标准规范 $\{u_{ij}=+1\}$ 下，哈密顿量

$$\begin{aligned}H(0)&=\frac{1}{2}\begin{pmatrix}a^\dagger & a^\top\end{pmatrix}h(0)\begin{pmatrix}a\\ \left(a^\dagger\right)^\top\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}a^\dagger & a^\top\end{pmatrix}U(0)D(0)U^\dagger(0)\begin{pmatrix}a\\ \left(a^\dagger\right)^\top\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}D(0)\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}\end{aligned}\tag{1}$$

$$U(0)=\begin{pmatrix}W(0) & V^*(0)\\ V(0) & W^*(0)\end{pmatrix},\quad D(0)=\mathrm{diag}(E_1(0),\cdots,E_N(0),-E_1(0),\cdots,-E_N(0))\tag{2}$$

$$\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}=U^\dagger(0)\begin{pmatrix}a\\ \left(a^\dagger\right)^\top\end{pmatrix},\quad\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}=\begin{pmatrix}a^\dagger & a^\top\end{pmatrix}U(0)\tag{3}$$

$$\begin{pmatrix}a\\ \left(a^\dagger\right)^\top\end{pmatrix}=U(0)\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix},\quad\begin{pmatrix}a^\dagger & a^\top\end{pmatrix}=\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}U^\dagger(0)\tag{4}$$

则 $H(0)$ 的基态是 $\alpha(0)$ 的真空态 $\left|0_{\alpha(0)}\right\rangle$，可以由 a 的真空 $|0_a\rangle$ 生成:

$$\left|0_{\alpha(0)}\right\rangle=\mathcal{N}(0)\exp\left(\frac{1}{2}\sum_{i,j}F_{i,j}(0)a_i^\dagger a_j^\dagger\right)|0_a\rangle\tag{5}$$

$$F(0)=V(0)^*\left[W^*(0)\right]^{-1}\tag{6}$$

$$\mathcal{N}(0)=\det{}^{-1/4}\left(I+F^\dagger(0)F(0)\right)\tag{7}$$

考虑构型 $\{u_{ij}\}$ 下，哈密顿量

$$\begin{aligned}H&=\frac{1}{2}\begin{pmatrix}a^\dagger & a^\top\end{pmatrix}h\begin{pmatrix}a\\ \left(a^\dagger\right)^\top\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}a^\dagger & a^\top\end{pmatrix}UDU^\dagger\begin{pmatrix}a\\ \left(a^\dagger\right)^\top\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}U^\dagger(0)UDU^\dagger U(0)\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}\\&\equiv\frac{1}{2}\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}\widetilde{U}D\widetilde{U}^\dagger\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}\\&\equiv\frac{1}{2}\begin{pmatrix}\alpha^\dagger & \alpha^\top\end{pmatrix}D\begin{pmatrix}\alpha\\ \left(\alpha^\dagger\right)^\top\end{pmatrix}\end{aligned}\tag{8}$$

$$\widetilde{U}\equiv U^\dagger(0)U,\quad\widetilde{U}^\dagger=U^\dagger U(0)\tag{9}$$

$$\widetilde{U}=\begin{pmatrix}\widetilde{W} & \widetilde{V}^*\\ \widetilde{V} & \widetilde{W}^*\end{pmatrix},\quad\widetilde{U}^\dagger=\begin{pmatrix}\widetilde{W}^\dagger & \widetilde{V}^\dagger\\ \widetilde{V}^\top & \widetilde{W}^\top\end{pmatrix}\tag{10}$$

$$\begin{pmatrix}\alpha^\dagger & \alpha^\top\end{pmatrix}=\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}\widetilde{U}=\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}\begin{pmatrix}\widetilde{W} & \widetilde{V}^*\\ \widetilde{V} & \widetilde{W}^*\end{pmatrix},\tag{11}$$

$$\begin{pmatrix}\alpha\\ \left(\alpha^\dagger\right)^\top\end{pmatrix}=\widetilde{U}^\dagger\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}=\begin{pmatrix}\widetilde{W}^\dagger & \widetilde{V}^\dagger\\ \widetilde{V}^\top & \widetilde{W}^\top\end{pmatrix}\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}\tag{12}$$

H 的基态是 α 的真空态 $|0_\alpha\rangle$，可以由 $\alpha(0)$ 的真空态 $\left|0_{\alpha(0)}\right\rangle$ 生成:

$$|0_\alpha\rangle = \widetilde{\mathcal{N}} \exp\left(\frac{1}{2}\sum_{i,j}\widetilde{F}_{i,j}\alpha_i^\dagger(0)\alpha_j^\dagger(0)\right)|0_{\alpha(0)}\rangle \tag{13}$$

$$\widetilde{F} = \widetilde{V}^* \left(\widetilde{W}^*\right)^{-1} \tag{14}$$

$$\widetilde{\mathcal{N}} = \det{}^{-1/4}\left(I + \widetilde{F}^\dagger \widetilde{F}\right) \tag{15}$$

实际上 $\widetilde{\mathcal{N}}$ 可以化简。

$$\widetilde{N} = \left|\det\left(\widetilde{W}\right)\right|^{1/2} \tag{16}$$

c-Majorana如何换成 $\alpha(0)$

$$c_{i,A} = a_i + a_i^\dagger, \quad c_{i,B} = \frac{1}{\mathrm{i}}\left(a_i - a_i^\dagger\right), \tag{17}$$

$$\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} = U(0)\begin{pmatrix} \alpha(0) \\ \left(\alpha^\dagger(0)\right)^\top \end{pmatrix} \tag{18}$$

$$U(0) = \begin{pmatrix} \mathbf{W}(0) & \mathbf{V}^*(0) \\ \mathbf{V}(0) & \mathbf{W}^*(0) \end{pmatrix} \tag{19}$$

$$a_i = \frac{1}{2}\left(c_{i,A} + \mathrm{i}c_{i,B}\right), \quad a_i^\dagger = \frac{1}{2}\left(c_{i,A} - \mathrm{i}c_{i,B}\right) \tag{20}$$

$$a \equiv \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \frac{1}{2}c_A + \frac{\mathrm{i}}{2}c_B \tag{21}$$

$$\left(a^\dagger\right)^\top = \frac{1}{2}c_A - \frac{\mathrm{i}}{2}c_B \tag{22}$$

一方面

$$\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I & \frac{\mathrm{i}}{2}I \\ \frac{1}{2}I & -\frac{\mathrm{i}}{2}I \end{pmatrix}\begin{pmatrix} c_A \\ c_B \end{pmatrix} \tag{23}$$

另一方面

$$\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} = U(0)\begin{pmatrix} \alpha(0) \\ \left(\alpha^\dagger(0)\right)^\top \end{pmatrix} \tag{24}$$

$$U(0) = \begin{pmatrix} \mathbf{W}(0) & \mathbf{V}^*(0) \\ \mathbf{V}(0) & \mathbf{W}^*(0) \end{pmatrix} \tag{25}$$

于是

$$\begin{aligned} \begin{pmatrix} c_A \\ c_B \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}I & \frac{\mathrm{i}}{2}I \\ \frac{1}{2}I & -\frac{\mathrm{i}}{2}I \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{W}(0) & \mathbf{V}^*(0) \\ \mathbf{V}(0) & \mathbf{W}^*(0) \end{pmatrix}\begin{pmatrix} \alpha(0) \\ \left(\alpha^\dagger(0)\right)^\top \end{pmatrix} = \begin{pmatrix} I & I \\ -\mathrm{i}I & \mathrm{i}I \end{pmatrix}\begin{pmatrix} \mathbf{W}(0) & \mathbf{V}^*(0) \\ \mathbf{V}(0) & \mathbf{W}^*(0) \end{pmatrix}\begin{pmatrix} \alpha(0) \\ \left(\alpha^\dagger(0)\right)^\top \end{pmatrix} \\ &\equiv U'(0)\begin{pmatrix} \alpha(0) \\ \left(\alpha^\dagger(0)\right)^\top \end{pmatrix} = \begin{pmatrix} W(0) + V(0) & V^*(0) + W^*(0) \\ \mathrm{i}\left(-W(0) + V(0)\right) & \mathrm{i}\left(-V^*(0) + W^*(0)\right) \end{pmatrix}\begin{pmatrix} \alpha(0) \\ \left(\alpha^\dagger(0)\right)^\top \end{pmatrix} \\ &\equiv \begin{pmatrix} \mathbf{U}'_{11}(0) & \mathbf{U}'_{12}(0) \\ \mathbf{U}'_{21}(0) & \mathbf{U}'_{22}(0) \end{pmatrix}\begin{pmatrix} \alpha(0) \\ \left(\alpha^\dagger(0)\right)^\top \end{pmatrix} \end{aligned} \tag{26}$$

$$\boxed{c_{i,A}=U'_{i,}(0)\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}=\mathbf{U}'_{11}(0)_{i,}\alpha(0)+\mathbf{U}'_{12}(0)_{i,}\left(\alpha^\dagger(0)\right)^\top} \tag{27}$$

$$\boxed{c_{i,A}=c_{i,A}^\dagger=\alpha^\dagger(0)\left[\mathbf{U}'_{11}(0)_{i,}\right]^\dagger+\alpha^\top(0)\left[\mathbf{U}'_{12}(0)_{i,}\right]^\dagger} \tag{28}$$

$$\boxed{c_{j,B}=U'_{j+N,}(0)\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}=\mathbf{U}'_{21}(0)_{j,}\alpha(0)+\mathbf{U}'_{22}(0)_{j,}\left(\alpha^\dagger(0)\right)^\top} \tag{29}$$

$$\boxed{c_{j,B}=c_{j,B}^\dagger=\alpha^\dagger(0)\left[\mathbf{U}'_{21}(0)_{j,}\right]^\dagger+\alpha^\top(0)\left[\mathbf{U}'_{22}(0)_{j,}\right]^\dagger} \tag{30}$$

跃迁振幅

$$\begin{aligned}
&\left\langle 0_\chi,0_{\alpha(1)}\,\middle|\,d_{\boldsymbol{r},x}(1)H_h d_{\boldsymbol{r},y}^\dagger(2)\,\middle|\,0_\chi,0_{\alpha(2)}\right\rangle\\
=&\left\langle \chi(\boldsymbol{r},x),0_{\alpha(1)}\,\middle|\,\alpha(1)\left[-h_z\left(\sigma_{\boldsymbol{r}}^z+\sigma_{\boldsymbol{r}+\boldsymbol{\delta}_z}^z\right)\right]\alpha^\dagger(2)\,\middle|\,\chi(\boldsymbol{r},y),0_{\alpha(2)}\right\rangle\\
=&-h_z\left\langle \chi(\boldsymbol{r},x),0_{\alpha(1)}\,\middle|\,\alpha(1)\left(\sigma_{\boldsymbol{r}}^z+\sigma_{\boldsymbol{r}+\boldsymbol{\delta}_z}^z\right)\alpha^\dagger(2)\,\middle|\,\chi(\boldsymbol{r},y),0_{\alpha(2)}\right\rangle\\
=&-h_z\left\langle \chi,0_{\alpha(1)}\,\middle|\,\alpha(1)\left[-\mathrm{i}\left(1+\mathrm{i}c_{i,A}c_{j,B}\right)\right]\alpha^\dagger(2)\,\middle|\,\chi,0_{\alpha(2)}\right\rangle\\
=&\mathrm{i}h_z\left\langle \chi,0_{\alpha(1)}\,\middle|\,\alpha(1)\left(1+\mathrm{i}c_{i,A}c_{j,B}\right)\alpha^\dagger(2)\,\middle|\,\chi,0_{\alpha(2)}\right\rangle\\
=&\mathrm{i}h_z\widetilde{\mathcal{N}}(1)\widetilde{\mathcal{N}}(2)\left\langle \chi,0_{\alpha(0)}\,\middle|\,\exp\left(\frac{1}{2}F_{i,j}^*(1)\alpha_i(0)\alpha_j(0)\right)\alpha(1)\left(1+\mathrm{i}c_{i,A}c_{j,B}\right)\alpha^\dagger(2)\exp\left(\frac{1}{2}\sum_{i,j}F_{i,j}(2)\alpha_i^\dagger(0)\alpha_j^\dagger(0)\right)\,\middle|\,\chi,0_{\alpha(0)}\right\rangle\\
\equiv&\mathrm{i}h_z\left\langle \Psi(1)\,\middle|\,\alpha(1)\left(1+\mathrm{i}c_{i,A}c_{j,B}\right)\alpha^\dagger(2)\,\middle|\,\Psi(2)\right\rangle\\
=&\mathrm{i}h_z\left\langle \Psi(1)\,\middle|\,\Psi(2)\right\rangle\left\langle\alpha(1)\left(1+\mathrm{i}c_{i,A}c_{j,B}\right)\alpha^\dagger(2)\right\rangle\\
=&h_z\langle\Psi(1)|\Psi(2)\rangle\left[\mathrm{i}\langle\alpha(1)\alpha^\dagger(2)\rangle-\langle\alpha(1)c_{i,A}c_{j,B}\alpha^\dagger(2)\rangle\right]
\end{aligned} \tag{31}$$

- $\langle\Psi(1)\,|\,\Psi(2)\rangle$

$$\langle\Psi(1)\,|\,\Psi(2)\rangle=\widetilde{N}(1)\widetilde{N}(2)\left(-1\right)^{N(N+1)/2}\operatorname{Pf}(M) \tag{32}$$

$$M\equiv\begin{pmatrix}\widetilde{F}(2) & -I\\ I & -\widetilde{F}^*(1)\end{pmatrix} \tag{33}$$

$$\widetilde{N}=\left|\det\left(\widetilde{W}\right)\right|^{1/2} \tag{34}$$

$$\widetilde{F}=\widetilde{V}^*\left(\widetilde{W}^*\right)^{-1} \tag{35}$$

$$\widetilde{U}\equiv U^\dagger(0)U,\quad \widetilde{U}^\dagger=U^\dagger U(0) \tag{36}$$

$$\widetilde{U}=\begin{pmatrix}\widetilde{W} & \widetilde{V}^*\\ \widetilde{V} & \widetilde{W}^*\end{pmatrix},\quad \widetilde{U}^\dagger=\begin{pmatrix}\widetilde{W}^\dagger & \widetilde{V}^\dagger\\ \widetilde{V}^\top & \widetilde{W}^\top\end{pmatrix} \tag{37}$$

- $\langle\alpha(1)\alpha^\dagger(2)\rangle$

$$\begin{pmatrix}\alpha(1)\\ \left(\alpha^\dagger(1)\right)^\top\end{pmatrix}=\widetilde{U}^\dagger(1)\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix}=\begin{pmatrix}\widetilde{W}^\dagger(1) & \widetilde{V}^\dagger(1)\\ \widetilde{V}^\top(1) & \widetilde{W}^\top(1)\end{pmatrix}\begin{pmatrix}\alpha(0)\\ \left(\alpha^\dagger(0)\right)^\top\end{pmatrix} \tag{38}$$

$$\boxed{\alpha(1)=\widetilde{W}^\dagger(1)\alpha(0)+\widetilde{V}^\dagger(1)\left(\alpha^\dagger(0)\right)^\top} \tag{39}$$

$$\begin{pmatrix}\alpha^\dagger(2) & \alpha^\top(2)\end{pmatrix}=\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}\widetilde{U}(2)=\begin{pmatrix}\alpha^\dagger(0) & \alpha^\top(0)\end{pmatrix}\begin{pmatrix}\widetilde{W}(2) & \widetilde{V}^*(2)\\ \widetilde{V}(2) & \widetilde{W}^*(2)\end{pmatrix}, \tag{40}$$

$$\boxed{\alpha^\dagger(2)=\alpha^\dagger(0)\widetilde{W}(2)+\alpha^\top(0)\widetilde{V}(2)} \tag{41}$$

$$
\begin{aligned}
&\left\langle\alpha(1)\alpha^{\dagger}(2)\right\rangle\\
&=\left\langle\left[\widetilde{W}^{\dagger}(1)\alpha(0)+\widetilde{V}^{\dagger}(1)\left(\alpha^{\dagger}(0)\right)^{\top}\right]\left[\alpha^{\dagger}(0)\widetilde{W}(2)+\alpha^{\top}(0)\widetilde{V}(2)\right]\right\rangle\\
&=\left\langle\begin{pmatrix}\widetilde{V}^{\dagger}(1) & \widetilde{W}^{\dagger}(1)\end{pmatrix}\begin{pmatrix}\left(\alpha^{\dagger}(0)\right)^{\top}\\ \alpha(0)\end{pmatrix}\begin{pmatrix}\alpha^{\dagger}(0) & \alpha^{\top}(0)\end{pmatrix}\begin{pmatrix}\widetilde{W}(2)\\ \widetilde{V}(2)\end{pmatrix}\right\rangle\\
&=\begin{pmatrix}\widetilde{V}^{\dagger}(1) & \widetilde{W}^{\dagger}(1)\end{pmatrix}\left\langle\begin{pmatrix}\left(\alpha^{\dagger}(0)\right)^{\top}\alpha^{\dagger}(0) & \left(\alpha^{\dagger}(0)\right)^{\top}\alpha^{\top}(0)\\ \alpha(0)\alpha^{\dagger}(0) & \alpha(0)\alpha^{\top}(0)\end{pmatrix}\right\rangle\begin{pmatrix}\widetilde{W}(2)\\ \widetilde{V}(2)\end{pmatrix}\\
&=\begin{pmatrix}\widetilde{V}^{\dagger}(1) & \widetilde{W}^{\dagger}(1)\end{pmatrix}\left(-M^{-1}+\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}\widetilde{W}(2)\\ \widetilde{V}(2)\end{pmatrix}
\end{aligned}
\tag{42}
$$

- $\left\langle\alpha(1)c_{i,A}c_{j,B}\alpha^{\dagger}(2)\right\rangle$

Wick定理给出

$$
\begin{aligned}
&\left\langle\alpha(1)c_{i,A}c_{j,B}\alpha^{\dagger}(2)\right\rangle\\
&=\left\langle\alpha(1)c_{i,A}\right\rangle\left\langle c_{j,B}\alpha^{\dagger}(2)\right\rangle-\left\langle\alpha(1)c_{j,B}\right\rangle\left\langle c_{i,A}\alpha^{\dagger}(2)\right\rangle+\left\langle\alpha(1)\alpha^{\dagger}(2)\right\rangle\left\langle c_{i,A}c_{j,B}\right\rangle
\end{aligned}
\tag{43}
$$

- - $\left\langle\alpha(1)c_{i,A}\right\rangle$

$$
\boxed{\alpha(1)=\widetilde{W}^{\dagger}(1)\alpha(0)+\widetilde{V}^{\dagger}(1)\left(\alpha^{\dagger}(0)\right)^{\top}}
\tag{44}
$$

$$
\boxed{c_{i,A}=c_{i,A}^{\dagger}=\alpha^{\dagger}(0)\left[\mathbf{U}_{11}'(0)_{i,}\right]^{\dagger}+\alpha^{\top}(0)\left[\mathbf{U}_{12}'(0)_{i,}\right]^{\dagger}}
\tag{45}
$$

$$
\begin{aligned}
&\left\langle\alpha(1)c_{i,A}\right\rangle\\
&=\left\langle\left\{\widetilde{W}^{\dagger}(1)\alpha(0)+\widetilde{V}^{\dagger}(1)\left(\alpha^{\dagger}(0)\right)^{\top}\right\}\left\{\alpha^{\dagger}(0)\left[\mathbf{U}_{11}'(0)_{i,}\right]^{\dagger}+\alpha(0)^{\top}\left[\mathbf{U}_{12}'(0)_{i,}\right]^{\dagger}\right\}\right\rangle\\
&=\begin{pmatrix}\widetilde{V}^{\dagger}(1) & \widetilde{W}^{\dagger}(1)\end{pmatrix}\left\langle\begin{pmatrix}\left(\alpha^{\dagger}(0)\right)^{\top}\\ \alpha(0)\end{pmatrix}\begin{pmatrix}\alpha^{\dagger}(0) & \alpha^{\top}(0)\end{pmatrix}\right\rangle\begin{pmatrix}\left[\mathbf{U}_{11}'(0)_{i,}\right]^{\dagger}\\ \left[\mathbf{U}_{12}'(0)_{i,}\right]^{\dagger}\end{pmatrix}\\
&=\begin{pmatrix}\widetilde{V}^{\dagger}(1) & \widetilde{W}^{\dagger}(1)\end{pmatrix}\left(-M^{-1}+\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}\left[\mathbf{U}_{11}'(0)_{i,}\right]^{\dagger}\\ \left[\mathbf{U}_{12}'(0)_{i,}\right]^{\dagger}\end{pmatrix}
\end{aligned}
\tag{46}
$$

- - $\left\langle\alpha(1)c_{j,B}\right\rangle$

$$
\boxed{\alpha(1)=\widetilde{W}^{\dagger}(1)\alpha(0)+\widetilde{V}^{\dagger}(1)\left(\alpha^{\dagger}(0)\right)^{\top}}
\tag{47}
$$

$$
\boxed{c_{j,B}=c_{j,B}^{\dagger}=\alpha^{\dagger}(0)\left[\mathbf{U}_{21}'(0)_{j,}\right]^{\dagger}+\alpha^{\top}(0)\left[\mathbf{U}_{22}'(0)_{j,}\right]^{\dagger}}
\tag{48}
$$

$$
\begin{aligned}
&\left\langle\alpha(1)c_{j,B}\right\rangle\\
&=\left\langle\left\{\widetilde{W}^{\dagger}(1)\alpha(0)+\widetilde{V}^{\dagger}(1)\left(\alpha^{\dagger}(0)\right)^{\top}\right\}\left\{\alpha^{\dagger}(0)\left[\mathbf{U}_{21}'(0)_{j,}\right]^{\dagger}+\alpha^{\top}(0)\left[\mathbf{U}_{22}'(0)_{j,}\right]^{\dagger}\right\}\right\rangle\\
&=\begin{pmatrix}\widetilde{V}^{\dagger}(1) & \widetilde{W}^{\dagger}(1)\end{pmatrix}\left\langle\begin{pmatrix}\left(\alpha^{\dagger}(0)\right)^{\top}\\ \alpha(0)\end{pmatrix}\begin{pmatrix}\alpha^{\dagger}(0) & \alpha^{\top}(0)\end{pmatrix}\right\rangle\begin{pmatrix}\left[\mathbf{U}_{21}'(0)_{j,}\right]^{\dagger}\\ \left[\mathbf{U}_{22}'(0)_{j,}\right]^{\dagger}\end{pmatrix}\\
&=\begin{pmatrix}\widetilde{V}^{\dagger}(1) & \widetilde{W}^{\dagger}(1)\end{pmatrix}\left(-M^{-1}+\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}\left[\mathbf{U}_{21}'(0)_{j,}\right]^{\dagger}\\ \left[\mathbf{U}_{22}'(0)_{j,}\right]^{\dagger}\end{pmatrix}
\end{aligned}
\tag{49}
$$

- - $\left\langle c_{i,A}\alpha^{\dagger}(2)\right\rangle$

$$
\boxed{\alpha^{\dagger}(2)=\alpha^{\dagger}(0)\widetilde{W}(2)+\alpha^{\top}(0)\widetilde{V}(2)}
\tag{50}
$$

$$\boxed{c_{i,A}=U'_{i,}(0)\begin{pmatrix}\alpha(0)\\ \left(\alpha^{\dagger}(0)\right)^{\top}\end{pmatrix}=\mathbf{U}'_{11}(0)_{i,}\alpha(0)+\mathbf{U}'_{12}(0)_{i,}\left(\alpha^{\dagger}(0)\right)^{\top}} \tag{51}$$

$$\begin{aligned}&\left\langle c_{i,A}\alpha^{\dagger}(2)\right\rangle\\ =&\left\langle\left\{\mathbf{U}'_{11}(0)_{i,}\alpha(0)+\mathbf{U}'_{12}(0)_{i,}\left(\alpha^{\dagger}(0)\right)^{\top}\right\}\left\{\alpha^{\dagger}(0)\widetilde{W}(2)+\alpha^{\top}(0)\widetilde{V}(2)\right\}\right\rangle\\ =&\begin{pmatrix}\mathbf{U}'_{12}(0)_{i,} & \mathbf{U}'_{11}(0)_{i,}\end{pmatrix}\left\langle\begin{pmatrix}\left(\alpha^{\dagger}(0)\right)^{\top}\\ \alpha(0)\end{pmatrix}\begin{pmatrix}\alpha^{\dagger}(0) & \alpha^{\top}(0)\end{pmatrix}\right\rangle\begin{pmatrix}\widetilde{W}(2)\\ \widetilde{V}(2)\end{pmatrix}\\ =&\begin{pmatrix}\mathbf{U}'_{12}(0)_{i,} & \mathbf{U}'_{11}(0)_{i,}\end{pmatrix}\left(-M^{-1}+\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}\widetilde{W}(2)\\ \widetilde{V}(2)\end{pmatrix}\end{aligned} \tag{52}$$

- ◦ $\left\langle c_{j,B}\alpha^{\dagger}(2)\right\rangle$

$$\boxed{\alpha^{\dagger}(2)=\alpha^{\dagger}(0)\widetilde{W}(2)+\alpha^{\top}(0)\widetilde{V}(2)} \tag{53}$$

$$\boxed{c_{j,B}=U'_{j+N,}(0)\begin{pmatrix}\alpha(0)\\ \left(\alpha^{\dagger}(0)\right)^{\top}\end{pmatrix}=\mathbf{U}'_{21}(0)_{j,}\alpha(0)+\mathbf{U}'_{22}(0)_{j,}\left(\alpha^{\dagger}(0)\right)^{\top}} \tag{54}$$

$$\begin{aligned}&\left\langle c_{j,B}\alpha^{\dagger}(2)\right\rangle\\ =&\left\langle\left\{\mathbf{U}'_{21}(0)_{j,}\alpha(0)+\mathbf{U}'_{22}(0)_{j,}\left(\alpha^{\dagger}(0)\right)^{\top}\right\}\left\{\alpha^{\dagger}(0)\widetilde{W}(2)+\alpha^{\top}(0)\widetilde{V}(2)\right\}\right\rangle\\ =&\begin{pmatrix}\mathbf{U}'_{22}(0)_{j,} & \mathbf{U}'_{21}(0)_{j,}\end{pmatrix}\left\langle\begin{pmatrix}\left(\alpha^{\dagger}(0)\right)^{\top}\\ \alpha(0)\end{pmatrix}\begin{pmatrix}\alpha^{\dagger}(0) & \alpha^{\top}(0)\end{pmatrix}\right\rangle\begin{pmatrix}\widetilde{W}(2)\\ \widetilde{V}(2)\end{pmatrix}\\ =&\begin{pmatrix}\mathbf{U}'_{22}(0)_{j,} & \mathbf{U}'_{21}(0)_{j,}\end{pmatrix}\left(-M^{-1}+\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}\widetilde{W}(2)\\ \widetilde{V}(2)\end{pmatrix}\end{aligned} \tag{55}$$

- ◦ $\left\langle c_{i,A}c_{j,B}\right\rangle$

$$\boxed{c_{i,A}=U'_{i,}(0)\begin{pmatrix}\alpha(0)\\ \left(\alpha^{\dagger}(0)\right)^{\top}\end{pmatrix}=\mathbf{U}'_{11}(0)_{i,}\alpha(0)+\mathbf{U}'_{12}(0)_{i,}\left(\alpha^{\dagger}(0)\right)^{\top}} \tag{56}$$

$$\boxed{c_{j,B}=c_{j,B}^{\dagger}=\alpha^{\dagger}(0)\left[\mathbf{U}'_{21}(0)_{j,}\right]^{\dagger}+\alpha^{\top}(0)\left[\mathbf{U}'_{22}(0)_{j,}\right]^{\dagger}} \tag{57}$$

$$\begin{aligned}&\left\langle c_{i,A}c_{j,B}\right\rangle\\ =&\left\langle\left\{\mathbf{U}'_{11}(0)_{i,}\alpha(0)+\mathbf{U}'_{12}(0)_{i,}\left(\alpha^{\dagger}(0)\right)^{\top}\right\}\left\{\alpha^{\dagger}(0)\left[\mathbf{U}'_{21}(0)_{j,}\right]^{\dagger}+\alpha^{\top}(0)\left[\mathbf{U}'_{22}(0)_{j,}\right]^{\dagger}\right\}\right\rangle\\ =&\begin{pmatrix}\mathbf{U}'_{12}(0)_{i,} & \mathbf{U}'_{11}(0)_{i,}\end{pmatrix}\left\langle\begin{pmatrix}\left(\alpha^{\dagger}(0)\right)^{\top}\alpha^{\dagger}(0) & \left(\alpha^{\dagger}(0)\right)^{\top}\alpha^{\top}(0)\\ \alpha(0)\alpha^{\dagger}(0) & \alpha(0)\alpha^{\top}(0)\end{pmatrix}\right\rangle\begin{pmatrix}\left[\mathbf{U}'_{21}(0)_{j,}\right]^{\dagger}\\ \left[\mathbf{U}'_{22}(0)_{j,}\right]^{\dagger}\end{pmatrix}\\ =&\begin{pmatrix}\mathbf{U}'_{12}(0)_{i,} & \mathbf{U}'_{11}(0)_{i,}\end{pmatrix}\left(-M^{-1}+\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}\left[\mathbf{U}'_{21}(0)_{j,}\right]^{\dagger}\\ \left[\mathbf{U}'_{22}(0)_{j,}\right]^{\dagger}\end{pmatrix}\end{aligned} \tag{58}$$

以B子格为中心

要算

$$\begin{aligned}&\left\langle 0_{\chi},0_{\alpha(1)}\left|\chi_{\boldsymbol{r}+\boldsymbol{a}_2,x}\alpha(1)H_h\alpha^{\dagger}(2)\chi^{\dagger}_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}\right|0_{\chi},0_{\alpha(2)}\right\rangle\\ =&\left\langle 0_{\chi},0_{\alpha(1)}\left|\chi_{\boldsymbol{r}+\boldsymbol{a}_2,x}\alpha(1)\left[-h_z\left(\sigma_{\boldsymbol{r}}+\sigma_{\boldsymbol{r}+\boldsymbol{\delta}_z}\right)\right]\alpha^{\dagger}(2)\chi^{\dagger}_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}\right|0_{\chi},0_{\alpha(2)}\right\rangle\end{aligned} \tag{59}$$

注意到

$$\chi_{\boldsymbol{r},\alpha} \equiv \frac{1}{2}\left(b_{\boldsymbol{r}}^{\alpha} + \mathrm{i} b_{\boldsymbol{r}+\boldsymbol{\delta}_\alpha}^{\alpha}\right), \quad \chi_{\boldsymbol{r},\alpha}^{\dagger} \equiv \frac{1}{2}\left(b_{\boldsymbol{r}}^{\alpha} - \mathrm{i} b_{\boldsymbol{r}+\boldsymbol{\delta}_\alpha}^{\alpha}\right), \quad \boldsymbol{r} \in A \tag{60}$$

$$b_{\boldsymbol{r}}^{\alpha} = \chi_{\boldsymbol{r},\alpha} + \chi_{\boldsymbol{r},\alpha}^{\dagger}, \quad b_{\boldsymbol{r}+\boldsymbol{\delta}_\alpha}^{\alpha} = \frac{1}{\mathrm{i}}\left(\chi_{\boldsymbol{r},\alpha} - \chi_{\boldsymbol{r},\alpha}^{\dagger}\right) \tag{61}$$

于是

$$\begin{aligned}
\sigma_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{z} &= \mathrm{i} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{z} c_{\boldsymbol{r}+\boldsymbol{\delta}_z} = -\mathrm{i} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{x} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{y} \\
&= -\mathrm{i} b_{\boldsymbol{r}+\boldsymbol{a_2}+\boldsymbol{\delta}_x}^{x} b_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a_2}+\boldsymbol{\delta}_y}^{y} \\
&= -\mathrm{i} \cdot \frac{1}{\mathrm{i}}\left(\chi_{\boldsymbol{r}+\boldsymbol{a_2},x} - \chi_{\boldsymbol{r}+\boldsymbol{a_2},x}^{\dagger}\right) \cdot \frac{1}{\mathrm{i}}\left(\chi_{\boldsymbol{r}-\boldsymbol{a_1}+\boldsymbol{a_2},y} - \chi_{\boldsymbol{r}-\boldsymbol{a_1}+\boldsymbol{a_2},y}^{\dagger}\right) \\
&= \mathrm{i}\left(\chi_{\boldsymbol{r}+\boldsymbol{a_2},x} - \chi_{\boldsymbol{r}+\boldsymbol{a_2},x}^{\dagger}\right)\left(\chi_{\boldsymbol{r}-\boldsymbol{a_1}+\boldsymbol{a_2},y} - \chi_{\boldsymbol{r}-\boldsymbol{a_1}+\boldsymbol{a_2},y}^{\dagger}\right)
\end{aligned} \tag{62}$$

$$\begin{aligned}
\sigma_{\boldsymbol{r}+\boldsymbol{\delta}_z} \chi_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}^{\dagger} &= \mathrm{i}\left(\chi_{\boldsymbol{r}+\boldsymbol{a_2},x} - \chi_{\boldsymbol{r}+\boldsymbol{a_2},x}^{\dagger}\right)\left(\chi_{\boldsymbol{r}-\boldsymbol{a_1}+\boldsymbol{a_2},y} - \chi_{\boldsymbol{r}-\boldsymbol{a_1}+\boldsymbol{a_2},y}^{\dagger}\right) \cdot \chi_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}^{\dagger} \\
&= \mathrm{i}\left(\chi_{\boldsymbol{r}+\boldsymbol{a_2},x} - \chi_{\boldsymbol{r}+\boldsymbol{a_2},x}^{\dagger}\right) \\
\text{在braket中} &= -\mathrm{i}\chi_{\boldsymbol{r}+\boldsymbol{a_2},x}^{\dagger}
\end{aligned} \tag{63}$$

$$\begin{aligned}
\sigma_{\boldsymbol{r}}^{z} = \mathrm{i} b_{\boldsymbol{r}}^{z} c_{\boldsymbol{r}} &= b_{\boldsymbol{r}+\boldsymbol{\delta_z}}^{x} b_{\boldsymbol{r}+\boldsymbol{\delta_z}}^{y} b_{\boldsymbol{r}+\boldsymbol{\delta_z}}^{z} c_{\boldsymbol{r}+\boldsymbol{\delta_z}}\left(\mathrm{i} b_{\boldsymbol{r}}^{z} c_{\boldsymbol{r}}\right) \\
&= \left(-\mathrm{i} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{x} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{y}\right)\left(\mathrm{i} c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta}_z}\right)\left(-\mathrm{i} b_{\boldsymbol{r}}^{z} b_{\boldsymbol{r}+\boldsymbol{\delta_z}}^{z}\right) \\
&= \left(-\mathrm{i} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{x} b_{\boldsymbol{r}+\boldsymbol{\delta_z}}^{y}\right)\left(\mathrm{i} c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta}_z}\right) \cdot 1
\end{aligned} \tag{64}$$

$$\begin{aligned}
\sigma_{\boldsymbol{r}}^{z} \chi_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}^{\dagger} &= \left(-\mathrm{i} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{x} b_{\boldsymbol{r}+\boldsymbol{\delta_z}}^{y}\right)\left(\mathrm{i} c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta_z}}\right) \chi_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}^{\dagger} \\
&= \left(\mathrm{i} c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta_z}}\right)\left(-\mathrm{i} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{x} b_{\boldsymbol{r}+\boldsymbol{\delta}_z}^{y}\right) \chi_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}^{\dagger} \\
&= \left(\mathrm{i} c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta_z}}\right)\left[\mathrm{i}\left(\chi_{\boldsymbol{r}+\boldsymbol{a_2},x} - \chi_{\boldsymbol{r}+\boldsymbol{a_2},x}^{\dagger}\right)\right] \\
&= -c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta}_z}\left(\chi_{\boldsymbol{r}+\boldsymbol{a_2},x} - \chi_{\boldsymbol{r}+\boldsymbol{a_2},x}^{\dagger}\right) \\
\text{在braket中} &= c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta}_z} \chi_{\boldsymbol{r}+\boldsymbol{a_2},x}^{\dagger}
\end{aligned} \tag{65}$$

因此

$$\begin{aligned}
&\left\langle 0_\chi, 0_{\alpha(1)} \middle| \chi_{\boldsymbol{r}+\boldsymbol{a}_2,x} \alpha(1) H_h \alpha^{\dagger}(2) \chi_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}^{\dagger} \middle| 0_\chi, 0_{\alpha(2)} \right\rangle \\
=& \left\langle 0_\chi, 0_{\alpha(1)} \middle| \chi_{\boldsymbol{r}+\boldsymbol{a}_2,x} \alpha(1)\left[-h_z\left(\sigma_{\boldsymbol{r}} + \sigma_{\boldsymbol{r}+\boldsymbol{\delta}_z}\right)\right] \alpha^{\dagger}(2) \chi_{\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y}^{\dagger} \middle| 0_\chi, 0_{\alpha(2)} \right\rangle \\
=& -h_z \left\langle 0_\chi, 0_{\alpha(1)} \middle| \chi_{\boldsymbol{r}+\boldsymbol{a}_2,x} \alpha(1)\left[\left(c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta_z}} - \mathrm{i}\right)\right] \alpha^{\dagger}(2) \chi_{\boldsymbol{r}+\boldsymbol{a}_2,x}^{\dagger} \middle| 0_\chi, 0_{\alpha(2)} \right\rangle \\
=& \mathrm{i} h_z \left\langle 0_\chi, 0_{\alpha(1)} \middle| \chi_{\boldsymbol{r}+\boldsymbol{a}_2,x} \alpha(1)\left(1 + \mathrm{i} c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta}_z}\right) \alpha^{\dagger}(2) \chi_{\boldsymbol{r}+\boldsymbol{a}_2,x}^{\dagger} \middle| 0_\chi, 0_{\alpha(2)} \right\rangle \\
=& \mathrm{i} h_z \left\langle 0_\chi, 0_{\alpha(1)} \middle| \alpha(1)\left(1 + \mathrm{i} c_{\boldsymbol{r}} c_{\boldsymbol{r}+\boldsymbol{\delta}_z}\right) \alpha^{\dagger}(2) \middle| 0_\chi, 0_{\alpha(2)} \right\rangle \\
=& \mathrm{i} h_z \left\langle 0_\chi, 0_{\alpha(1)} \middle| \alpha(1)\left(1 + \mathrm{i} c_{i,A} c_{j,B}\right) \alpha^{\dagger}(2) \middle| 0_\chi, 0_{\alpha(2)} \right\rangle
\end{aligned} \tag{66}$$

$T(\boldsymbol{a}_1)$ 的作用

$$\begin{aligned}
& T(\boldsymbol{a}_1)\begin{pmatrix} a_1 & a_2 & \cdots & a_{N_1} & a_{N_1+1} & a_{N_1+2} & \cdots & a_{2N_1} & \cdots & \cdots \end{pmatrix} T^{-1}(\boldsymbol{a}_1) \\
=& \begin{pmatrix} a_2 & a_3 & \cdots & -a_1 & a_{N_1+2} & a_{N_1+3} & \cdots & -a_{N_1} & \cdots & \cdots \end{pmatrix} \\
=& \begin{pmatrix} a_1 & a_2 & \cdots & a_{N_1} & a_{N_1+1} & a_{N_1+2} & \cdots & a_{2N_1} & \cdots & \cdots \end{pmatrix} \bigoplus_{i=1}^{N_2} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}
\end{aligned} \tag{67}$$

$$T_1 \equiv \bigoplus_{i=1}^{N_2} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{68}$$

$$T(\boldsymbol{a}_1) \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_1) = \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix} \tag{69}$$

$$\begin{aligned} T(\boldsymbol{a}_1) \begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_1) &= \begin{pmatrix} T_1^\dagger & 0 \\ 0 & T_1^\dagger \end{pmatrix} \begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} \\ &= \begin{pmatrix} T_1^\top & 0 \\ 0 & T_1^\top \end{pmatrix} \begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} \end{aligned} \tag{70}$$

$$\begin{aligned} \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} &= T(\boldsymbol{a}_1) \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_1) \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix}^{-1} \\ &= T(\boldsymbol{a}_1) \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_1) \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix}^{\dagger} \\ &= T(\boldsymbol{a}_1) \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_1) \begin{pmatrix} T_1^\top & 0 \\ 0 & T_1^\top \end{pmatrix} \end{aligned} \tag{71}$$

$$\boxed{\begin{aligned} T(-\boldsymbol{a}_1) \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T^{-1}(-\boldsymbol{a}_1) &= T^{-1}(\boldsymbol{a}_1) \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T(\boldsymbol{a}_1) \\ &= \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} \begin{pmatrix} T_1^\top & 0 \\ 0 & T_1^\top \end{pmatrix} \end{aligned}} \tag{72}$$

$$T(-\boldsymbol{a}_1) \begin{pmatrix} a^\top \\ \left(a^\dagger\right)^\top \end{pmatrix} T^{-1}(-\boldsymbol{a}_1) = \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} \tag{73}$$

$T(\boldsymbol{a}_2)$ 的作用

$$\begin{aligned} & T(\boldsymbol{a}_2) \begin{pmatrix} a_1 & a_2 & \cdots & a_{N_1} & a_{N_1+1} & a_{N_1+2} & \cdots & a_{2N_1} & \cdots & \cdots \end{pmatrix} T^{-1}(\boldsymbol{a}_2) \\ =& T(\boldsymbol{a}_1) \begin{pmatrix} a_{N_1+1} & a_{N_1+2} & \cdots & a_{2N_1} & a_{2N_1+1} & a_{2N_1+2} & \cdots & a_{3N_1} & \cdots & -a_1 & -a_2 & \cdots & -a_{N_1} \end{pmatrix} T^{-1}(\boldsymbol{a}_1) \\ =& \begin{pmatrix} a_1 & a_2 & \cdots & a_{N_1} & a_{N_1+1} & a_{N_1+2} & \cdots & a_{2N_1} & \cdots & \cdots \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & -I_{N_1} \\ I_{N_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & I_{N_1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & I_{N_1} & 0 \end{pmatrix} \end{aligned} \tag{74}$$

$$T_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & -I_{N_1} \\ I_{N_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & I_{N_1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & I_{N_1} & 0 \end{pmatrix} \tag{75}$$

$$T(\boldsymbol{a}_2) \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_2) = \begin{pmatrix} a^\dagger & a^\top \end{pmatrix} \begin{pmatrix} T_2 & 0 \\ 0 & T_2 \end{pmatrix} \tag{76}$$

$$T(\boldsymbol{a}_2)\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_2) = \begin{pmatrix} T_2^\dagger & 0 \\ 0 & T_2^\dagger \end{pmatrix}\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} = \begin{pmatrix} T_2^\top & 0 \\ 0 & T_2^\top \end{pmatrix}\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} \tag{77}$$

哈密顿量的关系

已知 $H(\boldsymbol{r},x/y/z)$，要求 $H(\boldsymbol{r}+\boldsymbol{a}_2,x)$ 和 $H(\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y)$

$$\begin{aligned} H(\boldsymbol{r}+\boldsymbol{a}_2,x) &= T(\boldsymbol{a}_2)H(\boldsymbol{r},x)T^{-1}(\boldsymbol{a}_2) \\ &= \frac{1}{2}T(\boldsymbol{a}_2)\begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_2)h(\boldsymbol{r},x)T(\boldsymbol{a}_2)\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_2) \\ &= \frac{1}{2}\begin{pmatrix} a^\dagger & a^\top \end{pmatrix}\begin{pmatrix} T_2 & 0 \\ 0 & T_2 \end{pmatrix} h(\boldsymbol{r},x)\begin{pmatrix} T_2^\top & 0 \\ 0 & T_2^\top \end{pmatrix}\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} \end{aligned} \tag{78}$$

$$\boxed{h(\boldsymbol{r}+\boldsymbol{a}_2) = \begin{pmatrix} T_2 & 0 \\ 0 & T_2 \end{pmatrix} h(\boldsymbol{r},x)\begin{pmatrix} T_2^\top & 0 \\ 0 & T_2^\top \end{pmatrix}} \tag{79}$$

$$h(\boldsymbol{r},x)U = UD \Longrightarrow h(\boldsymbol{r}+\boldsymbol{a}_2)\left(\mathbf{T}_2 U\right) = \left(\mathbf{T}_2 U\right) D \tag{80}$$

$$\begin{aligned} H(\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y) &= T(-\boldsymbol{a}_1)T(\boldsymbol{a}_2)H(\boldsymbol{r},y)T^{-1}(\boldsymbol{a}_2)T^{-1}(-\boldsymbol{a}_1) \\ &= \frac{1}{2}T(-\boldsymbol{a}_1)T(\boldsymbol{a}_2)\begin{pmatrix} a^\dagger & a^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_2)T^{-1}(-\boldsymbol{a}_1)h(\boldsymbol{r},y)T(-\boldsymbol{a}_1)T(\boldsymbol{a}_2)\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} T^{-1}(\boldsymbol{a}_2)T^{-1}(-\boldsymbol{a}_1) \\ &= \frac{1}{2}\begin{pmatrix} a^\dagger & a^\top \end{pmatrix}\begin{pmatrix} T_2 & 0 \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} T_1^\top & 0 \\ 0 & T_1^\top \end{pmatrix} h(\boldsymbol{r},y)\begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix}\begin{pmatrix} T_2^\top & 0 \\ 0 & T_2^\top \end{pmatrix}\begin{pmatrix} a \\ \left(a^\dagger\right)^\top \end{pmatrix} \end{aligned} \tag{81}$$

$$\boxed{h(\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2,y) = \begin{pmatrix} T_2 & 0 \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} T_1^\top & 0 \\ 0 & T_1^\top \end{pmatrix} h(\boldsymbol{r},y)\begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix}\begin{pmatrix} T_2^\top & 0 \\ 0 & T_2^\top \end{pmatrix}} \tag{82}$$

$$h(\boldsymbol{r},y)U = UD \Longrightarrow h(\boldsymbol{r}-\boldsymbol{a}_1+\boldsymbol{a}_2)\left(\mathbf{T}_2\mathbf{T}_1^\top U\right) = \left(\mathbf{T}_2\mathbf{T}_1^\top U\right) D \tag{83}$$