1(a)

$$\begin{cases}x=\sigma\tau\\ y=\frac{1}{2}(\tau^2-\sigma^2)\end{cases}\Longrightarrow\begin{cases}\mathrm{d}x=\sigma\mathrm{d}\tau+\tau\mathrm{d}\sigma\\ \mathrm{d}y=\tau\mathrm{d}\tau-\sigma\mathrm{d}\sigma\end{cases}$$

线元：

$$\begin{aligned}\mathrm{d}s^2&\equiv\mathrm{d}x^2+\mathrm{d}y^2\\&=(\sigma\mathrm{d}\tau+\tau\mathrm{d}\sigma)^2+(\tau\mathrm{d}\tau-\sigma\mathrm{d}\sigma)^2\\&=\sigma^2\mathrm{d}\tau^2+2\sigma\tau\mathrm{d}\tau\mathrm{d}\sigma+\tau^2\mathrm{d}\sigma^2+\tau^2\mathrm{d}\tau^2-2\tau\sigma\mathrm{d}\tau\mathrm{d}\sigma+\sigma^2\mathrm{d}\sigma^2\\&=(\sigma^2+\tau^2)(\mathrm{d}\sigma^2+\mathrm{d}\tau^2)\end{aligned}$$

计算偏导数：

$$\frac{\partial x}{\partial\sigma}=\tau,\frac{\partial x}{\partial\tau}=\sigma,\frac{\partial y}{\partial\sigma}=-\sigma,\frac{\partial y}{\partial\tau}=\tau$$

雅可比行列式：

$$\left|\frac{\partial(x,y)}{\partial(\sigma,\tau)}\right|=\begin{vmatrix}\frac{\partial x}{\partial\sigma}&\frac{\partial x}{\partial\tau}\\\frac{\partial y}{\partial\sigma}&\frac{\partial y}{\partial\tau}\end{vmatrix}=\tau^2+\sigma^2$$

面元：

$$\mathrm{d}A\equiv\mathrm{d}x\mathrm{d}y=\left|\frac{\partial(x,y)}{\partial(\sigma,\tau)}\right|\mathrm{d}\sigma\mathrm{d}\tau=(\tau^2+\sigma^2)\mathrm{d}\sigma\mathrm{d}\tau$$

1(b)

设 $u=u_1(x,y)=u_2(\sigma,\tau)$

由链式法则，有：

$$\begin{aligned}\partial_\sigma u&=\partial_x u\partial_\sigma x+\partial_y u\partial_\sigma y\\&=\tau\partial_x u-\sigma\partial_y u\end{aligned}$$

$$\begin{aligned}\partial_\tau u&=\partial_x u\partial_\tau x+\partial_y u\partial_\tau y\\&=\sigma\partial_x u+\tau\partial_y u\end{aligned}$$

于是：

$$\begin{cases}\partial_\sigma=\tau\partial_x-\sigma\partial_y\\ \partial_\tau=\sigma\partial_x+\tau\partial_y\end{cases}$$

2

自然坐标系下，

$$\begin{aligned}\vec{v}&=\dot{s}\vec{e}_\tau\\ \vec{a}&=\frac{\dot{s}^2}{\rho}\vec{e}_n+\ddot{s}\vec{e}_\tau\end{aligned}$$

于是：

$$\vec{a}\times\vec{v}=\frac{\dot{s}^3}{\rho}\vec{e}_k$$

得到：

$$\rho=\frac{\dot{s}^3}{|\vec{a}\times\vec{v}|}\tag{1}$$

这里，

$$\dot{s}\equiv\frac{\mathrm{d}s}{\mathrm{d}t}=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$$

$$\vec{v} \equiv \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = (\dot{x}, \dot{y}, \dot{z})$$

$$\vec{a} \equiv \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = (\ddot{x}, \ddot{y}, \ddot{z})$$

代入(1)，得:

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^{\frac{3}{2}}}{\sqrt{(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{x}\ddot{z} - \dot{z}\ddot{x})^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})^2}}$$

3(a)

$$\begin{cases} x' = \dfrac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - \frac{u}{c^2}x}{\sqrt{1 - \frac{u^2}{c^2}}} \end{cases}$$

3(b)

由于 $y' = y, z' = z$，要证明 $x^\mu x_\mu$ 是一个Lorentz标量，只要证明:

$$c^2t^2 - x^2 = c^2t'^2 - x'^2$$

而:

$$\begin{aligned} c^2t'^2 - x'^2 &= c^2\frac{(t - \frac{u}{c^2}x)^2}{1 - (\frac{u}{c})^2} - \frac{(x - ut)^2}{1 - (\frac{u}{c})^2} \\ &= \frac{(ct - \frac{u}{c}x)^2 - (x - ut)^2}{1 - (\frac{u}{c})^2} \\ &= \frac{c^2t^2 - u^2t^2 + \frac{u^2}{c^2}x^2 - x^2}{1 - (\frac{u}{c})^2} \\ &= \frac{c^2t^2(1 - \frac{u^2}{c^2}) + x^2(\frac{u^2}{c^2} - 1)}{1 - (\frac{u}{c})^2} \\ &= c^2t^2 - x^2 \end{aligned}$$

于是 $x^\mu x_\mu$ 是一个 Lorentz 标量

3(c)

此变换为线性变换，于是有:

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

结合条件(ii)，有:

$$\begin{cases} ct' = a_{11}ct + a_{12}x \\ x' = a_{21}ct + a_{22}x \\ c^2t^2 - x^2 = c^2t'^2 - x'^2 \end{cases}$$

消去 t', x'，得:

$$c^2t^2 - x^2 = (a_{11}ct + a_{12}x)^2 - (a_{21}ct + a_{22}x)^2$$

由对应项系数相等，得到:

$$\begin{cases} a_{11}^2 - a_{21}^2 = 1 \quad (1) \\ a_{12}^2 - a_{22}^2 = -1 (2) \\ a_{11}a_{12} - a_{21}a_{22} = 0 \quad (3) \end{cases}$$

$(1)(2)$ 代入 (3)，消去 a_{11}^2, a_{12}^2，得:

$$a_{22}^2 = a_{21}^2 + 1$$

令 $a_{11} = k$，则:

$$\begin{cases} a_{11} &= k \\ a_{21} &= \pm\sqrt{k^2-1} \\ a_{12} &= \pm\sqrt{k^2-1} \\ a_{22} &= \pm k \end{cases}$$

其中，正负号还要满足 (3)，于是，与此线性变换对应的矩阵的所有可能为:

$$\begin{bmatrix} k & \sqrt{k^2-1} \\ \sqrt{k^2-1} & k \end{bmatrix}, \begin{bmatrix} k & \sqrt{k^2-1} \\ -\sqrt{k^2-1} & -k \end{bmatrix}, \begin{bmatrix} k & -\sqrt{k^2-1} \\ \sqrt{k^2-1} & -k \end{bmatrix}, \begin{bmatrix} k & -\sqrt{k^2-1} \\ -\sqrt{k^2-1} & k \end{bmatrix}$$

从左往右数第四个矩阵即为四维时空$\mathrm{Lorentz}$变换的二维对应

3(d)

伽利略变换:

$$\begin{cases} x' = x - ut \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

对于伽利略变换:

$$(t, \vec{r}) \to (t, \vec{r} + \vec{v}t)$$

在 K 系内，

$$\vec{F} = m\ddot{\vec{r}}$$

在 K' 系内，

$$\vec{F'} = \vec{F} = m\ddot{\vec{r}} = m\frac{\mathrm{d}^2(\vec{r} + \vec{v}t)}{\mathrm{d}t^2}$$

两者形式一致，于是牛顿运动方程在伽利略变换 $(t, \vec{r}) \to (t, \vec{r} + \vec{v}t)$ 下保持不变

3(e)

在伽利略变换下，

$$x^\mu x_\mu \equiv c^2t^2 - x^2 - y^2 - z^2$$

$$x'^\mu x'_\mu \equiv c^2t^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - (x + v_xt)^2 - (y + v_yt)^2 - (z + v_zt)^2 \neq x^\mu x_\mu$$

3(f)

$$\begin{cases} x' = \dfrac{x - ut}{\sqrt{1 - (\frac{u}{c})^2}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - \frac{u}{c^2}x}{\sqrt{1 - (\frac{u}{c})^2}} \end{cases}$$

注意到，

$$\frac{\mathrm{d}x'}{\mathrm{d}t} = \frac{\mathrm{d}x'}{\mathrm{d}t'} \cdot \frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{1 - \frac{u}{c^2}v_x}{\sqrt{1 - (\frac{u}{c})^2}} v'_x$$

对第一行等式左右两边同时对 t 求导得:

$$v_x' = \frac{v_x - u}{1 - \frac{u}{c^2}v_x}$$

同理有:

$$v_y' = \frac{v_y\sqrt{1-(\frac{u}{c})^2}}{1 - \frac{u}{c^2}v_x}$$

$$v_z' = \frac{v_z\sqrt{1-(\frac{u}{c})^2}}{1 - \frac{u}{c^2}v_x}$$

3(g)

令:

$$v_x^2 + v_y^2 + v_z^2 = c^2$$

则:

$$\begin{aligned} v_x'^2 + v_y'^2 + v_z'^2 &= \frac{(v_x-u)^2 + v_y^2(1-\frac{u^2}{c^2}) + v_z^2(1-\frac{u^2}{c^2})}{(1-\frac{u}{c^2}v_x)^2} \\ &= \frac{(v_x^2+v_y^2+v_z^2)(1-\frac{u^2}{c^2}) - v_x^2(1-\frac{u^2}{c^2}) + (v_x-u)^2}{(1-\frac{u}{c^2}v_x)^2} \\ &= \frac{\frac{u^2}{c^2}v_x^2 - 2uv_x + c^2}{(1-\frac{u}{c^2}v_x)^2} \\ &= \frac{(\frac{u}{c}v_x - c)^2}{(1-\frac{u}{c^2}v_x)^2} \\ &= \frac{c^2(\frac{u}{c^2}v_x - 1)^2}{(1-\frac{u}{c^2}v_x)^2} \\ &= c^2 \end{aligned}$$

这就是说，光速不变

3(h)

由:

$$\begin{cases} v_x' = \dfrac{v_x - u}{1 - \frac{u}{c^2}v_x} \\ v_y' = \dfrac{v_y\sqrt{1-(\frac{u}{c})^2}}{1 - \frac{u}{c^2}v_x} \\ v_z' = \dfrac{v_z\sqrt{1-(\frac{u}{c})^2}}{1 - \frac{u}{c^2}v_x} \\ t' = \dfrac{t - \frac{u}{c^2}x}{\sqrt{1-(\frac{u}{c})^2}} \end{cases}$$

对所有等式两边同时对 t 求导，并结合链式法则，得:

$$\begin{cases} a_x' = \dfrac{(1-\frac{u^2}{c^2})^{\frac{3}{2}}a_x}{(1-\frac{u}{c^2}v_x)^3} \\ a_y' = \dfrac{(1-\frac{u^2}{c^2})[(1-\frac{u}{c^2}v_x)a_y + \frac{uv_y}{c^2}a_x]}{(1-\frac{u}{c^2}v_x)^3} \\ a_z' = \dfrac{(1-\frac{u^2}{c^2})[(1-\frac{u}{c^2}v_x)a_z + \frac{uv_z}{c^2}a_x]}{(1-\frac{u}{c^2}v_x)^3} \end{cases}$$