3

3.1

弱引力场近似下，度规表示为

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \tag{1}$$

线性近似理论中只保留 $h_{\mu\nu}$ 中的线性项（一阶小量）。定义

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad h \equiv \eta^{\mu\nu}h_{\mu\nu}, \tag{2}$$

试证明，它的逆变换是

$$\bar{\bar{h}}_{\mu\nu} \equiv \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = h_{\mu\nu} \tag{3}$$

证明:

$$\begin{aligned}
\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} &= \eta^{\mu\nu}\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) \\
&= \eta^{\mu\nu}h_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}h \\
&= h - \frac{1}{2}\delta^{\mu}_{\mu}h \\
&= -h
\end{aligned} \tag{4}$$

于是

$$\begin{aligned}
\bar{\bar{h}}_{\mu\nu} &\equiv \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \\
&= \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) - \frac{1}{2}\eta_{\mu\nu}\left(-h\right) \\
&= h_{\mu\nu}
\end{aligned} \tag{5}$$

3.2

线性近似理论中，证明克氏符

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\beta h_{\alpha\nu} + \partial_\alpha h_{\beta\nu} - \partial_\nu h_{\alpha\beta}\right) = \frac{1}{2}\left(h^{\mu}_{\alpha,\beta} + h^{\mu}_{\beta,\alpha} - h^{,\mu}_{\alpha\beta}\right) \tag{6}$$

线性近似理论中张量指标的升降借助 $\eta_{\mu\nu}$ 和 $\eta^{\mu\nu}$ 进行。并求线性化后的Ricci张量

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} = -\frac{1}{2}\left(h^{,\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\mu,\nu,\alpha} - h^{\alpha}_{\nu,\mu,\alpha}\right) \tag{7}$$

证明:

由于

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \tag{8}$$

线性近似理论只保留小量 $h_{\mu\nu}$ 的一阶项，于是

$$
\begin{aligned}
\Gamma^{\mu}_{\alpha\beta} &\equiv \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) \\
&\equiv \frac{1}{2}\left(\eta^{\mu\nu} + h^{\mu\nu}\right)\left[\partial_\alpha\left(\eta_{\beta\nu} + h_{\beta\nu}\right) + \partial_\beta\left(\eta_{\alpha\nu+h_{\alpha\nu}}\right) - \partial_\nu\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right)\right] \\
&= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\beta h_{\alpha\nu} + \partial_\alpha h_{\beta\nu} - \partial_\nu h_{\alpha\beta}\right) \\
&= \frac{1}{2}\left[\partial_\beta\left(\eta^{\mu\nu} h_{\alpha\nu}\right) + \partial_\alpha\left(\eta^{\mu\nu} h_{\beta\nu}\right) - \eta^{\mu\nu}\partial_\nu\left(h_{\alpha\beta}\right)\right] \\
&= \frac{1}{2}\left(\partial_\beta h^{\mu}_{\alpha} + \partial_\alpha h^{\mu}_{\beta} - \partial^{\mu} h_{\alpha\beta}\right) \\
&= \frac{1}{2}\left(h^{\mu}_{\alpha,\beta} + h^{\mu}_{\beta,\alpha} - h^{;\mu}_{\alpha\beta}\right)
\end{aligned} \tag{9}
$$

克氏符 Γ 是 $h_{\mu\nu}$ 一阶小量的叠加，因此 Riemann 曲率张量中的 $\Gamma\Gamma$ 项可以舍去，此时 Riemann 曲率张量为：

$$
\begin{aligned}
R^{\lambda}_{\mu\alpha\nu} &\equiv \partial_\alpha \Gamma^{\lambda}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}_{\alpha\mu} + \Gamma^{\lambda}_{\alpha\beta}\Gamma^{\beta}_{\nu\mu} - \Gamma^{\lambda}_{\nu\beta}\Gamma^{\beta}_{\alpha\mu} \\
&= \partial_\alpha \Gamma^{\lambda}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}_{\alpha\mu} \\
&= \frac{1}{2}\left[\partial_\alpha\left(h^{\lambda}_{\nu,\mu} + h^{\lambda}_{\mu,\nu} - h^{;\lambda}_{\nu\mu}\right) - \partial_\nu\left(h^{\lambda}_{\alpha,\mu} + h^{\lambda}_{\mu,\alpha} - h^{;\lambda}_{\alpha\mu}\right)\right]
\end{aligned} \tag{10}
$$

Ricci张量为

$$
\begin{aligned}
R_{\mu\nu} &= R^{\alpha}_{\mu\alpha\nu} \\
&= \frac{1}{2}\left[\partial_\alpha\left(h^{\alpha}_{\nu,\mu} + h^{\alpha}_{\mu,\nu} - h^{;\alpha}_{\nu\mu}\right) - \partial_\nu\left(h^{\alpha}_{\alpha,\mu} + h^{\alpha}_{\mu,\alpha} - h^{;\alpha}_{\alpha\mu}\right)\right] \\
&= \frac{1}{2}\left(\partial_\alpha h^{\alpha}_{\nu,\mu} + \partial_\nu h^{;\alpha}_{\alpha\mu} - \partial_\alpha h^{;\alpha}_{\nu\mu} - \partial_\nu h^{\alpha}_{\alpha,\mu}\right) \\
&= -\frac{1}{2}\left(h^{;\alpha}_{\nu\mu,\alpha} + h^{\alpha}_{\alpha,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{;\alpha}_{\alpha\mu,\nu}\right) \\
&= -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h^{\alpha}_{\alpha,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{;\alpha}_{\alpha\mu,\nu}\right)
\end{aligned} \tag{11}
$$

注意到

$$
h^{\alpha}_{\alpha,\mu,\nu} = \eta^{\lambda\alpha} h_{\lambda\alpha,\mu,\nu} = \left(\eta^{\lambda\alpha} h_{\lambda\alpha}\right)_{,\mu,\nu} = h_{,\mu,\nu} \tag{12}
$$

$$
h^{;\alpha}_{\alpha\mu,\nu} = \eta^{\alpha\beta} h_{\alpha\mu,\nu,\beta} = h^{\beta}_{\mu,\nu,\beta} = h^{\alpha}_{\mu,\nu,\alpha} \tag{13}
$$

因此Ricci张量可化为

$$
\begin{aligned}
R_{\mu\nu} &= -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h^{\alpha}_{\alpha,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{;\alpha}_{\alpha\mu,\nu}\right) \\
&= -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{\alpha}_{\mu,\nu,\alpha}\right)
\end{aligned} \tag{14}
$$

3.3

由以上公式，证明线性化Einstein引力场方程

$$
\bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{15}
$$

具体化为

$$
\bar{h}^{;\alpha}_{\mu\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{;\alpha,\beta}_{\alpha\beta} - \bar{h}^{;\alpha}_{\mu\alpha,\nu} - \bar{h}^{;\alpha}_{\nu\alpha,\mu} = -16\pi G T_{\mu\nu} \tag{16}
$$

证明：

先算Ricci标量：

$$
\begin{aligned}
R &= \eta^{\mu\nu} R_{\mu\nu} \\
&= -\frac{1}{2}\eta^{\mu\nu}\left(h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{\alpha}_{\mu,\nu,\alpha}\right) \\
&= -\frac{1}{2}\left(h^{;\alpha}_{,\alpha} + h^{;\nu}_{,\nu} - h^{\alpha,\nu}_{\nu,\alpha} - h^{\alpha,\mu}_{\mu,\alpha}\right) \\
&= -\frac{1}{2}\left(2h^{;\alpha}_{,\alpha} - 2h^{\alpha,\beta}_{\beta,\alpha}\right) \\
&= -h^{;\alpha}_{,\alpha} + h^{\alpha,\beta}_{\beta,\alpha}
\end{aligned}
\tag{17}
$$

利用

$$
\bar{h} = -h, \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h = h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{h} \Longrightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{18}
$$

$$
h^{\alpha}_{\nu} = \bar{h}^{\alpha}_{\nu} - \frac{1}{2}\delta^{\alpha}_{\nu}\bar{h} \tag{19}
$$

有

$$
\begin{aligned}
\bar{R}_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\
&= -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{\alpha}_{\mu,\nu,\alpha}\right) - \frac{1}{2}\eta_{\mu\nu}\left(-h^{;\alpha}_{,\alpha} + h^{\alpha,\beta}_{\beta,\alpha}\right) \\
&= -\frac{1}{2}\left[h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\nu,\mu,\alpha} - h^{\alpha}_{\mu,\nu,\alpha} - \eta_{\mu\nu}h^{;\alpha}_{,\alpha} + \eta_{\mu\nu}h^{\alpha,\beta}_{\beta,\alpha}\right] \\
&= -\frac{1}{2}\left[\left(\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{;\alpha}_{,\alpha}\right) + \left(-\bar{h}_{,\mu,\nu}\right) - \left(\bar{h}^{\alpha}_{\nu,\mu,\alpha} - \frac{1}{2}\delta^{\alpha}_{\nu}\bar{h}_{,\mu,\alpha}\right) - \left(\bar{h}^{\alpha}_{\mu,\nu,\alpha} - \frac{1}{2}\delta^{\alpha}_{\mu}\bar{h}_{,\nu,\alpha}\right) - \left(-\eta_{\mu\nu}\bar{h}^{;\alpha}_{,\alpha}\right) + \left(\eta_{\mu\nu}\left(\bar{h}^{\alpha,\beta}_{\beta,\alpha} - \frac{1}{2}\delta^{\alpha}_{\beta}\bar{h}^{;\beta}_{,\alpha}\right)\right)\right] \\
&= -\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{\alpha,\beta}_{\beta,\alpha}\right] \\
&= -\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha} + \eta_{\mu\nu}\eta^{\alpha\rho}\bar{h}^{;\beta}_{\rho\beta,\alpha}\right] \\
&= -\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{,\rho,\beta}_{\rho\beta}\right] \\
&= -\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{,\alpha,\beta}_{\alpha\beta} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha}\right]
\end{aligned}
\tag{20}
$$

最后，Einstein引力场方程

$$
\bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{21}
$$

就化为：

$$
-\frac{1}{2}\left[\bar{h}^{;\alpha}_{\mu\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{,\alpha,\beta}_{\alpha\beta} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha}\right] = 8\pi G T_{\mu\nu} \tag{22}
$$

也即：

$$
\bar{h}^{;\alpha}_{\mu\nu,\alpha} + \eta_{\mu\nu}\bar{h}^{,\alpha,\beta}_{\alpha\beta} - \bar{h}^{\alpha}_{\nu,\mu,\alpha} - \bar{h}^{\alpha}_{\mu,\nu,\alpha} = -16\pi G T_{\mu\nu} \tag{23}
$$

3.4

证明对于静态时空，线性化Ricci

$$
R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} = -\frac{1}{2}\left(h^{;\alpha}_{\mu\nu,\alpha} + h_{,\mu,\nu} - h^{\alpha}_{\mu,\nu,\alpha} - h^{\alpha}_{\nu,\mu,\alpha}\right) \tag{24}
$$

可化为

$$
R_{00} = -\frac{1}{2}h_{00,i,i}, \quad R_{0i} = \frac{1}{2}\left(h_{k0,i,k} - h_{0i,k,k}\right) \tag{25}
$$

$$
R_{ij} = -\frac{1}{2}\left(-h_{00,i,j} + h_{kk,i,j} - h_{ki,j,k} - h_{kj,i,k} + h_{ij,k,k}\right) \tag{26}
$$

证明：

静态时空满足

$$\partial_0 h_{\mu\nu} = 0 \tag{27}$$

也即

$$h_{\mu\nu,0} = 0 \tag{28}$$

$$h_{\mu\nu}^{,0} = \eta^{0\alpha} h_{\mu\nu,\alpha} = \eta^{00} h_{\mu\nu,0} = 0 \tag{29}$$

$$h_{,0} = \eta^{\mu\nu} h_{\mu\nu,0} = 0 \tag{30}$$

$$h^{,0} = \eta^{0\alpha} h_{,\alpha} = \eta^{00} h^{,0} = 0 \tag{31}$$

$$h^{\mu}_{\nu,0} = \eta^{\mu\alpha} h_{\alpha\nu,0} = 0 \tag{32}$$

于是可计算Ricci张量：

$$\begin{aligned} R_{00} &= -\frac{1}{2}\left(h_{00,\alpha}^{,\alpha} + h_{,0,0} - h^{\alpha}_{0,0,\alpha} - h^{\alpha}_{0,0,\alpha}\right) \\ &= -\frac{1}{2} h_{00,\alpha}^{,\alpha} \\ &= -\frac{1}{2} h_{00,i}^{,i} \end{aligned} \tag{33}$$

$$\begin{aligned} R_{0i} &= -\frac{1}{2}\left(h_{0i,\alpha}^{,\alpha} + h_{,0,i} - h^{\alpha}_{0,i,\alpha} - h^{\alpha}_{i,0,\alpha}\right) \\ &= \frac{1}{2}\left(h^{\alpha}_{0,i,\alpha} - h_{0i,\alpha}^{,\alpha}\right) \\ &= \frac{1}{2}\left(h^{k}_{0,i,k} - h_{0i,k}^{,k}\right) \end{aligned} \tag{34}$$

$$\begin{aligned} R_{ij} &= -\frac{1}{2}\left(h_{ij,\alpha}^{,\alpha} + h_{,i,j} - h^{\alpha}_{i,j,\alpha} - h^{\alpha}_{j,i,\alpha}\right) \\ &= -\frac{1}{2}\left(h_{ij,k}^{,k} + h_{,i,j} - h^{k}_{i,j,k} - h^{k}_{j,i,k}\right) \\ &= -\frac{1}{2}\left(h_{ij,k}^{,k} - h_{00,i,j} + h^{k}_{k,i,j} - h^{k}_{i,j,k} - h^{k}_{j,i,k}\right) \end{aligned} \tag{35}$$

由于

$$\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1) \tag{36}$$

若认为一对重复的上/下指标也代表求和，则：

$$\begin{aligned} R_{00} &= -\frac{1}{2} h_{00,i}^{,i} \\ &= -\frac{1}{2}\eta^{\alpha i} h_{00,i,\alpha} \\ &= -\frac{1}{2} h_{00,i,i} \end{aligned} \tag{37}$$

$$\begin{aligned} R_{0i} &= \frac{1}{2}\left(h^{k}_{0,i,k} - h_{0i,k}^{,k}\right) \\ &= \frac{1}{2}\left(\eta^{k\alpha} h_{\alpha 0,i,k} - \eta^{k\alpha} h_{0i,k,\alpha}\right) \\ &= \frac{1}{2}\left(h_{k0,i,k} - h_{0i,k,k}\right) \end{aligned} \tag{38}$$

$$\begin{aligned} R_{ij} &= -\frac{1}{2}\left(h_{ij,k}^{,k} - h_{00,i,j} + h^{k}_{k,i,j} - h^{k}_{i,j,k} - h^{k}_{j,i,k}\right) \\ &= -\frac{1}{2}\left(\eta^{k\alpha} h_{ij,k,\alpha} - h_{00,i,j} + \eta^{k\alpha} h_{\alpha k,i,j} - \eta^{k\alpha} h_{\alpha i,j,k} - \eta^{k\alpha} h_{kj,i,k}\right) \\ &= -\frac{1}{2}\left(-h_{00,i,j} + h_{kk,i,j} - h_{ki,j,k} - h_{kj,i,k} + h_{ij,k,k}\right) \end{aligned} \tag{39}$$