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> 证明正反粒子单位旋量正交关系
>
> $$u_i^\dagger(\vec{p})u_j(\vec{p})=\delta_{ij}$$
>
> $$v_i^\dagger(\vec{p})v_j(\vec{p})=\delta_{ij}$$
>
> $$u_i^\dagger(\vec{p})v_j(-\vec{p})=0$$
>
> $$v_i^\dagger(-\vec{p})u_j(\vec{p})=0$$
>
> $$u_i^\dagger(-\vec{p})v_j(\vec{p})=0$$
>
> $$v_i^\dagger(\vec{p})u_j(-\vec{p})=0$$

证明：

由于 $u_a(\vec{p}),a=1,2,3,4$ 是力学量完全集 $\{H,\vec{\sigma}\cdot\vec{n}\}$ 属于不同本征值的本征态，因此它们正交。若进一步要求正交归一，则有：

$$u_a^\dagger(\vec{p})u_b(\vec{p})=\delta_{ab},\quad a,b=1,2,3,4$$

因此：

$$u_i^\dagger(\vec{p})u_j(\vec{p})=\delta_{ij},\quad i,j=1,2$$

$v_a(\vec{p})$ 的正交性：

$$
\begin{aligned}
v_a^{\dagger} v_b &= \left(u_a^C\right)^{\dagger} u_b^C \\
&= \left(C \bar{u}_a^{\mathrm{T}}\right)^{\dagger} \left(C \bar{u}_b^{\mathrm{T}}\right) \\
&= \left(\bar{u}_a^{\mathrm{T}}\right)^{\dagger} C^{\dagger} C \bar{u}_b^{\dagger} \\
&= \left(\bar{u}_a^{\mathrm{T}}\right)^{\dagger} \bar{u}_b^{\mathrm{T}} \\
&= \left(\left(u_a^{\dagger} \gamma_4\right)^{\mathrm{T}}\right)^{\dagger} \left(u_b^{\dagger} \gamma_4\right)^{\mathrm{T}} \\
&= \left(\gamma_4 u_a\right)^{\mathrm{T}} \gamma_4^{\mathrm{T}} \left(u_b^{\dagger}\right)^{\mathrm{T}} \\
&= \left(u_b^{\dagger} \gamma_4 \gamma_4 u_a\right)^{\mathrm{T}} \\
&= \left(u_b^{\dagger} u_a\right)^{\mathrm{T}} \\
&= \delta_{ba} \\
&= \delta_{ab}
\end{aligned}
$$

因此

$$
v_i^{\dagger} v_j = \delta_{ij}, \quad i, j = 1, 2
$$

由于:

$$
v_1(\vec{p}) = \alpha_1 u_4(-\vec{p})
$$

$$
v_2(\vec{p}) = \alpha_2 u_3(-\vec{p})
$$

$$
v_3(\vec{p}) = \alpha_3 u_2(-\vec{p})
$$

$$
v_4(\vec{p}) = \alpha_4 u_1(-\vec{p})
$$

$$
|\alpha_a| = 1
$$

因此，由 $u_a(\vec{p})$ 的正交性可得:

$$
u_i^{\dagger}(\vec{p}) v_j(-\vec{p}) = 0
$$

$$
v_i^{\dagger}(-\vec{p}) u_j(\vec{p}) = 0
$$

由 $v_a(\vec{p})$ 的正交性可得:

$$
u_i^{\dagger}(-\vec{p}) v_j(\vec{p}) = 0
$$

$$
v_i^{\dagger}(\vec{p}) u_j(-\vec{p}) = 0
$$