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> 由自由旋量场的拉格朗日密度
>
> $$\mathcal{L}_0=-\frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi-\partial_\alpha\bar{\psi}\gamma_\alpha\psi\right)-m\bar{\psi}\psi$$
>
> 求其能量动量张量、能量密度和动量密度
>
> $$T_{\mu\nu}=\frac{1}{2}\left(\bar{\psi}\gamma_\nu\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma_\nu\psi\right)$$
>
> $$W=\frac{1}{2\mathrm{i}}\left(\partial_t\psi^\dagger\psi-\psi^\dagger\partial_t\psi\right)$$
>
> $$G_i=\frac{1}{2\mathrm{i}}\left(\psi^\dagger\partial_i\psi-\partial_i\psi^\dagger\psi\right)$$

旋量场拉格朗日密度：

$$L=-\frac{1}{2}\left(\bar{\psi}\gamma_\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma_\mu\psi\right)-m\bar{\psi}\psi$$

$$\frac{\partial L}{\partial\left(\partial_\nu\psi\right)}=-\frac{1}{2}\bar{\psi}\gamma_\nu$$

$$\frac{\partial L}{\partial\left(\partial_\nu\bar{\psi}\right)}=\frac{1}{2}\gamma_\nu\psi$$

能量动量张量：

$$\begin{aligned}
T_{\mu\nu}&=L\delta_{\mu\nu}-\frac{\partial L}{\partial\left(\partial_\nu\phi_A\right)}\partial_\mu\phi_A\\
&=L\delta_{\mu\nu}-\frac{\partial L}{\partial\left(\partial_\nu\psi\right)}\partial_\mu\psi-\partial_\mu\bar{\psi}\frac{\partial L}{\partial\left(\partial_\nu\bar{\psi}\right)}\\
&=\left[-\frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi-\partial_\alpha\bar{\psi}\gamma_\alpha\psi\right)-m\bar{\psi}\psi\right]\delta_{\mu\nu}-\left(-\frac{1}{2}\bar{\psi}\gamma_\nu\right)\partial_\mu\psi-\partial_\mu\bar{\psi}\left(\frac{1}{2}\gamma_\nu\psi\right)\\
&=\left[-\frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi-\partial_\alpha\bar{\psi}\gamma_\alpha\psi\right)-m\bar{\psi}\psi\right]\delta_{\mu\nu}+\frac{1}{2}\bar{\psi}\gamma_\nu\partial_\mu\psi-\frac{1}{2}\partial_\mu\bar{\psi}\gamma_\nu\psi
\end{aligned}$$

利用旋量场及共轭旋量场运动方程

$$\gamma_\mu\partial_\mu\psi+m\psi=0$$

$$\partial_\mu \bar{\psi} \gamma_\mu - m\bar{\psi} = 0$$

进一步有:

$$\begin{aligned} T_{\mu\nu} &= \left[-\frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\alpha\psi\right) - m\bar{\psi}\psi\right]\delta_{\mu\nu} + \frac{1}{2}\bar{\psi}\gamma_\nu\partial_\mu\psi - \frac{1}{2}\partial_\mu\bar{\psi}\gamma_\nu\psi \\ &= \left[\frac{m}{2}\bar{\psi}\psi + \frac{m}{2}\bar{\psi}\psi - m\bar{\psi}\psi\right]\delta_{\mu\nu} + \frac{1}{2}\left(\bar{\psi}\gamma_\nu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_\nu\psi\right) \\ &= \frac{1}{2}\left(\bar{\psi}\gamma_\nu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_\nu\psi\right) \end{aligned}$$

能量密度:

$$\begin{aligned} W &= -T_{44} \\ &= -\frac{1}{2}\left(\bar{\psi}\gamma_4\partial_4\psi - \partial_4\bar{\psi}\gamma_4\psi\right) \\ &= -\frac{1}{2}\left(-\mathrm{i}\psi^\dagger\gamma_4^2\partial_t\psi + \mathrm{i}\partial_t\psi^\dagger\gamma_4^2\psi\right) \\ &= \frac{\mathrm{i}}{2}\left(\psi^\dagger\partial_t\psi - \partial_t\psi^\dagger\psi\right) \\ &= \frac{1}{2\mathrm{i}}\left(\partial_t\psi^\dagger\psi - \psi^\dagger\partial_t\psi\right) \end{aligned}$$

动量密度:

$$\begin{aligned} G_i &= \frac{1}{\mathrm{i}}T_{i4} \\ &= \frac{1}{\mathrm{i}}\cdot\frac{1}{2}\left(\bar{\psi}\gamma_4\partial_i\psi - \partial_i\bar{\psi}\gamma_4\psi\right) \\ &= \frac{1}{2\mathrm{i}}\left(\psi^\dagger\gamma_4^2\partial_i\psi - \partial_i\psi^\dagger\gamma_4^2\psi\right) \\ &= \frac{1}{2\mathrm{i}}\left(\psi^\dagger\partial_i\psi - \partial_i\psi^\dagger\psi\right) \end{aligned}$$