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已知作用量

$$S[p,x,A]=S_0+S_1=\int_{t_1}^{t_2}\mathrm{d}t[p\dot{x}-H(p,x)-A(t)H(p,x)]+k\int_{t_1}^{t_2}\mathrm{d}tA(t)$$

求系统的运动方程。

解：

$$\begin{aligned}
\delta S\left[p,x,A\right]&=\int_{t_1}^{t_2}\mathrm{d}t\delta[p\dot{x}-H(p,x)-A(t)H(p,x)]+k\int_{t_1}^{t_2}k\mathrm{d}t\delta A\\
&=\int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p+p\delta\dot{x}-H(p,x)\delta A-[1+A]\left[\frac{\partial H}{\partial x}\delta x+\frac{\partial H}{\partial p}\delta p\right]\right\}+\int_{t_1}^{t_2}k\mathrm{d}t\delta A\\
&=\int_{t_1}^{t_2}\mathrm{d}tp\delta\dot{x}+\int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p-H(p,x)\delta A-[1+A]\left[\frac{\partial H}{\partial x}\delta x+\frac{\partial H}{\partial p}\delta p\right]\right\}+\int_{t_1}^{t_2}k\mathrm{d}t\delta A\\
&=\int_{t_1}^{t_2}\mathrm{d}tp\frac{\mathrm{d}\delta x}{\mathrm{d}t}+\int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p-H(p,x)\delta A-[1+A]\left[\frac{\partial H}{\partial x}\delta x+\frac{\partial H}{\partial p}\delta p\right]\right\}+\int_{t_1}^{t_2}k\mathrm{d}t\delta A\\
&=\int_{t_1}^{t_2}\mathrm{d}tp\mathrm{d}\delta x+\int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p-H(p,x)\delta A-[1+A]\left[\frac{\partial H}{\partial x}\delta x+\frac{\partial H}{\partial p}\delta p\right]\right\}+\int_{t_1}^{t_2}k\mathrm{d}t\delta A\\
&=p\delta x\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\delta x\dot{p}\mathrm{d}t+\int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p-H(p,x)\delta A-[1+A]\left[\frac{\partial H}{\partial x}\delta x+\frac{\partial H}{\partial p}\delta p\right]\right\}+\int_{t_1}^{t_2}k\mathrm{d}t\delta A\\
&=-\int_{t_1}^{t_2}\delta x\dot{p}\mathrm{d}t+\int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p-H(p,x)\delta A-[1+A]\left[\frac{\partial H}{\partial x}\delta x+\frac{\partial H}{\partial p}\delta p\right]\right\}+\int_{t_1}^{t_2}k\mathrm{d}t\delta A\\
&=\int_{t_1}^{t_2}\mathrm{d}t\left\{\delta p\left[\dot{x}-(1+A)\frac{\partial H}{\partial p}\right]+\delta x\left[-\dot{p}-(1+A)\frac{\partial H}{\partial x}\right]+\delta A\left[-H+k\right]\right\}
\end{aligned}$$

$\delta S=0$ 给出系统的运动方程：

$$\dot{x}-(1+A)\frac{\partial H}{\partial p}=0$$

$$-\dot{p}-(1+A)\frac{\partial H}{\partial x}=0$$

$$-H+k=0$$

或者：

$$\dot{x}=(1+A(t))\frac{\partial H(p,x)}{\partial p}$$

$$\dot{p}=-(1+A(t))\frac{\partial H(p,x)}{\partial x}$$

$$H(p,x)=k$$

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一维谐振子，其拉格朗日量为

$$L(x,\dot{x})=\frac{1}{2}\dot{x}^2-\frac{1}{2}\omega^2x^2$$

其中 x 为偏离平衡位的位移，$\dot{x}$ 为速度。求解系统的作用量 S，确定其与初末时刻及位置的关系。

解:

$$\frac{\partial L}{\partial x}=-\omega^2x,\ \ \frac{\partial L}{\partial \dot{x}}=\dot{x}$$

代入 E-L 方程，得:

$$\ddot{x}+\omega^2x=0$$

解得:

$$x(t)=A\cos(\omega t+\varphi_0)$$

设 $x(0)=x_0$，则:

$$A\cos\varphi_0=x_0$$

$$\dot{x}(t)=-\omega A\sin(\omega t+\varphi_0)$$

作用量为:

$$\begin{aligned}
S&=\int_{t_1}^{t_2}L(x,\dot{x})\mathrm{d}t\\
&=\frac{1}{2}\omega^2A^2\int_{t_1}^{t_2}[\sin^2(\omega t+\varphi_0)-\cos^2(\omega t+\varphi_0)]\mathrm{d}t\\
&=-\frac{1}{2}\omega^2A^2\int_{t_1}^{t_2}\cos(2\omega t+2\varphi_0)\mathrm{d}t\\
&=-\frac{1}{4}\omega A^2\int_{t=t_1}^{t=t_2}\cos(2\omega t+2\varphi_0)\mathrm{d}(2\omega t+2\varphi_0)\\
&=-\frac{1}{4}\omega A^2\sin(2\omega t+2\varphi_0)\Big|_{t=t_1}^{t=t_2}\\
&=-\frac{1}{4}\omega A^2\Big[\sin(2\omega t_2+2\varphi_0)-\sin(2\omega t_1+2\varphi_0)\Big]
\end{aligned}$$

其中，$A\cos\varphi_0=x_0$，x_0 是初始位置。