基态波函数

基态 $|\Omega\rangle$ 是 α 准粒子的真空态，满足：

$$\alpha_i |\Omega\rangle = 0 \tag{1}$$

也即

$$\sum_{j=1}^{N} \left(w_{ji}^* a_j + v_{ji}^* a_j^\dagger \right) |\Omega\rangle = 0 \tag{2}$$

基态形式

$$|\Omega\rangle = \mathcal{N} \exp\left(\frac{1}{2} \sum_{i,j} f_{i,j} a_i^\dagger a_j^\dagger \right) |0_a\rangle \tag{3}$$

其中 $|0_a\rangle$ 是 a 费米子真空态，$f_{i,j} = -f_{j,i}$

由于

$$\left[a_i^\dagger a_j^\dagger, a_k^\dagger a_l^\dagger \right] = 0 \tag{4}$$

而

$$[A, B] = 0 \Longrightarrow \exp(A + B) = \exp(A)\exp(B) \tag{5}$$

于是

$$\begin{aligned} |\Omega\rangle &= \mathcal{N} \exp\left(\frac{1}{2} \sum_{i,j} f_{i,j} a_i^\dagger a_j^\dagger \right) |0_a\rangle \\ &= \mathcal{N} \prod_{i,j} \exp\left(\frac{1}{2} f_{i,j} a_i^\dagger a_j^\dagger \right) |0_a\rangle \end{aligned} \tag{6}$$

注意到

$$\begin{aligned} \exp\left(\frac{1}{2} f_{i,j} a_i^\dagger a_j^\dagger \right) |0_a\rangle &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{2} f_{i,j} a_i^\dagger a_j^\dagger \right)^k |0_a\rangle \\ &= \left(1 + \frac{1}{1!} \cdot \frac{1}{2^1} f_{i,j} a_i^\dagger a_j^\dagger + \frac{1}{2!} \cdot \frac{1}{2^2} a_i^\dagger a_j^\dagger a_i^\dagger a_j^\dagger + \cdots \right) |0_a\rangle \\ &= \left(1 + \frac{1}{2} f_{i,j} a_i^\dagger a_j^\dagger \right) |0_a\rangle \end{aligned} \tag{7}$$

于是

$$\begin{aligned} |\Omega\rangle &= \mathcal{N} \prod_{i,j} \exp\left(\frac{1}{2} f_{i,j} a_i^\dagger a_j^\dagger \right) |0_a\rangle \\ &= \mathcal{N} \prod_{i,j} \left(1 + \frac{1}{2} f_{i,j} a_i^\dagger a_j^\dagger \right) |0_a\rangle \end{aligned} \tag{8}$$

求基态波函数 $f_{i,j}$

$$\alpha_i = \sum_{j=1}^{N} \left(w_{ji}^* a_j + v_{ji}^* a_j^\dagger \right)$$

$$|\Omega\rangle = \mathcal{N} \prod_{l,m} \left(1 + \frac{1}{2} f_{l,m} a_l^\dagger a_m^\dagger \right) |0_a\rangle \tag{9}$$

基态由

$$\alpha_i |\Omega\rangle = 0 \tag{10}$$

确定。

利用

$$AB = [A, B] + BA \tag{11}$$

$$\left[A, \prod_i B_i \right] = \sum_k \left(\prod_{j<k} B_j \right) [A, B_k] \left(\prod_{j>k} B_j \right) \tag{12}$$

$$\left[1 + \frac{1}{2} f_{i,j} a_i^\dagger a_j^\dagger, 1 + \frac{1}{2} f_{l,m} a_l^\dagger a_m^\dagger \right] = 0 \tag{13}$$

$$\begin{aligned}
\left[a_j, 1 + \frac{1}{2} f_{l',m'} a_{l'}^\dagger a_{m'}^\dagger \right] &= \frac{1}{2} f_{l',m'} \left[a_j, a_{l'}^\dagger a_{m'}^\dagger \right] \\
&= \frac{1}{2} f_{l',m'} \left(\left[a_j, a_{l'}^\dagger \right] a_{m'}^\dagger + a_{l'}^\dagger \left[a_j, a_{m'}^\dagger \right] \right) \\
&= \frac{1}{2} f_{l',m'} \left[\left(\left\{ a_j, a_{l'}^\dagger \right\} - 2 a_{l'}^\dagger a_j \right) a_{m'}^\dagger + a_{l'}^\dagger \left(\left\{ a_j, a_{m'}^\dagger \right\} - 2 a_{m'}^\dagger a_j \right) \right] \\
&= \frac{1}{2} f_{l',m'} \left[\left(\delta_{j,l'} - 2 a_{l'}^\dagger a_j \right) a_{m'}^\dagger + a_{l'}^\dagger \left(\delta_{j,m'} - 2 a_{m'}^\dagger a_j \right) \right] \\
&= \frac{1}{2} f_{l',m'} \left(\delta_{j,l'} a_{m'}^\dagger + \delta_{j,m'} a_{l'}^\dagger - 2 a_{l'}^\dagger a_j a_{m'}^\dagger - 2 a_{l'}^\dagger a_{m'}^\dagger a_j \right) \\
&= \frac{1}{2} f_{l',m'} \left[\delta_{j,l'} a_{m'}^\dagger + \delta_{j,m'} a_{l'}^\dagger - 2 a_{l'}^\dagger \left(\delta_{j,m'} - a_{m'}^\dagger a_j \right) - 2 a_{l'}^\dagger a_{m'}^\dagger a_j \right] \\
&= \frac{1}{2} f_{l',m'} \left(\delta_{j,l'} a_{m'}^\dagger - \delta_{j,m'} a_{l'}^\dagger \right)
\end{aligned} \tag{14}$$

$$\left[a_{m'}^\dagger, 1 + \frac{1}{2} f_{l,m} a_l^\dagger a_m^\dagger \right] = 0 \tag{15}$$

$$\begin{aligned}
&\frac{1}{2} f_{l',m'} \left(\delta_{j,l'} a_{m'}^\dagger - \delta_{j,m'} a_{l'}^\dagger \right) \cdot \left(1 + \frac{1}{2} f_{l',m'} a_{l'}^\dagger a_{m'}^\dagger \right) \\
=&\frac{1}{2} f_{l',m'} \left(\delta_{j,l'} a_{m'}^\dagger - \delta_{j,m'} a_{l'}^\dagger \right) + \frac{1}{4} f_{l',m'}^2 \left(\delta_{j,l'} a_{m'}^\dagger - \delta_{j,m'} a_{l'}^\dagger \right) a_{l'}^\dagger a_{m'}^\dagger \\
=&\frac{1}{2} f_{l',m'} \left(\delta_{j,l'} a_{m'}^\dagger - \delta_{j,m'} a_{l'}^\dagger \right)
\end{aligned} \tag{16}$$

有

$$
\begin{aligned}
a_j \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right)\left|0_a\right\rangle &= \left[a_j, \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right)\right]\left|0_a\right\rangle+\prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle \\
&= \left[a_j, \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right)\right]\left|0_a\right\rangle \\
&= \sum_{l',m'}\left[a_j, 1+\frac{1}{2} f_{l',m'} a_{l'}^{\dagger} a_{m'}^{\dagger}\right] \prod_{(l,m)\backslash(l',m')}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle \\
&= \sum_{l',m'} \frac{1}{2} f_{l',m'}\left(\delta_{j,l'} a_{m'}^{\dagger}-\delta_{j,m'} a_{l'}^{\dagger}\right) \prod_{(l,m)\backslash(l',m')}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle \\
&= \sum_{l',m'} \frac{1}{2} f_{l',m'}\left(\delta_{j,l'} a_{m'}^{\dagger}-\delta_{j,m'} a_{l'}^{\dagger}\right) \cdot\left(1+\frac{1}{2} f_{l',m'} a_{l'}^{\dagger} a_{m'}^{\dagger}\right) \prod_{(l,m)\backslash(l',m')}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle \\
&= \sum_{l',m'} \frac{1}{2} f_{l',m'}\left(\delta_{j,l'} a_{m'}^{\dagger}-\delta_{j,m'} a_{l'}^{\dagger}\right) \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle \\
&= \left[\sum_{m'} \frac{1}{2} f_{j,m'} a_{m'}^{\dagger}-\sum_{l'} \frac{1}{2} f_{l',j} a_{l'}^{\dagger}\right] \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle \\
&= \left[\sum_{m'} \frac{1}{2} f_{j,m'} a_{m'}^{\dagger}+\sum_{l'} \frac{1}{2} f_{j,l'} a_{l'}^{\dagger}\right] \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle \\
&= \sum_k f_{j,k} a_k^{\dagger} \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle
\end{aligned}
\tag{17}
$$

于是 a_j 对 $|\Omega\rangle$ 的作用为

$$
\begin{aligned}
a_j|\Omega\rangle &= \mathcal{N} a_j \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right)\left|0_a\right\rangle \\
&= \mathcal{N} \sum_k f_{j,k} a_k^{\dagger} \prod_{l,m}\left(1+\frac{1}{2} f_{l,m} a_l^{\dagger} a_m^{\dagger}\right) a_j\left|0_a\right\rangle \\
&= \sum_k f_{j,k} a_k^{\dagger}|\Omega\rangle
\end{aligned}
\tag{18}
$$

基态由

$$
\alpha_i|\Omega\rangle=0 \tag{19}
$$

确定，也即

$$
\sum_{j=1}^N\left(w_{ji}^* a_j+v_{ji}^* a_j^{\dagger}\right)|\Omega\rangle=0 \tag{20}
$$

把 $a_j|\Omega\rangle$ 代入：

$$
\begin{aligned}
0 &= \sum_{j=1}^{N}\left(w_{ji}^{*}a_j + v_{ji}^{*}a_j^{\dagger}\right)|\Omega\rangle \\
&= \sum_{j=1}^{N}\left(w_{ji}^{*}\sum_k f_{j,k}a_k^{\dagger} + v_{ji}^{*}a_j^{\dagger}\right)|\Omega\rangle \\
&= \sum_{j=1}^{N}\left(w_{ji}^{*}\sum_k f_{j,k}a_k^{\dagger} + v_{ji}^{*}\sum_k \delta_{j,k}a_k^{\dagger}\right)|\Omega\rangle \\
&= \sum_{j=1}^{N}\left(\sum_k w_{ji}^{*}f_{j,k}a_k^{\dagger} + \sum_k v_{ji}^{*}\delta_{j,k}a_k^{\dagger}\right)|\Omega\rangle \\
&= \sum_{j=1}^{N}\sum_k\left(w_{ji}^{*}f_{j,k}a_k^{\dagger} + v_{ji}^{*}\delta_{j,k}a_k^{\dagger}\right)|\Omega\rangle \\
&= \sum_k\sum_{j=1}^{N}\left(w_{ji}^{*}f_{j,k}a_k^{\dagger} + v_{ji}^{*}\delta_{j,k}a_k^{\dagger}\right)|\Omega\rangle \\
&= \sum_k\left[\sum_{j=1}^{N}\left(w_{ji}^{*}f_{j,k} + v_{ji}^{*}\delta_{j,k}\right)\right]a_k^{\dagger}|\Omega\rangle
\end{aligned}
\tag{21}
$$

因此有

$$
\sum_{j=1}^{N}\left(w_{ji}^{*}f_{j,k} + v_{ji}^{*}\delta_{j,k}\right) = 0, \quad \forall i, k \tag{22}
$$

利用 $f_{j,k}$ 的反对称性有

$$
\sum_{j=1}^{N}\left(-f_{k,j}w_{ji}^{*} + \delta_{k,j}v_{ji}^{*}\right) = 0, \quad \forall i, k \tag{23}
$$

构造 $\mathbf{f}$ 矩阵:

$$
\mathbf{f} = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots \\ f_{2,1} & \ddots & \vdots \\ \vdots & \cdots & \ddots \end{pmatrix} \tag{24}
$$

则化为

$$
-\left(\mathbf{f}\mathbf{w}^{*}\right)_{k,i} + \left(\mathbf{I}\mathbf{v}^{*}\right)_{k,i} = 0, \quad \forall i, k \tag{25}
$$

也即矩阵方程

$$
\mathbf{f}\mathbf{w}^{*} = \mathbf{v}^{*} \tag{26}
$$

于是

$$
\mathbf{f} = \mathbf{v}^{*}\left(\mathbf{w}^{*}\right)^{-1} \tag{27}
$$