推导复变函数可导的柯西-黎曼条件。

设 f(z) 在 z 点可导,则极限

$$\lim_{\Delta z o 0} rac{f(z+\Delta z) - f(z)}{\Delta z}$$

存在且与  $\Delta z$  趋于 0 的方式无关。

设 $z=x+\mathrm{i}y, f(z)=u(x,y)+\mathrm{i}v(x,y)$ , 则:

$$\lim_{\Delta z o 0} rac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z o 0} rac{\Delta u + \mathrm{i} \Delta v}{\Delta x + \mathrm{i} \Delta y}$$

特别地

(1) 令:

$$\mathrm{i}\Delta y=0, \Delta x o 0$$

此时,

$$\lim_{\Delta z o 0} rac{\Delta u + \mathrm{i} \Delta v}{\Delta x + \mathrm{i} \Delta y} = \lim_{\Delta x o 0} rac{\Delta u + \mathrm{i} \Delta v}{\Delta x} = rac{\partial u}{\partial x} + \mathrm{i} rac{\partial v}{\partial x}$$

(2) 令:

$$\Delta x = 0, \mathrm{i} \Delta y \to 0$$

此时,

$$\lim_{\Delta z o 0} rac{\Delta u + \mathrm{i} \Delta v}{\Delta x + \mathrm{i} \Delta y} = -\mathrm{i} rac{\partial u}{\partial y} + rac{\partial v}{\partial y}$$

由于 f(z) 在  $z_0$  点可导,则这两个导数值应该相等,于是:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

求  $f(z)=rac{1}{z(z-1)}$  在环形区域 0<|z|<1 和 |z|>1 内,在  $z_0=0$  处的展开式。

## 0<|z|<1 区域在 $z_0=0$ 处展开 f(z)

由于 |z| < 1,于是有几何级数:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

于是:

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$$

$$= -\frac{1}{1-z} - \frac{1}{z}$$

$$= -\sum_{n=0}^{\infty} z^n - z^{-1}$$

$$= \sum_{n=-1}^{\infty} -z^n$$

## |z|>1 区域在 $z_0=0$ 处展开 f(z)

注意到 |z| > 1,则 |1/z| < 1,于是:

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$$

$$= \frac{1}{z(1-\frac{1}{z})} - z^{-1}$$

$$= \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} - z^{-1}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n - z^{-1}$$

$$= \sum_{n=0}^{\infty} z^{-n-1} - z^{-1}$$

$$= \sum_{n=0}^{\infty} z^{-n-1}$$

3

计算回路积分 
$$I=\oint\limits_{l^+}rac{\mathrm{d}z}{(z^2+1)(z-1)^2}$$
 ,其中回路  $l$  的方程为  $x^2+y^2-2x-2y=0$ 

在回路  $l:(x-1)^2+(y-1)^2=\sqrt{2}$  内的孤立奇点有:  $z_1={\rm i},z_2=1,\ z_1$  为一阶极点,  $z_2$  为二阶极点。

计算 f(z) 在回路内孤立奇点处的留数:

$$egin{aligned} ext{Res} f(z_1) &= rac{1}{0!} \lim_{z o \mathrm{i}} rac{\mathrm{d}^0}{\mathrm{d}z^0} (z-\mathrm{i}) \cdot rac{1}{(z+\mathrm{i})(z-\mathrm{i})(z-1)^2} \ &= \lim_{z o \mathrm{i}} rac{1}{(z+\mathrm{i})(z-1)^2} \ &= rac{1}{2\mathrm{i}(\mathrm{i}-1)^2} \ &= rac{1}{4} \end{aligned}$$

$$egin{aligned} \operatorname{Res} & f(z_2) = rac{1}{1!} \lim_{z o 1} rac{\operatorname{d}^1}{\operatorname{d} z^1} (z-1)^2 \cdot rac{1}{(z+\operatorname{i})(z-\operatorname{i})(z-1)^2} \ & = \lim_{z o 1} rac{\operatorname{d}}{\operatorname{d} z} \left(rac{1}{z^2+1}
ight) \ & = \lim_{z o 1} rac{-2z}{(z^2+1)^2} \ & = -rac{1}{2} \end{aligned}$$

于是:

$$egin{aligned} I &= \oint\limits_{l} rac{\mathrm{d}z}{(z^2+1)(z-1)^2} \ &= 2\pi\mathrm{i}\left[\mathrm{Res}f(z_1) + \mathrm{Res}f(z_2)
ight] \ &= 2\pi\mathrm{i}\left(rac{1}{4} - rac{1}{2}
ight) \ &= -rac{\pi\mathrm{i}}{2} \end{aligned}$$

4

计算定积分 
$$I=\int_0^{2\pi} rac{\mathrm{d} heta}{1+arepsilon\cos heta}$$
,其中  $0$ 

令:

$$z=\mathrm{e}^{\mathrm{i} heta},\;\;z^{-1}=\mathrm{e}^{-\mathrm{i} heta},\;\;\mathrm{d}z=\mathrm{i}\mathrm{e}^{\mathrm{i} heta}\mathrm{d} heta\Longrightarrow\mathrm{d} heta=rac{\mathrm{d}z}{\mathrm{i}\mathrm{e}^{\mathrm{i} heta}}=rac{\mathrm{d}z}{\mathrm{i}z},\;\;\cos heta=rac{1}{2}\left(z+z^{-1}
ight)$$

于是:

$$I = \int_0^{2\pi} rac{\mathrm{d} heta}{1+arepsilon\cos heta} \ = rac{2}{\mathrm{i}} \oint\limits_{C^+} rac{1}{arepsilon z^2 + 2z + arepsilon} \mathrm{d}z$$

其中, C 是复平面上以原点为圆心的单位圆。

令 
$$f(z) = \frac{1}{\varepsilon z^2 + 2z + \varepsilon}$$
 , 被积函数的两个一阶极点为:

$$z_1 = rac{-1 + \sqrt{1 - arepsilon^2}}{arepsilon}, \ \ z_2 = rac{-1 - \sqrt{1 - arepsilon^2}}{arepsilon}$$

被积函数 f(z) 可写为:

$$f(z)=rac{1}{arepsilon(z-z_1)(z-z_2)}$$

只有  $z_1$  在积分回路内。

计算 f(z) 在回路内孤立奇点  $z_1$  处的留数:

$$egin{aligned} \operatorname{Res} &f(z_1) = rac{1}{0!} \lim_{z o z_1} rac{\operatorname{d}^0}{\operatorname{d} z^0} (z-z_1) f(z) \ &= \lim_{z o z_1} rac{1}{arepsilon (z-z_2)} \ &= rac{1}{arepsilon (z_1-z_2)} \ &= rac{1}{2\sqrt{1-arepsilon^2}} \end{aligned}$$

由留数定理,有:

$$egin{aligned} \oint\limits_{C^+} rac{1}{arepsilon z^2 + 2z + arepsilon} \mathrm{d}z &= 2\pi \mathrm{i} \mathrm{Res} f(z_1) \ &= 2\pi \mathrm{i} \cdot rac{1}{2\sqrt{1 - arepsilon^2}} \ &= rac{\pi \mathrm{i}}{\sqrt{1 - arepsilon^2}} \end{aligned}$$

于是积分为:

$$egin{aligned} I &= rac{2}{\mathrm{i}} \oint\limits_{C^+} rac{1}{arepsilon z^2 + 2z + arepsilon} \mathrm{d}z \ &= rac{2}{\mathrm{i}} \cdot rac{\pi \mathrm{i}}{\sqrt{1 - arepsilon^2}} \ &= rac{2\pi}{\sqrt{1 - arepsilon^2}} \end{aligned}$$

用拉普拉斯变换解下列 RL 串联电路方程, 其中 L, R, E 为常数:

$$\begin{cases} L \frac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) = E\\ i(0) = 0 \end{cases}$$

设i(t) = F(p)

微分定理给出:

$$rac{\mathrm{d}i(t)}{\mathrm{d}t}\coloneqq p^1F(p)-p^0i^{(0)}(0)=pF(p)-i(0)=pF(p)$$

常用拉普拉斯变换:

$$\mathcal{L}{1}(p) = \frac{1}{p}, \text{ Re } p > 0, \text{ or } 1 = \frac{1}{p}$$

对方程  $L rac{\mathrm{d}i(t)}{\mathrm{d}t} = +Ri(t) = E$  两边同时作拉普拉斯变换,得:

$$LpF(p)+RF(p)=rac{E}{p}$$

解出 F(p):

$$egin{aligned} F(p) &= rac{E}{Lp^2 + Rp} \ &= rac{E}{R} \left(rac{1}{p} - rac{p}{p + R/L}
ight) \end{aligned}$$

常用拉普拉斯变换的反演:

$$\frac{1}{p-\alpha} = \mathrm{e}^{\alpha t}$$

于是:

$$\frac{1}{p} \stackrel{.}{=} 1, \;\; \frac{1}{p+R/L} \stackrel{.}{=} \mathrm{e}^{-\frac{R}{L}t}$$

对方程  $F(p)=rac{E}{R}\left(rac{1}{p}-rac{p}{p+R/L}
ight)$  两边同时作拉普拉斯逆变换,得:

$$i(t) = rac{E}{R} \left( 1 - \mathrm{e}^{-rac{R}{L}t} 
ight)$$

6

证明 
$$\nabla \cdot \left( \varphi \vec{A} \right) = \vec{A} \cdot (\nabla \varphi) + \varphi \nabla \cdot \vec{A}$$
,并据此由 Gauss 公式  $\int\limits_{\Omega} \nabla \cdot \vec{A} \mathrm{d}V = \int\limits_{\partial \Omega} \vec{A} \cdot \mathrm{d}\vec{S}$  证明 Green 公式  $\int\limits_{\Omega} \psi \nabla^2 \varphi + \varphi \nabla^2 \psi \mathrm{d}V = \int\limits_{\partial \Omega} (\psi \nabla \varphi + \varphi \nabla \psi) \cdot \mathrm{d}\vec{S}$ 

$$\nabla \cdot (\varphi \vec{A}) = \partial_i (\varphi \vec{A})_i$$

$$= \partial_i (\varphi A_i)$$

$$= A_i \partial_i \varphi + \varphi \partial_i A_i$$

$$= A_i (\nabla \varphi)_i + \varphi \partial_i A_i$$

$$= \vec{A} \cdot (\nabla \varphi) + \varphi \nabla \cdot \vec{A}$$

高斯公式给出:

$$\int\limits_{\Omega} \nabla \cdot \vec{A} \mathrm{d}V = \int\limits_{\partial \Omega} \vec{A} \cdot \mathrm{d}\vec{S}$$

令  $\vec{A} = \nabla(\psi\varphi)$ ,代入高斯公式得:

$$\int\limits_{\Omega} 
abla \cdot 
abla (\psi arphi) \mathrm{d}V = \int\limits_{\partial \Omega} 
abla (\psi arphi) \cdot \mathrm{d}ec{S}$$

即:

$$\int\limits_{\Omega} \psi 
abla^2 arphi + arphi 
abla^2 \psi \mathrm{d}V = \int\limits_{\partial \Omega} (\psi 
abla arphi + arphi 
abla \psi) \cdot \mathrm{d}ec{S}$$

7

求定解问题:

$$egin{cases} u_{tt} - a^2 u_{xx} &= 0 \ u_x igg|_{x=0} &= 0 \ u_x igg|_{x=0} &= 0 \ u_{igg|_{t=0}} &= \cos\left(rac{\pi x}{l}
ight) + 0.3\cos\left(rac{3\pi x}{l}
ight) \ u_t igg|_{t=0} &= 0 \end{cases}$$

设:

$$u(x,t) = U(x)T(t)$$

代入一维波动方程  $u_{tt} - a^2 u_{xx} = 0$  可得:

$$egin{aligned} rac{T''(t)}{T(t)} &= a^2 rac{U''(x)}{U(x)} = -\omega^2 \ T''(t) + \omega^2 T(t) &= 0 \Longrightarrow T(t) = A\cos\omega t + B\sin\omega t \ T'(t) &= -\omega A\sin\omega t + \omega B\cos\omega t \ u_tigg|_{t=0} &= 0 \Longrightarrow T'(t)igg|_{t=0} &= 0 \Longrightarrow B = 0 \end{aligned}$$

因此:

$$T(t) = A\cos\omega t$$

令:

$$k\equiv rac{\omega}{a},~~k^2=rac{\omega^2}{a^2}$$
  $U''(x)+k^2U(x)=0\Longrightarrow U(x)=C\cos kx+D\sin kx$   $U'(x)=-kC\sin kx+kD\cos kx$ 

$$\left.u_x
ight|_{x=0}=0\Longrightarrow U'(x)
ight|_{x=0}=0\Longrightarrow D=0$$

因此:

$$U(x) = C\cos kx, \ U'(x) = -kC\sin kx$$

$$\left. u_x 
ight|_{x=l} = 0 \Longrightarrow U'(x) 
ight|_{x=l} = 0 \Longrightarrow -kC \sin kl = 0$$

因此, k 的本征值  $k_n$  为:

$$k_n=rac{n\pi}{l}, \;\; n=,1,2,\cdots$$

n=0 是平庸解,不考虑。

相应的本征函数  $U_n(x)$  为:

$$U_n(x)=\cos k_n x=\cos\left(rac{n\pi}{l}x
ight), \ \ n=1,2,\cdots$$

由  $k \equiv \omega/a$ , 得  $\omega$  的本征值  $\omega_n$  为:

$$\omega_n=ak_n=rac{n\pi a}{l}, \ \ n=1,2,\cdots$$

相应的本征函数  $T_n(x)$  为:

$$T_n(t)=\cos\omega_n t=\cos\left(rac{n\pi a}{l}t
ight), \;\; n=,1,2,\cdots$$

本征解  $u_n(x,t)$  为:

$$u_n(x,t) = U_n(t)T_n(t) = \cos\left(rac{n\pi}{l}x
ight)\cos\left(rac{n\pi a}{l}t
ight), \;\; n=,1,2,\cdots$$

定解问题的通解 u(x,t) 为:

$$u(x,t) = \sum_{n=1}^{\infty} E_n u_n(x,t) = \sum_{n=1}^{\infty} E_n \cos\left(rac{n\pi}{l}x
ight) \cos\left(rac{n\pi a}{l}t
ight)$$

最后结合初始条件

$$\left. u 
ight|_{t=0} = \cos \left( rac{\pi x}{l} 
ight) + 0.3 \cos \left( rac{3\pi x}{l} 
ight)$$

得到:

$$E_1 = 1, E_2 = 0, E_3 = 0.3, E_4 = E_5 = \cdots = 0$$

最终得到定解问题的解为:

8

在均匀电场  $ec{E}_0$  中放一半径为 a 的接地导体球,求球外电势、电场、导体球表面面电荷密度分布。

以球心 O 为坐标原点,选取  $\vec{E}_0$  方向为 z 轴正方向,则电势 u 关于 z 轴轴对称。

球外无自由电荷,于是球外电势分布  $u(\vec{r})$  满足拉普拉斯方程:

$$abla^2 u(\vec{r}) = 0, \quad r > a$$

特别地,这里电势 u 关于 z 轴对称, u 与  $\varphi$  无关,拉普拉斯方程可简化为:

$$abla^2 u(r, heta) = 0, \ \ r > a$$

导体球接地,得到一个边界条件:

$$\left. u(r, heta) 
ight|_{r=a} = 0$$

由电势的叠加原理,实际电势  $u(r,\theta)$  是导体球面上的感应电荷产生的电势和匀强电场  $\vec{E}_0$  导致的电势的代数和。把感应电荷在无穷远处产生的电势设为零,则当  $r\to +\infty$ ,电势只由匀强电场贡献。设匀强电场单独存在时在坐标原点产生的电势为  $u_0$ ,则:

$$u_0 - u(r, \theta) = E_0 r \cos \theta, \ \ r \to +\infty$$

定解问题为:

$$egin{cases} 
abla^2 u(r, heta) = 0 \ u(r, heta)igg|_{r=a} = 0 \ u(r, heta) = u_0 - E_0 r\cos heta, \ \ r o +\infty \end{cases}$$

套用结论, 轴对称问题的拉普拉斯方程在自然边界条件约束下的形式解为:

$$u(r, heta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}
ight) \mathrm{P}_l(\cos heta)$$

考虑边界条件  $u(r,\theta)igg|_{r o +\infty} = u_0 - E_0 r \cos \theta$ ,当  $r o +\infty$ ,有  $r^{-(l+1)} o 0$ ,于是:

$$egin{aligned} u_0 - E_0 r \cos heta &= \sum_{l=0}^\infty A_l r^l \mathrm{P}_l (\cos heta) \ &= A_0 + A_1 r \cos heta + \cdots \end{aligned}$$

左右两边都看作关于r的多项式,对比系数得:

$$A_0 = u_0, \ A_1 = -E_0, \ A_2 = A_3 = \cdots = 0$$

于是形式解可写为:

$$egin{aligned} u(r, heta) &= \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}
ight) \mathrm{P}_l(\cos heta) \ &= u_0 - E_0 r\cos heta + \sum_{l=0}^{\infty} B_l r^{-(l+1)} \mathrm{P}_l(\cos heta) \end{aligned}$$

再考虑边界条件  $u(r,\theta)\bigg|_{r=a}=0$ ,将形式解代入边界条件,得:

$$u_0-E_0a\cos heta+\sum_{l=0}^\infty B_la^{-(l+1)}\mathrm{P}_l(\cos heta)=0$$

即:

$$u_0\mathrm{P}_0(\cos heta)-E_0a\mathrm{P}_1(\cos heta)+\sum_{l=0}^\infty B_la^{-(l+1)}\mathrm{P}_l(\cos heta)=0$$

整理成以  $\cos \theta$  为自变量的各阶勒让德多项式  $P_l(\cos \theta)$  的线性叠加的形式:

$$\left(u_0 + B_0 a^{-1}
ight) \mathrm{P}_0(\cos heta) + \left(-E_0 a + B_1 a^{-2}
ight) \mathrm{P}_1(\cos heta) + \sum_{l=2}^{\infty} B_l a^{-(l+1)} \mathrm{P}_l(\cos heta) = 0$$

由各阶勒让德多项式的正交性,它们的线性叠加为零,当且仅当所有线性叠加系数为零,即:

$$B_0 = -au_0, \ B_1 = a^3 E_0, \ B_2 = B_3 = \dots = 0$$

综上,导体球外电势分布为:

$$egin{align} u(r, heta) &= u_0 - E_0 r\cos heta + \sum_{l=0}^\infty B_l r^{-(l+1)} \mathrm{P}_l(\cos heta) \ &= u_0 - E_0 r\cos heta - rac{u_0 a}{r} + E_0 a^3 rac{\cos heta}{r^2}, \ \ r\geqslant a \ \end{aligned}$$

其中, $u_0$  为匀强电场单独存在时在坐标原点产生的电势。

取  $u_0=0$ ,则导体球外电势分布为:

$$\left| u(r, heta) = -E_0 r\cos heta + E_0 a^3 rac{\cos heta}{r^2} 
ight|, \;\; r \geqslant a^3$$

球外电场与电势的关系为:

$$\begin{split} \vec{E}(\vec{r}) &= -\nabla u(\vec{r}) \\ &= -\left[\frac{\partial u}{\partial r}\vec{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\vec{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\vec{\mathbf{e}}_\varphi\right] \\ &= E_0\cos\theta\left(1 + \frac{2a^3}{r^3}\right)\vec{\mathbf{e}}_r + E_0\sin\theta\left(\frac{a^3}{r^3} - 1\right)\vec{\mathbf{e}}_\theta, \ \ r \geqslant a \end{split}$$

导体表面电场为:

$$\left.ec{E}(ec{r})
ight|_{r=a}=3E_{0}\cos hetaec{\mathrm{e}}_{r}$$

利用高斯定理,导体球表面面电荷密度分布为:

$$\left. \sigma(ec{r}) 
ight|_{r=a} = arepsilon_0 ec{E}(ec{r}) 
ight|_{r=a} \cdot ec{\mathrm{e}}_r = 3 arepsilon_0 E_0 \cos heta$$

9

求边缘固定半径为 b 的圆形膜的本征振动频率及本征振动模式。

以圆形膜的圆心为原点建立极坐标,设  $u(\rho,\varphi,t)$  是 t 时刻  $\rho,\varphi$  处质点偏离平衡位置的位移,则  $u(\rho,\varphi,t)$  满足二维波动方程:

$$u_{tt}(
ho,arphi,t)-a^2
abla_{(2)}^2u(
ho,arphi,t)=0$$

其中, $\nabla^2_{(2)}$  是二维拉普拉斯算子:

$$abla_{(2)}^2 \equiv rac{1}{
ho}rac{\partial}{\partial
ho}\left(
horac{\partial}{\partial
ho}
ight) + rac{1}{
ho^2}rac{\partial^2}{\partialarphi^2}$$

设  $u(\rho,\varphi,t)$  可分离变量为:

$$u(\rho, \varphi, t) = U(\rho, \varphi)T(t)$$

代入二维波动方程可得:

$$U(
ho,arphi)T''(t)-a^2T(t)\left[rac{1}{
ho}rac{\partial}{\partial
ho}\left(
horac{\partial}{\partial
ho}
ight)+rac{1}{
ho^2}rac{\partial^2}{\partialarphi^2}
ight]U(
ho,arphi)=0$$

上式两边同时除以  $U(\rho,\varphi)T(t)$ , 再移项, 得:

$$rac{T''(t)}{T(t)} = rac{a^2}{U(
ho,arphi)} \left[rac{1}{
ho}rac{\partial}{\partial
ho}\left(
horac{\partial}{\partial
ho}
ight) + rac{1}{
ho^2}rac{\partial^2}{\partialarphi^2}
ight] U(
ho,arphi)$$

注意到,  $\frac{T''(t)}{T(t)}$  只与 t 有关,而  $\frac{a^2}{U(\rho,\varphi)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right]U(\rho,\varphi)$  只与  $\rho,\varphi$  有关,二 者相等,因此二者均等于同一常数  $-\omega^2$ :

$$rac{T''(t)}{T(t)} = -\omega^2, \;\; rac{a^2}{U(
ho,arphi)} \left[rac{1}{
ho}rac{\partial}{\partial
ho}\left(
horac{\partial}{\partial
ho}
ight) + rac{1}{
ho^2}rac{\partial^2}{\partialarphi^2}
ight]U(
ho,arphi) = -\omega^2$$

由于要求本征振动频率和本征振动模式,因此只需要关注空间部分  $U(\rho,\varphi)$  满足的方程和边界条件。

对上式空间部分  $U(\rho,\varphi)$  满足的方程等号两边同乘  $\frac{U(\rho,\varphi)}{a^2}$  并移项,得:

$$rac{1}{
ho}rac{\partial}{\partial
ho}\left(
horac{\partial U(
ho,arphi)}{\partial
ho}
ight)+rac{1}{
ho^2}rac{\partial^2 U(
ho,arphi)}{\partialarphi^2}+rac{\omega^2}{a^2}U(
ho,arphi)=0$$

令:

$$k\equiv rac{\omega}{a}, ~~k^2=rac{\omega^2}{a^2}$$

则  $U(\rho,\varphi)$  满足的方程为:

$$rac{1}{
ho}rac{\partial}{\partial
ho}\left(
horac{\partial U(
ho,arphi)}{\partial
ho}
ight)+rac{1}{
ho^2}rac{\partial^2 U(
ho,arphi)}{\partialarphi^2}+k^2U(
ho,arphi)=0$$

由于圆形膜边界固定,因此得到一个边界条件:

$$U(
ho,arphi)igg|_{
ho=b}=0$$

且圆心处质点偏离平衡位置的位移应有限,因此得到一个自然边界条件:

$$|U(
ho,arphi)| igg|_{
ho=0} < +\infty$$

再结合  $\varphi$  作为角度这一物理量应使得  $U(\rho,\varphi)$  满足周期性边界条件:

$$U(
ho, arphi + 2\pi) = U(
ho, arphi)$$

综上,空间部分  $U(\rho,\varphi)$  要满足的所有条件为:

$$\begin{cases} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial U(\rho, \varphi)}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 U(\rho, \varphi)}{\partial \varphi^2} + k^2 U(\rho, \varphi) = 0 \\ \left. \left. \left| U(\rho, \varphi) \right| \right|_{\rho = b} = 0 \\ \left| \left| U(\rho, \varphi) \right| \right|_{\rho = 0} < +\infty \\ \left. \left| U(\rho, \varphi) \right| = U(\rho, \varphi) \end{cases} \end{cases}$$

设  $U(\rho,\varphi)$  可分离变量为:

$$U(\rho, \varphi) = R(\rho)\Phi(\varphi)$$

代入空间部分  $U(\rho,\varphi)$  要满足的方程,得:

$$rac{\Phi(arphi)}{
ho}rac{\mathrm{d}}{\mathrm{d}
ho}\left(
horac{\mathrm{d}R(
ho)}{\mathrm{d}
ho}
ight)+rac{R(
ho)}{
ho^2}rac{\mathrm{d}^2\Phi(arphi)}{\mathrm{d}arphi^2}+k^2R(
ho)\Phi(arphi)=0$$

上式等号两边同乘  $\frac{
ho^2}{R(
ho)\Phi(arphi)}$  , 整理得:

$$\frac{1}{\Phi(\varphi)} \frac{\mathrm{d}^2 \Phi(\varphi)}{\mathrm{d} \varphi^2} = -\left[ \frac{\rho}{R(\rho)} \frac{\mathrm{d}}{\mathrm{d} \rho} \left( \rho \frac{\mathrm{d} R(\rho)}{\mathrm{d} \rho} \right) + k^2 \rho^2 \right]$$

上式等号左边只与  $\varphi$  有关,等号右边只与  $\rho$  有关,因此二者均等于一个常数  $-m^2$ :

$$\frac{1}{\Phi(\varphi)}\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2} = -\left[\frac{\rho}{R(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}\right) + k^2\rho^2\right] = -m^2$$

因此,角度部分满足方程:

$$\Phi''(\varphi) + m^2 \Phi(\varphi) = 0$$

周期性边界条件:

$$U(
ho, \varphi + 2\pi) = U(
ho, \varphi) \Longrightarrow R(
ho)\Phi(\varphi + 2\pi) = R(
ho)\Phi(\varphi) \Longrightarrow \Phi(\varphi + 2\pi) = \Phi(\varphi)$$

$$egin{cases} \Phi''(arphi) + m^2 \Phi(arphi) = 0 \ \Phi(arphi + 2\pi) = \Phi(arphi) \end{cases}$$

从

$$\Phi''(\varphi) + m^2 \Phi(\varphi) = 0$$

可以解得:

$$\Phi(\varphi) = A\cos(m\varphi) + B\sin(m\varphi)$$

结合周期性边界条件

$$\Phi(\varphi+2\pi)=\Phi(\varphi)$$

可得:

$$m=0,1,2,\cdots$$

径向部分  $R(\rho)$  满足:

$$-\left[rac{
ho}{R(
ho)}rac{\mathrm{d}}{\mathrm{d}
ho}\left(
horac{\mathrm{d}R(
ho)}{\mathrm{d}
ho}
ight)+k^2
ho^2
ight]=-m^2$$

可以整理成:

$$\frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d}\rho} \left( \rho \frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho} \right) + \left( k^2 - \frac{m^2}{\rho^2} \right) R(\rho) = 0$$

令  $x=k
ho,
ho=x/k,R(
ho)igg|_{
ho=x/k}=R(x/k)\equiv y(x)$ ,则上面可方程化为 m 阶贝塞尔方程:

$$\begin{split} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x} \frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{m^2}{x^2}\right) y &= 0 \\ U(\rho, \varphi) \bigg|_{\rho = b} &= 0 \Longrightarrow R(\rho) \Phi(\varphi) \bigg|_{\rho = b} = 0 \Longrightarrow R(\rho) \bigg|_{\rho = b} = 0 \\ |U(\rho, \varphi)| \left|_{\rho = 0} < +\infty \Longrightarrow |R(\rho) \Phi(\varphi)| \left|_{\rho = 0} < +\infty \Longrightarrow |R(\rho)| \right|_{\rho = 0} < +\infty \end{split}$$

$$\begin{cases} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x} \frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{m^2}{x^2}\right) y = 0 \\ y(x) \equiv R(\rho) \Big|_{\rho = x/k} = R(x/k), \ R(\rho) = y(x) \Big|_{x = k\rho} = y(k\rho) \\ R(\rho) \Big|_{\rho = b} = 0 \\ |R(\rho)| \Big|_{\rho = 0} < +\infty \end{cases}$$

对于 m 阶贝塞尔方程

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x} \frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{m^2}{x^2}\right) y = 0$$

其通解为:

$$y^{(m)}(x) = C_m \mathbf{J}_m(x) + D_m \mathbf{N}_m(x)$$

考虑自然边界条件  $|R(
ho)| \left|_{
ho=0} < +\infty$ ,可得:

$$D_m = 0$$

因此:

$$y^{(m)}(x) = C_m \mathrm{J}_m(x)$$

对上面等式两边同取附加条件:

$$\left. y^{(m)}(x) 
ight|_{x=k
ho} = C_m \mathrm{J}_m(x) 
ight|_{x=k
ho}$$

结合  $x=k
ho, R(
ho)=y(x)igg|_{x=k
ho}=y(k
ho)$  可得:

$$R^{(m)}(
ho)=C_m {
m J}_m(k
ho)$$

设 m 阶贝塞尔函数  $\mathrm{J}_m(x)$  的第 n 个正零点为  $x_n^{(m)}$  ,即:

$${
m J}_m\left(x_n^{(m)}
ight)=0, \;\; m=0,1,2,\cdots; \;\; n=,1,2,\cdots$$

结合边界条件  $R(
ho)igg|_{
ho=b}=0$ ,即:

$$C_m \mathbf{J}_m(kb) = 0$$

因此 k 的本征值  $k_n^{(m)}$  为:

$$k_n^{(m)} = rac{x_n^{(m)}}{b}, \;\; m = 0, 1, 2, \cdots; \;\; n = 1, 2, \cdots$$

相应的本征振动模式  $R_n^{(m)}(\rho)$  为:

$$R_n^{(m)}(
ho)=\mathrm{J}_m\left(k_n^{(m)}
ho
ight)=\mathrm{J}_m\left(rac{x_n^{(m)}}{b}
ho
ight), \ \ m=0,1,2,\cdots; \ \ n=1,2,\cdots$$

再根据  $k \equiv \omega/a$ ,得到  $\omega$  的本征值,即圆形膜的本征频率  $\omega_n^{(m)}$  为 :

$$\omega_n^{(m)} = a k_n^{(m)} = rac{x_n^{(m)}}{b} \cdot a, \;\; m = 0, 1, 2, \cdots; \;\; n = 1, 2, \cdots$$

综上所述,边缘固定半径为 b 的圆形膜的本征振动频率  $\omega_n^{(m)}$  及本征振动模式  $R_n^{(m)}(\rho)$  为:

$$oxed{\omega_n^{(m)} = rac{x_n^{(m)}}{b} \cdot a}, \;\; m = 0, 1, 2, \cdots; \;\; n = 1, 2, \cdots$$

$$oxed{R_n^{(m)}(
ho)=\mathrm{J}_m\left(rac{x_n^{(m)}}{b}
ho
ight)}, \ \ m=0,1,2,\cdots; \ \ n=1,2,\cdots$$

其中,  $x_n^{(m)}$  是 m 阶贝塞尔函数  $J_m(x)$  的第 n 个正零点。

## 10

数学物理方程反映了同一类现象的共同规律,各个具体问题所处的特定"环境"(边界条件)决定了其特殊性的一面,试简述你边界条件的认识(从边界条件的分类、边界条件与本征值的关系、自然边界条件等方面阐述)。

大家自由发挥。