设一粒子沿 z 轴运动,其速度 $\vec{v} = v\vec{\mathrm{e}}_z, v \ll c$,

$$v=v(t) = egin{cases} \Delta v &, t < 0 \ \Delta v (1+\omega_0 t) \mathrm{e}^{-\omega_0 t} &, t \geqslant 0 \end{cases}$$

求其 $\frac{\mathrm{d}W_{\omega}}{\mathrm{d}\Omega}$

$$\dot{ec{v}}(t) = egin{cases} ec{0} & , & t < 0 \ ec{\mathrm{e}}_z \left[-\omega_0^2 \left(\Delta v
ight) t \mathrm{e}^{-\omega_0 t}
ight] & , & t \geqslant 0 \end{cases}$$

带电粒子产生的辐射场为:

$$ec{E}_{\mathrm{rad}} \equiv rac{q}{4\piarepsilon_{0}} \cdot rac{1}{r} rac{\hat{r} imes \left[\left(\hat{r}-ec{v}/c
ight) imes \dot{ec{v}}
ight]}{c^{2}\left(1-\hat{r}\cdotec{v}/c
ight)^{3}}$$

这里粒子沿直线运动, $\vec{v} \parallel \dot{\vec{v}}$, 因此:

$$\vec{v} imes \dot{\vec{v}} = \vec{0}$$

且粒子低速运动, 因此:

$$1-\hat{r}\cdotec{v}/cpprox 1$$

于是, 带电粒子的辐射场为:

$$ec{E} = rac{q}{4\piarepsilon_0 c^2 r} \hat{r} imes \left(\hat{r} imes \dot{ec{v}}
ight)$$

粒子低速运动, $t=t'+r/c, \mathrm{d}t/\mathrm{d}t'pprox 1,\ \mathrm{d}tpprox \mathrm{d}t'$

其频谱为:

$$\begin{split} \vec{E}_{\omega} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \vec{E} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{q}{4\pi\varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \dot{\vec{v}} \right) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \\ &\approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{q}{4\pi\varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \dot{\vec{v}} \right) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \\ &\approx \frac{1}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times (\hat{r} \times \vec{e}_z) \int_{0}^{+\infty} -\omega_0^2 \left(\Delta v \right) t \mathrm{e}^{-\omega_0 t} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times (\hat{r} \times \vec{e}_z) \int_{0}^{+\infty} t \mathrm{e}^{-\omega_0 t} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \\ &\approx \frac{-\omega_0^2 \left(\Delta v \right) q}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times (\hat{r} \times \vec{e}_z) \int_{0}^{+\infty} \left(t' + \frac{r}{c} \right) \mathrm{e}^{-\omega_0 \left(t' + r/c \right)} \mathrm{e}^{\mathrm{i}\omega \left(t' + r/c \right)} \mathrm{d}t' \\ &\approx \frac{-\omega_0^2 \left(\Delta v \right) q \mathrm{e}^{(-\omega_0 + \mathrm{i}\omega) r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times (\hat{r} \times \vec{e}_z) \int_{0}^{+\infty} \left(t' + \frac{r}{c} \right) \mathrm{e}^{(-\omega_0 + \mathrm{i}\omega) t'} \mathrm{d}t' \\ &\approx \frac{-\omega_0^2 \left(\Delta v \right) q \mathrm{e}^{(-\omega_0 + \mathrm{i}\omega) r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times (\hat{r} \times \vec{e}_z) \left[\frac{1}{\left(-\omega_0 + \mathrm{i}\omega_0 \right)^2} + \frac{r}{c} \frac{1}{\omega_0 - \mathrm{i}\omega} \right] \end{split}$$

辐射能量角密度为:

$$egin{aligned} rac{\mathrm{d}W_{\omega}}{\mathrm{d}\Omega} &= 4\piarepsilon_0 c \left|ec{E}_{\omega}
ight|^2 r^2 \ &= 4\piarepsilon_0 c r^2 imes \left|rac{-\omega_0^2 \left(\Delta v
ight) q \mathrm{e}^{\left(-\omega_0 + \mathrm{i}\omega
ight)r/c}}{8\pi^2arepsilon_0 c^2 r} \hat{r} imes \left(\hat{r} imes ec{\mathbf{e}}_z
ight) \left[rac{1}{\left(-\omega_0 + \mathrm{i}\omega_0
ight)^2} + rac{r}{c} rac{1}{\omega_0 - \mathrm{i}\omega}
ight]^2 \end{aligned}$$