兰州大学 2023～2024 学年第 一 学期

期末考试试卷

课程名称：力学基础 IV（广义相对论） 任课教师：杨捷

学院：________ 专业：________ 年级：________

姓名：________ 校园卡号：________

题号	一	二	三	四	五	六	七	总分
分数								

注：7 道任选 5 道，每题 20 分，共 100 分，第 7 题研究生必做

1.计算牛顿近似中的所以非零联络 $\Gamma^{\lambda}_{\mu\nu}$

2.写出逆变、协变矢量与二阶张量的的坐标变换式,（已知 $A^{\mu}_{\nu}=\frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}}$，$\bar{A}^{\mu}_{\nu}=\frac{\partial x^{\mu}}{\partial \bar{x}^{\nu}}$ 且 $\det[A^{\mu}_{\nu}]\neq 0$）并写出它们的协变微分表达式。

3.由毕安基恒等式证明 $\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)_{;\nu}=0$

4.写出爱因斯坦作用量，并变分得到场方程。

已知 Palatini 公式

$$\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}=\partial_{\mu}(\sqrt{-g}\varphi^{\mu})$$

且 $\delta g=gg^{\mu\nu}\delta g_{\mu\nu}=-gg_{\mu\nu}\delta g^{\mu\nu}$ 。

5.求弱场线性近似的里奇张量，并导出线性化的场方程。

6.推导光线在行星附近的轨道方程，并讨论其偏折，这证明了广义相对论的什么效应？

7.推导史瓦西时空与闵可夫斯基时空的类光测地线，简述并画出 krusal 沿拓图与彭若斯图。

参考提示

一、

我们在下列近似下化简运动方程(3.4.27)。

(1) 引力场是弱场,即

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (3.4.$$

其中 $\eta_{\mu\nu}$ 是闵可夫斯基度规(2.8.10),而

$$|h_{\mu\nu}| \ll 1 \qquad (3.4.$$

(2) 引力场是静态的,即

$$g_{\mu\nu,0} = h_{\mu\nu,0} = 0 \qquad (3.4.$$

(3) 引力场是空间缓变的,即

$$|g_{\mu\nu,i}| = |h_{\mu\nu,i}| \ll 1, \quad i = 1,2,3 \qquad (3.4.$$

(4) 粒子作低速运动

$$\left|\frac{\mathrm{d}x^i}{\mathrm{d}x^0}\right| \ll 1, \quad \left|\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right| \ll \left|\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right|, \quad \left|\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right| \ll c \qquad (3.4.$$

注意,在自然单位制下 $c=1$。这些近似能把运动方程化成牛顿形式,所以称上述化简为
顿近似。

利用近似条件(1)、(2)和(3),略去高于一阶的小量,联络可以化成

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}(h_{\rho\mu,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}) \qquad (3.6.$$

具体算得其分量为

$$\begin{cases} \Gamma^0_{00} = 0, \quad \Gamma^0_{0i} = -\dfrac{1}{2}h_{00,i}, \quad \Gamma^0_{ij} = -\dfrac{1}{2}(h_{0i,j} + h_{0j,i}), \quad \Gamma^i_{00} = -\dfrac{1}{2}h_{00,i} \\ \Gamma^i_{0j} = \dfrac{1}{2}(h_{0i,j} - h_{0j,i}), \quad \Gamma^i_{jk} = \dfrac{1}{2}(h_{ij,k} + h_{ik,j} - h_{jk,i}), \quad i,j,k = 1,2,3 \end{cases} \qquad (3.6.1$$

二、

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^\lambda_{\mu\nu}A_\lambda$$

$$A^\mu_{;\nu} = A^\mu_{,\nu} + \Gamma^\mu_{\lambda\nu}A^\lambda$$

$$T_{\mu\nu;\lambda} = T_{\mu\nu,\lambda} - \Gamma^\rho_{\mu\lambda}T_{\rho\nu} - \Gamma^\rho_{\nu\lambda}T_{\mu\rho}$$

$$T^{\mu\nu}_{;\lambda} = T^{\mu\nu}_{,\lambda} + \Gamma^\mu_{\rho\lambda}T^{\rho\nu} + \Gamma^\nu_{\rho\lambda}T^{\mu\rho}$$

$$T^\mu_{\nu;\lambda} = T^\mu_{\nu,\lambda} + \Gamma^\mu_{\rho\lambda}T^\rho_\nu - \Gamma^\rho_{\nu\lambda}T^\mu_\rho$$

三、

现在缩并毕安基恒等式(2.11.25)的指标 ρ 和 σ,得

$$R^\sigma_{\lambda\mu\nu;\sigma} - R_{\lambda\nu;\mu} + R_{\lambda\mu;\nu} = 0 \qquad (2.11.30)$$

乘以 $g^{\nu\lambda}$,并注意到 $g^{\nu\lambda}_{;\alpha}=0$,上式可以化成

$$R^\sigma_{\mu;\sigma} - R_{;\mu} + R^\nu_{\mu;\nu} = 0 \qquad (2.11.31)$$

即

$$R^\nu_{\mu;\nu} - \frac{1}{2}R_{;\mu} = 0 \qquad (2.11.32)$$

也即

$$\left(R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R\right)_{;\nu} = 0 \qquad (2.11.33)$$

(2.11.33)式还可以写成协变或逆变张量的形式

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)^{;\nu} = 0 \qquad (2.11.34)$$

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\nu} = 0 \qquad (2.11.35)$$

这三个式子表明爱因斯坦张量的协变散度为零

$$G^\nu_{\mu;\nu} = G^{;\nu}_{\mu\nu} = G^{\mu\nu}_{;\nu} = 0 \qquad (2.11.36)$$

爱因斯坦张量的这一性质,对于建立广义相对论的场方程极为重要。

四、

$$I_g = \int_M L_g \sqrt{-g} \mathrm{d}^4 x, \quad L_g = \frac{c^3}{16\pi G} R$$

$$I_n = \int_M L_n \sqrt{-g} \mathrm{d}^4 x$$

$$\delta I_m = \frac{1}{2c} \int_M T_{\mu\nu} \delta q_{\mu\nu} \sqrt{-g} \mathrm{d}^4 x$$

对前一式变分可利用公式得

$$\delta(R\sqrt{-g}) = (\delta g^{\mu\nu}) R_{\mu\nu} \sqrt{-g} + g^{\mu\nu} (\delta R_{\mu\nu}) \sqrt{-g} + g^{\mu\nu} R_{\mu\nu} (\delta \sqrt{-g})$$

$$\delta(R\sqrt{-g}) = \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \sqrt{-g} \delta g^{\mu\nu} + \partial_\mu \left(\sqrt{-g} \phi^\mu \right)$$

故

$$\delta \int_M R\sqrt{-g} \mathrm{d}^4 x = \int_M \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \sqrt{-g} \delta g^{\mu\nu} \mathrm{d}^4 x$$

由此得

$$\begin{aligned} \delta I_m &= \delta I_g + \delta I_m \\ &= \frac{c^3}{16\pi G} \int_M \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \sqrt{-g} \delta g^{\mu\nu} \mathrm{d}^4 x \\ &+ \frac{1}{2c} \int_M T_{\mu\nu} \delta g_{\mu\nu} \sqrt{-g} \mathrm{d}^4 x \end{aligned}$$

由最小作用量原理

$$\frac{c^3}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2c} T_{\mu\nu} = 0 \tag{241}$$

由此可得 Einstein 引力场方程

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{242}$$

五、作业 3.2,3.3

计算曲率张量时，忽略二阶小量，可得

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\lambda,\nu} - \Gamma^{\lambda}_{\mu\nu,\lambda} \equiv \frac{1}{2} \left(h^{,\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\mu,\nu,\alpha} - h^{\alpha}_{\nu,\mu,\alpha} \right),$$

场方程为

$$\overline{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

$$\Leftrightarrow 2R_{\mu\nu}-\eta_{\mu\nu}R=-16\pi GT_{\mu\nu}$$

$$\Leftrightarrow\left(h_{\mu\nu,\alpha}^{,\alpha}+h_{,\mu,\nu}-h_{\mu,\nu,\alpha}^{\alpha}-h_{\nu,\mu,\alpha}^{\alpha}\right)-\frac{1}{2}\eta_{\mu\nu}\left(h_{,\alpha}^{,\alpha}+h_{,\alpha}^{,\alpha}-h_{,\beta,\alpha}^{\beta\alpha}-h_{,\beta,\alpha}^{\beta\alpha}\right)=-16\pi GT_{\mu\nu}。$$

$$\because h_{\mu,\nu,\alpha}^{\alpha}=\eta^{\alpha\beta}h_{\alpha\mu,\nu,\beta}=h_{\mu\alpha,\nu}^{,\alpha},\quad h_{\nu,\mu,\alpha}^{\alpha}=\eta^{\alpha\beta}h_{\alpha\nu,\mu,\beta}=h_{\nu\alpha,\mu}^{,\alpha},$$

$$h_{,\beta,\alpha}^{\beta\alpha}=\eta^{\rho\alpha}\eta^{\varphi\beta}h_{\varphi\rho,\beta,\alpha}=h_{\varphi\rho}^{,\varphi,\rho}=h_{\alpha\beta}^{,\alpha,\beta},$$

$$\therefore -16\pi GT_{\mu\nu}=\left(h_{\mu\nu,\alpha}^{,\alpha}+h_{,\mu,\nu}-h_{\mu,\nu,\alpha}^{\alpha}-h_{\nu,\mu,\alpha}^{\alpha}\right)-\frac{1}{2}\eta_{\mu\nu}\left(h_{,\alpha}^{,\alpha}+h_{,\alpha}^{,\alpha}-h_{,\beta,\alpha}^{\beta\alpha}-h_{,\beta,\alpha}^{\beta\alpha}\right)$$

$$=\left(h_{\mu\nu,\alpha}^{,\alpha}-\frac{1}{2}\eta_{\mu\nu}h_{,\alpha}^{,\alpha}\right)+\frac{1}{2}h_{,\mu,\nu}+\frac{1}{2}h_{,\mu,\nu}-h_{\mu\alpha,\nu}^{,\alpha}-h_{\nu\alpha,\mu}^{,\alpha}-\frac{1}{2}\eta_{\mu\nu}\left(h_{,\alpha}^{,\alpha}-2h_{\alpha\beta}^{,\alpha,\beta}\right)$$

$$=\left(h_{\mu\nu,\alpha}^{,\alpha}-\frac{1}{2}\eta_{\mu\nu}h_{,\alpha}^{,\alpha}\right)-\left(h_{\mu\alpha,\nu}^{,\alpha}-\frac{1}{2}h_{,\mu,\nu}\right)-\left(h_{\nu\alpha,\mu}^{,\alpha}-\frac{1}{2}h_{,\nu,\mu}\right)+\eta_{\mu\nu}\left(h_{\alpha\beta}^{,\alpha,\beta}-\frac{1}{2}h_{,\alpha}^{,\alpha}\right)$$

$$=\left(h_{\mu\nu,\alpha}^{,\alpha}-\frac{1}{2}\eta_{\mu\nu}h_{,\alpha}^{,\alpha}\right)+\eta_{\mu\nu}\left(h_{\alpha\beta}^{,\alpha,\beta}-\frac{1}{2}\eta_{\alpha\beta}h^{,\alpha,\beta}\right)-\left(h_{\mu\alpha,\nu}^{,\alpha}-\frac{1}{2}\eta_{\mu\alpha}h_{,\nu}^{,\alpha}\right)-\left(h_{\nu\alpha,\mu}^{,\alpha}-\frac{1}{2}\eta_{\nu\alpha}h_{,\mu}^{,\alpha}\right)$$

$$=\bar{h}_{\mu\nu,\alpha}^{,\alpha}+\eta_{\mu\nu}\bar{h}_{\alpha\beta}^{,\alpha,\beta}-\bar{h}_{\mu\alpha,\nu}^{,\alpha}-\bar{h}_{\nu\alpha,\mu}^{,\alpha},$$

$$\therefore \bar{h}_{\mu\nu,\alpha}^{,\alpha}+\eta_{\mu\nu}\bar{h}_{\alpha\beta}^{,\alpha,\beta}-\bar{h}_{\mu\alpha,\nu}^{,\alpha}-\bar{h}_{\nu\alpha,\mu}^{,\alpha}=-16\pi GT_{\mu\nu}。$$

故线性化的场方程为

$$\bar{h}_{\mu\nu,\alpha}^{,\alpha}+\eta_{\mu\nu}\bar{h}_{\alpha\beta}^{,\alpha,\beta}-\bar{h}_{\mu\alpha,\nu}^{,\alpha}-\bar{h}_{\nu\alpha,\mu}^{,\alpha}=-16\pi GT_{\mu\nu}.$$

六、见赵峥第四章 P96-99 或第三章 PPT3.3

守恒量方程：

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda}=E\cdot\left(1-\frac{2M}{r}\right)^{-1}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}=\frac{\tilde{L}}{r^2}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2=E^2-\left(1-\frac{2M}{r}\right)\frac{\tilde{L}^2}{r^2}$$

利用上式可得可得轨道

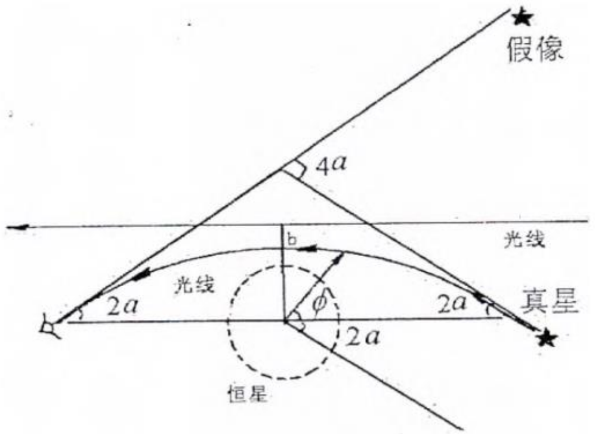
$$\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\varphi^2}+\tilde{u}=\frac{3GM}{c^2}\tilde{u}^2$$

作近似令修正项为 $A\cos^2\varphi+B$ 可得

$$u=\frac{1}{b}\sin\phi+\frac{GM}{c^2\cdot b^2}\left(\cos^2\phi+1\right)$$

利用泰勒展开可得偏转角

$$\delta=\frac{4GM}{c^2b}$$



七、

见赵峥 P131

$$\begin{cases} T = 4M\left(\dfrac{r}{2M}-1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \operatorname{sh} \dfrac{t}{4M}, \\ R = 4M\left(\dfrac{r}{2M}-1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \operatorname{ch} \dfrac{t}{4M}, \end{cases} \quad r > 2M, \text{Ⅰ 区}$$

$$\begin{cases} T = -4M\left(\dfrac{r}{2M}-1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \operatorname{sh} \dfrac{t}{4M}, \\ R = -4M\left(\dfrac{r}{2M}-1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \operatorname{ch} \dfrac{t}{4M}, \end{cases} \quad r > 2M, \text{Ⅱ 区}$$

$$\begin{cases} T = 4M\left(1-\dfrac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \operatorname{ch} \dfrac{t}{4M}, \\ R = 4M\left(1-\dfrac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \operatorname{sh} \dfrac{t}{4M}, \end{cases} \quad r < 2M, F \text{ 区}$$

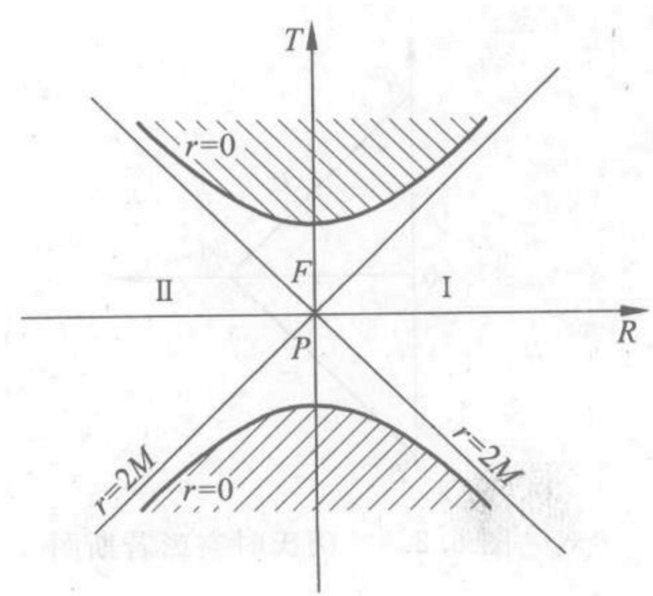
$$\begin{cases} T = -4M\left(1-\dfrac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \operatorname{ch} \dfrac{t}{4M}, \\ R = -4M\left(1-\dfrac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \operatorname{sh} \dfrac{t}{4M}, \end{cases} \quad r < 2M, P \text{ 区}$$

把史瓦西时空中的线元变成

$$ds^2 = \frac{2M}{r} e^{-r/2M}(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6.2.21}$$

此即克鲁斯卡坐标系下的线元表达式，(R, T) 即克鲁斯卡坐标。从(6.2.21)式的号差可以判定，T 是时间坐标，R 是空间坐标。其中 r 与 R、T 的关系由下式决定：

$$16M^2\left(\frac{r}{2M}-1\right)e^{\frac{r}{2M}} = R^2 - T^2 \tag{6.2.22}$$



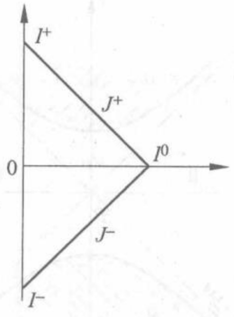


图 6.2.5　闵氏时空彭若斯图

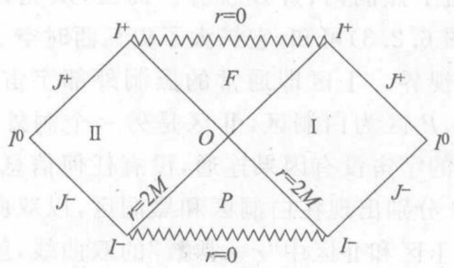


图 6.2.6　克鲁斯卡时空彭若斯图

(1) 类时未来无穷远 I^+：r 有限，$t\to+\infty$；

(2) 类时过去无穷远 I^-：r 有限，$t\to-\infty$；

(3) 类空无穷远 I^0：t 有限，$r\to\infty$；

(4) 类光未来无穷远 J^+：$(t-r)$ 有限，$(t+r)\to+\infty$；

(5) 类光过去无穷远 J^-：$(t+r)$ 有限，$(t-r)\to-\infty$。