3-4

证明正反粒子单位旋量正交关系:

$$egin{align} ar{u}_i(ec{p})u_j(ec{p}) &= rac{m}{E}\delta_{ij} \ ar{v}_i(ec{p})v_j(ec{p}) &= -rac{m}{E}\delta_{ij} \ ar{u}_i(ec{p})v_j(ec{p}) &= 0 \ ar{v}_j(ec{p})u_j(ec{p}) &= 0 \ \end{split}$$

当 i=1,2,正反粒子单位旋量满足动量表象 Dirac 方程

$$\left(\mathrm{i}\hat{p}+m
ight)u_{i}(ec{p})=0,\quad \left(\mathrm{i}\hat{p}-m
ight)v_{i}(ec{p})=0$$

可写成:

$$(i\vec{\gamma} \cdot \vec{p} - \gamma_4 E + m) u_i(\vec{p}) = 0$$

 $(i\vec{\gamma} \cdot \vec{p} - \gamma_4 E - m) v_i(\vec{p}) = 0$

取厄米共轭:

$$egin{aligned} u_i^\dagger(ec{p}) \left(-\mathrm{i}ec{\gamma}\cdotec{p} - \gamma_4 E + m
ight) &= 0 \ v_i^\dagger(ec{p}) \left(-\mathrm{i}ec{\gamma}\cdotec{p} - \gamma_4 E - m
ight) &= 0 \end{aligned}$$

因此:

$$u_i^{\dagger}(\vec{p}) \left(i \vec{\gamma} \cdot \vec{p} - \gamma_4 E + m \right) u_i(\vec{p}) = 0 \tag{1}$$

$$v_i^{\dagger}(\vec{p}) \left(i\vec{\gamma} \cdot \vec{p} - \gamma_4 E - m \right) v_i(\vec{p}) = 0 \tag{2}$$

$$u_i^{\dagger}(\vec{p}) \left(-i\vec{\gamma} \cdot \vec{p} - \gamma_4 E + m \right) u_j(\vec{p}) = 0 \tag{3}$$

$$v_i^{\dagger}(\vec{p}) \left(-i\vec{\gamma} \cdot \vec{p} - \gamma_4 E - m \right) v_i(\vec{p}) = 0 \tag{4}$$

(1),(3) 式相加, (2),(4) 式相加, 得:

$$egin{aligned} u_i^\dagger(ec{p}) \left(-2E\gamma_4 + 2m
ight) u_j(ec{p}) &= 0 \ v_i^\dagger(ec{p}) \left(-2E\gamma_4 - 2m
ight) v_j(ec{p}) &= 0 \end{aligned}$$

即:

$$egin{aligned} u_i^\dagger(ec{p})\gamma_4 u_j(ec{p}) &= rac{m}{E} u_i^\dagger(ec{p}) u_j(ec{p}) \ v_i^\dagger(ec{p})\gamma_4 v_j(ec{p}) &= -rac{m}{E} v_i^\dagger(ec{p}) v_j(ec{p}) \end{aligned}$$

利用正交性和定义

$$egin{aligned} ar{u}_i(ec{p}) &\equiv u_i^\dagger(ec{p}) \gamma_4, \quad ar{v}_i(ec{p}) \equiv v_i^\dagger(ec{p}) \gamma_4 \ & \ u_i^\dagger(ec{p}) u_j(ec{p}) = \delta_{ij}, \quad v_i^\dagger(ec{p}) v_j(ec{p}) = \delta_{ij} \end{aligned}$$

得到:

$$egin{aligned} ar{u}_i(ec{p})u_j(ec{p}) &= rac{m}{E}\delta_{ij} \ ar{v}_i(ec{p})v_j(ec{p}) &= -rac{m}{E}\delta_{ij} \end{aligned}$$