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> 写出协变形式的电荷守恒定律，并由 4 维电磁势写出协变形式的达朗贝尔方程，推导电磁场张量 $F_{\mu\nu}$ 的各分量，并利用 $F_{\mu\nu}$ 将真空中的麦克斯韦方程组写成协变形式。

协变形式的电荷守恒定律：

$$\partial_\mu J_\mu = 0$$

协变形式的达朗贝尔方程：

$$\Box A_\mu = -\mu_0 J_\mu$$

推导电磁场张量 $F_{\mu\nu}$ 的各分量：

$$A_\mu = \left(\vec{A}, \frac{\mathrm{i}}{c}\varphi\right),\ \ \vec{B} = \nabla \times \vec{A},\ \ \vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{11} = F_{22} = F_{33} = F_{44} = 0$$

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = (\nabla \times \vec{A})_3 = B_3,\ \ F_{21} = -F_{12} = -B_3$$

$$F_{13} = \partial_1 A_3 - \partial_3 A_1 = -(\nabla \times \vec{A})_2 = -B_2,\ \ F_{31} = -F_{13} = B_2$$

$$F_{14} = \partial_1 A_4 - \partial_4 A_1 = \frac{\mathrm{i}}{c}\partial_1\varphi - \frac{1}{\mathrm{i}c}\frac{\partial A_1}{\partial t} = \frac{1}{\mathrm{i}c}\left(-\partial_1\varphi - \frac{\partial A_1}{\partial t}\right) = \frac{1}{\mathrm{i}c}E_1 = -\frac{\mathrm{i}}{c}E_1,\ \ F_{41} = -F_{14} = \frac{\mathrm{i}}{c}E_1$$

$$F_{23} = \partial_2 A_3 - \partial_3 A_2 = (\nabla \times \vec{A})_1 = B_1,\ \ F_{32} = -F_{23} = -B_1$$

$$F_{24} = \partial_2 A_4 - \partial_4 A_2 = \frac{\mathrm{i}}{c}\partial_2\varphi - \frac{1}{\mathrm{i}c}\frac{\partial A_2}{\partial t} = \frac{1}{\mathrm{i}c}\left(-\partial_2\varphi - \frac{\partial A_2}{\partial t}\right) = -\frac{\mathrm{i}}{c}E_2,\ \ F_{42} = -F_{24} = \frac{\mathrm{i}}{c}E_2$$

$$F_{34} = \partial_3 A_4 - \partial_4 A_3 = \frac{\mathrm{i}}{c}\partial_3\varphi - \frac{1}{\mathrm{i}c}\frac{\partial A_3}{\partial t} = \frac{1}{\mathrm{i}c}\left(-\partial_3\varphi - \frac{\partial A_3}{\partial t}\right) = -\frac{\mathrm{i}}{c}E_3,\ \ F_{43} = -F_{34} = \frac{\mathrm{i}}{c}E_3$$

综上，

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -\frac{\mathrm{i}}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{\mathrm{i}}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{\mathrm{i}}{c}E_3 \\ \frac{\mathrm{i}}{c}E_1 & \frac{\mathrm{i}}{c}E_2 & \frac{\mathrm{i}}{c}E_3 & 0 \end{bmatrix}$$

真空中麦克斯韦方程组协变形式：

$$\partial_\nu F_{\mu\nu} = \mu_0 J_\mu$$

$$\partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0$$

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> 由 4 维动量推出相对论中能量、动量和质量间的关系式，并推导相对论力学方程，讨论各分量方程的意义。

一方面，四维动量 p_μ 是四维矢量，其与自己的内积是不变量：

$$\begin{aligned}p_\mu p_\mu &= m_0U_\mu\cdot m_0U_\mu=m_0\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}\cdot m_0\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}=m_0^2\gamma^2\frac{\mathrm{d}x_\mu}{\mathrm{d}t}\frac{\mathrm{d}x_\mu}{\mathrm{d}t}\\ &=m_0^2\gamma^2\left(v^2-c^2\right)=m_0^2\frac{1}{1-v^2/c^2}(v^2-c^2)=m_0^2\frac{c^2}{c^2-v^2}(v^2-c^2)\\ &=-m_0^2c^2\end{aligned}$$

另一方面，

$$p_\mu=\left(\vec{p},\frac{\mathrm{i}W}{c}\right)$$

于是：

$$p^2-\frac{W^2}{c^2}=-m_0^2c^2$$

即：

$$\boxed{W=\sqrt{p^2c^2+m_0^2c^4}}$$

一方面，四维力可表达为：

$$k_\mu=\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}=\gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{p},\frac{\mathrm{i}W}{c}\right)=\left(\gamma\frac{\mathrm{d}\vec{p}}{\mathrm{d}t},\gamma\frac{\mathrm{i}}{c}\frac{\mathrm{d}W}{\mathrm{d}t}\right)$$

另一方面，

$$k_\mu=\left(\gamma\vec{F},\frac{\mathrm{i}\gamma}{c}\vec{F}\cdot\vec{v}\right)$$

对比可得分量方程：

$$\begin{cases}\vec{F}=\dfrac{\mathrm{d}\vec{p}}{\mathrm{d}t}\\ \vec{F}\cdot\vec{v}=\dfrac{\mathrm{d}W}{\mathrm{d}t}\end{cases}$$

上面第一条方程就是牛顿第二定律，第二条方程表示外力做功导致系统能量变化，也就是能量守恒。

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> 写出协变形式的洛伦兹力公式（分别写出点电荷以及力密度形式的协变洛伦兹力公式）。

点电荷形式协变洛伦兹力公式：

$$K_\mu = qF_{\mu\nu}U_\nu$$

力密度形式协变洛伦兹力公式：

$$f_\mu = F_{\mu\nu}J_\nu$$

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> 由电磁场能动张量表达式 $T_{\mu\lambda} = \frac{1}{\mu_0}\left(F_{\mu\nu}F_{\nu\lambda} + \frac{1}{4}\delta_{\mu\lambda}F_{\nu\tau}F_{\nu\tau}\right)$，计算能动张量的各分量，并说明与 3 维电磁场能量密度、能流密度、动量密度、动量流密度之间的关系。

$$F_{\nu\tau}F_{\nu\tau} = 2\vec{B}^2 - \frac{2\vec{E}^2}{c^2}$$

$$T_{11} = \frac{1}{\mu_0}\left(F_{1\nu}F_{\nu1} + \frac{1}{4}\delta_{11}F_{\nu\tau}F_{\nu\tau}\right) = \frac{1}{2}\left[\frac{1}{\mu_0}\left(B_1^2 - B_2^2 - B_3^2\right) + \varepsilon_0(E_1^2 - E_2^2 - E_3^2)\right]$$

$$T_{12} = \frac{1}{\mu_0}\left(F_{1\nu}F_{\nu2} + \frac{1}{4}\delta_{12}F_{\nu\tau}F_{\nu\tau}\right) = \frac{1}{\mu_0}B_1B_2 + \varepsilon_0E_1E_2$$

$$T_{21} = T_{12} = \frac{1}{\mu_0}B_1B_2 + \varepsilon_0E_1E_2$$

$$T_{13} = \frac{1}{\mu_0}\left(F_{1\nu}F_{\nu3} + \frac{1}{4}\delta_{13}F_{\nu\tau}F_{\nu\tau}\right) = \frac{1}{\mu_0}B_1B_3 + \varepsilon_0E_1E_3$$

$$T_{31} = T_{13} = \frac{1}{\mu_0}B_1B_3 + \varepsilon_0E_1E_3$$

$$T_{14} = \frac{1}{\mu_0}\left(F_{1\nu}F_{\nu4} + \frac{1}{4}\delta_{14}F_{\nu\tau}F_{\nu\tau}\right) = \frac{\mathrm{i}}{\mu_0c}\left(B_2E_3 - B_3E_2\right)$$

$$T_{41} = T_{14} = \frac{\mathrm{i}}{\mu_0c}\left(B_2E_3 - B_3E_2\right)$$

$$T_{22} = \frac{1}{\mu_0}\left(F_{2\nu}F_{\nu2} + \frac{1}{4}\delta_{22}F_{\nu\tau}F_{\nu\tau}\right) = \frac{1}{2}\left[\frac{1}{\mu_0}\left(-B_1^2 + B_2^2 - B_3^2\right) + \varepsilon_0\left(-E_1^2 + E_2^2 - E_3^2\right)\right]$$

$$T_{23} = \frac{1}{\mu_0}\left(F_{2\nu}F_{\nu3} + \frac{1}{4}\delta_{23}F_{\nu\tau}F_{\nu\tau}\right) = \frac{1}{\mu_0}B_2B_3 + \varepsilon E_2E_3$$

$$T_{32}=T_{23}=\frac{1}{\mu_0}B_2B_3+\varepsilon E_2E_3$$

$$T_{24}=\frac{1}{\mu_0}\left(F_{2\nu}F_{\nu 4}+\frac{1}{4}\delta_{24}F_{\nu\tau}F_{\nu\tau}\right)=\frac{\mathrm{i}}{\mu_0 c}\left(B_3E_1-B_1E_3\right)$$

$$T_{42}=T_{24}=\frac{\mathrm{i}}{\mu_0 c}\left(B_3E_1-B_1E_3\right)$$

$$T_{33}=\frac{1}{\mu_0}\left(F_{3\nu}F_{\nu 3}+\frac{1}{4}\delta_{33}F_{\nu\tau}F_{\nu\tau}\right)=\frac{1}{2}\left[\frac{1}{\mu_0}\left(-B_1^2-B_2^2+B_3^2\right)+\varepsilon_0\left(-E_1^2-E_2^2+E_3^2\right)\right]$$

$$T_{34}=\frac{1}{\mu_0}\left(F_{3\nu}F_{\nu 4}+\frac{1}{4}\delta_{34}F_{\nu\tau}F_{\nu\tau}\right)=\frac{\mathrm{i}}{\mu_0 c}\left(B_1E_2-B_2E_1\right)$$

$$T_{43}=T_{34}=\frac{\mathrm{i}}{\mu_0 c}\left(B_1E_2-B_2E_1\right)$$

$$T_{44}=\frac{1}{\mu_0}\left(F_{4\nu}F_{\nu 4}+\frac{1}{4}\delta_{44}F_{\nu\tau}F_{\nu\tau}\right)=\frac{1}{2}\left(\frac{1}{\mu_0}B^2+\varepsilon_0E^2\right)$$

与 3 维电磁场能量密度、能流密度、动量密度、动量流密度之间的关系：

能量密度（真空中）：

$$w=\frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)=\frac{1}{2}\left(\varepsilon_0E+\frac{1}{\mu_0}B^2\right)$$

$$T_{44}=w$$

能流密度（真空中）：

$$\vec{S}=\vec{E}\times\vec{H}=\frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$T_{14}=T_{41}=\frac{\mathrm{i}}{\mu_0 c}\left(B_2E_3-B_3E_2\right)=-\frac{\mathrm{i}}{c}S_1$$

$$T_{24}=T_{42}=\frac{\mathrm{i}}{\mu_0 c}\left(B_3E_1-B_1E_3\right)=-\frac{\mathrm{i}}{c}S_2$$

$$T_{34}=T_{43}=\frac{\mathrm{i}}{\mu_0 c}\left(B_1E_2-B_2E_1\right)=-\frac{\mathrm{i}}{c}S_3$$

动量密度（真空中）：

$$\vec{g}=\vec{D}\times\vec{B}=\varepsilon_0\vec{E}\times\vec{B}$$

$$T_{14}=T_{41}=\frac{\mathrm{i}}{\mu_0 c}\left(B_2E_3-B_3E_2\right)=-\mathrm{i}cg_1$$

$$T_{24}=T_{42}=\frac{\mathrm{i}}{\mu_0 c}\left(B_3E_1-B_1E_3\right)=-\mathrm{i}cg_2$$

$$T_{34}=T_{43}=\frac{\mathrm{i}}{\mu_0 c}\left(B_1E_2-B_2E_1\right)=-\mathrm{i}cg_3$$

动量流密度（真空中）：

$$\overset{\leftrightarrow}{T}=-(\vec{D}\vec{E}+\vec{B}\vec{H})+\frac{1}{2}\overset{\leftrightarrow}{I}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})=-\left(\varepsilon_0\vec{E}\vec{E}+\frac{1}{\mu_0}\vec{B}\vec{B}\right)+\frac{1}{2}\overset{\leftrightarrow}{I}\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right)$$

$$T_{ij}=-\overset{\leftrightarrow}{T},\ \ i,j=1,2,3$$

综上，电磁场能动张量与能量密度、能流密度、动量密度、动量流密度的关系为：

$$T_{\mu\nu}=\begin{bmatrix}-\overset{\leftrightarrow}{T}_{3\times3} & -\mathrm{i}c\vec{g}_{3\times1}\\ -\mathrm{i}c\vec{g}^{\mathrm{T}}_{1\times3} & w_{1\times1}\end{bmatrix}=\begin{bmatrix}-\overset{\leftrightarrow}{T}_{3\times3} & -\frac{\mathrm{i}}{c}\vec{S}_{3\times1}\\ -\frac{\mathrm{i}}{c}\vec{S}^{\mathrm{T}}_{1\times3} & w_{1\times1}\end{bmatrix}$$

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> 分别写出自由电磁场与一般电磁场的拉格朗日密度，并变分得到场方程。

对于电磁场，拉格朗日密度是场量及其一阶导的函数：

$$\mathscr{L}=\mathscr{L}\left(\varphi_\sigma(x_\nu),\partial_\mu\varphi_\sigma(x_\nu)\right)$$

最小作用量原理：

$$\delta\int\mathscr{L}\left(\varphi_\sigma,\partial_\mu\varphi_\sigma\right)\mathrm{d}\Omega=0$$

拉格朗日密度的变分为：

$$\begin{aligned}\delta\mathscr{L}&=\mathscr{L}[\varphi_\sigma+\delta\varphi_\sigma,\partial_\mu\varphi_\sigma+\delta(\partial_\mu\varphi_\sigma)]-\mathscr{L}[\varphi_\sigma,\partial_\mu\varphi_\sigma]\\&=\frac{\partial\mathscr{L}}{\partial\varphi_\sigma}\delta\varphi_\sigma+\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_\sigma)}\delta(\partial_\mu\varphi_\sigma)\\&=\frac{\partial\mathscr{L}}{\partial\varphi_\sigma}\delta\varphi_\sigma-\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_\sigma)}\right)\delta\varphi_\sigma+\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_\sigma)}\delta\varphi_\sigma\right)\end{aligned}$$

由于：

$$\int\limits_{\Omega}\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_\sigma)}\delta\varphi_\sigma\right)\mathrm{d}\Omega=\int\limits_{\partial\Omega}\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_\sigma)}\delta\varphi_\sigma\mathrm{d}\Omega_\mu=0$$

代入最小最用量原理，得：

$$\int\left[\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_\sigma)}\right)-\frac{\partial\mathscr{L}}{\partial\varphi_\sigma}\right]\delta\varphi_\sigma\mathrm{d}\Omega=0$$

由 $\delta\varphi_\sigma$ 的任意性，得到场方程：

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi_\sigma)}\right)-\frac{\partial\mathscr{L}}{\partial\varphi_\sigma}=0$$

对于电磁场，

$$A_\nu(x_\mu)=\left(\vec{A}(x_\mu),\frac{\mathrm{i}}{c}\varphi(x_\mu)\right)$$

电磁场的拉格朗日方程为：

$$\frac{\partial\mathscr{L}}{\partial A_\nu}-\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\nu)}\right)=0$$

自由电磁场的拉格朗日密度：

$$\mathscr{L}_0(\partial_\mu A_\mu)=-\frac{1}{4\mu_0}F_{\mu\nu}F_{\mu\nu}$$

代入电磁场的拉格朗日方程，得：

$$\frac{\partial(F_{\alpha\beta}F_{\alpha\beta})}{\partial A_\nu}-\frac{\partial}{\partial x_\mu}\frac{\partial(F_{\alpha\beta}F_{\alpha\beta})}{\partial(\partial_\mu A_\nu)}=0$$

即：

$$\partial_\mu F_{\mu\nu}=0$$

一般电磁场的拉格朗日密度：

$$\mathscr{L}(A_\mu,\partial_\mu A_\nu)=-\frac{1}{4\mu_0}F_{\mu\nu}F_{\mu\nu}+J_\mu A_\mu$$

代入电磁场的拉格朗日方程，得：

$$\partial_\mu F_{\mu\nu}=-\mu_0 J_\nu$$