两点关联函数

用Grassmann积分表达 $\left\langle\Psi_1\,\middle|\,F\left(a,a^\dagger\right)\,\middle|\,\Psi_2\right\rangle$

构造

$$\tilde{z}\equiv\left(\bar{z}_1,\cdots,\bar{z}_N,z_1,\cdots,z_N\right)^\top, \tag{1}$$

完备性关系

$$\int\left(\prod_i \mathrm{d}\bar{z}_i\mathrm{d}z_i\right)\exp\left(-\sum_j\bar{z}_jz_j\right)|z\rangle\langle z|=1 \tag{2}$$

可化为

$$(-1)^{N(N-1)/2}\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\exp\left(-\sum_j\bar{z}_iz_j\right)|z\rangle\langle z|=1 \tag{3}$$

$\langle\Psi_1\mid a_la_m\mid\Psi_2\rangle$

$$\begin{aligned}
&\langle\Psi_1\mid a_la_m\mid\Psi_2\rangle\\
=&\langle\Psi_1\mid a_la_m1\mid\Psi_2\rangle\\
=&(-1)^{N(N+1)/2}\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\exp\left(-\sum_j\bar{z}_jz_j\right)\langle\Psi_1\mid a_la_m\mid z\rangle\langle z\mid\Psi_2\rangle\\
=&(-1)^{N(N+1)/2}\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\exp\left(-\sum_j\bar{z}_jz_j\right)\langle\Psi_1\mid a_la_m\mid z\rangle\langle z\mid\Psi_2\rangle\\
=&(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\exp\left(-\sum_j\bar{z}_jz_j\right)\left\langle 0\,\middle|\exp\left(\frac{1}{2}\sum_{i,j}f_{i,j}^{(1)*}a_ja_i\right)\middle|\,a_la_m\,\middle|\,z\right\rangle\left\langle z\,\middle|\exp\left(\frac{1}{2}\sum_{i,j}f_{i,j}^{(2)}a_i^\dagger a_j^\dagger\right)\middle|\,0\right\rangle\\
=&(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\exp\left(-\sum_j\bar{z}_jz_j\right)\exp\left(\frac{1}{2}\sum_{i,j}f_{i,j}^{(1)*}z_jz_i\right)z_lz_m\exp\left(\frac{1}{2}\sum_{i,j}f_{i,j}^{(2)}\bar{z}_i\bar{z}_j\right)\langle 0\mid z\rangle\langle z\mid 0\rangle\\
=&(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)z_lz_m\exp\left(-\sum_j\bar{z}_jz_j\right)\exp\left(-\frac{1}{2}\sum_{i,j}f_{i,j}^{(1)*}z_iz_j\right)\exp\left(\frac{1}{2}\sum_{i,j}f_{i,j}^{(2)}\bar{z}_i\bar{z}_j\right)\\
=&(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)z_lz_m\exp\left(\frac{1}{2}\begin{pmatrix}\bar{z} & z\end{pmatrix}\begin{pmatrix}f^{(2)} & -I\\ I & -f^{(1)*}\end{pmatrix}\begin{pmatrix}\bar{z}\\ z\end{pmatrix}\right)\\
=&(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)z_lz_m\exp\left(\frac{1}{2}\tilde{z}^\top M\tilde{z}\right)
\end{aligned} \tag{4}$$

其中

$$M\equiv\begin{pmatrix}f^{(2)} & -I\\ I & -f^{(1)*}\end{pmatrix} \tag{5}$$

$\left\langle \Psi_1 \,\middle|\, a_l a_m^\dagger \,\middle|\, \Psi_2 \right\rangle$

$$\begin{aligned}
&\left\langle \Psi_1 \,\middle|\, a_l a_m^\dagger \,\middle|\, \Psi_2 \right\rangle \\
=&\left\langle \Psi_1 \,\middle|\, a_l 1 a_m^\dagger \,\middle|\, \Psi_2 \right\rangle \\
=&(-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) \exp\left(-\sum_j \bar{z}_j z_j\right) \left\langle 0 \,\middle|\, \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(1)*} a_j a_i\right) \,\middle|\, a_l \,\middle|\, z \right\rangle \left\langle z \,\middle|\, a_m^\dagger \exp\left(\frac{1}{2}\sum_{i,j} f_{i,j}^{(2)} a_i^\dagger a_j^\dagger\right) \,\middle|\, 0 \right\rangle \\
=&(-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) z_l \bar{z}_m \exp\left(\frac{1}{2}\tilde{z}^\top M \tilde{z}\right)
\end{aligned} \tag{6}$$

$\left\langle \Psi_1 \,\middle|\, a_l^\dagger a_m \,\middle|\, \Psi_2 \right\rangle$

$$\begin{aligned}
\left\langle \Psi_1 \,\middle|\, a_l^\dagger a_m \,\middle|\, \Psi_2 \right\rangle &= \left\langle \Psi_1 \,\middle|\, \left(\delta_{l,m} - a_m a_l^\dagger\right) \,\middle|\, \Psi_2 \right\rangle \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) (\delta_{l,m} - z_m \bar{z}_l) \exp\left(\frac{1}{2}\tilde{z}^\top M \tilde{z}\right) \\
(l \neq m) &= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) \bar{z}_l z_m \exp\left(\frac{1}{2}\tilde{z}^\top M \tilde{z}\right)
\end{aligned} \tag{7}$$

$\left\langle \Psi_1 \,\middle|\, a_l^\dagger a_m^\dagger \,\middle|\, \Psi_2 \right\rangle$

$$\begin{aligned}
\left\langle \Psi_1 \,\middle|\, a_l^\dagger a_m^\dagger \,\middle|\, \Psi_2 \right\rangle &= \left\langle \Psi_1 \,\middle|\, 1 a_l^\dagger a_m^\dagger \,\middle|\, \Psi_2 \right\rangle \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) \bar{z}_l \bar{z}_m \exp\left(\frac{1}{2}\tilde{z}^\top M \tilde{z}\right)
\end{aligned} \tag{8}$$

如果定义

$$\tilde{a} \equiv \left(a_1, \cdots, a_N, a_1^\dagger, \cdots, a_N^\dagger\right)^\top, \tag{9}$$

则

- $1 \leqslant i \leqslant N, 1 \leqslant j \leqslant N$

$$\begin{aligned}
\langle \Psi_1 \,|\, \tilde{a}_i \tilde{a}_j \,|\, \Psi_2 \rangle &= \langle \Psi_1 \,|\, a_i a_j \,|\, \Psi_2 \rangle \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) z_i z_j \exp\left(\frac{1}{2}\tilde{z}^\top M \tilde{z}\right) \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) \tilde{z}_{i+N} \tilde{z}_{j+N} \exp\left(\frac{1}{2}\tilde{z}^\top M \tilde{z}\right)
\end{aligned} \tag{10}$$

- $1 \leqslant i \leqslant N, N+1 \leqslant j \leqslant 2N$

$$\begin{aligned}
\langle \Psi_1 \,|\, \tilde{a}_i \tilde{a}_j \,|\, \Psi_2 \rangle &= \left\langle \Psi_1 \,\middle|\, a_i a_{j-N}^\dagger \,\middle|\, \Psi_2 \right\rangle \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) z_i \bar{z}_{j-N} \exp\left(\frac{1}{2}\tilde{z}^\top M \tilde{z}\right) \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i\right) \tilde{z}_{i+N} \tilde{z}_{j-N} \exp\left(\frac{1}{2}\tilde{z}^\top M \tilde{z}\right)
\end{aligned} \tag{11}$$

- $N+1 \leqslant i \leqslant 2N, 1 \leqslant j \leqslant N, i-N \neq j$

$$
\begin{aligned}
\langle\Psi_1 \mid \tilde{a}_i \tilde{a}_j \mid \Psi_2\rangle &= \left\langle \Psi_1 \,\middle|\, a_{i-N}^\dagger a_j \,\middle|\, \Psi_2 \right\rangle \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i \right) \bar{z}_{i-N} z_j \exp\left(\frac{1}{2} \tilde{z}^\top M \tilde{z} \right) \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i \right) \tilde{z}_{i-N} \tilde{z}_{j+N} \exp\left(\frac{1}{2} \tilde{z}^\top M \tilde{z} \right)
\end{aligned} \tag{12}
$$

- $N+1 \leqslant i \leqslant 2N, N+1 \leqslant j \leqslant 2N$

$$
\begin{aligned}
\langle\Psi_1 \mid \tilde{a}_i \tilde{a}_j \mid \Psi_2\rangle &= \left\langle \Psi_1 \,\middle|\, a_{i-N}^\dagger a_{j-N}^\dagger \,\middle|\, \Psi_2 \right\rangle \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i \right) \bar{z}_{i-N} \bar{z}_{j-N} \exp\left(\frac{1}{2} \tilde{z}^\top M \tilde{z} \right) \\
&= (-1)^{N(N+1)/2} \mathcal{N}_1^* \mathcal{N}_2 \int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i \right) \tilde{z}_{i-N} \tilde{z}_{j-N} \exp\left(\frac{1}{2} \tilde{z}^\top M \tilde{z} \right)
\end{aligned} \tag{13}
$$

计算 $\exp$ 前有Grassmann数的积分

现在要计算形如

$$
\int \left(\prod_{i=1}^{2N} \mathrm{d}\tilde{z}_i \right) \tilde{z}_l \tilde{z}_m \exp\left(\frac{1}{2} \tilde{z}^\top M \tilde{z} \right) \tag{14}
$$

的积分。

根据

$$
\int \left(\prod_i \mathrm{d}\theta_i \right) \exp\left(\frac{1}{2} \Theta^\top A \Theta \right) \exp\left(\eta^\top \Theta \right) = \mathrm{Pf}(A) \exp\left(\frac{1}{2} \eta^\top A^{-1} \eta \right) \tag{15}
$$

两边同时求偏导 $\partial^2 / \partial\eta_m \partial\eta_l$，左边：

$$
\begin{aligned}
\mathrm{LHS} &= \frac{\partial^2}{\partial\eta_m \partial\eta_l} \int \left(\prod_i \mathrm{d}\theta_i \right) \exp\left(\frac{1}{2} \Theta^\top A \Theta \right) \exp\left(\eta^\top \Theta \right) \\
&= \int \left(\prod_i \mathrm{d}\theta_i \right) \exp\left(\frac{1}{2} \Theta^\top A \Theta \right) \frac{\partial^2}{\partial\eta_m \partial\eta_l} \exp\left(\eta^\top \Theta \right) \\
&= \int \left(\prod_i \mathrm{d}\theta_i \right) \exp\left(\frac{1}{2} \Theta^\top A \Theta \right) (-\theta_l \theta_m) \exp\left(\eta^\top \Theta \right)
\end{aligned} \tag{16}
$$

右边：

$$
\begin{aligned}
\mathrm{RHS} &= \frac{\partial^2}{\partial\eta_m \partial\eta_l} \mathrm{Pf}(A) \exp\left(\frac{1}{2} \eta^\top A^{-1} \eta \right) \\
&= \mathrm{Pf}(A) \frac{\partial^2}{\partial\eta_m \partial\eta_l} \exp\left(\frac{1}{2} \sum_{i,j} \left(A^{-1} \right)_{ij} \eta_i \eta_j \right) \\
&= \mathrm{Pf}(A) \left(A^{-1} \right)_{l,m} \exp\left(\frac{1}{2} \eta^\top A^{-1} \eta \right)
\end{aligned} \tag{17}
$$

于是

$$
\int \left(\prod_i \mathrm{d}\theta_i \right) \exp\left(\frac{1}{2} \Theta^\top A \Theta \right) (-\theta_l \theta_m) \exp\left(\eta^\top \Theta \right) = \mathrm{Pf}(A) \left(A^{-1} \right)_{l,m} \exp\left(\frac{1}{2} \eta^\top A^{-1} \eta \right) \tag{18}
$$

对比两边关于 η 的零次项，得

$$\int\left(\prod_i \mathrm{d}\theta_i\right)\exp\left(\frac{1}{2}\Theta^\top A\Theta\right)(-\theta_l\theta_m)=\mathrm{Pf}(A)\left(A^{-1}\right)_{l,m} \tag{19}$$

也即

$$\int\left(\prod_i \mathrm{d}\theta_i\right)\theta_l\theta_m\exp\left(\frac{1}{2}\Theta^\top A\Theta\right)=\mathrm{Pf}(A)(-1)\left(A^{-1}\right)_{l,m} \tag{20}$$

$$\langle\Psi_1\,|\,\mathrm{i}c_{i,A}c_{j,B}\,|\,\Psi_2\rangle=\left[\mathcal{N}_1^*\mathcal{N}_2(-1)^{N(N+1)/2}\mathrm{Pf}(M)\right]\left[-\left(M^{-1}\right)_{i+N,j+N}+\left(M^{-1}\right)_{i+N,j}-\left(M^{-1}\right)_{i,j+N}+\left(M^{-1}\right)_{i,j}\right] \tag{21}$$

$$\langle\Psi_1\,|\,c_{i,B}c_{j,B}\,|\,\Psi_2\rangle=\left[\mathcal{N}_1^*\mathcal{N}_2(-1)^{N(N+1)/2}\mathrm{Pf}(M)\right]\left[\left(M^{-1}\right)_{i+N,j+N}-\left(M^{-1}\right)_{i+N,j}-\left(M^{-1}\right)_{i,j+N}+\left(M^{-1}\right)_{i,j}\right] \tag{22}$$

- $1\leqslant i\leqslant N, 1\leqslant j\leqslant N$

$$\begin{aligned}\langle\Psi_1\,|\,\tilde{a}_i\tilde{a}_j\,|\,\Psi_2\rangle&=\langle\Psi_1\,|\,a_ia_j\,|\,\Psi_2\rangle\\&=(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\tilde{z}_{i+N}\tilde{z}_{j+N}\exp\left(\frac{1}{2}\tilde{z}^\top M\tilde{z}\right)\\&=(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\mathrm{Pf}(M)(-1)\left(M^{-1}\right)_{i+N,j+N}\end{aligned} \tag{23}$$

- $1\leqslant i\leqslant N, N+1\leqslant j\leqslant 2N$

$$\begin{aligned}\langle\Psi_1\,|\,\tilde{a}_i\tilde{a}_j\,|\,\Psi_2\rangle&=\left\langle\Psi_1\,\middle|\,a_ia_{j-N}^\dagger\,\middle|\,\Psi_2\right\rangle\\&=(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\tilde{z}_{i+N}\tilde{z}_{j-N}\exp\left(\frac{1}{2}\tilde{z}^\top M\tilde{z}\right)\\&=(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\mathrm{Pf}(M)(-1)\left(M^{-1}\right)_{i+N,j-N}\end{aligned} \tag{24}$$

- $N+1\leqslant i\leqslant 2N, 1\leqslant j\leqslant N, i-N\neq j$

$$\begin{aligned}\langle\Psi_1\,|\,\tilde{a}_i\tilde{a}_j\,|\,\Psi_2\rangle&=\left\langle\Psi_1\,\middle|\,a_{i-N}^\dagger a_j\,\middle|\,\Psi_2\right\rangle\\&=(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\tilde{z}_{i-N}\tilde{z}_{j+N}\exp\left(\frac{1}{2}\tilde{z}^\top M\tilde{z}\right)\\&=(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\mathrm{Pf}(M)(-1)\left(M^{-1}\right)_{i-N,j+N}\end{aligned} \tag{25}$$

- $N+1\leqslant i\leqslant 2N, N+1\leqslant j\leqslant 2N$

$$\begin{aligned}\langle\Psi_1\,|\,\tilde{a}_i\tilde{a}_j\,|\,\Psi_2\rangle&=\left\langle\Psi_1\,\middle|\,a_{i-N}^\dagger a_{j-N}^\dagger\,\middle|\,\Psi_2\right\rangle\\&=(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\int\left(\prod_{i=1}^{2N}\mathrm{d}\tilde{z}_i\right)\tilde{z}_{i-N}\tilde{z}_{j-N}\exp\left(\frac{1}{2}\tilde{z}^\top M\tilde{z}\right)\\&=(-1)^{N(N+1)/2}\mathcal{N}_1^*\mathcal{N}_2\mathrm{Pf}(M)(-1)\left(M^{-1}\right)_{i-N,j-N}\end{aligned} \tag{26}$$

两点关联函数

$$\langle\tilde{a}_i\tilde{a}_j\rangle\equiv\frac{\left\langle 0\,\middle|\,\exp\left(\frac{1}{2}\sum_{l,m}f_{l,m}^*(\lambda_1)a_ma_l\right)\tilde{a}_i\tilde{a}_j\exp\left(\frac{1}{2}\sum_{l,m}f_{l,m}(\lambda_2)a_l^\dagger a_m^\dagger\right)\,\middle|\,0\right\rangle}{\left\langle 0\,\middle|\,\exp\left(\frac{1}{2}\sum_{l,m}f_{l,m}^*(\lambda_1)a_ma_l\right)\exp\left(\frac{1}{2}\sum_{l,m}f_{l,m}(\lambda_2)a_l^\dagger a_m^\dagger\right)\,\middle|\,0\right\rangle} \tag{27}$$

$$\mathcal{Z} \equiv \left\langle 0 \left| \exp\left(\frac{1}{2}\sum_{l,m} f^*_{l,m}(\lambda_1) a_m a_l\right) \exp\left(\frac{1}{2}\sum_{l,m} f_{l,m}(\lambda_2) a_l^\dagger a_m^\dagger\right) \right| 0 \right\rangle = (-1)^{N(N+1)/2}\mathrm{Pf}(M) \tag{28}$$

- $1 \leqslant i \leqslant N, 1 \leqslant j \leqslant N$

$$\begin{aligned} \langle \tilde{a}_i \tilde{a}_j \rangle &= \frac{1}{(-1)^{N(N+1)/2}\mathrm{Pf}(M)} \cdot (-1)^{N(N+1)/2}\,\mathrm{Pf}(M)(-1)\left(M^{-1}\right)_{i+N,j+N} \\ &= (-1)\left(M^{-1}\right)_{i+N,j+N} \end{aligned} \tag{29}$$

- $1 \leqslant i \leqslant N, N+1 \leqslant j \leqslant 2N$

$$\langle \tilde{a}_i \tilde{a}_j \rangle = (-1)\left(M^{-1}\right)_{i+N,j-N} \tag{30}$$

- $N+1 \leqslant i \leqslant 2N, 1 \leqslant j \leqslant N, i-N \neq j$

$$\langle \tilde{a}_i \tilde{a}_j \rangle = (-1)\left(M^{-1}\right)_{i-N,j+N} \tag{31}$$

- $N+1 \leqslant i \leqslant 2N, N+1 \leqslant j \leqslant 2N$

$$\langle \tilde{a}_i \tilde{a}_j \rangle = (-1)\left(M^{-1}\right)_{i-N,j-N} \tag{32}$$