3-4

> 证明正反粒子单位旋量正交关系：
>
> $$\bar{u}_i(\vec{p})u_j(\vec{p}) = \frac{m}{E}\delta_{ij}$$
>
> $$\bar{v}_i(\vec{p})v_j(\vec{p}) = -\frac{m}{E}\delta_{ij}$$
>
> $$\bar{u}_i(\vec{p})v_j(\vec{p}) = 0$$
>
> $$\bar{v}_j(\vec{p})u_j(\vec{p}) = 0$$

当 $i = 1, 2$，正反粒子单位旋量满足动量表象 Dirac 方程

$$(\mathrm{i}\hat{p} + m)\, u_i(\vec{p}) = 0, \quad (\mathrm{i}\hat{p} - m)\, v_i(\vec{p}) = 0$$

可写成：

$$(\mathrm{i}\vec{\gamma} \cdot \vec{p} - \gamma_4 E + m)\, u_i(\vec{p}) = 0$$

$$(\mathrm{i}\vec{\gamma} \cdot \vec{p} - \gamma_4 E - m)\, v_i(\vec{p}) = 0$$

取厄米共轭：

$$u_i^\dagger(\vec{p})\, (-\mathrm{i}\vec{\gamma} \cdot \vec{p} - \gamma_4 E + m) = 0$$

$$v_i^\dagger(\vec{p})\, (-\mathrm{i}\vec{\gamma} \cdot \vec{p} - \gamma_4 E - m) = 0$$

因此：

$$u_i^\dagger(\vec{p})\, (\mathrm{i}\vec{\gamma} \cdot \vec{p} - \gamma_4 E + m)\, u_i(\vec{p}) = 0 \tag{1}$$

$$v_i^\dagger(\vec{p})\, (\mathrm{i}\vec{\gamma} \cdot \vec{p} - \gamma_4 E - m)\, v_i(\vec{p}) = 0 \tag{2}$$

$$u_i^\dagger(\vec{p})\, (-\mathrm{i}\vec{\gamma} \cdot \vec{p} - \gamma_4 E + m)\, u_j(\vec{p}) = 0 \tag{3}$$

$$v_i^\dagger(\vec{p})\, (-\mathrm{i}\vec{\gamma} \cdot \vec{p} - \gamma_4 E - m)\, v_j(\vec{p}) = 0 \tag{4}$$

$(1), (3)$ 式相加，$(2), (4)$ 式相加，得：

$$u_i^\dagger(\vec{p})\, (-2E\gamma_4 + 2m)\, u_j(\vec{p}) = 0$$

$$v_i^\dagger(\vec{p})\, (-2E\gamma_4 - 2m)\, v_j(\vec{p}) = 0$$

即：

$$u_i^\dagger(\vec{p})\gamma_4 u_j(\vec{p}) = \frac{m}{E} u_i^\dagger(\vec{p}) u_j(\vec{p})$$

$$v_i^\dagger(\vec{p})\gamma_4 v_j(\vec{p}) = -\frac{m}{E} v_i^\dagger(\vec{p}) v_j(\vec{p})$$

利用正交性和定义

$$\bar{u}_i(\vec{p}) \equiv u_i^\dagger(\vec{p})\gamma_4, \quad \bar{v}_i(\vec{p}) \equiv v_i^\dagger(\vec{p})\gamma_4$$

$$u_i^\dagger(\vec{p}) u_j(\vec{p}) = \delta_{ij}, \quad v_i^\dagger(\vec{p}) v_j(\vec{p}) = \delta_{ij}$$

得到：

$$\bar{u}_i(\vec{p}) u_j(\vec{p}) = \frac{m}{E}\delta_{ij}$$

$$\bar{v}_i(\vec{p}) v_j(\vec{p}) = -\frac{m}{E}\delta_{ij}$$