2

2.1

> 已知 $\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=-\mathrm{d}\tau^2$，从变分原理 $\delta\int_A^B\mathrm{d}s=0$ 或 $\delta\int_A^B(\mathrm{d}\tau/\mathrm{d}\lambda)^2\,\mathrm{d}\lambda=0$ 求出“短”程线方程。

段先生书上已经有对线长变分求短程线的过程，也就是从

$$\delta\int_{\lambda_1}^{\lambda_2}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\mathrm{d}\lambda=0 \tag{1}$$

出发找到 $x^\mu(\lambda)$ 要满足的方程。

这里

$$\mathrm{d}\tau=\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}=\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\mathrm{d}\lambda, \tag{2}$$

也就是从

$$\delta\int_{\lambda_1}^{\lambda_2}-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\mathrm{d}\lambda=0 \tag{3}$$

出发推导短程线方程。也即

$$\int_{\lambda_1}^{\lambda_2}\delta(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)\mathrm{d}\lambda=0 \tag{4}$$

令：

$$F(x,\dot{x})=-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{5}$$

$$
\begin{aligned}
\delta F &= -\delta\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)\\
&= -\left[\dot{x}^\mu\dot{x}^\nu\delta g_{\mu\nu} + g_{\mu\nu}\dot{x}^\nu\delta\dot{x}^\mu + g_{\mu\nu}\dot{x}^\mu\delta\dot{x}^\nu\right]\\
&= -\left[\dot{x}^\mu\dot{x}^\nu\partial_\alpha g_{\mu\nu}\delta x^\alpha + g_{\mu\nu}\dot{x}^\nu\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\mu + g_{\mu\nu}\dot{x}^\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\nu\right]\\
&= -\left[\dot{x}^\mu\dot{x}^\nu\partial_\alpha g_{\mu\nu}\delta x^\alpha + g_{\alpha\nu}\dot{x}^\nu\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\alpha + g_{\mu\alpha}\dot{x}^\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta x^\alpha\right]\\
&= -\left\{\dot{x}^\mu\dot{x}^\nu\partial_\alpha g_{\mu\nu}\delta x^\alpha + \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\delta x^\alpha\right] - \delta x^\alpha\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\right\}\\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\delta x^\alpha\right] + \delta x^\alpha\left[\left(\dot{x}^\nu\partial_\beta g_{\alpha\nu}\dot{x}^\beta + g_{\alpha\nu}\ddot{x}^\nu + \dot{x}^\mu\partial_\beta g_{\mu\alpha}\dot{x}^\beta + g_{\mu\alpha}\ddot{x}^\mu\right) - \dot{x}^\mu\dot{x}^\nu\partial_\alpha g_{\mu\nu}\right]\\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\delta x^\alpha\right] + \delta x^\alpha\left[\left(\dot{x}^\nu\partial_\mu g_{\alpha\nu}\dot{x}^\mu + g_{\alpha\mu}\ddot{x}^\mu + \dot{x}^\mu\partial_\nu g_{\mu\alpha}\dot{x}^\nu + g_{\alpha\mu}\ddot{x}^\mu\right) - \dot{x}^\mu\dot{x}^\nu\partial_\alpha g_{\mu\nu}\right]\\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right)\delta x^\alpha\right] + \delta x^\alpha\left[\dot{x}^\mu\dot{x}^\nu\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) + 2g_{\alpha\mu}\ddot{x}^\mu\right]
\end{aligned}
\tag{6}
$$

由 $\int_{\lambda_1}^{\lambda}\delta F\mathrm{d}\lambda$ 可得

$$
\dot{x}^\mu\dot{x}^\nu\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) + 2g_{\alpha\mu}\ddot{x}^\mu = 0 \tag{7}
$$

上式两边同乘 $g_{\lambda\alpha}$ 可得

$$
\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \ddot{x}^\lambda = 0 \tag{8}
$$

也即短程线方程:

$$
\frac{\mathrm{d}^2x^\lambda}{\mathrm{d}\lambda^2} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0. \tag{9}
$$

2.2

> 已知 $\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu}A_\lambda$，利用标量微分关系 $\nabla_\nu U = \partial_\nu U$ 以及莱布尼茨法则证明 $\nabla_\nu B^\mu = \partial_\nu B^\mu + \Gamma^\mu_{\nu\lambda}B^\lambda$.

一方面，协变微商满足莱布尼兹法则:

$$
\begin{aligned}
\nabla_\nu\left(A_\mu B^\mu\right) &= B^\mu\nabla_\nu A_\mu + A_\mu\nabla_\nu B^\mu\\
&= B^\mu\left(\partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu}A_\lambda\right) + A_\mu\nabla_\nu B^\mu\\
&= B^\mu\partial_\nu A_\mu - B^\mu\Gamma^\lambda_{\nu\mu}A_\lambda + A_\mu\nabla_\nu B^\mu
\end{aligned}
\tag{10}
$$

另一方面，$A_\mu B^\mu$ 是标量，其协变微商等于普通偏微分:

$$
\begin{aligned}
\nabla_\nu\left(A_\mu B^\mu\right) &= \partial_\nu\left(A_\mu B^\mu\right)\\
&= B^\mu\partial_\nu A_\mu + A_\mu\partial_\nu B^\mu
\end{aligned}
\tag{11}
$$

对比可得:

$$
\begin{aligned}
A_\mu \nabla_\nu B^\mu &= A_\mu \partial_\nu B^\mu + B^\mu \Gamma^\lambda_{\nu\mu} A_\lambda \\
&= A_\mu \partial_\nu B^\mu + A_\lambda \Gamma^\lambda_{\nu\mu} B^\mu \\
&= A_\mu \partial_\nu B^\mu + A_\mu \Gamma^\mu_{\nu\lambda} B^\lambda
\end{aligned} \tag{12}
$$

从而得到:

$$
\nabla_\nu B^\mu = \partial_\nu B^\mu + \Gamma^\mu_{\nu\lambda} B^\lambda \tag{13}
$$

2.3

> 一个嵌入三维欧氏空间的普通球面，选用球极坐标，其线元为 $\mathrm{d}s^2 = a^2 \mathrm{d}\theta^2 + a^2 \sin^2\theta \mathrm{d}\phi^2$，求：(1) $g^{\nu\mu}$；(2) 全部非零克氏符 $\Gamma^\lambda_{\mu\nu}$；(3) 全部非零 $R^\nu_{\mu\sigma\lambda}, R_{\mu\nu}, R$；(4) 写出该度规表示的球面空间的测地线方程。

度规

取 $\left(x^1, x^2\right) = (\theta, \phi)$，根据线元

$$
\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = a^2 \mathrm{d}\theta^2 + a^2 \sin^2\theta \mathrm{d}\phi^2 \tag{14}
$$

可得度规

$$
g_{11} = a^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = a^2 \sin^2\theta, \tag{15}
$$

$$
[g_{\mu\nu}] = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{bmatrix} \tag{16}
$$

以及逆度规

$$
[g^{\mu\nu}] = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2 \sin^2\theta} \end{bmatrix} \tag{17}
$$

$$
g^{11} = \frac{1}{a^2}, \quad g^{12} = g^{21} = 0, \quad g^{22} = \frac{1}{a^2 \sin^2\theta}, \tag{18}
$$

克氏符

由度规 $g_{\mu\nu}$ 以及逆度规 $g^{\mu\nu}$ 可得克氏符:

$$
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right), \tag{19}
$$

$$
\begin{aligned}
\Gamma^1_{\mu\nu} &= \frac{1}{2} g^{1\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \\
&= \frac{1}{2} g^{11}\left(\partial_\mu g_{1\nu} + \partial_\nu g_{1\mu} - \partial_1 g_{\mu\nu}\right) \\
&= \frac{1}{2a^2}\left(-\partial_1 g_{\mu\nu}\right) \\
&= -\frac{1}{2a^2}\partial_1 g_{\mu\nu}
\end{aligned} \tag{20}
$$

$$
\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = 0, \tag{21}
$$

$$
\Gamma^1_{22} = -\frac{1}{2a^2}\partial_1 g_{22} = -\frac{1}{2a^2}\partial_\theta\left(a^2\sin^2\theta\right) = -\sin\theta\cos\theta, \tag{22}
$$

$$
\begin{aligned}
\Gamma^2_{\mu\nu} &= \frac{1}{2} g^{2\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \\
&= \frac{1}{2} g^{22}\left(\partial_\mu g_{2\nu} + \partial_\nu g_{2\mu} - \partial_2 g_{\mu\nu}\right) \\
&= \frac{1}{2} g^{22}\left(\partial_\mu g_{2\nu} + \partial_\nu g_{2\mu}\right) \\
&= \frac{1}{2a^2\sin^2\theta}\left(\partial_\mu g_{2\nu} + \partial_\nu g_{2\mu}\right)
\end{aligned} \tag{23}
$$

$$
\Gamma^2_{11} = \Gamma^2_{22} = 0, \tag{24}
$$

$$
\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2a^2\sin^2\theta}\left(\partial_1 g_{22} + \partial_2 g_{21}\right) = \frac{1}{2a^2\sin^2\theta}\partial_\theta\left(a^2\sin^2\theta\right) = \frac{\cos\theta}{\sin\theta} = \cot\theta \tag{25}
$$

Riemann张量

$$
R^\lambda_{\sigma\mu\nu} = \partial_\mu\Gamma^\lambda_{\nu\sigma} - \partial_\nu\Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\mu\sigma} \tag{26}
$$

$$
R^1_{212} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1_{1\alpha}\Gamma^\alpha_{22} - \Gamma^1_{2\alpha}\Gamma^\alpha_{12} = \sin^2\theta \tag{27}
$$

- $\lambda = 1$
-
  - $(\mu, \nu) = (1, 2)$

$$
\begin{aligned}
R^1_{\sigma 12} &= \partial_1\Gamma^1_{2\sigma} - \partial_2\Gamma^1_{1\sigma} + \Gamma^1_{1\alpha}\Gamma^\alpha_{2\sigma} - \Gamma^1_{2\alpha}\Gamma^\alpha_{1\sigma} \\
&= \partial_1\Gamma^1_{2\sigma} - \Gamma^1_{22}\Gamma^2_{1\sigma}
\end{aligned} \tag{28}
$$

-
  -
    - $\sigma = 1$

$$
\begin{aligned}
R^1_{112} &= \partial_1\Gamma^1_{21} - \Gamma^1_{22}\Gamma^2_{11} \\
&= -\Gamma^1_{22}\Gamma^2_{11} \\
&= 0
\end{aligned} \tag{29}
$$

-
  -
    - $\sigma = 2$

$$\begin{aligned} R^1_{212} &= \partial_1 \Gamma^1_{22} - \Gamma^1_{22}\Gamma^2_{12} \\ &= \sin^2\theta \end{aligned} \tag{30}$$

$$R^1_{221} = -R^1_{212} = -\sin^2\theta \tag{31}$$

- $\lambda = 2$
- ◦ $(\mu, \nu) = (1, 2)$

$$\begin{aligned} R^2_{\sigma 12} &= \partial_1 \Gamma^2_{2\sigma} - \partial_2 \Gamma^2_{1\sigma} + \Gamma^2_{1\alpha}\Gamma^\alpha_{2\sigma} - \Gamma^2_{2\alpha}\Gamma^\alpha_{1\sigma} \\ &= \partial_1 \Gamma^2_{2\sigma} + \Gamma^2_{12}\Gamma^2_{2\sigma} - \Gamma^2_{21}\Gamma^1_{1\sigma} \\ &= \partial_1 \Gamma^2_{2\sigma} + \cot\theta \Gamma^2_{2\sigma} - \cot\theta \Gamma^1_{1\sigma} \end{aligned} \tag{32}$$

- ◦ ▪ $\sigma = 1$

$$\begin{aligned} R^2_{112} &= \partial_1 \Gamma^2_{21} + \cot\theta \Gamma^2_{21} - \cot\theta \Gamma^1_{11} \\ &= \partial_\theta \cot\theta + \cot\theta \cdot \cot\theta \\ &= -1 \end{aligned} \tag{33}$$

$$R^2_{121} = -R^2_{112} = 1 \tag{34}$$

- ◦ ▪ $\sigma = 2$

$$\begin{aligned} R^2_{212} &= \partial_1 \Gamma^2_{22} + \cot\theta \Gamma^2_{22} - \cot\theta \Gamma^1_{12} \\ &= 0 \end{aligned} \tag{35}$$

Ricci张量

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = R^1_{\mu 1\nu} + R^2_{\mu 2\nu} \tag{36}$$

- $(\mu, \nu) = (1, 1)$

$$R_{11} = R^1_{111} + R^2_{121} = 1 \tag{37}$$

- $(\mu, \nu) = (1, 2)$

$$R_{12} = R^1_{112} + R^2_{122} = 0 \tag{38}$$

- $(\mu, \nu) = (2, 1)$

$$R_{21} = R^1_{211} + R^2_{221} = 0 \tag{39}$$

- $(\mu, \nu) = (2, 2)$

$$R_{22} = R^1_{212} + R^2_{222} = \sin^2\theta \tag{40}$$

曲率标量

$$R=g^{\mu\nu}R_{\mu\nu}=g^{11}R_{11}+g^{22}R_{22}=\frac{2}{a^2} \tag{41}$$

测地线方程

测地线方程：

$$\ddot{x}^\lambda+\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0 \tag{42}$$

- $\lambda=1$

$$\ddot{x}^1+\Gamma^1_{22}\dot{x}^2\dot{x}^2=0 \tag{43}$$

$$\ddot{\theta}-\sin\theta\cos\theta\dot{\phi}^2=0 \tag{44}$$

- $\lambda=2$

$$\ddot{x}^2+\Gamma^2_{12}\dot{x}^1\dot{x}^2+\Gamma^2_{21}\dot{x}^2\dot{x}^1=0 \tag{45}$$

$$\ddot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}=0 \tag{46}$$

2.4

> 由协变矢量双重协变微商非对称部分 $\phi_{\lambda;[\mu;\nu]}$ 推导曲率张量与挠率张量 $\nabla_\mu\nabla_\nu\phi_\lambda-\nabla_\nu\nabla_\mu\phi_\lambda=-R^\sigma_{\lambda\mu\nu}\phi_\sigma-T^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda$.

$$\begin{aligned}
\nabla_\mu\nabla_\nu\phi_\lambda-\mu\leftrightarrow\nu&=\nabla_\mu\left(\nabla_\nu\phi_\lambda\right)-\mu\leftrightarrow\nu\\
&=\partial_\mu\left(\nabla_\nu\phi_\lambda\right)-\Gamma^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda-\Gamma^\alpha_{\mu\lambda}\nabla_\nu\phi_\alpha-\mu\leftrightarrow\nu\\
&=\partial_\mu\left(\partial_\nu\phi_\lambda-\Gamma^\alpha_{\nu\lambda}\phi_\alpha\right)-\Gamma^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda-\Gamma^\alpha_{\mu\lambda}\left(\partial_\nu\phi_\alpha-\Gamma^\beta_{\nu\alpha}\phi_\beta\right)-\mu\leftrightarrow\nu\\
&=\partial_\mu\partial_\nu\phi_\lambda-\left(\partial_\mu\Gamma^\alpha_{\nu\lambda}\right)\phi_\alpha-\Gamma^\alpha_{\nu\lambda}\partial_\mu\phi_\alpha-\Gamma^\alpha_{\mu\lambda}\partial_\nu\phi_\alpha+\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\nu\alpha}\phi_\beta-\Gamma^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda-\mu\leftrightarrow\nu\\
&=-\left(\partial_\mu\Gamma^\alpha_{\nu\lambda}-\partial_\nu\Gamma^\alpha_{\mu\lambda}\right)\phi_\alpha+\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\nu\alpha}\phi_\beta-\Gamma^\alpha_{\nu\lambda}\Gamma^\beta_{\mu\alpha}\phi_\beta-\left(\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}\right)\nabla_\alpha\phi_\lambda\\
&=-\left(\partial_\mu\Gamma^\beta_{\nu\lambda}-\partial_\nu\Gamma^\beta_{\mu\lambda}\right)\phi_\beta+\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\nu\alpha}\phi_\beta-\Gamma^\alpha_{\nu\lambda}\Gamma^\beta_{\mu\alpha}\phi_\beta-\left(\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}\right)\nabla_\alpha\phi_\lambda\\
&=-\left(\partial_\mu\Gamma^\beta_{\nu\lambda}-\partial_\nu\Gamma^\beta_{\mu\lambda}+\Gamma^\alpha_{\nu\lambda}\Gamma^\beta_{\mu\alpha}-\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\nu\alpha}\right)\phi_\beta-\left(\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}\right)\nabla_\alpha\phi_\lambda\\
&=-R^\beta_{\lambda\mu\nu}\phi_\beta-T^\alpha_{\mu\nu}\nabla_\alpha\phi_\lambda
\end{aligned} \tag{47}$$

2.5

> 试求出适用任意时轴正交时空、任意观者的光速表达式，并由此验证广义相对论中的光速不变原理。提示：利用类光线元，并只考虑光线沿径向传播的简单情形。

$$\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{48}$$

时轴正交时空，$g_{0i}=0$，线元可写为

$$\mathrm{d}s^2=g_{tt}c^2\mathrm{d}t^2+g_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{49}$$

考虑光线沿径向传播，则类光线元满足

$$\mathrm{d}s^2=g_{tt}c^2\mathrm{d}t^2+g_{rr}\mathrm{d}r^2=0 \tag{50}$$

其中

$$g_{tt}<0,\quad g_{rr}>0 \tag{51}$$

考虑任意观者 $x^\mu(\tau)$，其四速度 $u^\mu(\tau)$ 满足

$$g_{\mu\nu}u^\mu u^\nu=-c^2 \tag{52}$$

观者观测的时间间隔为

$$\mathrm{d}t_{\text{obs}}=-\frac{1}{c^2}u_\mu\mathrm{d}x^\mu=-\frac{1}{c^2}\left(u_0\mathrm{d}x^0+u_1\mathrm{d}x^1\right) \tag{53}$$

观者观测的空间距离为

$$\mathrm{d}l_{\text{obs}}=\sqrt{h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}=\sqrt{\left(g_{\mu\nu}+\frac{u_\mu u_\nu}{c^2}\right)\mathrm{d}x^\mu\mathrm{d}x^\nu} \tag{54}$$

对于类光线元，

$$\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=0 \tag{55}$$

则

$$\mathrm{d}l_{\text{obs}}=\sqrt{\frac{1}{c^2}\left(u_\mu\mathrm{d}x^\mu\right)^2}=-\frac{1}{c}u_\mu\mathrm{d}x^\mu \tag{56}$$

测量光速

$$c_{\text{obs}}=\frac{\mathrm{d}l_{\text{obs}}}{\mathrm{d}t_{\text{obs}}}=c \tag{57}$$