1

> 提示：用电势叠加原理可快速出答案

$$U_3=\frac{1}{4\pi\varepsilon_0}\frac{Q_a+Q_b}{r}$$

$$U_2=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q_a}{r}+\frac{Q_b}{R_b}\right)$$

$$U_1=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q_a}{R_a}+\frac{Q_b}{R_b}\right)$$

2

由对称性可知，球心处的电场强度只有沿 z 轴负方向的分量

$\theta\sim\theta+\mathrm{d}\theta,\varphi\sim\varphi+\mathrm{d}\varphi$ 小范围内的面元 $\mathrm{d}S$ 所带面电荷在球心处产生的电场强度大小：

$$\begin{aligned}\mathrm{d}E&=\frac{1}{4\pi\varepsilon_0}\frac{\sigma\mathrm{d}S}{R^2}\\&=\frac{1}{4\pi\varepsilon_0R^2}P\cos\theta\cdot R\mathrm{d}\theta\cdot R\sin\theta\mathrm{d}\varphi\\&=\frac{1}{4\pi\varepsilon_0}P\cos\theta\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\end{aligned}$$

积分可得球心处电场强度大小 E_O：

$$\begin{aligned}E_O&=\int\mathrm{d}E\cos\theta\\&=\frac{P}{4\pi\varepsilon_0}\int_{\varphi=0}^{\varphi=2\pi}\mathrm{d}\varphi\int_{\theta=0}^{\theta=\pi/2}\sin\theta\cos^2\theta\mathrm{d}\theta\\&=-\frac{P}{2\varepsilon_0}\int_{\theta=0}^{\theta=\pi/2}\cos^2\theta\mathrm{d}(\cos\theta)\\&=-\frac{P}{2\varepsilon_0}\cdot\frac{\cos^3\theta}{3}\bigg|_{\theta=0}^{\theta=\pi/2}\\&=\frac{P}{6\varepsilon_0}\end{aligned}$$

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(1)

$$\begin{cases}(\sigma_1+\sigma_2)S=Q_A\\(\sigma_3+\sigma_4)S=0\\(\sigma_5+\sigma_6)S=Q_C\end{cases}$$

每个导体内部电场为零：

$$\begin{cases}E_A=0\\E_B=0\\E_C=0\end{cases}$$

将每个导体板看成两个无限大均匀带电平面，选择水平向右为正方向，上面三个条件可化为：

$$\begin{cases}\frac{\sigma_1}{2\varepsilon_0}-\frac{\sigma_2}{2\varepsilon_0}-\frac{\sigma_3}{2\varepsilon_0}-\frac{\sigma_4}{2\varepsilon_0}-\frac{\sigma_5}{2\varepsilon_0}-\frac{\sigma_6}{2\varepsilon_0}=0\\ \frac{\sigma_1}{2\varepsilon_0}+\frac{\sigma_2}{2\varepsilon_0}+\frac{\sigma_3}{2\varepsilon_0}-\frac{\sigma_4}{2\varepsilon_0}-\frac{\sigma_5}{2\varepsilon_0}-\frac{\sigma_6}{2\varepsilon_0}=0\\ \frac{\sigma_1}{2\varepsilon_0}+\frac{\sigma_2}{2\varepsilon_0}+\frac{\sigma_3}{2\varepsilon_0}+\frac{\sigma_4}{2\varepsilon_0}+\frac{\sigma_5}{2\varepsilon_0}-\frac{\sigma_6}{2\varepsilon_0}=0\end{cases}$$

最终解得:

$$\begin{cases}\sigma_1=\frac{Q_A+Q_C}{2S}\\ \sigma_2=\frac{Q_A-Q_C}{2S}\\ \sigma_3=\frac{Q_C-Q_A}{2S}\\ \sigma_4=\frac{Q_A-Q_C}{2S}\\ \sigma_5=\frac{Q_C-Q_A}{2S}\\ \sigma_6=\frac{Q_A+Q_C}{2S}\end{cases}$$

(2)

A,B 间场强:

$$E_{AB}\Delta S=\frac{1}{\varepsilon_0}\Delta S\sigma_2\Longrightarrow E_{AB}=\frac{Q_A-Q_C}{2\varepsilon_0 S}$$

A,B 间电势:

$$U_{AB}=E_{AB}\cdot d_1=\frac{d_1(Q_A-Q_C)}{2\varepsilon_0 S}$$

B,C 间场强:

$$E_{BC}\Delta S=\frac{1}{\varepsilon_0}\Delta S\sigma_4\Longrightarrow E_{BC}=\frac{Q_A-Q_C}{2\varepsilon_0 S}$$

B,C 间电势:

$$U_{BC}=E_{BC}\cdot d_2=\frac{d_2(Q_A-Q_C)}{2\varepsilon_0 S}$$

(3)

合上电键后 A 板和 C 板间可能存在电荷转移，但总量不变。设稳定后 A 板带电量 Q'_A，C 板带电量 Q'_C，则:

$$Q'_A+Q'_C=Q_A+Q_C$$

$$U'_{AB}=\frac{d_1(Q'_A-Q'_C)}{2\varepsilon_0 S}$$

$$U'_{BC}=\frac{d_2(Q'_A-Q'_C)}{2\varepsilon_0 S}$$

一方面:

$$U'_{AC}=U'_{AB}+U'_{BC}=\frac{d_1(Q'_A-Q'_C)}{2\varepsilon_0 S}+\frac{d_2(Q'_A-Q'_C)}{2\varepsilon_0 S}$$

另一方面:

$$U'_{AC}=U_0$$

$$\begin{cases}Q'_A+Q'_C=Q_A+Q_C\\ \frac{d_1(Q'_A-Q'_C)}{2\varepsilon_0 S}+\frac{d_2(Q'_A-Q'_C)}{2\varepsilon_0 S}=U_0\end{cases}$$

解得:

$$\begin{cases}Q'_A=\dfrac{Q_A+Q_C}{2}+\dfrac{\varepsilon_0 SU_0}{d_1+d_2}\\Q'_C=\dfrac{Q_A+Q_C}{2}-\dfrac{\varepsilon_0 SU_0}{d_1+d_2}\end{cases}$$

> 其实没必要解出 Q'_A, Q'_C，需要的只是：
>
> $$\begin{cases}Q'_A+Q'_C=Q_A+Q_C\\\dfrac{Q'_A-Q'_C}{2S}=\dfrac{\varepsilon_0 U_0}{d_1+d_2}\end{cases}$$

于是：

$$\begin{cases}\sigma'_1=\dfrac{Q'_A+Q'_C}{2S}=\dfrac{Q_A+Q_C}{2S}\\\sigma'_2=\dfrac{Q'_A-Q'_C}{2S}=\dfrac{\varepsilon_0 U_0}{d_1+d_2}\\\sigma'_3=\dfrac{Q'_C-Q'_A}{2S}=-\dfrac{\varepsilon_0 U_0}{d_1+d_2}\\\sigma'_4=\dfrac{Q'_A-Q'_C}{2S}=\dfrac{\varepsilon_0 U_0}{d_1+d_2}\\\sigma'_5=\dfrac{Q'_C-Q'_A}{2S}=-\dfrac{\varepsilon_0 U_0}{d_1+d_2}\\\sigma'_6=\dfrac{Q'_A+Q'_C}{2S}=\dfrac{Q_A+Q_C}{2S}\end{cases}$$

$$U'_{AB}=\frac{d_1(Q'_A-Q'_C)}{2\varepsilon_0 S}=\frac{d_1}{d_1+d_2}U_0$$

$$U'_{BC}=\frac{d_2(Q'_A-Q'_C)}{2\varepsilon_0 S}=\frac{d_2}{d_1+d_2}U_0$$

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(1)

$$\oiint_{\partial V}\vec{D}\cdot \mathrm{d}\vec{S}=Q_0$$

其中，Q_0 是 V 内总自由电荷

选取竖直向下为正方向，

$$D_1\Delta S=\sigma_0\Delta S\implies D_1=\sigma_0$$

“好”介质：

$$\begin{aligned}\vec{D}&\equiv\varepsilon_0\vec{E}+\vec{P}\\&=\varepsilon_0\vec{E}+\chi_e\varepsilon_0\vec{E}\\&=\varepsilon_0(1+\chi_e)\vec{E}\\&=\varepsilon_0\varepsilon\vec{E}\end{aligned}$$

于是：

$$\begin{aligned}E_1&=\frac{D_1}{\varepsilon_0\varepsilon_1}\\&=\frac{\sigma_0}{\varepsilon_0\varepsilon_1}\end{aligned}$$

> 注意，D_1 是一维矢量，其方向由正负给出。上面的式子的正负号要服从同一正方向规定。

$$D_2(-\Delta S)=\Delta S(-\sigma_0)\implies D_2=\sigma_0$$

$$
\begin{aligned}
E_2 &= \frac{D_2}{\varepsilon_0\varepsilon_2} \\
&= \frac{\sigma_0}{\varepsilon_0\varepsilon_2}
\end{aligned}
$$

(2)

“好”介质:

$$
\begin{aligned}
\vec{P} &= \chi_e\varepsilon_0\vec{E} \\
&= (\varepsilon - 1)\varepsilon_0\vec{E}
\end{aligned}
$$

于是:

$$
\begin{aligned}
P_1 &= (\varepsilon_1 - 1)\varepsilon_0 E_1 \\
&= \frac{\sigma_0(\varepsilon_1 - 1)}{\varepsilon_1}
\end{aligned}
$$

$$
\begin{aligned}
P_2 &= (\varepsilon_2 - \varepsilon_0)E_2 \\
&= \frac{\sigma_0(\varepsilon_2 - 1)}{\varepsilon_2}
\end{aligned}
$$

(3)

$$
\begin{aligned}
U_{AB} &= E_1 d_1 + E_2 d_2 \\
&= \frac{\sigma_0 d_1}{\varepsilon_0\varepsilon_1} + \frac{\sigma_0 d_2}{\varepsilon_0\varepsilon_2} \\
&= \frac{\sigma_0(d_1\varepsilon_2 + d_2\varepsilon_1)}{\varepsilon_0\varepsilon_1\varepsilon_2}
\end{aligned}
$$

(4)

$$
\begin{aligned}
\sigma_1' &= P_{1n} \\
&= -P_1 \\
&= -\frac{\sigma_0(\varepsilon_1 - 1)}{\varepsilon_1}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_2' &= P_{2n} \\
&= P_2 \\
&= \frac{\sigma_0(\varepsilon_2 - 1)}{\varepsilon_2}
\end{aligned}
$$

两介质交界面处:

$$
\oiint_{\partial V} \vec{P} \cdot \mathrm{d}\vec{S} = -Q_{\mathrm{p}}
$$

其中，Q_{p} 是 ∂V 内总束缚电荷

$$
P_1(-\Delta S) + P_2(\Delta S) = -\sigma' \Delta S
$$

解得:

$$
\begin{aligned}
\sigma' &= P_1 - P_2 \\
&= \frac{\sigma_0(\varepsilon_1 - 1)}{\varepsilon_1} - \frac{\sigma_0(\varepsilon_2 - 1)}{\varepsilon_2} \\
&= \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1\varepsilon_2}\sigma_0
\end{aligned}
$$

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基尔霍夫第一、第二方程组:

$$\begin{cases} I_1 + I_2 = I_3 \\ -E_1 + R_1 I_1 + E_2 - R_2 I_2 = 0 \\ -E_2 + R_2 I_2 + R_3 I_3 = 0 \end{cases}$$

解得：

$$\begin{cases} I_1 = 6\ \mathrm{A} \\ I_2 = -3\ \mathrm{A} \\ I_3 = 3\ \mathrm{A} \end{cases}$$

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(1)

电流连续方程积分形式：

$$\oiint_{\partial V} \vec{j} \cdot \mathrm{d}\vec{S} = -\frac{\mathrm{d}Q}{\mathrm{d}t}$$

其中，Q 是 ∂V 内总电荷

电流连续方程微分形式：

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

(2)

欧姆定律微分形式：

$$\vec{j} = \sigma \vec{E}$$

其中，σ 是电导率

(3)

$$\begin{aligned} \oiint_{\partial V} \vec{E} \cdot \mathrm{d}\vec{S} &= \frac{Q}{\varepsilon_0} \\ &= \frac{1}{\varepsilon_0} \iiint_V \rho \mathrm{d}V \end{aligned}$$

高斯公式：

$$\oiint_{\partial V} \vec{E} \cdot \mathrm{d}\vec{S} = \iiint_V (\nabla \cdot \vec{E}) \mathrm{d}V$$

对比得静电场高斯定理的微分形式：

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

(4)

$$\begin{cases} \vec{j} = \sigma \vec{E} \\ \nabla \cdot \vec{j} + \dfrac{\partial \rho}{\partial t} = 0 \end{cases} \Longrightarrow \nabla \cdot \vec{E} = -\frac{1}{\sigma} \frac{\partial \rho}{\partial t}$$

$$\begin{cases} \nabla \cdot \vec{E} = -\dfrac{1}{\sigma} \dfrac{\partial \rho}{\partial t} \\ \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \end{cases} \Longrightarrow \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon_0} \rho = 0$$

考虑某一确定场点处的体电荷密度：

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\frac{\sigma}{\varepsilon_0}\rho=0$$

解得：

$$\rho=Ce^{-\frac{\sigma}{\varepsilon_0}t}$$

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安培环路定理：

$$\oint_{\partial S}\vec{B}\cdot\mathrm{d}\vec{l}=\mu_0\sum I_0$$

$$2B\Delta L=\mu_0 j\Delta L\Longrightarrow B=\frac{\mu j}{2}$$

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(1)

负电荷导电

(2)

电子所受洛伦兹力和电场力平衡：

$$evB=e\frac{U_{AA'}}{b}\Longrightarrow vB=\frac{U_{AA'}}{b}$$

其中，v 是电子速度

电流微观表达式：

$$\begin{aligned}I&\equiv\frac{\Delta Q}{\Delta t}\\&=\frac{enSv\Delta t}{\Delta t}\\&=enSv\\&=enabv\end{aligned}$$

其中，n 是单位体积内参与导电的电子数。

$$\begin{cases}vB=\dfrac{U_{AA'}}{b}\\I=enabv\end{cases}$$

解得：

$$\begin{aligned}n&=\frac{IB}{eaU_{AA'}}\\&=2.9\times10^{14}\ \mathrm{cm}^{-3}\end{aligned}$$