## 1

由于 f(z) 在 z 点可导,故极限

$$\lim_{\Delta z o 0} rac{f(z+\Delta z) - f(z)}{\Delta z}$$

存在且与  $\Delta z$  趋于 0 的方式无关

设 z = x + iy, f(z) = u(x, y) + iv(x, y), 则:

$$\lim_{\Delta z o 0} rac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z o 0} rac{\Delta u + \mathrm{i} \Delta v}{\Delta x + \mathrm{i} \Delta y}$$

特别地

1.令:

$$\mathrm{i}\Delta y=0, \Delta x o 0$$

此时,

$$\lim_{\Delta z o 0} rac{\Delta u + \mathrm{i} \Delta v}{\Delta x + \mathrm{i} \Delta y} = \lim_{\Delta x o 0} rac{\Delta u + \mathrm{i} \Delta v}{\Delta x} = rac{\partial u}{\partial x} + \mathrm{i} rac{\partial v}{\partial x}$$

2.令:

$$\Delta x=0, \mathrm{i}\Delta y o 0$$

此时,

$$\lim_{\Delta z o 0} rac{\Delta u + \mathrm{i} \Delta v}{\Delta x + \mathrm{i} \Delta y} = -\mathrm{i} rac{\partial u}{\partial y} + rac{\partial v}{\partial y}$$

由于 f(z) 在  $z_0$  点可导,则这两个导数值应该相等,于是:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

2

$$0 < |z| < 1$$
:

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$$

$$= -\frac{1}{1-z} - \frac{1}{z}$$

$$= -\sum_{n=0}^{\infty} z^n - z^{-1}$$

$$= \sum_{n=-1}^{\infty} -z^n$$

|z| > 1:

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$$

$$= \frac{1}{z(1-\frac{1}{z})} - z^{-1}$$

$$= \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} - z^{-1}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} (\frac{1}{z})^n - z^{-1}$$

$$= \sum_{n=0}^{\infty} z^{-n-1} - z^{-1}$$

$$= \sum_{n=1}^{\infty} z^{-n-1}$$

3

$$\Leftrightarrow f(z) = rac{1}{(z^2+1)(z-1)^2} = rac{1}{(z+\mathrm{i})(z-\mathrm{i})(z-1)^2}$$

在回路内的孤立奇点有:  $z_1={
m i}, z_2=1$ ,  $z_1$ 为一阶极点,  $z_2$ 二阶极点

计算回路内孤立奇点处的留数:

$$egin{aligned} \operatorname{Res} & f(z_1) = rac{1}{0!} \lim_{z o \mathrm{i}} rac{\mathrm{d}^0}{\mathrm{d}z^0} (z-\mathrm{i}) \cdot rac{1}{(z+\mathrm{i})(z-\mathrm{i})(z-1)^2} \ & = rac{1}{4} \end{aligned}$$

$$egin{split} ext{Res} f(z_2) &= rac{1}{1!} \lim_{z o 1} rac{ ext{d}^1}{ ext{d} z^1} (z-1)^2 \cdot rac{1}{(z+ ext{i})(z- ext{i})(z-1)^2} \ &= -rac{1}{2} \end{split}$$

于是:

$$egin{aligned} I &= \oint\limits_{l} rac{\mathrm{d}z}{(z^2+1)(z-1)^2} \ &= 2\pi\mathrm{i}(\mathrm{Res}f(z_1) + \mathrm{Res}f(z_2)) \ &= -rac{\pi\mathrm{i}}{2} \end{aligned}$$

4

$$\Rightarrow z = e^{i\theta}, z^{-1} = e^{i(-\theta)}, \theta = \frac{\ln z}{i}, d\theta = \frac{dz}{iz}, \cos \theta = \frac{1}{2}(z + z^{-1})$$

$$I = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon \cos \theta}$$

$$= \frac{2}{i} \oint_{C^+} \frac{1}{\varepsilon z^2 + 2z + \varepsilon} dz$$

其中,C 是复平面上的单位圆

令  $f(z)=rac{1}{arepsilon z^2+2z+arepsilon}$ ,被积函数的两个一阶极点为:

$$z_1 = rac{-1 + \sqrt{1 - arepsilon^2}}{arepsilon}, \ \ z_2 = rac{-1 - \sqrt{1 - arepsilon^2}}{arepsilon}$$

被积函数 f(z) 可写为:

$$f(z)=rac{1}{arepsilon(z-z_1)(z-z_2)}$$

只有  $z_1$  在回路内

计算回路内孤立奇点的留数:

$$egin{aligned} \operatorname{Res} &f(z_1) = rac{1}{0!} \lim_{z o z_1} rac{\operatorname{d}^0}{\operatorname{d} z^0} (z - z_1) f(z) \ &= \lim_{z o z_1} rac{1}{arepsilon (z - z_2)} \ &= rac{1}{arepsilon (z_1 - z_2)} \ &= rac{1}{2\sqrt{1 - arepsilon^2}} \end{aligned}$$

留数定理:

$$egin{aligned} \oint\limits_{C^+} rac{1}{arepsilon z^2 + 2z + arepsilon} \mathrm{d}z &= 2\pi \mathrm{i} \mathrm{Res} f(z_1) \ &= 2\pi \mathrm{i} \cdot rac{1}{2\sqrt{1 - arepsilon^2}} \ &= rac{\pi \mathrm{i}}{\sqrt{1 - arepsilon^2}} \end{aligned}$$

于是积分为:

$$egin{aligned} I &= rac{2}{\mathrm{i}} \oint\limits_{C^+} rac{1}{arepsilon z^2 + 2z + arepsilon} \mathrm{d}z \ &= rac{2}{\mathrm{i}} \cdot rac{\pi \mathrm{i}}{\sqrt{1 - arepsilon^2}} \ &= rac{2\pi}{\sqrt{1 - arepsilon^2}} \end{aligned}$$

5

设
$$i(t) = F(p)$$

对方程  $L rac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) = E$  两边同时作拉普拉斯变换,得:

$$LpF(p)+RF(p)=rac{E}{p}$$

解出 F(p):

$$F(p) = rac{E}{Lp^2 + Rp} \ = rac{E}{R}(rac{1}{p} - rac{1}{p + rac{R}{L}})$$

两边同时做拉普拉斯逆变换得:

$$i(t)=rac{E}{R}(1-e^{-rac{R}{L}t})$$

6

$$egin{aligned} 
abla \cdot (arphi ec{A}) &= \partial_i (arphi ec{A})_i \ &= \partial_i (arphi A_i) \ &= A_i \partial_i arphi + arphi \partial_i A_i \ &= A_i (
abla arphi)_i + arphi \partial_i A_i \ &= ec{A} \cdot (
abla arphi) + arphi 
abla \cdot ec{A} \end{aligned}$$

高斯公式给出:

$$\int\limits_{\Omega} 
abla \cdot ec{A} \mathrm{d}V = \int\limits_{\partial\Omega} ec{A} \cdot \mathrm{d}ec{S}$$

令  $ec{A}=
abla(\psiarphi)$ ,代入得:

$$\int\limits_{\Omega} 
abla \cdot 
abla (\psi arphi) \mathrm{d}V = \int\limits_{\partial \Omega} 
abla (\psi arphi) \cdot \mathrm{d}ec{S}$$

即:

$$\int\limits_{\Omega}\psi
abla^2arphi+arphi
abla^2\psi\mathrm{d}V=\int\limits_{\partial\Omega}(\psi
ablaarphi+arphi
abla\psi)\cdot\mathrm{d}ec{S}$$