1

$$U_{\text{eff}}(r)=\frac{l^2}{2mr^2}-\frac{1}{r^3}$$

$$\frac{\mathrm{d}U_{\text{eff}}(r)}{\mathrm{d}r}=-\frac{l^2}{mr^3}+\frac{3}{r^4}$$

令 $U'_{\text{eff}}(r_m)=0$，得：

$$r_m=\frac{3m}{l^2}>0$$

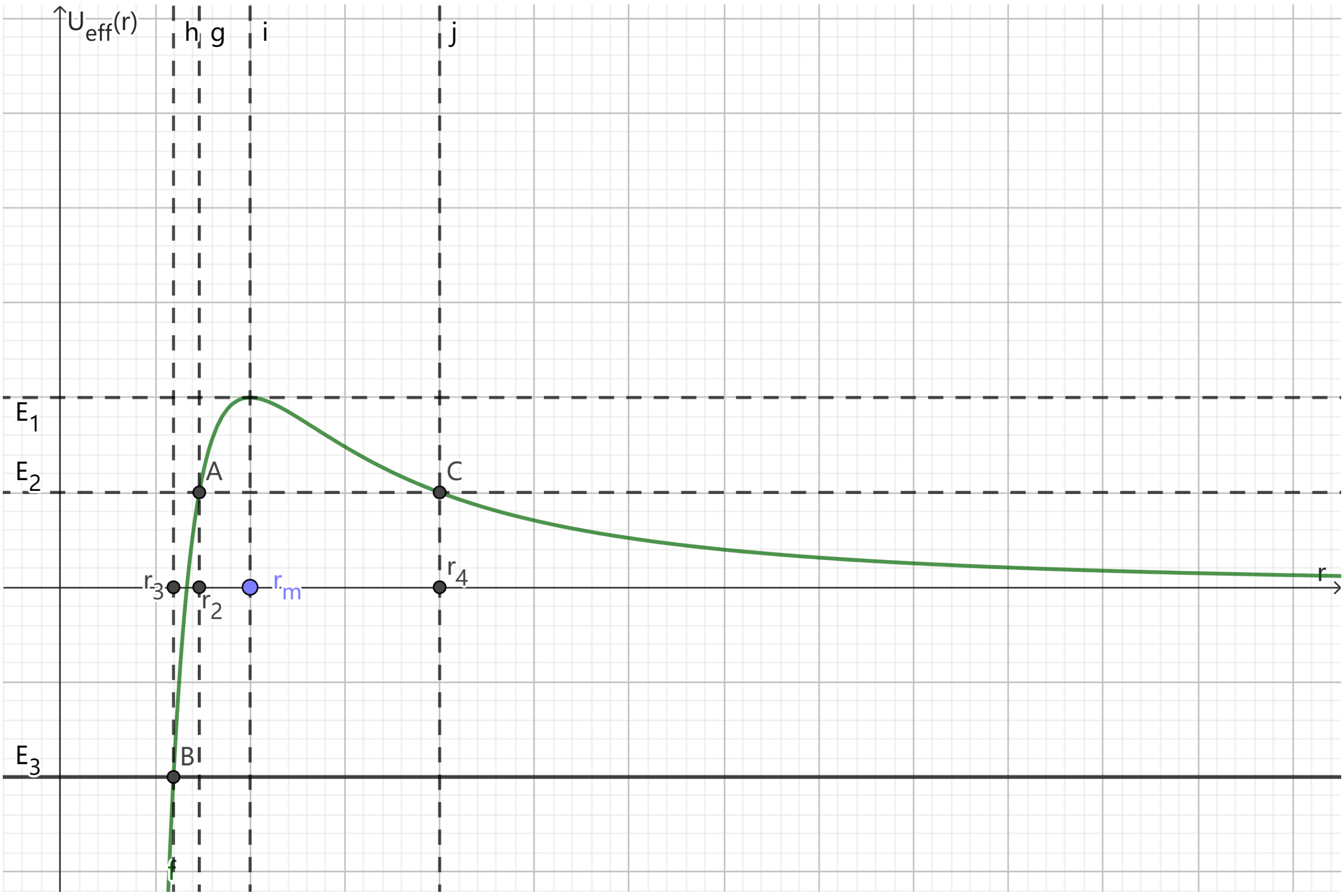
$$U''_{\text{eff}}(r_m)=\frac{1}{r_m^4}\cdot(\frac{-l^2}{m})<0$$

于是 r_m 是极小值点

$$\lim_{r\to 0^+}U_{\text{eff}}(r)=0$$

$$\lim_{r\to +\infty}U_{\text{eff}}(r)=-\infty$$

$U_{\text{eff}}(r)$ 关于 r 的函数关系图大致如下：



图中，$E_1=U_{\text{eff}}(r_m),0<E_2<U_{\text{eff}}(r_m),E_3\leqslant 0$

当能量 $E>E_1=U_{\text{eff}}(r_m)$ 时，质点可在全空间运动，轨道不闭合

当 $E=E_1$ 时，质点绕力心做圆周运动，轨道为圆形

当 $0<E=E_2<U_{\text{eff}}(r_m)$ 时，质点在 $0<r<r_2$ 和 $r>r_4$ 区域内运动

当 $E=E_3\leqslant 0$ 时，质点在 $0<r<r_3$ 的区域内运动

体系的拉氏量为：

$$L=\frac{m(\dot{r}^2+r^2\dot{\theta}^2)}{2}+\frac{1}{r^3}$$

关于广义坐标 r 的 E-L 方程为:

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{3}{r^4} = 0 \tag{1}$$

体系角动量守恒，而:

$$\begin{aligned} l &\equiv |\vec{r} \times \vec{p}| \\ &= m|\vec{r} \times \vec{v}| \\ &= m|r\vec{e}_r \times (\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta)| \\ &= mr^2\dot{\theta} \end{aligned}$$

于是:

$$\dot{\theta} = \frac{l}{mr^2}$$

将上式代入 (1)，消去 $\dot{\theta}$，得:

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{3}{r^4}$$

注意到:

$$\begin{aligned} \ddot{r} &= \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\dot{r}}{\mathrm{d}r} \\ &= \dot{r}\frac{\mathrm{d}\dot{r}}{\mathrm{d}r} \end{aligned}$$

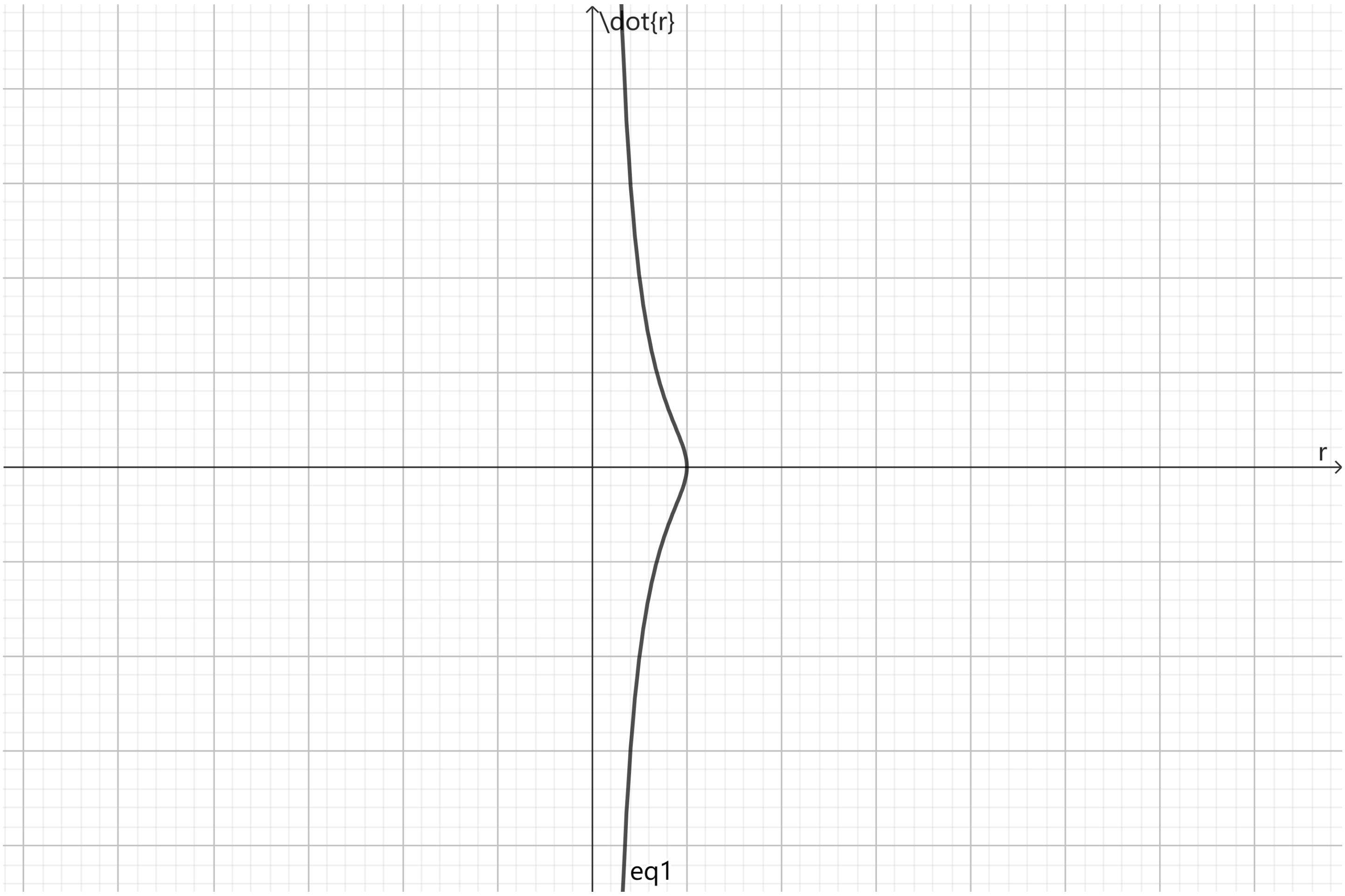
于是得到:

$$m\dot{r}\mathrm{d}\dot{r} = (\frac{l^2}{mr^3} - \frac{3}{r^4})\mathrm{d}r$$

积分得:

$$\frac{m\dot{r}^2}{2} = \frac{1}{r^3} - \frac{l^2}{2mr^2} + C$$

在 $(r, \dot{r})$ 空间的相流大致如下:



2

对于 $V(r)=-\frac{k}{r^n}$，

$$U_{\text{eff}}(r)=\frac{l^2}{2mr^2}-\frac{k}{r^n}$$

$$\frac{\mathrm{d}U_{\text{eff}}(r)}{\mathrm{d}r}=\frac{-l^2}{mr^3}+kn\frac{1}{r^{n+1}}$$

存在圆轨道要求存在 r_m 使得：

$$\left.\frac{\mathrm{d}U_{\text{eff}}(r)}{\mathrm{d}r}\right|_{r=r_m}=0$$

得到：

$$r_m^{2-n}=\frac{l^2}{mkn}\tag{1}$$

$$\frac{\mathrm{d}^2U_{\text{eff}}(r)}{\mathrm{d}r^2}=\frac{3l^2}{mr^4}-kn(n+1)\frac{1}{r^{n+2}}$$

存在稳定圆轨道要求：

$$\left.\frac{\mathrm{d}^2U_{\text{eff}}(r)}{\mathrm{d}r^2}\right|_{r=r_m}>0$$

结合 (1)，得到：

$$n<2$$

对于 $V=\alpha r^m$，

$$U_{\text{eff}}(r)=\frac{l^2}{2mr^2}+\alpha r^m$$

$$\frac{\mathrm{d}U_{\text{eff}}(r)}{\mathrm{d}r}=\frac{-l^2}{mr^3}+\alpha mr^{m-1}$$

存在圆轨道要求存在 r_k 使得:

$$\left.\frac{\mathrm{d}U_{\mathrm{eff}}(r)}{\mathrm{d}r}\right|_{r=r_k}=0$$

得到:

$$r_k^{2+m}=\frac{l^2}{\alpha m^2} \tag{2}$$

$$\frac{\mathrm{d}^2U_{\mathrm{eff}}(r)}{\mathrm{d}r^2}=\frac{3l^2}{mr^4}+\alpha m(m-1)r^{m-2}$$

存在稳定圆轨道要求:

$$\left.\frac{\mathrm{d}^2U_{\mathrm{eff}}(r)}{\mathrm{d}r^2}\right|_{r=r_k}>0$$

结合 (2)，得到:

$$m>-2$$

3

(a)

轨道方程为:

$$\theta=\int_{r_0}^{r}\frac{l}{r^2\sqrt{2m[E-V(r)]-\frac{l^2}{r^2}}}\mathrm{d}r+\theta_0$$

当 $V(r)=-\frac{k}{r}$，令 $u=\frac{1}{r}$，得:

$$\theta=\theta_0-\int\frac{\mathrm{d}u}{\sqrt{\frac{2mE}{l^2}+\frac{2mku}{l^2}-u^2}}$$

利用积分公式:

$$\int\frac{\mathrm{d}x}{\sqrt{\alpha+\beta x+\gamma x^2}}=\frac{1}{\sqrt{-\gamma}}\arccos\frac{-(\beta+2\gamma x)}{\sqrt{\beta^2-\alpha\gamma}}$$

得:

$$\theta=\theta_0-\arccos\frac{\frac{l^2u}{mk}-1}{\sqrt{1+\frac{2El^2}{mk^2}}}$$

再将 $u=\frac{1}{r}$ 代回得:

$$r=\frac{\frac{l^2}{mk}}{1+\sqrt{1+\frac{2El^2}{mk^2}}\cos(\theta-\theta_0)}$$

令 $p=\frac{l^2}{mk},e=\sqrt{1+\frac{2El^2}{mk^2}}$，$r,\theta$ 的关系可改写为:

$$r=\frac{p}{1+e\cos(\theta-\theta_0)}$$

$\cos(\theta-\theta_0)\in[-1,1]$，于是:

$$r_{\min}=\frac{p}{1+e},\ \ r_{\max}=\frac{p}{1-e}$$

于是:

$$
\begin{aligned}
\Delta\theta &= 2\int_{r_{\min}}^{r_{\max}} \frac{l\mathrm{d}r/r^2}{\sqrt{2m(E-V_{\text{eff}}(r))}} \\
&= -2\int_{u=\frac{1+e}{p}}^{u=\frac{1-e}{p}} \frac{\mathrm{d}u}{\sqrt{\frac{2mE}{l^2}+\frac{2mk}{l^2}u-u^2}} \\
&= 2\arccos\frac{\frac{l^2u}{mk}-1}{\sqrt{1+\frac{2El^2}{mk^2}}}\Bigg|_{u=\frac{1+e}{p}}^{u=\frac{1-e}{p}} \\
&= 2\cdot(\arccos(-1)-\arccos 1) \\
&= 2\pi
\end{aligned}
$$

(b)

令 $u=\frac{1}{r},\mathrm{d}u=-\frac{1}{r^2}\mathrm{d}r$，令 $x=u^2,\mathrm{d}x=2u\mathrm{d}u$

$$
\begin{aligned}
\mathrm{d}\theta &= \frac{l\mathrm{d}r/r^2}{\sqrt{2m(E-V_{\text{eff}}(r))}} \\
&= \frac{\frac{l}{r^2}\mathrm{d}r}{\sqrt{2m(E-\frac{l^2}{2mr^2}-\frac{1}{2}kr^2)}} \\
&= \frac{-l\mathrm{d}u}{\sqrt{2m(E-\frac{l^2}{2m}u^2-\frac{k}{2}\frac{1}{u^2})}} \\
&= \frac{-\frac{l}{2}\mathrm{d}(u^2)}{\sqrt{2m(Eu^2-\frac{l^2}{2m}u^4-\frac{k}{2})}} \\
&= \frac{-l}{2\sqrt{2m}}\frac{\mathrm{d}x}{\sqrt{-\frac{l^2}{2m}x^2+Ex-\frac{k}{2}}}
\end{aligned}
$$

积分得:

$$
\begin{aligned}
\theta-\theta_0 &= \frac{-l}{2\sqrt{2m}}\int_{x_0}^{x}\frac{\mathrm{d}x}{\sqrt{-\frac{l^2}{2m}x^2+Ex-\frac{k}{2}}} \\
&= -\frac{1}{2}\arccos\frac{\frac{l^2}{m}x-E}{\sqrt{E^2-\frac{kl^2}{m}}} \\
&= -\frac{1}{2}\arccos\frac{\frac{l^2}{mr^2}-E}{\sqrt{E^2-\frac{kl^2}{m}}}
\end{aligned}
$$

于是:

$$
\cos[2(\theta-\theta_0)] = \frac{\frac{l^2}{mr^2}-E}{\sqrt{E^2-\frac{kl^2}{m}}}
$$

得到:

$$
r_{\min} = \frac{l}{\sqrt{m}}\cdot\frac{1}{\sqrt{E+\sqrt{E^2-\frac{kl^2}{m}}}}
$$

$$
r_{\max} = \frac{l}{\sqrt{m}}\cdot\frac{1}{\sqrt{E-\sqrt{E^2-\frac{kl^2}{m}}}}
$$

于是:

当 $r=r_{\min},\theta_{\min}=\theta_0$

当 $r=r_{\max},\theta_{\max}=\theta_0+\frac{\pi}{2}$

于是:

$$
\begin{aligned}
\Delta\theta &= 2\cdot(\theta_{\max}-\theta_{\min}) \\
&= 2\cdot[(\theta_0+\frac{\pi}{2})-\theta_0] \\
&= \pi
\end{aligned}
$$

(c)

当 $E\to 0^-$，有:

$$r_{\max} = +\infty$$

$$V_{\text{eff}}(r_{\min}) = \frac{l^2}{2mr_{\min}^2} - kr_{\min}^{-\beta} = 0 \Longrightarrow r_{\min} = (\frac{l^2}{2mk})^{\frac{1}{2-\beta}}$$

$$
\begin{aligned}
\Delta\theta_0 &\equiv \lim_{E\to 0^-}\Delta\theta \\
&= \lim_{E\to 0^-} 2\int_{r_{\min}}^{r_{\max}} \frac{l\mathrm{d}r/r^2}{\sqrt{2m(E-V_{\text{eff}}(r))}} \\
&= \lim_{E\to 0^-} \frac{2}{\sqrt{2m}}\int_{r_{\min}}^{r_{\max}} \frac{l\mathrm{d}r/r^2}{\sqrt{E-\frac{l^2}{2mr^2}+kr^{-\beta}}} \\
&= \frac{2l}{\sqrt{2m}}\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r/r^2}{\sqrt{-\frac{l^2}{2mr^2}+kr^{-\beta}}} \\
&= \frac{2l}{\sqrt{2m}}\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{r\sqrt{-\frac{l^2}{2m}+kr^{2-\beta}}}
\end{aligned}
$$

令:

$$u = \sqrt{-\frac{l^2}{2m}+kr^{2-\beta}}$$

当 $r=r_{\min}, u=0$；当 $r=r_{\max}, u=+\infty$

$$r = (\frac{u^2}{k}+\frac{l^2}{2mk})^{\frac{1}{2-\beta}}$$

$$\ln r = \frac{1}{2-\beta}[\ln(u^2+\frac{l^2}{2m})-\ln k]$$

$$
\begin{aligned}
\frac{\mathrm{d}r}{r} &= \mathrm{d}\ln r \\
&= \frac{1}{2-\beta}\frac{2u\mathrm{d}u}{u^2+\frac{l^2}{2m}}
\end{aligned}
$$

代回积分表达式:

$$
\begin{aligned}
\Delta\theta_0 &= \frac{2l}{\sqrt{2m}}\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{r\sqrt{-\frac{l^2}{2m}+kr^{2-\beta}}} \\
&= \frac{2l}{\sqrt{2m}}\cdot\frac{2}{2-\beta}\int_0^{+\infty}\frac{\mathrm{d}u}{u^2+\frac{l^2}{2m}} \\
&= \frac{2l}{\sqrt{2m}}\cdot\frac{2}{2-\beta}\cdot\frac{2m}{l^2}\cdot\frac{l}{\sqrt{2m}}\int_0^{+\infty}\frac{\mathrm{d}(\frac{\sqrt{2m}}{l}u)}{1+(\frac{\sqrt{2m}}{l}u)^2} \\
&= \frac{2l}{\sqrt{2m}}\cdot\frac{2}{2-\beta}\cdot\frac{2m}{l^2}\cdot\frac{l}{\sqrt{2m}}\arctan\frac{\sqrt{2m}}{l}u\Big|_0^{+\infty} \\
&= \frac{2l}{\sqrt{2m}}\cdot\frac{2}{2-\beta}\cdot\frac{2m}{l^2}\cdot\frac{l}{\sqrt{2m}}\cdot\frac{\pi}{2} \\
&= \frac{2\pi}{2-\beta}
\end{aligned}
$$

4

(a)

体系能量守恒:

$$E=\frac{m\dot{r}^2}{2}+\frac{l^2}{2mr^2}-\frac{k}{r}$$

得到:

$$\mathrm{d}t=\frac{\mathrm{d}r}{\sqrt{\frac{2}{m}[E+\frac{k}{r}-\frac{l^2}{2mr^2}]}}$$

积分得:

$$\begin{aligned}\frac{\tau}{2}&=\int_{r_{\min}}^{r_{\max}}\frac{\mathrm{d}r}{\sqrt{\frac{2}{m}[E+\frac{k}{r}-\frac{l^2}{2mr^2}]}}\\&=\sqrt{\frac{m}{2}}\int_{r_{\min}}^{r_{\max}}\frac{\mathrm{d}r}{\sqrt{E-\frac{l^2}{2mr^2}+\frac{k}{r}}}\end{aligned}$$

于是:

$$\tau=\sqrt{2m}\int_{r_{\min}}^{r_{\max}}\frac{\mathrm{d}r}{\sqrt{E-\frac{l^2}{2mr^2}+\frac{k}{r}}}$$

(b)

半长轴 a 为:

$$\begin{aligned}a&\equiv\frac{r_{\max}+r_{\min}}{2}=-\frac{k}{2E}\\e&\equiv\sqrt{1+\frac{2El^2}{mk^2}}=\sqrt{1-\frac{l^2}{mka}}\end{aligned}$$

令 $r=a(1-e\cos\psi)$，当 $r=r_{\max},\psi=0$；当 $r=r_{\min},\psi=\pi$，于是:

$$\begin{aligned}\tau&=\sqrt{2m}\int_{r_{\min}}^{r_{\max}}\frac{\mathrm{d}r}{\sqrt{E-\frac{l^2}{2mr^2}+\frac{k}{r}}}\\&=\sqrt{2m}\int_0^{\pi}\frac{ae\sin\psi\mathrm{d}\psi}{\sqrt{-\frac{k}{2a}-\frac{(1-e^2)k}{2a(1-e\cos\psi)^2}+\frac{k}{a(1-e\cos\psi)}}}\\&=\sqrt{2m}\cdot\sqrt{\frac{2}{k}}a^{\frac{3}{2}}\cdot\int_0^{\pi}(1-e\cos\psi)\mathrm{d}\psi\\&=2\pi a^{\frac{3}{2}}\sqrt{\frac{m}{k}}\end{aligned}$$

5

$$\begin{aligned}\vec{J}=\vec{r}\times\vec{p}\Longrightarrow J_j&=\varepsilon_{lmj}x_lp_m\\\vec{A}=\vec{p}\times\vec{J}-mk\vec{e}_r\Longrightarrow A_a&=\varepsilon_{ija}p_iJ_j-mk\frac{x_a}{r}\\&=\varepsilon_{ija}\varepsilon_{lmj}p_ip_mx_l-mk\frac{x_a}{r}\\&=\varepsilon_{jia}\varepsilon_{jml}p_ip_mx_l-mk\frac{x_a}{r}\\&=(\delta_{im}\delta_{al}-\delta_{il}\delta_{am})p_ip_mx_l-mk\frac{x_a}{r}\\&=p^2x_a-p_lx_lp_a-mk\frac{x_a}{r}\end{aligned}$$

于是:

$$\{A_a,H\}=\{p^2x_a-p_lx_lp_a-mk\frac{x_a}{r},\frac{p^2}{2m}-\frac{k}{r}\}$$

注意到:

$$\begin{aligned}\{p^2x_a,\frac{p^2}{2m}\}&=\frac{p_ap^2}{m}\\\{p^2x_a,\frac{k}{r}\}&=2k\frac{p_bx_bx_a}{r^3}\end{aligned}$$

$$\{p_l x_l p_a, \frac{p^2}{2m}\} = \frac{p_a p^2}{m}$$

$$\{p_l x_l p_a, \frac{k}{r}\} = \frac{k x_l p_l x_a}{r^3} + \frac{k p_a}{r}$$

$$\{mk\frac{x_a}{r}, \frac{p^2}{2m}\} = -\frac{k x_b p_b x_a}{r^3} + \frac{k p_a}{r}$$

$$\{mk\frac{x_a}{r}, \frac{k}{r}\} = 0$$

于是:

$$\begin{aligned}
\{A_a, H\} &= \{p^2 x_a - p_l x_l p_a - mk\frac{x_a}{r}, \frac{p^2}{2m} - \frac{k}{r}\} \\
&= \{p^2 x_a, \frac{p^2}{2m}\} - \{p^2 x_a, \frac{k}{r}\} - \{p_l x_l p_a, \frac{p^2}{2m}\} + \{p_l x_l p_a, \frac{k}{r}\} - \{mk\frac{x_a}{r}, \frac{p^2}{2m}\} + \{mk\frac{x_a}{r}, \frac{k}{r}\} \\
&= 0
\end{aligned}$$

于是 $\vec{A}$ 是个守恒量

6

$$r = ae^{b\theta} \Longrightarrow \theta = \frac{1}{b}\ln\frac{r}{a} \Longrightarrow \frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{1}{br} \tag{1}$$

而轨道方程的微分形式为:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\frac{l}{r^2}}{\sqrt{2m[E - U(r)] - \frac{l^2}{r^2}}}$$

在有心力 $F = -\frac{k}{r^3}$ 作用下，势能为:

$$U(r) = \int_r^{+\infty} -\frac{k}{r^3}\mathrm{d}r = -\frac{k}{2r^2}$$

代入轨道方程的微分形式，得:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\frac{l}{r^2}}{\sqrt{2mE + \frac{mk}{r^2} - \frac{l^2}{r^2}}}$$

与 (1) 对比得，$E = 0$，进一步得到:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{l}{r\sqrt{mk - l^2}}$$

再与 (1) 对比，得:

$$\frac{l}{\sqrt{mk - l^2}} = \frac{1}{b}$$

解得:

$$l = \sqrt{\frac{mk}{b^2 + 1}}$$

综上，

$$\begin{cases} E = 0 \\ l = \sqrt{\dfrac{mk}{b^2 + 1}} \end{cases}$$

7

(a)

由对称性可知，散射角 Θ 与入射方向极角 χ_0 的关系为:

$$\Theta = \pi - 2\chi_0$$

轨道方程的微分形式为：

$$\mathrm{d}\theta = \frac{\frac{l}{r^2}\mathrm{d}r}{\sqrt{2m[E-V(r)]-\frac{l^2}{r^2}}} \tag{1}$$

整个运动过程机械能守恒，而在无穷远处势能为零，于是：

$$E = \frac{1}{2}mv_\infty^2 \tag{2}$$

在无穷远处，设 $\vec{r}$ 与 $\vec{p}$ 的夹角为 ϕ，有：

$$\begin{aligned} |\vec{l}| &= |\vec{r}\times\vec{p}| \\ &= |\frac{b}{\sin\phi}mv_\infty\sin\phi| \\ &= mbv_\infty \end{aligned} \tag{3}$$

$(2)(3)$ 代入 (1)，消去 E, l 得：

$$\mathrm{d}\theta = \frac{\frac{b}{r^2}\mathrm{d}r}{\sqrt{1-\frac{b^2}{r^2}-\frac{2V(r)}{mv_\infty^2}}}$$

积分得：

$$\chi_0 = \int_{r_{\min}}^{+\infty}\frac{\frac{b}{r^2}\mathrm{d}r}{\sqrt{1-\frac{b^2}{r^2}-\frac{2V(r)}{mv_\infty^2}}}$$

(b)

$$\begin{aligned} \chi_0 &= \int_R^{+\infty}\frac{b\mathrm{d}r/r^2}{\sqrt{1-\frac{b^2}{r^2}}} \\ &= \int_R^{+\infty}\frac{b\mathrm{d}r}{r\sqrt{r^2-b^2}} \end{aligned}$$

令 $u=\sqrt{r^2-b^2}$，

$$r = \sqrt{u^2+b^2}$$

$$\ln r = \frac{1}{2}\ln(u^2+b^2)$$

$$\begin{aligned} \frac{\mathrm{d}r}{r} &= \mathrm{d}(\ln r) \\ &= \frac{u\mathrm{d}u}{u^2+b^2} \end{aligned}$$

代回积分式，得：

$$\begin{aligned} \chi_0 &= \int_{r=R}^{r=+\infty}\frac{b\mathrm{d}r}{r\sqrt{r^2-b^2}} \\ &= \int_{u=\sqrt{R^2-b^2}}^{u=+\infty}\frac{b\mathrm{d}u}{u^2+b^2} \\ &= \int_{u=\sqrt{R^2-b^2}}^{u=+\infty}\frac{\mathrm{d}(\frac{u}{b})}{1+(\frac{u}{b})^2} \\ &= \arctan\frac{u}{b}\Big|_{u=\sqrt{R^2-b^2}}^{u=+\infty} \\ &= \frac{\pi}{2}-\arctan\frac{\sqrt{R^2-b^2}}{b} \end{aligned}$$

又 $\Theta = \pi - 2\chi_0$，与上式联立，消去 χ_0 得：

$$\tan\frac{\Theta}{2} = \frac{\sqrt{R^2-b^2}}{b}$$

即：

$$
\begin{aligned}
b &= \sqrt{\frac{R^2}{\tan^2\frac{\Theta}{2}+1}} \\
&= \frac{R}{\sqrt{\frac{\sin^2\frac{\Theta}{2}}{\cos^2\frac{\Theta}{2}}+1}} \\
&= R\cos\frac{\Theta}{2}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\sigma &= \frac{b}{\sin\Theta}|\frac{\mathrm{d}b}{\mathrm{d}\Theta}| \\
&= \frac{R^2}{4}
\end{aligned}
$$

(d)

$$
\sigma = \frac{R^2}{4}
$$

$$
\begin{cases} b = R\cos\frac{\Theta}{2} \\ b \in [0, R] \end{cases} \Longrightarrow \Theta \in [0, \pi]
$$

根据对称性，入射钢球的出射方向可以是空间中任意一个方向，即：

$$
\int \mathrm{d}\Omega = 4\pi
$$

于是总散射截面为：

$$
S = \int \sigma \mathrm{d}\Omega = 4\pi \cdot \frac{R^2}{4} = \pi R^2
$$