(a)

$$\vec{r}_1 = l_1 \cos \theta_1 \vec{e}_x + l_1 \sin \theta_1 \vec{e}_y$$

$$\vec{r}_2 = (l_1 \cos \theta_1 + l_2 \cos \theta_2) \vec{e}_x + (l_1 \sin \theta_1 + l_2 \sin \theta_2) \vec{e}_y$$

$$\dot{\vec{r}}_1 = -l_1 \dot{\theta}_1 \sin \theta_1 \vec{e}_x + l_1 \dot{\theta}_1 \cos \theta_1 \vec{e}_y$$

$$\dot{\vec{r}}_2 = (-l_1 \dot{\theta}_1 \sin \theta_1 - l_2 \dot{\theta}_2 \sin \theta_2) \vec{e}_x + (l_1 \dot{\theta}_1 \cos \theta_1 + l_2 \dot{\theta}_2 \cos \theta_2) \vec{e}_y$$

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2)]$$

$$V = -m_1 g l_1 \cos \theta_1 - m_2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2)$$

拉格朗日量为:

$$L = T - V$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2) \right] + m_1 g l_1 \cos \theta_1 + m_2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2)$$

对于微振动,

$$egin{aligned} &\sin heta_1\sin heta_2+\cos heta_1\cos heta_2=\cos(heta_1- heta_2)pprox 1\ &V=-m_1gl_1\cos heta_1-m_2g(l_1\cos heta_1+l_2\cos heta_2)\ &pprox V_0+rac{1}{2}(m_1+m_2)gl_1\cos heta_1igg|_{ heta_1=0}(heta_1)^2+rac{1}{2}m_2gl_2\cos heta_2igg|_{ heta_2=0}(heta_2)^2\ &=V_0+rac{1}{2}(m_1+m_2)gl_1 heta_1^2+rac{1}{2}m_2gl_2 heta_2^2 \end{aligned}$$

于是:

$$Lpprox rac{1}{2}(m_1+m_2)l_1^2\dot{ heta}_1^2 + rac{1}{2}m_2l_2^2\dot{ heta}_2^2 + m_2l_1l_2\dot{ heta}_1\dot{ heta}_2 - rac{1}{2}(m_1+m_2)gl_1 heta_1^2 - rac{1}{2}m_2gl_2 heta_2^2 + V_0$$

微振动运动方程为:

$$(m_1+m_2)l_1\ddot{ heta}_1+m_2l_2\ddot{ heta}_2+(m_1+m_2)g heta_1=0 \ m_2l_2\ddot{ heta}_2+m_2l_1\ddot{ heta}_1+m_2g heta_2=0$$

(b)

若 $m_1=m_2=m, l_1=l_2=l$,则方程简化为:

$$\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2 + \frac{g}{l}\theta_1 = 0$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 + \frac{g}{l}\theta_2 = 0$$

$$\mathbf{M} = \begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix}$$

$$L = \frac{1}{2}(\dot{\theta}^{\mathrm{T}}\mathbf{M}\dot{\theta} - \theta^{\mathrm{T}}\mathbf{V}\theta), \ \ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \ \ \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

久期方程为:

$$|-\mathbf{M}\omega^2+\mathbf{V}|=\mathbf{0}$$

解得:

$$\omega_1^2 = (2+\sqrt{2})rac{g}{l}, \;\; \omega_2^2 = (2-\sqrt{2})rac{g}{l}$$

于是简正频率为:

$$\omega_1=\sqrt{(2+\sqrt{2})rac{g}{l}},~~\omega_2=\sqrt{(2-\sqrt{2})rac{g}{l}}$$

将简正频率代回方程 $[-\mathbf{M}\omega^2 + \mathbf{V}]\eta(\omega^2) = \mathbf{0}$, 得:

$$\eta_1 = egin{bmatrix} rac{1}{\sqrt{2}} \ -rac{1}{\sqrt{2}} \end{bmatrix}, \ \ \eta_2 = egin{bmatrix} rac{1}{\sqrt{3}} \ rac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

2

(a)

$$\begin{split} \vec{r}_1 &= l\cos\theta\vec{e}_x + l\sin\theta\vec{e}_y, \ \ \dot{\vec{r}}_1 = -l\dot{\theta}\sin\theta\vec{e}_x + l\dot{\theta}\cos\theta\vec{e}_y \\ \vec{r}_2 &= l\cos\varphi\vec{e}_x + (y_0 + l\sin\varphi)\vec{e}_y, \ \ \dot{\vec{r}}_2 = -l\dot{\varphi}\sin\varphi\vec{e}_x + l\dot{\varphi}\cos\varphi\vec{e}_y \\ T &= \frac{1}{2}m\dot{\vec{r}}_1^2 + \frac{1}{2}m'\dot{\vec{r}}_2^2 \\ &= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}m'l^2\dot{\varphi}^2 \end{split}$$

设弹簧固有长度为 l_0 ,

$$V=-mgl\cos heta-m'gl\cosarphi+rac{1}{2}k(\sqrt{(ec{r}_1-ec{r}_2)^2}-l_0)^2$$

将 V 在平衡位置 $\theta=0, \varphi=0$ 处展开至二阶:

$$V=V_0+rac{1}{2}mgl heta^2+rac{1}{2}m'glarphi^2+rac{1}{2}kl^2 heta^2+rac{1}{2}kl^2arphi^2-kl^2 hetaarphi$$

拉格朗日量为:

$$\begin{split} L &= T - V \\ &= \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m' l^2 \dot{\varphi}^2 - V_0 - \frac{1}{2} m g l \theta^2 - \frac{1}{2} m' g l \varphi^2 - \frac{1}{2} k l^2 \theta^2 - \frac{1}{2} k l^2 \varphi^2 + k l^2 \theta \varphi \\ &= \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m' l^2 \dot{\varphi}^2 - V_0 - \frac{1}{2} l (m g + k l) \theta^2 - \frac{1}{2} l (m' g + k l) \varphi^2 + k l^2 \theta \varphi \end{split}$$

运动方程为:

$$ml\ddot{ heta} + (mg + kl)\theta - kl\varphi = 0$$

 $m'l\ddot{\varphi} + (m'q + kl)\varphi - kl\theta = 0$

(b)

$$\mathbf{M} = egin{bmatrix} ml^2 & 0 \ 0 & m'l^2 \end{bmatrix}$$

$$\mathbf{V} = egin{bmatrix} l(mg+kl) & -kl^2 \ -kl^2 & l(m'g+kl) \end{bmatrix}$$

拉格朗日量可写为 (舍去常数项):

$$L = rac{1}{2} (\dot{q}^{ ext{T}} \mathbf{M} \dot{q} - q^{ ext{T}} \mathbf{V} q), \;\; q = egin{bmatrix} heta \ arphi \end{bmatrix}, \;\; \dot{q} = egin{bmatrix} \dot{ heta} \ \dot{arphi} \end{bmatrix}$$

久期方程为:

$$|-\mathbf{M}\omega^2+\mathbf{V}|=\mathbf{0}$$

解得:

$$\omega_1^2=rac{g}{l},~~\omega_2^2=rac{g}{l}+rac{m+m'}{mm'}k$$

于是简正频率为:

$$\omega_1 = \sqrt{rac{g}{l}}, \;\; \omega_2 = \sqrt{rac{g}{l} + rac{m+m'}{mm'}k}$$

将简正频率代回方程 $[-\mathbf{M}\omega^2 + \mathbf{V}]\eta(\omega^2) = \mathbf{0}$,得:

$$\eta_1 = egin{bmatrix} rac{1}{\sqrt{2}} \ rac{1}{\sqrt{2}} \end{bmatrix}, \;\; \eta_2 = egin{bmatrix} rac{m'}{\sqrt{m^2+m'^2}} \ rac{-m}{\sqrt{m^2+m'^2}} \end{bmatrix}$$

(c)

代入初始条件 t=0 时, $\theta=\theta_0, \varphi=0$,得:

$$A_1 = rac{\sqrt{2}m heta_0}{m+m'}, \;\; A_2 = rac{\sqrt{m^2+m'^2} heta_0}{m+m'}$$

于是:

$$ilde{ heta} = rac{m heta_0}{m_1+m_2'}e^{\mathrm{i}\omega_1 t} + rac{m' heta_0}{m_1+m_2'}e^{\mathrm{i}\omega_2 t}, \;\; ilde{arphi} = rac{m heta_0}{m_1+m_2'}e^{\mathrm{i}\omega_1 t} - rac{m heta_0}{m_1+m_2'}e^{\mathrm{i}\omega_2 t}$$

取实部,得:

$$heta = rac{m heta_0}{m+m'}\cos\omega_1 t + rac{m' heta_0}{m+m'}\cos\omega_2 t, \;\; arphi = rac{m heta_0}{m+m'}\cos\omega_1 t - rac{m heta_0}{m+m'}\cos\omega_2 t$$

其中,

$$\omega_1 = \sqrt{rac{g}{l}}, \ \ \omega_2 = \sqrt{rac{g}{l} + rac{m+m'}{mm'}k}$$

当 $rac{g}{l}\ggrac{k}{m},rac{k}{m'}$, $\omega_1=\omega_2=\omega=\sqrt{rac{g}{l}}$,由辅助角公式,

 θ 摆幅:

$$A_ heta = \sqrt{\left(rac{m heta_0}{m+m'}
ight)^2 + \left(rac{m' heta_0}{m+m'}
ight)^2} = rac{\sqrt{m^2+m'^2} heta_0}{m+m'}$$

 θ 摆幅:

$$A_{arphi} = \sqrt{\left(rac{m heta_0}{m+m'}
ight)^2 + \left(-rac{m heta_0}{m+m'}
ight)^2} = rac{\sqrt{2}m heta_0}{m+m'}$$