- (1) 质点在约束下可能发生的微小位移 $\delta \vec{r}$ 称为虚位移。质点实际运动产生的位移称为实位移。实位移是众多虚位移之中的一个。
- (2) 若拉格朗日量中不显含某个广义坐标,则这个广义坐标称为循环坐标,其对应的广义动量是个守恒量。
- (3) 某一时刻, 刚体上速度为零的质点称为转动瞬心。
- (4)科里奥利力:

$$ec{F}_c = -2m \, (ec{\omega} imes ec{v})$$

其中, \vec{F}_c 为科里奥利力,m 为质点质量, $\vec{\omega}$ 为旋转参考系的角速度, \vec{v} 为质点相对于旋转参考系的速度。

例子: 傅科摆

(5)

限制在平面上的质点自由度: 2

傅科摆自由度: 2

水平面上作匀速纯滚动的刚体球自由度: 1

设粒子的静止质量为 m_0 , 粒子速度 v=0.9c

运动质量:

$$m = rac{m_0}{\sqrt{1 - v^2/c^2}} = rac{m_0}{\sqrt{1 - \left(0.9c
ight)^2/c^2}} = rac{m_0}{\sqrt{0.19}} = rac{10m_0}{\sqrt{19}}$$

总能量:

$$E = mc^2 = rac{10 m_0 c^2}{\sqrt{19}}$$

动能:

$$T=E-m_0c^2=\left(rac{10}{\sqrt{19}}-1
ight)m_0c^2$$

动量:

$$p = mv = \frac{10m_0}{\sqrt{19}} \cdot (0.9c) = \frac{9m_0c}{\sqrt{19}}$$

Ξ

虚功原理:受理想约束的力学体系平衡的充要条件是此力学体系的所有主动力在任何虚位移中所做的元功**之和**等于零,即 $\delta W = \sum_i \vec{F}_i^{(A)} \cdot \delta \vec{r}_i = 0$

三个受主动力的质点分别记为 1, 2, 3

选取 α , β 为广义坐标来描述所有质点的位置

$$egin{aligned} ec{r}_1 &= rac{l_1}{2}\sinlphaec{e}_x + rac{l_1}{2}\coslphaec{e}_y, \ ec{r}_2 &= \left(l_1\sinlpha + rac{l_2}{2}\sineta
ight)ec{e}_x + \left(l_1\coslpha + rac{l_2}{2}\coseta
ight)ec{e}_y, \ ec{r}_3 &= \left(l_1\sinlpha + l_2\sineta
ight)ec{e}_x + \left(l_1\coslpha + l_2\coseta
ight)ec{e}_y, \end{aligned}$$

$$\begin{cases} \delta \vec{r}_1 = \frac{l_1}{2} \cos \alpha \delta \alpha \vec{e}_x - \frac{l_1}{2} \sin \alpha \delta \alpha \vec{e}_y \\ \delta \vec{r}_2 = \left(l_1 \cos \alpha \delta \alpha + \frac{l_2}{2} \cos \beta \delta \beta \right) \vec{e}_x + \left(-l_1 \sin \alpha \delta \alpha - \frac{l_2}{2} \sin \beta \delta \beta \right) \vec{e}_y \\ \delta \vec{r}_3 = \left(l_1 \cos \alpha \delta \alpha + l_2 \cos \beta \delta \beta \right) \vec{e}_x + \left(-l_1 \sin \alpha \delta \alpha - l_2 \sin \beta \delta \beta \right) \vec{e}_y \end{cases}$$

主动力:

$$ec{F}_1^{(A)} = P_1 ec{e}_x, \ \ ec{F}_2^{(A)} = P_2 ec{e}_x, \ \ ec{F}_3^{(A)} = F ec{e}_y$$

虚功原理:

$$\sum_{i=1}^{3}ec{F}_{i}^{(A)}\cdot\deltaec{r}_{i}=0$$

即:

$$\delta lpha \left[rac{1}{2}l_1P_1\coslpha + l_1P_2\coslpha - Fl_1\sinlpha
ight] + \delta eta \left[rac{1}{2}l_2P_2\coseta - Fl_2\sineta
ight] = 0$$

 $\delta\alpha,\delta\beta$ 相互独立,上式成立,当且仅当:

$$\left\{egin{aligned} &rac{1}{2}l_1P_1\coslpha+l_1P_2\coslpha-Fl_1\sinlpha=0\ &rac{1}{2}l_2P_2\coseta-Fl_2\sineta=0 \end{aligned}
ight.$$

解得:

$$\begin{cases} \tan \alpha = \frac{P_1 + 2P_2}{2F} \\ \tan \beta = \frac{P_2}{2F} \end{cases}$$

四

ps:参考答案错了吧。。。

以质心为原点, 过原点垂直上底面为 z 轴建系

密度:

$$ho = rac{M}{\pi R^2 h}$$

柱坐标 r, θ, z 与直角坐标 x, y, z 关系:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

柱坐标体积元:

$$dV = r dr d\theta dz$$

质量元:

$$dm = \rho dV = \rho r dr d\theta dz$$

积分区域:

$$egin{cases} r \in [0,R] \ heta \in [0,2\pi] \ z \in \left[-rac{h}{2},rac{h}{2}
ight] \end{cases}$$

$$egin{align} I_{zz} &= \int (y^2+x^2)\mathrm{d}m \ &=
ho \int_{z=-rac{h}{2}}^{z=rac{h}{2}} \int_{ heta=0}^{ heta=2\pi} \int_{r=0}^{r=R} r^3 \mathrm{d}r \mathrm{d} heta \mathrm{d}z \ &= rac{1}{2}MR^2 \end{split}$$

$$egin{align*} I_{xx} &= I_{yy} \ &= \int (x^2 + z^2) \mathrm{d} m \ &=
ho \int_{z=-rac{h}{2}}^{z=rac{h}{2}} \int_{ heta=0}^{ heta=2\pi} \int_{r=0}^{r=R} (r^2 \cos^2 heta + z^2) r \mathrm{d} r \mathrm{d} heta \mathrm{d} z \ &= rac{1}{12} M \left(3R^2 + h^2
ight) \end{split}$$

(1)

泊松括号:

$$\{f,g\} \equiv \sum_lpha \left(rac{\partial f}{\partial q_lpha}rac{\partial g}{\partial p_lpha} - rac{\partial f}{\partial p_lpha}rac{\partial g}{\partial q_lpha}
ight)$$

物理量对时间的全导数:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}$$

(2)

$$\{x,L_x\} = 0$$
 $\{x,L_y\} = z$ $\{x,L_z\} = -y$ $\{p_x,L_x\} = 0$ $\{p_x,L_y\} = p_z$ $\{p_x,L_z\} = -p_y$

证明:

$$\Leftrightarrow x=x_1,y=x_2,z=x_3$$
; $ec{L}=ec{x} imesec{p},L_j=arepsilon_{jmn}x_mp_n$

$$egin{aligned} \{x_i,L_j\} &= rac{\partial x_i}{\partial x_k}rac{\partial L_j}{\partial p_k} - rac{\partial x_i}{\partial p_k}rac{\partial L_j}{\partial x_k} = \delta_{ik}rac{\partial \left(arepsilon_{jmn}x_mp_n
ight)}{\partial p_k} = \delta_{ik}\delta_{nk}arepsilon_{jmn}x_m = \delta_{in}arepsilon_{jmn}x_m = arepsilon_{jmn}x_m = arepsilon_{jmn}x_$$

$$\{p_i,L_j\}=rac{\partial p_i}{\partial x_k}rac{\partial L_j}{\partial p_k}-rac{\partial p_i}{\partial p_k}rac{\partial L_j}{\partial x_k}=-\delta_{ik}rac{\partial \left(arepsilon_{jmn}x_mp_n
ight)}{\partial x_k}=-\delta_{ik}\delta_{mk}arepsilon_{jmn}p_n=-\delta_{im}arepsilon_{jmn}p_n=-arepsilon_{jin}p_n=arepsilon_{ijn}p_n= \ \{p_x,L_x\}=\{p_1,L_1\}=arepsilon_{11n}p_n=0 \ \ \{p_x,L_y\}=arepsilon_{12n}p_n=p_3=p_z$$

$$\{p_x, L_z\} = \{p_1, L_3\} = \varepsilon_{13n}p_n = -p_2 = -p_y$$



(1)

$$egin{aligned} V_{ ext{eff}}(r) &= rac{l^2}{2mr^2} + V(r) = rac{l^2}{2mr^2} - kr^{-eta}, \quad k > 0, eta \in (0,2) \ &\lim_{r o 0^+} V_{ ext{eff}}(r) = +\infty \ &\lim_{r o +\infty} V_{ ext{eff}}(r) = 0^- \ &rac{ ext{d}V_{ ext{eff}}(r)}{ ext{d}r} = rac{-l^2}{mr^3} + keta r^{-eta - 1} \ &rac{ ext{d}V_{ ext{eff}}(r)}{ ext{d}r}igg|_{r = r_{-r}} = 0 \Longrightarrow r_m = \left(rac{l^2}{mketa}
ight)^{rac{1}{2-eta}} \end{aligned}$$

可以画出 $V_{\rm eff}(r)$ 的大致图像

质点只能在 $V_{\rm eff}(r) \leqslant E$ 的区域内运动,而 E < 0,于是从图像上可以看出 r 不能趋于无穷大。

(2)

$$V_{ ext{eff}}(r) = rac{L^2}{2mr^2} - kr^{-eta} \leqslant E$$

由方程 $rac{L^2}{2mr^2}-kr^{-eta}=E$ 可用图像法得 $r_{
m min}, r_{
m max}.$

角动量、能量守恒,把能量表达式中的 $\dot{ heta}$ 替换成角动量 L:

$$egin{cases} L = mr^2\dot{ heta} \ E = rac{1}{2}m\left(\dot{r}^2 + r^2\dot{ heta}^2
ight) - kr^{-eta} \Longrightarrow E = rac{1}{2}m\dot{r}^2 + rac{L^2}{2mr^2} - kr^{-eta} \end{cases}$$

从中可用两个守恒量 E, L 表达 $\mathrm{d}r/\mathrm{d}t$:

$$rac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{rac{2}{m}iggl[E-rac{L^2}{2mr^2}+kr^{-eta}iggr]}$$

结合角动量表达式 $L=mr^2\dot{ heta}\Longrightarrow rac{\mathrm{d} heta}{\mathrm{d}t}=rac{L}{mr^2}$,与上式相除,得:

$$\mathrm{d} heta = rac{\mathrm{d}r}{r\sqrt{rac{2mE}{L^2}r^2 + rac{2mk}{L^2}r^{2-eta} - 1}}$$

积分得:

$$egin{aligned} \Delta\Phi &= \int \mathrm{d} heta \ &= \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{\mathrm{d}r}{r\sqrt{rac{2mE}{L^2}r^2 + rac{2mk}{L^2}r^{2-eta} - 1}} \end{aligned}$$

(3)

$$V_{ ext{eff}}(r) = rac{L^2}{2mr^2} - kr^{-eta} \leqslant E = 0^-$$

由方程 $rac{L^2}{2mr^2}-kr^{-eta}=E$ 可用图像法得 $r_{
m min}, r_{
m max}.$

但当 $E=0^-$ 时, $r_{
m min}$ 的解析解很容易求得, $r_{
m max}$ 从有效势图像上也容易看出:

$$rac{L^2}{2mr^2}-kr^{-eta}=0\Longrightarrow egin{cases} r_{
m min}=\left(rac{L^2}{2mk}
ight)^{rac{1}{2-eta}}\ r_{
m max}=+\infty \end{cases}$$

将 $r_{
m min}, r_{
m max}, E=0^-$ 代入 $\Delta\Phi$ 的积分式,得:

$$\Delta\Phi = \int_{r=\left(rac{L^2}{2mk}
ight)^{rac{1}{2-eta}}}^{r=+\infty} rac{\mathrm{d}r}{r\sqrt{rac{2mk}{L^2}r^{2-eta}-1}}$$
 (1)

令 $u=\sqrt{rac{2mk}{L^2}r^{2-eta}-1},rac{L^2}{2mk}(u^2+1)=r^{2-eta}$,两边微分:

$$rac{L^2}{mk}u\mathrm{d}u=(2-eta)r^{1-eta}\mathrm{d}r\Longrightarrow r^{1-eta}\mathrm{d}r=rac{L^2}{(2-eta)mk}u\mathrm{d}u$$

两边同时除以 $r^{2-\beta}$:

$$egin{aligned} rac{\mathrm{d}r}{r} &= rac{L^2}{(2-eta)mk}u\mathrm{d}uigg/r^{2-eta} \ &= rac{L^2}{(2-eta)mk}u\mathrm{d}uigg/rac{L^2}{2mk}(u^2+1) \ &= rac{2}{2-eta}rac{u}{u^2+1}\mathrm{d}u \end{aligned}$$

$$\Delta \Phi = \int_{r = (\frac{L^2}{2mk})^{\frac{1}{2-\beta}}}^{r = +\infty} \frac{\mathrm{d}r}{r\sqrt{\frac{2mk}{L^2}r^{2-\beta} - 1}}
= \int_{u=0}^{u = +\infty} \frac{2}{2-\beta} \frac{1}{u^2 + 1} \mathrm{d}u
= \frac{2}{2-\beta} \cdot \arctan u \Big|_{u=0}^{u = +\infty}
= \frac{\pi}{2-\beta}$$
(1)

可见, $\Delta\Phi$ 与角动量无关。