# 1

## 1.1

> 证明弧元 $\mathrm{d}s^2 \equiv e g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ 是坐标变换下的不变量。

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu = A^\mu_\nu \mathrm{d}x^\nu \tag{1}$$

$$\begin{aligned}
\mathrm{d}s'^2 &\equiv e g'_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu \\
&= e\bar{A}^\alpha_\mu \bar{A}^\beta_\nu g_{\alpha\beta}\left(A^\mu_\lambda \mathrm{d}x^\lambda\right)\left(A^\nu_\rho \mathrm{d}x^\rho\right) \\
&= e g_{\alpha\beta}\left(\bar{A}^\alpha_\mu A^\mu_\lambda\right)\left(\bar{A}^\beta_\nu A^\nu_\rho\right)\mathrm{d}x^\lambda \mathrm{d}x^\rho \\
&= e g_{\alpha\beta}\delta^\alpha_\lambda \delta^\beta_\rho \mathrm{d}x^\lambda \mathrm{d}x^\rho \\
&= e g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta
\end{aligned} \tag{2}$$

## 1.2

> 由协变微商的协变性推导联络在坐标变换下的变换式。

由定义

$$\nabla_\mu \phi^\nu(x) \equiv \partial_\mu \phi^\nu(x) + \Gamma^\nu_{\mu\lambda}\phi^\lambda(x) \tag{3}$$

$$\nabla'_\mu \phi'^\nu(x') \equiv \partial'_\mu \phi'^\nu(x) + \Gamma'^\nu_{\mu\lambda}\phi'^\lambda(x') \tag{4}$$

我们知道 $\partial_\mu$ 是协变矢量，$\phi^\nu(x)$ 是逆变矢量，利用它们的变换规律

$$\partial'_\mu = \bar{A}^\alpha_\mu \partial_\alpha, \quad \phi'^\nu(x') = A^\nu_\beta \phi^\beta(x) \tag{5}$$

有

$$\begin{aligned}
\nabla'_\mu \phi'^\nu(x') &\equiv \partial'_\mu \phi'^\nu(x) + \Gamma'^\nu_{\mu\lambda}\phi'^\lambda(x') \\
&= \bar{A}^\alpha_\mu \partial_\alpha\left[A^\nu_\beta \phi^\beta(x)\right] + \Gamma'^\nu_{\mu\lambda}A^\lambda_\gamma \phi^\gamma(x) \\
&= \bar{A}^\alpha_\mu\left(\partial_\alpha A^\nu_\beta\right)\phi^\beta(x) + \bar{A}^\alpha_\mu A^\nu_\beta \partial_\alpha \phi^\beta(x) + \Gamma'^\nu_{\mu\lambda}A^\lambda_\gamma \phi^\gamma(x)
\end{aligned} \tag{6}$$

而我们希望逆变矢量的协变微商是一个张量，其满足张量的变换规律

$$
\begin{aligned}
\nabla'_\mu \phi'^\nu(x') &= \bar{A}^\alpha_\mu A^\nu_\beta \nabla_\alpha \phi^\beta(x) \\
&= \bar{A}^\alpha_\mu A^\nu_\beta \left[\partial_\alpha \phi^\beta(x) + \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x)\right] \\
&= \bar{A}^\alpha_\mu A^\nu_\beta \partial_\alpha \phi^\beta(x) + \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x)
\end{aligned}
\tag{7}
$$

有

$$
\bar{A}^\alpha_\mu \left(\partial_\alpha A^\nu_\beta\right)\phi^\beta(x) + \bar{A}^\alpha_\mu A^\nu_\beta \partial_\alpha \phi^\beta(x) + \Gamma'^\nu_{\mu\lambda} A^\lambda_\gamma \phi^\gamma(x) = \bar{A}^\alpha_\mu A^\nu_\beta \partial_\alpha \phi^\beta(x) + \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x) \tag{8}
$$

化简为

$$
\Gamma'^\nu_{\mu\lambda} A^\lambda_\gamma \phi^\gamma(x) = \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x) - \bar{A}^\alpha_\mu \left(\partial_\alpha A^\nu_\beta\right)\phi^\beta(x) \tag{9}
$$

替换哑标得

$$
\Gamma'^\nu_{\mu\lambda} A^\lambda_\gamma \phi^\gamma(x) = \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma}\phi^\gamma(x) - \bar{A}^\alpha_\mu \left(\partial_\alpha A^\nu_\gamma\right)\phi^\gamma(x) \tag{10}
$$

因此有

$$
\Gamma'^\nu_{\mu\lambda} A^\lambda_\gamma = \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma} - \bar{A}^\alpha_\mu \left(\partial_\alpha A^\nu_\gamma\right) \tag{11}
$$

两边同乘 $\bar{A}^\gamma_\rho$，并对 $\gamma$ 求和，(11) 式左边

$$
\begin{aligned}
\Gamma'^\nu_{\mu\lambda} A^\lambda_\gamma \bar{A}^\gamma_\rho &= \Gamma'^\nu_{\mu\lambda}\delta^\lambda_\rho \\
&= \Gamma'^\nu_{\mu\rho}
\end{aligned}
\tag{12}
$$

(11) 式右边

$$
\begin{aligned}
\bar{A}^\gamma_\rho \left[\bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma} - \bar{A}^\alpha_\mu \left(\partial_\alpha A^\nu_\gamma\right)\right] &= \bar{A}^\gamma_\rho \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma} - \bar{A}^\alpha_\mu \bar{A}^\gamma_\rho \partial_\alpha A^\nu_\gamma \\
&= \bar{A}^\gamma_\rho \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma} - \bar{A}^\alpha_\mu \left[\partial_\alpha \left(\bar{A}^\gamma_\rho A^\nu_\gamma\right) - A^\nu_\gamma \partial_\alpha \bar{A}^\gamma_\rho\right] \\
&= \bar{A}^\gamma_\rho \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma} - \bar{A}^\alpha_\mu \left[\partial_\alpha \delta^\nu_\rho - A^\nu_\gamma \partial_\alpha \bar{A}^\gamma_\rho\right] \\
&= \bar{A}^\gamma_\rho \bar{A}^\alpha_\mu A^\nu_\beta \Gamma^\beta_{\alpha\gamma} + \bar{A}^\alpha_\mu A^\nu_\gamma \partial_\alpha \bar{A}^\gamma_\rho \\
&= A^\nu_\beta \bar{A}^\alpha_\mu \bar{A}^\gamma_\rho \Gamma^\beta_{\alpha\gamma} + A^\nu_\gamma \bar{A}^\alpha_\mu \partial_\alpha \bar{A}^\gamma_\rho
\end{aligned}
\tag{13}
$$

于是得到联络的变换规律

$$
\Gamma'^\nu_{\mu\rho} = A^\nu_\beta \bar{A}^\alpha_\mu \bar{A}^\gamma_\rho \Gamma^\beta_{\alpha\gamma} + A^\nu_\gamma \bar{A}^\alpha_\mu \partial_\alpha \bar{A}^\gamma_\rho \tag{14}
$$

替换指标就得到

$$
\Gamma'^\mu_{\nu\lambda} = A^\mu_\alpha \bar{A}^\beta_\nu \bar{A}^\gamma_\lambda \Gamma^\alpha_{\beta\gamma} + A^\mu_\alpha \bar{A}^\beta_\nu \partial_\beta \bar{A}^\alpha_\lambda \tag{15}
$$

## 1.3

> 证明挠率 $\Gamma^{\lambda}_{[\mu,\nu]}$ 是一个张量。

$$\Gamma'^{\lambda}_{\mu\nu} = A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} + A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\partial_{\beta}\bar{A}^{\alpha}_{\nu} \tag{16}$$

$$\Gamma'^{\lambda}_{\nu\mu} = A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\bar{A}^{\gamma}_{\mu}\Gamma^{\alpha}_{\beta\gamma} + A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\mu} \tag{17}$$

注意到

$$\begin{aligned}
\bar{A}^{\beta}_{\mu}\partial_{\beta}\bar{A}^{\alpha}_{\nu} - \bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\mu} &\equiv \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}} - \frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \\
&= \frac{\partial}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}} - \frac{\partial}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \\
&= 0
\end{aligned} \tag{18}$$

因此

$$\begin{aligned}
\Gamma'^{\lambda}_{[\mu,\nu]} &\equiv \Gamma'^{\lambda}_{\mu\nu} - \Gamma'^{\lambda}_{\nu\mu} \\
&= \left(A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} + A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\partial_{\beta}\bar{A}^{\alpha}_{\nu}\right) - \left(A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\bar{A}^{\gamma}_{\mu}\Gamma^{\alpha}_{\beta\gamma} + A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\mu}\right) \\
&= \left(A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} - A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\nu}\bar{A}^{\gamma}_{\mu}\Gamma^{\alpha}_{\beta\gamma}\right) + A^{\lambda}_{\alpha}\left(\bar{A}^{\beta}_{\mu}\partial_{\beta}\bar{A}^{\alpha}_{\nu} - \bar{A}^{\beta}_{\nu}\partial_{\beta}\bar{A}^{\alpha}_{\mu}\right) \\
&= A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} - A^{\lambda}_{\alpha}\bar{A}^{\color{red}\beta}_{\nu}\bar{A}^{\color{green}\gamma}_{\mu}\Gamma^{\alpha}_{\color{red}\beta\color{green}\gamma} \\
&= A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{\beta\gamma} - A^{\lambda}_{\alpha}\bar{A}^{\color{red}\gamma}_{\nu}\bar{A}^{\color{green}\beta}_{\mu}\Gamma^{\alpha}_{\color{red}\gamma\color{green}\beta} \\
&= A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\left(\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}\right) \\
&= A^{\lambda}_{\alpha}\bar{A}^{\beta}_{\mu}\bar{A}^{\gamma}_{\nu}\Gamma^{\alpha}_{[\beta,\gamma]}
\end{aligned} \tag{19}$$

## 1.4

> 由 $\nabla_{\lambda}g_{\mu\nu} = 0$ 证明 $\nabla_{\lambda}g^{\mu\nu} = 0$.

$$g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu} \tag{20}$$

两边用协变微商作用

$$\nabla_{\lambda}\left(g^{\mu\alpha}g_{\alpha\nu}\right) = 0 \tag{21}$$

利用协变微商的莱布尼兹律

$$
\begin{aligned}
\nabla_\lambda\left(g^{\mu\alpha}g_{\alpha\nu}\right) &= \left(\nabla_\lambda g^{\mu\alpha}\right)g_{\alpha\nu}+g^{\mu\alpha}\left(\nabla_\lambda g_{\alpha\nu}\right)\\
&= \left(\nabla_\lambda g^{\mu\alpha}\right)g_{\alpha\nu}
\end{aligned} \tag{22}
$$

得到

$$
\left(\nabla_\lambda g^{\mu\alpha}\right)g_{\alpha\nu}=0 \tag{23}
$$

上式两边乘 $g^{\nu\beta}$ 并对 $\nu$ 求和

$$
\begin{aligned}
0 &= \left(\nabla_\lambda g^{\mu\alpha}\right)g_{\alpha\nu}g^{\nu\beta}\\
&= \left(\nabla_\lambda g^{\mu\alpha}\right)\delta_\alpha^\beta\\
&= \nabla_\lambda g^{\mu\beta}
\end{aligned} \tag{24}
$$

# 1.5

> 假设有一对称张量 $f_{\mu\nu}$ 及其逆 $f^{\mu\nu}$ 可对张量指标进行升降
>
> $$
> f_{\mu\nu}\phi^\nu=\phi_\mu,\quad f^{\mu\nu}\phi_\nu=\phi^\mu \tag{25}
> $$
>
> 由协变微商定义及其性质，证明此张量的协变微商为零，即
>
> $$
> \nabla_\lambda f_{\mu\nu}=0 \tag{26}
> $$

$$
\phi_\mu=f_{\mu\nu}\phi^\nu \tag{27}
$$

$$
\begin{aligned}
\nabla_\lambda\phi_\mu &= \nabla_\lambda\left(f_{\mu\nu}\phi^\nu\right)\\
&= \left(\nabla_\lambda f_{\mu\nu}\right)\phi^\nu+f_{\mu\nu}\left(\nabla_\lambda\phi^\nu\right)\\
&= \left(\nabla_\lambda f_{\mu\nu}\right)\phi^\nu+\nabla_\lambda\phi_\mu
\end{aligned} \tag{28}
$$

前后对比得

$$
\left(\nabla_\lambda f_{\mu\nu}\right)\phi^\nu=0 \tag{29}
$$

由 $\phi^\nu$ 的任意性有

$$
\nabla_\lambda f_{\mu\nu}=0 \tag{30}
$$