4(a)

$$\vec{p}c = cm\vec{v}$$

$$E\frac{\vec{v}}{c} = mc^2\frac{\vec{v}}{c} = cm\vec{v}$$

于是

$$\vec{p}c = E\frac{\vec{v}}{c}$$

4(b)

$$\begin{aligned} p^{\mu}p_{\mu} &= \frac{E^2}{c^2} - p^2 \\ &= \frac{m^2c^4}{c^2} - m^2v^2 \\ &= m^2(c^2 - v^2) \\ &= \frac{m_0^2}{1-\frac{v^2}{c^2}}(c^2 - v^2) \\ &= \frac{c^2m_0^2}{c^2 - v^2}(c^2 - v^2) \\ &= m_0^2c^2 \end{aligned}$$

4(c)

设 K' 惯性系相对 K 惯性系以 $\vec{u} = u\vec{e}_x$ 的速度运动，两惯性系坐标轴指向一致，在 $t = 0, t' = 0$ 时刻两惯性系原点重合

设某粒子在 K 系中的速度为 $\vec{v} = v_x\vec{e}_x + v_y\vec{e}_y + v_z\vec{e}_z$，在 K' 系中的速度为 $\vec{v}' = v_x'\vec{e}_x + v_y'\vec{e}_y + v_z'\vec{e}_z$，根据能量-动量矢量的定义，该粒子在 K 系中的能量-动量矢量为：

$$p^{\mu} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix}$$

该粒子在 K' 系中的能量-动量矢量为：

$$p'^{\mu} = \frac{m_0}{\sqrt{1-\frac{v'^2}{c^2}}}\begin{bmatrix} c \\ v_x' \\ v_y' \\ v_z' \end{bmatrix}$$

要验证 p^{μ} 是一个 $\mathrm{Lorentz}$ 四维矢量，只需要验证从 p^{μ} 到 p'^{μ} 的变换是洛伦兹变换，也就是要验证下面等式成立：

$$\frac{m_0}{\sqrt{1-\frac{v'^2}{c^2}}}\begin{bmatrix} c \\ v_x' \\ v_y' \\ v_z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix}$$

其中，$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}, \beta = \frac{u}{c}, v'^2 = v_x'^2 + v_y'^2 + v_z'^2, v^2 = v_x^2 + v_y^2 + v_z^2,$

也就是验证下面四条等式成立：

$$\frac{c}{\sqrt{1-\frac{v'^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{c - \frac{uv_x}{c}}{\sqrt{1-\frac{u^2}{c^2}}} \tag{1}$$

$$\frac{v_x'}{\sqrt{1-\frac{v'^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{v_x - u}{\sqrt{1-\frac{u^2}{c^2}}} \tag{2}$$

$$\frac{v_y'}{\sqrt{1-\frac{v'^2}{c^2}}} = \frac{v_y}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3}$$

$$\frac{v_z'}{\sqrt{1-\frac{v'^2}{c^2}}}=\frac{v_z}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4}$$

由3(f)知:

$$\begin{cases} v_x'=\dfrac{v_x-u}{1-\frac{u}{c^2}v_x} \\ v_y'=\dfrac{v_y\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{u}{c^2}v_x} \\ v_z'=\dfrac{v_z\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{u}{c^2}v_x} \end{cases}$$

再注意到重要结论 $(*)$:

$$\begin{aligned} \sqrt{1-\frac{v'^2}{c^2}} &= \sqrt{1-\frac{v_x'^2+v_y'^2+v_z'^2}{c^2}} \\ &= \sqrt{1-\frac{1}{c^2}\cdot\frac{1}{(1-\frac{u}{c^2}v_x)^2}\cdot\left[(v_x-u)^2+(1-\frac{u^2}{c^2})v_y^2+(1-\frac{u^2}{c^2})v_z^2\right]} \\ &= \sqrt{1-\frac{1}{c^2}\cdot\frac{1}{(1-\frac{u}{c^2}v_x)^2}\cdot\left[(1-\frac{u^2}{c^2})(v_x^2+v_y^2+v_z^2)+(v_x-u)^2-(1-\frac{u^2}{c^2})v_x^2\right]} \\ &= \sqrt{1-\frac{1}{c^2}\cdot\frac{1}{(1-\frac{u}{c^2}v_x)^2}\cdot\left[(1-\frac{u^2}{c^2})v^2+\frac{u^2v_x^2}{c^2}-2uv_x+u^2\right]} \\ &= \sqrt{1-\frac{1}{c^2}\cdot\frac{1}{(1-\frac{u}{c^2}v_x)^2}\cdot\left[(1-\frac{u^2}{c^2})v^2+\frac{u^2v_x^2}{c^2}-2uv_x+c^2+u^2-c^2\right]} \\ &= \sqrt{1-\frac{1}{c^2}\cdot\frac{1}{(1-\frac{u}{c^2}v_x)^2}\cdot\left[(1-\frac{u^2}{c^2})v^2+(\frac{uv_x}{c}-c)^2+u^2-c^2\right]} \\ &= \sqrt{1-\frac{1}{c^2}\cdot\frac{1}{(1-\frac{u}{c^2}v_x)^2}\cdot\left[(1-\frac{u^2}{c^2})v^2+u^2-c^2\right]-1} \\ &= \frac{1}{1-\frac{u}{c^2}v_x}\cdot\sqrt{(1-\frac{u^2}{c^2})-\frac{1}{c^2}\cdot(1-\frac{u^2}{c^2})v^2} \\ &= \frac{1}{1-\frac{u}{c^2}v_x}\cdot\sqrt{1-\frac{u^2}{c^2}}\cdot\sqrt{1-\frac{v^2}{c^2}} \end{aligned}$$

验证(1):

$$\begin{aligned} \text{左边} &= \frac{c}{\sqrt{1-\frac{v'^2}{c^2}}} \\ (\text{利用结论}*) &= \frac{c(1-\frac{u}{c^2}v_x)}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{c-\frac{uv_x}{c}}{\sqrt{1-\frac{u^2}{c^2}}} \\ &= \text{右边} \end{aligned}$$

于是等式 (1) 成立

验证(2):

$$\begin{aligned} \text{左边} &= \frac{v_x'}{\sqrt{1-\frac{v'^2}{c^2}}} \\ (\text{利用结论}*) &= \frac{v_x-u}{1-\frac{u}{c^2}v_x}\cdot\frac{1-\frac{u}{c^2}v_x}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{v_x-u}{\sqrt{1-\frac{u^2}{c^2}}} \\ &= \text{右边} \end{aligned}$$

于是等式 (2) 成立

验证(3)：

$$\begin{aligned}
\text{左边} &= \frac{v'_y}{\sqrt{1-\frac{v'^2}{c^2}}} \\
(\text{利用结论}*) &= \frac{v_y\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{u}{c^2}v_x}\cdot\frac{1-\frac{u}{c^2}v_x}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{v_y}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \text{右边}
\end{aligned}$$

于是等式 (3) 成立

验证(4)：

由于 y, z 地位相同，(3) 成立，则 (4) 也成立

综上所述，p^μ 是一个 Lorentz 四维矢量

5

证明：

设质点组由 N 个质点组成，第 i 个质点的质量记为 m_i，质心的质量记为 $M=\sum_i m_i$，第 i 个质点在惯性系 K 中的速度记为 $\vec{v}_i$，质心在惯性系 K 中的速度记为 $\vec{v}_c=\frac{1}{M}\sum_i m_i\vec{v}_i$,第 i 个质点在质心系中的速度记为 $\vec{v}_{ic}$

$$\begin{aligned}
\sum_i \frac{1}{2}m_i v_i^2 &= \sum_i \frac{1}{2}m_i(\vec{v}_c+\vec{v}_{ic})^2 \\
&= \sum_i \frac{1}{2}m_i(v_c^2+v_{ic}^2+2\vec{v}_c\cdot\vec{v}_{ic}) \\
&= \frac{1}{2}Mv_c^2+\sum_i\frac{1}{2}m_i v_{ic}^2+\sum_i m_i\vec{v}_c\cdot\vec{v}_{ic}
\end{aligned} \tag{1}$$

注意到，

$$\begin{aligned}
\sum_i m_i\vec{v}_c\cdot\vec{v}_{ic} &= \sum_i m_i\vec{v}_c\cdot(\vec{v}_i-\vec{v}_c) \\
&= \sum_i m_i\vec{v}_c\cdot\vec{v}_i-\sum_i m_i v_c^2 \\
&= \vec{v}_c\cdot\sum_i m_i\vec{v}_i - v_c^2\sum_i m_i \\
&= M\vec{v}_c\cdot\frac{1}{M}\sum_i m_i\vec{v}_i - Mv_c^2 \\
&= Mv_c^2-Mv_c^2 \\
&= 0
\end{aligned}$$

代回(1) 得：

$$\sum_i\frac{1}{2}m_i v_i^2=\frac{1}{2}Mv_c^2+\sum_i\frac{1}{2}m_i v_{ic}^2$$

这就是柯尼希定理

6

对于球坐标系，有：

$$\begin{cases}
x=r\sin\theta\cos\varphi \\
y=r\sin\theta\sin\varphi \\
z=r\cos\theta
\end{cases}$$

于是:

$$\begin{aligned}\vec{r} &\equiv x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \\ &= r\sin\theta\cos\varphi\vec{e}_x + r\sin\theta\sin\varphi\vec{e}_y + r\cos\theta\vec{e}_z\end{aligned}$$

而:

$$\begin{aligned}\vec{e}_r &= \frac{\partial\vec{r}}{\partial r}\bigg/\left|\frac{\partial\vec{r}}{\partial r}\right| \\ &= \sin\theta\cos\varphi\vec{e}_x + \sin\theta\sin\varphi\vec{e}_y + \cos\theta\vec{e}_z\end{aligned}$$

$$\begin{aligned}\vec{e}_\theta &= \frac{\partial\vec{r}}{\partial\theta}\bigg/\left|\frac{\partial\vec{r}}{\partial\theta}\right| \\ &= \cos\theta\cos\varphi\vec{e}_x + \cos\theta\sin\varphi\vec{e}_y - \sin\theta\vec{e}_z\end{aligned}$$

$$\begin{aligned}\vec{e}_\varphi &= \frac{\partial\vec{r}}{\partial\varphi}\bigg/\left|\frac{\partial\vec{r}}{\partial\varphi}\right| \\ &= -\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y + 0\vec{e}_z\end{aligned}$$

这等价于:

$$\begin{bmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0\end{bmatrix}\begin{bmatrix}\vec{e}_x \\ \vec{e}_y \\ \vec{e}_z\end{bmatrix} = \begin{bmatrix}\vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi\end{bmatrix}$$

由克拉默法则，得:

$$\begin{aligned}\vec{e}_x &= \sin\theta\cos\varphi\vec{e}_r + \cos\theta\cos\varphi\vec{e}_\theta - \sin\varphi\vec{e}_\varphi \\ \vec{e}_y &= \sin\theta\sin\varphi\vec{e}_r + \cos\theta\sin\varphi\vec{e}_\theta + \cos\varphi\vec{e}_\varphi \\ \vec{e}_z &= \cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta + 0\vec{e}_\varphi\end{aligned}$$

$\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$ 分别对 t 求导，得:

$$\begin{aligned}\dot{\vec{e}}_r &= \dot\theta(\cos\theta\cos\varphi\vec{e}_x + \cos\theta\sin\varphi\vec{e}_y - \sin\theta\vec{e}_z) + \dot\varphi\sin\theta(-\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y + 0\vec{e}_z) \\ &= \dot\theta\vec{e}_\theta + \dot\varphi\sin\theta\vec{e}_\varphi\end{aligned}$$

$$\begin{aligned}\dot{\vec{e}}_\theta &= -\dot\theta(\sin\theta\cos\varphi\vec{e}_x + \sin\theta\sin\varphi\vec{e}_y + \cos\theta\vec{e}_z) + \dot\varphi\cos\theta(-\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y + 0\vec{e}_z) \\ &= -\dot\theta\vec{e}_r + \dot\varphi\cos\theta\vec{e}_\varphi\end{aligned}$$

$$\begin{aligned}\dot{\vec{e}}_\varphi &= -\dot\varphi(\cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y + 0\vec{e}_z) \\ &= -\dot\varphi(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta) \\ &= -\dot\varphi\sin\theta\vec{e}_r - \dot\varphi\cos\theta\vec{e}_\theta\end{aligned}$$

于是:

$$\begin{aligned}\vec{v} &\equiv \frac{\mathrm{d}(r\vec{e}_r)}{\mathrm{d}t} \\ &= \dot{r}\vec{e}_r + r\dot{\vec{e}}_r \\ &= \dot{r}\vec{e}_r + r\dot\theta\vec{e}_\theta + r\dot\varphi\sin\theta\vec{e}_\varphi\end{aligned}$$

$$\begin{aligned}\vec{a} &\equiv \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \\ &= \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + (\dot{r}\dot\theta + r\ddot\theta)\vec{e}_\theta + r\dot\theta\dot{\vec{e}}_\theta + (\dot{r}\dot\varphi\sin\theta + r\ddot\varphi\sin\theta + r\dot\varphi\dot\theta\cos\theta)\vec{e}_\varphi + r\dot\varphi\sin\theta\dot{\vec{e}}_\varphi \\ &= (\ddot{r} - r\dot\theta^2 - r\dot\varphi^2\sin^2\theta)\vec{e}_r + (r\ddot\theta + 2\dot{r}\dot\theta - r\dot\varphi^2\sin\theta\cos\theta)\vec{e}_\theta + (r\ddot\varphi\sin\theta + 2\dot{r}\dot\varphi\sin\theta + 2r\dot\theta\dot\varphi\cos\theta)\vec{e}_\varphi\end{aligned}$$