基本概念与公式

黎曼几何基本物理量

张量

标量

$$\phi'(x') = \phi(x) \tag{1}$$

逆变矢量

$$\phi'^{\mu}(x') = A^{\mu}_{\nu}\phi^{\nu}(x) \tag{2}$$

协变矢量

$$\phi'_{\mu}(x') = \bar{A}^{\nu}_{\mu}\phi_{\nu}(x) \tag{3}$$

张量

$$\phi'^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}(x') = A^{\mu_1}_{\alpha_1}\cdots A^{\mu_n}_{\alpha_n}\bar{A}^{\beta_1}_{\nu_1}\cdots\bar{A}^{\beta_m}_{\nu_m}\phi^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_m}(x) \tag{4}$$

协变微商

逆变矢量的协变微商

$$\nabla_{\mu}\phi^{\nu}(x) \equiv \partial_{\mu}\phi^{\nu}(x) + \Gamma^{\nu}_{\mu\lambda}\phi^{\lambda}(x) \tag{5}$$

协变矢量的协变微商

$$\nabla_{\mu}\phi_{\nu}(x) = \partial_{\mu}\phi_{\nu}(x) - \Gamma^{\lambda}_{\mu\nu}\phi_{\lambda}(x) \tag{6}$$

二阶张量的协变微商

二阶张量的协变微商

二阶张量的协变微商的具体形式分别为

$$\nabla_{\mu}\phi^{\nu\lambda} = \partial_{\mu}\phi^{\nu\lambda} + \Gamma^{\nu}_{\mu\rho}\phi^{\rho\lambda} + \Gamma^{\lambda}_{\mu\rho}\phi^{\nu\rho} \tag{7}$$

$$\nabla_{\mu}\phi_{\nu\lambda} = \partial_{\mu}\phi_{\nu\lambda} - \Gamma^{\rho}_{\mu\nu}\phi_{\rho\lambda} - \Gamma^{\rho}_{\mu\lambda}\phi_{\nu\rho} \tag{8}$$

$$\nabla_\mu \phi^\nu_\lambda = \partial_\mu \phi^\nu_\lambda + \Gamma^\nu_{\mu\rho}\phi^\rho_\lambda - \Gamma^\rho_{\mu\lambda}\phi^\nu_\rho \tag{9}$$

联络在坐标变换下的变换规律

联络的变换规律

$$\Gamma'^\mu_{\nu\lambda} = A^\mu_\alpha \bar{A}^\beta_\nu \bar{A}^\gamma_\lambda \Gamma^\alpha_{\beta\gamma} + A^\mu_\alpha \bar{A}^\beta_\nu \partial_\beta \bar{A}^\alpha_\lambda \tag{10}$$

曲率张量

$$R^\lambda_{\alpha\mu\nu} \equiv \partial_\mu \Gamma^\lambda_{\nu\alpha} - \partial_\nu \Gamma^\lambda_{\mu\alpha} + \Gamma^\lambda_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\lambda_{\nu\beta}\Gamma^\beta_{\mu\alpha} \tag{11}$$

挠率张量

$$T^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} \tag{12}$$

黎曼联络（克氏符）

由 $\nabla_\lambda g_{\mu\nu} = 0$ 和 $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ 求得的联络 $\Gamma^\lambda_{\mu\nu}$ 称为黎曼联络，也称为克氏符。

可以证明，克氏符的具体表达式为：

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right) \tag{13}$$

黎曼曲率张量

当曲率张量

$$R^\lambda{}_{\sigma\mu\nu} \tag{14}$$

中的联络为黎曼联络（克氏符）时，其称为黎曼曲率张量。

$$R^\lambda{}_{\sigma\mu\nu} = -R^\lambda{}_{\sigma\nu\mu} \tag{15}$$

全部协变指标的黎曼曲率张量：

$$R_{\tau\sigma\mu\nu} \equiv g_{\tau\lambda} R^\lambda{}_{\sigma\mu\nu} \tag{16}$$

其满足：

$$R_{\lambda\sigma\mu\nu} = R_{\mu\nu\lambda\sigma} \tag{17}$$

$$R_{\lambda\sigma\mu\nu} = -R_{\lambda\sigma\nu\mu} \tag{18}$$

$$R_{\lambda\sigma\mu\nu} = -R_{\sigma\lambda\mu\nu} \tag{19}$$

$$R_{\lambda\sigma\mu\nu} + R_{\lambda\mu\nu\sigma} + R_{\lambda\nu\sigma\mu} = 0 \tag{20}$$

比安基恒等式

$$\nabla_\lambda R^\rho{}_{\sigma\mu\nu} + \nabla_\mu R^\rho{}_{\sigma\nu\lambda} + \nabla_\nu R^\rho{}_{\sigma\lambda\mu} = 0 \tag{21}$$

里奇张量

$$R_{\sigma\nu} \equiv R^\lambda{}_{\sigma\lambda\nu} \tag{22}$$

标曲率

$$R \equiv g^{\mu\nu} R_{\mu\nu} \tag{23}$$

爱因斯坦张量

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R \tag{24}$$

可以证明，混合指标的爱因斯坦张量的协变散度为零：

$$\nabla_\mu G^\mu{}_\nu = 0 \tag{25}$$

也可以证明：

$$\nabla^\mu G_{\mu\nu} = 0 \tag{26}$$

$$\nabla_\mu G^{\mu\nu} = 0 \tag{27}$$

证明：

比安基恒等式：

$$\nabla_\lambda R^\rho{}_{\sigma\mu\nu} + \nabla_\mu R^\rho{}_{\sigma\nu\lambda} + \nabla_\nu R^\rho{}_{\sigma\lambda\mu} = 0 \tag{28}$$

令 $\rho = \mu$ 得到

$$\nabla_\lambda R^\mu{}_{\sigma\mu\nu} + \nabla_\mu R^\mu{}_{\sigma\nu\lambda} + \nabla_\nu R^\mu{}_{\sigma\lambda\mu} = 0 \tag{29}$$

由 $R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu}$ 可知第一项化为

$$\nabla_\lambda R^\mu{}_{\sigma\mu\nu} = \nabla_\lambda R_{\sigma\nu} \tag{30}$$

由 $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$ 知第三项化为

$$\nabla_\nu R^\mu{}_{\sigma\lambda\mu} = -\nabla_\nu R^\mu{}_{\sigma\mu\lambda} = -\nabla_\nu R_{\sigma\lambda} \tag{31}$$

于是有

$$\nabla_\nu R_{\sigma\lambda} = \nabla_\lambda R_{\sigma\nu} + \nabla_\mu R^\mu{}_{\sigma\nu\lambda} \tag{32}$$

两边同乘 $g^{\sigma\lambda}$，利用 $\nabla_\nu g^{\sigma\lambda} = 0$ 、协变微分的莱布尼兹律以及 $R \equiv g^{\mu\nu} R_{\mu\nu}, R^\mu{}_\nu \equiv g^{\mu\rho} R_{\rho\nu}$ 有

$$\nabla_\nu R = \nabla_\lambda R^\lambda{}_\nu + \nabla_\mu \left(g^{\sigma\lambda} R^\mu{}_{\sigma\nu\lambda}\right) \tag{33}$$

上式右边第二项，利用 $R_{\mu\nu\lambda\rho}=-R_{\nu\mu\lambda\rho}, R^{\lambda}{}_{\sigma\mu\nu}=-R^{\lambda}{}_{\sigma\nu\mu}$ 有

$$\begin{aligned}g^{\sigma\lambda}R^{\mu}{}_{\sigma\nu\lambda}&=g^{\mu\rho}g^{\sigma\lambda}R_{\rho\sigma\nu\lambda}=g^{\mu\rho}g^{\sigma\lambda}R_{\nu\lambda\rho\sigma}\\&=-g^{\mu\rho}g^{\sigma\lambda}R_{\lambda\nu\rho\sigma}=-g^{\mu\rho}R^{\sigma}{}_{\nu\rho\sigma}=g^{\mu\rho}R^{\sigma}{}_{\nu\sigma\rho}\\&=g^{\mu\rho}R_{\nu\rho}=R^{\mu}{}_{\nu}\end{aligned}\tag{34}$$

于是得到

$$\nabla_{\nu}R=\nabla_{\lambda}R^{\lambda}{}_{\nu}+\nabla_{\mu}R^{\mu}{}_{\nu}\tag{35}$$

也即

$$\nabla_{\nu}R=2\nabla_{\mu}R^{\mu}{}_{\nu},\quad \nabla_{\mu}R^{\mu}{}_{\nu}=\frac{1}{2}\nabla_{\nu}R=\frac{1}{2}\delta^{\mu}{}_{\nu}\nabla_{\mu}R\tag{36}$$

即

$$\nabla_{\mu}\left(R^{\mu}{}_{\nu}-\frac{1}{2}\delta^{\mu}{}_{\nu}R\right)=0\tag{37}$$

两边同乘 $g^{\rho\nu}$ 得

$$\nabla_{\mu}\left(R^{\mu\rho}-\frac{1}{2}g^{\rho\mu}R\right)=0\tag{38}$$

又 $R^{\mu\nu}=R^{\nu\mu}$ 于是

$$\nabla_{\mu}\left(R^{\rho\mu}-\frac{1}{2}g^{\rho\mu}R\right)=0\tag{39}$$

解题必背物理量

引力场作用量、拉式密度以及变分

$$I_g=\int\limits_M L_g\sqrt{-g}\mathrm{d}^4x=\frac{c^3}{16\pi G}\int\limits_M R\sqrt{-g}\mathrm{d}^4x,\quad L_g=\frac{c^3}{16\pi G}R\tag{40}$$

想算 I_g 对度规的变分，要先算 $R\sqrt{-g}$ 对度规的变分。

$$\begin{aligned}\delta(R\sqrt{-g})&=\delta\left(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\right)\\&=\left(\delta g^{\mu\nu}\right)R_{\mu\nu}\sqrt{-g}+g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\sqrt{-g}+g^{\mu\nu}R_{\mu\nu}\delta\left(\sqrt{-g}\right)\end{aligned}\tag{41}$$

利用Palatini.II公式

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}=\partial_{\mu}\left(\sqrt{-g}\phi^{\mu}\right),\quad \phi^{\mu}\equiv g^{\lambda\nu}\delta\Gamma^{\mu}_{\lambda\nu}-g^{\mu\nu}\delta\Gamma^{\lambda}_{\lambda\nu}\tag{42}$$

以及

$$\delta g=gg^{\mu\nu}\delta g_{\mu\nu}=-gg_{\mu\nu}\delta g^{\mu\nu}\tag{43}$$

$$\delta\left(\sqrt{-g}\right)=-\frac{1}{2}\frac{\delta g}{\sqrt{-g}}=-\frac{1}{2}\frac{-gg_{\mu\nu}\delta g^{\mu\nu}}{\sqrt{-g}}=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{44}$$

于是有

$$\begin{aligned}\delta(R\sqrt{-g})&=\left(\delta g^{\mu\nu}\right)R_{\mu\nu}\sqrt{-g}+g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\sqrt{-g}+g^{\mu\nu}R_{\mu\nu}\delta\left(\sqrt{-g}\right)\\&=R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}+\partial_\mu\left(\sqrt{-g}\phi^\mu\right)-\frac{1}{2}\sqrt{-g}Rg_{\mu\nu}\delta g^{\mu\nu}\end{aligned} \tag{45}$$

因此引力场作用量对度规的变分为

$$\begin{aligned}\delta I_g&=\frac{c^3}{16\pi G}\int\limits_M\delta\left(R\sqrt{-g}\right)\mathrm{d}^4x\\&=\frac{c^3}{16\pi G}\int\limits_M\left(R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}+\partial_\mu\left(\sqrt{-g}\phi^\mu\right)-\frac{1}{2}\sqrt{-g}Rg_{\mu\nu}\delta g^{\mu\nu}\right)\mathrm{d}^4x\\&=\frac{c^3}{16\pi G}\int\limits_M\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x\end{aligned} \tag{46}$$

引力源物质作用量的变分

$$I_m=\frac{1}{c}\int\limits_M L_m\sqrt{-g}\mathrm{d}^4x \tag{47}$$

假定引力源物质拉式密度 L_m 只含 $g^{\mu\nu}$，而不含 $\partial_\lambda g^{\mu\nu}$，则

$$\begin{aligned}\delta I_m&=\frac{1}{c}\int\limits_M\delta\left(L_m\sqrt{-g}\right)\mathrm{d}^4x\\&=\frac{1}{c}\int\limits_M\frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\mathrm{d}^4x\end{aligned} \tag{48}$$

定义

$$T_{\mu\nu}\equiv-\frac{2}{\sqrt{-g}}\frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}},\quad\frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}=-\frac{1}{2}\sqrt{-g}T_{\mu\nu} \tag{49}$$

则

$$\begin{aligned}\delta I_m&=\frac{1}{c}\int\limits_M\frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\mathrm{d}^4x\\&=-\frac{1}{2c}\int\limits_M T_{\mu\nu}\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x\end{aligned} \tag{50}$$

史瓦西时空类时 $\theta=\pi/2$ 平面类时测地线方程

以线长 s 为参数，在 $\theta=\pi/2$ 平面内

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s}=0,\quad \text{or}\quad \mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{51}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{52}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{53}$$

史瓦西时空类光 $\theta=\pi/2$ 平面类光测地线方程

以 λ 为参数，在$\theta=\pi/2$ 平面内

$$\frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda}=0,\quad \text{or}\quad \mathrm{d}s^2=0=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{54}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0 \tag{55}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=0 \tag{56}$$

方程特解与形式解

水星近日点进动

u 的方程

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\frac{3GM}{c^2}u^2+\frac{GM}{h^2} \tag{57}$$

u_0 的方程

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}\phi^2}+u_0=\frac{GM}{h^2} \tag{58}$$

其解为

$$u_0=\frac{1}{p}\left(1+e\cos\phi\right),\quad p=\frac{h^2}{GM} \tag{59}$$

u_1 的方程

令 $\alpha = 3GM/c^2$，则

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \alpha u^2 + \frac{GM}{h^2} \tag{60}$$

设 $u = u_0 + \alpha u_1$，则

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 + \alpha \frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + \alpha u_1 = \alpha \left(u_0 + \alpha u_1\right)^2 + \frac{GM}{h^2} \tag{61}$$

利用 u_0 满足的方程就得到

$$\alpha \frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + \alpha u_1 = \alpha \left(u_0 + \alpha u_1\right)^2 \tag{62}$$

由于 α 是个小量，约去右边括号内的高阶小量，再利用 u_0 的表达式，就得到

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = u_0^2 = \frac{\left(1 + e\cos\phi\right)^2}{p^2} = \frac{1}{p^2}\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right] \tag{63}$$

u 的解

设 u_1 的形式解为

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi \tag{64}$$

$$u_1 = \frac{1}{p^2}\left(1 + \frac{e^2}{2}\right) + \frac{e}{p^2}\phi\sin\phi - \frac{e^2}{6p^2}\cos 2\phi \tag{65}$$

$$u = u_0 + \alpha u_1 = \frac{1}{p}\left[\left(1 + e\cos\phi\right) + \frac{\alpha}{p}\left(1 + \frac{e^2}{2}\right) + \frac{\alpha e}{p}\phi\sin\phi - \frac{1}{6}\frac{\alpha e^2}{p^2}\cos 2\phi\right] \tag{66}$$

上面只有 $\phi\sin\phi$ 项是累加的，只保留对轨道有长期影响的项：

$$u = \frac{1}{p}\left[\left(1 + e\cos\phi\right) + \frac{\alpha e}{p}\phi\sin\phi\right] \tag{67}$$

由于 α 是小量，则

$$1 \approx \cos\left(\frac{\alpha}{p}\phi\right), \quad \frac{\alpha}{p}\phi \approx \sin\left(\frac{\alpha}{p}\phi\right) \tag{68}$$

于是

$$
\begin{aligned}
u &= \frac{1}{p}\left[(1+e\cos\phi)+\frac{\alpha e}{p}\phi\sin\phi\right] \\
&= \frac{1}{p}\left[1+e\left(1\cdot\cos\phi+\frac{\alpha}{p}\phi\sin\phi\right)\right] \\
&\approx \frac{1}{p}\left[1+e\left(\cos\left(\frac{\alpha}{p}\phi\right)\cos\phi+\sin\left(\frac{\alpha}{p}\phi\right)\sin\phi\right)\right] \\
&= \frac{1}{p}\left[1+\cos\left(\left(1-\frac{\alpha}{p}\right)\phi\right)\right] \\
&\equiv \frac{1}{p}(1+e\cos\Phi)
\end{aligned} \tag{69}
$$

光线偏折

u 的方程

$$
\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2}+u=\frac{3GM}{c^2}u^2=\alpha u^2 \tag{70}
$$

设

$$
u=u_0+\alpha u_1 \tag{71}
$$

u_0 的方程

$$
\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2}+u_0=0 \Longrightarrow u_0=\frac{1}{b}\sin\phi \tag{72}
$$

u_1 的方程

$$
\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2}+u_0+\alpha\left(\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2}+u_1\right)=\alpha\left(u_0+\alpha u_1\right)^2 \tag{73}
$$

也即

$$
\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2}+u_1=\left(u_0+\alpha u_1\right)^2\approx u_0^2=\frac{1}{b^2}\sin^2\phi \tag{74}
$$

设 u_1 的形式解为

$$
u_1=A\sin^2\phi+B \tag{75}
$$

u 的解

$$
u_1=A\sin^2\phi+B=-\frac{1}{3b^2}\left(\sin^2\phi-2\right)=\frac{1}{3b^2}\left(\cos^2\phi+1\right) \tag{76}
$$

$$
\begin{aligned}
u = u_0 + \alpha u_1 &= \frac{1}{b}\sin\phi + \frac{\alpha}{3b^2}\left(\cos^2\phi + 1\right) \\
&= \frac{1}{b}\sin\phi + \frac{GM}{c^2b^2}\left(\cos^2\phi + 1\right)
\end{aligned} \tag{77}
$$

重点题目

张量在坐标变换下的变换规律

最小作用量原理推导场方程

总作用量

$$
I = I_g + I_m \tag{78}
$$

其中 I_g 是引力场作用量，I_m 是引力源物质作用量。

$$
I_g = \int\limits_M L_g\sqrt{-g}\mathrm{d}^4x = \frac{c^3}{16\pi G}\int\limits_M R\sqrt{-g}\mathrm{d}^4x, \quad L_g = \frac{c^3}{16\pi G}R \tag{79}
$$

想算 I_g 对度规的变分，要先算 $R\sqrt{-g}$ 对度规的变分。

$$
\begin{aligned}
\delta(R\sqrt{-g}) &= \delta\left(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\right) \\
&= \left(\delta g^{\mu\nu}\right)R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\sqrt{-g} + g^{\mu\nu}R_{\mu\nu}\delta\left(\sqrt{-g}\right)
\end{aligned} \tag{80}
$$

利用Palatini.II公式

$$
\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \partial_\mu\left(\sqrt{-g}\phi^\mu\right), \quad \phi^\mu \equiv g^{\lambda\nu}\delta\Gamma^\mu_{\lambda\nu} - g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\nu} \tag{81}
$$

以及

$$
\delta g = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu} \tag{82}
$$

$$
\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\frac{\delta g}{\sqrt{-g}} = -\frac{1}{2}\frac{-gg_{\mu\nu}\delta g^{\mu\nu}}{\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{83}
$$

于是有

$$
\begin{aligned}
\delta(R\sqrt{-g}) &= \left(\delta g^{\mu\nu}\right)R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\sqrt{-g} + g^{\mu\nu}R_{\mu\nu}\delta\left(\sqrt{-g}\right) \\
&= R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu} + \partial_\mu\left(\sqrt{-g}\phi^\mu\right) - \frac{1}{2}\sqrt{-g}Rg_{\mu\nu}\delta g^{\mu\nu}
\end{aligned} \tag{84}
$$

因此引力场作用量对度规的变分为

$$
\begin{aligned}
\delta I_g &= \frac{c^3}{16\pi G}\int\limits_M \delta\left(R\sqrt{-g}\right)\mathrm{d}^4x \\
&= \frac{c^3}{16\pi G}\int\limits_M \left(R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu} + \partial_\mu\left(\sqrt{-g}\phi^\mu\right) - \frac{1}{2}\sqrt{-g}Rg_{\mu\nu}\delta g^{\mu\nu}\right)\mathrm{d}^4x \\
&= \frac{c^3}{16\pi G}\int\limits_M \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x
\end{aligned} \tag{85}
$$

接下来还需要引力源物质作用量对度规的变分。

$$
I_m = \frac{1}{c}\int\limits_M L_m\sqrt{-g}\mathrm{d}^4x \tag{86}
$$

假定引力源物质拉式密度 L_m 只含 $g^{\mu\nu}$，而不含 $\partial_\lambda g^{\mu\nu}$，则

$$
\begin{aligned}
\delta I_m &= \frac{1}{c}\int\limits_M \delta\left(L_m\sqrt{-g}\right)\mathrm{d}^4x \\
&= \frac{1}{c}\int\limits_M \frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\mathrm{d}^4x
\end{aligned} \tag{87}
$$

定义

$$
T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}, \quad \frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}T_{\mu\nu} \tag{88}
$$

则

$$
\begin{aligned}
\delta I_m &= \frac{1}{c}\int\limits_M \frac{\partial\left(L_m\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\mathrm{d}^4x \\
&= -\frac{1}{2c}\int\limits_M T_{\mu\nu}\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x
\end{aligned} \tag{89}
$$

最小作用量原理说，$\delta I = 0$ 给出体系的运动方程，也即

$$
\begin{aligned}
0 = \delta I = \delta I_g + \delta I_m &= \frac{c^3}{16\pi G}\int\limits_M \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x - \frac{1}{2c}\int\limits_M T_{\mu\nu}\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x \\
&= \frac{c^3}{16\pi G}\int\limits_M \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{8\pi G}{c^4}T_{\mu\nu}\right)\left(\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x
\end{aligned} \tag{90}
$$

最终得到爱因斯坦引力场方程：

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{91}
$$

Kruskal图与Penrose图

闵氏时空线元

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{92}$$

类光测地线满足

$$\mathrm{d}s^2 = 0 \tag{93}$$

如果只考虑径向，则

$$-c^2\mathrm{d}t^2 + \mathrm{d}r^2 = 0 \tag{94}$$

取 $c=1$，则

$$\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = 1, \quad \frac{\mathrm{d}t}{\mathrm{d}r} = \pm 1, \tag{95}$$

径向类光测地线为

$$t = \pm r + \mathrm{const} \tag{96}$$

史瓦西时空线元

$$\mathrm{d}s^2 = -c^2\left(1-\frac{2GM}{c^2r}\right)\mathrm{d}t^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{97}$$

如果只考虑径向，并取 $c=G=1$，则

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 \tag{98}$$

类光测地线满足

$$\mathrm{d}s^2 = 0 \tag{99}$$

即

$$\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = \left(1-\frac{2M}{r}\right)^{-2}, \quad \frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1-\frac{2M}{r}\right)^{-1} \tag{100}$$

$$\mathrm{d}t = \pm\frac{1}{1-2M/r}\mathrm{d}r = \pm\frac{r}{r-2M}\mathrm{d}r = \pm\frac{r-2M+2M}{r-2M}\mathrm{d}r = \pm\left(1+\frac{2M}{r-2M}\right)\mathrm{d}(r-2M) \tag{101}$$

两边积分得

$$t = \pm\left(r + 2M\ln|r-2M|\right) + \mathrm{const} \tag{102}$$

史瓦西时空径向线元：

$$\mathrm{d}s^2 = -\left(1-2M/r\right)\mathrm{d}t^2 + \left(1-2M/r\right)^{-1}\mathrm{d}r^2 \tag{103}$$

考虑坐标变换

$$\begin{cases} T=4M\left(\frac{r}{2M}-1\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\sinh\left(\frac{t}{4M}\right) \\ R=4M\left(\frac{r}{2M}-1\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\cosh\left(\frac{t}{4M}\right) \end{cases},r>2M,\mathrm{I}$$

$$\begin{cases} T=-4M\left(\frac{r}{2M}-1\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\sinh\left(\frac{t}{4M}\right) \\ R=-4M\left(\frac{r}{2M}-1\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\cosh\left(\frac{t}{4M}\right) \end{cases},r>2M,\mathrm{II}$$

$$\begin{cases} T=4M\left(1-\frac{r}{2M}\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\cosh\left(\frac{t}{4M}\right) \\ R=4M\left(1-\frac{r}{2M}\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\sinh\left(\frac{t}{4M}\right) \end{cases},r<2M,F$$

$$\begin{cases} T=-4M\left(1-\frac{r}{2M}\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\cosh\left(\frac{t}{4M}\right) \\ R=-4M\left(1-\frac{r}{2M}\right)^{1/2}\mathrm{e}^{\frac{r}{4M}}\sinh\left(\frac{t}{4M}\right) \end{cases},r<2M,P$$

Kruskal坐标下的线元：

$$\mathrm{d}s^2=\frac{2M}{r}\mathrm{e}^{-r/2M}\left(-\mathrm{d}T^2+\mathrm{d}R^2\right)+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \tag{104}$$

r 与 R,T 的关系：

$$R^2-T^2=16M^2\left(\frac{r}{2M}-1\right)\mathrm{e}^{r/2M} \tag{105}$$

> 上式对四个区域都成立。

Kruskal坐标消除了度规分量在引力半径处的奇异性，但无法消除 $r=0$ 的奇点。

Kruskal度规具有最大解析区和最高完备性。

- 类时未来无穷远 I^+：r 有限，$t\to+\infty$
- 类时过去无穷远 I^-：r 有限，$t\to-\infty$
- 类空无穷远 I^0：t 有限，$r\to+\infty$
- 类光未来无穷远 J^+：$(t-r)$ 有限，$(t+r)\to+\infty$
- 类光过去无穷远 J^-：$(t+r)$ 有限，$(t-r)\to+\infty$

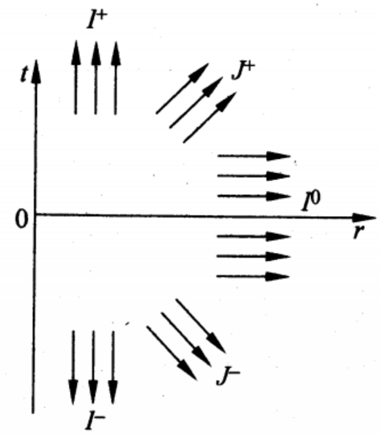

图 6.2.4　闵可夫斯基时空图

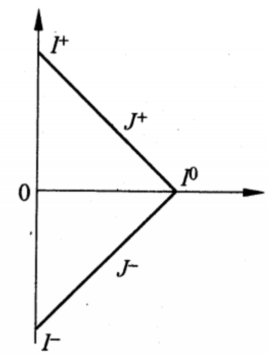

图 6.2.5　闵氏时空彭若斯图

克鲁斯卡坐标系可以统一描述整个史瓦西时空，它覆盖了黑洞内、外及视界。而且，从克鲁斯卡时空图(图 6.2.3)可知，它扩大了史瓦西时空。两条对角线是视界。Ⅰ区即通常的黑洞外部宇宙，F 区为黑洞区，P 区为白洞区，Ⅱ区是另一个洞外宇宙，它和我们的宇宙没有因果连通，没有任何信息交流。奇点 $r=0$ 分别出现在白洞区和黑洞区，以双曲线形式呈现。Ⅰ区和Ⅱ区中“$r=$常数”的双曲线，就是史瓦西时空中静止粒子的世界线。F 区和 P 区中“$r=$常数”的双曲线为等时线。应当注意，此图中的任何一点，都代表一个二维球面。光锥如图 6.2.3 所示，

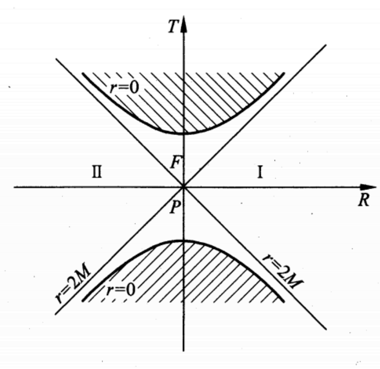

图 6.2.3　克鲁斯卡时空图

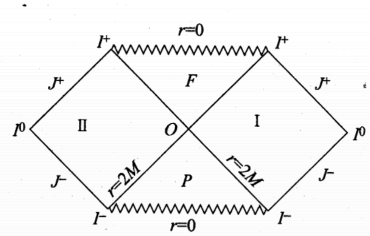

图 6.2.6　克鲁斯卡时空彭若斯图

水星近日点进动

考虑史瓦西时空，以线长 s 为参数，在 $\theta=\pi/2$ 平面内

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s}=0, \quad \text{or} \quad \mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{106}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2=0 \tag{107}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{108}$$

注意到上面第三条方程

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0, \quad r^2\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2}+2r\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0, \quad r^2\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}+\frac{\mathrm{d}\left(r^2\right)}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s}=0 \tag{109}$$

也即

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)=0 \tag{110}$$

因此

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}=\text{const}\equiv\frac{h}{c} \tag{111}$$

这是 GR 中的角动量守恒。

我们要找轨道方程，因此需要找到 r,ϕ 的微分方程。

由于 $\theta=\pi/2$，则线元

$$\mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right), \quad \mathrm{e}^{\nu}=1-\frac{2GM}{c^2r} \tag{112}$$

可化简为

$$\mathrm{d}s^2=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{113}$$

利用线元 $\mathrm{d}s^2$ 与线长 $\mathrm{d}s$ 的关系

$$\mathrm{d}s=\sqrt{-\mathrm{d}s^2}, \quad (\mathrm{d}s)^2=-\mathrm{d}s^2=c^2\mathrm{e}^{\nu}\mathrm{d}t^2-\mathrm{e}^{-\nu}\mathrm{d}r^2-r^2\mathrm{d}\phi^2 \tag{114}$$

两边同除 $(\mathrm{d}s)^2$，再同乘 e^{ν}，得到

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2=c^2\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2-r^2\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2-\mathrm{e}^{\nu} \tag{115}$$

上式代入 r 关于 s 的二阶偏微分方程，就消去 t:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2}+\frac{\nu'}{2}\mathrm{e}^{\nu}r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2+\frac{1}{2}\nu'\mathrm{e}^{\nu}=0 \tag{116}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2}+\left(\frac{1}{2}r^2\nu'\mathrm{e}^{\nu}-r\mathrm{e}^{\nu}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2+\frac{1}{2}\nu'\mathrm{e}^{\nu}=0 \tag{117}$$

这就是GR中参数形式的行星轨道方程。

利用 GR 中的角动量守恒

$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{c},\quad \frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{cr^2} \tag{118}$$

进一步化简为

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\left(\frac{\nu'}{2}r^2-r\right)\mathrm{e}^\nu\left(\frac{h}{cr^2}\right)^2+\frac{1}{2}\nu'\mathrm{e}^\nu=0 \tag{119}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2}+\frac{1}{2}\nu'\mathrm{e}^\nu\left(\frac{h^2}{c^2r^2}+1\right)-r\mathrm{e}^\nu\left(\frac{h}{cr^2}\right)^2=0 \tag{120}$$

令 $u=\frac{1}{r}$，注意到

$$\mathrm{e}^\nu=1-\frac{2GM}{c^2r}=1-\frac{2GM}{c^2}u \tag{121}$$

$$\nu'\mathrm{e}^\nu=(\mathrm{e}^\nu)'=\frac{2GM}{c^2r^2}=\frac{2GM}{c^2}u^2 \tag{122}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{h}{cr^2}=\frac{h}{c}u^2 \tag{123}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}=\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\phi}=\frac{h}{c}u^2\frac{\mathrm{d}}{\mathrm{d}\phi} \tag{124}$$

$$\begin{aligned}\frac{\mathrm{d}^2r}{\mathrm{d}s}&=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}(1/u)}{\mathrm{d}s}\right)=\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}s}\right)=\frac{\mathrm{d}}{\mathrm{d}s}\left[-\frac{1}{u^2}\cdot\left(\frac{h}{c}u^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\right]\\&=-\frac{h}{c}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)=-\frac{h}{c}\cdot\frac{h}{c}u^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\\&=-\frac{h^2}{c^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}\end{aligned} \tag{125}$$

于是轨道的参数方程就化为如下的轨道方程：

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}+u=\frac{3GM}{c^2}u^2+\frac{GM}{h^2} \tag{126}$$

上式就是GR中的Binet方程。

设 u_0 满足牛顿力学中的Binet方程

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}\phi^2}+u_0=\frac{GM}{h^2} \tag{127}$$

其解为

$$u_0=\frac{1}{p}\left(1+e\cos\phi\right),\quad p=\frac{h^2}{GM} \tag{128}$$

令 $\alpha = 3GM/c^2$，则

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \alpha u^2 + \frac{GM}{h^2} \tag{129}$$

设 $u = u_0 + \alpha u_1$，则

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 + \alpha \frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + \alpha u_1 = \alpha \left(u_0 + \alpha u_1\right)^2 + \frac{GM}{h^2} \tag{130}$$

利用 u_0 满足的方程就得到

$$\alpha \frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + \alpha u_1 = \alpha \left(u_0 + \alpha u_1\right)^2 \tag{131}$$

由于 α 是个小量，约去右边括号内的高阶小量，再利用 u_0 的表达式，就得到

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = u_0^2 = \frac{\left(1 + e\cos\phi\right)^2}{p^2} = \frac{1}{p^2}\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right] \tag{132}$$

设 u_1 的形式解为

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi \tag{133}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}\phi} = B\left(\sin\phi + \phi\cos\phi\right) - 2C\sin 2\phi \tag{134}$$

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} = B\left(2\cos\phi - \phi\sin\phi\right) - 4C\cos 2\phi \tag{135}$$

代回方程得到

$$B\left(2\cos\phi - \phi\sin\phi\right) - 4C\cos 2\phi + A + B\phi\sin\phi + C\cos 2\phi = \frac{1}{p^2}\left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right] \tag{136}$$

对比各项前的系数就得到

$$A = \frac{1}{p^2}\left(1 + \frac{e^2}{2}\right), \quad B = \frac{e}{p^2}, \quad C = -\frac{e^2}{6p^2} \tag{137}$$

因此

$$u_1 = \frac{1}{p^2}\left(1 + \frac{e^2}{2}\right) + \frac{e}{p^2}\phi\sin\phi - \frac{e^2}{6p^2}\cos 2\phi \tag{138}$$

$$u = u_0 + \alpha u_1 = \frac{1}{p}\left[\left(1 + e\cos\phi\right) + \frac{\alpha}{p}\left(1 + \frac{e^2}{2}\right) + \frac{\alpha e}{p}\phi\sin\phi - \frac{1}{6}\frac{\alpha e^2}{p^2}\cos 2\phi\right] \tag{139}$$

上面只有 $\phi\sin\phi$ 项是累加的，只保留对轨道有长期影响的项：

$$u = \frac{1}{p}\left[\left(1 + e\cos\phi\right) + \frac{\alpha e}{p}\phi\sin\phi\right] \tag{140}$$

由于 α 是小量，则

$$1 \approx \cos\left(\frac{\alpha}{p}\phi\right), \quad \frac{\alpha}{p}\phi \approx \sin\left(\frac{\alpha}{p}\phi\right) \tag{141}$$

于是

$$\begin{aligned} u &= \frac{1}{p}\left[(1+e\cos\phi)+\frac{\alpha e}{p}\phi\sin\phi\right] \\ &= \frac{1}{p}\left[1+e\left(1\cdot\cos\phi+\frac{\alpha}{p}\phi\sin\phi\right)\right] \\ &\approx \frac{1}{p}\left[1+e\left(\cos\left(\frac{\alpha}{p}\phi\right)\cos\phi+\sin\left(\frac{\alpha}{p}\phi\right)\sin\phi\right)\right] \\ &= \frac{1}{p}\left[1+\cos\left(\left(1-\frac{\alpha}{p}\right)\phi\right)\right] \\ &\equiv \frac{1}{p}\left(1+e\cos\Phi\right) \end{aligned} \tag{142}$$

$$\Phi \equiv \left(1-\frac{\alpha}{p}\right)\phi \tag{143}$$

当 $\Phi=\Phi_n=(2n+1)\pi$ 时，$\cos\Phi=-1$，此时 u 最小，r 最大，相应 ϕ_n 为

$$\phi_n=\frac{\Phi_n}{1-\alpha/p}\approx\Phi_n\left(1+\frac{\alpha}{p}\right) \tag{144}$$

$$\phi_{n+1}\approx\Phi_{n+1}\left(1+\frac{\alpha}{p}\right) \tag{145}$$

于是

$$\phi_{n+1}-\phi_n=2\pi\left(1+\frac{\alpha}{p}\right) \tag{146}$$

一周期进动角为

$$\Delta=\phi_{n+1}-\phi_n-2\pi=2\pi\frac{\alpha}{p}=\frac{6\pi GM}{c^2p} \tag{147}$$

由

$$T^2=\frac{4\pi^2}{GM}a^3, \quad GM=4\pi^2\frac{a^3}{T^2}, \quad p=a\left(1-e^2\right) \tag{148}$$

可得

$$\Delta=\frac{24\pi^3a^2}{c^2T^2\left(1-e^2\right)} \tag{149}$$

对于水星，其世纪进动角为

$$\Delta_c \approx 43'' \tag{150}$$

光线偏折

对于光有 $\mathrm{d}s^2 = 0$，因此测地线方程不能以线长 s 为参数。但可引入参数 λ 来定义光线的切矢：

$$K^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{151}$$

由于 $\mathrm{d}s^2 = 0$ 可得

$$g_{\mu\nu} K^\mu K^\nu = 0 \tag{152}$$

假设切矢 K^μ 在光的传播路线上是平行的，即

$$\nabla_\mu K^\sigma = 0, \quad K^\mu \nabla_\mu K^\sigma = 0 \tag{153}$$

也即

$$K^\mu \left(\partial_\mu K^\sigma + \Gamma^\sigma_{\mu\nu} K^\nu\right) = 0 \tag{154}$$

又

$$K^\mu \partial_\mu K^\sigma = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\partial K^\sigma}{\partial x^\mu} = \frac{\mathrm{d}K^\sigma}{\mathrm{d}\lambda} \tag{155}$$

因此有

$$\frac{\mathrm{d}K^\sigma}{\mathrm{d}\lambda} + \Gamma^\sigma_{\mu\nu} K^\mu K^\nu = 0, \quad K^\sigma \equiv \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} \tag{156}$$

也即

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0 \tag{157}$$

这就是光线传播的路径方程。相比有质量粒子的测地线方程，只是把线长 s 替换成参数 λ 而已。

我们知道，Schwarzschild解情况下有质量粒子在 $\theta = \pi/2$ 平面内的测地线方程有三条：

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} + \nu' \frac{\mathrm{d}t}{\mathrm{d}s} \frac{\mathrm{d}r}{\mathrm{d}s} = 0 \tag{158}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \frac{c^2 \nu'}{2} \mathrm{e}^{2\nu} \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{\nu'}{2} \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r\mathrm{e}^{\nu} \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{159}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}s^2} + \frac{2}{r} \frac{\mathrm{d}r}{\mathrm{d}s} \frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \tag{160}$$

对于光子，同样考虑Schwarzschild解，由于二者的联络都是一样，因此只需要把 s 替换成 λ 就得到条Schwarzschild解下光传播路径的参数方程：

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2}+\nu'\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda}=0 \tag{161}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2}+\frac{c^2\nu'}{2}\mathrm{e}^{2\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2-\frac{\nu'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2-r\mathrm{e}^{\nu}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0 \tag{162}$$

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\lambda^2}+\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=0 \tag{163}$$

对于第三条方程，同样有

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)=0,\quad r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=k \tag{164}$$

在 $\theta=\pi/2$ 平面上，线元

$$\mathrm{d}s^2=0=-c^2\mathrm{e}^{\nu}\mathrm{d}t^2+\mathrm{e}^{-\nu}\mathrm{d}r^2+r^2\mathrm{d}\phi^2 \tag{165}$$

可以证明，可以从上式和第三条方程推导出第一条方程。上式两边同除 $(\mathrm{d}\lambda)^2$，并移项，就得到

$$c^2\mathrm{e}^{\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2=\mathrm{e}^{-\nu}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2+r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 \tag{166}$$

上式代回 r 关于 λ 二阶导式子，就得到

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2}+\left(\frac{r^2}{2}\nu'\mathrm{e}^{\nu}-r\mathrm{e}^{\nu}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0 \tag{167}$$

令 $u=\frac{1}{r}$，注意到

$$\mathrm{e}^{\nu}=1-\frac{2GM}{c^2 r}=1-\frac{2GM}{c^2}u \tag{168}$$

$$\nu'\mathrm{e}^{\nu}=(\mathrm{e}^{\nu})'=\frac{2GM}{c^2r^2}=\frac{2GM}{c^2}u^2 \tag{169}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=\frac{k}{r^2}=ku^2 \tag{170}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}=\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\phi}=ku^2\frac{\mathrm{d}}{\mathrm{d}\phi} \tag{171}$$

$$\begin{aligned}\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2}&=\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}(1/u)}{\mathrm{d}\lambda}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\lambda}\right)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-\frac{1}{u^2}\cdot ku^2\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)\\&=-k\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}u}{\mathrm{d}\phi}=-k\cdot ku^2\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\mathrm{d}u}{\mathrm{d}\phi}\\&=-k^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}\end{aligned} \tag{172}$$

于是可以消去参数 λ，得到轨道微分方程

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{3GM}{c^2} u^2 \tag{173}$$

定义小量

$$\alpha \equiv \frac{3GM}{c^2} \tag{174}$$

则

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \alpha u^2 \tag{175}$$

设 u_0 满足

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 = 0 \tag{176}$$

其解为

$$u_0 = \frac{1}{b} \sin\phi \tag{177}$$

设

$$u = u_0 + \alpha u_1 \tag{178}$$

则

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\phi^2} + u_0 + \alpha\left(\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1\right) = \alpha\left(u_0 + \alpha u_1\right)^2 \tag{179}$$

也即

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} + u_1 = \left(u_0 + \alpha u_1\right)^2 \approx u_0^2 = \frac{1}{b^2} \sin^2\phi \tag{180}$$

设 u_1 的形式解为

$$u_1 = A \sin^2\phi + B \tag{181}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}\phi} = 2A \sin\phi \cos\phi = A \sin 2\phi \tag{182}$$

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\phi^2} = 2A \cos 2\phi = 2A\left(1 - 2\sin^2\phi\right) = 2A - 4A\sin^2\phi \tag{183}$$

代回 u_1 满足的微分方程，得到

$$2A - 4A\sin^2\phi + A\sin^2\phi + B = \frac{1}{b^2}\sin^2\phi \tag{184}$$

对比可得

$$3A = -\frac{1}{b^2}, \quad 2A + B = 0 \tag{185}$$

解得

$$A = -\frac{1}{3b^2}, \quad B = \frac{2}{3b^2} \tag{186}$$

$$u_1 = A\sin^2\phi + B = -\frac{1}{3b^2}\left(\sin^2\phi - 2\right) = \frac{1}{3b^2}\left(\cos^2\phi + 1\right) \tag{187}$$

$$\begin{aligned} u = u_0 + \alpha u_1 &= \frac{1}{b}\sin\phi + \frac{\alpha}{3b^2}\left(\cos^2\phi + 1\right) \\ &= \frac{1}{b}\sin\phi + \frac{GM}{c^2b^2}\left(\cos^2\phi + 1\right) \end{aligned} \tag{188}$$

定义小量 $a \equiv GM/c^2b$，当 $r \to +\infty, u = 0$，此时

$$\sin\phi + a\left(2 - \sin^2\phi\right) = 0 \tag{189}$$

$$\sin\phi = \frac{1 \pm \sqrt{1 + 8a^2}}{2a} \approx \frac{1 \pm \left(1 + 4a^2\right)}{2a} = -2a \quad \text{or} \quad \frac{1 + 2a^2}{a} \tag{190}$$

舍去 $\sin\phi = \frac{1+2a^2}{a} > 1$ 的解，考虑 $\phi \to 0$ 的那侧，则

$$\phi \approx \sin\phi = -2a, \quad r \to +\infty \tag{191}$$

偏折角为

$$\delta = 4a = \frac{4GM}{c^2b} \tag{192}$$