- (1) 质点在约束下可能发生的微小位移 $\delta \vec{r}$ 称为虚位移。质点实际运动产生的位移称为实位移。实位移是众多虚位移之中的一个。
- (2) 若拉格朗日量中不显含某个广义坐标,则这个广义坐标称为循环坐标,其对应的广义动量是个守恒量。
- (3) 某一时刻,刚体上速度为零的质点称为转动瞬心。
- (4)科里奥利力:

$$ec{F}_c = -2m \, (ec{\omega} imes ec{v})$$

其中, \vec{F}_c 为科里奥利力,m 为质点质量, $\vec{\omega}$ 为旋转参考系的角速度, \vec{v} 为质点相对于旋转参考系的速度。

例子: 傅科摆

(5)

限制在平面上的质点自由度: 2

傅科摆自由度: 2

水面上作匀速纯滚动的刚体球自由度: 3

设粒子的静止质量为 m_0 ,粒子速度 v=0.9c

运动质量:

$$m = rac{m_0}{\sqrt{1 - v^2/c^2}} = rac{m_0}{\sqrt{1 - \left(0.9c\right)^2/c^2}} = rac{m_0}{\sqrt{0.19}} = rac{10m_0}{\sqrt{19}}$$

总能量:

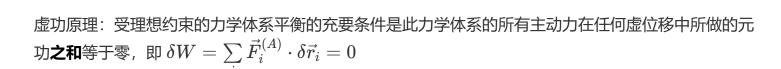
$$E=mc^2=rac{m_0c^2}{\sqrt{0.19}}=rac{10m_0c^2}{\sqrt{19}}$$

动能:

$$T=E-m_0c^2=\left(rac{10}{\sqrt{19}}-1
ight)m_0c^2$$

动量:

$$p = mv = \frac{10m_0}{\sqrt{19}} \cdot (0.9c) = \frac{9m_0c}{\sqrt{19}}$$



三个受主动力的质点分别记为 1, 2, 3

选取 α , β 为广义坐标来描述所有质点的位置

$$egin{aligned} ec{r}_1 &= rac{l_1}{2}\sinlphaec{e}_x + rac{l_1}{2}\coslphaec{e}_y, \ ec{r}_2 &= (l_1\sinlpha + rac{l_2}{2}\sineta)ec{e}_x + (l_1\coslpha + rac{l_2}{2}\coseta)ec{e}_y, \ ec{r}_3 &= (l_1\sinlpha + l_2\sineta)ec{e}_x + (l_1\coslpha + l_2\coseta)ec{e}_y, \end{aligned}$$

主动力:

$$\begin{split} \vec{F}_1^{(A)} &= P_1 \vec{e}_x, \ \, \vec{F}_2^{(A)} = P_2 \vec{e}_x, \ \, \vec{F}_3^{(A)} = F \vec{e}_y \\ \delta \vec{r}_1 &= \frac{l_1}{2} \cos \alpha \delta \alpha \vec{e}_x - \frac{l_1}{2} \sin \alpha \delta \alpha \vec{e}_y \\ \delta \vec{r}_2 &= (l_1 \cos \alpha \delta \alpha + \frac{l_2}{2} \cos \beta \delta \beta) \vec{e}_x + (-l_1 \sin \alpha \delta \alpha - \frac{l_2}{2} \sin \beta \delta \beta) \vec{e}_y \\ \delta \vec{r}_3 &= (l_1 \cos \alpha \delta \alpha + l_2 \cos \beta \delta \beta) \vec{e}_x + (-l_1 \sin \alpha \delta \alpha - l_2 \sin \beta \delta \beta) \vec{e}_y \end{split}$$

虚功原理:

$$\sum_{i=1}^{3} ec{F}_i^{(A)} \cdot \delta ec{r}_i = 0$$

即:

$$\delta lpha iggl[rac{1}{2} l_1 P_1 \cos lpha + l_1 P_2 \cos lpha - F l_1 \sin lpha iggr] + \delta eta iggl[rac{1}{2} l_2 P_2 \cos eta - F l_2 \sin eta iggr] = 0$$

 $\delta\alpha$, $\delta\beta$ 相互独立,上式成立,当且仅当:

$$\left\{ egin{aligned} &rac{1}{2}l_1P_1\coslpha+l_1P_2\coslpha-Fl_1\sinlpha=0\ &rac{1}{2}l_2P_2\coseta-Fl_2\sineta=0 \end{aligned}
ight.$$

解得:

$$\left\{egin{aligned} an lpha &= rac{P_1 + 2P_2}{2F} \ an eta &= rac{P_2}{2F} \end{aligned}
ight.$$

ps: 也可以用广义力来写

兀

ps:参考答案错了

以质心为原点, 过原点垂直上底面为 z 轴建系

密度:

$$\rho = \frac{M}{\pi R^2 h}$$

柱坐标 r, θ, z 与直角坐标 x, y, z 关系:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

柱坐标体积元:

$$dV = r dr d\theta dz$$

质量元:

$$\mathrm{d}m = \rho \mathrm{d}V = \rho r \mathrm{d}r \mathrm{d}\theta \mathrm{d}z$$

积分区域:

$$egin{aligned} r \in [0,R] \ heta \in [0,2\pi] \ z \in [-rac{h}{2},rac{h}{2}] \end{aligned} \ I_{zz} = \int (y^2+x^2)\mathrm{d}m \ &=
ho \int_{z=-rac{h}{2}}^{z=rac{h}{2}} \int_{ heta=0}^{ heta=2\pi} \int_{r=0}^{r=R} r^3 \mathrm{d}r \mathrm{d} heta \mathrm{d}z \ &= rac{1}{2}MR^2 \end{aligned}$$

$$egin{aligned} I_{xx} &= I_{yy} \ &= \int (x^2 + z^2) \mathrm{d} m \ &=
ho \int_{z=-rac{h}{2}}^{z=rac{h}{2}} \int_{ heta=0}^{ heta=2\pi} \int_{r=0}^{r=R} (r^2 \cos^2 heta + z^2) r \mathrm{d} r \mathrm{d} heta \mathrm{d} z \ &= rac{1}{12} M (3R^2 + h^2) \end{aligned}$$

五

(1)

泊松括号:

$$\{f,g\} \equiv \sum_{lpha} \left(rac{\partial f}{\partial q_lpha} rac{\partial g}{\partial p_lpha} - rac{\partial f}{\partial p_lpha} rac{\partial g}{\partial q_lpha}
ight)$$

物理量对时间的全导数:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}$$

(2)

证明:

$$\{x,L_x\} = 0$$
 $\{x,L_y\} = z$ $\{x,L_z\} = -y$ $\{p_x,L_x\} = 0$ $\{p_x,L_y\} = p_z$ $\{p_x,L_z\} = -p_y$



(1)

$$egin{aligned} V_{ ext{eff}}(r) &= rac{l^2}{2mr^2} + V(r) = rac{l^2}{2mr^2} - kr^{-eta} \ &\lim_{r o 0^+} V_{ ext{eff}}(r) = +\infty \ &\lim_{r o +\infty} V_{ ext{eff}}(r) = 0^- \ &rac{\mathrm{d} V_{ ext{eff}}(r)}{\mathrm{d} r}igg|_{r=r_m} = 0 \Longrightarrow r_m = \left(rac{2l^2}{mketa}
ight)^{rac{1}{2-eta}} \end{aligned}$$

可以画出 $V_{\rm eff}(r)$ 的大致图像

质点只能在 $V_{\rm eff}(r)\leqslant E$ 的区域内运动,而 E<0,于是从图像上可以看出 r 不能趋于无穷大 (2)

$$V_{ ext{eff}}(r) = rac{L^2}{2mr^2} - kr^{-eta} \leqslant E$$

由方程 $rac{L^2}{2mr^2}-kr^{-eta}=E$ 可得 $r_{
m min}, r_{
m max}$

角动量、能量守恒:

$$egin{cases} L = mr^2\dot{ heta} \ E = rac{1}{2}m(\dot{r}^2 + r^2\dot{ heta}^2) - kr^{-eta} \Longrightarrow E = rac{1}{2}m\dot{r}^2 + rac{L^2}{2mr^2} - kr^{-eta} \end{cases}$$

$$rac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{rac{2}{m}iggl[E - rac{L^2}{2mr^2} + kr^{-eta}iggr]}$$

结合角动量表达式 $L=mr^2\dot{ heta}$, $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{L}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}$, 代入上式, 得:

$$\mathrm{d} heta = rac{\mathrm{d}r}{r\sqrt{rac{2mE}{L^2}r^2 + rac{2mk}{L^2}r^{2-eta} - 1}}$$

积分得:

$$egin{aligned} \Delta\Phi &= \int \mathrm{d} heta \ &= \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{\mathrm{d}r}{r\sqrt{rac{2mE}{L^2}r^2 + rac{2mk}{L^2}r^{2-eta} - 1}} \end{aligned}$$

(3)

$$V_{ ext{eff}}(r) = rac{L^2}{2mr^2} - kr^{-eta} \leqslant E = 0^-$$

当 $E=0^-$,可得:

$$egin{cases} r_{ ext{min}} = \left(rac{L^2}{2mk}
ight)^{rac{1}{2-eta}} \ r_{ ext{max}} = +\infty \end{cases}$$

将 $r_{\min}, r_{\max}, E = 0^-$ 代入 (2) 的积分式, 得:

$$\Delta \Phi = \int_{r=(rac{L^2}{2mk})^{rac{1}{2-eta}}}^{r=+\infty} rac{\mathrm{d}r}{r\sqrt{rac{2mk}{L^2}r^{2-eta}-1}}$$
 (1)

令 $u=\sqrt{rac{2mk}{L^2}r^{2-eta}-1}, rac{L^2}{2mk}(u^2+1)=r^{2-eta}$,两边微分:

$$rac{L^2}{mk}u\mathrm{d}u=(2-eta)r^{1-eta}\mathrm{d}r\Longrightarrow r^{1-eta}\mathrm{d}r=rac{L^2}{(2-eta)mk}u\mathrm{d}u$$

两边同时除以 $r^{2-\beta}$:

$$\frac{\mathrm{d}r}{r} = \frac{L^2}{(2-\beta)mk} u \mathrm{d}u / r^{2-\beta}$$

$$= \frac{L^2}{(2-\beta)mk} u \mathrm{d}u / \frac{L^2}{2mk} (u^2 + 1)$$

$$= \frac{2}{2-\beta} \frac{u}{u^2 + 1} \mathrm{d}u$$

$$\Delta \Phi = \int_{r=(\frac{L^2}{2mk})^{\frac{1}{2-\beta}}}^{r=+\infty} \frac{\mathrm{d}r}{r\sqrt{\frac{2mk}{L^2}r^{2-\beta} - 1}}$$

$$= \int_{u=0}^{u=+\infty} \frac{2}{2-\beta} \frac{1}{u^2 + 1} \mathrm{d}u$$

$$= \frac{2}{2-\beta} \cdot \arctan u \Big|_{u=0}^{u=+\infty}$$

$$= \frac{\pi}{2-\beta}$$
(1)

可见, $\Delta\Phi$ 与角动量无关