#### 格林公式

$$egin{aligned} 
abla \cdot (\psi 
abla arphi) &= \partial_i [\psi (
abla arphi)_i] \ &= (\partial_i \psi) (
abla arphi)_i + \psi \partial_i (
abla arphi)_i \ &= (
abla \psi)_i (
abla arphi)_i + \psi 
abla \cdot (
abla arphi) \ &= (
abla \psi) \cdot (
abla arphi) + \psi 
abla^2 arphi \end{aligned}$$

于是:

$$\begin{split} \int\limits_{\partial V} (\psi \nabla \varphi) \cdot \mathrm{d} \vec{S} &= \int\limits_{V} \nabla \cdot (\psi \nabla \varphi) \mathrm{d} V \\ &= \int\limits_{V} \left[ (\nabla \psi) \cdot (\nabla \varphi) + \psi \nabla^2 \varphi \right] \mathrm{d} V \end{split}$$

一方面:

$$\begin{split} \int\limits_{\partial V} (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \mathrm{d}\vec{S} &= \int\limits_{\partial V} (\psi \nabla \varphi) \cdot \mathrm{d}\vec{S} - \int\limits_{\partial V} (\varphi \nabla \psi) \cdot \mathrm{d}\vec{S} \\ &= \int\limits_{V} \left[ (\nabla \psi) \cdot (\nabla \varphi) + \psi \nabla^2 \varphi \right] \mathrm{d}V - \int\limits_{V} \left[ (\nabla \varphi) \cdot (\nabla \psi) + \varphi \nabla^2 \psi \right] \mathrm{d}V \\ &= \int\limits_{V} (\psi \nabla^2 \varphi - \varphi \nabla^2 \psi) \mathrm{d}V \end{split}$$

另一方面:

$$\begin{split} \int_{\partial V} (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \mathrm{d}\vec{S} &= \int_{\partial V} (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \vec{n} \mathrm{d}S \\ &= \int_{\partial V} (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}r} \mathrm{d}S \\ &= \int_{\partial V} (\psi \frac{\nabla \varphi \cdot \mathrm{d}\vec{r}}{\mathrm{d}r} - \varphi \frac{\nabla \psi \cdot \mathrm{d}\vec{r}}{\mathrm{d}r}) \mathrm{d}S \\ &= \int_{\partial V} (\psi \frac{\mathrm{d}\varphi}{\mathrm{d}r} - \varphi \frac{\mathrm{d}\psi}{\mathrm{d}r}) \mathrm{d}S \\ &= \int_{\partial V} (\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n}) \mathrm{d}S \end{split}$$

于是得到格林公式:

$$\int\limits_V (\psi 
abla^2 arphi - arphi 
abla^2 \psi) \mathrm{d}V = \int\limits_{\partial V} (\psi rac{\partial arphi}{\partial n} - arphi rac{\partial \psi}{\partial n}) \mathrm{d}S$$

# 线元的模方

设  $\mathbb{R}^3$  空间的位矢  $\vec{r}$  可由三个参数  $u_1, u_2, u_3$  描述,即:

$$\vec{r} = \vec{r}(u_1, u_2, u_3)$$

若参数  $u_1, u_2, u_3$  分别产生一个小变化  $du_1, du_2, du_3$ , 由此导致  $\vec{r}$  产生的小变化  $d\vec{r}$  应满足:

$$\mathrm{d}ec{r}=rac{\partialec{r}}{\partial u_i}\mathrm{d}u_i$$

 $\mathbb{R}^3$  空间中的线元的模方,记为  $\mathrm{d} r^2$ ,定义为:

$$\mathrm{d}r^2 \equiv \mathrm{d}\vec{r}\cdot\mathrm{d}\vec{r}$$

于是:

$$\begin{split} \mathrm{d}r^2 &\equiv \mathrm{d}\vec{r} \cdot \mathrm{d}\vec{r} \\ &= \left(\frac{\partial \vec{r}}{\partial u_i} \mathrm{d}u_i\right) \cdot \left(\frac{\partial \vec{r}}{\partial u_j} \mathrm{d}u_j\right) \\ &= \frac{\partial \vec{r}}{\partial u_i} \cdot \frac{\partial \vec{r}}{\partial u_j} \mathrm{d}u_i \mathrm{d}u_j \end{split}$$

## 度量系数

曲线坐标系中的度量系数,记为 $g_{ij}$ ,定义为:

$$g_{ij} \equiv rac{\partial ec{r}}{\partial u_i} \cdot rac{\partial ec{r}}{\partial u_j}$$

线元模方  $\mathrm{d}r^2$  可利用度量系数写为:

$$\mathrm{d}r^2 = g_{ij}\mathrm{d}u_i\mathrm{d}u_j$$

# 正交曲线坐标系

称以  $u_1, u_2, u_3$  为参数描述空间位置的坐标系是一个正交曲线坐标系,若:

$$g_{ij}=0,\ i\neq j$$

度量系数  $g_{ij}$  的矩阵表示,记为  $(g_{ij})$ ,定义为:

$$(g_{ij}) \equiv egin{pmatrix} g_{11} & g_{12} & g_{13} \ g_{21} & g_{22} & g_{23} \ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

利用系数度量矩阵  $(g_{ij})$ , 线元的模方  $\mathrm{d}r^2$  在  $(u_1,u_2,u_3)$  曲线坐标系下可表示为:

$$egin{aligned} \mathrm{d} r^2 &= egin{pmatrix} \mathrm{d} u_1 & \mathrm{d} u_2 & \mathrm{d} u_3 \end{pmatrix} egin{pmatrix} \mathrm{d} u_1 \ \mathrm{d} u_2 \ \mathrm{d} u_3 \end{pmatrix} \ &= egin{pmatrix} \mathrm{d} u_1 & \mathrm{d} u_2 & \mathrm{d} u_3 \end{pmatrix} egin{pmatrix} g_{11} & g_{12} & g_{13} \ g_{21} & g_{22} & g_{23} \ g_{31} & g_{32} & g_{33} \end{pmatrix} egin{pmatrix} \mathrm{d} u_1 \ \mathrm{d} u_2 \ \mathrm{d} u_3 \end{pmatrix} \end{aligned}$$

正交曲线坐标系的度量系数矩阵  $(g_{ij})$  只有对角元非零。笛卡尔坐标系、球坐标系和柱坐标系都是正交曲线坐标系,因此度量系数矩阵都只有对角元非零。也就是说,对于正交曲线坐标系  $(u_1,u_2,u_3)$ ,有:

$$\mathrm{d}r^2 = g_{11}(\mathrm{d}u_1)^2 + g_{22}(\mathrm{d}u_2)^2 + g_{33}(\mathrm{d}u_3)^2$$

# 度量分量

上面说到,对于正交曲线坐标系  $(u_1, u_2, u_3)$ ,线元的模方  $dr^2$  可以表示为:

$$dr^2 = g_{11}(du_1)^2 + g_{22}(du_2)^2 + g_{33}(du_3)^2$$

特别地,若  $\mathrm{d}u_2=\mathrm{d}u_3=0$ ,即  $\vec{r}$  只沿  $u_1$  参数曲线作微小变化时,有:

$$\mathrm{d}r^2 = g_{11}(\mathrm{d}u_1)^2$$

把此时的微小弧长记为  $ds_1$ , 即:

$$\mathrm{d}s_1 = \sqrt{g_{11}} \mathrm{d}u_1$$

同理有:

$$\mathrm{d}s_2 \equiv \sqrt{g_{22}} \mathrm{d}u_2$$

$$\mathrm{d}s_3 = \sqrt{g_{33}} \mathrm{d}u_3$$

## 直角坐标系

对于直角坐标系, $(u_1,u_2,u_3)=(x_1,x_2,x_3); \vec{r}=\vec{r}(x_1,x_2,x_3)$ 

$$egin{aligned} rac{\partial ec{r}}{\partial x_1} &\equiv \lim_{\Delta x_1 o 0} rac{ec{r}(x_1 + \Delta x_1, x_2, x_3) - ec{r}(x_1, x_2, x_3)}{\Delta x_1} \ &= \lim_{\Delta x_1 o 0} rac{\Delta x_1 ec{e}_1}{\Delta x_1} \ &= ec{e}_1 \end{aligned}$$

$$rac{\partial ec{r}}{\partial x_2} = ec{e}_2$$

$$rac{\partial ec{r}}{\partial x_3} = ec{e}_3$$

于是笛卡尔坐标系的度量系数  $g_{ij}$  为:

$$g_{ij} \equiv rac{\partial ec{r}}{\partial u_i} \cdot rac{\partial ec{r}}{\partial u_j} \ = rac{\partial ec{r}}{\partial x_i} \cdot rac{\partial ec{r}}{\partial x_j} \ = ec{e}_i \cdot ec{e}_j \ = \delta_{ij}$$

度量系数的矩阵表示为:

$$(g_{ij}) = egin{pmatrix} 1 & 0 & 0 \ 0 & 1 & 0 \ 0 & 0 & 1 \end{pmatrix}$$

线元的模方  $\mathrm{d} r^2$  在笛卡尔坐标系下的表示为:

$$\mathrm{d}r^2 = g_{ij}\mathrm{d}u_i\mathrm{d}u_j \ = \delta_{ij}\mathrm{d}x_i\mathrm{d}x_j \ = \mathrm{d}x_j\mathrm{d}x_j \ = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$$

度量分量分别为:

$$h_1 = h_2 = h_3 = 1$$

#### 球坐标系

对于球坐标系,  $(u_1,u_2,u_3)=(r,\theta,\varphi); \vec{r}=\vec{r}(r,\theta,\varphi)$ 

$$\begin{split} \frac{\partial \vec{r}}{\partial r} &\equiv \lim_{\Delta r \to 0} \frac{\vec{r}(r + \Delta r, \theta, \varphi) - \vec{r}(r, \theta, \varphi)}{\Delta r} \\ &= \lim_{\Delta r \to 0} \frac{\Delta r \vec{e}_r}{\Delta r} \\ &= \vec{e}_r \end{split}$$

$$\frac{\partial \vec{r}}{\partial \theta} &\equiv \lim_{\Delta \theta \to 0} \frac{\vec{r}(r, \theta + \Delta \theta, \varphi) - \vec{r}(r, \theta, \varphi)}{\Delta \theta} \\ &= \lim_{\Delta \theta \to 0} \frac{r \Delta \theta \vec{e}_{\theta}}{\Delta \theta} \\ &= r \vec{e}_{\theta} \end{split}$$

$$\frac{\partial \vec{r}}{\partial \varphi} &\equiv \lim_{\Delta \varphi \to 0} \frac{\vec{r}(r, \theta, \varphi + \Delta \varphi) - \vec{r}(r, \theta, \varphi)}{\Delta \varphi} \\ &= \lim_{\Delta \varphi \to 0} \frac{r \sin \theta \Delta \varphi \vec{e}_{\varphi}}{\Delta \varphi} \end{split}$$

于是球坐标系的度量系数  $g_{ij}$  的矩阵表示为:

$$(g_{ij}) = egin{pmatrix} 1 & 0 & 0 \ 0 & r^2 & 0 \ 0 & 0 & r^2 \sin^2 heta \end{pmatrix}$$

 $=r\sin hetaec{e}_{\omega}$ 

度量分量分别为:

$$h_1 = 1, \ h_2 = r, \ h_3 = r \sin \theta$$

## 柱坐标系下的线元表示

对于柱坐标系,  $(u_1, u_2, u_3) = (\rho, \varphi, z)$ ;  $\vec{r} = \vec{r}(\rho, \varphi, z)$ 

$$egin{aligned} rac{\partial ec{r}}{\partial 
ho} &\equiv \lim_{\Delta 
ho o 0} rac{ec{r}(
ho + \Delta 
ho, arphi, z) - ec{r}(
ho, arphi, z)}{\Delta 
ho} \ &= \lim_{\Delta 
ho o 0} rac{\Delta 
ho ec{e}_
ho}{\Delta 
ho} \ &= ec{e}_
ho \end{aligned}$$

$$egin{aligned} rac{\partial ec{r}}{\partial arphi} &\equiv \lim_{\Delta arphi o 0} rac{ec{r}(
ho, arphi + \Delta arphi, z) - ec{r}(
ho, arphi, z)}{\Delta arphi} \ &= \lim_{\Delta arphi o 0} rac{
ho \Delta arphi ec{e}_{arphi}}{\Delta arphi} \ &= 
ho ec{e}_{arphi} \end{aligned}$$

$$\begin{split} \frac{\partial \vec{r}}{\partial z} &\equiv \lim_{\Delta z \to 0} \frac{\vec{r}(\rho, \varphi, z + \Delta z) - \vec{r}(\rho, \varphi, z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\Delta z \vec{e}_z}{\Delta z} \\ &= \vec{e}_z \end{split}$$

于是柱坐标的度量系数矩阵  $(g_{ij})$  为:

$$(g_{ij}) = egin{pmatrix} 1 & 0 & 0 \ 0 & 
ho^2 & 0 \ 0 & 0 & 1 \end{pmatrix}$$

度量分量分别为:

$$h_1 = 1, h_2 = \rho, h_3 = 1$$

# 梯度、散度、旋度

## 梯度

#### 梯度的定义

设  $\psi$  是  $\mathbb{R}^3$  空间中的标量场, $\mathrm{d}\vec{r}$  是  $\vec{r}$  处的任意有向线元, $\mathrm{d}\psi$  是位矢  $\vec{r}$  产生小变化  $\mathrm{d}\vec{r}$  所导致的  $\psi$  产生的小变化; $\psi$  的梯度,记为  $\nabla\psi$ ,定义为满足下式的矢量场:

$$\nabla \psi \cdot d\vec{r} = d\psi$$

## 梯度与方向导数的关系

对于梯度的定义式:

$$\nabla \psi \cdot \mathrm{d}\vec{r} = \mathrm{d}\psi$$

取:

$$\mathrm{d}\vec{r} = \vec{n}_l \mathrm{d}l$$

其中,l 是以  $\vec{r}$  为端点的射线,标记了一个方向,dl 是这条  $\vec{r}$  的端点沿射线 l 方向延伸出的小线元, $\vec{n}_l$  是射线 l 方向上的单位向量,则:

$$\nabla \psi \cdot \vec{n}_l dl = d\psi$$

即:

$$oxed{
abla\psi\cdotec{n}_l=rac{\partial\psi}{\partial n}igg|_l}$$

其中, $\left. \frac{\partial \psi}{\partial n} \right|_l$ 是标量场  $\psi$  沿射线 l 方向上的方向导数。

#### 正交曲线坐标系下梯度的一般表达式

对于正交曲线坐标系  $(u_1,u_2,u_3)$ , 设坐标基向量为  $\vec{e}_1,\vec{e}_2,\vec{e}_3$ , 梯度  $\nabla \psi$  和有限线元  $d\vec{r}$  可在坐标基向量上展开为:

$$abla \psi = \sum_i (
abla \psi)_i ec{e}_i, \; \, \mathrm{d}ec{r} = \sum_i \mathrm{d} s_i ec{e}_i$$

由梯度的定义式  $abla\psi\cdot\mathrm{d}\vec{r}=\mathrm{d}\psi$  ,有:

$$\sum_i (
abla \psi)_i \mathrm{d} s_i = \mathrm{d} \psi$$

一方面,前面的推导给出:

$$\mathrm{d}s_i = h_i \mathrm{d}u_i$$

另一方面,

$$\mathrm{d}\psi(u_1,u_2,u_3) = \sum_i \frac{\partial \psi}{\partial u_i} \mathrm{d}u_i$$

两者代入,得:

$$\sum_{i} (\nabla \psi)_{i} h_{i} du_{i} = \sum_{i} \frac{\partial \psi}{\partial u_{i}} du_{i}$$

对比可得:

$$(
abla\psi)_i h_i = rac{\partial \psi}{\partial u_i}$$

于是得到矢量  $\nabla \psi$  在正交曲线坐标系  $(u_1,u_2,u_3)$ 下的分量表示:

$$(
abla\psi)_i=rac{1}{h_i}rac{\partial\psi}{\partial u_i},\;\;(i$$
不求和)

以及:

$$abla\psi=\sum_{i=1}^{3}rac{1}{h_{i}}rac{\partial\psi}{\partial u_{i}}ec{e}_{i}$$

### 直角坐标系下的梯度

$$abla = ec{e}_x rac{\partial}{\partial x} + ec{e}_y rac{\partial}{\partial y} + ec{e}_z rac{\partial}{\partial z}$$

#### 球坐标系下的梯度

$$(u_1,u_2,u_3)=(r,\theta,\varphi), h=(1,r,r\sin\theta)$$

$$abla \ = ec{e}_r rac{\partial}{\partial r} + ec{e}_ heta rac{1}{r} rac{\partial}{\partial heta} + ec{e}_arphi rac{1}{r \sin heta} rac{\partial}{\partial arphi}$$

#### 柱坐标系下的梯度

$$(u_1, u_2, u_3) = (\rho, \varphi, z), h = (1, \rho, 1)$$

$$abla = ec{e}_
ho rac{\partial}{\partial 
ho} + ec{e}_arphi rac{1}{
ho} rac{\partial}{\partial arphi} + ec{e}_z rac{\partial}{\partial z}$$

# 散度

矢量场  $\vec{A}$  的散度,记为  $\nabla \cdot \vec{A}$ ,定义为:

$$abla \cdot ec{A} \equiv \lim_{\Delta V 
ightarrow 0^+} rac{1}{\Delta V} \oint\limits_{\partial V^+} ec{A} \cdot \mathrm{d}ec{S}$$

其中,  $\Delta V$  是区域 V 的体积,  $\partial V$  是区域 V 的边界,  $\partial V^+$  表明面元的方向为边界外法向

$$\begin{split} \Delta V &= \mathrm{d} s_1 \mathrm{d} s_2 \mathrm{d} s_3 \\ &= h_1 h_2 h_3 \mathrm{d} u_1 \mathrm{d} u_2 \mathrm{d} u_3 \\ \mathrm{d} \vec{\sigma} \big|_{u_1, u_2, u_3} &= -\vec{e}_1 \mathrm{d} s_2 \big|_{u_1, u_2, u_3} \mathrm{d} s_3 \big|_{u_1, u_2, u_3} \\ &= -\vec{e}_1 (h_2 h_3) \big|_{u_1, u_2, u_3} \mathrm{d} u_2 \mathrm{d} u_3 \\ \mathrm{d} \vec{\sigma} \big|_{u_1 + \mathrm{d} u_1, u_2, u_3} &= \vec{e}_1 \mathrm{d} s_2 \big|_{u_1 + \mathrm{d} u_1, u_2, u_3} \mathrm{d} s_3 \big|_{u_1 + \mathrm{d} u_1, u_2, u_3} \\ &= \vec{e}_1 (h_2 h_3) \big|_{u_1 + \mathrm{d} u_1, u_2, u_3} \mathrm{d} u_2 \mathrm{d} u_3 \end{split}$$

$$egin{aligned} ec{A}ig|_{u_1,u_2,u_3} \cdot \mathrm{d}ec{\sigma}ig|_{u_1,u_2,u_3} + ec{A}ig|_{u_1+\mathrm{d}u_1,u_2,u_3} \cdot \mathrm{d}ec{\sigma}ig|_{u_1,u_2,u_3} &= -(A_1h_2h_3)ig|_{u_1,u_2,u_3} \mathrm{d}u_2\mathrm{d}u_3 + (A_1h_2h_3)ig|_{u_1+\mathrm{d}u_1,u_2,u_3} \mathrm{d}u_2\mathrm{d}u_3 \\ &= rac{\partial (A_1h_2h_3)}{\partial u_1}\mathrm{d}u_2\mathrm{d}u_3 \end{aligned}$$

设坐标  $u_1, u_2, u_3$  各有一个小增量  $du_1, du_2, du_3$ , 此过程中会在空间中围成一个体积元 dV, dV 可表达为:

$$egin{aligned} \mathrm{d} V &= \mathrm{d} s_1 \mathrm{d} s_2 \mathrm{d} s_3 \ &= (h_1 \mathrm{d} u_1) (h_2 \mathrm{d} u_2) (h_3 \mathrm{d} u_3) \ &= h_1 h_2 h_3 \mathrm{d} u_1 \mathrm{d} u_2 \mathrm{d} u_3 \end{aligned}$$

$$oxed{
abla\cdotec{A}=rac{1}{h_1h_2h_3}igg[rac{\partial}{\partial u_1}(A_1h_2h_3)+rac{\partial}{\partial u_2}(A_2h_3h_1)+rac{\partial}{\partial u_3}(A_3h_1h_2)igg]}$$

#### 直角坐标系下的散度

$$abla \cdot ec{A} = rac{\partial A_x}{\partial x} + rac{\partial A_y}{\partial y} + rac{\partial A_z}{\partial z}$$

#### 球坐标系下的散度

$$abla \cdot ec{A} = rac{1}{r^2 \sin heta} iggl[ rac{\partial}{\partial r} (A_r r^2 \sin heta) + rac{\partial}{\partial heta} (A_ heta r \sin heta) + rac{\partial}{\partial arphi} (A_arphi r) iggr]$$

## 柱坐标系下的散度

$$abla \cdot ec{A} = rac{1}{
ho} igg[ rac{\partial}{\partial 
ho} (A_{
ho} 
ho) + rac{\partial}{\partial arphi} A_{arphi} + rac{\partial}{\partial z} (A_{z} 
ho) igg]$$

## 旋度

$$(
abla imes ec{A}) \cdot ec{e}_n = \lim_{\sigma o 0^+} rac{1}{\sigma} \oint\limits_{\partial \sigma} ec{A} \cdot \mathrm{d}ec{l}$$

其中,  $\sigma$  是垂直于  $\vec{e}_n$  的面元

$$egin{align*} (
abla imes ec{A})_1 &= \lim_{\sigma o 0^+} rac{1}{\sigma} \oint_{\partial \sigma} ec{A} \cdot \mathrm{d}ec{l} \ &= rac{1}{h_2 h_3 \mathrm{d} u_2 \mathrm{d} u_3} \cdot \left[ \left( A_2 h_2 
ight) ig|_{u_1, u_2, u_3} \mathrm{d} u_2 - \left( A_2 h_2 
ight) ig|_{u_1, u_2, u_3 + \mathrm{d} u_3} \mathrm{d} u_2 - \left( A_3 h_3 
ight) ig|_{u_1, u_2, u_3} + \left( A_3 h_3 
ight) ig|_{u_1, u_2 + \mathrm{d} u_2, u_3} \mathrm{d} u_3 
ight] \ &= rac{1}{h_2 h_3} \left[ rac{\partial (A_3 h_3)}{\partial u_2} - rac{\partial (A_2 h_2)}{\partial u_3} 
ight] \end{split}$$

$$\boxed{\nabla \times \vec{A} = \frac{1}{h_2 h_3} \left[ \frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right] \vec{e}_1 + \frac{1}{h_3 h_1} \left[ \frac{\partial}{\partial u_3} (A_1 h_1) - \frac{\partial}{\partial u_1} (A_3 h_3) \right] \vec{e}_2 + \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial u_1} (A_2 h_2) - \frac{\partial}{\partial u_2} (A_1 h_1) \right] \vec{e}_3}$$

#### 直角坐标系下的旋度

$$abla imes ec{A} = ec{e}_k arepsilon_{ijk} \partial_i A_j$$

## 球坐标系下的旋度

$$abla imes ec{A} = rac{1}{r^2 \sin heta} iggl[ rac{\partial}{\partial heta} (r \sin heta A_arphi) - rac{\partial}{\partial arphi} (r A_ heta) iggr] ec{e}_r + rac{1}{r \sin heta} iggl[ rac{\partial}{\partial arphi} A_r - rac{\partial}{\partial r} (r \sin heta A_arphi) iggr] ec{e}_ heta + rac{1}{r} iggl[ rac{\partial}{\partial r} (r A_ heta) - rac{\partial}{\partial heta} A_r iggr] ec{e}_arphi$$

#### 柱坐标系下的旋度

$$abla imes ec{A} = rac{1}{
ho} iggl[ rac{\partial}{\partial arphi} A_z - rac{\partial}{\partial z} (
ho A_arphi) iggr] ec{e}_
ho + iggl[ rac{\partial}{\partial z} A_
ho - rac{\partial}{\partial 
ho} A_z iggr] ec{e}_arphi + rac{1}{
ho} iggl[ rac{\partial}{\partial 
ho} (
ho A_arphi) - rac{\partial}{\partial arphi} A_arphi iggr] ec{e}_z$$

#### 斯托克斯公式

$$\int\limits_{\Sigma} (\nabla \times \vec{A}) \cdot \mathrm{d}\vec{\sigma} = \oint\limits_{\partial \Sigma} \vec{A} \cdot \mathrm{d}\vec{l}$$

# 拉普拉斯算符 $\nabla^2$

$$abla^2\psi=rac{1}{h_1h_2h_3}igg[rac{\partial}{\partial u_1}igg(rac{h_2h_3}{h_1}rac{\partial\psi}{\partial u_1}igg)+rac{\partial}{\partial u_2}igg(rac{h_3h_1}{h_2}rac{\partial\psi}{\partial u_2}igg)+rac{\partial}{\partial u_3}igg(rac{h_1h_2}{h_3}rac{\partial\psi}{\partial u_3}igg)igg]$$

## 直角坐标系下的拉普拉斯算符

#### 球坐标系下的拉普拉斯算符

$$abla^2 \psi = rac{1}{r^2 \sin heta} \left[ rac{\partial}{\partial r} \left( r^2 \sin heta rac{\partial \psi}{\partial r} 
ight) + rac{\partial}{\partial heta} \left( \sin heta rac{\partial \psi}{\partial heta} 
ight) + rac{\partial}{\partial arphi} \left( rac{1}{\sin heta} rac{\partial \psi}{\partial arphi} 
ight) 
ight]$$

## 柱坐标系下的拉普拉斯算符

$$\nabla^2 \psi = \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{\partial}{\partial \varphi} \left( \frac{1}{\rho} \frac{\partial \psi}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left( \rho \frac{\partial \psi}{\partial z} \right) \right]$$

# 线性空间

#### 线性空间的内积

定义在数域 № 和线性空间 № 上的内积是一个映射:

$$\langle \cdot, \cdot \rangle : \mathbb{L} \times \mathbb{L} \to \mathbb{K}$$

其满足:

(1)  $\forall \psi, \chi \in \mathbb{L}$ , 有:

$$\langle \psi, \chi \rangle = \langle \chi, \psi \rangle^*$$

其中, \*表示复共轭

(2)  $\forall a, b \in \mathbb{K}, \forall \psi, \chi, \varphi \in \mathbb{L}$ , 有:

$$\left\langle \psi,a\chi+b\varphi\right\rangle =a\left\langle \psi,\chi\right\rangle +b\left\langle \psi,\varphi\right\rangle$$

$$\langle a\chi+b\varphi,\psi\rangle=a^*\left\langle \chi,\psi\right\rangle+b^*\left\langle \varphi,\psi\right\rangle$$

(3)

$$\langle \psi, \psi \rangle \geqslant 0$$

## 线性空间向量的模

$$|\psi| \equiv \sqrt{\langle \psi, \psi 
angle}$$

正交

$$\langle \psi, \chi \rangle = 0$$

归一化

$$\frac{\psi}{|\psi|}$$

## 施密特正交化

#### 完备性

# $\delta$ 函数

## $\delta$ 函数定义

 $\delta$  函数是一个定义在  $\mathbb R$  上的函数,其满足:

$$\delta(x-x_0) = egin{cases} 0 &, x 
eq x_0 \ +\infty &, x = x_0 \end{cases}, oxtless \int_a^b \delta(x-x_0) \mathrm{d}x = egin{cases} 1 &, x_0 \in (a,b) \ 0 &, x_0 
otin (a,b) \end{cases}$$

#### $\delta$ 函数各种形式

$$egin{aligned} &\lim_{lpha o 0}rac{1}{\pi}rac{lpha}{lpha^2+x^2}=\delta(x)\ &\lim_{n o\infty}\sqrt{rac{n}{\pi}}\mathrm{e}^{-nx^2}=\delta(x)\ &\lim_{n o\infty}rac{\sin nx}{\pi x}=\delta(x) \end{aligned}$$

#### $\delta$ 函数的傅里叶展开

$$\delta(x-x') = rac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}k(x-x')} \mathrm{d}k$$

## $\delta$ 函数的性质

(1) 筛选性质

设 f(x) 为连续函数,则:

$$\int_{-\infty}^{+\infty} f(x) \delta(x-x_0) \mathrm{d}x = f(x_0)$$

证明:

取 $\varepsilon > 0$ 

$$egin{aligned} \int_{-\infty}^{+\infty}f(x)\delta(x-x_0)\mathrm{d}x &= \int_{x_0-arepsilon}^{x_0+arepsilon}f(x)\delta(x-x_0)\mathrm{d}x \ &= f(\xi)\int_{x_0-arepsilon}^{x_0+arepsilon}\delta(x-x_0)\mathrm{d}x \ &= f(\xi) \end{aligned}$$

其中,  $\xi \in (x_0 - \varepsilon, x_0 + \varepsilon)$ 

取极限得:

$$\int_{-\infty}^{+\infty} f(x)\delta(x-x_0)\mathrm{d}x = f(x_0)$$

(2)  $\delta(x)$  是偶函数:

$$\delta(-x) = \delta(x)$$

(3):

$$f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$$

(4):

$$x\delta(x) = 0$$

(5):

$$\int_{-\infty}^{+\infty} \delta(x-x_2) \delta(x-x_1) \mathrm{d}x = \delta(x_1-x_2)$$

(6) : 设 $x_i$ 为 $\varphi(x)$ 的单根,则:

$$\delta(arphi(x)) = \sum_i rac{1}{|arphi'(x_i)|} \delta(x-x_i)$$

# 三维 $\delta$ 函数

## 直角坐标

$$\delta(ec{r}-ec{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

球坐标

$$\delta(ec{r}-ec{r}_0) = rac{1}{r^2\sin heta}\delta(r-r_0)\delta( heta- heta_0)\delta(arphi-arphi_0)$$

柱坐标

$$\delta(ec{r}-ec{r}_0)=rac{1}{
ho}\delta(
ho-
ho_0)\delta(arphi-arphi_0)\delta(z-z_0)$$

结论

$$\delta(ec{r}) = -rac{1}{4\pi}
abla^2rac{1}{r}$$

三维  $\delta$  函数傅里叶分解

$$\delta(ec{r}-ec{r}_0)=rac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}ec{k}\cdot(ec{r}-ec{r}_0)}\mathrm{d}^3ec{k}$$

# $\delta$ 函数广义傅里叶级数展开

 $\{\varphi_j(x)\}$  是一组完备正交归一基,即:

$$egin{aligned} \langle arphi_i(x), arphi_j(x) 
angle &= \int arphi_i^*(x) arphi_j(x) \mathrm{d}x = \delta_{ij} \ &I = \sum_i \ket{arphi_j} raket{arphi_j} &= \sum_i arphi_j arphi_j^\dagger \end{aligned}$$

$$egin{aligned} \delta(x-x') &= |\delta(x-x')
angle \ &= I \cdot |\delta(x-x')
angle \ &= \left(\sum_j |arphi_j
angle \left\langle arphi_j 
ight| \left\langle arphi_j 
ight| \left\langle arphi_j 
ight| \left\langle arphi_j 
ight| 
ight) |\delta(x-x')
angle \ &= \sum_j \left\langle arphi_j |\delta(x-x')
angle \left| arphi_j 
ight
angle \ &= \sum_j \left(\int arphi_j^*(x) \delta(x-x') \mathrm{d}x 
ight) arphi_j(x) \ &= \sum_j arphi_j^*(x') arphi_j(x) \end{aligned}$$

# $\delta$ 函数在格林函数中的应用

# Sturm-Liouville 本征值问题

具有如下形式带参数  $\lambda$  的二阶常微分方程称为 Sturm-Liouville 方程(简称 S-L 方程):

$$rac{\mathrm{d}}{\mathrm{d}x}igg[k(x)rac{\mathrm{d}}{\mathrm{d}x}y(x)igg]-q(x)y(x)+\lambda q(x)y(x)=0$$

若定义线性算子

$$L \equiv -rac{\mathrm{d}}{\mathrm{d}x}igg[k(x)rac{\mathrm{d}}{\mathrm{d}x}igg] + q(x)$$

则 S-L 方程可写为:

$$Ly(x) = \lambda \rho(x)y(x)$$

为方便, 取  $k(x) \ge 0, q(x) \ge 0, \rho(x) > 0$ 

# 本征值和本征函数的性质

若方程的边界条件限制为如下三种边界条件:

1) 三类齐次边界条件,即在x=a,x=b的边界点上,有:

$$[\alpha_1 y - \beta_1 y']_{r=a} = 0$$

$$[\alpha_2 y + \beta_2 y']_{x=b} = 0$$

其中,  $\alpha_{1,2}, \beta_{1,2} \geq 0$ 

- 2) k(a)=k(b)=0,称为自然边界条件,其等价于  $y(a) 
  eq \infty, y(b) 
  eq \infty$
- 3) 周期性边界条件

则有结论:

- (1) S-L 方程存在本征解。每一个本征值有唯一的本征函数  $y_n(x)$ ,所有的本征解  $\{y_n(x)\}$  构成一个正交的函数系。
- (2) S-L 问题有无穷多个非负的本征值,所有的本征值组成一个单调递增以无穷远点为凝聚点的序列

$$0\leqslant \lambda_1<\lambda_2<\cdots<\lambda_n<\cdots, \lim_{n o\infty}\lambda_n=+\infty$$

(3) 无穷多个本征值  $\lambda_n$  对应的无穷多个本征函数  $y_n(x)$  构成一个完备的正交函数系  $\{y_n(x)\}$ ,任何一个定义在  $x\in[a,b]$  上的满足 Direchlet 条件的函数 f(x) 都可以在函数系  $\{y_n(x)\}$  上作广义 Fourier 展开:

$$f(x) = \sum_{n=1}^{\infty} C_n y_n(x)$$

展开系数为:

$$C_n = rac{1}{\int_a^b |y_n(x)|^2 
ho(x) \mathrm{d}x} \int_a^b f(x) 
ho(x) y_n^*(x) \mathrm{d}x$$

# 格林函数

二阶偏微分方程的普遍形式为:

$$Lu(x_0, x_1, x_2, x_3) = f(x_0, x_1, x_2, x_3)$$

其中,

$$L=a_{ij}rac{\partial^{2}}{\partial x_{i}\partial x_{j}}+b_{i}rac{\partial}{\partial x_{i}}+c$$

考虑无边界条件,即在无穷空间中求解微分方程,

$$Lu(x) = f(x), \ x \in \mathbb{R}^{n+1}, \ n \leqslant 3$$

算子 L 的无界空间的格林函数,记为  $G_0(x,x')$ ,定义为:

$$LG_0(x,x')=\delta(x-x'), \;\; x,x'\in\mathbb{R}^{n+1}, \;\; n\leqslant 3$$

 $G_0(x,x')$  可以写成:

$$G_0(x,x') = L^{-1}\delta(x-x') + u_0(x)$$

其中,  $u_0(x)$  是相应的齐次方程的解, 即:

$$Lu_{0}(x) = 0$$

 $\delta$  函数的傅里叶变换式:

$$\delta(x-x') = rac{1}{(2\pi)^{n+1}} \int e^{\mathrm{i}k_lpha(x_lpha-x'_lpha)} \mathrm{d}^{n+1}k$$

于是:

$$egin{aligned} G_0(x,x') &= L^{-1}\delta(x-x') \ &= rac{1}{(2\pi)^{n+1}} \int L^{-1}e^{\mathrm{i}k_lpha(x_lpha-x'_lpha)}\mathrm{d}^{n+1} \end{aligned}$$

可以验证,利用格林函数,方程 Lu(x) = f(x) 的解可表达为:

$$u(x)=u_0(x)+\int f(x')G_0(x,x')\mathrm{d}^{n+1}x'$$

代入验证:

$$egin{split} Ligg[u_0(x)+\int f(x')G_0(x,x')\mathrm{d}^{n+1}x'igg] &=\int f(x')LG_0(x,x')\mathrm{d}^{n+1}x'\ &=\int f(x')LL^{-1}\delta(x-x')\mathrm{d}^{n+1}x'\ &=\int f(x')\delta(x-x')\mathrm{d}^{n+1}x'\ &=f(x) \end{split}$$

刚好满足原方程

例:

在半空间 z > 0 内求解 Poisson 方程的第一类边值问题:

$$\left\{egin{aligned} 
abla^2 u(x,y,z) &= f(x,y,z), & z > 0 \ u(x,y,z)igg|_{z=0} &= arphi(x,y), & z = 0 \end{aligned}
ight.$$

解:

$$\left\{egin{aligned} 
abla^2 G(ec{r},ec{r}') &= \delta(ec{r}-ec{r}') \ G(ec{r})igg|_{z=0} &= 0 \end{aligned}
ight.$$

somehow derive:

$$G(x,y,z;x',y',z') = -rac{1}{4\pi}rac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + rac{1}{4\pi}rac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

 $G_0(x,x')$  求法

$$G_0(x,x') = rac{1}{(2\pi)^{n+1}} \int rac{\exp\left[\mathrm{i}\sum_lpha k_lpha(x_lpha-x_lpha')
ight]}{-a_{ij}k_ik_j + \mathrm{i}b_jk_j + c} \mathrm{d}^{n+1}k$$

 $abla^2$  算子基本解

$$abla^2 G_0(ec{r},ec{r}') = \delta(ec{r}-ec{r}')$$

利用结论:

$$\delta(ec{r}-ec{r}')=
abla^2(-rac{1}{4\pi}rac{1}{|ec{r}-ec{r}'|})$$

对比可得拉普拉斯算子的基本解:

$$G_0(ec{r},ec{r}') = -rac{1}{4\pi}rac{1}{|ec{r}-ec{r}'|}$$

# 拉普拉斯算子的格林函数

$$\left\{egin{aligned} 
abla^2 G(x,x') &= \delta(x-x') \ (G+etarac{\partial G}{\partial n})igg|_{x\in\partial\Omega} &= 0 \end{aligned}
ight.$$

求 G(x, x')

构造一个光滑的辅助函数 g(x,x') 使得:

$$\left\{ egin{aligned} 
abla^2 g(x,x') &= 0 \ gigg|_{x\in\partial\Omega} &= G_0(x,x')igg|_{x\in\partial\Omega} \end{aligned} 
ight.$$

则

$$G(x,x')=G_0(x,x')-g(x,x')$$

验证:

$$egin{aligned} 
abla^2 G(x,x') &= G_0(x,x') - g(x,x') \ &= 
abla^2 G_0(x,x') \ &= \delta(x-x') \end{aligned}$$

$$G(x,x')igg|_{x\in\partial\Omega} = G_0igg|_{x\in\partial\Omega} - gigg|_{\partial\Omega}$$

# 三维亥姆霍兹方程基本解

$$L = \nabla^2 + k^2$$

基本解:

$$(
abla^2 + k^2)G_0(ec{r}, ec{r}') = \delta(ec{r} - ec{r}')$$
 $G_0(ec{r}, ec{r}') = (
abla^2 + k^2)^{-1}\delta(ec{r} - ec{r}') = rac{1}{(2\pi)^3}\int rac{\exp(\mathrm{i}ec{q}\cdot(ec{r} - ec{r}'))}{k^2 - q^2}\mathrm{d}^3ec{q}$ 

令  $\vec{x} = \vec{r} - \vec{r}'$ , 使  $\vec{x}$  轴作为  $\vec{q}$  的 z 轴

$$\vec{q} \cdot (\vec{r} - \vec{r}') = qx \cos \theta$$

# \$\$

# **G\_0(\vec{r}-\vec{r}')**

\$\$