

S.-T. Yau College Student Mathematics Contests 2011

## Algebra, Number Theory and Combinatorics

Individual

2:30{5:00 pm, July 10, 2011

(Please select 5 problems to solve)

For the following problems, every example and statement must be backed up by proof. Examples and statements without proof will receive no-credit.

1. Let  $K = \mathbb{Q}(\sqrt{-3})$ , an imaginary quadratic field.
  - (a) Does there exist a finite Galois extension  $L/\mathbb{Q}$  which contains  $K$  such that  $\text{Gal}(L/\mathbb{Q}) \cong S_3$ ? (Here  $S_3$  is the symmetric group in 3 letters.)
  - (b) Does there exist a finite Galois extension  $L/\mathbb{Q}$  which contains  $K$  such that  $\text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ ?
  - (c) Does there exist a finite Galois extension  $L/\mathbb{Q}$  which contains  $K$  such that  $\text{Gal}(L/\mathbb{Q}) \cong Q$ ? Here  $Q$  is the quaternion group with 8 elements  $\{1, i, j, k, -1, -i, -j, -k\}$ , a finite subgroup of the group of units  $H^\times$  of the ring  $H$  of all Hamiltonian quaternions.
2. Let  $\rho$  be a two-dimensional (complex) representation of a finite group  $G$  such that 1 is an eigenvalue of  $\rho(g)$  for every  $g \in G$ . Prove that  $\rho$  is a direct sum of two one-dimensional representations of  $G$ .
3. Let  $F \subseteq \mathbb{R}$  be the subset of all real numbers that are roots of monic polynomials  $f(X) \in \mathbb{Q}[X]$ .
  - (1) Show that  $F$  is a field.
  - (2) Show that the only field automorphisms of  $F$  are the identity automorphism  $\sigma(x) = x$  for all  $x \in F$ .
4. Let  $V$  be a finite-dimensional vector space over  $\mathbb{R}$  and  $T : V \rightarrow V$  be a linear transformation such that
  - (1) the minimal polynomial of  $T$  is irreducible;
  - (2) there exists a vector  $v \in V$  such that  $\{T^i v \mid i \geq 0\}$  spans  $V$ .
 Show that  $V$  contains no non-trivial proper  $T$ -invariant subspace.
5. Given a commutative diagram

$$\begin{array}{ccccc}
 A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & E \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' & \rightarrow & E'
 \end{array}$$

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*Algebra, Number Theory and Combinatorics, 2011-Individual* 2

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of Abelian groups, such that (i) both rows are exact sequences and (ii) every vertical map, except the middle one, is an isomorphism. Show that the middle map  $C \rightarrow C'$  is also an isomorphism.

6. Prove that a group of order 150 is not simple.