1(a)

选取 α, β 为广义坐标

以 O 为原点，过 O 竖直向下为 x 轴正方向，过 O 水平向右为 y 轴正方向建系

各质点的位矢：

$$\begin{cases} \vec{r}_1 = l\cos\alpha\vec{e}_x + l\sin\alpha\vec{e}_y \\ \vec{r}_2 = l\cos\alpha\vec{e}_x - l\sin\alpha\vec{e}_y \\ \vec{r}_3 = (l\cos\alpha - l\sin\alpha\cdot\dfrac{1}{\tan\beta})\vec{e}_x \end{cases}$$

各质点位矢对各广义坐标的偏导：

$$\begin{cases} \dfrac{\partial\vec{r}_1}{\partial\alpha} = -l\sin\alpha\vec{e}_x + l\cos\alpha\vec{e}_y \\ \dfrac{\partial\vec{r}_2}{\partial\alpha} = -l\sin\alpha\vec{e}_x - l\cos\alpha\vec{e}_y \\ \dfrac{\partial\vec{r}_3}{\partial\alpha} = (-l\sin\alpha - l\cos\alpha\cdot\dfrac{1}{\tan\beta})\vec{e}_x \end{cases}$$

$$\begin{cases} \dfrac{\partial\vec{r}_1}{\partial\beta} = \vec{0} \\ \dfrac{\partial\vec{r}_2}{\partial\beta} = \vec{0} \\ \dfrac{\partial\vec{r}_3}{\partial\beta} = (l\sin\alpha\cdot\dfrac{1}{\sin^2\beta})\vec{e}_x \end{cases}$$

主动力：

$$\begin{cases} \vec{F}_1^{(A)} = (mg + N\cos\beta)\vec{e}_x + N\sin\beta\vec{e}_y \\ \vec{F}_2^{(A)} = (mg + N\cos\beta)\vec{e}_x - N\sin\beta\vec{e}_y \\ \vec{F}_3^{(A)} = (mg - 2N\cos\beta)\vec{e}_x \end{cases}$$

广义力：

$$\begin{aligned} Q_\alpha &\equiv \vec{F}_i^{(A)}\cdot\frac{\partial\vec{r}_i}{\partial\alpha} \\ &= -3mgl\sin\alpha + 2lN\cos\alpha\sin\beta - \frac{mgl\cos\alpha}{\tan\beta} + \frac{2lN\cos\alpha\cos\beta}{\tan\beta} \end{aligned}$$

$$\begin{aligned} Q_\beta &\equiv \vec{F}_i^{(A)}\cdot\frac{\partial\vec{r}_i}{\partial\beta} \\ &= \frac{l\sin\alpha}{\sin^2\beta}\cdot(mg - 2N\cos\beta) \end{aligned}$$

虚功原理要求：

$$\begin{cases} Q_\alpha = 0 \\ Q_\beta = 0 \end{cases}$$

由 $Q_\beta = 0$ 得到：

$$N = \frac{mg}{2\cos\beta}$$

代入方程 $Q_\alpha = 0$ 得到：

$$\tan\beta = 3\tan\alpha$$

1(b)

利用 $\tan\beta = 3\tan\alpha$：

$$
\begin{cases}
\vec{r}_1 = l\cos\alpha\vec{e}_x + l\sin\alpha\vec{e}_y \\
\vec{r}_2 = l\cos\alpha\vec{e}_x - l\sin\alpha\vec{e}_y \\
\vec{r}_3 = (l\cos\alpha - l\sin\alpha\cdot\dfrac{1}{\tan\beta})\vec{e}_x = \dfrac{2l\cos\alpha}{3}\vec{e}_x
\end{cases}
$$

于是:

$$
\begin{cases}
\dot{\vec{r}}_1 = -l\dot{\alpha}\sin\alpha\vec{e}_x + l\dot{\alpha}\cos\alpha\vec{e}_y \\
\dot{\vec{r}}_2 = -l\dot{\alpha}\sin\alpha\vec{e}_x - l\dot{\alpha}\cos\alpha\vec{e}_y \\
\dot{\vec{r}}_3 = -\dfrac{2l\dot{\alpha}\sin\alpha}{3}\vec{e}_x
\end{cases}
$$

取原点 O 所在平面为零势能面，则:

$$
\begin{aligned}
L &\equiv T - V \\
&= \frac{1}{2}m(\dot{\vec{r}}_1^2 + \dot{\vec{r}}_2^2 + \dot{\vec{r}}_3^2) - mg\left[2\cdot l\cos\alpha + \frac{2l\cos\alpha}{3}\right] \\
&= ml^2\dot{\alpha}^2(1 + \frac{2\sin^2\alpha}{9}) - \frac{8}{3}mgl\cos\alpha
\end{aligned}
$$

欧拉-拉格朗日方程:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\alpha}} - \frac{\partial L}{\partial\alpha} = 0
$$

即:

$$
(2 + \frac{4\sin^2\alpha}{9})l\ddot{\alpha} + (\frac{8l}{9}\sin\alpha\cos\alpha)\dot{\alpha}^2 - \frac{8}{3}g\sin\alpha = 0
$$

2

选取质心坐标 x_c, y_c 作为广义坐标

$$
\vec{r}_c = x_c\vec{e}_x + y_c\vec{e}_y
$$

$$
\delta\vec{r}_c = \delta x_c\vec{e}_x + \delta y_c\vec{e}_y
$$

主动力:

$$
\vec{F}^{(A)} = -mg\vec{e}_y
$$

理想约束下的虚功:

$$
\delta W = \vec{F}^{(A)}\cdot\delta\vec{r}_c = -mg\delta y_c
$$

虚功定理要求:

$$
\delta W = 0
$$

即:

$$
-mg\delta y_c = 0
$$

得到:

$$
y_c = \mathrm{const}
$$

由初始状态杆贴墙知:

$$
y_c = \frac{L}{2}
$$

设 $B(x,y)$，由几何关系知:

$$
\begin{cases}
x = L\sin\theta \\
y = \dfrac{L}{2} - \dfrac{L}{2}\cos\theta
\end{cases}
$$

利用 $\sin^2\theta + \cos^2\theta = 1$ 消去 θ 得:

$$(\frac{x}{L})^2+(\frac{2y}{L}-1)^2=1$$

这就是杆下端 B 点所在约束面的形状

3

设 $\vec{r}_i=\vec{r}_i(q_1,q_2,\cdots,q_s;t)$，由链式法则，有：

$$\dot{\vec{r}}_i=\frac{\partial\vec{r}_i}{\partial t}+\sum_{\alpha=1}^{s}\frac{\partial\vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha$$

于是：

$$\frac{\partial\dot{\vec{r}}_i}{\partial\dot{q}_\alpha}=\frac{\partial\vec{r}_i}{\partial q_\alpha}\tag{1}$$

其中，$\dot{\vec{r}}_i=\dot{\vec{r}}_i(q_1,q_2,\cdots,q_s;\dot{q}_1,\dot{q}_2,\cdots,\dot{q}_s;t)$

注意到，$\frac{\partial\vec{r}_i}{\partial q_\alpha}=\frac{\partial\vec{r}_i}{\partial q_\alpha}(q_1,q_2,\cdots,q_\alpha;t)$，于是：

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\vec{r}_i}{\partial q_\alpha}&=\frac{\partial}{\partial t}(\frac{\partial\vec{r}_i}{\partial q_\alpha})+\sum_{\beta=1}^{s}\left[\frac{\partial}{\partial q_\beta}(\frac{\partial\vec{r}_i}{\partial q_\alpha})\right]\dot{q}_\beta\\
&=\frac{\partial}{\partial q_\alpha}(\frac{\partial\vec{r}_i}{\partial t})+\frac{\partial}{\partial q_\alpha}\sum_{\beta=1}^{s}\frac{\partial\vec{r}_i}{\partial q_\beta}\dot{q}_\beta\\
&=\frac{\partial}{\partial q_\alpha}\left[\frac{\partial\vec{r}_i}{\partial t}+\sum_{\beta=1}^{s}\frac{\partial\vec{r}_i}{\partial q_\beta}\dot{q}_\beta\right]\\
&=\frac{\partial}{\partial q_\alpha}(\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t})
\end{aligned}$$

即：

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\vec{r}_i}{\partial q_\alpha}=\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}\tag{2}$$

于是：

$$\begin{aligned}
\sum_{i=1}^{N}m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i&=\sum_{i=1}^{N}m_i\ddot{\vec{r}}_i\cdot\sum_{\alpha=1}^{s}\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha\\
&=\sum_{i=1}^{N}\sum_{\alpha=1}^{s}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha\\
&=\sum_{\alpha=1}^{s}\sum_{i=1}^{N}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\delta q_\alpha\\
&=\sum_{\alpha=1}^{s}\delta q_\alpha\sum_{i=1}^{N}m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\\
&=\sum_{\alpha=1}^{s}\delta q_\alpha\sum_{i=1}^{N}\frac{\mathrm{d}(m_i\dot{\vec{r}}_i)}{\mathrm{d}t}\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha}\\
&=\sum_{\alpha=1}^{s}\delta q_\alpha\sum_{i=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t}(m_i\dot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_\alpha})-m_i\dot{\vec{r}}_i\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\vec{r}_i}{\partial q_\alpha}\right]\\
[(1)(2)\text{代入}]&=\sum_{\alpha=1}^{s}\delta q_\alpha\sum_{i=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t}(m_i\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial\dot{q}_\alpha})-m_i\dot{\vec{r}}_i\cdot\frac{\partial\dot{\vec{r}}_i}{\partial q_\alpha}\right]\\
&=\sum_{\alpha=1}^{s}\delta q_\alpha\sum_{i=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\frac{1}{2}m_i\dot{\vec{r}}_i^2)}{\partial\dot{q}_\alpha}-\frac{\partial(\frac{1}{2}m_i\dot{\vec{r}}_i^2)}{\partial q_\alpha}\right]\\
&=\sum_{\alpha=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{q}_\alpha}-\frac{\partial T}{\partial q_\alpha}\right]\delta q_\alpha
\end{aligned}$$

其中，$T\equiv\sum\limits_{i=1}^{N}\frac{1}{2}m_i\dot{\vec{r}}_i^2$

4

取图中的 x,y 为广义坐标，设斜面高度为 h

各质点坐标：

$$\begin{cases}\vec{r}_1=(x+y\cos\theta)\vec{e}_x+(h-y\sin\theta)\vec{e}_y\\ \vec{r}_2=x\vec{e}_x\end{cases}$$

各质点速度：

$$\begin{cases}\dot{\vec{r}}_1=(\dot{x}+\dot{y}\cos\theta)\vec{e}_x-\dot{y}\sin\theta\vec{e}_y\\ \dot{\vec{r}}_2=\dot{x}\vec{e}_x\end{cases}$$

以斜面顶端水平面为零势能面，有：

$$\begin{aligned}L&\equiv T-V\\ &=\frac{1}{2}m_1\dot{\vec{r}}_1^2+\frac{1}{2}m_2\dot{\vec{r}}_2^2-m_1gy\sin\theta\\ &=\frac{1}{2}(m_1+m_2)\dot{x}^2+\frac{1}{2}m_1\dot{y}^2+m_1\cos\theta\cdot\dot{x}\dot{y}-m_1g\sin\theta\cdot y\end{aligned}$$

欧拉-拉格朗日方程：

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=0$$

代入得：

$$(m_1+m_2)\ddot{x}+m_1\cos\theta\cdot\ddot{y}=0$$

$$m_1\ddot{y}+m_1\cos\theta\cdot\ddot{x}-m_1g\sin\theta=0$$

解得：

$$\begin{aligned}\ddot{x}&=\frac{m_1g\sin\theta\cos\theta}{-m_1\sin^2\theta-m_2}\\ \ddot{y}&=\frac{(m_1+m_2)g\sin\theta}{m_1\sin^2\theta+m_2}\end{aligned}$$

斜面加速度：

$$\vec{a}_2\equiv\ddot{\vec{r}}_2=\frac{m_1g\sin\theta\cos\theta}{-m_1\sin^2\theta-m_2}\vec{e}_x$$

方块加速度：

$$\vec{a}_1\equiv\ddot{\vec{r}}_1=\frac{m_2g\sin\theta\cos\theta}{m_1\sin^2\theta+m_2}\vec{e}_x-\frac{m_1g\sin^2\theta\cos\theta}{m_1\sin^2\theta-m_2}\vec{e}_y$$

方块相对斜面的加速度：

$$\vec{a}_{12}=\vec{a}_1-\vec{a}_2=\frac{(m_1+m_2)g\sin\theta\cos\theta}{m_1\sin^2\theta+m_2}\vec{e}_x-\frac{m_1g\sin^2\theta\cos\theta}{m_1\sin^2\theta-m_2}\vec{e}_y$$

5

$$\begin{aligned}\vec{a}_1&=(\ddot{x}+\ddot{y}\cos\theta)\vec{e}_x-\ddot{y}\sin\theta\vec{e}_y\\ \vec{a}_2&=\ddot{x}\vec{e}_x\end{aligned}$$

$$\begin{aligned}\vec{F}_1^{(A)}&=-m_1g\vec{e}_y\\ \vec{F}_2^{(A)}&=-m_2g\vec{e}_y\end{aligned}$$

$$
\begin{aligned}
Z &\equiv \frac{1}{2}\sum_i m_i(\vec{a}_i - \frac{\vec{F}_i^{(A)}}{m_i})^2 \\
&= \frac{m_1+m_2}{2}\ddot{x}^2 + \frac{m_1}{2}\ddot{y}^2 + m_1\cos\theta\cdot\ddot{x}\ddot{y} - m_1 g\sin\theta\cdot\ddot{y} + \frac{m_1+m_2}{2}g^2
\end{aligned}
$$

高斯原理要求:

$$
\frac{\partial Z}{\partial \ddot{x}} = 0
$$
$$
\frac{\partial Z}{\partial \ddot{y}} = 0
$$

即:

$$
\begin{aligned}
(m_1+m_2)\ddot{x} + m_1\cos\theta\cdot\ddot{y} = 0 \\
m_1\ddot{y} + m_1\cos\theta\cdot\ddot{x} - m_1 g\sin\theta = 0
\end{aligned}
$$

解得:

$$
\ddot{x} = \frac{m_1 g\sin\theta\cos\theta}{-m_1\sin^2\theta - m_2}
$$

$$
\ddot{y} = \frac{(m_1+m_2)g\sin\theta}{m_1\sin^2\theta + m_2}
$$

6

齐次方程的解

运动方程对应的齐次方程 $\ddot{x}+\lambda\dot{x}+\omega^2x=0$ 的特征方程为:

$$
r^2+\lambda r+\omega^2=0
$$

1）若 $\lambda^2-4\omega^2=0$:

特征根为:

$$
r_{1,2} = -\frac{\lambda}{2}
$$

齐次方程 $\ddot{x}+\lambda\dot{x}+\omega^2x=0$ 的通解为:

$$
X(t) = e^{-\frac{\lambda}{2}t}(C_1+C_2t)
$$

2）若 $\lambda^2-4\omega^2>0$:

特征根为:

$$
r_{1,2} = \frac{-\lambda\pm\sqrt{\lambda^2-4\omega^2}}{2}
$$

齐次方程 $\ddot{x}+\lambda\dot{x}+\omega^2x=0$ 的通解为:

$$
X(t) = C_1e^{r_1t}+C_2e^{r_2t}
$$

3）若 $\lambda^2-4\omega^2<0$:

特征根为:

$$
r_{1,2} = \frac{-\lambda\pm \mathrm{i}\sqrt{4\omega^2-\lambda^2}}{2}
$$

齐次方程 $\ddot{x}+\lambda\dot{x}+\omega^2x=0$ 的通解为:

$$
X(t) = e^{-\frac{\lambda}{2}t}(C_1\cos\frac{\sqrt{4\omega^2-\lambda^2}}{2}t + C_2\sin\frac{\sqrt{4\omega^2-\lambda^2}}{2}t)
$$

利用辅助角公式可以将 $X(t)$ 改写为:

$$
X(t) = C_3e^{-\frac{\lambda}{2}t}\cos(\frac{\sqrt{4\omega^2-\lambda^2}}{2}t+\phi)
$$

其中，$C_3=\sqrt{C_1^2+C_2^2}, \cos\phi = \frac{C_1}{\sqrt{C_1^2+C_2^2}}, \sin\phi = -\frac{C_2}{\sqrt{C_1^2+C_2^2}}$

非齐次方程的形式特解

设方程 $\ddot{x}+\lambda\dot{x}+\omega^2x=F\cos\Omega t$ 的形式特解为：

$$\chi(t)=A\cos\Omega t+B\sin\Omega t$$

则：

$$\dot{\chi}(t)=\Omega B\cos\Omega t-\Omega A\sin\Omega t$$

$$\ddot{\chi}(t)=-\Omega^2A\cos\Omega t-\Omega^2B\sin\Omega t$$

代入方程 $\ddot{x}+\lambda\dot{x}+\omega^2x=F\cos\Omega t$ 得：

$$[-\Omega^2A+\lambda\Omega B+\omega^2A]\cos\Omega t+[-\Omega^2B-\lambda\Omega A+\omega^2B]\sin\Omega t=F\cos\Omega t$$

对应项系数相等，得到：

$$-\Omega^2A+\lambda\Omega B+\omega^2A=F$$
$$-\Omega^2B-\lambda\Omega A+\omega^2B=0$$

解得：

$$A=\frac{(\omega^2-\Omega^2)F}{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}$$

$$B=\frac{\lambda\Omega F}{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}$$

于是形式特解为：

$$\chi(t)=\frac{(\omega^2-\Omega^2)F}{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}\cos\Omega t+\frac{\lambda\Omega F}{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}\sin\Omega t$$

利用辅助角公式，可以把 $\chi(t)$ 化为：

$$\chi(t)=\frac{F}{\sqrt{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}}\cos(\Omega t+\varphi)$$

其中，$\cos\varphi=\frac{\omega^2-\Omega^2}{\sqrt{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}},\sin\varphi=-\frac{\lambda\Omega}{\sqrt{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}}$

原方程的解为：

$$\begin{aligned}x(t)&=X(t)+\chi(t)\\&=\begin{cases}e^{-\frac{\lambda}{2}t}(C_1+C_2t)+\dfrac{F}{\sqrt{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}}\cos(\Omega t+\varphi) & ,\lambda^2=4\omega^2\\C_1e^{r_1t}+C_2e^{r_2t}+\dfrac{F}{\sqrt{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}}\cos(\Omega t+\varphi) & ,\lambda^2>4\omega^2\\C_3e^{-\frac{\lambda}{2}t}\cos(\dfrac{\sqrt{4\omega^2-\lambda^2}}{2}t+\phi)+\dfrac{F}{\sqrt{\lambda^2\Omega^2+(\omega^2-\Omega^2)^2}}\cos(\Omega t+\varphi) & ,\lambda^2<4\omega^2\end{cases}\end{aligned}$$

1）当 $\lambda\to0$ 时，振幅不随时间发生变化

注意到：

$$\lambda^2\Omega^2+(\omega^2-\Omega^2)^2=(\Omega^2-\frac{2\omega^2-\lambda^2}{2})^2+\lambda^2\omega^2-\frac{\lambda^4}{4}$$

于是当 $\Omega=\sqrt{\omega^2-\frac{\lambda^2}{2}}\approx\omega$ 时发生共振，

2）当 $\Omega\to\omega$ 时，振幅随时间的增加而减小，当时间 $t\to+\infty$ 时振幅保持不变，其值为：

$$A=\frac{F}{\lambda\omega}$$

3）当$\lambda=0,\Omega=\omega$ 时，振幅不随时间改变，且发生共振，振幅为：

$$A=+\infty$$

7

以 O 为原点，过 O 竖直向下为 x 轴正方向，过 O 水平向右为 y 轴正方向建系

能量守恒：

$$\frac{1}{2}mv^2 = mgx \Longrightarrow v = \sqrt{2gx}$$

而：

$$v \equiv \frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{1+(\frac{\mathrm{d}x}{\mathrm{d}y})^2}\frac{\mathrm{d}y}{\mathrm{d}t}$$

于是：

$$\mathrm{d}t = \sqrt{\frac{1+x'^2}{2gx}}\mathrm{d}y$$

积分得：

$$t = \frac{1}{\sqrt{2g}}\int_0^y \sqrt{\frac{1+x'^2}{x}}\mathrm{d}y$$

最小作用量原理：

$$S \equiv \int_{t_0}^t L(q,\dot{q},t)\mathrm{d}t$$

S 取极小值要求 L 满足：

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0$$

类比可知，t 取极小值要求：

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial}{\partial x'}\sqrt{\frac{1+x'^2}{x}} - \frac{\partial}{\partial x}\sqrt{\frac{1+x'^2}{x}} = 0$$

整理得：

$$x'^2 + 2xx'' + 1 = 0$$

上面等式等号左右两边同乘 x' 得：

$$x'\cdot(1+x'^2) + x\cdot(2x'x'') = 0$$

即：

$$(1+x'^2)\cdot\frac{\mathrm{d}}{\mathrm{d}y}(x) + x\cdot\frac{\mathrm{d}}{\mathrm{d}y}(1+x'^2) = 0$$

即：

$$\frac{\mathrm{d}}{\mathrm{d}y}[x(1+x'^2)] = 0$$

于是：

$$x(1+x'^2) = C_1$$

即：

$$\mathrm{d}y = \sqrt{\frac{x}{C_1 - x}}\mathrm{d}x$$

积分得：

$$y = \int\sqrt{\frac{x}{C_1-x}}\mathrm{d}x$$

令 $x = C_1\sin^2\theta$，则：

$$\begin{aligned}\int\sqrt{\frac{x}{C_1-x}}\mathrm{d}x &= 2C_1\int\sin^2\theta\mathrm{d}\theta \\ &= C_1\theta-\frac{C_1}{2}\sin 2\theta+C_2 \\ &= C_1\arcsin\sqrt{\frac{x}{C_1}}-C_1\sqrt{x(C_1-x)}+C_3\end{aligned}$$

于是:

$$y=C_1\arcsin\sqrt{\frac{x}{C_1}}-C_1\sqrt{x(C_1-x)}+C_4$$

当 $x=0$ 时，$y=0$，于是 $C_4=0$

设轨道下降终点坐标为 $x=a,y=b$，则曲线方程可表达为:

$$y=C_1\arcsin\sqrt{\frac{x}{C_1}}-C_1\sqrt{x(C_1-x)}$$

其中，C_1 满足:

$$b=C_1\arcsin\sqrt{\frac{a}{C_1}}-C_1\sqrt{a(C_1-a)}$$

8

设球面半径为 R

$$\begin{cases}x=R\sin\theta\cos\varphi \\ y=R\sin\theta\sin\varphi \\ z=R\cos\theta\end{cases}$$

$$\begin{aligned}\mathrm{d}s &\equiv \sqrt{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2} \\ &= R\sqrt{\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2} \\ &= R\sqrt{1+\sin^2\theta(\frac{\mathrm{d}\varphi}{\mathrm{d}\theta})^2}\mathrm{d}\theta \\ &= R\sqrt{1+\sin^2\theta\varphi'^2}\mathrm{d}\theta\end{aligned}$$

设球面上两点 A,B 坐标的球坐标描述为:

$$\theta_A=0$$

$$\theta_B=\Theta,\varphi_B=0$$

其中，$\Theta\in[0,\pi]$ 是常数

球面上连结 A,B 的曲线方程设为 $\varphi=\varphi(\theta)$，其长度为:

$$\begin{aligned}S &= \int\mathrm{d}s \\ &= R\int_0^{\Theta}\sqrt{1+\sin^2\theta\varphi'^2}\mathrm{d}\theta\end{aligned}$$

S 取极小值要求:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial\sqrt{1+\sin^2\theta\varphi'^2}}{\partial\varphi'}-\frac{\partial\sqrt{1+\sin^2\theta\varphi'^2}}{\partial\varphi}=0$$

即:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial\sqrt{1+\sin^2\theta\varphi'^2}}{\partial\varphi'}=0$$

于是:

$$\frac{\partial\sqrt{1+\sin^2\theta\varphi'^2}}{\partial\varphi'}=C_1$$

即:

$$\frac{\varphi' \sin^2 \theta}{\sqrt{1+\varphi' \sin^2 \theta}} = C_1$$

构造函数:

$$f(x) = \frac{x}{\sqrt{1+x}}$$

其在定义域上单调，故

$$\frac{\varphi' \sin^2 \theta}{\sqrt{1+\varphi' \sin^2 \theta}} = C_1$$

当且仅当 $\varphi' \sin^2 \theta = C_2$

曲线过 A 点，而 $\theta_A = 0$，于是:

$$C_2 = 0$$

θ 是变量，$\varphi' \sin^2 \theta$ 恒为零，则:

$$\varphi' = 0$$

即:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\theta} = 0$$

这就是说，A, B 两点之间的短程线是大圆在两点间的劣弧段