4-1 量子场论习题四

4-1-1

> 写出 Noether 定理。它主要包含哪两个部分？

Noether 定理包括广义守恒定理1和广义守恒定理2。

广义守恒定理1

设 $\theta_{\mu\cdots\nu\lambda}(x)$ 是 n 阶张量函数，且满足：

$$\theta_{\mu\cdots\nu\lambda}(x)\Big|_{\vec{x}\to\infty}=0$$

若

$$\partial_\lambda\theta_{\mu\cdots\nu\lambda}=0$$

则存在一个 $(n-1)$ 阶守恒张量：

$$T_{\mu\cdots\nu}(x_4)\equiv\frac{1}{\mathrm{i}}\int\limits_{\vec{x}\in\mathbb{R}^3}\theta_{\mu\cdots\nu4}(\vec{x},x_4)\mathrm{d}^3\vec{x}=\mathrm{const}$$

广义守恒定理2

若场的作用量

$$I=\int\limits_G\mathcal{L}\left(\phi_A,\partial_\mu\phi_A\right)\mathrm{d}^4x$$

对微量变换

$$x\to x'=x+\delta x,\quad \phi_A\to\phi_A'=\phi_A+\delta_0\phi_A$$
$$\phi_A(x)\to\phi_A'(x')=\phi_A(x)+\delta\phi_A(x)$$

保持不变，则存在一个矢量

$$\theta_\mu=\left(\mathcal{L}\delta_{\mu\nu}-\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_A\right)}\partial_\nu\phi_A\right)\delta x_\nu+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_A\right)}\delta\phi_A$$

满足关系式：

$$\partial_\mu\theta_\mu+[\mathcal{L}]_{\phi_A}\delta_0\phi_A=0$$

4-1-2

> 分别讨论下述变换的 $\delta x_\nu,\delta\phi_A$
>
> （1）四维时空平移
>
> （2）相因子变换
>
> （3）无穷小 Lorentz 固有转动

四维时空平移

$$x_\mu \to x'_\mu = x_\mu + \alpha_\mu, \quad \phi_A(x) \to \phi'_A(x') = \phi_A(x)$$

$$\delta x_\mu = x'_\mu - x_\mu = \alpha_\mu$$

$$\delta \phi_A(x) = \phi'_A(x') - \phi_A(x) = 0$$

相因子变换

$$x_\mu \to x'_\mu = x_\mu, \quad \phi_A(x) \to \phi'_A(x') = \mathrm{e}^{\mathrm{i}\alpha}\phi_A(x)$$

$$\delta x_\mu = x'_\mu - x_\mu = 0$$

$$\begin{aligned}\delta \phi_A(x) &= \phi'_A(x') - \phi_A(x) \\ &= \mathrm{e}^{\mathrm{i}\alpha}\phi_A(x) - \phi_A(x) \\ &\approx \mathrm{i}\alpha\phi_A(x)\end{aligned}$$

无穷小 Lorentz 固有转动

$$x_\mu \to x'_\mu = (\delta_{\mu\nu} + \alpha_{\mu\nu})x_\nu, \quad \phi(x) \to \phi'(x') = \phi(x) + \frac{1}{2}\alpha_{\mu\nu}I_{\mu\nu}\phi(x)$$

其中，$I_{\mu\nu} = \left.\dfrac{\partial D(\alpha)}{\partial \alpha_{\mu\nu}}\right|_{\alpha=0}$，$D(\alpha)$ 为固有 Lorentz 群的某种线性表示。

$$\delta x_\mu = x'_\mu - x_\mu = \alpha_{\mu\nu}x_\nu$$

$$\begin{aligned}\delta \phi(x) &= \phi'(x') - \phi(x) \\ &= \frac{1}{2}\alpha_{\mu\nu}I_{\mu\nu}\phi(x)\end{aligned}$$

4-1-3

> 由 Lorentz 原理，推导分量形式 Dirac 方程。

x 系 Dirac 方程：

$$(\gamma_\mu\partial_\mu + m)\,\psi(x) = 0$$

考虑时空坐标进行广义 Lorentz 变换 $x_\mu \to x'_\mu = A_{\mu\nu}x_\nu + b_\mu$，由于 Dirac 方程应当具有 Lorentz 协变性，则 x' 系 Dirac 方程形式为：

$$\left(\gamma_\mu\partial'_\mu + m\right)\psi'(x') = 0$$

为了使 Dirac 方程具有 Lorentz 协变性，当时空坐标进行广义 Lorentz 变换时，$\psi(x)$ 也应当进行变换。设：

$$x'_\mu = A_{\mu\nu}x_\nu + b_\mu, \quad \psi'(x') = \Lambda(A)\psi(x)$$

注意到：

$$x'_\mu = A_{\mu\nu}x_\nu + b_\mu \Longrightarrow A_{\mu\lambda}\mathrm{d}x'_\mu = \mathrm{d}x_\lambda$$

$$\partial'_\mu = \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu} = A_{\mu\nu}\partial_\nu$$

则 x' 系 Dirac 方程化为：

$$
\begin{aligned}
0 &= \left(\gamma_\mu \partial'_\mu + m\right) \psi'(x') \\
&= \left(\gamma_\mu A_{\mu\nu} \partial_\nu + m\right) \Lambda(A) \psi(x)
\end{aligned}
$$

上式左乘 $\Lambda^{-1}(A)$ 得:

$$
\left[\Lambda^{-1}(A)\gamma_\mu \Lambda(A) A_{\mu\nu}\partial_\nu + m\right]\psi(x) = 0
$$

与 x 系 Dirac 方程

$$
\left(\gamma_\nu \partial_\nu + m\right)\psi(x) = 0
$$

对比可得:

$$
\Lambda^{-1}(A)\gamma_\mu \Lambda(A) A_{\mu\nu} = \gamma_\nu
$$

上式乘 $A_{\lambda\nu}$，对 ν 求和，并利用正交关系，得:

$$
\Lambda^{-1}(A)\gamma_\lambda \Lambda(A) = A_{\lambda\nu}\gamma_\nu
$$

满足上式的 $\Lambda(A)$ 必定是矩阵，且构成 Lorentz 群的旋量表示。因此 $\psi(x)$ 是一个四元列矩阵，记为:

$$
\psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}
$$

Dirac 表象 γ_μ 矩阵:

$$
\sigma_1^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2^0 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
$$

$$
\gamma_i = \begin{bmatrix} 0 & -\mathrm{i}\sigma_i^0 \\ \mathrm{i}\sigma_i^0 & 0 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} I^0 & 0 \\ 0 & -I^0 \end{bmatrix}
$$

把旋量 $\psi(x)$ 写成二分量形式:

$$
\psi(x) = \begin{bmatrix} \varphi \\ \chi \end{bmatrix}, \quad \varphi = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}, \quad \chi = \begin{bmatrix} \psi_3(x) \\ \psi_4(x) \end{bmatrix}
$$

代入 Dirac 方程

$$
\left(\gamma_\mu \partial_\mu + m\right)\psi(x) = 0
$$

得到二分量形式 Dirac 方程:

$$
-\mathrm{i}\vec{\sigma}^0 \cdot \nabla\chi - \mathrm{i}\partial_t\varphi + m\varphi = 0
$$

$$
\mathrm{i}\vec{\sigma}^0 \cdot \nabla\varphi + \mathrm{i}\partial_t\chi + m\chi = 0
$$

4-1-4

> 求复标量场 $T_{\mu\nu}, L_{[\alpha\beta]\mu}, j_\mu$ 以及 $W, G, \vec{L}, \rho$

$$
\mathcal{L}_0 = -\partial_\alpha\phi^*\partial_\alpha\phi - m^2\phi^*\phi
$$

$$
\begin{aligned}
T_{\mu\nu} &= \mathcal{L}_0\delta_{\mu\nu} - \frac{\partial\mathcal{L}_0}{\partial\left(\partial_\nu\phi\right)}\partial_\mu\phi - \frac{\partial\mathcal{L}_0}{\partial\left(\partial_\nu\phi^*\right)}\partial_\mu\phi^* \\
&= \mathcal{L}_0\delta_{\mu\nu} + \partial_\nu\phi^*\partial_\mu\phi + \partial_\nu\phi\partial_\mu\phi^* \\
&= \left(-\partial_\alpha\phi^*\partial_\alpha\phi - m^2\phi^*\phi\right)\delta_{\mu\nu} + \partial_\nu\phi^*\partial_\mu\phi + \partial_\nu\phi\partial_\mu\phi^*
\end{aligned}
$$

$$T_{i4}=\partial_i\phi\partial_4\phi^*+\partial_i\phi^*\partial_4\phi$$

$$\begin{aligned}W&=-T_{44}\\&=-\left[\left(-\partial_\alpha\phi^*\partial_\alpha\phi-m^2\phi^*\phi\right)\delta_{44}+\partial_4\phi^*\partial_4\phi+\partial_4\phi\partial_4\phi^*\right]\\&=\partial_i\phi^*\partial_i\phi+\partial_4\phi^*\partial_4\phi+m^2\phi^*\phi-\partial_4\phi^*\partial_4\phi-\partial_4\phi\partial_4\phi^*\\&=\partial_i\phi^*\partial_i\phi+m^2\phi^*\phi-\left(\frac{1}{\mathrm{i}}\right)^2\partial_t\phi\partial_t\phi^*\\&=\nabla\phi^*\cdot\nabla\phi+\partial_t\phi^*\partial_t\phi+m^2\phi^*\phi\end{aligned}$$

$$\begin{aligned}G_i&=\frac{1}{\mathrm{i}}T_{i4}\\&=\frac{1}{\mathrm{i}}\left[\partial_i\phi\partial_4\phi^*+\partial_i\phi^*\partial_4\phi\right]\\&=-\partial_i\phi^*\partial_t\phi-\partial_i\phi\partial_t\phi^*\end{aligned}$$

$$\vec{G}=G_i\vec{\mathrm{e}}_i=-\nabla\phi^*\partial_t\phi-\nabla\phi\partial_t\phi^*$$

$$\begin{aligned}L_{[\alpha\beta]\mu}&=T_{\alpha\mu}x_\beta-T_{\beta\mu}x_\alpha\\&=\left[\left(-\partial_\rho\phi^*\partial_\rho\phi-m^2\phi^*\phi\right)\delta_{\alpha\mu}+\partial_\mu\phi^*\partial_\alpha\phi+\partial_\mu\phi\partial_\alpha\phi^*\right]x_\beta-\left[\left(-\partial_\rho\phi^*\partial_\rho\phi-m^2\phi^*\phi\right)\delta_{\beta\mu}+\partial_\mu\phi^*\partial_\beta\phi+\partial_\mu\phi\partial_\beta\phi^*\right]x_\alpha\end{aligned}$$

$$\begin{aligned}\vec{l}&=\vec{x}\times\vec{G}\\&=-\vec{x}\times\left(\nabla\phi^*\partial_t\phi+\nabla\phi\partial_t\phi^*\right)\end{aligned}$$

$$\begin{aligned}\vec{L}&=\int\vec{l}\mathrm{d}^3\vec{x}\\&=-\int\vec{x}\times\left(\nabla\phi^*\partial_t\phi+\nabla\phi\partial_t\phi^*\right)\mathrm{d}^3\vec{x}\end{aligned}$$

$$\begin{aligned}j_\mu&=-\mathrm{i}e\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\phi_A-\phi_A^*\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A^*}\right)\\&=\mathrm{i}e\left(\phi\partial_\mu\phi^*-\phi^*\partial_\mu\phi\right)\end{aligned}$$

$$j_4\equiv\mathrm{i}\rho$$

$$\begin{aligned}\rho&=\frac{1}{\mathrm{i}}j_4\\&=-\mathrm{i}e\left(\phi\partial_t\phi^*-\phi^*\partial_t\phi\right)\end{aligned}$$

4-1-5

> 设
>
> $$L=-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}-\frac{1}{2}\left(\partial_\mu-\mathrm{i}eA_\mu\right)\phi\cdot\left(\partial_\mu-\mathrm{i}eA_\mu\right)\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4}\phi^4$$
>
> 求（1）$A_\mu,\phi(x)$ 的方程；（2）$T_{\mu\nu}$

$$F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha$$

$$\begin{aligned}L&=-\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta}-\frac{1}{2}\left(\partial_\alpha-\mathrm{i}eA_\alpha\right)\phi\cdot\left(\partial_\alpha-\mathrm{i}eA_\alpha\right)\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4}\phi^4\\&=-\frac{1}{4}\left(\partial_\alpha A_\beta-\partial_\beta A_\alpha\right)\left(\partial_\alpha A_\beta-\partial_\beta A_\alpha\right)-\frac{1}{2}\left[\partial_\alpha\phi\partial_\alpha\phi-2\mathrm{i}eA_\alpha\phi\partial_\alpha\phi-e^2A_\alpha A_\alpha\phi^2\right]-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4}\phi^4\\&=-\frac{1}{2}\partial_\alpha A_\beta\partial_\alpha A_\beta+\frac{1}{2}\partial_\alpha A_\beta\partial_\beta A_\alpha-\frac{1}{2}\partial_\alpha\phi\partial_\alpha\phi+\mathrm{i}eA_\alpha\phi\partial_\alpha\phi+\frac{1}{2}e^2A_\alpha A_\alpha\phi^2-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4}\phi^4\end{aligned}$$

A_μ 的运动方程

$$\frac{\partial L}{\partial A_\mu} = \mathrm{i}e\phi\partial_\mu\phi + e^2 A_\mu \phi^2$$

$$\begin{aligned}\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)} &= -\partial_\nu A_\mu + \partial_\mu A_\nu \\ &= F_{\mu\nu}\end{aligned}$$

把 L 代入 E-L 方程

$$\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)} = 0$$

可得 A_μ 的运动方程:

$$\mathrm{i}e\phi\partial_\mu\phi + e^2 A_\mu\phi^2 - \partial_\nu\left(F_{\mu\nu}\right) = 0$$

即:

$$\partial_\nu F_{\mu\nu} = \mathrm{i}e\phi\partial_\mu\phi + e^2 A_\mu\phi^2$$

$\phi(x)$ 的运动方程

$$\frac{\partial L}{\partial\phi} = \mathrm{i}eA_\alpha\partial_\alpha\phi + e^2 A_\alpha A_\alpha\phi - m^2\phi - \lambda\phi^3$$

$$\frac{\partial L}{\partial\left(\partial_\mu\phi\right)} = -\partial_\mu\phi + \mathrm{i}eA_\mu\phi$$

把 L 代入 E-L 方程

$$\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial\left(\partial_\mu\phi\right)} = 0$$

可得:

$$\mathrm{i}eA_\alpha\partial_\alpha\phi + e^2 A_\alpha A_\alpha\phi - m^2\phi - \lambda\phi^3 - \partial_\mu\left(-\partial_\mu\phi + \mathrm{i}eA_\mu\phi\right) = 0$$

即:

$$\partial_\mu\partial_\mu\phi - \mathrm{i}e\phi\partial_\mu A_\mu + e^2 A_\alpha A_\alpha\phi - m^2\phi - \lambda\phi^3 = 0$$

计算 $T_{\mu\nu}$

能动张量 $T_{\mu\nu}$ 的定义:

$$\begin{aligned}T_{\mu\nu} &\equiv L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_\nu\phi_A\right)}\partial_\mu\phi_A \\ &\equiv L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_\nu\phi\right)}\partial_\mu\phi - \frac{\partial L}{\partial\left(\partial_\nu A_\alpha\right)}\partial_\mu A_\alpha \\ &= \left(-\frac{1}{2}\partial_\alpha A_\beta\partial_\alpha A_\beta + \frac{1}{2}\partial_\alpha A_\beta\partial_\beta A_\alpha - \frac{1}{2}\partial_\alpha\phi\partial_\alpha\phi + \mathrm{i}eA_\alpha\phi\partial_\alpha\phi + \frac{1}{2}e^2A_\alpha A_\alpha\phi^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4\right)\delta_{\mu\nu} \\ &- \left(-\partial_\nu\phi + \mathrm{i}eA_\nu\phi\right)\partial_\mu\phi - F_{\alpha\nu}\partial_\mu A_\alpha\end{aligned}$$

4-1-6

> 由下列包含相互作用体系的拉氏函数，分别求其运动方程、能量密度和电荷密度。

> （1）
>
> $$L=-\frac{1}{2}\partial_\mu\tilde{\phi}\partial_\mu\tilde{\phi}-\frac{1}{2}m^2\tilde{\phi}^2-\frac{1}{2}\left(\bar{\psi}\gamma_\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma_\mu\psi\right)-M\bar{\psi}\psi+\mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}$$
>
> （2）
>
> $$L=-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}-\bar{\psi}\left(\gamma_\mu\partial_\mu+m\right)\psi+\mathrm{i}e\bar{\psi}\gamma_\mu\psi A_\mu$$

（1）

$$L=-\frac{1}{2}\partial_\mu\tilde{\phi}\partial_\mu\tilde{\phi}-\frac{1}{2}m^2\tilde{\phi}^2-\frac{1}{2}\left(\bar{\psi}\gamma_\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma_\mu\psi\right)-M\bar{\psi}\psi+\mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}$$

$\tilde{\phi}$ 的运动方程

$$L=-\frac{1}{2}\partial_\mu\tilde{\phi}\partial_\mu\tilde{\phi}-\frac{1}{2}m^2\tilde{\phi}^2-\frac{1}{2}\left(\bar{\psi}\gamma_\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma_\mu\psi\right)-M\bar{\psi}\psi+\mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}$$

$$\frac{\partial L}{\partial\tilde{\phi}}=-m^2\tilde{\phi}+\mathrm{i}G\bar{\psi}\gamma_5\psi$$

$$\frac{\partial L}{\partial\left(\partial_\mu\tilde{\phi}\right)}=-\partial_\mu\tilde{\phi}$$

代入 E-L 方程

$$\frac{\partial L}{\partial\tilde{\phi}}-\partial_\mu\frac{\partial L}{\partial\left(\partial_\mu\tilde{\phi}\right)}=0$$

可得：

$$\left(-m^2\tilde{\phi}+\mathrm{i}G\bar{\psi}\gamma_5\psi\right)-\partial_\mu\left(-\partial_\mu\tilde{\phi}\right)=0$$

即：

$$\boxed{\left(\partial_\mu\partial_\mu-m^2\right)\tilde{\phi}=-\mathrm{i}G\bar{\psi}\gamma_5\psi}$$

$\bar{\psi}$ 的运动方程（对 ψ 变分）

$$L=-\frac{1}{2}\partial_\mu\tilde{\phi}\partial_\mu\tilde{\phi}-\frac{1}{2}m^2\tilde{\phi}^2-\frac{1}{2}\left(\bar{\psi}\gamma_\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma_\mu\psi\right)-M\bar{\psi}\psi+\mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}$$

$$\frac{\partial L}{\partial\psi}=\frac{1}{2}\partial_\alpha\bar{\psi}\gamma_\alpha-M\bar{\psi}+\mathrm{i}G\bar{\psi}\gamma_5\tilde{\phi}$$

$$\frac{\partial L}{\partial\left(\partial_\mu\psi\right)}=-\frac{1}{2}\bar{\psi}\gamma_\mu$$

代入 E-L 方程

$$\frac{\partial L}{\partial\psi}-\partial_\mu\frac{\partial L}{\partial\left(\partial_\mu\psi\right)}=0$$

可得：

$$\frac{1}{2}\partial_\alpha\bar{\psi}\gamma_\alpha-M\bar{\psi}+\mathrm{i}G\bar{\psi}\gamma_5\tilde{\phi}-\partial_\mu\left(-\frac{1}{2}\bar{\psi}\gamma_\mu\right)=0$$

即：

$$\boxed{\partial_\mu\bar{\psi}\gamma_\mu - M\bar{\psi} + \mathrm{i}G\bar{\psi}\gamma_5\tilde{\phi} = 0}$$

ψ 的运动方程（对 $\bar{\psi}$ 变分）

$$L = -\frac{1}{2}\partial_\mu\tilde{\phi}\partial_\mu\tilde{\phi} - \frac{1}{2}m^2\tilde{\phi}^2 - \frac{1}{2}\left(\bar{\psi}\gamma_\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_\mu\psi\right) - M\bar{\psi}\psi + \mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}$$

$$\frac{\partial L}{\partial\bar{\psi}} = -\frac{1}{2}\gamma_\alpha\partial_\alpha - M\psi + \mathrm{i}G\gamma_5\psi\tilde{\phi}$$

$$\frac{\partial L}{\partial\left(\partial_\mu\bar{\psi}\right)} = \frac{1}{2}\gamma_\mu\psi$$

代入 E-L 方程

$$\frac{\partial L}{\partial\bar{\psi}} - \partial_\mu\frac{\partial L}{\partial\left(\partial_\mu\bar{\psi}\right)} = 0$$

可得：

$$-\frac{1}{2}\gamma_\alpha\partial_\alpha\psi - M\psi + \mathrm{i}G\gamma_5\psi\tilde{\phi} - \partial_\mu\left(\frac{1}{2}\gamma_\mu\psi\right) = 0$$

即：

$$\boxed{-\gamma_\mu\partial_\mu\psi - M\psi + \mathrm{i}G\gamma_5\psi\tilde{\phi} = 0}$$

能量密度

能量动量张量：

$$\begin{aligned}
T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_\nu\phi_A\right)}\partial_\mu\phi_A \\
&= L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_\nu\tilde{\phi}\right)}\partial_\mu\tilde{\phi} - \frac{\partial L}{\partial\left(\partial_\nu\psi\right)}\partial_\mu\psi - \partial_\mu\bar{\psi}\frac{\partial L}{\partial\left(\partial_\nu\bar{\psi}\right)} \\
&= \left[-\frac{1}{2}\partial_\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - \frac{1}{2}m^2\tilde{\phi}^2 - \frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\alpha\psi\right) - M\bar{\psi}\psi + \mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}\right]\delta_{\mu\nu} \\
&- \left(-\partial_\nu\tilde{\phi}\right)\partial_\mu\tilde{\phi} - \left(-\frac{1}{2}\bar{\psi}\gamma_\nu\right)\partial_\mu\psi - \partial_\mu\bar{\psi}\left(\frac{1}{2}\gamma_\nu\psi\right) \\
&= \left[-\frac{1}{2}\partial_\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} - \frac{1}{2}m^2\tilde{\phi}^2 - \frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\alpha\psi\right) - M\bar{\psi}\psi + \mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}\right]\delta_{\mu\nu} \\
&+ \partial_\nu\tilde{\phi}\partial_\mu\tilde{\phi} + \frac{1}{2}\bar{\psi}\gamma_\nu\partial_\mu\psi + \frac{1}{2}\partial_\mu\bar{\psi}\gamma_\nu\psi
\end{aligned}$$

能量密度：

$$\begin{aligned}
W &= -T_{44} \\
&= \left[\frac{1}{2}\partial_\alpha\tilde{\phi}\partial_\alpha\tilde{\phi} + \frac{1}{2}m^2\tilde{\phi}^2 + \frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\alpha\psi\right) + M\bar{\psi}\psi - \mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}\right] \\
&+ \partial_t\tilde{\phi}\partial_t\tilde{\phi} + \frac{\mathrm{i}}{2}\bar{\psi}\gamma_4\partial_t\psi + \frac{\mathrm{i}}{2}\partial_t\bar{\psi}\gamma_4\psi
\end{aligned}$$

电荷密度

$$L=-\frac{1}{2}\partial_\mu\tilde{\phi}\partial_\mu\tilde{\phi}-\frac{1}{2}m^2\tilde{\phi}^2-\frac{1}{2}\left(\bar{\psi}\gamma_\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma_\mu\psi\right)-M\bar{\psi}\psi+\mathrm{i}G\bar{\psi}\gamma_5\psi\tilde{\phi}$$

$$\frac{\partial L}{\partial\left(\partial_\mu\psi\right)}=-\frac{1}{2}\bar{\psi}\gamma_\mu$$

$$\frac{\partial L}{\partial\left(\partial_\mu\bar{\psi}\right)}=\frac{1}{2}\gamma_\mu\psi$$

由于赝标量场 $\tilde{\phi}^*=\tilde{\phi}$，因此赝标量场对电流密度矢量无贡献。

电流密度矢量：

$$\begin{aligned}
j_\mu &\equiv -\mathrm{ie}\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_A\right)}\phi_A-\phi_A^*\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_A^*\right)}\right]\\
&=-\mathrm{ie}\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\psi-\bar{\psi}\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\bar{\psi}\right)}\right]\\
&=-\mathrm{ie}\left[\left(-\frac{1}{2}\bar{\psi}\gamma_\mu\right)\psi-\bar{\psi}\left(\frac{1}{2}\gamma_\mu\psi\right)\right]\\
&=\mathrm{ie}\bar{\psi}\gamma_\mu\psi
\end{aligned}$$

电荷密度：

$$\begin{aligned}
\rho &=\frac{1}{\mathrm{i}}j_4\\
&=e\bar{\psi}\gamma_4\psi\\
&=e\psi^\dagger\psi
\end{aligned}$$

（2）

$$\begin{aligned}
L &=-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}-\bar{\psi}\left(\gamma_\mu\partial_\mu+m\right)\psi+\mathrm{ie}\bar{\psi}\gamma_\mu\psi A_\mu\\
&=-\frac{1}{2}\partial_\alpha A_\beta\partial_\alpha A_\beta+\frac{1}{2}\partial_\alpha A_\beta\partial_\beta A_\alpha-\bar{\psi}\left(\gamma_\mu\partial_\mu+m\right)\psi+\mathrm{ie}\bar{\psi}\gamma_\mu\psi A_\mu
\end{aligned}$$

A_μ 的运动方程

$$L=-\frac{1}{2}\partial_\alpha A_\beta\partial_\alpha A_\beta+\frac{1}{2}\partial_\alpha A_\beta\partial_\beta A_\alpha-\bar{\psi}\left(\gamma_\mu\partial_\mu+m\right)\psi+\mathrm{ie}\bar{\psi}\gamma_\mu\psi A_\mu$$

$$\frac{\partial L}{\partial A_\mu}=\mathrm{ie}\bar{\psi}\gamma_\mu\psi$$

$$\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)}=-\partial_\nu A_\mu+\partial_\mu A_\nu=F_{\mu\nu}$$

代入 E-L 方程

$$\frac{\partial L}{\partial A_\mu}-\partial_\nu\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)}=0$$

可得：

$$\mathrm{ie}\bar{\psi}\gamma_\mu\psi-\partial_\nu F_{\mu\nu}=0$$

即：

$$\boxed{\partial_\nu F_{\mu\nu} = \mathrm{i}e\bar{\psi}\gamma_\mu\psi}$$

$\bar{\psi}$ 的运动方程（对 ψ 变分）

$$L = -\frac{1}{2}\partial_\alpha A_\beta \partial_\alpha A_\beta + \frac{1}{2}\partial_\alpha A_\beta \partial_\beta A_\alpha - \bar{\psi}\left(\gamma_\mu\partial_\mu + m\right)\psi + \mathrm{i}e\bar{\psi}\gamma_\mu\psi A_\mu$$

$$\frac{\partial L}{\partial \psi} = -m\bar{\psi} + \mathrm{i}e\bar{\psi}\gamma_\mu A_\mu$$

$$\frac{\partial L}{\partial\left(\partial_\mu\psi\right)} = -\bar{\psi}\gamma_\mu$$

代入 E-L 方程

$$\frac{\partial L}{\partial \psi} - \partial_\mu \frac{\partial L}{\partial\left(\partial_\mu\psi\right)} = 0$$

可得:

$$-m\bar{\psi} + \mathrm{i}e\bar{\psi}\gamma_\mu A_\mu - \partial_\mu\left(-\bar{\psi}\gamma_\mu\right) = 0$$

即:

$$\boxed{\partial_\mu\bar{\psi}\gamma_\mu - m\bar{\psi} + \mathrm{i}e\bar{\psi}\gamma_\mu A_\mu = 0}$$

ψ 的运动方程（对 $\bar{\psi}$ 变分）

$$L = -\frac{1}{2}\partial_\alpha A_\beta \partial_\alpha A_\beta + \frac{1}{2}\partial_\alpha A_\beta \partial_\beta A_\alpha - \bar{\psi}\left(\gamma_\mu\partial_\mu + m\right)\psi + \mathrm{i}e\bar{\psi}\gamma_\mu\psi A_\mu$$

$$\frac{\partial L}{\partial \bar{\psi}} = -\gamma_\mu\partial_\mu\psi - m\psi + \mathrm{i}e\gamma_\mu\psi A_\mu$$

$$\frac{\partial L}{\partial\left(\partial_\mu\bar{\psi}\right)} = 0$$

代入 E-L 方程

$$\frac{\partial L}{\partial \bar{\psi}} - \partial_\mu \frac{\partial L}{\partial\left(\partial_\mu\bar{\psi}\right)} = 0$$

可得:

$$-\gamma_\mu\partial_\mu\psi - m\psi + \mathrm{i}e\gamma_\mu\psi A_\mu = 0$$

即:

$$\left(\gamma_\mu\partial_\mu + m\right)\psi = \mathrm{i}e\gamma_\mu\psi A_\mu$$

能量密度

$$L = -\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta} - \bar{\psi}\left(\gamma_\alpha\partial_\beta + m\right)\psi + \mathrm{i}e\bar{\psi}\gamma_\alpha\psi A_\alpha$$

$$\frac{\partial L}{\partial\left(\partial_\nu A_\alpha\right)} = -\partial_\nu A_\alpha + \partial_\mu A_\alpha = F_{\alpha\nu}$$

$$\frac{\partial L}{\partial\left(\partial_\nu\psi\right)} = -\bar{\psi}\gamma_\nu$$

$$\frac{\partial L}{\partial\left(\partial_\nu \bar{\psi}\right)}=0$$

能量动量张量:

$$\begin{aligned}
T_{\mu\nu} &= L\delta_{\mu\nu}-\frac{\partial L}{\partial\left(\partial_\nu \phi_A\right)}\partial_\mu \phi_A \\
&= L\delta_{\mu\nu}-\frac{\partial L}{\partial\left(\partial_\nu A_\alpha\right)}\partial_\mu A_\alpha-\frac{\partial L}{\partial\left(\partial_\nu \psi\right)}\partial_\mu \psi-\frac{\partial L}{\partial\left(\partial_\nu \bar{\psi}\right)}\partial_\mu \bar{\psi} \\
&= \left[-\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta}-\bar{\psi}\left(\gamma_\alpha\partial_\beta+m\right)\psi+\mathrm{i}e\bar{\psi}\gamma_\alpha\psi A_\alpha\right]\delta_{\mu\nu}-\left(-\partial_\nu A_\alpha+\partial_\mu A_\alpha\right)\partial_\mu A_\alpha-\left(-\bar{\psi}\gamma_\nu\right)\partial_\mu\psi \\
&= \left[-\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta}-\bar{\psi}\left(\gamma_\alpha\partial_\beta+m\right)\psi+\mathrm{i}e\bar{\psi}\gamma_\alpha\psi A_\alpha\right]\delta_{\mu\nu}+\left(\partial_\nu A_\alpha-\partial_\mu A_\alpha\right)\partial_\mu A_\alpha+\bar{\psi}\gamma_\nu\partial_\mu\psi
\end{aligned}$$

能量密度:

$$\begin{aligned}
W &= -T_{44} \\
&= \left[\frac{1}{4}F_{\alpha\beta}F_{\alpha\beta}+\bar{\psi}\left(\gamma_\alpha\partial_\beta+m\right)\psi-\mathrm{i}e\bar{\psi}\gamma_\alpha\psi A_\alpha\right]-\mathrm{i}\bar{\psi}\gamma_4\partial_t\psi
\end{aligned}$$

电荷密度

$$\frac{\partial L}{\partial\left(\partial_\mu \psi\right)}=-\bar{\psi}\gamma_\mu$$

$$\frac{\partial L}{\partial\left(\partial_\mu \bar{\psi}\right)}=0$$

电流密度矢量:

$$\begin{aligned}
j_\mu &\equiv -\mathrm{i}e\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_A\right)}\phi_A-\phi_A^*\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_A^*\right)}\right] \\
&= -\mathrm{i}e\left[\frac{\partial L}{\partial\left(\partial_\mu A_\alpha\right)}A_\alpha-A_\alpha^*\frac{\partial L}{\partial\left(\partial_\mu A_\alpha^*\right)}+\frac{\partial L}{\partial\left(\partial_\mu\psi\right)}\psi-\bar{\psi}\frac{\partial L}{\partial\left(\partial_\mu\bar{\psi}\right)}\right] \\
&= \mathrm{i}e\bar{\psi}\gamma_\mu\psi
\end{aligned}$$

电荷密度:

$$\begin{aligned}
\rho &= \frac{1}{\mathrm{i}}j_4 \\
&= e\bar{\psi}\gamma_4\psi \\
&= e\psi^\dagger\psi
\end{aligned}$$

4-1-7

> 利用 $\phi(x),\psi(x),A_\mu$ 的运动方程，分别验证 $\partial_\mu T_{\mu\nu}=0$

标量场 $\phi(x)$

标量场拉格朗日密度:

$$L=-\frac{1}{2}\partial_\alpha\phi\partial_\alpha\phi-\frac{1}{2}m^2\phi^2$$

$$\frac{\partial L}{\partial\left(\partial_\nu\phi\right)}=-\partial_\nu\phi$$

能量动量张量：

$$
\begin{aligned}
T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_\nu \phi_A\right)}\partial_\mu \phi_A \\
&= L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_\nu \phi\right)}\partial_\mu \phi \\
&= \left(-\frac{1}{2}\partial_\alpha\phi\partial_\alpha\phi - \frac{1}{2}m^2\phi^2\right)\delta_{\mu\nu} - \left(-\partial_\nu\phi\right)\partial_\mu\phi \\
&= \left(-\frac{1}{2}\partial_\alpha\phi\partial_\alpha\phi - \frac{1}{2}m^2\phi^2\right)\delta_{\mu\nu} + \partial_\nu\phi\partial_\mu\phi
\end{aligned}
$$

利用运动方程

$$
\left(\partial_\mu\partial_\mu - m^2\right)\phi = 0
$$

计算 $T_{\mu\nu}$ 的散度：

$$
\begin{aligned}
\partial_\nu T_{\mu\nu} &= \partial_\nu\left[\left(-\frac{1}{2}\partial_\alpha\phi\partial_\alpha\phi - \frac{1}{2}m^2\phi^2\right)\delta_{\mu\nu} + \partial_\nu\phi\partial_\mu\phi\right] \\
&= -\frac{1}{2}\partial_\mu\left(\partial_\alpha\phi\partial_\alpha\phi\right) - m^2\phi\partial_\mu\phi + \partial_\nu\left(\partial_\nu\phi\partial_\mu\phi\right) \\
&= -\frac{1}{2}\partial_\mu\left(\partial_\alpha\phi\partial_\alpha\phi\right) - \left(\partial_\alpha\partial_\alpha\phi\right)\partial_\mu\phi + \partial_\nu\left(\partial_\nu\phi\partial_\mu\phi\right) \\
&= -\frac{1}{2}\left(\partial_\mu\partial_\alpha\phi\right)\partial_\alpha\phi - \frac{1}{2}\partial_\alpha\phi\left(\partial_\mu\partial_\alpha\phi\right) - \left(\partial_\alpha\partial_\alpha\phi\right)\partial_\mu\phi + \left(\partial_\nu\partial_\nu\phi\right)\partial_\mu\phi + \partial_\nu\phi\left(\partial_\nu\partial_\mu\phi\right) \\
&= 0
\end{aligned}
$$

旋量场 $\psi(x)$

旋量场拉格朗日密度：

$$
L = -\frac{1}{2}\left(\bar{\psi}\gamma_\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_\mu\psi\right) - m\bar{\psi}\psi
$$

$$
\frac{\partial L}{\partial\left(\partial_\nu\psi\right)} = -\frac{1}{2}\bar{\psi}\gamma_\nu
$$

$$
\frac{\partial L}{\partial\left(\partial_\nu\bar{\psi}\right)} = \frac{1}{2}\gamma_\nu\psi
$$

能量动量张量：

$$
\begin{aligned}
T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_\nu \phi_A\right)}\partial_\mu \phi_A \\
&= L\delta_{\mu\nu} - \frac{\partial L}{\partial\left(\partial_\nu \psi\right)}\partial_\mu \psi - \partial_\mu\bar{\psi}\frac{\partial L}{\partial\left(\partial_\nu\bar{\psi}\right)} \\
&= \left[-\frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\alpha\psi\right) - m\bar{\psi}\psi\right]\delta_{\mu\nu} - \left(-\frac{1}{2}\bar{\psi}\gamma_\nu\right)\partial_\mu\psi - \partial_\mu\bar{\psi}\left(\frac{1}{2}\gamma_\nu\psi\right) \\
&= \left[-\frac{1}{2}\left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\alpha\psi\right) - m\bar{\psi}\psi\right]\delta_{\mu\nu} + \frac{1}{2}\bar{\psi}\gamma_\nu\partial_\mu\psi - \frac{1}{2}\partial_\mu\bar{\psi}\gamma_\nu\psi
\end{aligned}
$$

利用旋量场及共轭旋量场运动方程

$$
\gamma_\mu\partial_\mu\psi + m\psi = 0
$$

$$
\partial_\mu\bar{\psi}\gamma_\mu - m\bar{\psi} = 0
$$

可计算 $T_{\mu\nu}$ 的散度：

$$
\begin{aligned}
\partial_\nu T_{\mu\nu} &= \partial_\nu \left\{ \left[-\frac{1}{2} \left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\alpha\psi \right) - m\bar{\psi}\psi \right] \delta_{\mu\nu} + \frac{1}{2}\bar{\psi}\gamma_\nu\partial_\mu\psi - \frac{1}{2}\partial_\mu\bar{\psi}\gamma_\nu\psi \right\} \\
&= -\frac{1}{2}\partial_\mu \left(\bar{\psi}\gamma_\alpha\partial_\alpha\psi \right) + \frac{1}{2}\partial_\mu \left(\partial_\alpha\bar{\psi}\gamma_\alpha\psi \right) - m\partial_\mu \left(\bar{\psi}\psi \right) + \frac{1}{2}\partial_\nu \left(\bar{\psi}\gamma_\nu\partial_\mu\psi \right) - \frac{1}{2}\partial_\nu \left(\partial_\mu\bar{\psi}\gamma_\nu\psi \right) \\
&= \frac{m}{2}\partial_\mu \left(\bar{\psi}\psi \right) + \frac{m}{2}\partial_\mu \left(m\bar{\psi}\psi \right) - m\partial_\mu \left(\bar{\psi}\psi \right) + \frac{1}{2}\partial_\nu\bar{\psi}\gamma_\nu\partial_\mu\psi + \frac{1}{2}\bar{\psi}\gamma_\nu\partial_\nu\partial_\mu\psi - \frac{1}{2} \left(\partial_\nu\partial_\mu\bar{\psi} \right) \gamma_\nu\psi - \frac{1}{2}\partial_\mu\bar{\psi}\gamma_\nu\partial_\nu\psi \\
&= \frac{1}{2}\partial_\nu\bar{\psi}\gamma_\nu\partial_\mu\psi + \frac{1}{2}\bar{\psi}\partial_\mu \left(\gamma_\nu\partial_\nu\psi \right) - \frac{1}{2}\partial_\mu \left(\partial_\nu\bar{\psi}\gamma_\nu \right) \psi - \frac{1}{2}\partial_\mu\bar{\psi}\gamma_\nu\partial_\nu\psi \\
&= \frac{1}{2} \left(m\bar{\psi} \right) \partial_\mu\psi + \frac{1}{2}\bar{\psi}\partial_\mu \left(-m\psi \right) - \frac{1}{2}\partial_\mu \left(m\bar{\psi} \right) \psi - \frac{1}{2}\partial_\mu\bar{\psi} \left(-m\psi \right) \\
&= 0
\end{aligned}
$$

矢量场 $A_\mu(x)$

矢量场拉格朗日密度：

$$
L = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu\partial_\mu A_\nu
$$

$$
\frac{\partial L}{\partial \left(\partial_\nu A_\alpha \right)} = -\partial_\nu A_\alpha
$$

能量动量张量：

$$
\begin{aligned}
T_{\mu\nu} &= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_\nu \phi_A \right)}\partial_\mu\phi_A \\
&= L\delta_{\mu\nu} - \frac{\partial L}{\partial \left(\partial_\nu A_\alpha \right)}\partial_\mu A_\alpha \\
&= \left[-\frac{1}{2}\partial_\alpha A_\beta\partial_\alpha A_\beta \right] \delta_{\mu\nu} - \left(-\partial_\nu A_\alpha \right) \partial_\mu A_\alpha \\
&= \left[-\frac{1}{2}\partial_\alpha A_\beta\partial_\alpha A_\beta \right] \delta_{\mu\nu} + \partial_\nu A_\alpha\partial_\mu A_\alpha
\end{aligned}
$$

利用矢量场运动方程

$$
\partial_\alpha\partial_\alpha A_\mu = 0
$$

可计算 $T_{\mu\nu}$ 的散度：

$$
\begin{aligned}
\partial_\nu T_{\mu\nu} &= \partial_\nu \left\{ \left[-\frac{1}{2}\partial_\alpha A_\beta\partial_\alpha A_\beta \right] \delta_{\mu\nu} + \partial_\nu A_\alpha\partial_\mu A_\alpha \right\} \\
&= -\frac{1}{2}\partial_\mu \left(\partial_\alpha A_\beta\partial_\alpha A_\beta \right) + \partial_\nu \left(\partial_\nu A_\alpha\partial_\mu A_\alpha \right) \\
&= -\frac{1}{2} \left(\partial_\mu\partial_\alpha A_\beta \right) \partial_\alpha A_\beta - \frac{1}{2}\partial_\alpha A_\beta \left(\partial_\mu\partial_\alpha A_\beta \right) + \left(\partial_\nu\partial_\nu A_\alpha \right) \partial_\mu A_\alpha + \partial_\nu A_\alpha \left(\partial_\nu\partial_\mu A_\alpha \right) \\
&= -\partial_\alpha A_\beta \left(\partial_\mu\partial_\alpha A_\beta \right) + \partial_\nu A_\alpha \left(\partial_\mu\partial_\nu A_\alpha \right) \\
&= 0
\end{aligned}
$$