2-1

> 推导 $\hat{H}=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)$，完成腔内单模电磁场的量子化。

腔内电场满足波动方程：

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=\vec{0}$$

考虑沿 z 轴传播的 x 方向偏振的线偏振光，波动方程化为：

$$\frac{\partial^2 E_x(z,t)}{\partial z^2}-\frac{1}{c^2}\frac{\partial E_x(z,t)}{\partial t^2}=0$$

设 $E_x(z,t)=E_x(z)q(t)$，分离变量得：

$$\frac{\mathrm{d}^2E_x(z)}{\mathrm{d}z^2}+k^2E_x(z)=0$$

$$\frac{\mathrm{d}^2q(t)}{\mathrm{d}t^2}+\omega^2q(t)=0,\ \ \omega=ck$$

可以解得通解：

$$E_x(z)=A\sin(kz+\alpha)$$

$$q(t)=c_1\mathrm{e}^{-\mathrm{i}\omega t}+c_2\mathrm{e}^{\mathrm{i}\omega t}$$

边界条件：

$$E_x(0)=E_x(L)=0$$

得到：

$$\alpha=0,\ \ k=\frac{n\pi}{L}$$

因此：

$$\vec{E}=\vec{\mathrm{e}}_xAq(t)\sin(kz),\ \ k=\frac{n\pi}{L}$$

由 $\nabla\times\vec{B}=\mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ 可得：

$$\vec{B}=\vec{\mathrm{e}}_y\left(\frac{\mu_0\varepsilon_0}{k}\right)A\dot{q}(t)\cos(kz)$$

取基模 $n=1$，电磁场的能量为：

$$\begin{aligned}
H&=\frac{1}{2}\int\mathrm{d}^3\vec{r}\left[\varepsilon_0\left|E_x(z,t)\right|^2+\frac{1}{\mu_0}\left|B_y(z,t)\right|^2\right]\\
&=\frac{\varepsilon_0VA^2}{4}\left[q^2(t)+\frac{\dot{q}^2(t)}{c^2k^2}\right]\\
&=\frac{\varepsilon_0VA^2}{2m\omega^2}\left[\frac{m\omega^2q^2(t)}{2}+\frac{p^2(t)}{2m}\right]\\
&=\frac{p^2(t)}{2m}+\frac{m\omega^2q^2(t)}{2},\ \ A=\left(\frac{2m\omega^2}{V\varepsilon_0}\right)^{1/2}
\end{aligned}$$

$q\rightarrow\hat{q},p\rightarrow\hat{p},[\hat{q}(t),\hat{p}(t')]=\mathrm{i}\hbar\delta_{t,t'}$

$$\hat{H}=\frac{\hat{p}^2(t)}{2m}+\frac{m\omega^2\hat{q}^2(t)}{2}$$

$$\hat{a}\mathrm{e}^{-\mathrm{i}\omega t}=\frac{1}{\sqrt{2m\hbar\omega}}\left[m\omega\hat{q}(t)+\mathrm{i}\hat{p}(t)\right]$$

则：

$$\hat{H}=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)$$

$$\hat{\vec{E}}(z,t)=\vec{\mathrm{e}}_x\mathcal{E}\left(\hat{a}\mathrm{e}^{-\mathrm{i}\omega t}+\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t}\right)\sin(kz)$$

2-2

> 推导 $\hat{H}=\sum_{\vec{k}s}\hbar\omega_k\left(\hat{a}_{\vec{k}s}^{\dagger}\hat{a}_{\vec{k}s}+\frac{1}{2}\right)$，完成自由空间多模电磁场的量子化。

光场由矢势确定：

$$\vec{E}(\vec{r},t)=-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

$$\vec{B}(\vec{r},t)=\nabla\times\vec{A}(\vec{r},t)$$

麦克斯韦方程：

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

自由空间中 $\vec{J}=\vec{0},\vec{D}=\varepsilon_0\vec{E},\vec{H}=\vec{B}/\mu_0$，代入上式得：

$$\nabla\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

将上式可化为关于矢势 $\vec{A}$ 的方程：

$$\nabla\times(\nabla\times\vec{A})=-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

利用公式 $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$，并采用库伦规范 $\nabla\cdot\vec{A}=0$ 可得：

$$\nabla^2\vec{A}(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2}=0$$

设 $\vec{A}(\vec{r},t)=\vec{A}(\vec{r})A(t)$，分离变量可得：

$$\begin{cases}\nabla^2\vec{A}(\vec{r})+k^2\vec{A}(\vec{r})=\vec{0}\\ \ddot{A}(t)+\omega^2A(t)=0\\ \omega^2=c^2k^2\end{cases}$$

把自由空间看作边长为 L 的立方腔，可得通解：

$$\vec{A}(\vec{r},t)=\sum_{\vec{k},s}\vec{\mathrm{e}}_{\vec{k}s}\left[A_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}+A_{\vec{k}s}^*\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right]$$

采用周期性边界条件可得：

$$\vec{k}=\frac{2\pi}{L}(m_x,m_y,m_z),\ \ m_x,m_y,m_z\text{为整数}$$

由于 $\nabla\cdot\vec{A}=0$，因此 $\vec{k}\cdot\vec{\mathrm{e}}_{\vec{k}s}=0$，于是电磁波为横波，有两个独立的偏振方向，$s=1,2$

电场：

$$\begin{aligned}\vec{E}(\vec{r},t) &= -\frac{\partial\vec{A}(\vec{r},t)}{\partial t}\\ &= -\frac{\partial}{\partial t}\left\{\sum_{\vec{k},s}\vec{\mathrm{e}}_{\vec{k}s}\left[A_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}+A^*_{\vec{k}s}\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right]\right\}\\ &= \mathrm{i}\sum_{\vec{k},s}\omega_k\vec{\mathrm{e}}_{\vec{k}s}\left[A_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}-A^*_{\vec{k}s}\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right]\end{aligned}$$

结合 $\nabla\times(\varphi\vec{A})=(\nabla\varphi)\times\vec{A}+\varphi(\nabla\times\vec{A})$，可得磁场：

$$\begin{aligned}\vec{B}(\vec{r},t) &= \nabla\times\vec{A}(\vec{r},t)\\ &= \nabla\times\left\{\sum_{\vec{k},s}\vec{\mathrm{e}}_{\vec{k}s}\left[A_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}+A^*_{\vec{k}s}\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right]\right\}\\ &= \sum_{\vec{k},s}\left[\left(A_{\vec{k}s}\nabla\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right)\times\vec{\mathrm{e}}_{\vec{k}s}+\left(A^*_{\vec{k}s}\nabla\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right)\times\vec{\mathrm{e}}_{\vec{k}s}\right]\\ &= \sum_{\vec{k},s}\left[\left(\mathrm{i}A_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right)\vec{k}\times\vec{\mathrm{e}}_{\vec{k}s}-\left(\mathrm{i}A^*_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right)\vec{k}\times\vec{\mathrm{e}}_{\vec{k}s}\right]\\ &= \mathrm{i}\sum_{\vec{k},s}\left(\vec{k}\times\vec{\mathrm{e}}_{\vec{k}s}\right)\left[A_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}-A^*_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right]\\ &= \frac{\mathrm{i}}{c}\sum_{\vec{k},s}\omega_k\left(\hat{k}\times\vec{\mathrm{e}}_{\vec{k}s}\right)\left[A_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}-A^*_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right]\end{aligned}$$

哈密顿量为：

$$\begin{aligned}H &= \frac{1}{2}\int\limits_V\left(\varepsilon_0\vec{E}\cdot\vec{E}+\frac{1}{\mu_0}\vec{B}\cdot\vec{B}\right)\mathrm{d}V\\ &= 2\varepsilon_0 V\sum_{\vec{k},s}\omega_k^2 A^*_{\vec{k}s}A_{\vec{k}s}\end{aligned}$$

引入 $p_{\vec{k}s}, q_{\vec{k}s}$ 使得：

$$A_{\vec{k}s}=\frac{1}{2\omega_k\left(\varepsilon_0 V\right)^{1/2}}\left[\omega_k q_{\vec{k}s}+\mathrm{i}p_{\vec{k}s}\right]$$

则：

$$H=\frac{1}{2}\sum_{\vec{k},s}\left(p^2_{\vec{k}s}+\omega_k q^2_{\vec{k}s}\right)$$

量子化：

$$\left[\hat{q}_{\vec{k}s},\hat{p}_{\vec{k}'s'}\right]=\mathrm{i}\hbar\delta_{\vec{k}\vec{k}'}\delta_{ss'}$$

令：

$$\hat{a}_{\vec{k}s}=\frac{1}{\left(2\hbar\omega_k\right)^{1/2}}\left(\omega_k\hat{q}_{\vec{k}s}+\mathrm{i}\hat{p}_{\vec{k}s}\right)$$

则：

$$\hat{H}=\sum_{\vec{k},s}\hbar\omega_k\left(\hat{a}^\dagger_{\vec{k}s}\hat{a}_{\vec{k}s}+\frac{1}{2}\right)$$

$$\hat{\vec{E}}(\vec{r},t)=\mathrm{i}\sum_{\vec{k},s}\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2}\vec{\mathrm{e}}_{\vec{k}s}\left[\hat{a}_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}-\hat{a}^\dagger_{\vec{k}s}\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right]$$

$$\hat{\vec{B}}(\vec{r},t)=\frac{\mathrm{i}}{c}\sum_{\vec{k},s}\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2}\left(\hat{k}\times\vec{\mathrm{e}}_{\vec{k}s}\right)\left[\hat{a}_{\vec{k}s}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}-\hat{a}_{\vec{k}s}^{\dagger}\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)}\right]$$

2-3

> 求证，对单模自由光场 $\hat{H}=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+1/2\right)$，其本征态满足：（1）$|n\rangle=\frac{\left(\hat{a}^{\dagger}\right)^n}{\sqrt{n!}}|0\rangle$（2）$\hat{U}(t)=\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)=\sum_n\exp\left(-\frac{\mathrm{i}}{\hbar}E_n t\right)|n\rangle\langle n|$，其中 $E_n=\hbar\omega\,(n+1/2)$

(1)

$$\hat{a}^{\dagger}|n\rangle=\sqrt{n+1}\,|n+1\rangle\Longrightarrow|n+1\rangle=\frac{\hat{a}^{\dagger}}{\sqrt{n+1}}|n\rangle$$

于是：

$$\begin{aligned}|n\rangle&=\frac{\hat{a}^{\dagger}}{\sqrt{n}}|n-1\rangle\\&=\frac{\left(\hat{a}^{\dagger}\right)^2}{\sqrt{n}\sqrt{n-1}}|n-2\rangle\\&=\vdots\\&=\frac{\left(\hat{a}^{\dagger}\right)^n}{\sqrt{n!}}|0\rangle\end{aligned}$$

(2)

$$\hat{H}|n\rangle=E_n|n\rangle=\hbar\omega\,(n+1/2)\,|n\rangle$$

完备性：

$$|n\rangle\langle n|=I$$

$$\begin{aligned}\hat{U}(t)&=\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\\&=\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\sum_n|n\rangle\langle n|\\&=\sum_n\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)|n\rangle\langle n|\\&=\sum_n\exp\left(-\frac{\mathrm{i}}{\hbar}E_n t\right)|n\rangle\langle n|\end{aligned}$$

2-4

> 推导方程：$\rho=\frac{\mathrm{e}^{-\beta\hat{H}}}{Z}=\sum_{n=0}^{\infty}\frac{\bar{n}^n}{(1+\bar{n})^{n+1}}|n\rangle\langle n|$， $Z=\mathrm{Tr}\left(\mathrm{e}^{-\beta\hat{H}}\right)$

$$\beta=\frac{1}{k_{\mathrm{B}}T},\ \ \hat{H}=\hbar\omega(\hat{a}^{\dagger}\hat{a}+1/2)$$

热场的密度矩阵：

$$\rho=\frac{\exp\left(-\hat{H}/k_{\mathrm{B}}T\right)}{\mathrm{Tr}\left[\exp\left(-\hat{H}/k_{\mathrm{B}}T\right)\right]},\quad \hat{H}=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+1/2\right)$$

$$\begin{aligned}\mathrm{Tr}\left[\exp\left(-\hat{H}/k_{\mathrm{B}}T\right)\right]&=\sum_{n=0}^{\infty}\left\langle n\left|\exp\left(-\hat{H}/k_{\mathrm{B}}T\right)\right|n\right\rangle\\&=\sum_{n=0}^{\infty}\exp\left(-E_n/k_{\mathrm{B}}T\right)\\&\equiv Z\end{aligned}$$

其中 $Z\equiv\sum\limits_{n=0}^{\infty}\exp\left(-E_n/k_{\mathrm{B}}T\right)$ 称为配分函数。

$$\begin{aligned}Z&\equiv\sum_{n=0}^{\infty}\exp\left(-E_n/k_{\mathrm{B}}T\right)\\&=\exp\left(-\hbar\omega/2k_{\mathrm{B}}T\right)\sum_{n=0}^{\infty}\exp\left(-\hbar\omega n/k_{\mathrm{B}}T\right)\\&=\exp\left(-\hbar\omega/2k_{\mathrm{B}}T\right)\cdot\frac{1}{1-\exp\left(-\hbar\omega/k_{\mathrm{B}}T\right)}\\&=\frac{\exp\left(-\hbar\omega/2k_{\mathrm{B}}T\right)}{1-\exp\left(-\hbar\omega/k_{\mathrm{B}}T\right)}\end{aligned}$$

设：

$$\rho=\sum_{n=0}^{\infty}P_n\left|n\right\rangle\left\langle n\right|$$

则：

$$\begin{aligned}P_n&=\left\langle n\,|\,\rho\,|\,n\right\rangle\\&=\frac{1}{Z}\exp\left(-E_n/k_{\mathrm{B}}T\right)\end{aligned}$$

注意到：

$$\begin{aligned}\sum_{n=0}^{\infty}n\mathrm{e}^{-nx}&=-\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\left(\mathrm{e}^{-nx}\right)\\&=-\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}\mathrm{e}^{-nx}\\&=\frac{\mathrm{e}^{-x}}{\left(1-\mathrm{e}^{-x}\right)^2}\end{aligned}$$

热场的平均光子数：

$$\begin{aligned}\bar{n}&=\left\langle\hat{n}\right\rangle\\&=\mathrm{Tr}\left(\hat{n}\rho\right)\\&=\sum_{n=0}^{\infty}\left\langle n\,|\,\hat{n}\rho\,|\,n\right\rangle\\&=\sum_{n=0}^{\infty}nP_n\\&=\frac{\exp\left(-\hbar\omega/2k_{\mathrm{B}}T\right)}{Z}\sum_{n=0}^{\infty}n\exp\left(-\hbar\omega n/k_{\mathrm{B}}T\right)\\&=\frac{1}{\exp\left(\hbar\omega/k_{\mathrm{B}}T\right)-1}\end{aligned}$$

因此：

$$\exp\left(-\hbar\omega/k_{\mathrm{B}}T\right)=\frac{\bar{n}}{\bar{n}+1}$$

于是:

$$\begin{aligned}\rho&=\sum_{n=0}^{\infty}P_n\left|n\right\rangle\left\langle n\right|\\&=\sum_{n=0}^{\infty}\frac{1}{Z}\exp\left(-E_n/k_{\mathrm{B}}T\right)\left|n\right\rangle\left\langle n\right|\\&=\sum_{n=0}^{\infty}\frac{\left(\frac{\bar{n}}{\bar{n}+1}\right)^{1/2}\left(\frac{\bar{n}}{\bar{n}+1}\right)^{n}}{\left(\frac{\bar{n}}{\bar{n}+1}\right)^{1/2}\Big/\left(1-\frac{\bar{n}}{\bar{n}+1}\right)}\left|n\right\rangle\left\langle n\right|\\&=\sum_{n=0}^{\infty}\frac{\bar{n}^n}{\left(1+\bar{n}\right)^{n+1}}\left|n\right\rangle\left\langle n\right|\end{aligned}$$

2-5

> 推导黑体辐射公式：$\bar{U}(\omega)=\hbar\omega\bar{n}\rho(\omega)=\dfrac{\hbar\omega^3}{\pi^2c^3}\dfrac{1}{\mathrm{e}^{\beta\hbar\omega}-1}$

由 $\vec{k}=\dfrac{2\pi}{L}(m_x\vec{\mathrm{e}}_x+m_y\vec{\mathrm{e}}_y+m_z\vec{\mathrm{e}}_z)$，$m_x,m_y,m_z\in\mathbb{N}$ 可得:

动量空间中 $k\sim k+\mathrm{d}k$ 动量范围内的电磁模式数:

$$\frac{2\times4\pi k^2\mathrm{d}k}{\left(2\pi/L\right)^3}=\frac{Vk^2}{\pi^2}\mathrm{d}k$$

由色散关系 $\omega_k=ck,\mathrm{d}\omega_k=c\mathrm{d}k$，$\omega_k\sim\omega_k+\mathrm{d}\omega_k$ 圆频率范围内的电磁模式数:

$$\frac{V\omega_k^2}{\pi^2c^3}\mathrm{d}\omega_k$$

模密度（单位体积内 ω 附近单位圆频率区间内的电磁模式数）:

$$\rho(\omega)=\frac{\omega^2}{\pi^2c^3}$$

$$\begin{aligned}\bar{U}(\omega)&=\hbar\omega\bar{n}\rho(\omega)\\&=\frac{\hbar\omega}{\exp\left(\hbar\omega/k_{\mathrm{B}}T\right)-1}\cdot\frac{\omega^2}{\pi^2c^3}\\&=\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\mathrm{e}^{\beta\hbar\omega}-1}\end{aligned}$$

2-6

> 对相干态，求证：（1）$\hat{a}^\dagger\left|\alpha\right\rangle\left\langle\alpha\right|=\left(\alpha^*+\partial_\alpha\right)\left|\alpha\right\rangle\left\langle\alpha\right|$（2）$\left|\alpha\right\rangle\left\langle\alpha\right|\hat{a}=\left(\alpha+\partial_{\alpha^*}\right)\left|\alpha\right\rangle\left\langle\alpha\right|$

(1)

$$\left|\alpha\right\rangle\left\langle\alpha\right|=\mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|$$

一方面:

$$
\begin{aligned}
\hat{a}^\dagger |\alpha\rangle\langle\alpha| &= \hat{a}^\dagger \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \right) \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \hat{a}^\dagger |n\rangle\langle m| \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \sqrt{n+1} |n+1\rangle\langle m| \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{(n+1)!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n+1) |n+1\rangle\langle m| \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{n-1}}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n) |n\rangle\langle m|
\end{aligned}
$$

另一方面:

$$
\begin{aligned}
(\alpha^* + \partial_\alpha) |\alpha\rangle\langle\alpha| &= (\alpha^* + \partial_\alpha) \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \right) \\
&= \alpha^* \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \right) - \alpha^* \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \right) + \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{n\alpha^{n-1}}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \right) \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^{n-1}}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n) |n\rangle\langle m| \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{n-1}}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n) |n\rangle\langle m|
\end{aligned}
$$

对比可得:

$$
\hat{a}^\dagger |\alpha\rangle\langle\alpha| = (\alpha^* + \partial_\alpha) |\alpha\rangle\langle\alpha|
$$

(2)

$$
|\alpha\rangle\langle\alpha| = \mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m|
$$

产生算符作用于 Fock 态:

$$
\hat{a}^\dagger |m\rangle = \sqrt{m+1} |m+1\rangle
$$

取厄米共轭得:

$$
\langle m| \hat{a} = \sqrt{m+1} \langle m+1|
$$

一方面:

$$
\begin{aligned}
|\alpha\rangle\langle\alpha| \hat{a} &= \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \right) \hat{a} \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \hat{a} \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \sqrt{m+1} |n\rangle\langle m+1| \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{(m+1)!}} (m+1) |n\rangle\langle m+1| \\
&= \mathrm{e}^{-\alpha\alpha^*} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*(m-1)}}{\sqrt{m!}} (m) |n\rangle\langle m|
\end{aligned}
$$

另一方面：

$$
\begin{aligned}
(\alpha+\partial_{\alpha^*})\left|\alpha\right\rangle\left\langle\alpha\right| &= (\alpha+\partial_{\alpha^*})\left(\mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right) \\
&= \alpha\left(\mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right)-\alpha\left(\mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right)+\left(\mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{m\alpha^{*(m-1)}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right) \\
&= \mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{m\alpha^{*(m-1)}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right| \\
&= \mathrm{e}^{-\alpha\alpha^*}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{m\alpha^{*(m-1)}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right| \\
&= \mathrm{e}^{-\alpha\alpha^*}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*(m-1)}}{\sqrt{m!}}(m)\left|n\right\rangle\left\langle m\right|
\end{aligned}
$$

对比可得：

$$
\left|\alpha\right\rangle\left\langle\alpha\right|\hat{a}=(\alpha+\partial_{\alpha^*})\left|\alpha\right\rangle\left\langle\alpha\right|
$$

2-7

> 若单模自由光场初态为 $\left|\psi(0)\right\rangle$ 为相干态，求证 $\left|\psi(t)\right\rangle$ 也是相干态。

单模光场哈密顿量：

$$
\hat{H}=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)=\hbar\omega\left(\hat{n}+\frac{1}{2}\right)
$$

其中，$\hat{n}\equiv\hat{a}^\dagger\hat{a}$ 是粒子数算符。Fock 态是 $\hat{n}$ 的本征态：

$$
\hat{n}\left|n\right\rangle=n\left|n\right\rangle
$$

设 $\left|\psi(0)\right\rangle=\left|\alpha\right\rangle$，则：

$$
\begin{aligned}
\left|\psi(t)\right\rangle &= \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\left|\psi(0)\right\rangle \\
&= \mathrm{e}^{-\mathrm{i}(\hat{n}+1/2)\omega t}\left|\alpha\right\rangle \\
&= \mathrm{e}^{-\mathrm{i}\omega t/2}\mathrm{e}^{-\mathrm{i}\omega t\hat{n}}\mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \\
&= \mathrm{e}^{-\mathrm{i}\omega t/2}\mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\mathrm{e}^{-\mathrm{i}\omega t\hat{n}}\left|n\right\rangle \\
&= \mathrm{e}^{-\mathrm{i}\omega t/2}\mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\mathrm{e}^{-\mathrm{i}\omega tn}\left|n\right\rangle \\
&= \mathrm{e}^{-\mathrm{i}\omega t/2}\mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha\mathrm{e}^{-\mathrm{i}\omega t}\right)^n}{\sqrt{n!}}\left|n\right\rangle \\
&= \mathrm{e}^{-\mathrm{i}\omega t/2}\mathrm{e}^{-|\alpha\mathrm{e}^{-\mathrm{i}\omega t}|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha\mathrm{e}^{-\mathrm{i}\omega t}\right)^n}{\sqrt{n!}}\left|n\right\rangle \\
&= \mathrm{e}^{-\mathrm{i}\omega t/2}\left|\alpha\mathrm{e}^{-\mathrm{i}\omega t}\right\rangle
\end{aligned}
$$

相位因子的冗余是允许的，因此 $\left|\psi(t)\right\rangle$ 也是相干态。

2-8

> 推导压缩态不确定关系 $\Delta Y_1=\mathrm{e}^{-r}/2,\Delta Y_2=\mathrm{e}^{r}/2$

$$|\xi,\alpha\rangle = \hat{S}(\xi)\hat{D}(\alpha)\,|0\rangle\,,\quad \hat{S}(\xi) = \exp\left[\frac{1}{2}\left(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger^2}\right)\right],\quad \xi = r\mathrm{e}^{\mathrm{i}\theta}$$

$$\hat{Y}_1 = \exp\left(\frac{\mathrm{i}\theta\hat{a}^\dagger\hat{a}}{2}\right)\hat{X}_1\exp\left(\frac{-\mathrm{i}\theta\hat{a}^\dagger\hat{a}}{2}\right) = \frac{1}{2}\left[\exp\left(-\frac{\mathrm{i}\theta}{2}\right)\hat{a} + \exp\left(\frac{\mathrm{i}\theta}{2}\right)\hat{a}^\dagger\right]$$

$$\hat{Y}_2 = \exp\left(\frac{\mathrm{i}\theta\hat{a}^\dagger\hat{a}}{2}\right)\hat{X}_2\exp\left(\frac{-\mathrm{i}\theta\hat{a}^\dagger\hat{a}}{2}\right) = \frac{1}{2\mathrm{i}}\left[\exp\left(-\frac{\mathrm{i}\theta}{2}\right)\hat{a} - \exp\left(\frac{\mathrm{i}\theta}{2}\right)\hat{a}^\dagger\right]$$

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\cosh r - \hat{a}^\dagger\mathrm{e}^{\mathrm{i}\theta}\sinh r$$

$$\begin{aligned}\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) &= \left[\left(\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi)\right)^\dagger\right]^\dagger \\ &= \left(\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)\right)^\dagger \\ &= -\hat{a}\mathrm{e}^{-\mathrm{i}\theta}\sinh r + \hat{a}^\dagger\cosh r\end{aligned}$$

$$\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{a}\hat{S}(\xi) = \hat{a}^\dagger\hat{a}\left(\cosh^2 r + \sinh^2 r\right) + \sinh^2 r - \left(\hat{a}^{\dagger 2}\mathrm{e}^{\mathrm{i}\theta} + \hat{a}^2\mathrm{e}^{-\mathrm{i}\theta}\right)\cosh r\sinh r$$

$$\left\langle\xi,\alpha\left|\hat{Y}_1\right|\xi,\alpha\right\rangle = \mathrm{e}^{-r}\left(\alpha\mathrm{e}^{-\mathrm{i}\theta/2} + \alpha^*\mathrm{e}^{\mathrm{i}\theta/2}\right)/2$$

$$\left\langle\xi,\alpha\left|\hat{Y}_1^2\right|\xi,\alpha\right\rangle = \mathrm{e}^{-2r}\left[1 + \left(\alpha\mathrm{e}^{-\mathrm{i}\theta/2} + \alpha^*\mathrm{e}^{\mathrm{i}\theta/2}\right)^2\right]\Big/4$$

$$\left\langle\xi,\alpha\left|\hat{Y}_2\right|\xi,\alpha\right\rangle = \mathrm{e}^{r}\left(\alpha\mathrm{e}^{-\mathrm{i}\theta/2} - \alpha^*\mathrm{e}^{\mathrm{i}\theta/2}\right)/(2\mathrm{i})$$

$$\left\langle\xi,\alpha\left|\hat{Y}_2^2\right|\xi,\alpha\right\rangle = \mathrm{e}^{2r}\left[1 - \left(\alpha\mathrm{e}^{-\mathrm{i}\theta/2} - \alpha^*\mathrm{e}^{\mathrm{i}\theta/2}\right)^2\right]\Big/4$$

于是：

$$\begin{aligned}\Delta\hat{Y}_1 &= \sqrt{\left\langle\xi,\alpha\left|\hat{Y}_1^2\right|\xi,\alpha\right\rangle - \left\langle\xi,\alpha\left|\hat{Y}_1\right|\xi,\alpha\right\rangle^2} \\ &= \frac{\mathrm{e}^{-r}}{2}\end{aligned}$$

$$\begin{aligned}\Delta\hat{Y}_2 &= \sqrt{\left\langle\xi,\alpha\left|\hat{Y}_2^2\right|\xi,\alpha\right\rangle - \left\langle\xi,\alpha\left|\hat{Y}_2\right|\xi,\alpha\right\rangle^2} \\ &= \frac{\mathrm{e}^{r}}{2}\end{aligned}$$

2-9

> 求证：$\exp\left(\xi^*\hat{a}_A\hat{a}_B - \xi\hat{a}_A^\dagger\hat{a}_B^\dagger\right) = \frac{1}{\cosh r}\exp\left(-\Omega\hat{a}_A^\dagger\hat{a}_B^\dagger\right)\exp\left[\ln\cosh r\left(\hat{a}_A^\dagger\hat{a}_A + \hat{a}_B^\dagger\hat{a}_B\right)\right]\exp\left(\Omega^*\hat{a}_A\hat{a}_B\right)$

$$\begin{aligned}\left[\xi^*\hat{a}_A\hat{a}_B, -\xi\hat{a}_A^\dagger\hat{a}_B^\dagger\right] &= -\left|\xi\right|^2\left[\hat{a}_A\hat{a}_B, \hat{a}_A^\dagger\hat{a}_B^\dagger\right] \\ &= -\left|\xi\right|^2\left(\hat{a}_A\hat{a}_A^\dagger + \hat{a}_B^\dagger\hat{a}_B\right)\end{aligned}$$

$$\left[\xi^*\hat{a}_A\hat{a}_B, \hat{a}_A\hat{a}_A^\dagger + \hat{a}_B^\dagger\hat{a}_B\right] = 0$$

$$\left[-\xi\hat{a}_A^\dagger\hat{a}_B^\dagger, \hat{a}_A\hat{a}_A^\dagger + \hat{a}_B^\dagger\hat{a}_B\right] = 0$$

因此由 BCH 公式：

$$
\begin{aligned}
\exp\left(\xi^* \hat{a}_A \hat{a}_B - \xi \hat{a}_A^\dagger \hat{a}_B^\dagger\right) &= \exp\left(\xi^* \hat{a}_A \hat{a}_B\right) \exp\left(|\xi|^2 \left(\hat{a}_A \hat{a}_A^\dagger + \hat{a}_B^\dagger \hat{a}_B\right)/2\right) \exp\left(-\xi \hat{a}_A^\dagger \hat{a}_B^\dagger\right) \\
&= \frac{1}{\cosh r} \exp\left(-\Omega \hat{a}_A^\dagger \hat{a}_B^\dagger\right) \exp\left[\ln\cosh r \left(\hat{a}_A^\dagger \hat{a}_A + \hat{a}_B^\dagger \hat{a}_B\right)\right] \exp\left(\Omega^* \hat{a}_A \hat{a}_B\right)
\end{aligned}
$$

2-10

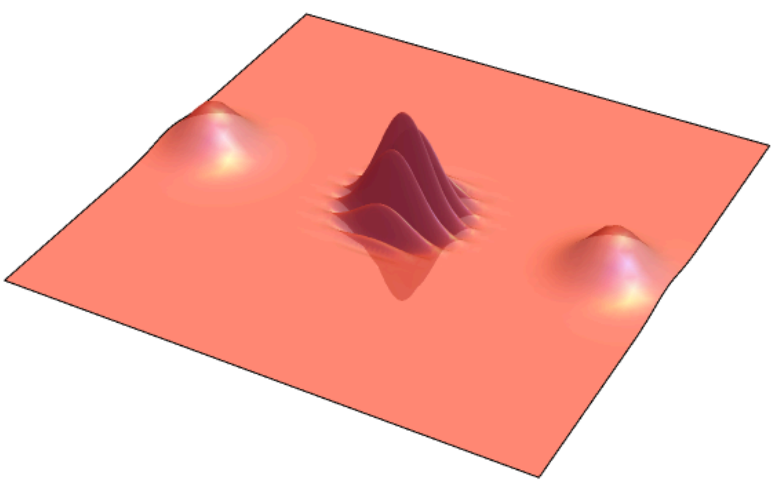
> 求薛定谔猫态 $|\Psi\rangle = \frac{1}{\sqrt{N}}\left(|\alpha_0\rangle + |-\alpha_0\rangle\right)$，其中 $N = 2\left(1 + \mathrm{e}^{-2|\alpha_0|^2}\right)$ 的 $W(\alpha, \alpha^*)$ 函数，并以 $\mathrm{Re}[\alpha]$ 和 $\mathrm{Im}[\alpha]$ 为两个水平轴，所得的 $W(\alpha, \alpha^*)$ 函数为竖直轴画出其三维图。

$$
\begin{aligned}
\rho &= |\Psi\rangle\langle\Psi| \\
&= \frac{1}{N}\left(|\alpha_0\rangle + |-\alpha_0\rangle\right)\left(\langle\alpha_0| + \langle-\alpha_0|\right)
\end{aligned}
$$

$$
\begin{aligned}
C_{\mathrm{W}}(\lambda) &= \mathrm{Tr}\left(\rho \hat{D}(\lambda)\right) \\
&= \mathrm{Tr}\left[\frac{1}{N}\left(|\alpha_0\rangle + |-\alpha_0\rangle\right)\left(\langle\alpha_0| + \langle-\alpha_0|\right)\hat{D}(\lambda)\right] \\
&= \frac{1}{N}\mathrm{Tr}\left[\left(\langle\alpha_0| + \langle-\alpha_0|\right)\hat{D}(\lambda)\left(|\alpha_0\rangle + |-\alpha_0\rangle\right)\right] \\
&= \frac{1}{N}\left(\langle\alpha_0| + \langle-\alpha_0|\right)\hat{D}(\lambda)\left(|\alpha_0\rangle + |-\alpha_0\rangle\right) \\
&= \frac{1}{N}\left\{\left(\langle\alpha_0| + \langle-\alpha_0|\right)\left(\mathrm{e}^{\mathrm{i}\Im(\lambda\alpha_0^*)}|\lambda+\alpha_0\rangle + \mathrm{e}^{-\mathrm{i}\Im(\lambda\alpha_0^*)}|\lambda-\alpha_0\rangle\right)\right\} \\
&= \frac{1}{N}\mathrm{e}^{-|\alpha_0|^2}\mathrm{e}^{-|\lambda|^2/2}\left[\mathrm{e}^{-\lambda^*\alpha_0}\left(\mathrm{e}^{-|\alpha_0|^2-\alpha_0^*\lambda} + \mathrm{e}^{|\alpha_0|^2+\alpha_0^*\lambda}\right) + \mathrm{e}^{\lambda^*\alpha_0}\left(\mathrm{e}^{|\alpha_0|^2-\alpha_0^*\lambda} + \mathrm{e}^{-|\alpha_0|^2+\alpha_0^*\lambda}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
W(\alpha) &= \frac{1}{\pi^2}\int C_{\mathrm{W}}(\lambda)\mathrm{e}^{\lambda^*\alpha-\lambda\alpha^*}\mathrm{d}^2\lambda \\
&= \frac{2}{\pi}\left[\mathrm{e}^{-2|\alpha-\alpha_0|^2} + \mathrm{e}^{-2|\alpha+\alpha_0|^2}\mathrm{e}^{-2|\alpha|^2}\left(\mathrm{e}^{-2(\alpha_0\alpha^*-\alpha\alpha_0^*)} + \mathrm{e}^{-2(-\alpha_0\alpha^*+\alpha\alpha_0^*)}\right)\right]
\end{aligned}
$$

Out[90]=



2-11

> 求相干压缩态 $|\Psi\rangle = \hat{S}(\xi)\hat{D}(\alpha_0)|0\rangle$ 的 $W(\alpha, \alpha^*)$ 函数，并以 $\mathrm{Re}[\alpha]$ 和 $\mathrm{Im}[\alpha]$ 为两个水平轴，所得的 $W(\alpha, \alpha^*)$ 函数为竖直轴画出其三维图。

$$
\rho = \hat{S}(\xi)\hat{D}(\alpha_0)|0\rangle\langle 0|\hat{D}^\dagger(\alpha_0)\hat{S}^\dagger(\xi)
$$

$$
\begin{aligned}
W(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int \mathrm{d}^2\lambda \mathrm{Tr}\left[\rho\hat{D}(\lambda)\right]\mathrm{e}^{-\lambda\alpha^*+\lambda^*\alpha} \\
&= \frac{1}{\pi^2}\int \mathrm{d}^2\lambda \left\langle 0\left|\hat{D}^\dagger(\alpha_0)\hat{S}^\dagger(\xi)\hat{D}(\lambda)\hat{S}(\xi)\hat{D}(\alpha_0)\right|0\right\rangle \mathrm{e}^{-\lambda\alpha^*+\lambda^*\alpha}
\end{aligned}
$$

利用公式：

$$
\hat{D}^\dagger(\alpha_0)\hat{D}(\lambda)\hat{D}(\alpha_0) = \hat{D}(\lambda)\mathrm{e}^{\lambda\alpha_0^*-\lambda^*\alpha_0}
$$

$$
\hat{S}^\dagger(\xi)\hat{D}(\lambda)\hat{S}(\xi) = \hat{D}\left(\lambda\cosh r+\lambda^*\mathrm{e}^{\mathrm{i}\theta}\sinh r\right)
$$

可得：

$$
\begin{aligned}
W(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int \mathrm{d}^2\lambda \left\langle 0\left|\hat{D}^\dagger(\alpha_0)\hat{S}^\dagger(\xi)\hat{D}(\lambda)\hat{S}(\xi)\hat{D}(\alpha_0)\right|0\right\rangle \mathrm{e}^{-\lambda\alpha^*+\lambda^*\alpha} \\
&= \frac{1}{\pi^2}\int \mathrm{d}^2\lambda \left\langle 0\left|\hat{D}\left(\lambda\cosh r+\lambda^*\mathrm{e}^{\mathrm{i}\theta}\sinh r\right)\right|0\right\rangle \mathrm{e}^{-\lambda(\alpha^*-\alpha_0^*)+\lambda^*(\alpha-\alpha_0)}
\end{aligned}
$$

作变量代换：

$$
\chi = \lambda\cosh r+\lambda^*\mathrm{e}^{\mathrm{i}\theta}\sinh r
$$

积分可化为：

$$
\begin{aligned}
W(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int \mathrm{d}^2\lambda \left\langle 0\left|\hat{D}\left(\lambda\cosh r+\lambda^*\mathrm{e}^{\mathrm{i}\theta}\sinh r\right)\right|0\right\rangle \mathrm{e}^{-\lambda(\alpha^*-\alpha_0^*)+\lambda^*(\alpha-\alpha_0)} \\
&= \frac{1}{\pi^2}\int \mathrm{d}^2\chi \left\langle 0\left|\hat{D}(\chi)\right|0\right\rangle \mathrm{e}^{-\chi A^*+\chi^* A} \\
&= \frac{2}{\pi}\mathrm{e}^{-2|A|^2} \\
&= \frac{2}{\pi}\mathrm{e}^{-2\left|(\alpha^*-\alpha_0^*)\mathrm{e}^{\mathrm{i}\theta}\sinh r+(\alpha-\alpha_0)\cosh r\right|^2}
\end{aligned}
$$

其中，

$$
A \equiv (\alpha^*-\alpha_0^*)\,\mathrm{e}^{\mathrm{i}\theta}\sinh r+(\alpha-\alpha_0)\cosh r
$$

2-12

> 从方程 $\hat{H} = \frac{1}{2m}\left[\hat{\vec{p}}-\frac{e}{c}\vec{A}(\vec{r},t)\right]^2+eU(\vec{r})+V(r)$ 出发，（1）推导偶极近似、二能级近似和旋波近似哈密顿量 $\hat{H}(t) = \frac{\hbar\omega_0}{2}\hat{\sigma}_z+\frac{\hbar\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}_+ + \mathrm{H.c.}\right)$（2）求二能级原子的布居反转的 Rabi 振荡，并用 Mathematica 软件画出结果。

$$
\hat{H} = \frac{1}{2m}\left[\hat{\vec{p}}-\frac{e}{c}\vec{A}(\vec{r},t)\right]^2+eU(\vec{r})+V(r)
$$

$$
\hat{\mathcal{U}} \equiv \exp\left(\frac{\mathrm{i}e}{\hbar c}\int_{\vec{r}_0}^{\vec{r}}\vec{A}(\vec{r}',t)\cdot\mathrm{d}\vec{r}'\right)
$$

$$
\begin{aligned}
\hat{\tilde{H}} &\equiv \hat{\mathcal{U}}^\dagger\hat{H}\hat{\mathcal{U}}-\mathrm{i}\hbar\hat{\mathcal{U}}^\dagger\dot{\hat{\mathcal{U}}} \\
&= \frac{\hat{\vec{p}}^2}{2m}+eU(\vec{r})+V(r)-\frac{e}{c}\int_{\vec{r}_0}^{\vec{r}}\vec{E}(\vec{r}',t)\cdot\mathrm{d}\vec{r}'
\end{aligned}
$$

偶极近似：

$$
\vec{A}(\vec{r},t) \approx \vec{A}(\vec{r}_0,t)
$$

二能级近似：

$$
\hat{\vec{p}}^2/2m+V(r) \approx E_e\left|e\right\rangle\left\langle e\right|+E_g\left|g\right\rangle\left\langle g\right| = \frac{\hbar\omega_0}{2}\hat{\sigma}_z,\ \ \hbar\omega_0 = E_e-E_g, \hat{\sigma_z} = \left|e\right\rangle\left\langle e\right|-\left|g\right\rangle\left\langle g\right|
$$

$$\hat{H} \approx \frac{\hbar\omega_0}{2}\hat{\sigma}_z - \frac{e}{c}\vec{r}\cdot\vec{E}(t) = \frac{\hbar\omega_0}{2}\hat{\sigma}_z - \frac{e}{c}\vec{1}_r\cdot\vec{E}(t)\left(r_{ge}\hat{\sigma}_+ + \mathrm{H.c.}\right)$$

$$\vec{E}(t) = \vec{1}_E E_0\cos(\omega t)$$

$$\begin{aligned}\hat{H} &= \frac{\hbar\omega_0}{2}\hat{\sigma}_z - \frac{e}{c}\vec{1}_r\cdot\vec{E}(t)\left(r_{ge}\hat{\sigma}_+ + \mathrm{H.c.}\right)\\ &= \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}\right)\left(\mathrm{e}^{\mathrm{i}\phi}\hat{\sigma}_+ + \mathrm{H.c.}\right)\end{aligned}$$

其中，

$$r_{ge} = \left|r_{ge}\right|\mathrm{e}^{\mathrm{i}\phi},\ \ \Omega = -\frac{e\vec{1}_r\cdot\vec{1}_E E_0\left|r_{ge}\right|}{\hbar c}$$

定义：

$$\hat{H}_0 = \frac{\hbar\omega_0}{2}\hat{\sigma}_z,\ \ \hat{H}_I = \hat{H} - \hat{H}_0$$

相互作用绘景：

$$\hat{H}_I^{(I)}(t) = \frac{\hbar\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}\right)\left(\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\omega_0 t}\hat{\sigma}_+ + \mathrm{H.c.}\right)$$

旋波近似，忽略快速振荡项 $\mathrm{e}^{\pm\mathrm{i}(\omega_0+\omega)t}$，

$$\begin{aligned}\hat{H}_I^{(I)}(t) &= \frac{\hbar\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}\right)\left(\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\omega_0 t}\hat{\sigma}_+ + \mathrm{H.c.}\right)\\ &\approx \frac{\bar{\Omega}}{2}\left[\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\omega_0 t}\hat{\sigma}_+ + \mathrm{H.c.}\right]\end{aligned}$$

回到薛定谔绘景：

$$\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}_+ + \mathrm{H.c}\right)$$

设 t 时刻体系的状态可展开为：

$$|\Psi(t)\rangle = c_g(t)\mathrm{e}^{\mathrm{i}\omega t/2}|g\rangle + c_e(t)\mathrm{e}^{-\mathrm{i}\omega_0 t/2}|e\rangle$$

由薛定谔方程可得：

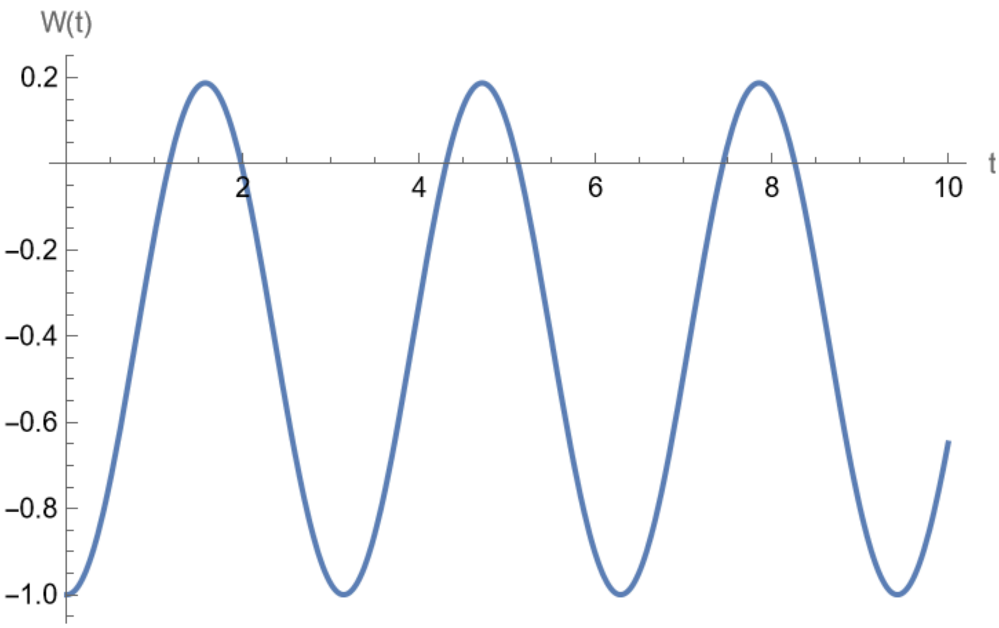
$$\mathrm{i}\dot{c}_e(t) = \frac{\Omega}{2}c_g(t)\mathrm{e}^{-\mathrm{i}\varDelta t},\ \ \mathrm{i}\dot{c}_g(t) = \frac{\Omega}{2}c_e(t)\mathrm{e}^{\mathrm{i}\varDelta t}$$

可以解得：

$$\begin{aligned}c_e(t) &= \mathrm{e}^{-\mathrm{i}\varDelta t/2}\left[c_e(0)\left(\cos\frac{\mu t}{2} + \frac{\mathrm{i}\varDelta}{\mu}\sin\frac{\mu t}{2}\right) - \mathrm{i}c_g(0)\frac{\Omega}{\mu}\sin\frac{\mu t}{2}\right]\\ c_g(t) &= \mathrm{e}^{\mathrm{i}\varDelta t/2}\left[c_g(0)\left(\cos\frac{\mu t}{2} - \frac{\mathrm{i}\varDelta}{\mu}\sin\frac{\mu t}{2}\right) - \mathrm{i}c_e(0)\frac{\Omega}{\mu}\sin\frac{\mu t}{2}\right]\end{aligned}$$

选择初始条件 $c_g(0) = 1, c_e(0) = 1$，布居反转为：

$$\begin{aligned}W(t) &= \langle\hat{\sigma}_z\rangle\\ &= \frac{\Omega^2 - \varDelta^2}{\mu^2}\sin^2\frac{\mu t}{2} - \cos^2\frac{\mu t}{2}\end{aligned}$$



2-13

> 在光与物质相互作用的全量子描述中，光的初态为相干态时，（1）求二能级原子布居反转的动力学演化；（2）用 Mathematica 画出你的结果；（3）与问题 12 得到的 Rabi 振荡相对比，体会量子光-物质相互作用的效应。

设体系状态可展为：

$$|\tilde{\Psi}_{n+1}(t)\rangle = c_{n+1,g}(t)\mathrm{e}^{-\mathrm{i}[(n+1)\omega-\omega_0/2]t}\,|n+1,g\rangle + c_{n,e}(t)\mathrm{e}^{-\mathrm{i}[n\omega+\omega_0/2]}\,|n,e\rangle$$

代入薛定谔方程可得：

$$\mathrm{i}\dot{c}_{n+1,g}(t) = g\sqrt{n+1}\mathrm{e}^{-\mathrm{i}\varDelta t}c_{n,e}(t)$$
$$\mathrm{i}\dot{c}_{n,e}(t) = g\sqrt{n+1}\mathrm{e}^{\mathrm{i}\varDelta t}c_{n+1,g}(t)$$

解得：

$$c_{n+1,g}(t) = \mathrm{e}^{-\mathrm{i}\varDelta t/2}\left[c_{n+1,g}(0)\left(\cos\frac{\Omega_n(\varDelta)t}{2} + \mathrm{i}\frac{\varDelta\sin\frac{\Omega_n(\varDelta)t}{2}}{\Omega_n(\varDelta)}\right) + c_{n,e}(0)\frac{-2\mathrm{i}g\sqrt{n+1}\sin\frac{\Omega_n(\varDelta t)}{2}}{\Omega_n(\varDelta)}\right]$$
$$c_{n,e}(t) = \mathrm{e}^{\mathrm{i}\varDelta t/2}\left[c_{n+1,g}(0)\frac{-2\mathrm{i}g\sqrt{n+1}\sin\frac{\Omega_n(\varDelta t)}{2}}{\Omega_n(\varDelta)} + c_{n,e}(0)\left(\cos\frac{\Omega_n(\varDelta)t}{2} - \mathrm{i}\frac{\varDelta\sin\frac{\Omega_n(\varDelta)t}{2}}{\Omega_n(\varDelta)}\right)\right]$$

布居反转：

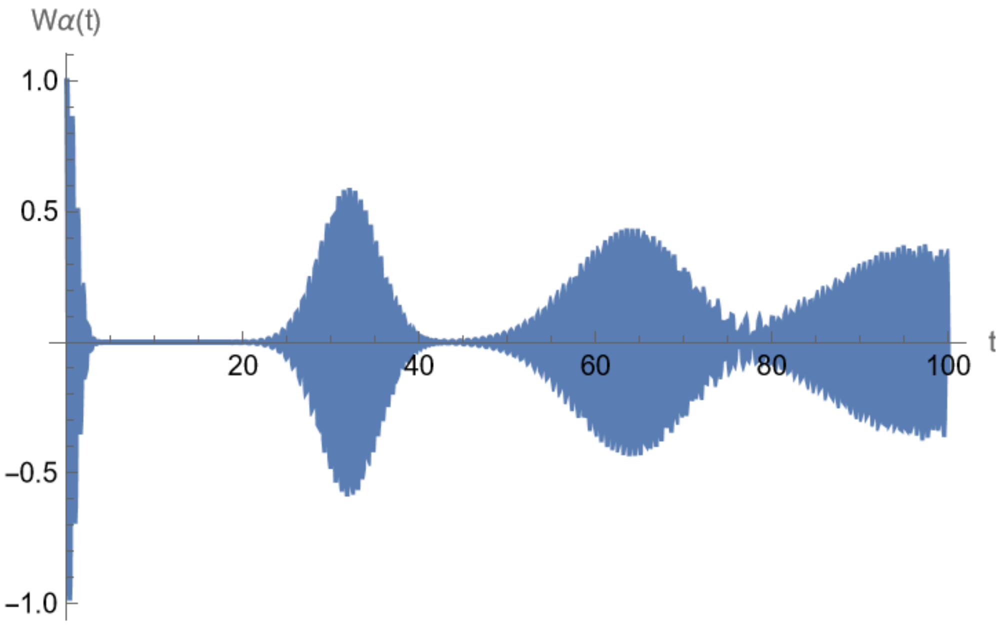
$$W(t) = \langle\Psi_{n+1}(t)\,|\,\hat{\sigma}_z\,|\,\Psi_{n+1}(t)\rangle = |c_{n,e}(t)|^2 - |c_{n+1,g}(t)|^2$$

当初态为相干态，即：

$$|\Psi(0)\rangle = |\alpha,e\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n,e\rangle$$
$$\Psi(t) = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|\tilde{\Psi}_{n+1}(t)\rangle$$

布居反转：

$$
\begin{aligned}
W_\alpha(t) &= \mathrm{e}^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}W_n(t)}{n!} \\
&= \mathrm{e}^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\left[\frac{\mathit{\Delta}^2}{\Omega_n^2(\mathit{\Delta})}+\frac{4g^2(n+1)}{\Omega_n^2(\mathit{\Delta})}\cos\left[\Omega_n(\mathit{\Delta})t\right]\right],\quad \Omega_n(\mathit{\Delta})=\sqrt{\mathit{\Delta}^2+4g^2(n+1)}
\end{aligned}
$$



2-14

> 在大 ω_0/ω 极限下研究量子 Rabi 模型，并用 Mathematica 画出你的结果（1）数值求解其本征能量随 g/ω 的变化；（2）数值求解基态能量 $E_g, \dfrac{\mathrm{d}E_g}{\mathrm{d}g}, \dfrac{\mathrm{d}^2E_g}{\mathrm{d}g^2}$ 随 g/ω 的变化，揭示在 $g=\sqrt{\omega\omega_0}/2$ 处发生量子相变。

量子 Rabi 模型哈密顿量:

$$
\hat{H}=\omega\hat{a}^\dagger\hat{a}+\frac{\omega_0}{2}\hat{\sigma}_z-g\left(\hat{a}+\hat{a}^\dagger\right)\hat{\sigma}_x
$$

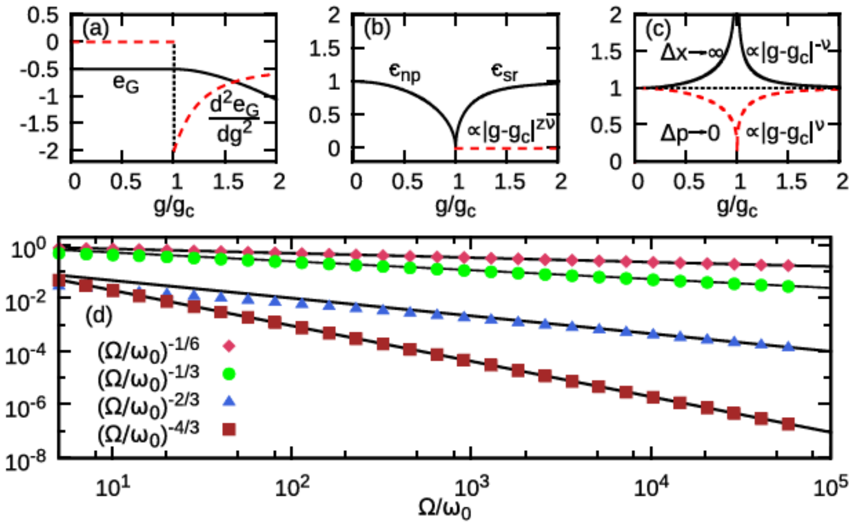
其中，ω 是光场圆频率，ω_0 是跃迁圆频率，g 是耦合强度。

当:

$$
\frac{2g}{\sqrt{\omega\omega_0}}=1
$$

时，发生量子相变，即:

$$
g=\frac{\sqrt{\omega\omega_0}}{2}
$$



(a)
e_G
$\frac{d^2e_G}{dg^2}$
g/g_c
(b)
ϵ_{np}
ϵ_{sr}
$\propto|g-g_c|^{z\bar{\nu}}$
g/g_c
(c)
$\Delta x \to \infty$
$\propto|g-g_c|^{-\nu}$
$\Delta p \to 0$
$\propto|g-g_c|^{\nu}$
g/g_c
(d)
$(\Omega/\omega_0)^{-1/6}$
$(\Omega/\omega_0)^{-1/3}$
$(\Omega/\omega_0)^{-2/3}$
$(\Omega/\omega_0)^{-4/3}$
Ω/ω_0