# 4

## 4.1

由 $f(R)$ 引力作用量

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} f(R) + S_m \tag{1}$$

变分，推出 $f(R)$ 引力中的引力场方程

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\nabla^\lambda \nabla_\lambda\right) f'(R) = 8\pi G T_{\mu\nu} \tag{2}$$

先看引力作用量

$$S_g = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} f(R) \tag{3}$$

对度规的变分。计算

$$\delta\left[\sqrt{-g}f(R)\right] = \delta\left(\sqrt{-g}\right)f(R) + \sqrt{-g}\delta\left[f(R)\right] \tag{4}$$

对于第一项，利用 $\delta g = -gg_{\mu\nu}\delta g^{\mu\nu}$ 有

$$\begin{aligned}
\delta\left(\sqrt{-g}\right)f(R) &= -\frac{1}{2}\frac{\delta g}{\sqrt{-g}}f(R) \\
&= -\frac{1}{2}\frac{-gg_{\mu\nu}\delta g^{\mu\nu}}{\sqrt{-g}}f(R) \\
&= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}f(R)\delta g^{\mu\nu}
\end{aligned} \tag{5}$$

对于第二项，利用 $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu \phi^\mu$

$$\begin{aligned}
\sqrt{-g}\delta\left[f(R)\right] &= \sqrt{-g}f'(R)\delta R \\
&= \sqrt{-g}f'(R)\delta\left(g^{\mu\nu}R_{\mu\nu}\right) \\
&= \sqrt{-g}f'(R)\left[\left(\delta g^{\mu\nu}\right)R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right] \\
&= \sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}f'(R)\nabla_\mu \phi^\mu \\
&= \sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\left\{\nabla_\mu\left[f'(R)\phi^\mu\right] - \phi^\mu \nabla_\mu f'(R)\right\} \\
&= \sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\nabla_\mu\left[f'(R)\phi^\mu\right] - \sqrt{-g}\phi^\mu \nabla_\mu f'(R) \\
&= \sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu} - \sqrt{-g}\phi^\mu \nabla_\mu f'(R) + [\partial M \text{ term}]
\end{aligned} \tag{6}$$

其中 $\partial M$ term 在体积分中可化为边界面积分，为零。为了进一步计算，注意到

$$\phi^\mu \equiv g^{\lambda\nu}\delta\Gamma^\mu_{\lambda\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\nu} \tag{7}$$

利用联络对 $g_{\mu\nu}$ 的变分

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\nu\sigma} - \nabla_\sigma \delta g_{\mu\nu}\right) \tag{8}$$

可以计算

$$\delta\Gamma^\mu_{\lambda\nu} = \frac{1}{2}g^{\mu\sigma}\left(\nabla_\lambda \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\lambda\sigma} - \nabla_\sigma \delta g_{\lambda\nu}\right) \tag{9}$$

$$\delta\Gamma^\lambda_{\lambda\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_\lambda \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\lambda\sigma} - \nabla_\sigma \delta g_{\lambda\nu}\right) \tag{10}$$

于是可以进一步表达 $\phi^\mu$

$$\phi^\mu \equiv g^{\lambda\nu}\delta\Gamma^\mu_{\lambda\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\nu}$$

$$= g^{\lambda\nu}\left[\frac{1}{2}g^{\mu\sigma}\left(\nabla_\lambda\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\lambda\sigma} - \nabla_\sigma\delta g_{\lambda\nu}\right)\right] - g^{\mu\nu}\left[\frac{1}{2}g^{\lambda\sigma}\left(\nabla_\lambda\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\lambda\sigma} - \nabla_\sigma\delta g_{\lambda\nu}\right)\right]$$

$$= \frac{1}{2}\left(\nabla^\nu\delta g^\mu_\nu + \nabla^\lambda\delta g^\mu_\lambda - \nabla^\mu\delta g^\nu_\nu\right) - \frac{1}{2}\left(\nabla^\sigma\delta g^\mu_\sigma + \nabla^\mu\delta g^\sigma_\sigma - \nabla^\lambda\delta g^\mu_\lambda\right) \tag{11}$$

$$= \frac{1}{2}\left(2\nabla^\nu\delta g^\mu_\nu - \nabla^\mu\delta g^\nu_\nu\right) - \frac{1}{2}\nabla^\mu\delta g^\sigma_\sigma$$

$$= \nabla^\nu\delta g^\mu_\nu - \nabla^\mu\delta g^\nu_\nu$$

于是

$$-\sqrt{-g}\phi^\mu\nabla_\mu f'(R) = -\sqrt{-g}\left[\nabla^\nu\left(\delta g^\mu_\nu\right) - \nabla^\mu\left(\delta g^\nu_\nu\right)\right]\nabla_\mu f'(R)$$

$$= \sqrt{-g}\left[\nabla^\mu\left(\delta g^\nu_\nu\right)\right]\nabla_\mu f'(R) - \sqrt{-g}\left[\nabla^\nu\left(\delta g^\mu_\nu\right)\right]\nabla_\mu f'(R)$$

$$= \sqrt{-g}\left[-\left(\delta g^\nu_\nu\right)\nabla^\mu\nabla_\mu f'(R)\right] - \sqrt{-g}\left[-\left(\delta g^\mu_\nu\right)\nabla^\nu\nabla_\mu f'(R)\right] + [\partial M \text{ term}]$$

$$= \sqrt{-g}g_{\nu\sigma}\left(\delta g^{\mu\sigma}\right)\nabla^\nu\nabla_\mu f'(R) - \sqrt{-g}g_{\nu\sigma}\left(\delta g^{\nu\sigma}\right)\nabla^\mu\nabla_\mu f'(R) + [\partial M \text{ term}]$$

$$= \sqrt{-g}\left(\delta g^{\mu\sigma}\right)\nabla_\sigma\nabla_\mu f'(R) - \sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^\lambda\nabla_\lambda f'(R) + [\partial M \text{ term}] \tag{12}$$

$$= \sqrt{-g}\left(\delta g^{\mu\nu}\right)\nabla_\nu\nabla_\mu f'(R) - \sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^\lambda\nabla_\lambda f'(R) + [\partial M \text{ term}]$$

$$= \sqrt{-g}\left(\delta g^{\nu\mu}\right)\nabla_\nu\nabla_\mu f'(R) - \sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^\lambda\nabla_\lambda f'(R) + [\partial M \text{ term}]$$

$$= \sqrt{-g}\left(\delta g^{\mu\nu}\right)\nabla_\mu\nabla_\nu f'(R) - \sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^\lambda\nabla_\lambda f'(R) + [\partial M \text{ term}]$$

总之，

$$\delta\left[\sqrt{-g}f(R)\right] = \delta\left(\sqrt{-g}\right)f(R) + \sqrt{-g}\delta\left[f(R)\right]$$

$$= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}f(R)\delta g^{\mu\nu} + \sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu} - \sqrt{-g}\phi^\mu\nabla_\mu f'(R) + [\partial M \text{ term}]$$

$$= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}f(R)\delta g^{\mu\nu} + \sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\left(\delta g^{\mu\nu}\right)\nabla_\mu\nabla_\nu f'(R) - \sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right)\nabla^\lambda\nabla_\lambda f'(R) + [\partial M \text{ term}] \tag{13}$$

$$= \sqrt{-g}\left[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\lambda\nabla_\lambda\right)f'(R)\right]\delta g^{\mu\nu} + [\partial M \text{ term}]$$

则引力作用量的变分为

$$\delta S_g = \frac{1}{16\pi G}\int \mathrm{d}^4 x\,\delta\left[\sqrt{-g}f(R)\right]$$

$$= \frac{1}{16\pi G}\int \mathrm{d}^4 x\sqrt{-g}\left[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\lambda\nabla_\lambda\right)f'(R)\right]\delta g^{\mu\nu} + \int [\partial M \text{ term}] \tag{14}$$

$$= \frac{1}{16\pi G}\int \mathrm{d}^4 x\sqrt{-g}\left[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\lambda\nabla_\lambda\right)f'(R)\right]\delta g^{\mu\nu}$$

对于引力源物质作用量，能动张量满足

$$\delta S_m = -\frac{1}{2}\int \mathrm{d}^4 x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \tag{15}$$

由最小作用量原理

$$\delta S = \delta S_g + \delta S_m = 0 \tag{16}$$

可得

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\lambda\nabla_\lambda\right)f'(R) - 8\pi G T_{\mu\nu} = 0 \tag{17}$$

也即 $f(R)$ 引力中的引力场方程

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^\lambda\nabla_\lambda\right)f'(R) = 8\pi G T_{\mu\nu} \tag{18}$$

## 4.2

计算Schwarzchild解的空间部分线元

$$ds^2 = r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{dr^2}{1 - 2GM/c^2 r} \tag{19}$$

当坐标半径 $r = a$ 时的球面面积；$r = a$ 时的球体体积；从半径为 $r = 2GM/c^2$ 的球面到 $r = 3GM/c^2$ 的球面的径向距离。

- $r = a$ 时的球面面积

在 $r = a$ 的二维球面上，

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2 \tag{20}$$

度规：

$$[g_{ij}] = \text{diag}\left(a^2, a^2\sin^2\theta\right) \tag{21}$$

面元为

$$dA = \sqrt{\det[g_{ij}]}dx^i dx^j = a^2\sin\theta d\theta d\phi \tag{22}$$

球面面积：

$$A = \int dA = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi} a^2\sin\theta d\theta d\phi = 4\pi a^2 \tag{23}$$

- $r = a$ 时的球体体积

令

$$r_s \equiv \frac{2GM}{c^2} \tag{24}$$

则

$$\begin{aligned} ds^2 &= r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{dr^2}{1 - 2GM/c^2 r}\\ &= ds^2 = r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{dr^2}{1 - r_s/r} \end{aligned} \tag{25}$$

度规：

$$[g_{ij}] = \text{diag}\left(1 - r_s/r, r^2, r^2\sin^2\theta\right) \tag{26}$$

体元：

$$dV = \sqrt{\det[g_{ij}]}dx^i dx^j = \frac{r^2\sin\theta}{\sqrt{1 - r_s/r}}dr d\theta d\phi \tag{27}$$

球体体积：

$$\begin{aligned} V &= \int dV = \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\frac{r^2\sin\theta}{\sqrt{1 - r_s/r}}dr d\theta d\phi\\ &= 4\pi\int_{r=0}^{r=a}\frac{r^2}{\sqrt{1 - r_s/r}}dr \end{aligned} \tag{28}$$

- 从半径为 $r = 2GM/c^2 = r_s$ 的球面到 $r = 3GM/c^2 = 3r_s/2$ 的球面的径向距离

$$\Delta r = \int_{r=r_s}^{r=3r_s/2} \frac{\mathrm{d}r}{\sqrt{1-r_s/r}} = \left[\sqrt{r(r-r_s)} + r_s \ln\left|\frac{\sqrt{r-r_s}+\sqrt{r}}{\sqrt{r_s}}\right|\right]\Bigg|_{r=r_s}^{r=3r_s/2}$$

$$= \frac{\sqrt{3}}{2}r_s + r_s \ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right) \tag{29}$$

$$= \left[\frac{\sqrt{3}}{2} + \ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)\right] r_s$$

## 4.3

计算Schwarzchild度规

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu}c^2\mathrm{d}t^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) + \mathrm{e}^{-\nu}\mathrm{d}r^2 \tag{30}$$

$$\mathrm{e}^{\nu} = 1 - \frac{2GM}{c^2 r} \tag{31}$$

情况下所有联络 $\Gamma_{\mu\nu}^{\lambda}$.

$$\left(x^0, x^1, x^2, x^3\right) \equiv (ct, r, \theta, \phi) \tag{32}$$

线元

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu}c^2\mathrm{d}t^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) + \mathrm{e}^{-\nu}\mathrm{d}r^2$$
$$= g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} \tag{33}$$

度规

$$[g_{\mu\nu}] = \mathrm{diag}\left(-\mathrm{e}^{\nu}, \mathrm{e}^{-\nu}, r^2, r^2\sin^2\theta\right) \tag{34}$$

逆度规

$$[g^{\mu\nu}] = \mathrm{diag}\left(-\mathrm{e}^{-\nu}, \mathrm{e}^{\nu}, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right) \tag{35}$$

下面计算联络。

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\right) \tag{36}$$

- $\lambda = 0$

$$\Gamma_{\mu\nu}^0 = \frac{1}{2}g^{0\sigma}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\right)$$
$$= \frac{1}{2}g^{00}\left(\partial_{\mu}g_{\nu 0} + \partial_{\nu}g_{\mu 0} - \partial_0 g_{\mu\nu}\right) \tag{37}$$
$$= -\frac{1}{2}\mathrm{e}^{-\nu}\left(\partial_{\mu}g_{\nu 0} + \partial_{\nu}g_{\mu 0}\right)$$

当 $(\mu,\nu) = (i,j)$ 时, 联络恒为零; 当 $\mu = 0$ 且 $\nu = 0$ 时联络也为零。

非零联络:

$$\Gamma_{10}^0 = \Gamma_{01}^0 = -\frac{1}{2}\mathrm{e}^{-\nu}\left(\partial_0 g_{10} + \partial_1 g_{00}\right) = -\frac{1}{2}\mathrm{e}^{-\nu}\partial_1\left(-\mathrm{e}^{\nu}\right) = \frac{1}{2}\nu' \tag{38}$$

- $\lambda = 1$

$$\Gamma_{\mu\nu}^1 = \frac{1}{2}g^{1\sigma}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\right)$$
$$= \frac{1}{2}g^{11}\left(\partial_{\mu}g_{\nu 1} + \partial_{\nu}g_{\mu 1} - \partial_1 g_{\mu\nu}\right) \tag{39}$$
$$= \frac{1}{2}\mathrm{e}^{\nu}\left(\partial_{\mu}g_{\nu 1} + \partial_{\nu}g_{\mu 1} - \partial_1 g_{\mu\nu}\right)$$

当 $\mu, \nu \neq 1$ 时，非零的联络为

$$\Gamma_{00}^1 = \frac{1}{2}\mathrm{e}^\nu\left(-\partial_r\left(-\mathrm{e}^\nu\right)\right) = \frac{\nu'}{2}\mathrm{e}^{2\nu} \tag{40}$$

$$\Gamma_{22}^1 = \frac{1}{2}\mathrm{e}^\nu\left(-\partial_r\left(r^2\right)\right) = -r\mathrm{e}^\nu \tag{41}$$

$$\Gamma_{33}^1 = \frac{1}{2}\mathrm{e}^\nu\left(-\partial_r\left(r^2\sin^2\theta\right)\right) = -r\mathrm{e}^\nu\sin^2\theta \tag{42}$$

当 $\mu = \nu = 1$ 时，

$$\Gamma_{11}^1 = \frac{1}{2}\mathrm{e}^\nu\left(\partial_1 g_{11}\right) = \frac{1}{2}\mathrm{e}^\nu\left(\partial_r\left(\mathrm{e}^{-\nu}\right)\right) = -\frac{\nu'}{2} \tag{43}$$

当 $\mu, \nu$ 中只有一个为 1 时，

$$\Gamma_{01}^1 = \Gamma_{10}^1 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{13}^1 = \Gamma_{31}^1 = 0 \tag{44}$$

- $\lambda = 2$

$$\begin{aligned}
\Gamma_{\mu\nu}^2 &= \frac{1}{2}g^{2\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\
&= \frac{1}{2}g^{22}\left(\partial_\mu g_{\nu 2} + \partial_\nu g_{\mu 2} - \partial_2 g_{\mu\nu}\right) \\
&= \frac{1}{2r^2}\left(\partial_\mu g_{\nu 2} + \partial_\nu g_{\mu 2} - \partial_2 g_{\mu\nu}\right)
\end{aligned} \tag{45}$$

当 $\mu, \nu \neq 2$ 时，

$$\Gamma_{33}^2 = \frac{1}{2r^2}\left(-\partial_\theta\left(r^2\sin^2\theta\right)\right) = -\sin\theta\cos\theta \tag{46}$$

当 $\mu, \nu$ 中只有一个为 2 时，

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2r^2}\left(\partial_1 g_{22}\right) = \frac{1}{2r^2}\left(\partial_r\left(r^2\right)\right) = \frac{1}{r} \tag{47}$$

当 $(\mu, \nu) = (2, 2)$ 时，

$$\Gamma_{22}^2 = 0 \tag{48}$$

- $\lambda = 3$

$$\begin{aligned}
\Gamma_{\mu\nu}^3 &= \frac{1}{2}g^{3\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\
&= \frac{1}{2}g^{33}\left(\partial_\mu g_{\nu 3} + \partial_\nu g_{\mu 3} - \partial_3 g_{\mu\nu}\right) \\
&= \frac{1}{2r^2\sin^2\theta}\left(\partial_\mu g_{\nu 3} + \partial_\nu g_{\mu 3}\right)
\end{aligned} \tag{49}$$

当 $\mu, \nu \neq 3$ 时，所有联络都为零。

当 $\mu, \nu$ 中只有一个为 3 时，

$$\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{2r^2\sin^2\theta}\left(\partial_1 g_{33}\right) = \frac{1}{2r^2\sin^2\theta}\left(\partial_r\left(r^2\sin^2\theta\right)\right) = \frac{1}{r} \tag{50}$$

$$\Gamma_{23}^3 = \Gamma_{32}^3 = \frac{1}{2r^2\sin^2\theta}\left(\partial_2 g_{33}\right) = \frac{1}{2r^2\sin^2\theta}\left(\partial_\theta\left(r^2\sin^2\theta\right)\right) = \cot\theta \tag{51}$$

当 $\mu, \nu = 3$ 时，

$$\Gamma_{33}^3 = 0 \tag{52}$$

# 4.4

据上题结果，由Schwarzchild时空粒子运动方程出发，导出在平面极坐标系下轨道满足的方程（GR中的Binet方程），并讨论水星进动问题。

Schwarzchild线元：

$$\mathrm{d}s^2 = -c^2\mathrm{e}^{\nu}\mathrm{d}t^2 + \mathrm{e}^{-\nu}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \quad \mathrm{e}^{\nu} = 1 - \frac{2GM}{c^2 r} \tag{53}$$

Schwarzschild度规下的非零联络：

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{\nu'}{2}, \tag{54}$$

$$\Gamma^1_{00} = \frac{\nu'}{2}\mathrm{e}^{2\nu}, \quad \Gamma^1_{11} = -\frac{\nu'}{2}, \quad \Gamma^1_{22} = -r\mathrm{e}^{\nu}, \quad \Gamma^1_{33} = -r\mathrm{e}^{\nu}\sin^2\theta \tag{55}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \tag{56}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \tag{57}$$

以线长 $s$ 为参量，测地线方程：

$$\frac{\mathrm{d}^2 x^{\sigma}}{\mathrm{d}s^2} + \Gamma^{\sigma}_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s} = 0 \tag{58}$$

$\sigma$ 分别取 $0, 1, 2, 3$ 就得到四条关于坐标的参数方程。

- $\sigma = 0$

非零联络：

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{\nu'}{2}, \tag{59}$$

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}s^2} + \Gamma^0_{01}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^0_{10}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s} = 0 \tag{60}$$

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} = 0 \tag{61}$$

- $\sigma = 1$

非零联络：

$$\Gamma^1_{00} = \frac{\nu'}{2}\mathrm{e}^{2\nu}, \quad \Gamma^1_{11} = -\frac{\nu'}{2}, \quad \Gamma^1_{22} = -r\mathrm{e}^{\nu}, \quad \Gamma^1_{33} = -r\mathrm{e}^{\nu}\sin^2\theta \tag{62}$$

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}s^2} + \Gamma^1_{00}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s} + \Gamma^1_{11}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^1_{22}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} + \Gamma^1_{33}\frac{\mathrm{d}x^3}{\mathrm{d}s}\frac{\mathrm{d}x^3}{\mathrm{d}s} = 0 \tag{63}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 - r\mathrm{e}^{\nu}\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{64}$$

- $\sigma = 2$

非零联络：

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \tag{65}$$

$$\frac{\mathrm{d}^2 x^2}{\mathrm{d}s^2} + \Gamma^2_{12}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} + \Gamma^2_{21}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^2_{33}\frac{\mathrm{d}x^3}{\mathrm{d}s}\frac{\mathrm{d}x^3}{\mathrm{d}s} = 0 \tag{66}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{67}$$

- $\sigma = 3$

非零联络：

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \tag{68}$$

$$\frac{\mathrm{d}^2 x^3}{\mathrm{d}s^2} + \Gamma^3_{13}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^3}{\mathrm{d}s} + \Gamma^3_{31}\frac{\mathrm{d}x^3}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^3_{23}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^3}{\mathrm{d}s} + \Gamma^3_{32}\frac{\mathrm{d}x^3}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} = 0 \tag{69}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \tag{70}$$

总之，四条参数方程为

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} = 0 \tag{71}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r\mathrm{e}^\nu\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 - r\mathrm{e}^\nu\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{72}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{73}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \tag{74}$$

取轨道面 $\theta = \pi/2$，则方程简化为

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} = 0 \tag{75}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r\mathrm{e}^\nu\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{76}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \tag{77}$$

注意到上面第三条方程

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0, \quad r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2r\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0, \quad r^2\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} + \frac{\mathrm{d}\left(r^2\right)}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \tag{78}$$

也即

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}\right) = 0 \tag{79}$$

因此

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}s} = \mathrm{const} \equiv \frac{h}{c} \tag{80}$$

这是 GR 中的角动量守恒。

我们要找轨道方程，因此需要找到 $r, \phi$ 的微分方程。

由于 $\theta = \pi/2$，则线元

$$\mathrm{d}s^2 = -c^2\mathrm{e}^\nu\mathrm{d}t^2 + \mathrm{e}^{-\nu}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \quad \mathrm{e}^\nu = 1 - \frac{2GM}{c^2 r} \tag{81}$$

可化简为

$$\mathrm{d}s^2 = -c^2\mathrm{e}^\nu\mathrm{d}t^2 + \mathrm{e}^{-\nu}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \tag{82}$$

利用线元 $\mathrm{d}s^2$ 与线长 $\mathrm{d}s$ 的关系

$$\mathrm{d}s = \sqrt{-\mathrm{d}s^2}, \quad (\mathrm{d}s)^2 = -\mathrm{d}s^2 = c^2\mathrm{e}^\nu\mathrm{d}t^2 - \mathrm{e}^{-\nu}\mathrm{d}r^2 - r^2\mathrm{d}\phi^2 \tag{83}$$

两边同除 $(\mathrm{d}s)^2$，再同乘 $\mathrm{e}^\nu$，得到

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = c^2 \mathrm{e}^{2\nu} \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - r^2 \mathrm{e}^{\nu} \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 - \mathrm{e}^{\nu} \tag{84}$$

上式代入 $r$ 关于 $s$ 的二阶偏微分方程，就消去 $t$：

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \frac{\nu'}{2} \mathrm{e}^{\nu} r^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 - r\mathrm{e}^{\nu} \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 + \frac{1}{2}\nu' \mathrm{e}^{\nu} = 0 \tag{85}$$

这就是GR中参数形式的行星轨道方程。

利用 GR 中的角动量守恒

$$r^2 \frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{h}{c}, \quad \frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{h}{cr^2} \tag{86}$$

进一步化简为

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \left(\frac{\nu'}{2}r^2 - r\right) \mathrm{e}^{\nu} \left(\frac{h}{cr^2}\right)^2 + \frac{1}{2}\nu' \mathrm{e}^{\nu} = 0 \tag{87}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \frac{1}{2}\nu' \mathrm{e}^{\nu} \left(\frac{h^2}{c^2 r^2} + 1\right) - r\mathrm{e}^{\nu} \left(\frac{h}{cr^2}\right)^2 = 0 \tag{88}$$

令 $u = \frac{1}{r}$，注意到

$$\mathrm{e}^{\nu} = 1 - \frac{2GM}{c^2 r} = 1 - \frac{2GM}{c^2}u \tag{89}$$

$$\nu' \mathrm{e}^{\nu} = (\mathrm{e}^{\nu})' = \frac{2GM}{c^2 r^2} = \frac{2GM}{c^2}u^2 \tag{90}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{h}{cr^2} = \frac{h}{c}u^2 \tag{91}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{h}{c}u^2\frac{\mathrm{d}}{\mathrm{d}\phi} \tag{92}$$

$$\begin{aligned}
\frac{\mathrm{d}^2 r}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}(1/u)}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left[-\frac{1}{u^2}\cdot\left(\frac{h}{c}u^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\right] \\
&= -\frac{h}{c}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right) = -\frac{h}{c}\cdot\frac{h}{c}u^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right) \\
&= -\frac{h^2}{c^2}u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2}
\end{aligned} \tag{93}$$

于是轨道的参数方程就化为如下的轨道方程：

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{3GM}{c^2}u^2 + \frac{GM}{h^2} \tag{94}$$

上式就是GR中的Binet方程。

设 $u_0$ 满足牛顿力学中的Binet方程

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 = \frac{GM}{h^2} \tag{95}$$

其解为

$$u_0 = \frac{1}{p}\left(1 + e\cos\phi\right), \quad p = \frac{h^2}{GM} \tag{96}$$

令 $\alpha = 3GM/c^2$，则

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \alpha u^2 + \frac{GM}{h^2} \tag{97}$$

设 $u = u_0 + \alpha u_1$，则

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 + \alpha \frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + \alpha u_1 = \alpha \left(u_0 + \alpha u_1\right)^2 + \frac{GM}{h^2} \tag{98}$$

利用 $u_0$ 满足的方程就得到

$$\alpha \frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + \alpha u_1 = \alpha \left(u_0 + \alpha u_1\right)^2 \tag{99}$$

由于 $\alpha$ 是个小量，约去右边括号内的高阶小量，再利用 $u_0$ 的表达式，就得到

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = u_0^2 = \frac{(1 + e\cos\phi)^2}{p^2} = \frac{1}{p^2}\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right] \tag{100}$$

设 $u_1$ 的形式解为

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi \tag{101}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}\phi} = B\left(\sin\phi + \phi\cos\phi\right) - 2C\sin 2\phi \tag{102}$$

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} = B\left(2\cos\phi - \phi\sin\phi\right) - 4C\cos 2\phi \tag{103}$$

代回方程得到

$$B\left(2\cos\phi - \phi\sin\phi\right) - 4C\cos 2\phi + A + B\phi\sin\phi + C\cos 2\phi = \frac{1}{p^2}\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right] \tag{104}$$

对比各项前的系数就得到

$$A = \frac{1}{p^2}\left(1 + \frac{e^2}{2}\right), \quad B = \frac{e}{p^2}, \quad C = -\frac{e^2}{6p^2} \tag{105}$$

因此

$$u_1 = \frac{1}{p^2}\left(1 + \frac{e^2}{2}\right) + \frac{e}{p^2}\phi\sin\phi - \frac{e^2}{6p^2}\cos 2\phi \tag{106}$$

$$u = u_0 + \alpha u_1 = \frac{1}{p}\left[(1 + e\cos\phi) + \frac{\alpha}{p}\left(1 + \frac{e^2}{2}\right) + \frac{\alpha e}{p}\phi\sin\phi - \frac{1}{6}\frac{\alpha e^2}{p^2}\cos 2\phi\right] \tag{107}$$

上面只有 $\phi\sin\phi$ 项是累加的，只保留对轨道有长期影响的项：

$$u = \frac{1}{p}\left[(1 + e\cos\phi) + \frac{\alpha e}{p}\phi\sin\phi\right] \tag{108}$$

由于 $\alpha$ 是小量，则

$$1 \approx \cos\left(\frac{\alpha}{p}\phi\right), \quad \frac{\alpha}{p}\phi \approx \sin\left(\frac{\alpha}{p}\phi\right) \tag{109}$$

于是

$$\begin{aligned}
u &= \frac{1}{p}\left[(1 + e\cos\phi) + \frac{\alpha e}{p}\phi\sin\phi\right] \\
&= \frac{1}{p}\left[1 + e\left(1 \cdot \cos\phi + \frac{\alpha}{p}\phi\sin\phi\right)\right] \\
&\approx \frac{1}{p}\left[1 + e\left(\cos\left(\frac{\alpha}{p}\phi\right)\cos\phi + \sin\left(\frac{\alpha}{p}\phi\right)\sin\phi\right)\right] \\
&= \frac{1}{p}\left[1 + \cos\left(\left(1 - \frac{\alpha}{p}\right)\phi\right)\right] \\
&\equiv \frac{1}{p}\left(1 + e\cos\Phi\right)
\end{aligned} \tag{110}$$

$$\Phi \equiv \left(1 - \frac{\alpha}{p}\right) \phi \tag{111}$$

当 $\Phi = \Phi_n = (2n+1)\pi$ 时, $\cos\Phi = -1$, 此时 $u$ 最小, $r$ 最大, 相应 $\phi_n$ 为

$$\phi_n = \frac{\Phi_n}{1 - \alpha/p} \approx \Phi_n \left(1 + \frac{\alpha}{p}\right) \tag{112}$$

$$\phi_{n+1} \approx \Phi_{n+1} \left(1 + \frac{\alpha}{p}\right) \tag{113}$$

于是

$$\phi_{n+1} - \phi_n = 2\pi \left(1 + \frac{\alpha}{p}\right) \tag{114}$$

一周期进动角为

$$\Delta = \phi_{n+1} - \phi_n - 2\pi = 2\pi \frac{\alpha}{p} = \frac{6\pi GM}{c^2 p} \tag{115}$$

由

$$T^2 = \frac{4\pi^2}{GM} a^3, \quad GM = 4\pi^2 \frac{a^3}{T^2}, \quad p = a\left(1 - e^2\right) \tag{116}$$

可得

$$\Delta = \frac{24\pi^3 a^2}{c^2 T^2 \left(1 - e^2\right)} \tag{117}$$

对于水星, 其世纪进动角为

$$\Delta_c \approx 43'' \tag{118}$$

## 4.5

导出光子在平面极坐标系下轨道所满足的方程, 并讨论光线在太阳附近偏折的问题。

对于光有 $\mathrm{d}s^2 = 0$, 因此测地线方程不能以线长 $s$ 为参数。但可引入参数 $\lambda$ 来定义光线的切矢:

$$K^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{119}$$

由于 $\mathrm{d}s^2 = 0$ 可得

$$g_{\mu\nu} K^\mu K^\nu = 0 \tag{120}$$

假设切矢 $K^\mu$ 在光的传播路线上是平行的, 即

$$\nabla_\mu K^\sigma = 0, \quad K^\mu \nabla_\mu K^\sigma = 0 \tag{121}$$

也即

$$K^\mu \left(\partial_\mu K^\sigma + \Gamma^\sigma_{\mu\nu} K^\nu\right) = 0 \tag{122}$$

又

$$K^\mu \partial_\mu K^\sigma = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\partial K^\sigma}{\partial x^\mu} = \frac{\mathrm{d}K^\sigma}{\mathrm{d}\lambda} \tag{123}$$

因此有

$$\frac{\mathrm{d}K^\sigma}{\mathrm{d}\lambda} + \Gamma^\sigma_{\mu\nu} K^\mu K^\nu = 0, \quad K^\sigma \equiv \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} \tag{124}$$

也即

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0 \tag{125}$$

我们知道，Schwarzschild解情况下有质量粒子在 $\theta = \pi/2$ 平面内的测地线方程有三条：

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \nu' \frac{\mathrm{d}t}{\mathrm{d}s} \frac{\mathrm{d}r}{\mathrm{d}s} = 0 \tag{126}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \frac{c^2 \nu'}{2} \mathrm{e}^{2\nu} \left( \frac{\mathrm{d}t}{\mathrm{d}s} \right)^2 - \frac{\nu'}{2} \left( \frac{\mathrm{d}r}{\mathrm{d}s} \right)^2 - r \mathrm{e}^\nu \left( \frac{\mathrm{d}\phi}{\mathrm{d}s} \right)^2 = 0 \tag{127}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2} + \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}s} \frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \tag{128}$$

对于光子，同样考虑Schwarzschild解，由于二者的联络都是一样，因此只需要把 $s$ 替换成 $\lambda$ 就得到条Schwarzschild解下光传播路径的参数方程：

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \nu' \frac{\mathrm{d}t}{\mathrm{d}\lambda} \frac{\mathrm{d}r}{\mathrm{d}\lambda} = 0 \tag{129}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + \frac{c^2 \nu'}{2} \mathrm{e}^{2\nu} \left( \frac{\mathrm{d}t}{\mathrm{d}\lambda} \right)^2 - \frac{\nu'}{2} \left( \frac{\mathrm{d}r}{\mathrm{d}\lambda} \right)^2 - r \mathrm{e}^\nu \left( \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \right)^2 = 0 \tag{130}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\lambda^2} + \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}\lambda} \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0 \tag{131}$$

对于第三条方程，同样有

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left( r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \right) = 0, \quad r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = k \tag{132}$$

在 $\theta = \pi/2$ 平面上，线元

$$\mathrm{d}s^2 = 0 = -c^2 \mathrm{e}^\nu \mathrm{d}t^2 + \mathrm{e}^{-\nu} \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2 \tag{133}$$

可以证明，可以从上式和第三条方程推导出第一条方程。上式两边同除 $(\mathrm{d}\lambda)^2$，并移项，就得到

$$c^2 \mathrm{e}^\nu \left( \frac{\mathrm{d}t}{\mathrm{d}\lambda} \right)^2 = \mathrm{e}^{-\nu} \left( \frac{\mathrm{d}r}{\mathrm{d}\lambda} \right)^2 + r^2 \left( \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \right)^2 \tag{134}$$

上式代回 $r$ 关于 $\lambda$ 二阶导式子，就得到

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + \left( \frac{r^2}{2} \nu' \mathrm{e}^\nu - r \mathrm{e}^\nu \right) \left( \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \right)^2 = 0 \tag{135}$$

令 $u = \frac{1}{r}$，注意到

$$\mathrm{e}^\nu = 1 - \frac{2GM}{c^2 r} = 1 - \frac{2GM}{c^2} u \tag{136}$$

$$\nu' \mathrm{e}^\nu = (\mathrm{e}^\nu)' = \frac{2GM}{c^2 r^2} = \frac{2GM}{c^2} u^2 \tag{137}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{k}{r^2} = k u^2 \tag{138}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \frac{\mathrm{d}}{\mathrm{d}\phi} = k u^2 \frac{\mathrm{d}}{\mathrm{d}\phi} \tag{139}$$

$$\begin{aligned} \frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\mathrm{d}(1/u)}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left( -\frac{1}{u^2} \frac{\mathrm{d}u}{\mathrm{d}\lambda} \right) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left( -\frac{1}{u^2} \cdot k u^2 \frac{\mathrm{d}u}{\mathrm{d}\phi} \right) \\ &= -k \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\mathrm{d}u}{\mathrm{d}\phi} = -k \cdot k u^2 \frac{\mathrm{d}}{\mathrm{d}\phi} \frac{\mathrm{d}u}{\mathrm{d}\phi} \\ &= -k^2 u^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} \end{aligned} \tag{140}$$

于是可以消去参数 $\lambda$，得到轨道微分方程

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{3GM}{c^2} u^2 \tag{141}$$

定义小量

$$\alpha \equiv \frac{3GM}{c^2} \tag{142}$$

则

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \alpha u^2 \tag{143}$$

设 $u_0$ 满足

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 = 0 \tag{144}$$

其解为

$$u_0 = \frac{1}{b} \sin \phi \tag{145}$$

设

$$u = u_0 + \alpha u_1 \tag{146}$$

则

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 + \alpha \left( \frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 \right) = \alpha \left( u_0 + \alpha u_1 \right)^2 \tag{147}$$

也即

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = \left( u_0 + \alpha u_1 \right)^2 \approx u_0^2 = \frac{1}{b^2} \sin^2 \phi \tag{148}$$

设 $u_1$ 的形式解为

$$u_1 = A \sin^2 \phi + B \tag{149}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}\phi} = 2A \sin\phi \cos\phi = A \sin 2\phi \tag{150}$$

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} = 2A \cos 2\phi = 2A \left( 1 - 2\sin^2 \phi \right) = 2A - 4A \sin^2 \phi \tag{151}$$

代回 $u_1$ 满足的微分方程，得到

$$2A - 4A \sin^2 \phi + A \sin^2 \phi + B = \frac{1}{b^2} \sin^2 \phi \tag{152}$$

对比可得

$$3A = -\frac{1}{b^2}, \quad 2A + B = 0 \tag{153}$$

解得

$$A = -\frac{1}{3b^2}, \quad B = \frac{2}{3b^2} \tag{154}$$

$$u_1 = A \sin^2 \phi + B = -\frac{1}{3b^2} \left( \sin^2 \phi - 2 \right) = \frac{1}{3b^2} \left( \cos^2 \phi + 1 \right) \tag{155}$$

$$\begin{aligned} u = u_0 + \alpha u_1 &= \frac{1}{b} \sin\phi + \frac{\alpha}{3b^2} \left( \cos^2 \phi + 1 \right) \\ &= \frac{1}{b} \sin\phi + \frac{GM}{c^2 b^2} \left( \cos^2 \phi + 1 \right) \end{aligned} \tag{156}$$

定义小量 $a \equiv GM/c^2b$，当 $r \to +\infty, u = 0$，此时

$$\sin\phi + a\left(2 - \sin^2\phi\right) = 0 \tag{157}$$

$$\sin\phi = \frac{1 \pm \sqrt{1 + 8a^2}}{2a} \approx \frac{1 \pm \left(1 + 4a^2\right)}{2a} = -2a \quad \text{or} \quad \frac{1 + 2a^2}{a} \tag{158}$$

舍去 $\sin\phi = \frac{1+2a^2}{a} > 1$ 的解，考虑 $\phi \to 0$ 的那侧，则

$$\phi \approx \sin\phi = -2a, \quad r \to +\infty \tag{159}$$

偏折角为

$$\delta = 4a = \frac{4GM}{c^2b} \tag{160}$$