3-1

3-1-1

由 $\gamma_i = -\mathrm{i}etalpha_i \Longrightarrow lpha_i = \mathrm{i}eta\gamma_i$

 α_i, β 满足:

$$lpha_ilpha_j+lpha_jlpha_i=2\delta_{ij}I,\quad i,j=1,2,3$$
 $lpha_ieta+etalpha_i=0$ $eta^2=I$

考虑:

$$\gamma_i = -\mathrm{i}etalpha_i$$

上式左乘 β 得:

$$eta \gamma_i = -\mathrm{i} eta^2 lpha_i = -\mathrm{i} lpha_i$$

因此:

$$lpha_i = rac{eta \gamma_i}{-\mathrm{i}} = \mathrm{i}eta \gamma_i$$

3-1-2

利用 Pauli 矩阵,写出 Dirac-Pauli 表象中 γ_μ 矩阵以及 $\gamma_5, C=\mathrm{i}\gamma_2\gamma_4$ 的具体形式。

二阶 Pauli 矩阵:

$$egin{aligned} \sigma_1^0 &= egin{bmatrix} 0 & 1 \ 1 & 0 \end{bmatrix}, \quad \sigma_2^0 &= egin{bmatrix} 0 & -\mathrm{i} \ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3^0 &= egin{bmatrix} 1 & 0 \ 0 & -1 \end{bmatrix} \ & lpha_i &= egin{bmatrix} 0 & \sigma_i^0 \ \sigma_i^0 & 0 \end{bmatrix}, \quad eta &= egin{bmatrix} I^0 & 0 \ 0 & -I^0 \end{bmatrix} \ & \gamma_i &\equiv -\mathrm{i}etalpha_i, \quad \gamma_4 &\equiv eta, \quad \gamma_5 &\equiv \gamma_1\gamma_2\gamma_3\gamma_4 \ & \gamma_i &= egin{bmatrix} 0 & -\mathrm{i}\sigma_i^0 \ \mathrm{i}\sigma_i^0 & 0 \end{bmatrix}, \quad \gamma_4 &= egin{bmatrix} I^0 & 0 \ 0 & -I^0 \end{bmatrix} \end{aligned}$$

$$C=\mathrm{i}\gamma_2\gamma_4$$

具体形式为:

$$\gamma_1 = egin{bmatrix} 0 & -\mathrm{i}\sigma_1^0 \ \mathrm{i}\sigma_1^0 & 0 \end{bmatrix} = egin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \ 0 & 0 & -\mathrm{i} & 0 \ 0 & \mathrm{i} & 0 & 0 \ \mathrm{i} & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_2 = egin{bmatrix} 0 & -\mathrm{i}\sigma_2^0 \ \mathrm{i}\sigma_2^0 & 0 \end{bmatrix} = egin{bmatrix} 0 & 0 & 0 & -1 \ 0 & 0 & 1 & 0 \ 0 & 1 & 0 & 0 \ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_3 = egin{bmatrix} 0 & -\mathrm{i}\sigma_3^0 \ \mathrm{i}\sigma_3^0 & 0 \end{bmatrix} = egin{bmatrix} 0 & 0 & -\mathrm{i} & 0 \ 0 & 0 & 0 & \mathrm{i} \ \mathrm{i} & 0 & 0 & 0 \ 0 & -\mathrm{i} & 0 & 0 \end{bmatrix}$$

$$\gamma_4 = egin{bmatrix} I^0 & 0 \ 0 & -I^0 \end{bmatrix} = egin{bmatrix} 1 & 0 & 0 & 0 \ 0 & 1 & 0 & 0 \ 0 & 0 & -1 & 0 \ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\gamma_5 \equiv \gamma_1 \gamma_2 \gamma_3 \gamma_4 = egin{bmatrix} 0 & 0 & -1 & 0 \ 0 & 0 & 0 & -1 \ -1 & 0 & 0 & 0 \ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$C=\mathrm{i}\gamma_2\gamma_4=egin{bmatrix} 0 & 0 & 0 & \mathrm{i} \ 0 & 0 & -\mathrm{i} & 0 \ 0 & \mathrm{i} & 0 & 0 \ -\mathrm{i} & 0 & 0 & 0 \end{bmatrix}$$

3-1-3

定义

$$\sigma_i = egin{bmatrix} \sigma_i^0 & 0 \ 0 & \sigma_i^0 \end{bmatrix} = rac{1}{2\mathrm{i}} arepsilon_{ijk} \gamma_j \gamma_k$$

其中, σ_i^0 具有以下性质:

$$\sigma_i^0\sigma_j^0+\sigma_j^0\sigma_i^0=2\delta_{ij}I^0$$

$$\sigma_i^0\sigma_j^0=\delta_{ij}I^0+\mathrm{i}arepsilon_{ijk}\sigma_k^0$$

证明:

$$egin{aligned} \sigma_i\sigma_j+\sigma_j\sigma_i&=2\delta_{ij}I\ \gamma_i\gamma_j&=\sigma_i\sigma_j=\delta_{ij}I+\mathrm{i}arepsilon_{ijk}\sigma_k\ \sigma_1\sigma_2&=\mathrm{i}\sigma_3\ \sigma_2\sigma_3&=\mathrm{i}\sigma_1\ \sigma_3\sigma_1&=\mathrm{i}\sigma_2\ ec\sigma=-eclpha\gamma_5 \end{aligned}$$

证明 $\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}I$

$$egin{aligned} \sigma_i \sigma_j + \sigma_j \sigma_i &= egin{bmatrix} \sigma_i^0 & 0 \ 0 & \sigma_i^0 \end{bmatrix} egin{bmatrix} \sigma_j^0 & 0 \ 0 & \sigma_i^0 \end{bmatrix} + egin{bmatrix} \sigma_j^0 & 0 \ 0 & \sigma_j^0 \end{bmatrix} egin{bmatrix} \sigma_i^0 & 0 \ 0 & \sigma_i^0 \end{bmatrix} \ &= egin{bmatrix} \sigma_i^0 \sigma_j^0 + \sigma_j^0 \sigma_i^0 & 0 \ 0 & \sigma_i^0 \sigma_j^0 + \sigma_j^0 \sigma_i^0 \end{bmatrix} \ &= egin{bmatrix} 2\delta_{ij} I^0 & 0 \ 0 & 2\delta_{ij} I^0 \end{bmatrix} \ &= 2\delta_{ij} I \end{aligned}$$

证明 $\gamma_i\gamma_j=\sigma_i\sigma_j=\delta_{ij}I+\mathrm{i}arepsilon_{ijk}\sigma_k$

二阶泡利矩阵满足:

$$\sigma_i^0\sigma_j^0=\delta_{ij}I^0+\mathrm{i}arepsilon_{ijk}\sigma_k^0$$

四阶泡利矩阵:

$$egin{aligned} \sigma_i \sigma_j &= egin{bmatrix} \sigma_i^0 & 0 \ 0 & \sigma_i^0 \end{bmatrix} egin{bmatrix} \sigma_j^0 & 0 \ 0 & \sigma_i^0 \sigma_j^0 \end{bmatrix} \ &= egin{bmatrix} \sigma_i^0 \sigma_j^0 & 0 \ 0 & \sigma_i^0 \sigma_j^0 \end{bmatrix} \ &= egin{bmatrix} \delta_{ij} I^0 + \mathrm{i} arepsilon_{ijk} \sigma_k^0 & 0 \ 0 & \delta_{ij} I^0 + \mathrm{i} arepsilon_{ijk} \sigma_k^0 \end{bmatrix} \ &= \delta_{ij} egin{bmatrix} I^0 & 0 \ 0 & I^0 \end{bmatrix} + \mathrm{i} arepsilon_{ijk} egin{bmatrix} \sigma_k^0 & 0 \ 0 & \sigma_k^0 \end{bmatrix} \ &= \delta_{ij} I + \mathrm{i} arepsilon_{ijk} \sigma_k \end{aligned}$$

$$\sigma_i = rac{1}{2 \mathrm{i}} arepsilon_{ijk} \gamma_j \gamma_k$$

同乘 ε_{ilm} 并对 i 求和:

$$egin{aligned} arepsilon_{ilm} \sigma_i &= rac{1}{2\mathrm{i}} arepsilon_{ilm} arepsilon_{ijk} \gamma_j \gamma_k \ &= rac{1}{2\mathrm{i}} \left(\delta_{lj} \delta_{mk} - \delta_{lk} \delta_{mj}
ight) \gamma_j \gamma_k \ &= rac{1}{2\mathrm{i}} \left(\gamma_l \gamma_m - \gamma_m \gamma_l
ight) \end{aligned}$$

即:

$$\gamma_l \gamma_m - \gamma_m \gamma_l = 2 \mathrm{i} arepsilon_{ilm} \sigma_i$$

另一方面,利用 γ_{μ} 矩阵反对易关系

$$\gamma_l\gamma_m+\gamma_m\gamma_l=2\delta_{lm}I$$

两式相加可得:

$$\gamma_l \gamma_m = \delta_{lm} I + \mathrm{i} arepsilon_{ilm} \sigma_i = \delta_{lm} I + \mathrm{i} arepsilon_{lmi} \sigma_i$$

替换哑标得:

$$egin{aligned} \gamma_i \gamma_j &= \delta_{ij} I + \mathrm{i} arepsilon_{ijk} \sigma_k \ \\ \gamma_i \gamma_j &= \sigma_i \sigma_j = \delta_{ij} I + \mathrm{i} arepsilon_{ijk} \sigma_k \end{aligned}$$

证明 $\sigma_1\sigma_2=\mathrm{i}\sigma_3,\sigma_2\sigma_3=\mathrm{i}\sigma_1,\sigma_3\sigma_1=\mathrm{i}\sigma_2$

由于 $\sigma_i \sigma_j = \delta_{ij} I + \mathrm{i} \varepsilon_{ijk} \sigma_k$, 因此:

$$egin{aligned} \sigma_1\sigma_2 &= \mathrm{i}arepsilon_{12k}\sigma_k = \mathrm{i}\sigma_3 \ & \ \sigma_2\sigma_3 &= \mathrm{i}arepsilon_{23k}\sigma_k = \mathrm{i}\sigma_1 \ & \ \sigma_3\sigma_1 &= \mathrm{i}arepsilon_{31k}\sigma_k = \mathrm{i}\sigma_2 \end{aligned}$$

证明 $\vec{\sigma} = -\vec{\alpha}\gamma_5$

$$egin{aligned} lpha_i &= egin{bmatrix} 0 & \sigma_i^0 \ \sigma_i^0 & 0 \end{bmatrix}, \quad eta &= egin{bmatrix} I^0 & 0 \ 0 & -I^0 \end{bmatrix} \ \gamma_i &= -\mathrm{i}etalpha_i &= -\mathrm{i}egin{bmatrix} I^0 & 0 \ 0 & -I^0 \end{bmatrix} egin{bmatrix} 0 & \sigma_i^0 \ \sigma_i^0 & 0 \end{bmatrix} &= egin{bmatrix} 0 & -\mathrm{i}\sigma_i^0 \ \mathrm{i}\sigma_i^0 & 0 \end{bmatrix} \ \gamma_4 &= eta &= egin{bmatrix} I^0 & 0 \ 0 & -I^0 \end{bmatrix} \end{aligned}$$

于是:

$$\begin{split} -\vec{\alpha}\gamma_5 &= -\alpha_i \vec{e}_i \gamma_1 \gamma_2 \gamma_3 \gamma_4 \\ &= -\alpha_i \vec{e}_i \begin{bmatrix} 0 & -\mathrm{i}\sigma_1^0 \\ \mathrm{i}\sigma_1^0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\mathrm{i}\sigma_2^0 \\ \mathrm{i}\sigma_2^0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\mathrm{i}\sigma_3^0 \\ \mathrm{i}\sigma_3^0 & 0 \end{bmatrix} \begin{bmatrix} I^0 & 0 \\ 0 & -I^0 \end{bmatrix} \\ &= -\alpha_i \vec{e}_i \begin{bmatrix} 0 & \mathrm{i}\sigma_1^0 \sigma_2^0 \sigma_3^0 \\ \mathrm{i}\sigma_1^0 \sigma_2^0 \sigma_3^0 & 0 \end{bmatrix} \\ &= -\alpha_i \vec{e}_i \begin{bmatrix} 0 & -I^0 \\ -I^0 & 0 \end{bmatrix} \\ &= -\begin{bmatrix} 0 & \sigma_i^0 \\ \sigma_i^0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -I^0 \\ -I^0 & 0 \end{bmatrix} \vec{e}_i \\ &= \begin{bmatrix} \sigma_i^0 & 0 \\ 0 & \sigma_i^0 \end{bmatrix} \vec{e}_i \\ &= \sigma_i \vec{e}_i \\ &= \vec{\sigma} \end{split}$$

3-1-4

已知
$$H=ec{lpha}\cdotec{p}+eta m, ec{L}=ec{r} imesec{p}$$
,证明: $\left[H,ec{L}
ight]=-\mathrm{i}ec{lpha} imesec{p}, \left[H,ec{\sigma}
ight]=2\mathrm{i}ec{lpha} imesec{p}$ $L_i=\left(ec{r} imesec{p}
ight)_i=arepsilon_{ijk}x_jp_k$

$$egin{aligned} [H,L_i] &= [lpha_l p_l + eta m, arepsilon_{ijk} x_j p_k] \ &= arepsilon_{ijk} lpha_l \left[p_l, x_j p_k
ight] \ &= arepsilon_{ijk} lpha_l \left(x_j \left[p_l, p_k
ight] + \left[p_l, x_j
ight] p_k
ight) \ &= arepsilon_{ijk} lpha_l \left(-\mathrm{i}\hbar \delta_{lj}
ight) p_k \ &= -\mathrm{i}\hbar arepsilon_{ijk} lpha_j p_k \ &= -\mathrm{i}\hbar \left(ec{lpha} imes ec{p}
ight)_i \ &= -\mathrm{i} \left(ec{lpha} imes ec{p}
ight)_i \end{aligned}$$

因此:

$$\left[H,ec{L}
ight]=-\mathrm{i}ec{lpha} imesec{p}$$

$$\begin{split} [\alpha_i,\alpha_j] &= \alpha_i\alpha_j - \alpha_j\alpha_i \\ &= \begin{bmatrix} 0 & \sigma_i^0 \\ \sigma_i^0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_j^0 \\ \sigma_j^0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma_j^0 \\ \sigma_j^0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_i^0 \\ \sigma_i^0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_i^0\sigma_j^0 - \sigma_j^0\sigma_i^0 & 0 \\ 0 & \sigma_i^0\sigma_j^0 - \sigma_j^0\sigma_i^0 \end{bmatrix} \\ &= 2\mathrm{i}\varepsilon_{ijk}\sigma_k \end{split}$$

$$\begin{split} [H,\sigma_i] &= \left[\alpha_l p_l + \beta m, \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \gamma_j \gamma_k\right] \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left[\alpha_l p_l + \beta m, \gamma_j \gamma_k\right] \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left[\alpha_l p_l + \beta m, (-\mathrm{i}\beta\alpha_j) \left(-\mathrm{i}\beta\alpha_k\right)\right] \\ &= -\frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left[\alpha_l p_l + \beta m, \beta\alpha_j \beta\alpha_k\right] \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left[\alpha_l p_l + \beta m, \alpha_j \beta\beta\alpha_k\right] \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left[\alpha_l p_l + \beta m, \alpha_j \beta\beta\alpha_k\right] \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left([\alpha_l p_l + \beta m, \alpha_j \alpha_k\right] \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left([\alpha_l p_l, \alpha_j \alpha_k] + m \left[\beta, \alpha_j \alpha_k\right]\right) \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left\{\alpha_l \left[p_l, \alpha_j \alpha_k\right] + \left[\alpha_l, \alpha_j \alpha_k\right] p_l + m \left(\beta\alpha_j \alpha_k - \alpha_j \alpha_k \beta\right)\right\} \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left\{(\alpha_j \left[\alpha_l, \alpha_k\right] + \left[\alpha_l, \alpha_j\right] \alpha_k\right) p_l + m \beta\alpha_j \alpha_k - m \beta\alpha_j \alpha_k\right\} \\ &= \frac{1}{2\mathrm{i}} \varepsilon_{ijk} \left(2\mathrm{i}\varepsilon_{lkn}\alpha_j \alpha_n + 2\mathrm{i}\varepsilon_{ljq}\alpha_q \alpha_k\right) p_l \\ &= \left(\varepsilon_{kij}\varepsilon_{knl}\alpha_j \alpha_n + \varepsilon_{jki}\varepsilon_{jql}\alpha_q \alpha_k\right) p_l \\ &= \left(\varepsilon_{kij}\varepsilon_{knl}\alpha_j \alpha_n + \varepsilon_{jki}\varepsilon_{jql}\alpha_q \alpha_k\right) p_l \\ &= \left(\varepsilon_{l\alpha_i} - \delta_{il}\alpha_n \alpha_n + \delta_{il}\alpha_k \alpha_k - \alpha_i \alpha_l\right) p_l \\ &= \left[\alpha_l, \alpha_i\right] p_l \\ &= 2\mathrm{i}\varepsilon_{ilt}\alpha_t p_l \\ &= 2\mathrm{i}\varepsilon_{ilt}\alpha_t p_l \\ &= 2\mathrm{i} \left(\vec{\alpha} \times \vec{p}\right)_i \end{split}$$

因此:

$$[H,ec{\sigma}]=2\mathrm{i}ec{lpha} imesec{p}$$

3-1-5

证明 $\vec{\sigma}$ 的本征值为 +1, -1.

利用公式

$$\left(ec{\sigma} \cdot ec{A}
ight) \left(ec{\sigma} \cdot ec{B}
ight) = ec{A} \cdot ec{B} + \mathrm{i} ec{\sigma} \cdot \left(ec{A} imes ec{B}
ight)$$

有:

$$\left(ec{\sigma} \cdot ec{n}
ight)^2 = ec{n} \cdot ec{n} + \mathrm{i} ec{\sigma} \cdot \left(ec{n} imes ec{n}
ight) = 1$$

设 $(\vec{\sigma} \cdot \vec{n})$ 的本征方程为:

$$(\vec{\sigma} \cdot \vec{n}) u(p) = \lambda u(p)$$

则:

$$\left(ec{\sigma}\cdot ec{n}
ight) ^{2}u(p)=\lambda^{2}u(p)$$

又

$$\left(\vec{\sigma} \cdot \vec{n} \right)^2 u(p) = u(p)$$

对比可得 $(\vec{\sigma} \cdot \vec{n})$ 的本征值为:

$$\lambda = \pm 1$$

3-1-6

证明有电磁场存在时, Dirac 方程是 Lorentz 协变的。

电磁场存在时 x' 系的 Dirac 方程为:

$$\left(\gamma_{\mu}\partial_{\mu}^{\prime}-\mathrm{i}eA_{\mu}^{\prime}\gamma_{\mu}+m\right)\psi^{\prime}(x^{\prime})=0$$

时空坐标进行 Lorentz 变换:

$$x_{\mu}
ightarrow x_{\mu}' = A_{\mu
u} x_{
u}, \quad A_{\mu \lambda} x_{\mu}' = x_{\lambda}$$

x' 系的物理量用 x 系的物理量表达:

$$egin{align} \partial'_{\mu} &\equiv rac{\partial}{\partial x'_{\mu}} = rac{\partial x_{
u}}{\partial x'_{\mu}} rac{\partial}{\partial x_{
u}} = A_{\mu
u}\partial_{
u} \ & \ \psi'(x') = \Lambda \psi(x) \ & \ A'_{\mu} = A_{\mu
u}A_{
u} \ & \ \end{pmatrix}$$

则 Dirac 方程化为:

$$egin{aligned} 0 &= \left(\gamma_{\mu} \partial_{\mu}' - \mathrm{i} e A_{\mu}' \gamma_{\mu} + m
ight) \psi'(x') \ &= \left(\gamma_{\mu} A_{\mu
u} \partial_{
u} - \mathrm{i} e A_{\mu
u} A_{
u} \gamma_{\mu} + m
ight) \Lambda \psi(x) \end{aligned}$$

左乘 Λ^{-1} ,并利用

$$\Lambda^{-1}\gamma_{\mu}\Lambda=A_{\mu
ho}\gamma_{
ho}$$

可得:

$$egin{aligned} 0 &= \Lambda^{-1} \left(\gamma_{\mu} A_{\mu
u} \partial_{
u} - \mathrm{i} e A_{\mu
u} A_{
u} \gamma_{\mu} + m
ight) \Lambda \psi(x) \ &= \left(A_{\mu
ho} \gamma_{
ho} A_{\mu
u} \partial_{
u} - \mathrm{i} e A_{\mu
u} A_{
u} A_{\mu
ho} \gamma_{
ho} + m
ight) \psi(x) \ &= \left(\delta_{
ho
u} \gamma_{
ho} \partial_{
u} - \mathrm{i} e \delta_{
u
ho} A_{
u} \gamma_{
ho} + m
ight) \psi(x) \ &= \left(\gamma_{
u} \partial_{
u} - \mathrm{i} e A_{
ho} \gamma_{
ho} + m
ight) \psi(x) \end{aligned}$$

因此, x 系中的 Dirac 方程为:

$$\left(\gamma_{\mu}\partial_{\mu}-\mathrm{i}eA_{\mu}\gamma_{\mu}+m
ight)\psi(x)=0$$

与 x' 系中的 Dirac 方程

$$\left(\gamma_{\mu}\partial_{\mu}^{\prime}-\mathrm{i}eA_{\mu}^{\prime}\gamma_{\mu}+m
ight)\psi^{\prime}(x^{\prime})=0$$

对比可知, 电磁场存在时 Dirac 方程具有 Lorentz 协变性。

3-1-7

证明 $\bar{\psi}\gamma_{\mu}\gamma_{5}\psi$ 是 Lorentz 赝矢量。

$$egin{aligned} ar{\psi}'\gamma_{\mu}\gamma_{5}\psi' &= kar{\psi}\Lambda^{-1}\gamma_{\mu}\gamma_{5}\Lambda\psi \ &= ar{\psi}\Lambda^{-1}\gamma_{\mu}\gamma_{5}\Lambda\psi \ &= ar{\psi}\Lambda^{-1}\gamma_{\mu}\Lambda\Lambda^{-1}\gamma_{5}\Lambda\psi \ &= ar{\psi}A_{\mu
u}\gamma_{
u}\left|A\right|\gamma_{5}\psi \ &= \left|A\right|A_{\mu
u}ar{\psi}\gamma_{
u}\gamma_{5}\psi \end{aligned}$$

即 $\bar{\psi}\gamma_{\mu}\gamma_{5}\psi$ 服从赝矢量的变换规律,因此 $\bar{\psi}\gamma_{\mu}\gamma_{5}\psi$ 是赝矢量。

3-1-8

利用 $u_a(ec p)$ 的正交完备性,证明 $u_a(ec p')$ 的正交完备性(空间反射变换)。

 $u_a(\vec{p})$ 的正交完备性给出:

 $u_a(-\vec{p})$ 的正交性:

$$egin{aligned} u_a^\dagger(-ec{p})u_b(-ec{p}) &= \left[\eta_P\gamma_4u_a(ec{p})
ight]^\dagger\eta_P\gamma_4u_b(ec{p}) \ &= \eta_P^\dagger\eta_Pu_a^\dagger(ec{p})\gamma_4^\dagger\gamma_4u_b(ec{p}) \ &= u_a^\dagger(ec{p})u_b(ec{p}) \ &= \delta_{ab} \end{aligned}$$

 $u_a(-\vec{p})$ 的完备性:

$$egin{aligned} u_a(-ec{p})u_a^\dagger(-ec{p}) &= \eta_P\gamma_4 u_a(ec{p}) \left[\eta_P\gamma_4 u_a(ec{p})
ight]^\dagger \ &= \eta_P\eta_P^\dagger\gamma_4 u_a(ec{p})u_a^\dagger(ec{p})\gamma_4^\dagger \ &= \gamma_4 I\gamma_4^\dagger \ &= I \end{aligned}$$