

S.-T. Yau College Student Mathematics Contests 2011

Geometry and Topology

Individual

9:30{12:00 am, July 10, 2011
(Please select 5 problems to solve)

- Suppose M is a closed smooth n -manifold.
 - Does there always exist a smooth map $f: M \rightarrow S^n$ from M into the n -sphere, such that f is essential (i.e. f is not homotopic to a constant map)? Justify your answer.
 - Same question, replacing S^n by the n -torus T^n .
- Suppose $(X; d)$ is a compact metric space and $f: X \rightarrow X$ is a map so that $d(f(x), f(y)) = d(x, y)$ for all x, y in X . Show that f is an onto map.
- Let C_1, C_2 be two linked circles in \mathbb{R}^3 . Show that C_1 cannot be homotopic to a point in $\mathbb{R}^3 - C_2$.
- Let $M = \mathbb{R}^2 / \mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in \mathbb{R}^2 , and $S = p_*(L) \subset M$ where $p: \mathbb{R}^2 \rightarrow M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S .
- A regular curve C in \mathbb{R}^3 is called a *Bertrand Curve*, if there exists a diffeomorphism $f: C \rightarrow D$ from C onto a different regular curve D in \mathbb{R}^3 such that $N_x C = N_{f(x)} D$ for any $x \in C$. Here $N_x C$ denotes the principal normal line of the curve C passing through x , and $T_x C$ will denote the tangent line of C at x . Prove that:
 - The distance $|x - f(x)|$ is constant for $x \in C$; and the angle made between the directions of the two tangent lines $T_x C$ and $T_{f(x)} D$ is also constant.
 - If the curvature k and torsion τ of C are nowhere zero, then there must be constants μ and ν such that $\mu k + \nu \tau = 1$.
- Let M be the closed surface generated by carrying a small circle with radius r around a closed curve C embedded in \mathbb{R}^3 such that the center moves along C and the circle is in the normal plane to C at each point. Prove that

$$\int_M H^2 dA \geq 2\pi^2 r^2;$$

and the equality holds if and only if C is a circle with radius $\sqrt{2}r$. Here H is the mean curvature of M and dA is the area element of M .