1. $A=\begin{pmatrix}\lambda & 1 & 1\\ 1 & \lambda & 1\\ 1 & 1 & \lambda\end{pmatrix}$ λ取何值 $Ax=\begin{pmatrix}1\\1\\1\end{pmatrix}$ 有解?

写出此时 col A 的基，并求出通解的参数向量形式

$$\left(\begin{array}{ccc|c}\lambda & 1 & 1 & 1\\ 1 & \lambda & 1 & 1\\ 1 & 1 & \lambda & 1\end{array}\right)\to\begin{pmatrix}1 & 1 & \lambda & 1\\ 1 & \lambda & 1 & 1\\ \lambda & 1 & 1 & 1\end{pmatrix}\to\begin{pmatrix}1 & 1 & \lambda & 1\\ 0 & \lambda-1 & 1-\lambda & 0\\ 0 & 1-\lambda & 1-\lambda^2 & 1-\lambda\end{pmatrix}$$

$$\to\begin{pmatrix}1 & 1 & \lambda & 1\\ 0 & \lambda-1 & 1-\lambda & 0\\ 0 & 0 & 2-\lambda-\lambda^2 & 1-\lambda\end{pmatrix}\qquad (2-\lambda-\lambda^2)=(2+\lambda)(1-\lambda)$$

① 当 $\lambda=-2$，无解

② 当 $\lambda\neq-2$ 且 $\lambda\neq1$ 时，唯一解。

$$\begin{pmatrix}1 & 1 & \lambda & 1\\ 0 & 1 & -1 & 0\\ 0 & 0 & 2+\lambda & 1\end{pmatrix}\qquad x_3=\frac{1}{\lambda+2}\qquad x_2=x_3=\frac{1}{\lambda+2}\qquad x_1+x_2+\lambda x_3=1$$

$$\Rightarrow x_1=\frac{1}{\lambda+2}$$

$$x_1=x_2=x_3=\frac{1}{\lambda+2}$$

$$\text{col}\,A=\text{span}\left\{\begin{pmatrix}\lambda\\1\\1\end{pmatrix}\begin{pmatrix}1\\\lambda\\1\end{pmatrix}\begin{pmatrix}1\\1\\\lambda\end{pmatrix}\right\}=\mathbb{R}^3$$

③ 当 $\lambda=1$ 时，$\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$ 有两个自由变量.

$$x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1-x_2-x_3\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}+x_2\begin{pmatrix}-1\\1\\0\end{pmatrix}+x_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

此时 $\text{col}\,A=\text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$

2. $T(x_1, x_2, x_3)=(2x_3-x_1,\ x_2+x_3)$

$$T=\begin{pmatrix}T(e_1) & T(e_2) & T(e_3)\end{pmatrix}=\begin{pmatrix}-1 & 0 & 2\\ 0 & 1 & 1\end{pmatrix}$$

3. $AB \neq BA$

4. $(AB)C = A(BC)$

$$((AB)C)_{ij} = \sum_{l} (AB)_{il} (C)_{lj} = \sum_{l} \sum_{k} A_{ik} B_{kl} C_{lj}$$

$$(A(BC))_{ij} = \sum_{k} A_{ik} (BC)_{kj} = \sum_{k} \sum_{l} A_{ik} B_{kl} C_{lj}$$

5. (a) $2A^{-1}B = B - 4I$.

$2B = AB - 4A$

$\underline{2B} = \underline{AB} - 4(A-2I) - 8I$.

$(2I - A)B = -4(A-2I) - 8I \Rightarrow (A-2I)(B+4I) = 8I$.

(b). $B = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ 求 $A$.

$A = 8(B-4I)^{-1} + 2I$.

$(B-4I)^{-1} = \dfrac{(B-4I)^*}{|B-4I|}$

$B - 4I = \begin{pmatrix} -3 & -2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow$

$= \dfrac{\begin{pmatrix} 4 & 2 & 0 \\ -4 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}^T}{-16} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{8} & -\frac{3}{8} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$

$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & C \end{array}\right) \Rightarrow \begin{pmatrix} A^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} = \left(\begin{array}{cc|c} -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{8} & -\frac{3}{8} & 0 \\ \hline 0 & 0 & -\frac{1}{2} \end{array}\right)$

有. $A = 8(B-4I)^{-1} + 2I = \begin{pmatrix} 0 & 2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

6. $\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 4 & -2 & 0 & 1 & 0 \\ 5 & -4 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & -13 & 6 & -1 \\ 0 & 0 & 1 & -29 & 13 & -2 \end{array}\right)$

$A^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ -13 & 6 & -1 \\ -29 & 13 & -2 \end{pmatrix}$ $\det A = 1$

$A^* = A^{-1}$

7. $(AB)^T_{23} = (AB)_{32} = \begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}$

8. (a) $\begin{vmatrix} 3 & 1 & 3 \\ 5 & 2 & 0 \\ 2 & 4 & 6 \end{vmatrix} = 54$

(b) $\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & b & 0 & 0 \\ 1 & 0 & c & 0 \\ 1 & 0 & 0 & d \end{vmatrix} = abcd - bc - bd - cd$

$= abcd - 1\times \begin{vmatrix} 1 & 1 & 1 \\ 0 & c & 0 \\ 0 & 0 & d \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ b & 0 & 0 \\ 0 & 0 & d \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ b & 0 & 0 \\ 0 & c & 0 \end{vmatrix}$

$= abcd - cd - bd - bc$

(c) $\begin{vmatrix} a_1+1 & a_2 & \cdots & a_n \\ a_1 & a_2+1 & \cdots & a_n \\ \vdots & \vdots & \ddots & \\ a_1 & a_2 & \cdots & a_n+1 \end{vmatrix} = 1 + (a_1 + \cdots + a_n)$

$= \begin{vmatrix} a_1+1 & a_2 & \cdots & a_n \\ -1 & 1 & \cdots & 0 \\ -1 & 0 & 1 & \vdots \\ \vdots & \vdots & \ddots \; 0 & \vdots \\ -1 & 0 \;\; 0 & 1 & 1 \end{vmatrix}$

考虑 $\begin{vmatrix} a_1 \cdots\cdots a_n \\ \hline e_1 \\ e_2 \\ \vdots \\ e_{j-1} \\ e_{j+1} \\ e_n \end{vmatrix}$ $\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ $n-1$. 只能生成 $e_1 \cdots e_n$ 其中 $n-1$ 个.

$e_j$ 无了

按第$j$列展开 $= a_j \times (-1)^{1+j}$

$= 1 + a_1 + a_2 + \dots + a_n$