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> 已知作用量
>
> $$S[p,x,A]=S_0+S_1=\int_{t_1}^{t_2}\mathrm{d}t[p\dot{x}-H(p,x)-A(t)H(p,x)]+k\int_{t_1}^{t_2}\mathrm{d}tA(t)$$
>
> 求系统的运动方程。

注意到:

$$\begin{aligned}\int_{t_1}^{t_2}\mathrm{d}tp\delta\dot{x}&=\int_{t_1}^{t_2}\mathrm{d}tp\frac{\mathrm{d}\left(\delta x\right)}{\mathrm{d}t}\\&=\int_{t_1}^{t_2}\mathrm{d}t\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(p\delta x\right)-\dot{p}\delta x\right]\\&=p\delta x\Big|_{t=t_1}^{t=t_2}-\int_{t_1}^{t_2}\mathrm{d}t\dot{p}\delta x\\&=-\int_{t_1}^{t_2}\mathrm{d}t\dot{p}\delta x\end{aligned}$$

$$\begin{aligned}\delta S\left[p,x,A\right]&=\int_{t_1}^{t_2}\mathrm{d}t\delta[p\dot{x}-H(p,x)-A(t)H(p,x)]+\int_{t_1}^{t_2}k\mathrm{d}t\delta A\\&=\int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p+p\delta\dot{x}-H(p,x)\delta A-[1+A]\left[\frac{\partial H}{\partial x}\delta x+\frac{\partial H}{\partial p}\delta p\right]\right\}+\int_{t_1}^{t_2}k\mathrm{d}t\delta A\end{aligned}$$

注意到:

$$\begin{aligned}\int_{t_1}^{t_2}\mathrm{d}tp\delta\dot{x}&=\int_{t_1}^{t_2}\mathrm{d}tp\frac{\mathrm{d}\left(\delta x\right)}{\mathrm{d}t}\\&=\int_{t_1}^{t_2}\mathrm{d}t\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(p\delta x\right)-\dot{p}\delta x\right]\\&=p\delta x\Big|_{t=t_1}^{t=t_2}-\int_{t_1}^{t_2}\mathrm{d}t\dot{p}\delta x\\&=-\int_{t_1}^{t_2}\mathrm{d}t\dot{p}\delta x\end{aligned}$$

因此:

$$\begin{aligned}\delta S[p,x,A] &= \int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p + p\delta\dot{x} - H(p,x)\delta A - [1+A]\left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial p}\delta p\right]\right\} + \int_{t_1}^{t_2} k\mathrm{d}t\delta A \\ &= -\int_{t_1}^{t_2}\mathrm{d}t\dot{p}\delta x + \int_{t_1}^{t_2}\mathrm{d}t\left\{\dot{x}\delta p - H(p,x)\delta A - [1+A]\left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial p}\delta p\right]\right\} + \int_{t_1}^{t_2} k\mathrm{d}t\delta A \\ &= \int_{t_1}^{t_2}\mathrm{d}t\left\{\delta p\left[\dot{x} - (1+A)\frac{\partial H}{\partial p}\right] + \delta x\left[-\dot{p} - (1+A)\frac{\partial H}{\partial x}\right] + \delta A[-H+k]\right\}\end{aligned}$$

$\delta S = 0$ 给出系统的运动方程：

$$\dot{x} - (1+A)\frac{\partial H}{\partial p} = 0$$

$$-\dot{p} - (1+A)\frac{\partial H}{\partial x} = 0$$

$$-H + k = 0$$

或者：

$$\dot{x} = [1+A(t)]\frac{\partial H(p,x)}{\partial p}$$

$$\dot{p} = -[1+A(t)]\frac{\partial H(p,x)}{\partial x}$$

$$H(p,x) = k$$

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> 一维谐振子，其拉格朗日量为
>
> $$L(x,\dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$$
>
> 其中 x 为偏离平衡位的位移，$\dot{x}$ 为速度。求解系统的作用量 S，确定其与初末时刻及位置的关系。

$$\frac{\partial L}{\partial x} = -\omega^2 x,\ \ \frac{\partial L}{\partial \dot{x}} = \dot{x}$$

代入 E-L 方程，得：

$$\ddot{x} + \omega^2 x = 0$$

解得：

$$x(t) = A\cos(\omega t + \varphi_0)$$

设 $x(0) = x_0$，则：

$$A\cos\varphi_0 = x_0$$

$$\dot{x}(t) = -\omega A \sin(\omega t + \varphi_0)$$

作用量为：

$$\begin{aligned}
S &= \int_{t_1}^{t_2} L(x, \dot{x})\mathrm{d}t \\
&= \frac{1}{2}\omega^2 A^2 \int_{t_1}^{t_2} [\sin^2(\omega t + \varphi_0) - \cos^2(\omega t + \varphi_0)]\mathrm{d}t \\
&= -\frac{1}{2}\omega^2 A^2 \int_{t_1}^{t_2} \cos(2\omega t + 2\varphi_0)\mathrm{d}t \\
&= -\frac{1}{4}\omega A^2 \int_{t=t_1}^{t=t_2} \cos(2\omega t + 2\varphi_0)\mathrm{d}(2\omega t + 2\varphi_0) \\
&= -\frac{1}{4}\omega A^2 \sin(2\omega t + 2\varphi_0)\Big|_{t=t_1}^{t=t_2} \\
&= -\frac{1}{4}\omega A^2 \Big[\sin(2\omega t_2 + 2\varphi_0) - \sin(2\omega t_1 + 2\varphi_0)\Big]
\end{aligned}$$

其中，$A\cos\varphi_0 = x_0$，x_0 是初始位置。