4-2 量子场论习题五

4-2-1

由表象变换 $\psi'=S\psi, \hat{F}'=S\hat{F}S^\dagger, SS^\dagger=I$,证明力学量 \hat{F} 的本征值、平均值和态矢量的内积均为不变量。

本征值不变

设 \hat{F} 的本征方程为:

$$\hat{F}\psi_f=f\psi_f$$

由定义,有:

$$\hat{F}' \equiv S \hat{F} S^\dagger, \quad \psi_f' \equiv S \psi_f$$

对本征方程作如下的变形:

$$S\hat{F}S^{\dagger}S\psi_f=fS\psi_f$$

即:

$$\hat{F}'\psi_f'=f\psi_f'$$

因此本征值不变。

平均值不变

 \hat{F} 在 $|\psi\rangle$ 态下的平均值:

$$egin{aligned} \left\langle \psi \,\middle|\, \hat{F} \,\middle|\, \psi \right\rangle &= \left\langle \psi \,\middle|\, S^\dagger S \hat{F} S^\dagger S \,\middle|\, \psi \right\rangle \\ &= \left\langle \psi' \,\middle|\, \hat{F}' \,\middle|\, \psi' \right\rangle \end{aligned}$$

因此平均值不变。

态矢量的内积不变

$$\ket{lpha'} \equiv S \ket{lpha}, \quad ra{lpha'} = ra{lpha} S^\dagger$$
 $\ket{eta'} \equiv S \ket{eta}$

态矢量内积:

$$\langle \alpha \, | \, \beta \rangle = \langle \alpha \, | \, S^{\dagger} S \, | \, \beta \rangle$$
$$= \langle \alpha' \, | \, \beta' \rangle$$

因此态矢量的内积不变。

4-2-2

证明

$$\theta_-(-x)=\theta_+(x)$$

$$\theta_+(-x)=\theta_-(x)$$

$$\varepsilon(x) = \theta_+(x) - \theta_-(x)$$

由定义:

$$heta_+(x) \equiv egin{cases} 1,x>0 \ 0,x<0 \end{cases}, \quad heta_-(x) \equiv egin{cases} 0,x>0 \ 1,x<0 \end{cases}, \quad arepsilon(x) \equiv egin{cases} 1,x>0 \ -1,x<0 \end{cases}$$

因此:

$$egin{aligned} heta_-(-x) &\equiv egin{cases} 0, -x > 0 \ 1, -x < 0 \end{cases} = egin{cases} 1, x > 0 \ 0, x < 0 \end{cases} = heta_+(x) \ heta_+(-x) &\equiv egin{cases} 1, -x > 0 \ 0, -x < 0 \end{cases} = egin{cases} 0, x > 0 \ 1, x < 0 \end{cases} = heta_-(x) \ heta_+(x) - heta_-(x) &= egin{cases} 1 - 0, x > 0 \ 0 - 1, x < 0 \end{cases} = egin{cases} 1, x > 0 \ - 1, x < 0 \end{cases} = arepsilon(x) \end{aligned}$$

4-2-3

推导 $\hat{H}_{--} = 0$

实标量场的能量密度:

$$W = rac{1}{2} \left(
abla \phi \cdot
abla \phi + \partial_t \phi \partial_t \phi + m^2 \phi^2
ight)$$

实标量场的傅里叶分解:

$$\hat{\phi}(x) = \hat{\phi}^{(+)}(x) + \hat{\phi}^{(-)}(x)$$
 $\hat{\phi}^{(+)}(x) = \left(rac{1}{2\pi}
ight)^{3/2}\int\limits_{k_0=arepsilon_{ec{k}}} \mathrm{e}^{-\mathrm{i}kx}rac{1}{\sqrt{2arepsilon_{ec{k}}}}\hat{a}^{(+)}\left(ec{k}
ight)\mathrm{d}^3ec{k}$ $\hat{\phi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2}\int\limits_{k_0=arepsilon_{ec{k}}} \mathrm{e}^{\mathrm{i}kx}rac{1}{\sqrt{2arepsilon_{ec{k}}}}\hat{a}^{(-)}\left(ec{k}
ight)\mathrm{d}^3ec{k}$

哈密顿算符:

$$egin{aligned} \hat{H} &= rac{1}{2} \int \left[\left(
abla \hat{\phi}
ight)^2 + \left(rac{\partial \hat{\phi}}{\partial t}
ight)^2 + m^2 \hat{\phi}^2
ight] \mathrm{d}V \ &= rac{1}{2} \int \left[\left(
abla \hat{\phi}^{(+)}(x) +
abla \hat{\phi}^{(-)}(x)
ight)^2 + \left(\partial_t \hat{\phi}^{(+)}(x) + \partial_t \hat{\phi}^{(-)}(x)
ight)^2 + m^2 \left(\hat{\phi}^{(+)}(x) + \hat{\phi}^{(-)}(x)
ight)^2
ight] \mathrm{d}V \end{aligned}$$

注意到,展开式中

$$\begin{split} &\int \left(\nabla \hat{\phi}^{(\pm)}(x) \cdot \nabla \hat{\phi}^{(\pm)}(x) + \partial_t \hat{\phi}^{(\pm)}(x) \partial_t \hat{\phi}^{(\pm)}(x) + m^2 \hat{\phi}^{(\pm)}(x) \hat{\phi}^{(\pm)}(x)\right) \mathrm{d}^3\vec{x} \\ &= \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3\vec{x} \int_{k_0 = \varepsilon_{\vec{k}}} \mathrm{d}^3\vec{k} \int_{k'_0 = \varepsilon_{\vec{k}'}} \mathrm{d}^3\vec{k}' \left(-\vec{k} \cdot \vec{k}' - \varepsilon_{\vec{k}} \varepsilon_{\vec{k}'} + m^2\right) \mathrm{e}^{\mp\mathrm{i}\left(\vec{k} + \vec{k}'\right) \cdot \vec{x}} \mathrm{e}^{\pm\mathrm{i}\left(\varepsilon_{\vec{k}} + \varepsilon_{\vec{k}'}\right) t} \frac{\hat{a}^{(\pm)} \left(\vec{k}\right) \hat{a}^{(\pm)} \left(\vec{k}'\right)}{2\sqrt{\varepsilon_{\vec{k}} \varepsilon_{\vec{k}'}}} \\ &= \int_{k_0 = \varepsilon_{\vec{k}}} \mathrm{d}^3\vec{k} \int_{k'_0 = \varepsilon_{\vec{k}'}} \mathrm{d}^3\vec{k}' \left(-\vec{k} \cdot \vec{k}' - \varepsilon_{\vec{k}} \varepsilon_{\vec{k}'} + m^2\right) \delta \left(\vec{k} + \vec{k}'\right) \mathrm{e}^{\pm\mathrm{i}\left(\varepsilon_{\vec{k}} + \varepsilon_{\vec{k}'}\right) t} \frac{\hat{a}^{(\pm)} \left(\vec{k}\right) \hat{a}^{(\pm)} \left(\vec{k}'\right)}{2\sqrt{\varepsilon_{\vec{k}} \varepsilon_{\vec{k}'}}} \\ &= \int_{k_0 = \varepsilon_{\vec{k}}} \mathrm{d}^3\vec{k} \left(\vec{k}^2 - \varepsilon_{\vec{k}}^2 + m^2\right) \mathrm{e}^{\pm\mathrm{i}\left(\varepsilon_{\vec{k}} + \varepsilon_{\vec{k}'}\right) t} \frac{\hat{a}^{(\pm)} \left(\vec{k}\right) \hat{a}^{(\pm)} \left(-\vec{k}\right)}{2\varepsilon_{\vec{k}}} \\ &= 0 \end{split}$$

其中包括:

$$\hat{H}_{--} = rac{1}{2} \int \left(
abla \hat{\phi}^{(-)}(x) \cdot
abla \hat{\phi}^{(-)}(x) + \partial_t \hat{\phi}^{(-)}(x) \partial_t \hat{\phi}^{(-)}(x) + m^2 \hat{\phi}^{(-)}(x) \hat{\phi}^{(-)}(x) \right) \mathrm{d}^3 \vec{x} = 0$$

4-2-4

对标量场,由

$$ec{p} = -\int
abla \phi \partial_t \phi \mathrm{d}^3 ec{x} \Longrightarrow ec{p} = rac{1}{2} \sum_{ec{k}} ec{k} \left\{ a^{(+)}_{ec{k}}, a^{(-)}_{ec{k}}
ight\}$$

实标量场的傅里叶分解:

$$\hat{\phi}(x) = \hat{\phi}^{(+)}(x) + \hat{\phi}^{(-)}(x)$$
 $\hat{\phi}^{(+)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{k_0 = arepsilon_{ec{k}}} \mathrm{e}^{-\mathrm{i}kx} rac{1}{\sqrt{2arepsilon_{ec{k}}}} \hat{a}^{(+)} \left(ec{k}
ight) \mathrm{d}^3ec{k}$ $\hat{\phi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{k_0 = arepsilon_{ec{k}}} \mathrm{e}^{\mathrm{i}kx} rac{1}{\sqrt{2arepsilon_{ec{k}}}} \hat{a}^{(-)} \left(ec{k}
ight) \mathrm{d}^3ec{k}$ $\hat{ec{p}} = -\int \left(
abla \hat{\phi}^{(+)}(x) +
abla \hat{\phi}^{(-)}(x)\right) \left(\partial_t \hat{\phi}^{(+)}(x) + \partial_t \hat{\phi}^{(-)}(x)\right) \mathrm{d}^3ec{x}$

注意到:

$$\int
abla \hat{\phi}^{(\pm)}(x) \partial_t \hat{\phi}^{(\pm)}(x) \mathrm{d}^3 ec{x} = \int \mathrm{d}^3 ec{k} \int \mathrm{d}^3 ec{k}' arepsilon_{ec{k}} ec{k}' \mathrm{e}^{\pm \mathrm{i} \left(arepsilon_{ec{k}} + arepsilon_{ec{k}'}
ight) t} \delta\left(ec{k} + ec{k}'
ight) rac{\hat{a}^{(\pm)}\left(ec{k}
ight) \hat{a}^{(\pm)}\left(ec{k}
ight)}{2\sqrt{arepsilon_{ec{k}} arepsilon_{ec{k}'}}} = -rac{1}{2} \int \mathrm{d}^3 ec{k} ec{k} \mathrm{e}^{\pm 2\mathrm{i} arepsilon_{ec{k}} t} \hat{a}^{(\pm)}\left(ec{k}
ight) \hat{a}^{(\pm)}\left(-ec{k}
ight) = 0$$

有贡献的仅为交叉项:

$$\begin{split} \hat{\vec{p}} &= -\int \mathrm{d}^{3}\vec{x} \left(\nabla \hat{\phi}^{(+)}(x) \partial_{t} \hat{\phi}^{(-)}(x) + \nabla \hat{\phi}^{(-)}(x) \partial_{t} \hat{\phi}^{(+)}(x) \right) \\ &= \left(\frac{1}{2\pi} \right)^{3} \int \mathrm{d}^{3}\vec{x} \int \mathrm{d}\vec{k} \int \mathrm{d}\vec{k}' \left[\vec{k} \varepsilon_{\vec{k}'} \mathrm{e}^{\mathrm{i} \left(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} \right) t} \mathrm{e}^{-\mathrm{i} \left(\vec{k} - \vec{k}' \right) \cdot \vec{x}} \frac{\hat{a}^{(+)} \left(\vec{k} \right) \hat{a}^{(-)} \left(\vec{k} \right)}{2 \sqrt{\varepsilon_{\vec{k}} \varepsilon_{\vec{k}'}}} + \vec{k} \varepsilon_{\vec{k}'} \mathrm{e}^{-\mathrm{i} \left(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} \right) t} \mathrm{e}^{\mathrm{i} \left(\vec{k} - \vec{k}' \right) \cdot \vec{x}} \frac{\hat{a}^{(-)} \left(\vec{k} \right) \hat{a}^{(+)} \left(\vec{k} \right)}{2 \sqrt{\varepsilon_{\vec{k}} \varepsilon_{\vec{k}'}}} \right] \\ &= \int \mathrm{d}^{3}\vec{k} \int \mathrm{d}^{3}\vec{k}' \delta \left(\vec{k} - \vec{k}' \right) \vec{k} \varepsilon_{\vec{k}'} \frac{1}{2 \sqrt{\varepsilon_{\vec{k}} \varepsilon_{\vec{k}'}}} \left[\mathrm{e}^{\mathrm{i} \left(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} \right) t} \hat{a}^{(+)} \left(\vec{k} \right) \hat{a}^{(-)} \left(\vec{k} \right) + \mathrm{e}^{-\mathrm{i} \left(\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}'} \right) t} \hat{a}^{(-)} \left(\vec{k} \right) \hat{a}^{(+)} \left(\vec{k} \right) \right] \\ &= \frac{1}{2} \int \vec{k} \left\{ \hat{a}^{(+)} \left(\vec{k} \right), \hat{a}^{(-)} \left(\vec{k} \right) \right\} \mathrm{d}^{3}\vec{k} \end{split}$$

利用:

$$\int \mathrm{d}^3 ec{k} \cdots = rac{(2\pi)^3}{V} \sum_{ec{k}} \cdots \ \hat{a}^{(\pm)} \equiv rac{(2\pi)^{3/2}}{\sqrt{V}} \hat{a}^{(\pm)} \left(ec{k}
ight), \quad \hat{a}^{(\pm)} \left(ec{k}
ight) = rac{\sqrt{V}}{(2\pi)^{3/2}} \hat{a}^{(\pm)}_{ec{k}}$$

有:

$$\begin{split} \hat{\vec{p}} &= \frac{1}{2} \int \vec{k} \left\{ \hat{a}^{(+)} \left(\vec{k} \right), \hat{a}^{(-)} \left(\vec{k} \right) \right\} \mathrm{d}^{3} \vec{k} \\ &= \frac{1}{2} \frac{(2\pi)^{3}}{V} \sum_{\vec{k}} \left\{ \frac{\sqrt{V}}{(2\pi)^{3/2}} \hat{a}_{\vec{k}}^{(+)}, \frac{\sqrt{V}}{(2\pi)^{3/2}} \hat{a}_{\vec{k}}^{(-)} \right\} \\ &= \frac{1}{2} \sum_{\vec{k}} \vec{k} \left\{ \hat{a}_{\vec{k}}^{(+)}, \hat{a}_{\vec{k}}^{(-)} \right\} \end{split}$$

4-2-5

证明

$$egin{aligned} \Delta^{(\pm)}(x-x') &= -\Delta^{(\mp)}(x'-x) \ \Delta^{(\pm)}\left(ec{x},t
ight) &= -\Delta^{(\pm)}\left(-ec{x},t
ight) \ \Delta^{(+)}(ec{x},t) &= \Delta^{(-)}(ec{x},-t) \ \Delta^{(+)}(x-x') &\equiv rac{\mathrm{i}}{\left(2\pi
ight)^3} \int \mathrm{d}^4k \mathrm{e}^{-\mathrm{i}kig(x-x')} \delta\left(k^2+m^2
ight) heta_+(k_0) \ \Delta^{(-)}(x-x') &\equiv -rac{\mathrm{i}}{\left(2\pi
ight)^3} \int \mathrm{d}^4k \mathrm{e}^{\mathrm{i}kig(x-x')} \delta\left(k^2+m^2
ight) heta_+(k_0) \end{aligned}$$

合写为:

$$egin{aligned} \Delta^{(\pm)}(x-x') &\equiv rac{\pm \mathrm{i}}{(2\pi)^3} \int \mathrm{d}^4 k \mathrm{e}^{\mp \mathrm{i} k \left(x-x'
ight)} \delta\left(k^2+m^2
ight) heta_+(k_0) \ \Delta^{(\pm)}(x-x') &\equiv rac{\pm \mathrm{i}}{(2\pi)^3} \int \mathrm{d}^4 k \mathrm{e}^{\mp \mathrm{i} k \left(x-x'
ight)} \delta\left(k^2+m^2
ight) heta_+(k_0) \ &= -rac{\mp \mathrm{i}}{\left(2\pi
ight)^3} \int \mathrm{d}^4 k \mathrm{e}^{\pm \mathrm{i} k \left(x'-x
ight)} \delta\left(k^2+m^2
ight) heta_+(k_0) \ &= -\Delta^{(\mp)}\left(x'-x
ight) \end{aligned}$$

$$\begin{split} \Delta^{(\pm)}\left(\vec{x},t\right) &\equiv \frac{\pm \mathrm{i}}{\left(2\pi\right)^3} \int \mathrm{d}^4k \mathrm{e}^{\mp \mathrm{i}kx} \delta\left(k^2 + m^2\right) \theta_+(k_0) \\ &\equiv \frac{\pm \mathrm{i}}{\left(2\pi\right)^3} \int \mathrm{d}^3\vec{k} \int \mathrm{d}k_0 \mathrm{e}^{\mp \mathrm{i}\left(\vec{k}\cdot\vec{x} - k_0t\right)} \delta\left(\vec{k}^2 + m^2 - k_0^2\right) \theta_+(k_0) \\ &\equiv \frac{\pm \mathrm{i}}{\left(2\pi\right)^3} \int \mathrm{d}^3\vec{k} \int \mathrm{d}k_0 \mathrm{e}^{\mp \mathrm{i}\left(\vec{k}\cdot\vec{x} - k_0t\right)} \frac{1}{2\varepsilon_{\vec{k}}} \left(\delta\left(k_0 + \varepsilon_{\vec{k}}\right) + \delta\left(k_0 - \varepsilon_{\vec{k}}\right)\right) \theta_+(k_0) \\ &= \frac{\pm \mathrm{i}}{\left(2\pi\right)^3} \int \mathrm{d}^3\vec{k} \mathrm{e}^{\mp \mathrm{i}\left(\vec{k}\cdot\vec{x} - \varepsilon_{\vec{k}}t\right)} \frac{1}{2\varepsilon_{\vec{k}}} \\ &= \frac{\pm \mathrm{i}}{\left(2\pi\right)^3} \int -\mathrm{d}^3\vec{k}' \mathrm{e}^{\mp \mathrm{i}\left(\vec{k}'\cdot(-\vec{x}) - \varepsilon_{\vec{k}'}t\right)} \frac{1}{2\varepsilon_{\vec{k}'}} \\ &= -\Delta^{(\pm)}\left(-\vec{x},t\right) \\ &\Delta^{(\pm)}\left(\vec{x},t\right) = \frac{\pm \mathrm{i}}{\left(2\pi\right)^3} \int \mathrm{d}^3\vec{k} \mathrm{e}^{\mp \mathrm{i}\left(\vec{k}\cdot\vec{x} - \varepsilon_{\vec{k}}t\right)} \frac{1}{2\varepsilon_{\vec{k}}} \\ &= \frac{+\mathrm{i}}{\left(2\pi\right)^3} \int -\mathrm{d}^3\vec{k}' \mathrm{e}^{-\mathrm{i}\left(\vec{k}\cdot\vec{x} - \varepsilon_{\vec{k}'}t\right)} \frac{1}{2\varepsilon_{\vec{k}'}} \\ &= \frac{-\mathrm{i}}{\left(2\pi\right)^3} \int \mathrm{d}^3\vec{k}' \mathrm{e}^{\mathrm{i}\left(\vec{k}'\cdot\vec{x} - \varepsilon_{\vec{k}'}(-t)\right)} \frac{1}{2\varepsilon_{\vec{k}'}} \\ &= \Delta^{(-)}\left(\vec{x},-t\right) \end{split}$$

4-2-6

利用标量场算符满足 Heisenberg 方程和对易关系,证明它仍满足标量场方程,即

$$egin{align} \left(\Box-m^2
ight)\hat{\phi}&=0\ \hat{\phi}(x)&=\hat{\phi}^{(+)}(x)+\hat{\phi}^{(-)}(x)\ \hat{\phi}^{(\pm)}(x)&=\left(rac{1}{2\pi}
ight)^{3/2}\int\limits_{k_0=arepsilon_{ec{k}}}\mathrm{e}^{\mp\mathrm{i}kx}rac{1}{\sqrt{2arepsilon_{ec{k}}}}\hat{a}^{(\pm)}\left(ec{k}
ight)\mathrm{d}^3ec{k}\ \left[\hat{H},\hat{a}^{(-)}\left(ec{k}
ight)
ight]&=-arepsilon_k\hat{a}^{(-)}\left(ec{k}
ight)\ \left[\hat{H},\hat{a}^{(+)}\left(ec{k}
ight)
ight]&=arepsilon_k\hat{a}^{(+)}\left(ec{k}
ight) \end{aligned}$$

海森堡方程:

$$\begin{split} \frac{\partial \hat{\phi}^{(\pm)}(x)}{\partial t} &= \mathrm{i} \left[\hat{H}, \hat{\phi}^{(\pm)}(x) \right] \\ &= \mathrm{i} \left(\frac{1}{2\pi} \right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mp \mathrm{i} k x} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \left[\hat{H}, \hat{a}^{(\pm)} \left(\vec{k} \right) \right] \mathrm{d}^3 \vec{k} \\ &= \mathrm{i} \left(\frac{1}{2\pi} \right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mp \mathrm{i} k x} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \left(\pm \varepsilon_{\vec{k}} \hat{a}^{(\pm)} \left(\vec{k} \right) \right) \mathrm{d}^3 \vec{k} \\ &= \pm \mathrm{i} \left(\frac{1}{2\pi} \right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mp \mathrm{i} k x} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \varepsilon_{\vec{k}} \hat{a}^{(\pm)} \left(\vec{k} \right) \mathrm{d}^3 \vec{k} \end{split}$$

$$\begin{split} \partial_t^2 \hat{\phi}^{(\pm)}(x) &= \mathrm{i} \left[\hat{H}, \partial_t \hat{\phi}^{(\pm)}(x) \right] \\ &= \mp \left(\frac{1}{2\pi} \right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mp \mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \varepsilon_{\vec{k}} \left[\hat{H}, \hat{a}^{(\pm)} \left(\vec{k} \right) \right] \mathrm{d}^3 \vec{k} \\ &= \mp \left(\frac{1}{2\pi} \right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mp \mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \varepsilon_{\vec{k}} \left(\pm \varepsilon_{\vec{k}} \hat{a}^{(\pm)} \left(\vec{k} \right) \right) \mathrm{d}^3 \vec{k} \\ &= - \left(\frac{1}{2\pi} \right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mp \mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \varepsilon_{\vec{k}}^2 \hat{a}^{(\pm)} \left(\vec{k} \right) \mathrm{d}^3 \vec{k} \\ \partial_i \partial_i \hat{\phi}^{(\pm)}(x) &= \left(\frac{1}{2\pi} \right)^{3/2} \partial_i \partial_i \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mp \mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \hat{a}^{(\pm)} \left(\vec{k} \right) \mathrm{d}^3 \vec{k} \\ &= \left(\frac{1}{2\pi} \right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} -k_i k_i \mathrm{e}^{\mp \mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \hat{a}^{(\pm)} \left(\vec{k} \right) \mathrm{d}^3 \vec{k} \\ &= - \left(\frac{1}{2\pi} \right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \vec{k}^2 \mathrm{e}^{\mp \mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \hat{a}^{(\pm)} \left(\vec{k} \right) \mathrm{d}^3 \vec{k} \\ \left(\Box - m^2 \right) \hat{\phi}^{(\pm)}(x) &= \left(\partial_i \partial_i - \partial_t^2 - m^2 \right) \hat{\phi}^{(\pm)}(x) \\ &= \left(\frac{1}{2\pi} \right)^{3/2} \left\{ \int\limits_{k_0 = \varepsilon_{\vec{k}}} \left(-\vec{k}^2 + \varepsilon_{\vec{k}}^2 - m^2 \right) \mathrm{e}^{\mp \mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} \hat{a}^{(\pm)} \left(\vec{k} \right) \mathrm{d}^3 \vec{k} \right\} \\ &= 0 \end{split}$$

因此:

$$\left(\Box - m^2\right)\hat{\phi}(x) = \left(\Box - m^2\right)\left(\hat{\phi}^{(+)}(x) + \hat{\phi}^{(-)}(x)\right)$$

4-2-7

讨论复标量场的量子化。

Fourier 表示

$$\begin{split} \hat{\phi}^{(+)}(x) &= \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{-\mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} b^{(+)} \left(\vec{k}\right) \mathrm{d}^3 \vec{k} \\ \hat{\phi}^{(-)}(x) &= \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} a^{(-)} \left(\vec{k}\right) \mathrm{d}^3 \vec{k} \\ \hat{\phi}^{*(+)}(x) &= \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{-\mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} a^{(+)} \left(\vec{k}\right) \mathrm{d}^3 \vec{k} \\ \hat{\phi}^{*(-)}(x) &= \left(\frac{1}{2\pi}\right)^{3/2} \int\limits_{k_0 = \varepsilon_{\vec{k}}} \mathrm{e}^{\mathrm{i}kx} \frac{1}{\sqrt{2\varepsilon_{\vec{k}}}} b^{(-)} \left(\vec{k}\right) \mathrm{d}^3 \vec{k} \end{split}$$

物理量算符表示

$$\begin{split} \hat{H} &= \int \varepsilon_{\vec{k}} \left[\hat{a}^{(+)} \left(\vec{k} \right) \hat{a}^{(-)} \left(\vec{k} \right) + b^{(-)} \left(\vec{k} \right) b^{(+)} \left(\vec{k} \right) \right] \mathrm{d}^3 \vec{k} \\ \hat{\vec{p}} &= \int \vec{k} \left[\hat{a}^{(+)} \left(\vec{k} \right) \hat{a}^{(-)} \left(\vec{k} \right) + b^{(-)} \left(\vec{k} \right) b^{(+)} \left(\vec{k} \right) \right] \mathrm{d}^3 \vec{k} \\ \hat{Q} &= \int e \left[\hat{a}^{(+)} \left(\vec{k} \right) \hat{a}^{(-)} \left(\vec{k} \right) - b^{(-)} \left(\vec{k} \right) b^{(+)} \left(\vec{k} \right) \right] \mathrm{d}^3 \vec{k} \end{split}$$

基本对易关系

$$\begin{split} \left[\hat{H},\hat{a}^{(-)}\left(\vec{k}\right)\right] &= -\varepsilon_{\vec{k}}\hat{a}^{(-)}\left(\vec{k}\right) \\ \left[\hat{H},\hat{a}^{(+)}\left(\vec{k}\right)\right] &= \varepsilon_{\vec{k}}\hat{a}^{(+)}\left(\vec{k}\right) \\ \left[\hat{H},\hat{b}^{(-)}\left(\vec{k}\right)\right] &= -\varepsilon_{\vec{k}}\hat{b}^{(-)}\left(\vec{k}\right) \\ \left[\hat{H},\hat{b}^{(+)}\left(\vec{k}\right)\right] &= \varepsilon_{\vec{k}}\hat{b}^{(+)}\left(\vec{k}\right) \\ \left[\hat{a}^{(-)}\left(\vec{k}\right),\hat{a}^{(+)}\left(\vec{k}'\right)\right] &= \delta\left(\vec{k}-\vec{k}'\right) \\ \left[\hat{b}^{(-)}\left(\vec{k}\right),\hat{b}^{(+)}\left(\vec{k}'\right)\right] &= \delta\left(\vec{k}-\vec{k}'\right) \end{split}$$

复标量场对易关系

$$egin{aligned} \left[\hat{\phi}^{*(+)}(x),\hat{\phi}^{(-)}(x')
ight] &= \mathrm{i}\Delta^{(+)}(x-x') \ \left[\hat{\phi}^{*(-)}(x),\hat{\phi}^{(+)}(x')
ight] &= \mathrm{i}\Delta^{(+)}(x-x') \ \left[\hat{\phi}^{*}(x),\hat{\phi}(x')
ight] &= \mathrm{i}\Delta(x-x') \end{aligned}$$

4-2-8

证明

$$\Delta^F(x - x') = \Delta^F(x' - x)$$
$$D^F(x - x') = D^F(x' - x)$$

由于

$$egin{align} \Delta^F(x) &= 2\mathrm{i}\left[\Delta^{(-)}(x) heta_+(t) - \Delta^+(x) heta_-(t)
ight] \ D^F(x) &= 2\mathrm{i}\left[D^{(-)}(x) heta_+(t) - D^+(x) heta_-(t)
ight] \end{aligned}$$

利用

$$egin{split} \Delta^{(\pm)}(x-x') &= -\Delta^{(\mp)}(x'-x) \ D^{(\pm)}(x-x') &= -D^{(\mp)}(x'-x) \ & heta_{\pm}(t) &= heta_{\mp}(-t) \end{split}$$

有:

$$\begin{split} \Delta^F(x-x') &= 2\mathrm{i} \left[\Delta^{(-)}(x-x')\theta_+(t-t') - \Delta^{(+)}(x-x')\theta_-(t-t') \right] \\ &= 2\mathrm{i} \left[-\Delta^{(+)}(x'-x)\theta_+(t-t') + \Delta^{(-)}(x'-x)\theta_-(t-t') \right] \\ &= 2\mathrm{i} \left[-\Delta^{(+)}(x'-x)\theta_-(t'-t) + \Delta^{(-)}(x'-x)\theta_+(t'-t) \right] \\ &= \Delta^F(x'-x) \end{split}$$

同理有:

$$D^F(x - x') = D^F(x' - x)$$

4-2-9

讨论 $\overline{A_{\mu}(x_1)A_{
u}(x_2)},\overline{\psi(x_1)\psi(x_2)}$ 的物理意义。

 $\overline{A_{\mu}(x_1)A_{\nu}(x_2)} \equiv \langle 0 \mid P\left[A_{\mu}(x_1)A_{\nu}(x_2)\right] \mid 0 \rangle$ 是矢量场 $A_{\mu}(x)$ 两点的关联函数,代表光子的传播振幅,即从时空点 x_2 发出一个光子,在不确定的中间过程中经过演化,在时空点 x_1 被探测到的概率振幅。

 $\overline{\psi(x_1)\psi(x_2)}$ 是旋量场的两点关联函数,描述电子(或正电子)从 x_2 传播到 x_1 的概率幅。

4-2-10

利用旋量场算符满足的 Heisenberg 方程,证明

$$egin{align} \left[H, a_i^{(+)}(ec{p})
ight] &= E_{ec{p}} a_i^{(+)}(ec{p}) \ \\ \left[H, b_i^{(-)}(ec{p})
ight] &= - E_{ec{p}} b_i^{(-)}(ec{p}) \ \end{aligned}$$

旋量场 Fourier 表示:

$$\hat{\psi}^{(+)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{-\mathrm{i}px} \hat{b}_i^{(+)}\left(ec{p}
ight) v_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{a}_i^{(-)}\left(ec{p}
ight) u_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(+)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{-\mathrm{i}px} \hat{a}_i^{(+)}\left(ec{p}
ight) ar{u}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}\left(ec{p}
ight) ar{v}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}\left(ec{p}
ight) ar{v}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}\left(ec{p}
ight) ar{v}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}\left(ec{p}
ight) ar{v}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}\left(ec{p}
ight) \dot{v}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}\left(ec{p}
ight) \dot{v}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \ \hat{\psi}^{(-)}(x) = \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}(x) + \left(\frac{1}{2\pi}
ight)^{3/2} \int\limits_{ec{p}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}(x) \, \hat{\psi}^{(-)}(x) \, \hat{\psi}^$$

海森堡方程:

$$\partial_t \hat{ar{\psi}}^{(+)}(x) = \mathrm{i} \left[\hat{H}, \hat{ar{\psi}}(x)
ight]$$

左边:

$$egin{aligned} \partial_t \hat{ar{\psi}}^{(+)}(x) &= \partial_t \left\{ \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{-\mathrm{i}px} \hat{a}_i^{(+)}\left(ec{p}
ight) ar{u}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p}
ight\} \ &= \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{i} E_{ec{p}} \mathrm{e}^{-\mathrm{i}px} \hat{a}_i^{(+)}\left(ec{p}
ight) ar{u}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \end{aligned}$$

右边:

$$egin{aligned} \mathrm{i}\left[\hat{H},\hat{\overline{\psi}}^{(+)}(x)
ight] &= \mathrm{i}\left[\hat{H},\left(rac{1}{2\pi}
ight)^{3/2}\int\limits_{p_0=E_{ec{p}}}\mathrm{e}^{-\mathrm{i}px}\hat{a}_i^{(+)}\left(ec{p}
ight)ar{u}_i\left(ec{p}
ight)\mathrm{d}^3ec{p}
ight] \ &= \mathrm{i}\left(rac{1}{2\pi}
ight)^{3/2}\int\limits_{p_0=E_{ec{p}}}\mathrm{e}^{-\mathrm{i}px}\left[\hat{H},\hat{a}_i^{(+)}\left(ec{p}
ight)
ight]ar{u}_i\left(ec{p}
ight)\mathrm{d}^3ec{p} \end{aligned}$$

对比可得:

$$\left[\hat{H},\hat{a}_{i}^{\left(+
ight)}\left(ec{p}
ight)
ight]=E_{ec{p}}a_{i}^{\left(+
ight)}(ec{p})$$

海森堡方程:

$$\partial_t \hat{ar{\psi}}^{(-)}(x) = \mathrm{i} \left[\hat{H}, \hat{ar{\psi}}^{(-)}(x)
ight]$$

左边:

$$egin{aligned} \partial_t \hat{ar{\psi}}^{(-)}(x) &= \partial_t \left\{ \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}\left(ec{p}
ight) ar{v}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p}
ight\} \ &= \left(rac{1}{2\pi}
ight)^{3/2} \int\limits_{p_0 = E_{ec{p}}} -\mathrm{i} E_{ec{p}} \mathrm{e}^{\mathrm{i}px} \hat{b}_i^{(-)}\left(ec{p}
ight) ar{v}_i\left(ec{p}
ight) \mathrm{d}^3 ec{p} \end{aligned}$$

右边:

$$\begin{split} \mathrm{i}\left[\hat{H},\hat{\bar{\psi}}^{(-)}(x)\right] &= \mathrm{i}\left[\hat{H},\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{p_0=E_{\vec{p}}}\mathrm{e}^{\mathrm{i}px}\hat{b}_i^{(-)}\left(\vec{p}\right)\bar{v}_i\left(\vec{p}\right)\mathrm{d}^3\vec{p}\right] \\ &= \mathrm{i}\left(\frac{1}{2\pi}\right)^{3/2}\int\limits_{p_0=E_{\vec{p}}}\mathrm{e}^{\mathrm{i}px}\left[\hat{H},\hat{b}_i^{(-)}\left(\vec{p}\right)\right]\bar{v}_i\left(\vec{p}\right)\mathrm{d}^3\vec{p} \end{split}$$

对比可得:

$$\left[\hat{H},\hat{b}_{i}^{\left(-
ight)}\left(ec{p}
ight)
ight]=-E_{ec{p}}\hat{b}_{i}^{\left(-
ight)}\left(ec{p}
ight)$$