lambda\sigma}-\nabla_{\sigma}\delta g_{\lambda\nu}\right) \tag{136}$$

于是可以进一步表达 ϕ^{μ}

$$\begin{aligned}\phi^{\mu}&\equiv g^{\lambda\nu}\delta\Gamma^{\mu}_{\lambda\nu}-g^{\mu\nu}\delta\Gamma^{\lambda}_{\lambda\nu}\\&=g^{\lambda\nu}\left[\frac{1}{2}g^{\mu\sigma}\left(\nabla_{\lambda}\delta g_{\nu\sigma}+\nabla_{\nu}\delta g_{\lambda\sigma}-\nabla_{\sigma}\delta g_{\lambda\nu}\right)\right]-g^{\mu\nu}\left[\frac{1}{2}g^{\lambda\sigma}\left(\nabla_{\lambda}\delta g_{\nu\sigma}+\nabla_{\nu}\delta g_{\lambda\sigma}-\nabla_{\sigma}\delta g_{\lambda\nu}\right)\right]\\&=\frac{1}{2}\left(\nabla^{\nu}\delta g^{\mu}_{\nu}+\nabla^{\lambda}\delta g^{\mu}_{\lambda}-\nabla^{\mu}\delta g^{\nu}_{\nu}\right)-\frac{1}{2}\left(\nabla^{\sigma}\delta g^{\mu}_{\sigma}+\nabla^{\mu}\delta g^{\sigma}_{\sigma}-\nabla^{\lambda}\delta g^{\mu}_{\lambda}\right)\\&=\frac{1}{2}\left(2\nabla^{\nu}\delta g^{\mu}_{\nu}-\nabla^{\mu}\delta g^{\nu}_{\nu}\right)-\frac{1}{2}\nabla^{\mu}\delta g^{\sigma}_{\sigma}\\&=\nabla^{\nu}\delta g^{\mu}_{\nu}-\nabla^{\mu}\delta g^{\nu}_{\nu}\end{aligned} \tag{137}$$

于是

$$\begin{aligned}-\sqrt{-g}\phi^{\mu}\nabla_{\mu}f'(R)&=-\sqrt{-g}\left[\nabla^{\nu}\left(\delta g^{\mu}_{\nu}\right)-\nabla^{\mu}\left(\delta g^{\nu}_{\nu}\right)\right]\nabla_{\mu}f'(R)\\&=\sqrt{-g}\left[\nabla^{\mu}\left(\delta g^{\nu}_{\nu}\right)\right]\nabla_{\mu}f'(R)-\sqrt{-g}\left[\nabla^{\nu}\left(\delta g^{\mu}_{\nu}\right)\right]\nabla_{\mu}f'(R)\\&=\sqrt{-g}\left[-\left(\delta g^{\nu}_{\nu}\right)\nabla^{\mu}\nabla_{\mu}f'(R)\right]-\sqrt{-g}\left[-\left(\delta g^{\mu}_{\nu}\right)\nabla^{\nu}\nabla_{\mu}f'(R)\right]+[\partial M\text{ term}]\\&=\sqrt{-g}g_{\nu\sigma}\left(\delta g^{\mu\sigma}\right)\nabla^{\nu}\nabla_{\mu}f'(R)-\sqrt{-g}g_{\nu\sigma}\left(\delta g^{\nu\sigma}\right)\nabla^{\mu}\nabla_{\mu}f'(R)+[\partial M\text{ term}]\\&=\sqrt{-g}\left(\delta g^{\mu\sigma}\right)\nabla_{\sigma}\nabla_{\mu}f'(R)-\sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^{\lambda}\nabla_{\lambda}f'(R)+[\partial M\text{ term}]\\&=\sqrt{-g}\left(\delta g^{\mu\nu}\right)\nabla_{\nu}\nabla_{\mu}f'(R)-\sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^{\lambda}\nabla_{\lambda}f'(R)+[\partial M\text{ term}]\\&=\sqrt{-g}\left(\delta g^{\nu\mu}\right)\nabla_{\nu}\nabla_{\mu}f'(R)-\sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^{\lambda}\nabla_{\lambda}f'(R)+[\partial M\text{ term}]\\&=\sqrt{-g}\left(\delta g^{\mu\nu}\right)\nabla_{\mu}\nabla_{\nu}f'(R)-\sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^{\lambda}\nabla_{\lambda}f'(R)+[\partial M\text{ term}]\end{aligned} \tag{138}$$

总之，

$$\begin{aligned}\delta\left[\sqrt{-g}f(R)\right]&=\delta\left(\sqrt{-g}\right)f(R)+\sqrt{-g}\delta\left[f(R)\right]\\&=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}f(R)\delta g^{\mu\nu}+\sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu}-\sqrt{-g}\phi^{\mu}\nabla_{\mu}f'(R)+[\partial M\text{ term}]\\&=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}f(R)\delta g^{\mu\nu}+\sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu}+\sqrt{-g}\left(\delta g^{\mu\nu}\right)\nabla_{\mu}\nabla_{\nu}f'(R)-\sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^{\lambda}\nabla_{\lambda}f'(R)+[\partial M\text{ term}]\\&=\sqrt{-g}\left[f'(R)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}f(R)+\left(\nabla_{\mu}\nabla_{\nu}-g_{\mu\nu}\nabla^{\lambda}\nabla_{\lambda}\right)f'(R)\right]\delta g^{\mu\nu}+[\partial M\text{ term}]\end{aligned} \tag{139}$$

则引力作用量的变分为

$$\begin{aligned}\delta S_g&=\frac{1}{16\pi G}\int\mathrm{d}^4x\,\delta\left[\sqrt{-g}f(R)\right]\\&=\frac{1}{16\pi G}\int\mathrm{d}^4x\sqrt{-g}\left[f'(R)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}f(R)+\left(\nabla_{\mu}\nabla_{\nu}-g_{\mu\nu}\nabla^{\lambda}\nabla_{\lambda}\right)f'(R)\right]\delta g^{\mu\nu}+\int[\partial M\text{ term}]\\&=\frac{1}{16\pi G}\int\mathrm{d}^4x\sqrt{-g}\left[f'(R)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}f(R)+\left(\nabla_{\mu}\nabla_{\nu}-g_{\mu\nu}\nabla^{\lambda}\nabla_{\lambda}\right)f'(R)\right]\delta g^{\mu\nu}\end{aligned} \tag{140}$$

对于引力源物质作用量，能动张量满足

$$\delta S_m=-\frac{1}{2}\int\mathrm{d}^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \tag{141}$$

由最小作用量原理

$$\delta S=\delta S_g+\delta S_m=0 \tag{142}$$

可得

$$f'(R)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}f(R)+\left(\nabla_\mu\nabla_\nu-g_{\mu\nu}\nabla^\lambda\nabla_\lambda\right)f'(R)-8\pi GT_{\mu\nu}=0 \tag{143}$$

也即 $f(R)$ 引力中的引力场方程

$$f'(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}-\left(\nabla_\mu\nabla_\nu-g_{\mu\nu}\nabla^\lambda\nabla_\lambda\right)f'(R)=8\pi GT_{\mu\nu} \tag{144}$$

4.2

计算Schwarzchild解的空间部分线元

$$\mathrm{d}s^2=r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)+\frac{\mathrm{d}r^2}{1-2GM/c^2r} \tag{145}$$

当坐标半径 $r=a$ 时的球面面积；$r=a$ 时的球体体积；从半径为 $r=2GM/c^2$ 的球面到 $r=3GM/c^2$ 的球面的径向距离。

- $r=a$ 时的球面面积

在 $r=a$ 的二维球面上，

$$\mathrm{d}s^2=a^2\mathrm{d}\theta^2+a^2\sin^2\theta\mathrm{d}\phi^2 \tag{146}$$

度规：

$$[g_{ij}]=\mathrm{diag}\left(a^2,a^2\sin^2\theta\right) \tag{147}$$

面元为

$$\mathrm{d}A=\sqrt{\det[g_{ij}]}\mathrm{d}x^i\mathrm{d}x^j=a^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi \tag{148}$$

球面面积：

$$A=\int\mathrm{d}A=\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}a^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi=4\pi a^2 \tag{149}$$

- $r=a$ 时的球体体积

令

$$r_s\equiv\frac{2GM}{c^2} \tag{150}$$

则

$$\begin{aligned}\mathrm{d}s^2&=r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)+\frac{\mathrm{d}r^2}{1-2GM/c^2r}\\&=\mathrm{d}s^2=r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)+\frac{\mathrm{d}r^2}{1-r_s/r}\end{aligned} \tag{151}$$

度规：

$$[g_{ij}]=\mathrm{diag}\left(1-r_s/r,r^2,r^2\sin^2\theta\right) \tag{152}$$

体元：

$$\mathrm{d}V=\sqrt{\det[g_{ij}]}\mathrm{d}x^i\mathrm{d}x^j=\frac{r^2\sin\theta}{\sqrt{1-r_s/r}}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{153}$$

球体体积：

$$
\begin{aligned}
V=\int \mathrm{d}V &= \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\frac{r^2\sin\theta}{\sqrt{1-r_s/r}}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= 4\pi\int_{r=0}^{r=a}\frac{r^2}{\sqrt{1-r_s/r}}\mathrm{d}r
\end{aligned} \tag{154}
$$

- 从半径为 $r=2GM/c^2=r_s$ 的球面到 $r=3GM/c^2=3r_s/2$ 的球面的径向距离

$$
\begin{aligned}
\Delta r &= \int_{r=r_s}^{r=3r_s/2}\frac{\mathrm{d}r}{\sqrt{1-r_s/r}} = \left[\sqrt{r(r-r_s)}+r_s\ln\left|\frac{\sqrt{r-r_s}+\sqrt{r}}{\sqrt{r_s}}\right|\right]\Bigg|_{r=r_s}^{r=3r_s/2} \\
&= \frac{\sqrt{3}}{2}r_s + r_s\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right) \\
&= \left[\frac{\sqrt{3}}{2}+\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)\right]r_s
\end{aligned} \tag{155}
$$

4.3

计算Schwarzchild度规

$$
\mathrm{d}s^2 = -\mathrm{e}^{\nu}c^2\mathrm{d}t^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)+\mathrm{e}^{-\nu}\mathrm{d}r^2 \tag{156}
$$

$$
\mathrm{e}^{\nu} = 1-\frac{2GM}{c^2r} \tag{157}
$$

情况下所有联络 $\Gamma^{\lambda}_{\mu\nu}$.

$$
\left(x^0,x^1,x^2,x^3\right)\equiv(ct,r,\theta,\phi) \tag{158}
$$

线元

$$
\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{e}^{\nu}c^2\mathrm{d}t^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)+\mathrm{e}^{-\nu}\mathrm{d}r^2 \\
&= g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}
\end{aligned} \tag{159}
$$

度规

$$
[g_{\mu\nu}] = \mathrm{diag}\left(-\mathrm{e}^{\nu},\mathrm{e}^{-\nu},r^2,r^2\sin^2\theta\right) \tag{160}
$$

逆度规

$$
[g^{\mu\nu}] = \mathrm{diag}\left(-\mathrm{e}^{-\nu},\mathrm{e}^{\nu},\frac{1}{r^2},\frac{1}{r^2\sin^2\theta}\right) \tag{161}
$$

下面计算联络。

$$
\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_{\mu}g_{\nu\sigma}+\partial_{\nu}g_{\mu\sigma}-\partial_{\sigma}g_{\mu\nu}\right) \tag{162}
$$

- $\lambda=0$

$$
\begin{aligned}
\Gamma^{0}_{\mu\nu} &= \frac{1}{2}g^{0\sigma}\left(\partial_{\mu}g_{\nu\sigma}+\partial_{\nu}g_{\mu\sigma}-\partial_{\sigma}g_{\mu\nu}\right) \\
&= \frac{1}{2}g^{00}\left(\partial_{\mu}g_{\nu0}+\partial_{\nu}g_{\mu0}-\partial_{0}g_{\mu\nu}\right) \\
&= -\frac{1}{2}\mathrm{e}^{-\nu}\left(\partial_{\mu}g_{\nu0}+\partial_{\nu}g_{\mu0}\right)
\end{aligned} \tag{163}
$$

当 $(\mu,\nu)=(i,j)$ 时，联络恒为零；当 $\mu=0$ 且 $\nu=0$ 时联络也为零。

非零联络：

$$
\Gamma^{0}_{10}=\Gamma^{0}_{01} = -\frac{1}{2}\mathrm{e}^{-\nu}\left(\partial_0 g_{10}+\partial_1 g_{00}\right) = -\frac{1}{2}\mathrm{e}^{-\nu}\partial_1\left(-\mathrm{e}^{\nu}\right) = \frac{1}{2}\nu' \tag{164}
$$

- $\lambda=1$

$$\begin{aligned}\Gamma^1_{\mu\nu} &= \frac{1}{2}g^{1\sigma}\left(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}\right)\\ &= \frac{1}{2}g^{11}\left(\partial_\mu g_{\nu 1}+\partial_\nu g_{\mu 1}-\partial_1 g_{\mu\nu}\right)\\ &= \frac{1}{2}\mathrm{e}^{\nu}\left(\partial_\mu g_{\nu 1}+\partial_\nu g_{\mu 1}-\partial_1 g_{\mu\nu}\right)\end{aligned} \tag{165}$$

当 $\mu,\nu\neq 1$ 时，非零的联络为

$$\Gamma^1_{00}=\frac{1}{2}\mathrm{e}^{\nu}\left(-\partial_r\left(-\mathrm{e}^{\nu}\right)\right)=\frac{\nu'}{2}\mathrm{e}^{2\nu} \tag{166}$$

$$\Gamma^1_{22}=\frac{1}{2}\mathrm{e}^{\nu}\left(-\partial_r\left(r^2\right)\right)=-r\mathrm{e}^{\nu} \tag{167}$$

$$\Gamma^1_{33}=\frac{1}{2}\mathrm{e}^{\nu}\left(-\partial_r\left(r^2\sin^2\theta\right)\right)=-r\mathrm{e}^{\nu}\sin^2\theta \tag{168}$$

当 $\mu=\nu=1$ 时，

$$\Gamma^1_{11}=\frac{1}{2}\mathrm{e}^{\nu}\left(\partial_1 g_{11}\right)=\frac{1}{2}\mathrm{e}^{\nu}\left(\partial_r\left(\mathrm{e}^{-\nu}\right)\right)=-\frac{\nu'}{2} \tag{169}$$

当 μ,ν 中只有一个为 1 时，

$$\Gamma^1_{01}=\Gamma^1_{10}=\Gamma^1_{12}=\Gamma^1_{21}=\Gamma^1_{13}=\Gamma^1_{31}=0 \tag{170}$$

- $\lambda=2$

$$\begin{aligned}\Gamma^2_{\mu\nu} &= \frac{1}{2}g^{2\sigma}\left(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}\right)\\ &= \frac{1}{2}g^{22}\left(\partial_\mu g_{\nu 2}+\partial_\nu g_{\mu 2}-\partial_2 g_{\mu\nu}\right)\\ &= \frac{1}{2r^2}\left(\partial_\mu g_{\nu 2}+\partial_\nu g_{\mu 2}-\partial_2 g_{\mu\nu}\right)\end{aligned} \tag{171}$$

当 $\mu,\nu\neq 2$ 时，

$$\Gamma^2_{33}=\frac{1}{2r^2}\left(-\partial_\theta\left(r^2\sin^2\theta\right)\right)=-\sin\theta\cos\theta \tag{172}$$

当 μ,ν 中只有一个为 2 时，

$$\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{2r^2}\left(\partial_1 g_{22}\right)=\frac{1}{2r^2}\left(\partial_r\left(r^2\right)\right)=\frac{1}{r} \tag{173}$$

当 $(\mu,\nu)=(2,2)$ 时，

$$\Gamma^2_{22}=0 \tag{174}$$

- $\lambda=3$

$$\begin{aligned}\Gamma^3_{\mu\nu} &= \frac{1}{2}g^{3\sigma}\left(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}\right)\\ &= \frac{1}{2}g^{33}\left(\partial_\mu g_{\nu 3}+\partial_\nu g_{\mu 3}-\partial_3 g_{\mu\nu}\right)\\ &= \frac{1}{2r^2\sin^2\theta}\left(\partial_\mu g_{\nu 3}+\partial_\nu g_{\mu 3}\right)\end{aligned} \tag{175}$$

当 $\mu,\nu\neq 3$ 时，所有联络都为零。

当 μ,ν 中只有一个为 3 时，

$$\Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{2r^2\sin^2\theta}\left(\partial_1 g_{33}\right)=\frac{1}{2r^2\sin^2\theta}\left(\partial_r\left(r^2\sin^2\theta\right)\right)=\frac{1}{r} \tag{176}$$

$$\Gamma^3_{23}=\Gamma^3_{32}=\frac{1}{2r^2\sin^2\theta}\left(\partial_2 g_{33}\right)=\frac{1}{2r^2\sin^2\theta}\left(\partial_\theta\left(r^2\sin^2\theta\right)\right)=\cot\theta \tag{177}$$

当 $\mu,\nu=3$ 时，

$$\Gamma_{33}^{3}=0 \tag{178}$$

$$
\begin{equation}

\end{equation}
$$

4.4

据上题结果，由Schwarzchild时空粒子运动方程出发，导出在平面极坐标系下轨道满足的方程（GR中的Binet方程），并讨论水星进动问题。

Schwarzchild线元：

$$\mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right),\quad \mathrm{e}^{\nu}=1-\frac{2GM}{c^2r} \tag{179}$$

Schwarzschild度规下的非零联络：

$$\Gamma_{01}^{0}=\Gamma_{10}^{0}=\frac{\nu'}{2}, \tag{180}$$

$$\Gamma_{00}^{1}=\frac{\nu'}{2}\mathrm{e}^{2\nu},\quad \Gamma_{11}^{1}=-\frac{\nu'}{2},\quad \Gamma_{22}^{1}=-r\mathrm{e}^{\nu},\quad \Gamma_{33}^{1}=-r\mathrm{e}^{\nu}\sin^2\theta \tag{181}$$

$$\Gamma_{12}^{2}=\Gamma_{21}^{2}=\frac{1}{r},\quad \Gamma_{33}^{2}=-\sin\theta\cos\theta, \tag{182}$$

$$\Gamma_{13}^{3}=\Gamma_{31}^{3}=\frac{1}{r},\quad \Gamma_{23}^{3}=\Gamma_{32}^{3}=\cot\theta \tag{183}$$

以线长 s 为参量，测地线方程：

$$\frac{\mathrm{d}^2x^{\sigma}}{\mathrm{d}s^2}+\Gamma_{\mu\nu}^{\sigma}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}=0 \tag{184}$$

σ 分别取 $0,1,2,3$ 就得到四条关于坐标的参数方程。

- $\sigma=0$

非零联络：

$$\Gamma_{01}^{0}=\Gamma_{10}^{0}=\frac{\nu'}{2}, \tag{185}$$

$$\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2}+\Gamma_{01}^{0}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}+\Gamma_{10}^{0}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s}=0 \tag{186}$$

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s}=0 \tag{187}$$

- $\sigma=1$

非零联络：

$$\Gamma_{00}^{1}=\frac{\nu'}{2}\mathrm{e}^{2\nu},\quad \Gamma_{11}^{1}=-\frac{\nu'}{2},\quad \Gamma_{22}^{1}=-r\mathrm{e}^{\nu},\quad \Gamma_{33}^{1}=-r\mathrm{e}^{\nu}\sin^2\theta \tag{188}$$

$$\frac{\mathrm{d}^2x^1}{\mathrm{d}s^2}+\Gamma_{00}^{1}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s}+\Gamma_{11}^{1}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}+\Gamma_{22}^{1}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s}+\Gamma_{33}^{1}\frac{\mathrm{d}x^3}{\mathrm{d}s}\frac{\mathrm{d}x^3}{\mathrm{d}s}=0 \tag{189}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{190}$$

- $\sigma=2$

非零联络：

$$\Gamma_{12}^2=\Gamma_{21}^2=\frac{1}{r},\quad \Gamma_{33}^2=-\sin\theta\cos\theta, \tag{191}$$

$$\frac{\mathrm{d}^2x^2}{\mathrm{d}s^2}+\Gamma_{12}^2\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s}+\Gamma_{21}^2\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}+\Gamma_{33}^2\frac{\mathrm{d}x^3}{\mathrm{d}s}\frac{\mathrm{d}x^3}{\mathrm{d}s}=0 \tag{192}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s}-\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{193}$$

- $\sigma=3$

非零联络：

$$\Gamma_{13}^3=\Gamma_{31}^3=\frac{1}{r},\quad \Gamma_{23}^3=\Gamma_{32}^3=\cot\theta \tag{194}$$

$$\frac{\mathrm{d}^2x^3}{\mathrm{d}s^2}+\Gamma_{13}^3\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^3}{\mathrm{d}s}+\Gamma_{31}^3\frac{\mathrm{d}x^3}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}+\Gamma_{23}^3\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^3}{\mathrm{d}s}+\Gamma_{32}^3\frac{\mathrm{d}x^3}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s}=0 \tag{195}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}+2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{196}$$

总之，四条参数方程为

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s}=0 \tag{197}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{198}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s}-\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{199}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}+2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{200}$$

取轨道面 $\theta=\pi/2$，则方程简化为

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s}=0 \tag{201}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{202}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{203}$$

注意到上面第三条方程

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0,\quad r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+2r\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0,\quad r^2\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}+\frac{\mathrm{d}\left(r^2\right)}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{204}$$

也即

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)=0 \tag{205}$$

因此

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}=\mathrm{const}\equiv\frac{h}{c} \tag{206}$$

这是 GR 中的角动量守恒。

我们要找轨道方程，因此需要找到 r,ϕ 的微分方程。

由于 $\theta=\pi/2$，则线元

$$\mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right),\quad \mathrm{e}^{\nu}=1-\frac{2GM}{c^2r} \tag{207}$$

可化简为

$$\mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{208}$$

利用线元 $\mathrm{d}s^2$ 与线长 $\mathrm{d}s$ 的关系

$$\mathrm{d}s=\sqrt{-\mathrm{d}s^2},\quad (\mathrm{d}s)^2=-\mathrm{d}s^2=c^2\mathrm{e}^{\nu}\mathrm{d}t^2-\mathrm{e}^{-\nu}\mathrm{d}r^2-r^2\mathrm{d}\phi^2 \tag{209}$$

两边同除 $(\mathrm{d}s)^2$，再同乘 e^{ν}，得到

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2=c^2\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-r^2\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2-\mathrm{e}^{\nu} \tag{210}$$

上式代入 r 关于 s 的二阶偏微分方程，就消去 t:

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{\nu'}{2}\mathrm{e}^{\nu}r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2+\frac{1}{2}\nu'\mathrm{e}^{\nu}=0 \tag{211}$$

这就是GR中参数形式的行星轨道方程。

利用 GR 中的角动量守恒

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{c},\quad \frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{cr^2} \tag{212}$$

进一步化简为

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\left(\frac{\nu'}{2}r^2-r\right)\mathrm{e}^{\nu}\left(\frac{h}{cr^2}\right)^2+\frac{1}{2}\nu'\mathrm{e}^{\nu}=0 \tag{213}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{1}{2}\nu'\mathrm{e}^{\nu}\left(\frac{h^2}{c^2r^2}+1\right)-r\mathrm{e}^{\nu}\left(\frac{h}{cr^2}\right)^2=0 \tag{214}$$

令 $u=\frac{1}{r}$，注意到

$$\mathrm{e}^{\nu}=1-\frac{2GM}{c^2r}=1-\frac{2GM}{c^2}u \tag{215}$$

$$\nu'\mathrm{e}^{\nu}=(\mathrm{e}^{\nu})'=\frac{2GM}{c^2r^2}=\frac{2GM}{c^2}u^2 \tag{216}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{cr^2}=\frac{h}{c}u^2 \tag{217}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}=\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\phi}=\frac{h}{c}u^2\frac{\mathrm{d}}{\mathrm{d}\phi} \tag{218}$$

$$\begin{aligned}\frac{\mathrm{d}^2r}{\mathrm{d}s}&=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}(1/u)}{\mathrm{d}s}\right)=\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}s}\right)=\frac{\mathrm{d}}{\mathrm{d}s}\left[-\frac{1}{u^2}\cdot\left(\frac{h}{c}u^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\right]\\&=-\frac{h}{c}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)=-\frac{h}{c}\cdot\frac{h}{c}u^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\\&=-\frac{h^2}{c^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}\end{aligned} \tag{219}$$

于是轨道的参数方程就化为如下的轨道方程:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\frac{3GM}{c^2}u^2+\frac{GM}{h^2} \tag{220}$$

上式就是GR中的Binet方程。

设 u_0 满足牛顿力学中的Binet方程

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}\phi^2}+u_0=\frac{GM}{h^2} \tag{221}$$

其解为

$$u_0 = \frac{1}{p}\left(1 + e\cos\phi\right), \quad p = \frac{h^2}{GM} \tag{222}$$

令 $\alpha = 3GM/c^2$，则

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \alpha u^2 + \frac{GM}{h^2} \tag{223}$$

设 $u = u_0 + \alpha u_1$，则

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 + \alpha\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + \alpha u_1 = \alpha\left(u_0 + \alpha u_1\right)^2 + \frac{GM}{h^2} \tag{224}$$

利用 u_0 满足的方程就得到

$$\alpha\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + \alpha u_1 = \alpha\left(u_0 + \alpha u_1\right)^2 \tag{225}$$

由于 α 是个小量，约去右边括号内的高阶小量，再利用 u_0 的表达式，就得到

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = u_0^2 = \frac{\left(1 + e\cos\phi\right)^2}{p^2} = \frac{1}{p^2}\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right] \tag{226}$$

设 u_1 的形式解为

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi \tag{227}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}\phi} = B\left(\sin\phi + \phi\cos\phi\right) - 2C\sin 2\phi \tag{228}$$

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} = B\left(2\cos\phi - \phi\sin\phi\right) - 4C\cos 2\phi \tag{229}$$

代回方程得到

$$B\left(2\cos\phi - \phi\sin\phi\right) - 4C\cos 2\phi + A + B\phi\sin\phi + C\cos 2\phi = \frac{1}{p^2}\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right] \tag{230}$$

对比各项前的系数就得到

$$A = \frac{1}{p^2}\left(1 + \frac{e^2}{2}\right), \quad B = \frac{e}{p^2}, \quad C = -\frac{e^2}{6p^2} \tag{231}$$

因此

$$u_1 = \frac{1}{p^2}\left(1 + \frac{e^2}{2}\right) + \frac{e}{p^2}\phi\sin\phi - \frac{e^2}{6p^2}\cos 2\phi \tag{232}$$

$$u = u_0 + \alpha u_1 = \frac{1}{p}\left[\left(1 + e\cos\phi\right) + \frac{\alpha}{p}\left(1 + \frac{e^2}{2}\right) + \frac{\alpha e}{p}\phi\sin\phi - \frac{1}{6}\frac{\alpha e^2}{p^2}\cos 2\phi\right] \tag{233}$$

上面只有 $\phi\sin\phi$ 项是累加的，只保留对轨道有长期影响的项：

$$u = \frac{1}{p}\left[\left(1 + e\cos\phi\right) + \frac{\alpha e}{p}\phi\sin\phi\right] \tag{234}$$

由于 α 是小量，则

$$1 \approx \cos\left(\frac{\alpha}{p}\phi\right), \quad \frac{\alpha}{p}\phi \approx \sin\left(\frac{\alpha}{p}\phi\right) \tag{235}$$

于是

$$
\begin{aligned}
u &= \frac{1}{p}\left[(1+e\cos\phi)+\frac{\alpha e}{p}\phi\sin\phi\right] \\
&= \frac{1}{p}\left[1+e\left(1\cdot\cos\phi+\frac{\alpha}{p}\phi\sin\phi\right)\right] \\
&\approx \frac{1}{p}\left[1+e\left(\cos\left(\frac{\alpha}{p}\phi\right)\cos\phi+\sin\left(\frac{\alpha}{p}\phi\right)\sin\phi\right)\right] \\
&= \frac{1}{p}\left[1+\cos\left(\left(1-\frac{\alpha}{p}\right)\phi\right)\right] \\
&\equiv \frac{1}{p}\left(1+e\cos\Phi\right)
\end{aligned} \tag{236}
$$

$$
\Phi \equiv \left(1-\frac{\alpha}{p}\right)\phi \tag{237}
$$

当 $\Phi=\Phi_n=(2n+1)\pi$ 时，$\cos\Phi=-1$，此时 u 最小，r 最大，相应 ϕ_n 为

$$
\phi_n=\frac{\Phi_n}{1-\alpha/p}\approx\Phi_n\left(1+\frac{\alpha}{p}\right) \tag{238}
$$

$$
\phi_{n+1}\approx\Phi_{n+1}\left(1+\frac{\alpha}{p}\right) \tag{239}
$$

于是

$$
\phi_{n+1}-\phi_n=2\pi\left(1+\frac{\alpha}{p}\right) \tag{240}
$$

一周期进动角为

$$
\Delta=\phi_{n+1}-\phi_n-2\pi=2\pi\frac{\alpha}{p}=\frac{6\pi GM}{c^2p} \tag{241}
$$

由

$$
T^2=\frac{4\pi^2}{GM}a^3,\quad GM=4\pi^2\frac{a^3}{T^2},\quad p=a\left(1-e^2\right) \tag{242}
$$

可得

$$
\Delta=\frac{24\pi^3a^2}{c^2T^2\left(1-e^2\right)} \tag{243}
$$

对于水星，其世纪进动角为

$$
\Delta_c\approx 43'' \tag{244}
$$

4.5

导出光子在平面极坐标系下轨道所满足的方程，并讨论光线在太阳附近偏折的问题。

对于光有 $\mathrm{d}s^2=0$，因此测地线方程不能以线长 s 为参数。但可引入参数 λ 来定义光线的切矢：

$$
K^\mu\equiv\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{245}
$$

由于 $\mathrm{d}s^2=0$ 可得

$$
g_{\mu\nu}K^\mu K^\nu=0 \tag{246}
$$

假设切矢 K^μ 在光的传播路线上是平行的，即

$$
\nabla_\mu K^\sigma=0,\quad K^\mu\nabla_\mu K^\sigma=0 \tag{247}
$$

也即

$$K^{\mu}\left(\partial_{\mu}K^{\sigma}+\Gamma^{\sigma}_{\mu\nu}K^{\nu}\right)=0 \tag{248}$$

又

$$K^{\mu}\partial_{\mu}K^{\sigma}=\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\partial K^{\sigma}}{\partial x^{\mu}}=\frac{\mathrm{d}K^{\sigma}}{\mathrm{d}\lambda} \tag{249}$$

因此有

$$\frac{\mathrm{d}K^{\sigma}}{\mathrm{d}\lambda}+\Gamma^{\sigma}_{\mu\nu}K^{\mu}K^{\nu}=0,\quad K^{\sigma}\equiv\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}\lambda} \tag{250}$$

也即

$$\frac{\mathrm{d}^2x^{\sigma}}{\mathrm{d}\lambda^2}+\Gamma^{\sigma}_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda}=0 \tag{251}$$

我们知道，Schwarzschild解情况下有质量粒子在 $\theta=\pi/2$ 平面内的测地线方程有三条：

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s}=0 \tag{252}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{253}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{254}$$

对于光子，同样考虑Schwarzschild解，由于二者的联络都是一样，因此只需要把 s 替换成 λ 就得到条Schwarzschild解下光传播路径的参数方程：

$$\frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda}=0 \tag{255}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0 \tag{256}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=0 \tag{257}$$

对于第三条方程，同样有

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)=0,\quad r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=k \tag{258}$$

在 $\theta=\pi/2$ 平面上，线元

$$\mathrm{d}s^2=0=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{259}$$

可以证明，可以从上式和第三条方程推导出第一条方程。上式两边同除 $(\mathrm{d}\lambda)^2$，并移项，就得到

$$c^2\mathrm{e}^{\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2=\mathrm{e}^{-\nu}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2+r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 \tag{260}$$

上式代回 r 关于 λ 二阶导式子，就得到

$$\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2}+\left(\frac{r^2}{2}\nu'\mathrm{e}^{\nu}-r\mathrm{e}^{\nu}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0 \tag{261}$$

令 $u=\frac{1}{r}$，注意到

$$\mathrm{e}^{\nu}=1-\frac{2GM}{c^2r}=1-\frac{2GM}{c^2}u \tag{262}$$

$$\nu'\mathrm{e}^{\nu}=(\mathrm{e}^{\nu})'=\frac{2GM}{c^2r^2}=\frac{2GM}{c^2}u^2 \tag{263}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=\frac{k}{r^2}=ku^2 \tag{264}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}=\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\phi}=ku^2\frac{\mathrm{d}}{\mathrm{d}\phi} \tag{265}$$

$$\begin{aligned}\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}(1/u)}{\mathrm{d}\lambda}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\lambda}\right)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\frac{1}{u^2}\cdot ku^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\\ =-k\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}u}{\mathrm{d}\phi}=-k\cdot ku^2\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\mathrm{d}u}{\mathrm{d}\phi}\\ =-k^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}\end{aligned} \tag{266}$$

于是可以消去参数 λ，得到轨道微分方程

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\frac{3GM}{c^2}u^2 \tag{267}$$

定义小量

$$\alpha\equiv\frac{3GM}{c^2} \tag{268}$$

则

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\alpha u^2 \tag{269}$$

设 u_0 满足

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}\phi^2}+u_0=0 \tag{270}$$

其解为

$$u_0=\frac{1}{b}\sin\phi \tag{271}$$

设

$$u=u_0+\alpha u_1 \tag{272}$$

则

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}\phi^2}+u_0+\alpha\left(\frac{\mathrm{d}^2u_1}{\mathrm{d}\phi^2}+u_1\right)=\alpha\left(u_0+\alpha u_1\right)^2 \tag{273}$$

也即

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\phi^2}+u_1=\left(u_0+\alpha u_1\right)^2\approx u_0^2=\frac{1}{b^2}\sin^2\phi \tag{274}$$

设 u_1 的形式解为

$$u_1=A\sin^2\phi+B \tag{275}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}\phi}=2A\sin\phi\cos\phi=A\sin 2\phi \tag{276}$$

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}\phi^2}=2A\cos 2\phi=2A\left(1-2\sin^2\phi\right)=2A-4A\sin^2\phi \tag{277}$$

代回 u_1 满足的微分方程，得到

$$2A-4A\sin^2\phi+A\sin^2\phi+B=\frac{1}{b^2}\sin^2\phi \tag{278}$$

对比可得

$$3A=-\frac{1}{b^2},\quad 2A+B=0 \tag{279}$$

解得

$$A=-\frac{1}{3b^2},\quad B=\frac{2}{3b^2} \tag{280}$$

$$u_1=A\sin^2\phi+B=-\frac{1}{3b^2}\left(\sin^2\phi-2\right)=\frac{1}{3b^2}\left(\cos^2\phi+1\right) \tag{281}$$

$$\begin{aligned} u=u_0+\alpha u_1&=\frac{1}{b}\sin\phi+\frac{\alpha}{3b^2}\left(\cos^2\phi+1\right)\\ &=\frac{1}{b}\sin\phi+\frac{GM}{c^2b^2}\left(\cos^2\phi+1\right)\end{aligned} \tag{282}$$

定义小量 $a\equiv GM/c^2b$，当 $r\to+\infty, u=0$，此时

$$\sin\phi+a\left(2-\sin^2\phi\right)=0 \tag{283}$$

$$\sin\phi=\frac{1\pm\sqrt{1+8a^2}}{2a}\approx\frac{1\pm\left(1+4a^2\right)}{2a}=-2a\quad\text{or}\quad\frac{1+2a^2}{a} \tag{284}$$

舍去 $\sin\phi=\frac{1+2a^2}{a}>1$ 的解，考虑 $\phi\to 0$ 的那侧，则

$$\phi\approx\sin\phi=-2a,\quad r\to+\infty \tag{285}$$

偏折角为

$$\delta=4a=\frac{4GM}{c^2b} \tag{286}$$