# 2019 - 2020 Spring Term: Midterm Exam for Linear Algebra

Please provide a detailed derivation or explanation of each of your solutions. Otherwise it will be considered as not solved.

提供的解答应该有过程和细节, 否则会被认定为无效解答.

## I. LINEAR EQUATIONS (线性方程组)

1. Find the solutions of $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}. \tag{1}$$

Read off the inverse of $\boldsymbol{A}$ from the solutions. (3 points)

求解线性方程组 $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{A}$ 如 (1) 式所示. 然后根据解写出 $\boldsymbol{A}$ 的逆矩阵. (3 分)

2. The three column vectors of a matrix $\boldsymbol{A}$ are given by

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \boldsymbol{a}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \ \boldsymbol{a}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}. \tag{2}$$

(a) Determine the rank of $\boldsymbol{A}$. (2 points)
(b) Each of the three equations obtained from $\boldsymbol{Ax} = \boldsymbol{0}$ describes a plane. Find the crossing points $\boldsymbol{x}$ of the three planes for $x_1 = 1$. (2 points)
(c) Is $\boldsymbol{A}$ invertible? (1 point)

构成矩阵 $\boldsymbol{A}$ 的三个列向量如 (2) 式所示,
(a) 求 rank $\boldsymbol{A}$. (2 分)
(b) 方程组 $\boldsymbol{Ax} = \boldsymbol{0}$ 中的每个方程均对应一个平面, 找出这三个平面的一个交点, 其第一个坐标分量满足 $x_1 = 1$. (2 分)
(c) $\boldsymbol{A}$ 是可逆的吗? (1 分)

3. For which values of $c$ does $\boldsymbol{Ax} = \boldsymbol{0}$ with

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & c \end{bmatrix} \tag{3}$$

have non-trivial solutions? (5 points)

确定能够使齐次方程组 $\boldsymbol{Ax} = \boldsymbol{0}$ 有非平凡解的参数 $c$ 的值. (5 分)

## II. LINEAR TRANSFORMATIONS AND MATRIX OPERATIONS (线性变换和矩阵运算)

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates each point in $\mathbb{R}^2$ counterclockwise about the origin by an angle $\theta$.

(a) Determine the standard matrix $\boldsymbol{A} = [T(\boldsymbol{e}_1), T(\boldsymbol{e}_2)]$ for the transformation. (6 points)
(b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates each point in $\mathbb{R}^2$ about the origin by an angle $-\theta$. Show that $TS = \boldsymbol{I_2}$. (2 points)
(c) Show that applying $T$ twice to a vector $\boldsymbol{u}$ corresponds to a rotation of $\boldsymbol{u}$ by an angle $2\theta$ about the origin. (2 points)

$T : \mathbb{R}^2 \to \mathbb{R}^2$ 是一个线性变换, 它将平面上的各个点绕原点沿逆时针方向旋转 $\theta$ 角.
(a) 确定该变换的标准矩阵. (6 分)
(b) 设 $S : \mathbb{R}^2 \to \mathbb{R}^2$ 是将 $\mathbb{R}^2$ 上的点绕原点沿逆时针方向旋转 $-\theta$ 角的线性变换, 证明 $TS = \boldsymbol{I_2}$. (2 分)
(c) 通过计算指出线性变换 $T$ 作用两次在某一向量 $\boldsymbol{u}$ 上相当于把 $\boldsymbol{u}$ 绕原点沿逆时针方向旋转了 $2\theta$ 角. (2 分)

## III. BASIS TRANSFORMATIONS (基变换)

5. Let $\mathcal{H} = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where

$$\boldsymbol{v}_1 = \begin{bmatrix} -6 \\ 4 \\ -9 \\ 4 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 8 \\ -3 \\ 7 \\ -3 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} -9 \\ 5 \\ -8 \\ 3 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} 4 \\ 7 \\ -8 \\ 3 \end{bmatrix}. \tag{4}$$

Show that $\mathcal{B}$ is a basis of $\mathcal{H}$ and find the $\mathcal{B}$ coordinate vector of $\boldsymbol{x}$. (10 points)

有子空间 $\mathcal{H} = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ 和向量组 $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, 证明 $\mathcal{B}$ 是 $\mathcal{H}$ 的一个基, 并确定 (4) 式中的向量 $\boldsymbol{x}$ 在 $\mathcal{B}$ 下的坐标向量. (10 分)

6. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ be basis vectors in $\mathbb{R}^3$. Suppose that $T(\boldsymbol{v}_1) = \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3$, $T(\boldsymbol{v}_2) = \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $T(\boldsymbol{v}_3) = \boldsymbol{w}_3$.
(a) Find the matrix $\boldsymbol{A}$ of $T$ for this basis transformation. (8 points)
(b) Which vector $\boldsymbol{v}$ yields $T(\boldsymbol{v}) = \boldsymbol{w}_1$? (4 points)
(c) Invert $\boldsymbol{A}$ to obtain $T^{-1}(\boldsymbol{w}_1)$, $T^{-1}(\boldsymbol{w}_2)$ and $T^{-1}(\boldsymbol{w}_3)$. (5 points)

有 $\mathbb{R}^3$ 的两组基向量 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ 和 $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$, 两个基之间的对应关系由 $T$ 给出 (如上所示).
(a) 找出基变换 $T$ 对应的矩阵 $\boldsymbol{A}$. (8 分)
(b) 什么向量满足 $T(\boldsymbol{v}) = \boldsymbol{w}_1$? (4 分)
(c) 对 $\boldsymbol{A}$ 求逆, 从而确定 $T^{-1}(\boldsymbol{w}_1)$, $T^{-1}(\boldsymbol{w}_2)$ 和 $T^{-1}(\boldsymbol{w}_3)$ 的表达式. (5 分)

## IV. DIMENSION OF THE FUNDAMENTAL SUBSPACES OF A MATRIX (矩阵子空间的维数)

7. Without computing $\boldsymbol{A}$ find the bases of the row space, the column space and the null space,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \tag{5}$$

(5 points)

如 (5) 式所示, $\boldsymbol{A}$ 由两个矩阵相乘得到. 试从该式直接确定 $\boldsymbol{A}$ 的行空间, 列空间和零空间的基. (5 分)

## V. DETERMINANTS (行列式)

8. Compute the determinant of

$$\boldsymbol{A} = \begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix}. \tag{6}$$

Hint: Use row replacements to bring the matrix $\boldsymbol{A}$ to a triangular form. (3 points)

计算 $\boldsymbol{A}$ 的行列式.
提示: 可以先利用行变换把 $\boldsymbol{A}$ 变成三角矩阵. (3 分)