3-1

3-1-1

> 由 $\gamma_i=-\mathrm{i}\beta\alpha_i \Longrightarrow \alpha_i=\mathrm{i}\beta\gamma_i$

α_i,β 满足:

$$\alpha_i\alpha_j+\alpha_j\alpha_i=2\delta_{ij}I,\quad i,j=1,2,3$$

$$\alpha_i\beta+\beta\alpha_i=0$$

$$\beta^2=I$$

考虑:

$$\gamma_i=-\mathrm{i}\beta\alpha_i$$

上式左乘 β 得:

$$\beta\gamma_i=-\mathrm{i}\beta^2\alpha_i=-\mathrm{i}\alpha_i$$

因此:

$$\alpha_i=\frac{\beta\gamma_i}{-\mathrm{i}}=\mathrm{i}\beta\gamma_i$$

3-1-2

> 利用 Pauli 矩阵，写出 Dirac-Pauli 表象中 γ_μ 矩阵以及 $\gamma_5, C=\mathrm{i}\gamma_2\gamma_4$ 的具体形式。

二阶 Pauli 矩阵:

$$\sigma_1^0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \sigma_2^0=\begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix},\quad \sigma_3^0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\alpha_i=\begin{bmatrix}0 & \sigma_i^0\\ \sigma_i^0 & 0\end{bmatrix},\quad \beta=\begin{bmatrix}I^0 & 0\\ 0 & -I^0\end{bmatrix}$$

$$\gamma_i\equiv-\mathrm{i}\beta\alpha_i,\quad \gamma_4\equiv\beta,\quad \gamma_5\equiv\gamma_1\gamma_2\gamma_3\gamma_4$$

$$\gamma_i=\begin{bmatrix}0 & -\mathrm{i}\sigma_i^0\\ \mathrm{i}\sigma_i^0 & 0\end{bmatrix},\quad \gamma_4=\begin{bmatrix}I^0 & 0\\ 0 & -I^0\end{bmatrix}$$

$$C = \mathrm{i}\gamma_2\gamma_4$$

具体形式为：

$$\gamma_1 = \begin{bmatrix} 0 & -\mathrm{i}\sigma_1^0 \\ \mathrm{i}\sigma_1^0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_2 = \begin{bmatrix} 0 & -\mathrm{i}\sigma_2^0 \\ \mathrm{i}\sigma_2^0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma_3 = \begin{bmatrix} 0 & -\mathrm{i}\sigma_3^0 \\ \mathrm{i}\sigma_3^0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{bmatrix}$$

$$\gamma_4 = \begin{bmatrix} I^0 & 0 \\ 0 & -I^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$C = \mathrm{i}\gamma_2\gamma_4 = \begin{bmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{bmatrix}$$

3-1-3

> 定义
>
> $$\sigma_i = \begin{bmatrix} \sigma_i^0 & 0 \\ 0 & \sigma_i^0 \end{bmatrix} = \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\gamma_j\gamma_k$$
>
> 其中，σ_i^0 具有以下性质：
>
> $$\sigma_i^0\sigma_j^0 + \sigma_j^0\sigma_i^0 = 2\delta_{ij}I^0$$

> $$\sigma_i^0\sigma_j^0=\delta_{ij}I^0+\mathrm{i}\varepsilon_{ijk}\sigma_k^0$$
>
> 证明:
>
> $$\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}I$$
>
> $$\gamma_i\gamma_j=\sigma_i\sigma_j=\delta_{ij}I+\mathrm{i}\varepsilon_{ijk}\sigma_k$$
>
> $$\sigma_1\sigma_2=\mathrm{i}\sigma_3$$
>
> $$\sigma_2\sigma_3=\mathrm{i}\sigma_1$$
>
> $$\sigma_3\sigma_1=\mathrm{i}\sigma_2$$
>
> $$\vec{\sigma}=-\vec{\alpha}\gamma_5$$

证明 $\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}I$

$$\begin{aligned}\sigma_i\sigma_j+\sigma_j\sigma_i&=\begin{bmatrix}\sigma_i^0&0\\0&\sigma_i^0\end{bmatrix}\begin{bmatrix}\sigma_j^0&0\\0&\sigma_j^0\end{bmatrix}+\begin{bmatrix}\sigma_j^0&0\\0&\sigma_j^0\end{bmatrix}\begin{bmatrix}\sigma_i^0&0\\0&\sigma_i^0\end{bmatrix}\\&=\begin{bmatrix}\sigma_i^0\sigma_j^0+\sigma_j^0\sigma_i^0&0\\0&\sigma_i^0\sigma_j^0+\sigma_j^0\sigma_i^0\end{bmatrix}\\&=\begin{bmatrix}2\delta_{ij}I^0&0\\0&2\delta_{ij}I^0\end{bmatrix}\\&=2\delta_{ij}I\end{aligned}$$

证明 $\gamma_i\gamma_j=\sigma_i\sigma_j=\delta_{ij}I+\mathrm{i}\varepsilon_{ijk}\sigma_k$

二阶泡利矩阵满足:

$$\sigma_i^0\sigma_j^0=\delta_{ij}I^0+\mathrm{i}\varepsilon_{ijk}\sigma_k^0$$

四阶泡利矩阵:

$$\begin{aligned}
\sigma_i\sigma_j &= \begin{bmatrix}\sigma_i^0 & 0\\ 0 & \sigma_i^0\end{bmatrix}\begin{bmatrix}\sigma_j^0 & 0\\ 0 & \sigma_j^0\end{bmatrix}\\
&= \begin{bmatrix}\sigma_i^0\sigma_j^0 & 0\\ 0 & \sigma_i^0\sigma_j^0\end{bmatrix}\\
&= \begin{bmatrix}\delta_{ij}I^0+\mathrm{i}\varepsilon_{ijk}\sigma_k^0 & 0\\ 0 & \delta_{ij}I^0+\mathrm{i}\varepsilon_{ijk}\sigma_k^0\end{bmatrix}\\
&= \delta_{ij}\begin{bmatrix}I^0 & 0\\ 0 & I^0\end{bmatrix}+\mathrm{i}\varepsilon_{ijk}\begin{bmatrix}\sigma_k^0 & 0\\ 0 & \sigma_k^0\end{bmatrix}\\
&= \delta_{ij}I+\mathrm{i}\varepsilon_{ijk}\sigma_k
\end{aligned}$$

$$\sigma_i=\frac{1}{2\mathrm{i}}\varepsilon_{ijk}\gamma_j\gamma_k$$

同乘 ε_{ilm} 并对 i 求和:

$$\begin{aligned}
\varepsilon_{ilm}\sigma_i &= \frac{1}{2\mathrm{i}}\varepsilon_{ilm}\varepsilon_{ijk}\gamma_j\gamma_k\\
&= \frac{1}{2\mathrm{i}}\left(\delta_{lj}\delta_{mk}-\delta_{lk}\delta_{mj}\right)\gamma_j\gamma_k\\
&= \frac{1}{2\mathrm{i}}\left(\gamma_l\gamma_m-\gamma_m\gamma_l\right)
\end{aligned}$$

即:

$$\gamma_l\gamma_m-\gamma_m\gamma_l=2\mathrm{i}\varepsilon_{ilm}\sigma_i$$

另一方面，利用 γ_μ 矩阵反对易关系

$$\gamma_l\gamma_m+\gamma_m\gamma_l=2\delta_{lm}I$$

两式相加可得:

$$\gamma_l\gamma_m=\delta_{lm}I+\mathrm{i}\varepsilon_{ilm}\sigma_i=\delta_{lm}I+\mathrm{i}\varepsilon_{lmi}\sigma_i$$

替换哑标得:

$$\gamma_i\gamma_j=\delta_{ij}I+\mathrm{i}\varepsilon_{ijk}\sigma_k$$

$$\gamma_i\gamma_j=\sigma_i\sigma_j=\delta_{ij}I+\mathrm{i}\varepsilon_{ijk}\sigma_k$$

证明 $\sigma_1\sigma_2=\mathrm{i}\sigma_3,\sigma_2\sigma_3=\mathrm{i}\sigma_1,\sigma_3\sigma_1=\mathrm{i}\sigma_2$

由于 $\sigma_i\sigma_j=\delta_{ij}I+\mathrm{i}\varepsilon_{ijk}\sigma_k$，因此:

$$\sigma_1\sigma_2=\mathrm{i}\varepsilon_{12k}\sigma_k=\mathrm{i}\sigma_3$$

$$\sigma_2\sigma_3=\mathrm{i}\varepsilon_{23k}\sigma_k=\mathrm{i}\sigma_1$$

$$\sigma_3\sigma_1=\mathrm{i}\varepsilon_{31k}\sigma_k=\mathrm{i}\sigma_2$$

证明 $\vec{\sigma}=-\vec{\alpha}\gamma_5$

$$\alpha_i=\begin{bmatrix}0 & \sigma_i^0\\ \sigma_i^0 & 0\end{bmatrix},\quad \beta=\begin{bmatrix}I^0 & 0\\ 0 & -I^0\end{bmatrix}$$

$$\gamma_i=-\mathrm{i}\beta\alpha_i=-\mathrm{i}\begin{bmatrix}I^0 & 0\\ 0 & -I^0\end{bmatrix}\begin{bmatrix}0 & \sigma_i^0\\ \sigma_i^0 & 0\end{bmatrix}=\begin{bmatrix}0 & -\mathrm{i}\sigma_i^0\\ \mathrm{i}\sigma_i^0 & 0\end{bmatrix}$$

$$\gamma_4=\beta=\begin{bmatrix}I^0 & 0\\ 0 & -I^0\end{bmatrix}$$

于是:

$$\begin{aligned}
-\vec{\alpha}\gamma_5&=-\alpha_i\vec{\mathrm{e}}_i\gamma_1\gamma_2\gamma_3\gamma_4\\
&=-\alpha_i\vec{\mathrm{e}}_i\begin{bmatrix}0 & -\mathrm{i}\sigma_1^0\\ \mathrm{i}\sigma_1^0 & 0\end{bmatrix}\begin{bmatrix}0 & -\mathrm{i}\sigma_2^0\\ \mathrm{i}\sigma_2^0 & 0\end{bmatrix}\begin{bmatrix}0 & -\mathrm{i}\sigma_3^0\\ \mathrm{i}\sigma_3^0 & 0\end{bmatrix}\begin{bmatrix}I^0 & 0\\ 0 & -I^0\end{bmatrix}\\
&=-\alpha_i\vec{\mathrm{e}}_i\begin{bmatrix}0 & \mathrm{i}\sigma_1^0\sigma_2^0\sigma_3^0\\ \mathrm{i}\sigma_1^0\sigma_2^0\sigma_3^0 & 0\end{bmatrix}\\
&=-\alpha_i\vec{\mathrm{e}}_i\begin{bmatrix}0 & -I^0\\ -I^0 & 0\end{bmatrix}\\
&=-\begin{bmatrix}0 & \sigma_i^0\\ \sigma_i^0 & 0\end{bmatrix}\begin{bmatrix}0 & -I^0\\ -I^0 & 0\end{bmatrix}\vec{\mathrm{e}}_i\\
&=\begin{bmatrix}\sigma_i^0 & 0\\ 0 & \sigma_i^0\end{bmatrix}\vec{\mathrm{e}}_i\\
&=\sigma_i\vec{\mathrm{e}}_i\\
&=\vec{\sigma}
\end{aligned}$$

3-1-4

> 已知 $H=\vec{\alpha}\cdot\vec{p}+\beta m,\vec{L}=\vec{r}\times\vec{p}$，证明：$\left[H,\vec{L}\right]=-\mathrm{i}\vec{\alpha}\times\vec{p},[H,\vec{\sigma}]=2\mathrm{i}\vec{\alpha}\times\vec{p}$

$$L_i=(\vec{r}\times\vec{p})_i=\varepsilon_{ijk}x_jp_k$$

$$
\begin{aligned}
[H, L_i] &= [\alpha_l p_l + \beta m, \varepsilon_{ijk} x_j p_k] \\
&= \varepsilon_{ijk} \alpha_l \left[p_l, x_j p_k\right] \\
&= \varepsilon_{ijk} \alpha_l \left(x_j \left[p_l, p_k\right] + \left[p_l, x_j\right] p_k\right) \\
&= \varepsilon_{ijk} \alpha_l \left(-\mathrm{i}\hbar \delta_{lj}\right) p_k \\
&= -\mathrm{i}\hbar \varepsilon_{ijk} \alpha_j p_k \\
&= -\mathrm{i}\hbar \left(\vec{\alpha} \times \vec{p}\right)_i \\
&= -\mathrm{i} \left(\vec{\alpha} \times \vec{p}\right)_i
\end{aligned}
$$

因此:

$$
\left[H, \vec{L}\right] = -\mathrm{i} \vec{\alpha} \times \vec{p}
$$

$$
\begin{aligned}
[\alpha_i, \alpha_j] &= \alpha_i \alpha_j - \alpha_j \alpha_i \\
&= \begin{bmatrix} 0 & \sigma_i^0 \\ \sigma_i^0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_j^0 \\ \sigma_j^0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma_j^0 \\ \sigma_j^0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_i^0 \\ \sigma_i^0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_i^0 \sigma_j^0 - \sigma_j^0 \sigma_i^0 & 0 \\ 0 & \sigma_i^0 \sigma_j^0 - \sigma_j^0 \sigma_i^0 \end{bmatrix} \\
&= 2\mathrm{i} \varepsilon_{ijk} \sigma_k
\end{aligned}
$$

$$
\begin{aligned}
[H,\sigma_i] &= \left[\alpha_l p_l+\beta m,\frac{1}{2\mathrm{i}}\varepsilon_{ijk}\gamma_j\gamma_k\right]\\
&= \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left[\alpha_l p_l+\beta m,\gamma_j\gamma_k\right]\\
&= \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left[\alpha_l p_l+\beta m,\left(-\mathrm{i}\beta\alpha_j\right)\left(-\mathrm{i}\beta\alpha_k\right)\right]\\
&= -\frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left[\alpha_l p_l+\beta m,\beta\alpha_j\beta\alpha_k\right]\\
&= \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left[\alpha_l p_l+\beta m,\alpha_j\beta\beta\alpha_k\right]\\
&= \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left[\alpha_l p_l+\beta m,\alpha_j\alpha_k\right]\\
&= \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left(\left[\alpha_l p_l,\alpha_j\alpha_k\right]+m\left[\beta,\alpha_j\alpha_k\right]\right)\\
&= \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left\{\alpha_l\left[p_l,\alpha_j\alpha_k\right]+\left[\alpha_l,\alpha_j\alpha_k\right]p_l+m\left(\beta\alpha_j\alpha_k-\alpha_j\alpha_k\beta\right)\right\}\\
&= \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left\{\left(\alpha_j\left[\alpha_l,\alpha_k\right]+\left[\alpha_l,\alpha_j\right]\alpha_k\right)p_l+m\beta\alpha_j\alpha_k-m\beta\alpha_j\alpha_k\right\}\\
&= \frac{1}{2\mathrm{i}}\varepsilon_{ijk}\left(2\mathrm{i}\varepsilon_{lkn}\alpha_j\alpha_n+2\mathrm{i}\varepsilon_{ljq}\alpha_q\alpha_k\right)p_l\\
&= \left(\varepsilon_{kij}\varepsilon_{knl}\alpha_j\alpha_n+\varepsilon_{jki}\varepsilon_{jql}\alpha_q\alpha_k\right)p_l\\
&= \left\{\left(\delta_{in}\delta_{jl}-\delta_{il}\delta_{jn}\right)\alpha_j\alpha_n+\left(\delta_{kq}\delta_{il}-\delta_{kl}\delta_{iq}\right)\alpha_q\alpha_k\right\}p_l\\
&= \left(\alpha_l\alpha_i-\delta_{il}\alpha_n\alpha_n+\delta_{il}\alpha_k\alpha_k-\alpha_i\alpha_l\right)p_l\\
&= \left[\alpha_l,\alpha_i\right]p_l\\
&= 2\mathrm{i}\varepsilon_{lit}\alpha_t p_l\\
&= 2\mathrm{i}\varepsilon_{itl}\alpha_t p_l\\
&= 2\mathrm{i}\left(\vec{\alpha}\times\vec{p}\right)_i
\end{aligned}
$$

因此:

$$
[H,\vec{\sigma}]=2\mathrm{i}\vec{\alpha}\times\vec{p}
$$

3-1-5

> 证明 $\vec{\sigma}$ 的本征值为 $+1,-1.$

利用公式

$$
\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right)=\vec{A}\cdot\vec{B}+\mathrm{i}\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right)
$$

有:

$$(\vec{\sigma}\cdot\vec{n})^2=\vec{n}\cdot\vec{n}+\mathrm{i}\vec{\sigma}\cdot(\vec{n}\times\vec{n})=1$$

设 $(\vec{\sigma}\cdot\vec{n})$ 的本征方程为：

$$(\vec{\sigma}\cdot\vec{n})\,u(p)=\lambda u(p)$$

则：

$$(\vec{\sigma}\cdot\vec{n})^2\,u(p)=\lambda^2 u(p)$$

又

$$(\vec{\sigma}\cdot\vec{n})^2\,u(p)=u(p)$$

对比可得 $(\vec{\sigma}\cdot\vec{n})$ 的本征值为：

$$\lambda=\pm 1$$

3-1-6

> 证明有电磁场存在时，Dirac 方程是 Lorentz 协变的。

电磁场存在时 x' 系的 Dirac 方程为：

$$\left(\gamma_\mu\partial'_\mu-\mathrm{i}eA'_\mu\gamma_\mu+m\right)\psi'(x')=0$$

时空坐标进行 Lorentz 变换：

$$x_\mu\to x'_\mu=A_{\mu\nu}x_\nu,\quad A_{\mu\lambda}x'_\mu=x_\lambda$$

x' 系的物理量用 x 系的物理量表达：

$$\partial'_\mu\equiv\frac{\partial}{\partial x'_\mu}=\frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu}=A_{\mu\nu}\partial_\nu$$

$$\psi'(x')=\Lambda\psi(x)$$

$$A'_\mu=A_{\mu\nu}A_\nu$$

则 Dirac 方程化为：

$$\begin{aligned}0&=\left(\gamma_\mu\partial'_\mu-\mathrm{i}eA'_\mu\gamma_\mu+m\right)\psi'(x')\\&=\left(\gamma_\mu A_{\mu\nu}\partial_\nu-\mathrm{i}eA_{\mu\nu}A_\nu\gamma_\mu+m\right)\Lambda\psi(x)\end{aligned}$$

左乘 Λ^{-1}，并利用

$$\Lambda^{-1}\gamma_\mu\Lambda = A_{\mu\rho}\gamma_\rho$$

可得:

$$\begin{aligned}0 &= \Lambda^{-1}\left(\gamma_\mu A_{\mu\nu}\partial_\nu - \mathrm{i}eA_{\mu\nu}A_\nu\gamma_\mu + m\right)\Lambda\psi(x)\\ &= \left(A_{\mu\rho}\gamma_\rho A_{\mu\nu}\partial_\nu - \mathrm{i}eA_{\mu\nu}A_\nu A_{\mu\rho}\gamma_\rho + m\right)\psi(x)\\ &= \left(\delta_{\rho\nu}\gamma_\rho\partial_\nu - \mathrm{i}e\delta_{\nu\rho}A_\nu\gamma_\rho + m\right)\psi(x)\\ &= \left(\gamma_\nu\partial_\nu - \mathrm{i}eA_\rho\gamma_\rho + m\right)\psi(x)\end{aligned}$$

因此，x 系中的 Dirac 方程为:

$$\left(\gamma_\mu\partial_\mu - \mathrm{i}eA_\mu\gamma_\mu + m\right)\psi(x) = 0$$

与 x' 系中的 Dirac 方程

$$\left(\gamma_\mu\partial'_\mu - \mathrm{i}eA'_\mu\gamma_\mu + m\right)\psi'(x') = 0$$

对比可知，电磁场存在时 Dirac 方程具有 Lorentz 协变性。

3-1-7

> 证明 $\bar{\psi}\gamma_\mu\gamma_5\psi$ 是 Lorentz 赝矢量。

$$\begin{aligned}\bar{\psi}'\gamma_\mu\gamma_5\psi' &= k\bar{\psi}\Lambda^{-1}\gamma_\mu\gamma_5\Lambda\psi\\ &= \bar{\psi}\Lambda^{-1}\gamma_\mu\gamma_5\Lambda\psi\\ &= \bar{\psi}\Lambda^{-1}\gamma_\mu\Lambda\Lambda^{-1}\gamma_5\Lambda\psi\\ &= \bar{\psi}A_{\mu\nu}\gamma_\nu\left|A\right|\gamma_5\psi\\ &= \left|A\right|A_{\mu\nu}\bar{\psi}\gamma_\nu\gamma_5\psi\end{aligned}$$

即 $\bar{\psi}\gamma_\mu\gamma_5\psi$ 服从赝矢量的变换规律，因此 $\bar{\psi}\gamma_\mu\gamma_5\psi$ 是赝矢量。

3-1-8

> 利用 $u_a(\vec{p})$ 的正交完备性，证明 $u_a(\vec{p}')$ 的正交完备性（空间反射变换）。

$u_a(\vec{p})$ 的正交完备性给出:

$$u_a^\dagger(\vec{p})u_b(\vec{p}) = \delta_{ab}$$

$$u_a(\vec{p})u_a^\dagger(\vec{p}) = I$$

$u_a(-\vec{p})$ 的正交性:

$$
\begin{aligned}
u_a^\dagger(-\vec{p})u_b(-\vec{p}) &= \left[\eta_P\gamma_4 u_a(\vec{p})\right]^\dagger \eta_P\gamma_4 u_b(\vec{p}) \\
&= \eta_P^\dagger\eta_P u_a^\dagger(\vec{p})\gamma_4^\dagger\gamma_4 u_b(\vec{p}) \\
&= u_a^\dagger(\vec{p})u_b(\vec{p}) \\
&= \delta_{ab}
\end{aligned}
$$

$u_a(-\vec{p})$ 的完备性:

$$
\begin{aligned}
u_a(-\vec{p})u_a^\dagger(-\vec{p}) &= \eta_P\gamma_4 u_a(\vec{p}) \left[\eta_P\gamma_4 u_a(\vec{p})\right]^\dagger \\
&= \eta_P\eta_P^\dagger\gamma_4 u_a(\vec{p})u_a^\dagger(\vec{p})\gamma_4^\dagger \\
&= \gamma_4 I\gamma_4^\dagger \\
&= I
\end{aligned}
$$