1

由于 $f(z)$ 在 z 点可导，故极限

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

存在且与 Δz 趋于 0 的方式无关

设 $z=x+\mathrm{i}y, f(z)=u(x,y)+\mathrm{i}v(x,y)$，则:

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\Delta u+\mathrm{i}\Delta v}{\Delta x+\mathrm{i}\Delta y}$$

特别地

1.令:

$$\mathrm{i}\Delta y=0,\Delta x\to 0$$

此时，

$$\lim_{\Delta z\to 0}\frac{\Delta u+\mathrm{i}\Delta v}{\Delta x+\mathrm{i}\Delta y}=\lim_{\Delta x\to 0}\frac{\Delta u+\mathrm{i}\Delta v}{\Delta x}=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}$$

2.令:

$$\Delta x=0,\mathrm{i}\Delta y\to 0$$

此时，

$$\lim_{\Delta z\to 0}\frac{\Delta u+\mathrm{i}\Delta v}{\Delta x+\mathrm{i}\Delta y}=-\mathrm{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

由于 $f(z)$ 在 z_0 点可导，则这两个导数值应该相等，于是:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

2

$0<|z|<1$:

$$
\begin{aligned}
\frac{1}{z(z-1)} &= \frac{1}{z-1} - \frac{1}{z} \\
&= -\frac{1}{1-z} - \frac{1}{z} \\
&= -\sum_{n=0}^{\infty} z^n - z^{-1} \\
&= \sum_{n=-1}^{\infty} -z^n
\end{aligned}
$$

$|z| > 1$:

$$
\begin{aligned}
\frac{1}{z(z-1)} &= \frac{1}{z-1} - \frac{1}{z} \\
&= \frac{1}{z(1-\frac{1}{z})} - z^{-1} \\
&= \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} - z^{-1} \\
&= \frac{1}{z} \sum_{n=0}^{\infty} (\frac{1}{z})^n - z^{-1} \\
&= \sum_{n=0}^{\infty} z^{-n-1} - z^{-1} \\
&= \sum_{n=1}^{\infty} z^{-n-1}
\end{aligned}
$$

3

令 $f(z) = \frac{1}{(z^2+1)(z-1)^2} = \frac{1}{(z+\mathrm{i})(z-\mathrm{i})(z-1)^2}$

在回路内的孤立奇点有：$z_1 = \mathrm{i}, z_2 = 1$，$z_1$ 为一阶极点，z_2 二阶极点

计算回路内孤立奇点处的留数：

$$
\begin{aligned}
\mathrm{Res} f(z_1) &= \frac{1}{0!} \lim_{z \to \mathrm{i}} \frac{\mathrm{d}^0}{\mathrm{d}z^0} (z-\mathrm{i}) \cdot \frac{1}{(z+\mathrm{i})(z-\mathrm{i})(z-1)^2} \\
&= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Res} f(z_2) &= \frac{1}{1!} \lim_{z \to 1} \frac{\mathrm{d}^1}{\mathrm{d}z^1} (z-1)^2 \cdot \frac{1}{(z+\mathrm{i})(z-\mathrm{i})(z-1)^2} \\
&= -\frac{1}{2}
\end{aligned}
$$

于是：

$$
\begin{aligned}
I &= \oint\limits_{l} \frac{\mathrm{d}z}{(z^2+1)(z-1)^2} \\
&= 2\pi\mathrm{i}(\mathrm{Res}f(z_1)+\mathrm{Res}f(z_2)) \\
&= -\frac{\pi\mathrm{i}}{2}
\end{aligned}
$$

4

令 $z=e^{\mathrm{i}\theta}, z^{-1}=e^{\mathrm{i}(-\theta)}, \theta=\frac{\ln z}{\mathrm{i}}, \mathrm{d}\theta=\frac{\mathrm{d}z}{\mathrm{i}z}, \cos\theta=\frac{1}{2}(z+z^{-1})$

$$
\begin{aligned}
I &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{1+\varepsilon\cos\theta} \\
&= \frac{2}{\mathrm{i}} \oint\limits_{C^+} \frac{1}{\varepsilon z^2+2z+\varepsilon}\mathrm{d}z
\end{aligned}
$$

其中，C 是复平面上的单位圆

令 $f(z)=\frac{1}{\varepsilon z^2+2z+\varepsilon}$，被积函数的两个一阶极点为：

$$
z_1=\frac{-1+\sqrt{1-\varepsilon^2}}{\varepsilon},\quad z_2=\frac{-1-\sqrt{1-\varepsilon^2}}{\varepsilon}
$$

被积函数 $f(z)$ 可写为：

$$
f(z)=\frac{1}{\varepsilon(z-z_1)(z-z_2)}
$$

只有 z_1 在回路内

计算回路内孤立奇点的留数：

$$
\begin{aligned}
\mathrm{Res}f(z_1) &= \frac{1}{0!}\lim_{z\to z_1}\frac{\mathrm{d}^0}{\mathrm{d}z^0}(z-z_1)f(z) \\
&= \lim_{z\to z_1}\frac{1}{\varepsilon(z-z_2)} \\
&= \frac{1}{\varepsilon(z_1-z_2)} \\
&= \frac{1}{2\sqrt{1-\varepsilon^2}}
\end{aligned}
$$

留数定理：

$$
\begin{aligned}
\oint\limits_{C^+} \frac{1}{\varepsilon z^2+2z+\varepsilon}\mathrm{d}z &= 2\pi\mathrm{i}\mathrm{Res}f(z_1) \\
&= 2\pi\mathrm{i}\cdot\frac{1}{2\sqrt{1-\varepsilon^2}} \\
&= \frac{\pi\mathrm{i}}{\sqrt{1-\varepsilon^2}}
\end{aligned}
$$

于是积分为:

$$
\begin{aligned}
I &= \frac{2}{\mathrm{i}}\oint\limits_{C^+} \frac{1}{\varepsilon z^2+2z+\varepsilon}\mathrm{d}z \\
&= \frac{2}{\mathrm{i}}\cdot\frac{\pi\mathrm{i}}{\sqrt{1-\varepsilon^2}} \\
&= \frac{2\pi}{\sqrt{1-\varepsilon^2}}
\end{aligned}
$$

5

设 $i(t) \risingdotseq F(p)$

对方程 $L\frac{\mathrm{d}i(t)}{\mathrm{d}t}+Ri(t)=E$ 两边同时作拉普拉斯变换，得:

$$
LpF(p)+RF(p)=\frac{E}{p}
$$

解出 $F(p)$:

$$
\begin{aligned}
F(p) &= \frac{E}{Lp^2+Rp} \\
&= \frac{E}{R}(\frac{1}{p}-\frac{1}{p+\frac{R}{L}})
\end{aligned}
$$

两边同时做拉普拉斯逆变换得:

$$
i(t)=\frac{E}{R}(1-e^{-\frac{R}{L}t})
$$

6

$$
\begin{aligned}
\nabla\cdot(\varphi\vec{A}) &= \partial_i(\varphi\vec{A})_i \\
&= \partial_i(\varphi A_i) \\
&= A_i\partial_i\varphi+\varphi\partial_i A_i \\
&= A_i(\nabla\varphi)_i+\varphi\partial_i A_i \\
&= \vec{A}\cdot(\nabla\varphi)+\varphi\nabla\cdot\vec{A}
\end{aligned}
$$

高斯公式给出：

$$\int\limits_{\Omega} \nabla \cdot \vec{A} \mathrm{d}V = \int\limits_{\partial\Omega} \vec{A} \cdot \mathrm{d}\vec{S}$$

令 $\vec{A} = \nabla(\psi\varphi)$，代入得：

$$\int\limits_{\Omega} \nabla \cdot \nabla(\psi\varphi) \mathrm{d}V = \int\limits_{\partial\Omega} \nabla(\psi\varphi) \cdot \mathrm{d}\vec{S}$$

即：

$$\int\limits_{\Omega} \psi\nabla^2\varphi + \varphi\nabla^2\psi \mathrm{d}V = \int\limits_{\partial\Omega} (\psi\nabla\varphi + \varphi\nabla\psi) \cdot \mathrm{d}\vec{S}$$