3-1

> 重复量子密集编码的过程。

分立变量系统

四个 Bell 态：

$$|\Phi^{\pm}\rangle=\frac{1}{\sqrt{2}}(|00\rangle\pm|11\rangle)$$

$$|\Psi^{\pm}\rangle=\frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle)$$

假设 Alice 和 Bob 共享一对光子纠缠态 $|\Phi^{+}\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$

注意到：

$$\hat{I}^{A}|\Phi^{+}\rangle=|\Phi^{+}\rangle$$

$$\hat{\sigma}_{x}^{A}|\Phi^{+}\rangle=\hat{\sigma}_{x}^{A}\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)=\frac{1}{\sqrt{2}}(|10\rangle+|01\rangle)=|\Psi^{+}\rangle$$

$$\hat{\sigma}_{z}^{A}|\Phi^{+}\rangle=\hat{\sigma}_{z}^{A}\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)=\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)=|\Phi^{-}\rangle$$

$$\mathrm{i}\hat{\sigma}_{y}^{A}|\Phi^{+}\rangle=\mathrm{i}\hat{\sigma}_{y}^{A}\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)=\frac{1}{\sqrt{2}}(-|10\rangle+|01\rangle)=|\Psi^{-}\rangle$$

Alice 想告诉 Bob 的信息	Alice 让 Bob 对他手上的光子进行的操作	操作后整个系统的状态
00	$\hat{I}^{A}$	$\|\Phi^{+}\rangle$
10	$\hat{\sigma}_{x}^{A}$	$\|\Psi^{+}\rangle$
01	$\hat{\sigma}_{z}^{A}$	$\|\Phi^{-}\rangle$
11	$\mathrm{i}\hat{\sigma}_{y}^{A}$	$\|\Psi^{-}\rangle$

- Alice 将她的光子发送给 Bob
- Bob 通过如下的 Bell 态测量读取 Alice 想传递给他的信息：
- - 施加控制非门：

$$\hat{C}_{\mathrm{NOT}}|\Phi^{\pm}\rangle=\hat{C}_{\mathrm{NOT}}\frac{1}{\sqrt{2}}(|00\rangle\pm|11\rangle)=\frac{1}{\sqrt{2}}(|00\rangle\pm|10\rangle)=\frac{1}{\sqrt{2}}(|0\rangle\pm|1\rangle)|0\rangle$$

$$\hat{C}_{\mathrm{NOT}}|\Psi^{\pm}\rangle=\hat{C}_{\mathrm{NOT}}\frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle)=\frac{1}{\sqrt{2}}(|01\rangle\pm|11\rangle)=\frac{1}{\sqrt{2}}(|0\rangle\pm|1\rangle)|1\rangle$$

- - 对 $\hat{\sigma}_{z}^{B}$ 测量可区分 $|\Phi\rangle$ 和 $|\Psi\rangle$：测得 $+1$ 对应 $|\Phi^{\pm}\rangle$，测得 -1 对应 $|\Psi^{\pm}\rangle$
- - 对 A 施加 Hadmard 门：

$$H^{A}\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)=\frac{1}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}+\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)=|0\rangle$$

$$H^{A}\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)=\frac{1}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}-\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)=|1\rangle$$

- - 对 $\hat{\sigma}_{z}^{A}$ 测量可区分 $\pm$ 分量。

连续变量系统

- 制备双模压缩真空态并传递给 Alice 和 Bob

$$|\Psi_{AB}\rangle = \frac{\exp\left(\tanh r \hat{a}^\dagger \hat{b}^\dagger\right)}{\cosh r}|00\rangle$$

- 假设要传递的信息是一个复数 $\mu_0 = (x_0 + \mathrm{i}p_0)/\sqrt{2}$，那么 Alice 对她手上的光场进行幺正操作 $\hat{D}(\mu_0)$:

$$\hat{D}(\mu_0) = \mathrm{e}^{\mu_0 \hat{a}^\dagger - \mu_0^* \hat{a}}$$

$$\mu_0 = \frac{x_0 + \mathrm{i}p_0}{\sqrt{2}}$$

$$|\tilde{\Psi}_{AB}\rangle = \hat{D}(\mu_0)|\Psi_{AB}\rangle$$

- Alice 将她的光场传递给 Bob
- Bob 对双模态测 $\hat{x}^A - \hat{x}^B$ 与 $\hat{p}^A + \hat{p}^B$，其中:

$$\hat{x}^A = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \quad \hat{x}^B = \frac{\hat{b} + \hat{b}^\dagger}{\sqrt{2}}$$

$$\hat{p}^A = \frac{\hat{a} - \hat{a}^\dagger}{\mathrm{i}\sqrt{2}}, \quad \hat{p}^B = \frac{\hat{b} - \hat{b}^\dagger}{\mathrm{i}\sqrt{2}}$$

$\hat{x}^A - \hat{x}^B$ 和 $\hat{p}^A + \hat{p}^B$ 的共同本征态 $\left|\Phi_{x,p}^{AB}\right\rangle$:

$$\left|\Phi_{x,p}^{AB}\right\rangle = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{|\mu|^2}{2} - \frac{\mathrm{i}px}{2} + \mu\hat{a} - \mu^*\hat{b}^\dagger + \hat{a}^\dagger\hat{b}^\dagger\right)|00\rangle$$

其中,

$$\mu = \frac{x + \mathrm{i}p}{\sqrt{2}}$$

- 测得 $\mu = (x + \mathrm{i}p)/\sqrt{2}$ 的概率:

$$P_{x,p} = \frac{\mathrm{e}^{2r}}{2\pi}\exp\left[-\frac{\mathrm{e}^{2r}}{2}\left[(x - x_0)^2 + (y - y_0)^2\right]\right]$$

当 $r \to +\infty$ 时,

$$\lim_{r \to +\infty} P_{x,p} = \delta(x - x_0)\delta(p - p_0)$$

只有 $r \to +\infty$ 信息的传递才准确。

3-2

> 重复理想情况下分立变量量子隐形传态的过程并推导其平均保真度 $\bar{F} = 1$

分立变量量子隐形传态

四个 Bell 态:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

可以反解出:

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle + |\Phi^-\rangle\right)$$

$$|11\rangle = \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle - |\Phi^-\rangle\right)$$

$$|01\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle + |\Psi^-\rangle\right)$$

$$|10\rangle=\frac{1}{\sqrt{2}}\left(\left|\Psi^{+}\right\rangle-\left|\Psi^{-}\right\rangle\right)$$

Alice 手里有粒子 $1,2$，Bob 手里有粒子 3，Alice 想把未知状态 1 传给 Bob.

- Alice 和 Bob 共享处于 $\left|\Phi^{+}\right\rangle_{23}$ 的纠缠粒子对，他们与粒子 1 形成的总系统态：

$$\begin{aligned}
|\varphi\rangle_1 \otimes\left|\Phi^{+}\right\rangle_{23} & =\left(a|0\rangle_1+b|1\rangle_1\right) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{23}+|11\rangle_{23}\right) \\
& =\frac{1}{\sqrt{2}}\left(a|00\rangle_{12}|0\rangle_3+a|01\rangle_{12}|1\rangle_3+b|10\rangle_{12}|0\rangle_3+b|11\rangle_{12}|1\rangle_3\right) \\
& =\frac{1}{2}\left[a\left(\left|\Phi^{+}\right\rangle_{12}+\left|\Phi^{-}\right\rangle_{12}\right)|0\rangle_3+a\left(\left|\Psi^{+}\right\rangle_{12}+\left|\Psi^{-}\right\rangle_{12}\right)|1\rangle_3+b\left(\left|\Psi^{+}\right\rangle_{12}-\left|\Psi^{-}\right\rangle_{12}\right)|0\rangle_3+b\left(\left|\Phi^{+}\right\rangle_{12}-\left|\Phi^{-}\right\rangle_{12}\right)|1\rangle_3\right] \\
& =\frac{1}{2}\left[\left|\Phi^{+}\right\rangle_{12} \otimes\left(a|0\rangle_3+b|1\rangle_3\right)+\left|\Phi^{-}\right\rangle_{12} \otimes\left(a|0\rangle_3-b|1\rangle_3\right)+\left|\Psi^{+}\right\rangle_{12} \otimes\left(b|0\rangle_3+a|1\rangle_3\right)+\left|\Psi^{-}\right\rangle\left(-b|0\rangle_3+a|1\rangle_3\right)\right]
\end{aligned}$$

- Alice 对 $1,2$ 粒子 Bell 态测量，使粒子 3 状态塌缩：

Alice 测量结果	粒子 3 塌缩至	Bob 采取的操作 $\hat{U}_k^{(3)}$
$\left\|\Phi^{+}\right\rangle$	$a\|0\rangle_3+b\|1\rangle_3$	$\hat{I}^{(3)}$
$\left\|\Phi^{-}\right\rangle$	$a\|0\rangle_3-b\|1\rangle_3$	$\hat{\sigma}_z^{(3)}$
$\left\|\Psi^{+}\right\rangle$	$b\|0\rangle_3+a\|1\rangle_3$	$\hat{\sigma}_x^{(3)}$
$\left\|\Psi^{-}\right\rangle$	$-b\|0\rangle_3+a\|1\rangle_3$	$\mathrm{i}\hat{\sigma}_y^{(3)}$

- Alice 通过经典信道告诉 Bob 她的测量结果。
- Bob 对粒子 3 执行相应的局域操作，使 $a|0\rangle_1+b|1\rangle_1$ 态出现在 Bob 的粒子 3 上。

平均保真度

分立变量隐形传态：

$$\left|\Psi_{\text{out},k}^{(3)}\right\rangle \equiv \frac{\hat{U}_k^{(3)}\left\langle\operatorname{Bell}_k^{(12)} \middle| \varphi_1, \Phi_{23}^{+}\right\rangle}{\sqrt{p_k}}$$

$$p_k=\left\langle\varphi_1, \Phi_{23}^{+} \middle| \operatorname{Bell}_k^{(12)}\right\rangle\left\langle\operatorname{Bell}_k^{(12)} \middle| \varphi_1, \Phi_{23}^{+}\right\rangle$$

设想要传递的未知态 $|\varphi\rangle_1=a|0\rangle_1+b|1\rangle_1, |a|^2+|b|^2=1$

之前给出，整个系统的初始状态 $\left|\varphi_1, \Phi_{23}^{+}\right\rangle$ 写为：

$$\left|\varphi_1, \Phi_{23}^{+}\right\rangle=\frac{1}{2}\left[\left|\Phi^{+}\right\rangle_{12} \otimes\left(a|0\rangle_3+b|1\rangle_3\right)+\left|\Phi^{-}\right\rangle_{12} \otimes\left(a|0\rangle_3-b|1\rangle_3\right)+\left|\Psi^{+}\right\rangle_{12} \otimes\left(b|0\rangle_3+a|1\rangle_3\right)+\left|\Psi^{-}\right\rangle\left(-b|0\rangle_3+a|1\rangle_3\right)\right]$$

可以计算：

$$\left\langle\operatorname{Bell}_1^{(12)} \middle| \varphi_1, \Phi_{23}^{+}\right\rangle=\left\langle\Phi_{12}^{+} \middle| \varphi_1, \Phi_{23}^{+}\right\rangle=\frac{1}{2}\left(a|0\rangle_3+|1\rangle_3\right)$$

$$\hat{U}_1^{(3)}\left\langle\operatorname{Bell}_1^{(12)} \middle| \varphi_1, \Phi_{23}^{+}\right\rangle=\hat{I}^{(3)}\left[\frac{1}{2}\left(a|0\rangle_3+|1\rangle_3\right)\right]=\frac{1}{2}\left(a|0\rangle_3+|1\rangle_3\right)$$

$$p_1=\left\langle\varphi_1, \Phi_{23}^{+} \middle| \operatorname{Bell}_1^{(12)}\right\rangle\left\langle\operatorname{Bell}_1^{(12)} \middle| \varphi_1, \Phi_{23}^{+}\right\rangle=\frac{1}{4}\left(a^*\langle 0|_3+b^*|1\rangle_3\right)\left(a|0\rangle_3+b|1\rangle_3\right)=\frac{1}{4}\left(|a|^2+|b|^2\right)=\frac{1}{4}$$

$$\left|\Psi_{\text{out},1}^{(3)}\right\rangle=\frac{\hat{U}_1^{(3)}\left\langle\operatorname{Bell}_1^{(12)} \middle| \varphi_1, \Phi_{23}^{+}\right\rangle}{\sqrt{p_1}}=\frac{\frac{1}{2}\left(a|0\rangle_3+|1\rangle_3\right)}{\sqrt{\frac{1}{4}}}=a|0\rangle_3+b|1\rangle_3$$

类似可得：

$$p_1=p_2=p_3=p_4=\frac{1}{4}$$

$$\left|\Psi_{\text{out},1}^{(3)}\right\rangle=\left|\Psi_{\text{out},2}^{(3)}\right\rangle=\left|\Psi_{\text{out},3}^{(3)}\right\rangle=\left|\Psi_{\text{out},4}^{(3)}\right\rangle=a\left|0\right\rangle_3+b\left|1\right\rangle_3$$

令:

$$a=\cos\left(\frac{\theta}{2}\right),\quad b=\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi}$$

$$\left|\varphi^{(3)}\right\rangle\equiv\cos\left(\frac{\theta}{2}\right)\left|0\right\rangle_3+\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi}\left|1\right\rangle_3$$

$$\left|\Psi_{\text{out},1}^{(3)}\right\rangle=\left|\Psi_{\text{out},2}^{(3)}\right\rangle=\left|\Psi_{\text{out},3}^{(3)}\right\rangle=\left|\Psi_{\text{out},4}^{(3)}\right\rangle=a\left|0\right\rangle_3+b\left|1\right\rangle_3=\cos\left(\frac{\theta}{2}\right)\left|0\right\rangle_3+\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi}\left|1\right\rangle_3$$

平均保真度定义为:

$$\bar{F}\equiv\frac{1}{4\pi}\int_{\theta=0}^{\theta=\pi}\sin\theta\mathrm{d}\theta\int_{\phi=0}^{\phi=2\pi}\mathrm{d}\phi\sum_{k=1}^{4}p_k\left\langle\varphi^{(3)}\middle|\Psi_{\text{out},k}^{(3)}\right\rangle\left\langle\Psi_{\text{out},k}^{(3)}\middle|\varphi^{(3)}\right\rangle$$

计算平均保真度:

$$\begin{aligned}\bar{F}&\equiv\frac{1}{4\pi}\int_{\theta=0}^{\theta=\pi}\sin\theta\mathrm{d}\theta\int_{\phi=0}^{\phi=2\pi}\mathrm{d}\phi\sum_{k=1}^{4}p_k\left\langle\varphi^{(3)}\middle|\Psi_{\text{out},k}^{(3)}\right\rangle\left\langle\Psi_{\text{out},k}^{(3)}\middle|\varphi^{(3)}\right\rangle\\&=\frac{1}{\pi}\int_{\theta=0}^{\theta=\pi}\sin\theta\mathrm{d}\theta\int_{\phi=0}^{\phi=2\pi}\mathrm{d}\phi p_1\left\langle\varphi^{(3)}\middle|\Psi_{\text{out},1}^{(3)}\right\rangle\left\langle\Psi_{\text{out},1}^{(3)}\middle|\varphi^{(3)}\right\rangle\\&=\frac{1}{4\pi}\int_{\theta=0}^{\theta=\pi}\sin\theta\mathrm{d}\theta\int_{\phi=0}^{\phi=2\pi}\mathrm{d}\phi\left\langle\varphi^{(3)}\middle|\Psi_{\text{out},1}^{(3)}\right\rangle\left\langle\Psi_{\text{out},1}^{(3)}\middle|\varphi^{(3)}\right\rangle\\&=\frac{1}{4\pi}\int_{\theta=0}^{\theta=\pi}\sin\theta\mathrm{d}\theta\int_{\phi=0}^{\phi=2\pi}\mathrm{d}\phi\\&=\frac{1}{2}\int_{\theta=0}^{\theta=\pi}\sin\theta\mathrm{d}\theta\\&=1\end{aligned}$$

3-3

> 推导噪声情况下分立变量量子隐形传态的平均保真度。

若纠缠通道 $\left|\Phi^+\right\rangle_{23}$ 有噪声，其动力学由主方程决定:

$$\dot{\rho}_{23}(t)=-\mathrm{i}\sum_{j=2,3}\left\{\left[\frac{\omega_0}{2}\hat{\sigma}_z^{(j)},\rho_{23}(t)\right]+\frac{\gamma}{2}\left[2\hat{\sigma}_-^{(j)}\rho_{23}(t)\hat{\sigma}_+^{(j)}-\hat{\sigma}_+^{(j)}\hat{\sigma}_-^{(j)}\rho_{23}(t)-\rho_{23}(t)\hat{\sigma}_+^{(j)}\hat{\sigma}_-^{(j)}\right]\right\}$$

若初态为 $\rho_{23}(0)=\left|\Phi_{23}^+\right\rangle\left\langle\Phi_{23}^+\right|$，方程的解为:

$$\rho_{23}(t)=\frac{1}{2}\left\{\left[P_t\left|e\right\rangle\left\langle e\right|+(1-P_t)\left|g\right\rangle\left\langle g\right|\right]^{\otimes2}+\left|g\right\rangle\left\langle g\right|^{\otimes2}+u^2(t)\left|e\right\rangle\left\langle g\right|^{\otimes2}+\text{H.c.}\right\}$$

其中,

$$u(t)=\mathrm{e}^{-(\mathrm{i}\omega_0+\gamma/2)t},\quad P_t=\left|u(t)\right|^2$$

$$\begin{aligned}\rho_{23}(t)&=\frac{1}{2}\left\{\left[P_t\left|e\right\rangle\left\langle e\right|+(1-P_t)\left|g\right\rangle\left\langle g\right|\right]^{\otimes2}+\left|g\right\rangle\left\langle g\right|^{\otimes2}+u^2(t)\left|e\right\rangle\left\langle g\right|^{\otimes2}+\text{H.c.}\right\}\\&=\frac{1}{2}\left\{P_t^2\left|e_2e_3\right\rangle\left\langle e_2e_3\right|+\left[1+(1-P_t)^2\right]\left|g_2g_3\right\rangle\left\langle g_2g_3\right|+P_t\left(1-P_t\right)\left(\left|e_2g_3\right\rangle\left\langle e_2g_3\right|+\left|g_2e_3\right\rangle\left\langle g_2e_3\right|\right)+u_t^2\left|e_2g_3\right\rangle\left\langle g_2g_3\right|+u_t^{*2}\left|g_2g_3\right\rangle\left\langle e_2e_3\right.\end{aligned}$$

$$\begin{aligned}\rho_1&=\left(a\left|e_1\right\rangle+b\left|g_1\right\rangle\right)\left(a^*\left\langle e_1\right|+b^*\left\langle g_1\right|\right)\\&=\left|a\right|^2\left|e_1\right\rangle\left\langle e_1\right|+\left|b\right|^2\left|g_1\right\rangle\left\langle g_1\right|+ab^*\left|e_1\right\rangle\left\langle g_1\right|+a^*b\left|g_1\right\rangle\left\langle e_1\right|\end{aligned}$$

$$
\begin{aligned}
&\left\langle \mathrm{Bell}_{12}^{(1)} \middle| \rho_1 \otimes \rho_{23}(t) \middle| \mathrm{Bell}_{12}^{(1)} \right\rangle \\
=&\frac{1}{2}\left(\langle e_1 e_2| + \langle g_1 g_2|\right) \rho_1 \otimes \rho_{23}(t) \left(|e_1 e_2\rangle + |g_1 g_2\rangle\right) \\
=&\frac{|a|^2}{4}\left[P_t^2 |e_3\rangle\langle e_3| + P_t(1-P_t)|g_3\rangle\langle g_3|\right] + \frac{|b|^2}{4}\left\{\left[1+(1-P_t)^2\right]|g_3\rangle\langle g_3| + P_t(1-P_t)|e_3\rangle\langle e_3|\right\} + \frac{ab^*}{4}u_t^2|e_3\rangle\langle g_3| + \frac{a^* b}{4}u_t^{*2}|g_3\rangle\langle e_3|
\end{aligned}
$$

$$
\begin{aligned}
&\langle\varphi|\left\langle \mathrm{Bell}_{12}^{(1)} \middle| \rho_1 \otimes \rho_{23}(t) \middle| \mathrm{Bell}_{12}^{(1)} \right\rangle|\varphi\rangle \\
=&\left(a^*\langle e| + b^*\langle g|\right)\left\langle \mathrm{Bell}_{12}^{(1)} \middle| \rho_1 \otimes \rho_{23}(t) \middle| \mathrm{Bell}_{12}^{(1)} \right\rangle\left(a|e\rangle + b|g\rangle\right) \\
=&\frac{1}{4}\left\{|a|^4 P_t^2 + 2|ab|^2 P_t(1-P_t) + |b|^4\left[1+(1-P_t)^2\right] + |ab|^2\left(u_t^2 + u_t^{*2}\right)\right\}
\end{aligned}
$$

类似可得：

$$
\begin{aligned}
&\langle\varphi|\sigma_z^{(3)}\left\langle \mathrm{Bell}_{12}^{(2)} \middle| \rho_1 \otimes \rho_{23}(t) \middle| \mathrm{Bell}_{12}^{(2)} \right\rangle\sigma_z^{(3)}|\varphi\rangle \\
=&\frac{1}{4}\left\{|a|^4 P_t^2 + 2|ab|^2 P_t(1-P_t) + |b|^4\left[1+(1-P_t)^2\right] + |ab|^2\left(u_t^2 + u_t^{*2}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\langle\varphi|\sigma_x^{(3)}\left\langle \mathrm{Bell}_{12}^{(3)} \middle| \rho_1 \otimes \rho_{23}(t) \middle| \mathrm{Bell}_{12}^{(3)} \right\rangle\sigma_x^{(3)}|\varphi\rangle \\
=&\frac{1}{4}\left\{|a|^4\left[1+(1-P_t)^2\right] + 2|ab|^2 P_t(1-P_t) + |b|^4 P_t^2 + |ab|^2\left(u_t^2 + u_t^{*2}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\langle\varphi|\sigma_y^{(3)}\left\langle \mathrm{Bell}_{12}^{(4)} \middle| \rho_1 \otimes \rho_{23}(t) \middle| \mathrm{Bell}_{12}^{(4)} \right\rangle\sigma_y^{(3)}|\varphi\rangle \\
=&\frac{1}{4}\left\{|a|^4\left[1+(1-P_t)^2\right] + 2|ab|^2 P_t(1-P_t) + |b|^4 P_t^2 + |ab|^2\left(u_t^2 + u_t^{*2}\right)\right\}
\end{aligned}
$$

于是：

$$
\begin{aligned}
&\sum_{k=1}^{4}\langle\varphi|\sigma_k^{(3)}\left\langle \mathrm{Bell}_{12}^{(4)} \middle| \rho_1 \otimes \rho_{23}(t) \middle| \mathrm{Bell}_{12}^{(4)} \right\rangle\sigma_k^{(3)}|\varphi\rangle \\
=&2|ab|^2 P_t(1-P_t) + \left(|a|^4 + |b|^4\right)\left(1-P_t+P_t^2\right) + |ab|^2\left(u_t^2 + u_t^{*2}\right)
\end{aligned}
$$

平均保真度：

$$
\begin{aligned}
\bar{F} &= \frac{1}{4\pi}\int_{\theta=0}^{\theta=2\pi}\sin\theta\mathrm{d}\theta\int_{\phi=0}^{\phi=2\pi}\mathrm{d}\phi\sum_{k=1}^{4}\langle\varphi|\sigma_k^{(3)}\left\langle \mathrm{Bell}_{12}^{(4)} \middle| \rho_1 \otimes \rho_{23}(t) \middle| \mathrm{Bell}_{12}^{(4)} \right\rangle\sigma_k^{(3)}|\varphi\rangle \\
&= \frac{1}{4\pi}\int_{\theta=0}^{\theta=2\pi}\sin\theta\mathrm{d}\theta\int_{\phi=0}^{\phi=2\pi}\mathrm{d}\phi\left\{2\left|\cos\frac{\theta}{2}\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi}\right|^2 P_t(1-P_t) + \left(\left|\cos\frac{\theta}{2}\right|^4 + \left|\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi}\right|^4\right)\left(1-P_t+P_t^2\right) + \left|\cos\frac{\theta}{2}\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi}\right|^2\left(u_t^2+u_t^{*2}\right)\right. \\
&= \frac{1}{2}\int_{\theta=0}^{\theta=\pi}\sin\theta\times\left\{\frac{\sin^2\theta}{2}P_t(1-P_t) + \left(\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}\right)\left(1-P_t+P_t^2\right) + \frac{\sin^2\theta}{4}\left(u_t^2+u_t^{*2}\right)\right\}\mathrm{d}\theta \\
&= \frac{1}{3}\left[P_t(1-P_t) + 2\left(1-P_t+P_t^2\right) + \frac{1}{2}\left(u_t^2+u_t^{*2}\right)\right] \\
&= \frac{1}{3}\left[2 + |u(t)|^2\left(|u(t)|^2 - 1\right) + \mathrm{Re}\left(u_t^2\right)\right]
\end{aligned}
$$

3-4

> 推导连续变量量子隐形传态的平均保真度 $F = \dfrac{1}{1+\mathrm{e}^{-2r}}$

- Alice 和 Bob 建立处于双模压缩真空态的光场纠缠通道，它们与待传输态形成的总状态为：

$$
|\Psi_0\rangle = \frac{\mathrm{e}^{-|\alpha|^2/2}}{\cosh r}\mathrm{e}^{\alpha\hat{a}_1^\dagger - \tanh r\hat{a}_2^\dagger\hat{a}_3^\dagger}|000\rangle
$$

- Alice 用 $50:50$ 分束仪耦合 $\hat{a}_1$ 和 $\hat{a}_2$，作用 $\hat{V} = \exp\left[\frac{\pi}{4}\left(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1\right)\right]$，态变为：

$$|\Psi_1\rangle = \hat{V}|\Psi_0\rangle = \frac{\mathrm{e}^{-|\alpha|^2/2}}{\cosh r}\exp\left[\frac{\alpha}{\sqrt{2}}\left(\hat{a}_1^\dagger - \hat{a}_2^\dagger\right) - \frac{\tanh r}{\sqrt{2}}\left(\hat{a}_1^\dagger + \hat{a}_2^\dagger\right)\hat{a}_3^\dagger\right]|000\rangle$$

- Alice 测 $\hat{X}_1 = \frac{1}{2}\left(\hat{a}_1 + \hat{a}_1^\dagger\right), \hat{P}_2 = \frac{1}{2\mathrm{i}}\left(\hat{a}_2 - \hat{a}_2^\dagger\right)$，测得 x_1, p_2，$|\Psi_1\rangle$ 塌缩至 $|\Psi_2\rangle \propto \langle x_1, p_2 \mid \Psi_1\rangle$

$$|\Psi_2\rangle \propto \langle x_1, p_2 \mid \Psi_1\rangle = \frac{\sqrt{2}\mathrm{e}^{-|\alpha|^2/2-|z|^2+\sqrt{2}z^*}}{\sqrt{\pi}\cosh r}\mathrm{e}^{\left(\alpha-\sqrt{2}z\right)\tanh r\hat{a}_2^\dagger}|0\rangle$$

其中，$z = x_1 - \mathrm{i}p_2$

- Alice 通过经典信道告诉 Bob 测量值 $z = x_1 - \mathrm{i}p_2$
- Bob 对手头得光场进行平移变换 $\hat{D}^{(3)}(\sqrt{2}z)$，态 $|\Psi_2\rangle$ 变为：

$$\begin{aligned}|\Psi_3\rangle &= \hat{D}^{(3)}(\sqrt{2}z)|\Psi_2\rangle \\ &= \frac{\sqrt{2}}{\sqrt{\pi}\cosh r}\exp\left[\frac{|(\alpha-\sqrt{z})\tanh r|^2 - |\alpha|^2}{2} - |z|^2 + \sqrt{2}z^*\alpha + \frac{(z\alpha^* - \alpha z^*)\tanh r}{\sqrt{2}}\right]\left|\sqrt{2}z + \left(\alpha - \sqrt{2}z\right)\tanh r\right\rangle\end{aligned}$$

在 $r \to +\infty$ 极限下，此态趋于 $|\alpha\rangle$，就实现了隐形传态。

平均保真度

连续变量隐形传态可以表示为：

$$\left|\Psi^{(3)}_{\mathrm{out},x_1,p_1}\right\rangle = \frac{\hat{D}^{(3)}(\sqrt{2}z)\left\langle x_1, p_2\left|\hat{V}^{(12)}\right|\Psi_0\right\rangle}{\sqrt{p_{x_1,p_2}}}$$

其中，

$$p_{x_1,p_2} = \left\langle\Psi_0\left|\hat{V}^{(12)}\right|x_1, p_2\right\rangle\left\langle x_1, p_2\left|\hat{V}^{(12)}\right|\Psi_0\right\rangle$$

平均保真度：

$$\begin{aligned}F &= \int \mathrm{d}x_1\mathrm{d}p_2 p_{x_1,p_2}\left\langle\alpha\left|\Psi^{(3)}_{\mathrm{out},x_1,p_1}\right.\right\rangle\left\langle\Psi^{(3)}_{\mathrm{out},x_1,p_1}\left|\alpha\right.\right\rangle \\ &= \int\frac{2\mathrm{d}x_1\mathrm{d}p_2}{\pi\cosh^2 r}\exp\left[\left|\left(\alpha-\sqrt{2}z\right)\tanh r\right|^2 - |\alpha|^2 - 2|z|^2 + \sqrt{2}\left(z^*\alpha + \alpha^* z\right)\right]\times \\ &\left\langle\alpha\left|\sqrt{2}z + \left(\alpha-\sqrt{2}z\right)\tanh r\right.\right\rangle\left\langle\sqrt{2}z + \left(\alpha-\sqrt{2}z\right)\tanh r\left|\alpha\right.\right\rangle \\ &= \int\frac{2\mathrm{d}x_1\mathrm{d}p_2}{\pi\cosh^2 r}\exp\left[\left|\left(\alpha-\sqrt{2}z\right)\tanh r\right|^2 - |\alpha|^2 - 2|z|^2 + \sqrt{2}\left(z^*\alpha + \alpha^* z\right)\right]\times \\ &\exp\left[-|\alpha|^2 - \left|\sqrt{2}z + \left(\alpha - \sqrt{2}z\tanh r\right)\right|^2 + \alpha^*\left(\sqrt{2}z + \left(\alpha - \sqrt{2}z\tanh r\right)\right) + \alpha\left(\sqrt{2}z^* + \left(\alpha - \sqrt{2}z^*\tanh r\right)\right)\right] \\ &= \int\frac{2\mathrm{d}x_1\mathrm{d}p_2}{\pi\cosh^2 r}\exp\left\{-4(1-\tanh r)\left(p_2^2 + x_1^2\right) + 2\sqrt{2}(1-\tanh r)\left[x_1(\alpha+\alpha^*) + \mathrm{i}p_2(\alpha-\alpha^*)\right] - 2(1-\tanh r)|\alpha|^2\right\} \\ &= \frac{2}{\pi\cosh^2 r}\frac{\pi}{4(1-\tanh r)}\exp\left[\frac{8(1-\tanh r)^2\cdot 4|\alpha|^2}{16(1-\tanh r)} - 2(1-\tanh r)|\alpha|^2\right] \\ &= \frac{1+\tanh r}{2} \\ &= \frac{1}{1+\mathrm{e}^{-2r}}\end{aligned}$$

$$\lim_{r\to+\infty} F = 1$$

3-5

> 结合必要的公式，重复量子密钥分发的 BB84 协议并证明其安全性保证。

BB84协议

- Alice 随机地将一组信息串编码在两组偏振态基矢任选的一组（线偏振或圆偏振）上。
- Alice 将光子传递给Bob
- Bob 随机选择基矢对光子状态进行测量。
- Alice 和 Bob 在经典信道中交流所采用的基矢。
- 他们将具有相同基矢的信息串保留下来作为密码，将不同基矢的舍弃。

BB84协议安全保证

若 Eve 试图窃取密度，他会：

- 截取光子。
- 随机选择基矢对光子进行测量以读取信息。
- 根据测量结果复制光子拷贝，并发送给 Bob.

Alice 与 Eve，Alice 与 Bob 各有 50% 概率采用了相同基矢且具有相同编码。Bob 与 Alice 有 25% 的概率采用相同基矢且具有相同编码。

Alice 和 Bob 可取一部分已定密钥比较，若有错，则放弃此次密钥分发。

3-6

> 结合必要的公式，重复量子密钥分发的 Ekert91 协议并证明其安全性保证。

Ekert91协议

- Alice 和 Bob 共享 $|\Psi\rangle_{AB}=\left(|+_z\rangle_A|-_z\rangle_B-|-_z\rangle_A|+_z\rangle_B\right)/\sqrt{2}$
- ALice 测 $\hat{\vec{\sigma}}^A\cdot\vec{a}_i,\vec{a}_i=\left(\cos\phi_i^A,\sin\phi_i^A\right),\phi_{1,2,3}^A\in\{0,\pi/4,\pi/2\}$
- Bob 测 $\hat{\vec{\sigma}}^B\cdot\vec{b}_i,\vec{b}_i=\left\{\cos\phi_i^B,\sin\phi_i^B\right\},\phi_{1,2,3}^B\in\{0,\pi/4,\pi/2\}$
- 二者公布测量方向，并将相同测量方向的结果保留为密钥。

Ekert91协议安全保证

- 假设 Eve 要窃取密码，则他要拦截发送给 Alice 和 Bob 的两个粒子。

Eve 测量 $\hat{\vec{\sigma}}^A\cdot\vec{\mathrm{e}}^A$ 和 $\hat{\vec{\sigma}}^B\cdot\vec{\mathrm{e}}^B$，其中 $\vec{\mathrm{e}}^A=\left(\cos\phi_a^E,\sin\phi_a^E\right),\vec{\mathrm{e}}^B=\left(\cos\phi_b^E,\sin\phi_b^E\right)$

- Eve 制备相同的态发送给 Alice 和 Bob
- Alice 和 Bob 重复以上过程后得到的 S 应为：

$$
\begin{aligned}
S\left(\vec{\mathrm{e}}_a,\vec{\mathrm{e}}_b\right)&=E\left(\vec{a}_1,\vec{\mathrm{e}}^A\right)E\left(\vec{b}_1,\vec{\mathrm{e}}^B\right)-E\left(\vec{a}_1,\vec{\mathrm{e}}^A\right)E\left(\vec{b}_3,\vec{\mathrm{e}}^B\right)+E\left(\vec{a}_3,\vec{\mathrm{e}}^A\right)E\left(\vec{b}_1,\vec{\mathrm{e}}^B\right)+E\left(\vec{a}_3,\vec{\mathrm{e}}^A\right)E\left(\vec{b}_3,\vec{\mathrm{e}}^B\right)\\
&=-\vec{a}_1\cdot\vec{\mathrm{e}}^A\left(\vec{b}_1-\vec{b}_3\right)\cdot\vec{\mathrm{e}}^B-\vec{a}_3\cdot\vec{\mathrm{e}}^A\left(\vec{b}_1+\vec{b}_3\right)\cdot\vec{\mathrm{e}}^B\\
&=-\sqrt{2}\cos\left(\phi_a^E-\phi_b^E\right)\\
&=\sqrt{2}\vec{\mathrm{e}}^A\cdot\vec{\mathrm{e}}^B
\end{aligned}
$$

- Eve 采用特定策略 $\rho\left(\vec{\mathrm{e}}^A,\vec{\mathrm{e}}^B\right)$ 选择 $\vec{\mathrm{e}}^A$ 和 $\vec{\mathrm{e}}^B$，但无论 Eve 采用什么策略，都有：

$$
|S|=\left|\int\mathrm{d}\vec{\mathrm{e}}^A\mathrm{d}\vec{\mathrm{e}}^B\rho\left(\vec{\mathrm{e}}^A,\vec{\mathrm{e}}^B\right)S\left(\vec{\mathrm{e}}^A,\vec{\mathrm{e}}^B\right)\right|\leqslant\sqrt{2}
$$

- Alice 和 Bob 二人公布不同测量方向的测值，若发现 $|S|\leqslant\sqrt{2}$，则放弃本轮密钥分发。