设一粒子沿 z 轴运动,其速度 $\vec{v} = v\vec{\mathrm{e}}_z, v \ll c$,

$$v=v(t) = egin{cases} \Delta v &, t < 0 \ \Delta v (1+\omega_0 t) \mathrm{e}^{-\omega_0 t} &, t \geqslant 0 \end{cases}$$

求其 $\frac{\mathrm{d}W_{\omega}}{\mathrm{d}\Omega}$

$$\dot{ec{v}}(t) = egin{cases} ec{0} & , & t < 0 \ ec{\mathrm{e}}_z \left[-\omega_0^2 \left(\Delta v
ight) t \mathrm{e}^{-\omega_0 t}
ight] & , & t \geqslant 0 \end{cases}$$

带电粒子产生的辐射场为:

$$ec{E}_{\mathrm{rad}} = rac{q}{4\piarepsilon_{0}} \cdot rac{1}{r} rac{\hat{r} imes \left[\left(\hat{r}-ec{v}/c
ight) imes \dot{ec{v}}
ight]}{c^{2}\left(1-\hat{r}\cdotec{v}/c
ight)^{3}}$$

这里粒子沿直线运动, $\vec{v} \parallel \dot{\vec{v}}$,因此:

$$\vec{v} imes \dot{\vec{v}} = \vec{0}$$

且粒子低速运动, 因此:

$$1-\hat{r}\cdotec{v}/cpprox 1$$

于是, 带电粒子的辐射场为:

$$ec{E} = rac{q}{4\piarepsilon_0 c^2 r} \hat{r} imes \left(\hat{r} imes \dot{ec{v}}
ight)$$

粒子低速运动, $t=t'+r/c, \mathrm{d}t/\mathrm{d}t'pprox 1,\ \mathrm{d}tpprox \mathrm{d}t'$

其频谱为:

$$\begin{split} \vec{E}_{\omega} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \vec{E} e^{i\omega t} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{q}{4\pi\varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \dot{\vec{v}} \right) e^{i\omega t} dt \\ &\approx \frac{q}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \int_{t=-\infty}^{t=+\infty} \dot{\vec{v}} e^{i\omega t} dt \right) \\ &\approx \frac{q}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \int_{t'=-\infty}^{t'=+\infty} \dot{\vec{v}} e^{i\omega t'} dt' \right) \\ &\approx \frac{q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \int_{t'=-\infty}^{t'=+\infty} \dot{\vec{v}} e^{i\omega t'} dt' \right) \\ &= \frac{q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \int_{t'=-\infty}^{t'=+\infty} \dot{\vec{v}} e^{i\omega t'} dt' \right) \\ &= \frac{q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \int_{t'=0}^{t'=+\infty} -\omega_0^2 \left(\Delta v \right) t' e^{-\omega_0 t'} \cdot e^{i\omega t'} dt' \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \int_{t'=0}^{t'=+\infty} t' e^{\left(-\omega_0 + i\omega\right) t'} dt' \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \int_{t'=0}^{t'=+\infty} \frac{\partial}{\partial \left(-\omega_0 + i\omega \right)} e^{\left(-\omega_0 + i\omega\right) t'} dt' \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \frac{d}{d \left(-\omega_0 + i\omega \right)} \int_{t'=0}^{t'=+\infty} e^{\left(-\omega_0 + i\omega\right) t'} dt' \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \frac{d}{d \left(-\omega_0 + i\omega \right)} \left[\frac{1}{-\omega_0 + i\omega} \cdot e^{\left(-\omega_0 + i\omega\right) t'} \right]_{t'=0}^{t'=+\infty} \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \frac{d}{d \left(-\omega_0 + i\omega \right)} \left[\frac{1}{-\omega_0 + i\omega} \cdot e^{\left(-\omega_0 + i\omega\right) t'} \right]_{t'=0}^{t'=+\infty} \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \frac{d}{d \left(-\omega_0 + i\omega \right)} \left[\frac{1}{-\omega_0 + i\omega} \cdot e^{\left(-\omega_0 + i\omega\right) t'} \right]_{t'=0}^{t'=+\infty} \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \frac{d}{d \left(-\omega_0 + i\omega \right)} \left[\frac{1}{-\omega_0 + i\omega} \cdot e^{\left(-\omega_0 + i\omega\right) t'} \right]_{t'=0}^{t'=+\infty} \\ &= \frac{-\omega_0^2 \left(\Delta v \right) q e^{i\omega r/c}}{8\pi^2 \varepsilon_0 c^2 r} \hat{r} \times \left(\hat{r} \times \vec{e}_z \right) \frac{d}{d \left(-\omega_0 + i\omega \right)} \left[\frac{1}{-\omega_0 + i\omega} \cdot e^{\left(-\omega_0 + i\omega\right) t'} \right]_{t'=0}^{t'=+\infty} \end{aligned}$$

设 \hat{r} 与 \vec{e}_z 的夹角为 θ , 辐射能量密度角分布为:

$$egin{aligned} rac{\mathrm{d}W_{\omega}}{\mathrm{d}\Omega} &= 4\piarepsilon_0 c \left|ec{E}_{\omega}
ight|^2 r^2 \ &= 4\piarepsilon_0 c r^2 imes \left|rac{-\omega_0^2 \left(\Delta v
ight) q \mathrm{e}^{\mathrm{i}\omega r/c}}{8\pi^2arepsilon_0 c^2 r} \hat{r} imes \left(\hat{r} imes ec{\mathbf{e}}_z
ight) rac{1}{\left(-\omega_0 + \mathrm{i}\omega
ight)^2}
ight|^2 \ &= 4\piarepsilon_0 c \cdot \left|rac{\omega_0^2 \left(\Delta v
ight) q}{8\pi^2arepsilon_0 c^2}
ight|^2 \cdot \sin^2 heta \cdot rac{1}{\left(\omega_0^2 - \omega^2
ight)^2 + 4\omega_0^2\omega^2} \end{aligned}$$