4-1 量子场论习题四

4-1-1

写出 Noether 定理。它主要包含哪两个部分?

Noether 定理包括广义守恒定理1和广义守恒定理2。

广义守恒定理1

设 $\theta_{\mu\cdots\nu\lambda}(x)$ 是 n 阶张量函数, 且满足:

$$\left. heta_{\mu \cdots
u \lambda}(x)
ight|_{ec{x}
ightarrow \infty} = 0$$

若

$$\partial_{\lambda}\theta_{\mu\cdots
u\lambda}=0$$

则存在一个 (n-1) 阶守恒张量:

$$T_{\mu\cdots
u}(x_4)\equivrac{1}{\mathrm{i}}\int\limits_{ec x\in\mathbb{R}^3} heta_{\mu\cdots
u4}(ec x,x_4)\mathrm{d}^3ec x=\mathrm{const}$$

广义守恒定理2

若场的作用量

$$I = \int\limits_{G} \mathcal{L}\left(\phi_{A},\partial_{\mu}\phi_{A}
ight) \mathrm{d}^{4}x$$

对微量变换

$$x
ightarrow x'=x+\delta x, \quad \phi_A
ightarrow \phi_A'=\phi_A+\delta_0\phi_A \ \phi_A(x)
ightarrow \phi_A'(x')=\phi_A(x)+\delta\phi_A(x)$$

保持不变,则存在一个矢量

$$heta_{\mu} = \left(\mathcal{L}\delta_{\mu
u} - rac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi_{A}
ight)}\partial_{
u}\phi_{A}
ight)\delta x_{
u} + rac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi_{A}
ight)}\delta\phi_{A}$$

满足关系式:

$$\partial_{\mu} heta_{\mu}+\left[\mathcal{L}
ight]_{\phi_{A}}\delta_{0}\phi_{A}=0$$

4-1-2

分别讨论下述变换的 $\delta x_{\nu}, \delta \phi_A$

- (1) 四维时空平移
- (2) 相因子变换
- (3) 无穷小 Lorentz 固有转动

四维时空平移

$$x_{\mu}
ightarrow x'_{\mu}=x_{\mu}+lpha_{\mu}, \quad \phi_A(x)
ightarrow \phi'_A(x')=\phi_A(x) \ \delta x_{\mu}=x'_{\mu}-x_{\mu}=lpha_{\mu} \ \delta \phi_A(x)=\phi'_A(x')-\phi_A(x)=0$$

相因子变换

$$x_{\mu}
ightarrow x_{\mu}' = x_{\mu}, \quad \phi_A(x)
ightarrow \phi_A'(x') = \mathrm{e}^{\mathrm{i}lpha}\phi_A(x) \ \delta x_{\mu} = x_{\mu}' - x_{\mu} = 0 \ \delta \phi_A(x) = \phi_A'(x') - \phi_A(x) \ = \mathrm{e}^{\mathrm{i}lpha}\phi_A(x) - \phi_A(x) \ pprox \mathrm{i}lpha\phi_A(x) \ pprox \mathrm{i}lpha\phi_A(x)$$

无穷小 Lorentz 固有转动

$$x_{\mu}
ightarrow x_{\mu}'=(\delta_{\mu
u}+lpha_{\mu
u})x_{
u}, \quad \phi(x)
ightarrow \phi'(x')=\phi(x)+rac{1}{2}lpha_{\mu
u}I_{\mu
u}\phi(x)$$

其中, $I_{\mu\nu}=\left.rac{\partial D(lpha)}{\partial lpha_{\mu\nu}}\right|_{lpha=0}$,D(lpha) 为固有 Lorentz 群的某种线性表示。

$$\delta x_{\mu} = x'_{\mu} - x_{\mu} = \alpha_{\mu\nu}x_{\nu}$$

$$\delta \phi(x) = \phi'(x') - \phi(x)$$

$$= \frac{1}{2}\alpha_{\mu\nu}I_{\mu\nu}\phi(x)$$

4-1-3

由 Lorentz 原理,推导分量形式 Dirac 方程。

x 系 Dirac 方程:

$$(\gamma_{\mu}\partial_{\mu}+m)\,\psi(x)=0$$

考虑时空坐标进行广义 Lorentz 变换 $x_\mu \to x'_\mu = A_{\mu\nu} x_\nu + b_\mu$,由于 Dirac 方程应当具有 Lorentz 协变性,则 x' 系 Dirac 方程形式为:

$$\left(\gamma_{\mu}\partial_{\mu}'+m
ight)\psi'(x')=0$$

为了使 Dirac 方程具有 Lorentz 协变性,当时空坐标进行广义 Lorentz 变换时, $\psi(x)$ 也应当进行变换。设:

$$x'_{\mu}=A_{\mu
u}x_{
u}+b_{\mu},\quad \psi'(x')=\Lambda(A)\psi(x)$$

注意到:

$$egin{aligned} x'_{\mu} &= A_{\mu
u} x_{
u} + b_{\mu} \Longrightarrow A_{\mu\lambda} \mathrm{d} x'_{\mu} = \mathrm{d} x_{\lambda} \ \ \partial'_{\mu} &= rac{\partial x_{
u}}{\partial x'_{\mu}} rac{\partial}{\partial x_{
u}} = A_{\mu
u} \partial_{
u} \end{aligned}$$

则 x' 系 Dirac 方程化为:

$$egin{aligned} 0 &= \left(\gamma_{\mu} \partial_{\mu}' + m
ight) \psi'(x') \ &= \left(\gamma_{\mu} A_{\mu
u} \partial_{
u} + m
ight) \Lambda(A) \psi(x) \end{aligned}$$

上式左乘 $\Lambda^{-1}(A)$ 得:

$$\left[\Lambda^{-1}(A)\gamma_{\mu}\Lambda(A)A_{\mu
u}\partial_{
u}+m
ight]\psi(x)=0$$

与x系 Dirac 方程

$$(\gamma_{\nu}\partial_{\nu}+m)\,\psi(x)=0$$

对比可得:

$$\Lambda^{-1}(A)\gamma_{\mu}\Lambda(A)A_{\mu
u}=\gamma_{
u}$$

上式乘 $A_{\lambda\nu}$, 对 ν 求和, 并利用正交关系, 得:

$$\Lambda^{-1}(A)\gamma_{\lambda}\Lambda(A)=A_{\lambda
u}\gamma_{
u}$$

满足上式的 $\Lambda(A)$ 必定是矩阵,且构成 Lorentz 群的旋量表示。因此 $\psi(x)$ 是一个四元列矩阵,记为:

$$\psi(x) = egin{bmatrix} \psi_1(x) \ \psi_2(x) \ \psi_3(x) \ \psi_4(x) \end{bmatrix}$$

Dirac 表象 γ_{μ} 矩阵:

$$egin{aligned} \sigma_1^0 &= egin{bmatrix} 0 & 1 \ 1 & 0 \end{bmatrix}, & \sigma_2^0 &= egin{bmatrix} 0 & -\mathrm{i} \ \mathrm{i} & 0 \end{bmatrix}, & \sigma_3^0 &= egin{bmatrix} 1 & 0 \ 0 & -1 \end{bmatrix} \ \gamma_i &= egin{bmatrix} 0 & -\mathrm{i}\sigma_i^0 \ \mathrm{i}\sigma_i^0 & 0 \end{bmatrix}, & \gamma_4 &= egin{bmatrix} I^0 & 0 \ 0 & -I^0 \end{bmatrix} \end{aligned}$$

把旋量 $\psi(x)$ 写成二分量形式:

$$\psi(x) = egin{bmatrix} arphi \ \chi \end{bmatrix}, \quad arphi = egin{bmatrix} \psi_1(x) \ \psi_2(x) \end{bmatrix}, \quad \chi = egin{bmatrix} \psi_3(x) \ \psi_4(x) \end{bmatrix}$$

代入 Dirac 方程

$$\left(\gamma_{\mu}\partial_{\mu}+m
ight)\psi(x)=0$$

得到二分量形式 Dirac 方程:

$$-\mathrm{i}\vec{\sigma}^{0} \cdot \nabla \chi - \mathrm{i}\partial_{t}\varphi + m\varphi = 0$$
$$\mathrm{i}\vec{\sigma}^{0} \cdot \nabla \varphi + \mathrm{i}\partial_{t}\chi + m\chi = 0$$

4-1-4

求复标量场 $T_{\mu
u}, L_{[lpha eta] \mu}, j_{\mu}$ 以及 $W, G, ec{L},
ho$

$$\mathcal{L}_{0} = -\partial_{\alpha}\phi^{*}\partial_{\alpha}\phi - m^{2}\phi^{*}\phi$$

$$T_{\mu\nu} = \mathcal{L}_{0}\delta_{\mu\nu} - \frac{\partial \mathcal{L}_{0}}{\partial (\partial_{\nu}\phi)}\partial_{\mu}\phi - \frac{\partial \mathcal{L}_{0}}{\partial (\partial_{\nu}\phi^{*})}\partial_{\mu}\phi^{*}$$

$$= \mathcal{L}_{0}\delta_{\mu\nu} + \partial_{\nu}\phi^{*}\partial_{\mu}\phi + \partial_{\nu}\phi\partial_{\mu}\phi^{*}$$

$$= \left(-\partial_{\alpha}\phi^{*}\partial_{\alpha}\phi - m^{2}\phi^{*}\phi\right)\delta_{\mu\nu} + \partial_{\nu}\phi^{*}\partial_{\mu}\phi + \partial_{\nu}\phi\partial_{\mu}\phi^{*}$$

$$T_{i4} = \partial_i \phi \partial_4 \phi^* + \partial_i \phi^* \partial_4 \phi$$

$$W = -T_{44}$$

$$= -\left[\left(-\partial_{\alpha}\phi^{*}\partial_{\alpha}\phi - m^{2}\phi^{*}\phi\right)\delta_{44} + \partial_{4}\phi^{*}\partial_{4}\phi + \partial_{4}\phi\partial_{4}\phi^{*}\right]$$

$$= \partial_{i}\phi^{*}\partial_{i}\phi + \partial_{4}\phi^{*}\partial_{4}\phi + m^{2}\phi^{*}\phi - \partial_{4}\phi^{*}\partial_{4}\phi - \partial_{4}\phi\partial_{4}\phi^{*}$$

$$= \partial_{i}\phi^{*}\partial_{i}\phi + m^{2}\phi^{*}\phi - \left(\frac{1}{i}\right)^{2}\partial_{t}\phi\partial_{t}\phi^{*}$$

$$= \nabla\phi^{*} \cdot \nabla\phi + \partial_{t}\phi^{*}\partial_{t}\phi + m^{2}\phi^{*}\phi$$

$$G_{i} = \frac{1}{i}T_{i4}$$

$$= \frac{1}{i}\left[\partial_{i}\phi\partial_{4}\phi^{*} + \partial_{i}\phi^{*}\partial_{4}\phi\right]$$

$$ec{G} = G_i ec{\mathrm{e}}_i = -
abla \phi^* \partial_t \phi -
abla \phi \partial_t \phi^*$$

 $= -\partial_i \phi^* \partial_t \phi - \partial_i \phi \partial_t \phi^*$

$$\begin{split} L_{[\alpha\beta]\mu} &= T_{\alpha\mu}x_{\beta} - T_{\beta\mu}x_{\alpha} \\ &= \left[\left(-\partial_{\rho}\phi^{*}\partial_{\rho}\phi - m^{2}\phi^{*}\phi \right) \delta_{\alpha\mu} + \partial_{\mu}\phi^{*}\partial_{\alpha}\phi + \partial_{\mu}\phi\partial_{\alpha}\phi^{*} \right] x_{\beta} - \left[\left(-\partial_{\rho}\phi^{*}\partial_{\rho}\phi - m^{2}\phi^{*}\phi \right) \delta_{\beta\mu} + \partial_{\mu}\phi^{*}\partial_{\beta}\phi + \partial_{\mu}\phi\partial_{\beta}\phi^{*} \right] x_{\alpha} \\ \vec{l} &= \vec{x} \times \vec{G} \\ &= -\vec{x} \times (\nabla\phi^{*}\partial_{t}\phi + \nabla\phi\partial_{t}\phi^{*}) \end{split}$$

$$\vec{L} = \int \vec{l} d^{3}\vec{x} \\ &= -\int \vec{x} \times (\nabla\phi^{*}\partial_{t}\phi + \nabla\phi\partial_{t}\phi^{*}) d^{3}\vec{x} \\ j_{\mu} &= -ie \left(\frac{\partial \mathcal{L}}{\partial \partial_{\mu}\phi_{A}} \phi_{A} - \phi_{A}^{*} \frac{\partial \mathcal{L}}{\partial \partial_{\mu}\phi_{A}^{*}} \right) \\ &= ie \left(\phi\partial_{\mu}\phi^{*} - \phi^{*}\partial_{\mu}\phi \right) \end{split}$$

$$\rho = \frac{1}{i}j_4$$

$$= -ie\left(\phi\partial_t\phi^* - \phi^*\partial_t\phi\right)$$

4-1-5

设

$$L=-rac{1}{4}F_{\mu
u}F_{\mu
u}-rac{1}{2}\left(\partial_{\mu}-\mathrm{i}eA_{\mu}
ight)\phi\cdot\left(\partial_{\mu}-\mathrm{i}eA_{\mu}
ight)\phi-rac{1}{2}m^{2}\phi^{2}-rac{\lambda}{4}\phi^{4}$$

求(1) $A_{\mu},\phi(x)$ 的方程;(2) $T_{\mu\nu}$

$$F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$$

$$\begin{split} L &= -\frac{1}{4} F_{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} \left(\partial_{\alpha} - \mathrm{i} e A_{\alpha} \right) \phi \cdot \left(\partial_{\alpha} - \mathrm{i} e A_{\alpha} \right) \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4 \\ &= -\frac{1}{4} \left(\partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} \right) \left(\partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} \right) - \frac{1}{2} \left[\partial_{\alpha} \phi \partial_{\alpha} \phi - 2 \mathrm{i} e A_{\alpha} \phi \partial_{\alpha} \phi - e^2 A_{\alpha} A_{\alpha} \phi^2 \right] - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4 \\ &= -\frac{1}{2} \partial_{\alpha} A_{\beta} \partial_{\alpha} A_{\beta} + \frac{1}{2} \partial_{\alpha} A_{\beta} \partial_{\beta} A_{\alpha} - \frac{1}{2} \partial_{\alpha} \phi \partial_{\alpha} \phi + \mathrm{i} e A_{\alpha} \phi \partial_{\alpha} \phi + \frac{1}{2} e^2 A_{\alpha} A_{\alpha} \phi^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4 \end{split}$$

A_{μ} 的运动方程

$$rac{\partial L}{\partial A_{\mu}}=\mathrm{i} e\phi\partial_{\mu}\phi+e^{2}A_{\mu}\phi^{2}$$

$$egin{aligned} rac{\partial L}{\partial \left(\partial_{
u}A_{\mu}
ight)} &= -\partial_{
u}A_{\mu} + \partial_{\mu}A_{
u} \ &= F_{\mu
u} \end{aligned}$$

把L代入E-L方程

$$rac{\partial L}{\partial A_{\mu}} - \partial_{
u} rac{\partial L}{\partial \left(\partial_{
u} A_{\mu}
ight)} = 0$$

可得 A_{μ} 的运动方程:

$$\mathrm{i}e\phi\partial_{\mu}\phi+e^{2}A_{\mu}\phi^{2}-\partial_{
u}\left(F_{\mu
u}
ight)=0$$

即:

$$\partial_{
u}F_{\mu
u} = \mathrm{i}e\phi\partial_{\mu}\phi + e^2A_{\mu}\phi^2$$

$\phi(x)$ 的运动方程

$$egin{aligned} rac{\partial L}{\partial \phi} &= \mathrm{i} e A_{lpha} \partial_{lpha} \phi + e^2 A_{lpha} A_{lpha} \phi - m^2 \phi - \lambda \phi^3 \ & rac{\partial L}{\partial \left(\partial_{\mu} \phi
ight)} &= - \partial_{\mu} \phi + \mathrm{i} e A_{\mu} \phi \end{aligned}$$

把L代入E-L方程

$$\frac{\partial L}{\partial \phi} - \partial_{\mu} \frac{\partial L}{\partial \left(\partial_{\mu} \phi\right)} = 0$$

可得:

$$\mathrm{i} e A_{lpha} \partial_{lpha} \phi + e^2 A_{lpha} A_{lpha} \phi - m^2 \phi - \lambda \phi^3 - \partial_{\mu} \left(-\partial_{\mu} \phi + \mathrm{i} e A_{\mu} \phi \right) = 0$$

即:

$$\partial_{\mu}\partial_{\mu}\phi - \mathrm{i}e\phi\partial_{\mu}A_{\mu} + e^{2}A_{\alpha}A_{\alpha}\phi - m^{2}\phi - \lambda\phi^{3} = 0$$

计算 $T_{\mu\nu}$

能动张量 $T_{\mu\nu}$ 的定义:

$$\begin{split} T_{\mu\nu} & \equiv L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_{\nu}\phi_{A}\right)}\partial_{\mu}\phi_{A} \\ & \equiv L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_{\nu}\phi\right)}\partial_{\mu}\phi - \frac{\partial L}{\partial\left(\partial_{\nu}A_{\alpha}\right)}\partial_{\mu}A_{\alpha} \\ & = \left(-\frac{1}{2}\partial_{\alpha}A_{\beta}\partial_{\alpha}A_{\beta} + \frac{1}{2}\partial_{\alpha}A_{\beta}\partial_{\beta}A_{\alpha} - \frac{1}{2}\partial_{\alpha}\phi\partial_{\alpha}\phi + \mathrm{i}eA_{\alpha}\phi\partial_{\alpha}\phi + \frac{1}{2}e^{2}A_{\alpha}A_{\alpha}\phi^{2} - \frac{1}{2}m^{2}\phi^{2} - \frac{\lambda}{4}\phi^{4}\right)\delta_{\mu\nu} \\ & - \left(-\partial_{\nu}\phi + \mathrm{i}eA_{\nu}\phi\right)\partial_{\mu}\phi - F_{\alpha\nu}\partial_{\mu}A_{\alpha} \end{split}$$

4-1-6

(1)

$$L = -rac{1}{2}\partial_{\mu} ilde{\phi}\partial_{\mu} ilde{\phi} - rac{1}{2}m^2 ilde{m{\phi}}^2 - rac{1}{2}\left(ar{\psi}\gamma_{\mu}\partial_{\mu}\psi - \partial_{\mu}ar{\psi}\gamma_{\mu}\psi
ight) - Mar{\psi}\psi + \mathrm{i}Gar{\psi}\gamma_5\psi ilde{\phi}$$

(2)

$$L=-rac{1}{4}F_{\mu
u}F_{\mu
u}-ar{\psi}\left(\gamma_{\mu}\partial_{\mu}+m
ight)\psi+\mathrm{i}ear{\psi}\gamma_{\mu}\psi A_{\mu}$$

(1)

$$L = -rac{1}{2}\partial_{\mu} ilde{\phi}\partial_{\mu} ilde{\phi} - rac{1}{2}m^2 ilde{m{\phi}}^2 - rac{1}{2}\left(ar{\psi}\gamma_{\mu}\partial_{\mu}\psi - \partial_{\mu}ar{\psi}\gamma_{\mu}\psi
ight) - Mar{\psi}\psi + \mathrm{i}Gar{\psi}\gamma_5\psiar{\phi}$$

$ilde{\phi}$ 的运动方程

$$egin{aligned} L = -rac{1}{2}\partial_{\mu} ilde{\phi}\partial_{\mu} ilde{\phi} - rac{1}{2}m^2 ilde{\phi}^2 - rac{1}{2}\left(ar{\psi}\gamma_{\mu}\partial_{\mu}\psi - \partial_{\mu}ar{\psi}\gamma_{\mu}\psi
ight) - Mar{\psi}\psi + \mathrm{i}Gar{\psi}\gamma_5\psi ilde{\phi} \ & rac{\partial L}{\partial ilde{\phi}} = -m^2 ilde{\phi} + \mathrm{i}Gar{\psi}\gamma_5\psi \ & rac{\partial L}{\partial\left(\partial_{\mu} ilde{\phi}
ight)} = -\partial_{\mu} ilde{\phi} \end{aligned}$$

代入 E-L 方程

$$rac{\partial L}{\partial ilde{\phi}} - \partial_{\mu} rac{\partial L}{\partial \left(\partial_{\mu} ilde{\phi}
ight)} = 0$$

可得:

$$\left(-m^2 ilde{\phi}+\mathrm{i}Gar{\psi}\gamma_5\psi
ight)-\partial_\mu\left(-\partial_\mu ilde{\phi}
ight)=0$$

即:

$$\left(\partial_{\mu}\partial_{\mu}-m^{2}
ight) ilde{\phi}=-\mathrm{i}Gar{\psi}\gamma_{5}\psi$$

$ar{\psi}$ 的运动方程 (对 ψ 变分)

$$\begin{split} L &= -\frac{1}{2} \partial_{\mu} \tilde{\phi} \partial_{\mu} \tilde{\phi} - \frac{1}{2} m^{2} \tilde{\phi}^{2} - \frac{1}{2} \left(\bar{\psi} \gamma_{\mu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma_{\mu} \psi \right) - M \bar{\psi} \psi + \mathrm{i} G \bar{\psi} \gamma_{5} \psi \tilde{\phi} \\ & \frac{\partial L}{\partial \psi} = \frac{1}{2} \partial_{\alpha} \bar{\psi} \gamma_{\alpha} - M \bar{\psi} + \mathrm{i} G \bar{\psi} \gamma_{5} \tilde{\phi} \\ & \frac{\partial L}{\partial \left(\partial_{\mu} \psi \right)} = -\frac{1}{2} \bar{\psi} \gamma_{\mu} \end{split}$$

代入 E-L 方程

$$\frac{\partial L}{\partial \psi} - \partial_{\mu} \frac{\partial L}{\partial \left(\partial_{\mu} \psi\right)} = 0$$

可得:

$$rac{1}{2}\partial_{lpha}ar{\psi}\gamma_{lpha}-Mar{\psi}+\mathrm{i}Gar{\psi}\gamma_{5} ilde{\phi}-\partial_{\mu}\left(-rac{1}{2}ar{\psi}\gamma_{\mu}
ight)=0$$

即:

$$\overline{\left[\partial_{\mu}ar{\psi}\gamma_{\mu}-Mar{\psi}+\mathrm{i}Gar{\psi}\gamma_{5} ilde{\phi}=0
ight]}$$

ψ 的运动方程 (对 $ar{\psi}$ 变分)

$$L = -rac{1}{2}\partial_{\mu} ilde{\phi}\partial_{\mu} ilde{\phi} - rac{1}{2}m^{2} ilde{\phi}^{2} - rac{1}{2}\left(ar{\psi}\gamma_{\mu}\partial_{\mu}\psi - \partial_{\mu}ar{\psi}\gamma_{\mu}\psi\right) - Mar{\psi}\psi + \mathrm{i}Gar{\psi}\gamma_{5}\psi ilde{\phi}$$
 $rac{\partial L}{\partialar{\psi}} = -rac{1}{2}\gamma_{lpha}\partial_{lpha} - M\psi + \mathrm{i}G\gamma_{5}\psi ilde{\phi}$ $rac{\partial L}{\partial\left(\partial_{\mu}ar{\psi}
ight)} = rac{1}{2}\gamma_{\mu}\psi$

代入 E-L 方程

$$\frac{\partial L}{\partial \bar{\psi}} - \partial_{\mu} \frac{\partial L}{\partial \left(\partial_{\mu} \bar{\psi}\right)} = 0$$

可得:

$$-rac{1}{2}\gamma_lpha\partial_lpha\psi-M\psi+\mathrm{i}G\gamma_5\psi ilde{\phi}-\partial_\mu\left(rac{1}{2}\gamma_\mu\psi
ight)=0$$

即:

$$-\gamma_{\mu}\partial_{\mu}\psi-M\psi+\mathrm{i}G\gamma_{5}\psi ilde{\phi}=0$$

能量密度

能量动量张量:

$$\begin{split} T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\phi_{A}\right)}\partial_{\mu}\phi_{A} \\ &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\tilde{\phi}\right)}\partial_{\mu}\tilde{\phi} - \frac{\partial L}{\partial \left(\partial_{\nu}\psi\right)}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\frac{\partial L}{\partial \left(\partial_{\nu}\bar{\psi}\right)} \\ &= \left[-\frac{1}{2}\partial_{\alpha}\tilde{\phi}\partial_{\alpha}\tilde{\phi} - \frac{1}{2}m^{2}\tilde{\phi}^{2} - \frac{1}{2}\left(\bar{\psi}\gamma_{\alpha}\partial_{\alpha}\psi - \partial_{\alpha}\bar{\psi}\gamma_{\alpha}\psi\right) - M\bar{\psi}\psi + iG\bar{\psi}\gamma_{5}\psi\tilde{\phi} \right]\delta_{\mu\nu} \\ &- \left(-\partial_{\nu}\tilde{\phi} \right)\partial_{\mu}\tilde{\phi} - \left(-\frac{1}{2}\bar{\psi}\gamma_{\nu} \right)\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\left(\frac{1}{2}\gamma_{\nu}\psi\right) \\ &= \left[-\frac{1}{2}\partial_{\alpha}\tilde{\phi}\partial_{\alpha}\tilde{\phi} - \frac{1}{2}m^{2}\tilde{\phi}^{2} - \frac{1}{2}\left(\bar{\psi}\gamma_{\alpha}\partial_{\alpha}\psi - \partial_{\alpha}\bar{\psi}\gamma_{\alpha}\psi\right) - M\bar{\psi}\psi + iG\bar{\psi}\gamma_{5}\psi\tilde{\phi} \right]\delta_{\mu\nu} \\ &+ \partial_{\nu}\tilde{\phi}\partial_{\mu}\tilde{\phi} + \frac{1}{2}\bar{\psi}\gamma_{\nu}\partial_{\mu}\psi + \frac{1}{2}\partial_{\mu}\bar{\psi}\gamma_{\nu}\psi \end{split}$$

能量密度:

$$egin{aligned} W &= -T_{44} \ &= \left[rac{1}{2}\partial_{lpha} ilde{\phi}\partial_{lpha} ilde{\phi} + rac{1}{2}m^2 ilde{\phi}^2 + rac{1}{2}\left(ar{\psi}\gamma_{lpha}\partial_{lpha}\psi - \partial_{lpha}ar{\psi}\gamma_{lpha}\psi
ight) + Mar{\psi}\psi - \mathrm{i}Gar{\psi}\gamma_5\psi ilde{\phi}
ight] \ &+ \partial_t ilde{\phi}\partial_t ilde{\phi} + rac{\mathrm{i}}{2}ar{\psi}\gamma_4\partial_t\psi + rac{\mathrm{i}}{2}\partial_tar{\psi}\gamma_4\psi \end{aligned}$$

电荷密度

$$egin{aligned} L &= -rac{1}{2}\partial_{\mu} ilde{\phi}\partial_{\mu} ilde{\phi} - rac{1}{2}m^2 ilde{\phi}^2 - rac{1}{2}\left(ar{\psi}\gamma_{\mu}\partial_{\mu}\psi - \partial_{\mu}ar{\psi}\gamma_{\mu}\psi
ight) - Mar{\psi}\psi + \mathrm{i}Gar{\psi}\gamma_5\psi ilde{\phi} \ & rac{\partial L}{\partial\left(\partial_{\mu}\psi
ight)} = -rac{1}{2}ar{\psi}\gamma_{\mu} \ & rac{\partial L}{\partial\left(\partial_{\mu}ar{\psi}
ight)} = rac{1}{2}\gamma_{\mu}\psi \end{aligned}$$

由于赝标量场 $ilde{\phi}^* = ilde{\phi}$,因此赝标量场对电流密度矢量无贡献。

电流密度矢量:

$$egin{aligned} j_{\mu} &\equiv -\mathrm{i}e\left[rac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi_{A}
ight)}\phi_{A} - \phi_{A}^{*}rac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi_{A}^{*}
ight)}
ight] \ &= -\mathrm{i}e\left[rac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\psi
ight)}\psi - ar{\psi}rac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}ar{\psi}
ight)}
ight] \ &= -\mathrm{i}e\left[\left(-rac{1}{2}ar{\psi}\gamma_{\mu}
ight)\psi - ar{\psi}\left(rac{1}{2}\gamma_{\mu}\psi
ight)
ight] \ &= \mathrm{i}ear{\psi}\gamma_{\mu}\psi \end{aligned}$$

电荷密度:

$$ho = rac{1}{\mathrm{i}} j_4 \ = e ar{\psi} \gamma_4 \psi \ = e \psi^\dagger \psi$$

(2)

$$egin{aligned} L &= -rac{1}{4}F_{\mu
u}F_{\mu
u} - ar{\psi}\left(\gamma_{\mu}\partial_{\mu} + m
ight)\psi + \mathrm{i}ear{\psi}\gamma_{\mu}\psi A_{\mu} \ &= -rac{1}{2}\partial_{lpha}A_{eta}\partial_{lpha}A_{eta} + rac{1}{2}\partial_{lpha}A_{eta}\partial_{eta}A_{lpha} - ar{\psi}\left(\gamma_{\mu}\partial_{\mu} + m
ight)\psi + \mathrm{i}ear{\psi}\gamma_{\mu}\psi A_{\mu} \end{aligned}$$

A_{μ} 的运动方程

$$egin{aligned} L &= -rac{1}{2}\partial_{lpha}A_{eta}\partial_{lpha}A_{eta} + rac{1}{2}\partial_{lpha}A_{eta}\partial_{eta}A_{lpha} - ar{\psi}\left(\gamma_{\mu}\partial_{\mu} + m
ight)\psi + \mathrm{i}ear{\psi}\gamma_{\mu}\psi A_{\mu} \ & rac{\partial L}{\partial A_{\mu}} = \mathrm{i}ear{\psi}\gamma_{\mu}\psi \ & rac{\partial L}{\partial\left(\partial_{
u}A_{\mu}
ight)} = -\partial_{
u}A_{\mu} + \partial_{\mu}A_{
u} = F_{\mu
u} \end{aligned}$$

代入 E-L 方程

$$rac{\partial L}{\partial A_{\mu}} - \partial_{
u} rac{\partial L}{\partial \left(\partial_{
u} A_{\mu}
ight)} = 0$$

可得:

$$\mathrm{i} e ar{\psi} \gamma_\mu \psi - \partial_
u F_{\mu
u} = 0$$

即:

$$\overline{\partial_
u F_{\mu
u} = \mathrm{i} e ar{\psi} \gamma_\mu \psi}$$

$ar{\psi}$ 的运动方程 (对 ψ 变分)

$$egin{aligned} L &= -rac{1}{2}\partial_{lpha}A_{eta}\partial_{lpha}A_{eta} + rac{1}{2}\partial_{lpha}A_{eta}\partial_{eta}A_{lpha} - ar{\psi}\left(\gamma_{\mu}\partial_{\mu} + m
ight)\psi + \mathrm{i}ear{\psi}\gamma_{\mu}\psi A_{\mu} \ & rac{\partial L}{\partial\psi} = -mar{\psi} + \mathrm{i}ear{\psi}\gamma_{\mu}A_{\mu} \ & rac{\partial L}{\partial\left(\partial_{\mu}\psi
ight)} = -ar{\psi}\gamma_{\mu} \end{aligned}$$

代入 E-L 方程

$$\frac{\partial L}{\partial \psi} - \partial_{\mu} \frac{\partial L}{\partial \left(\partial_{\mu} \psi\right)} = 0$$

可得:

$$-mar{\psi}+\mathrm{i}ear{\psi}\gamma_{\mu}A_{\mu}-\partial_{\mu}\left(-ar{\psi}\gamma_{\mu}
ight)=0$$

即:

$$\overline{\left[\partial_{\mu}ar{\psi}\gamma_{\mu}-mar{\psi}+\mathrm{i}ear{\psi}\gamma_{\mu}A_{\mu}=0
ight]}$$

ψ 的运动方程 (对 $ar{\psi}$ 变分)

$$egin{aligned} L &= -rac{1}{2}\partial_{lpha}A_{eta}\partial_{lpha}A_{eta} + rac{1}{2}\partial_{lpha}A_{eta}\partial_{eta}A_{lpha} - ar{\psi}\left(\gamma_{\mu}\partial_{\mu} + m
ight)\psi + \mathrm{i}ear{\psi}\gamma_{\mu}\psi A_{\mu} \ & rac{\partial L}{\partialar{\psi}} = -\gamma_{\mu}\partial_{\mu}\psi - m\psi + \mathrm{i}e\gamma_{\mu}\psi A_{\mu} \ & rac{\partial L}{\partial\left(\partial_{\mu}ar{\psi}
ight)} = 0 \end{aligned}$$

代入 E-L 方程

$$rac{\partial L}{\partial ar{\psi}} - \partial_{\mu} rac{\partial L}{\partial \left(\partial_{\mu} ar{\psi}
ight)} = 0$$

可得:

$$-\gamma_{\mu}\partial_{\mu}\psi-m\psi+\mathrm{i}e\gamma_{\mu}\psi A_{\mu}=0$$

即:

$$\left(\gamma_{\mu}\partial_{\mu}+m
ight)\psi=\mathrm{i}e\gamma_{\mu}\psi A_{\mu}$$

能量密度

$$egin{aligned} L &= -rac{1}{4}F_{lphaeta}F_{lphaeta} - ar{\psi}\left(\gamma_lpha\partial_eta + m
ight)\psi + \mathrm{i}ear{\psi}\gamma_lpha\psi A_lpha \ & rac{\partial L}{\partial\left(\partial_
u A_lpha
ight)} = -\partial_
u A_lpha + \partial_\mu A_lpha = F_{lpha
u} \ & rac{\partial L}{\partial\left(\partial_
u\psi
ight)} = -ar{\psi}\gamma_
u \end{aligned}$$

$$\frac{\partial L}{\partial \left(\partial_{\nu}\bar{\psi}\right)} = 0$$

能量动量张量:

$$\begin{split} T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\phi_{A}\right)}\partial_{\mu}\phi_{A} \\ &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}A_{\alpha}\right)}\partial_{\mu}A_{\alpha} - \frac{\partial L}{\partial \left(\partial_{\nu}\psi\right)}\partial_{\mu}\psi - \frac{\partial L}{\partial \left(\partial_{\nu}\bar{\psi}\right)}\partial_{\mu}\bar{\psi} \\ &= \left[-\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta} - \bar{\psi}\left(\gamma_{\alpha}\partial_{\beta} + m\right)\psi + \mathrm{i}e\bar{\psi}\gamma_{\alpha}\psi A_{\alpha} \right]\delta_{\mu\nu} - \left(-\partial_{\nu}A_{\alpha} + \partial_{\mu}A_{\alpha}\right)\partial_{\mu}A_{\alpha} - \left(-\bar{\psi}\gamma_{\nu}\right)\partial_{\mu}\psi \\ &= \left[-\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta} - \bar{\psi}\left(\gamma_{\alpha}\partial_{\beta} + m\right)\psi + \mathrm{i}e\bar{\psi}\gamma_{\alpha}\psi A_{\alpha} \right]\delta_{\mu\nu} + \left(\partial_{\nu}A_{\alpha} - \partial_{\mu}A_{\alpha}\right)\partial_{\mu}A_{\alpha} + \bar{\psi}\gamma_{\nu}\partial_{\mu}\psi \end{split}$$

能量密度:

$$egin{aligned} W &= -T_{44} \ &= \left[rac{1}{4}F_{lphaeta}F_{lphaeta} + ar{\psi}\left(\gamma_lpha\partial_eta + m
ight)\psi - \mathrm{i}ear{\psi}\gamma_lpha\psi A_lpha
ight] - \mathrm{i}ar{\psi}\gamma_4\partial_t\psi \end{aligned}$$

电荷密度

$$\begin{split} \frac{\partial L}{\partial \left(\partial_{\mu}\psi\right)} &= -\bar{\psi}\gamma_{\mu} \\ \frac{\partial L}{\partial \left(\partial_{\mu}\bar{\psi}\right)} &= 0 \end{split}$$

电流密度矢量:

$$egin{aligned} j_{\mu} &\equiv -\mathrm{i}e\left[rac{\partial \mathcal{L}}{\partial \left(\partial_{\mu}\phi_{A}
ight)}\phi_{A} - \phi_{A}^{*}rac{\partial \mathcal{L}}{\partial \left(\partial_{\mu}\phi_{A}^{*}
ight)}
ight] \ &= -\mathrm{i}e\left[rac{\partial L}{\partial \left(\partial_{\mu}A_{lpha}
ight)}A_{lpha} - A_{lpha}^{*}rac{\partial L}{\partial \left(\partial_{\mu}A_{lpha}^{*}
ight)} + rac{\partial L}{\partial \left(\partial_{\mu}\psi
ight)}\psi - ar{\psi}rac{\partial L}{\partial \left(\partial_{\mu}ar{\psi}
ight)}
ight] \ &= \mathrm{i}ear{\psi}\gamma_{\mu}\psi \end{aligned}$$

电荷密度:

$$ho = rac{1}{\mathrm{i}} j_4 \ = e ar{\psi} \gamma_4 \psi \ = e \psi^\dagger \psi$$

4-1-7

利用 $\phi(x), \psi(x), A_{\mu}$ 的运动方程,分别验证 $\partial_{\mu}T_{\mu
u} = 0$

标量场 $\phi(x)$

标量场拉格朗日密度:

$$egin{align} L = -rac{1}{2}\partial_lpha\phi\partial_lpha\phi - rac{1}{2}m^2\phi^2 \ & rac{\partial L}{\partial\left(\partial_
u\phi
ight)} = -\partial_
u\phi \ & \end{gathered}$$

能量动量张量:

$$\begin{split} T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\phi_{A}\right)}\partial_{\mu}\phi_{A} \\ &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\phi\right)}\partial_{\mu}\phi \\ &= \left(-\frac{1}{2}\partial_{\alpha}\phi\partial_{\alpha}\phi - \frac{1}{2}m^{2}\phi^{2}\right)\delta_{\mu\nu} - \left(-\partial_{\nu}\phi\right)\partial_{\mu}\phi \\ &= \left(-\frac{1}{2}\partial_{\alpha}\phi\partial_{\alpha}\phi - \frac{1}{2}m^{2}\phi^{2}\right)\delta_{\mu\nu} + \partial_{\nu}\phi\partial_{\mu}\phi \end{split}$$

利用运动方程

$$\left(\partial_{\mu}\partial_{\mu}-m^{2}
ight)\phi=0$$

计算 $T_{\mu\nu}$ 的散度:

$$\begin{split} \partial_{\nu}T_{\mu\nu} &= \partial_{\nu} \left[\left(-\frac{1}{2} \partial_{\alpha} \phi \partial_{\alpha} \phi - \frac{1}{2} m^{2} \phi^{2} \right) \delta_{\mu\nu} + \partial_{\nu} \phi \partial_{\mu} \phi \right] \\ &= -\frac{1}{2} \partial_{\mu} \left(\partial_{\alpha} \phi \partial_{\alpha} \phi \right) - m^{2} \phi \partial_{\mu} \phi + \partial_{\nu} \left(\partial_{\nu} \phi \partial_{\mu} \phi \right) \\ &= -\frac{1}{2} \partial_{\mu} \left(\partial_{\alpha} \phi \partial_{\alpha} \phi \right) - \left(\partial_{\alpha} \partial_{\alpha} \phi \right) \partial_{\mu} \phi + \partial_{\nu} \left(\partial_{\nu} \phi \partial_{\mu} \phi \right) \\ &= -\frac{1}{2} \left(\partial_{\mu} \partial_{\alpha} \phi \right) \partial_{\alpha} \phi - \frac{1}{2} \partial_{\alpha} \phi \left(\partial_{\mu} \partial_{\alpha} \phi \right) - \left(\partial_{\alpha} \partial_{\alpha} \phi \right) \partial_{\mu} \phi + \left(\partial_{\nu} \partial_{\nu} \phi \right) \partial_{\mu} \phi + \partial_{\nu} \phi \left(\partial_{\nu} \partial_{\mu} \phi \right) \\ &= 0 \end{split}$$

旋量场 $\psi(x)$

旋量场拉格朗日密度:

$$egin{align} L = -rac{1}{2} \left(ar{\psi} \gamma_{\mu} \partial_{\mu} \psi - \partial_{\mu} ar{\psi} \gamma_{\mu} \psi
ight) - m ar{\psi} \psi \ & rac{\partial L}{\partial \left(\partial_{
u} \psi
ight)} = -rac{1}{2} ar{\psi} \gamma_{
u} \ & rac{\partial L}{\partial \left(\partial_{
u} ar{\psi}
ight)} = rac{1}{2} \gamma_{
u} \psi \ & \end{split}$$

能量动量张量:

$$\begin{split} T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\phi_{A}\right)}\partial_{\mu}\phi_{A} \\ &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\psi\right)}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\frac{\partial L}{\partial \left(\partial_{\nu}\bar{\psi}\right)} \\ &= \left[-\frac{1}{2}\left(\bar{\psi}\gamma_{\alpha}\partial_{\alpha}\psi - \partial_{\alpha}\bar{\psi}\gamma_{\alpha}\psi\right) - m\bar{\psi}\psi\right]\delta_{\mu\nu} - \left(-\frac{1}{2}\bar{\psi}\gamma_{\nu}\right)\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\left(\frac{1}{2}\gamma_{\nu}\psi\right) \\ &= \left[-\frac{1}{2}\left(\bar{\psi}\gamma_{\alpha}\partial_{\alpha}\psi - \partial_{\alpha}\bar{\psi}\gamma_{\alpha}\psi\right) - m\bar{\psi}\psi\right]\delta_{\mu\nu} + \frac{1}{2}\bar{\psi}\gamma_{\nu}\partial_{\mu}\psi - \frac{1}{2}\partial_{\mu}\bar{\psi}\gamma_{\nu}\psi \end{split}$$

利用旋量场及共轭旋量场运动方程

$$\gamma_{\mu}\partial_{\mu}\psi+m\psi=0$$
 $\partial_{\mu}ar{\psi}\gamma_{\mu}-mar{\psi}=0$

可计算 $T_{\mu\nu}$ 的散度:

$$\begin{split} \partial_{\nu}T_{\mu\nu} &= \partial_{\nu} \left\{ \left[-\frac{1}{2} \left(\bar{\psi} \gamma_{\alpha} \partial_{\alpha} \psi - \partial_{\alpha} \bar{\psi} \gamma_{\alpha} \psi \right) - m \bar{\psi} \psi \right] \delta_{\mu\nu} + \frac{1}{2} \bar{\psi} \gamma_{\nu} \partial_{\mu} \psi - \frac{1}{2} \partial_{\mu} \bar{\psi} \gamma_{\nu} \psi \right\} \\ &= -\frac{1}{2} \partial_{\mu} \left(\bar{\psi} \gamma_{\alpha} \partial_{\alpha} \psi \right) + \frac{1}{2} \partial_{\mu} \left(\partial_{\alpha} \bar{\psi} \gamma_{\alpha} \psi \right) - m \partial_{\mu} \left(\bar{\psi} \psi \right) + \frac{1}{2} \partial_{\nu} \left(\bar{\psi} \gamma_{\nu} \partial_{\mu} \psi \right) - \frac{1}{2} \partial_{\nu} \left(\partial_{\mu} \bar{\psi} \gamma_{\nu} \psi \right) \\ &= \frac{m}{2} \partial_{\mu} \left(\bar{\psi} \psi \right) + \frac{m}{2} \partial_{\mu} \left(m \bar{\psi} \psi \right) - m \partial_{\mu} \left(\bar{\psi} \psi \right) + \frac{1}{2} \partial_{\nu} \bar{\psi} \gamma_{\nu} \partial_{\mu} \psi + \frac{1}{2} \bar{\psi} \gamma_{\nu} \partial_{\nu} \psi - \frac{1}{2} \partial_{\mu} \bar{\psi} \gamma_{\nu} \partial_{\nu} \psi \\ &= \frac{1}{2} \partial_{\nu} \bar{\psi} \gamma_{\nu} \partial_{\mu} \psi + \frac{1}{2} \bar{\psi} \partial_{\mu} \left(\gamma_{\nu} \partial_{\nu} \psi \right) - \frac{1}{2} \partial_{\mu} \left(\partial_{\nu} \bar{\psi} \gamma_{\nu} \right) \psi - \frac{1}{2} \partial_{\mu} \bar{\psi} \gamma_{\nu} \partial_{\nu} \psi \\ &= \frac{1}{2} \left(m \bar{\psi} \right) \partial_{\mu} \psi + \frac{1}{2} \bar{\psi} \partial_{\mu} \left(-m \psi \right) - \frac{1}{2} \partial_{\mu} \left(m \bar{\psi} \right) \psi - \frac{1}{2} \partial_{\mu} \bar{\psi} \left(-m \psi \right) \\ &= 0 \end{split}$$

矢量场 $A_{\mu}(x)$

矢量场拉格朗日密度:

$$egin{align} L = -rac{1}{4}F_{\mu
u}F_{\mu
u} = -rac{1}{2}\partial_{\mu}A_{
u}\partial_{\mu}A_{
u} \ & \ rac{\partial L}{\partial\left(\partial_{
u}A_{lpha}
ight)} = -\partial_{
u}A_{lpha} \ & \ \end{array}$$

能量动量张量:

$$\begin{split} T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}\phi_{A}\right)} \partial_{\mu}\phi_{A} \\ &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_{\nu}A_{\alpha}\right)} \partial_{\mu}A_{\alpha} \\ &= \left[-\frac{1}{2}\partial_{\alpha}A_{\beta}\partial_{\alpha}A_{\beta} \right] \delta_{\mu\nu} - \left(-\partial_{\nu}A_{\alpha}\right) \partial_{\mu}A_{\alpha} \\ &= \left[-\frac{1}{2}\partial_{\alpha}A_{\beta}\partial_{\alpha}A_{\beta} \right] \delta_{\mu\nu} + \partial_{\nu}A_{\alpha}\partial_{\mu}A_{\alpha} \end{split}$$

利用矢量场运动方程

$$\partial_{\alpha}\partial_{\alpha}A_{\mu}=0$$

可计算 $T_{\mu\nu}$ 的散度:

$$\begin{split} \partial_{\nu}T_{\mu\nu} &= \partial_{\nu} \left\{ \left[-\frac{1}{2} \partial_{\alpha}A_{\beta} \partial_{\alpha}A_{\beta} \right] \delta_{\mu\nu} + \partial_{\nu}A_{\alpha} \partial_{\mu}A_{\alpha} \right\} \\ &= -\frac{1}{2} \partial_{\mu} \left(\partial_{\alpha}A_{\beta} \partial_{\alpha}A_{\beta} \right) + \partial_{\nu} \left(\partial_{\nu}A_{\alpha} \partial_{\mu}A_{\alpha} \right) \\ &= -\frac{1}{2} \left(\partial_{\mu} \partial_{\alpha}A_{\beta} \right) \partial_{\alpha}A_{\beta} - \frac{1}{2} \partial_{\alpha}A_{\beta} \left(\partial_{\mu} \partial_{\alpha}A_{\beta} \right) + \left(\partial_{\nu} \partial_{\nu}A_{\alpha} \right) \partial_{\mu}A_{\alpha} + \partial_{\nu}A_{\alpha} \left(\partial_{\nu} \partial_{\mu}A_{\alpha} \right) \\ &= -\partial_{\alpha}A_{\beta} \left(\partial_{\mu} \partial_{\alpha}A_{\beta} \right) + \partial_{\nu}A_{\alpha} \left(\partial_{\mu} \partial_{\nu}A_{\alpha} \right) \\ &= 0 \end{split}$$