1

(a)

$$\vec{r}_1 = l_1\cos\theta_1\vec{e}_x + l_1\sin\theta_1\vec{e}_y$$

$$\vec{r}_2 = (l_1\cos\theta_1 + l_2\cos\theta_2)\vec{e}_x + (l_1\sin\theta_1 + l_2\sin\theta_2)\vec{e}_y$$

$$\dot{\vec{r}}_1 = -l_1\dot{\theta}_1\sin\theta_1\vec{e}_x + l_1\dot{\theta}_1\cos\theta_1\vec{e}_y$$

$$\dot{\vec{r}}_2 = (-l_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2)\vec{e}_x + (l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2)\vec{e}_y$$

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)\right] \end{aligned}$$

$$V = -m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

拉格朗日量为：

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)\right] + m_1gl_1\cos\theta_1 + m_2g(l_1\cos\theta_1 + l_2\cos\theta_2) \end{aligned}$$

对于微振动，

$$\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 = \cos(\theta_1 - \theta_2) \approx 1$$

$$\begin{aligned} V &= -m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2) \\ &\approx V_0 + \frac{1}{2}(m_1+m_2)gl_1\cos\theta_1\Big|_{\theta_1=0}(\theta_1)^2 + \frac{1}{2}m_2gl_2\cos\theta_2\Big|_{\theta_2=0}(\theta_2)^2 \\ &= V_0 + \frac{1}{2}(m_1+m_2)gl_1\theta_1^2 + \frac{1}{2}m_2gl_2\theta_2^2 \end{aligned}$$

于是：

$$L \approx \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 - \frac{1}{2}(m_1+m_2)gl_1\theta_1^2 - \frac{1}{2}m_2gl_2\theta_2^2 + V_0$$

微振动运动方程为：

$$(m_1+m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2 + (m_1+m_2)g\theta_1 = 0$$

$$m_2l_2\ddot{\theta}_2 + m_2l_1\ddot{\theta}_1 + m_2g\theta_2 = 0$$

(b)

若 $m_1 = m_2 = m, l_1 = l_2 = l$，则方程简化为：

$$\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2 + \frac{g}{l}\theta_1 = 0$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 + \frac{g}{l}\theta_2 = 0$$

$$\mathbf{M} = \begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 2mgl & 0 \\ 0 & mgl \end{bmatrix}$$

拉格朗日量可写为（舍去常数项）：

$$L=\frac{1}{2}(\dot\theta^{\mathrm{T}}\mathbf{M}\dot\theta-\theta^{\mathrm{T}}\mathbf{V}\theta),\ \ \theta=\begin{bmatrix}\theta_1\\\theta_2\end{bmatrix},\ \ \dot\theta=\begin{bmatrix}\dot\theta_1\\\dot\theta_2\end{bmatrix}$$

久期方程为:

$$|-\mathbf{M}\omega^2+\mathbf{V}|=\mathbf{0}$$

解得:

$$\omega_1^2=(2+\sqrt{2})\frac{g}{l},\ \ \omega_2^2=(2-\sqrt{2})\frac{g}{l}$$

于是简正频率为:

$$\omega_1=\sqrt{(2+\sqrt{2})\frac{g}{l}},\ \ \omega_2=\sqrt{(2-\sqrt{2})\frac{g}{l}}$$

将简正频率代回方程 $[-\mathbf{M}\omega^2+\mathbf{V}]\eta(\omega^2)=\mathbf{0}$,得:

$$\eta_1=\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix},\ \ \eta_2=\begin{bmatrix}\frac{1}{\sqrt{3}}\\\frac{\sqrt{2}}{\sqrt{3}}\end{bmatrix}$$

2

(a)

$$\vec r_1=l\cos\theta\vec e_x+l\sin\theta\vec e_y,\ \ \dot{\vec r}_1=-l\dot\theta\sin\theta\vec e_x+l\dot\theta\cos\theta\vec e_y$$

$$\vec r_2=l\cos\varphi\vec e_x+(y_0+l\sin\varphi)\vec e_y,\ \ \dot{\vec r}_2=-l\dot\varphi\sin\varphi\vec e_x+l\dot\varphi\cos\varphi\vec e_y$$

$$\begin{aligned}T&=\frac{1}{2}m\dot{\vec r}_1^{\,2}+\frac{1}{2}m'\dot{\vec r}_2^{\,2}\\&=\frac{1}{2}ml^2\dot\theta^2+\frac{1}{2}m'l^2\dot\varphi^2\end{aligned}$$

设弹簧固有长度为 l_0,

$$V=-mgl\cos\theta-m'gl\cos\varphi+\frac{1}{2}k(\sqrt{(\vec r_1-\vec r_2)^2}-l_0)^2$$

将 V 在平衡位置 $\theta=0,\varphi=0$ 处展开至二阶:

$$V=V_0+\frac{1}{2}mgl\theta^2+\frac{1}{2}m'gl\varphi^2+\frac{1}{2}kl^2\theta^2+\frac{1}{2}kl^2\varphi^2-kl^2\theta\varphi$$

拉格朗日量为:

$$\begin{aligned}L&=T-V\\&=\frac{1}{2}ml^2\dot\theta^2+\frac{1}{2}m'l^2\dot\varphi^2-V_0-\frac{1}{2}mgl\theta^2-\frac{1}{2}m'gl\varphi^2-\frac{1}{2}kl^2\theta^2-\frac{1}{2}kl^2\varphi^2+kl^2\theta\varphi\\&=\frac{1}{2}ml^2\dot\theta^2+\frac{1}{2}m'l^2\dot\varphi^2-V_0-\frac{1}{2}l(mg+kl)\theta^2-\frac{1}{2}l(m'g+kl)\varphi^2+kl^2\theta\varphi\end{aligned}$$

运动方程为:

$$ml\ddot\theta+(mg+kl)\theta-kl\varphi=0$$

$$m'l\ddot\varphi+(m'g+kl)\varphi-kl\theta=0$$

(b)

$$\mathbf{M}=\begin{bmatrix}ml^2&0\\0&m'l^2\end{bmatrix}$$

$$\mathbf{V}=\begin{bmatrix} l(mg+kl) & -kl^2 \\ -kl^2 & l(m'g+kl) \end{bmatrix}$$

拉格朗日量可写为（舍去常数项）：

$$L=\frac{1}{2}(\dot{q}^{\mathrm{T}}\mathbf{M}\dot{q}-q^{\mathrm{T}}\mathbf{V}q),\ \ q=\begin{bmatrix}\theta \\ \varphi\end{bmatrix},\ \ \dot{q}=\begin{bmatrix}\dot{\theta} \\ \dot{\varphi}\end{bmatrix}$$

久期方程为:

$$|-\mathbf{M}\omega^2+\mathbf{V}|=\mathbf{0}$$

解得:

$$\omega_1^2=\frac{g}{l},\ \ \omega_2^2=\frac{g}{l}+\frac{m+m'}{mm'}k$$

于是简正频率为:

$$\omega_1=\sqrt{\frac{g}{l}},\ \ \omega_2=\sqrt{\frac{g}{l}+\frac{m+m'}{mm'}k}$$

将简正频率代回方程 $[-\mathbf{M}\omega^2+\mathbf{V}]\eta(\omega^2)=\mathbf{0}$，得:

$$\eta_1=\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix},\ \ \eta_2=\begin{bmatrix}\frac{m'}{\sqrt{m^2+m'^2}} \\ \frac{-m}{\sqrt{m^2+m'^2}}\end{bmatrix}$$

(c)

$$\begin{bmatrix}\tilde{\theta} \\ \tilde{\varphi}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{m'}{\sqrt{m^2+m'^2}} \\ \frac{1}{\sqrt{2}} & \frac{-m}{\sqrt{m^2+m'^2}}\end{bmatrix}\begin{bmatrix}A_1e^{\mathrm{i}\omega_1 t} \\ A_2e^{\mathrm{i}\omega_2 t}\end{bmatrix}$$

代入初始条件 $t=0$ 时，$\theta=\theta_0,\varphi=0$，得:

$$A_1=\frac{\sqrt{2}m\theta_0}{m+m'},\ \ A_2=\frac{\sqrt{m^2+m'^2}\theta_0}{m+m'}$$

于是:

$$\tilde{\theta}=\frac{m\theta_0}{m+m'}e^{\mathrm{i}\omega_1 t}+\frac{m'\theta_0}{m+m'}e^{\mathrm{i}\omega_2 t},\ \ \tilde{\varphi}=\frac{m\theta_0}{m+m'}e^{\mathrm{i}\omega_1 t}-\frac{m\theta_0}{m+m'}e^{\mathrm{i}\omega_2 t}$$

取实部，得:

$$\theta=\frac{m\theta_0}{m+m'}\cos\omega_1 t+\frac{m'\theta_0}{m+m'}\cos\omega_2 t,\ \ \varphi=\frac{m\theta_0}{m+m'}\cos\omega_1 t-\frac{m\theta_0}{m+m'}\cos\omega_2 t$$

其中，

$$\omega_1=\sqrt{\frac{g}{l}},\ \ \omega_2=\sqrt{\frac{g}{l}+\frac{m+m'}{mm'}k}$$

当 $\frac{g}{l}\gg\frac{k}{m},\frac{k}{m'}$，$\omega_1=\omega_2=\omega=\sqrt{\frac{g}{l}}$，由辅助角公式，

θ 摆幅:

$$A_\theta=\sqrt{\left(\frac{m\theta_0}{m+m'}\right)^2+\left(\frac{m'\theta_0}{m+m'}\right)^2}=\frac{\sqrt{m^2+m'^2}\theta_0}{m+m'}$$

θ 摆幅:

$$A_\varphi=\sqrt{\left(\frac{m\theta_0}{m+m'}\right)^2+\left(-\frac{m\theta_0}{m+m'}\right)^2}=\frac{\sqrt{2}m\theta_0}{m+m'}$$