1

$$egin{aligned} U_{ ext{eff}}(r) &= rac{l^2}{2mr^2} - rac{1}{r^3} \ rac{\mathrm{d} U_{ ext{eff}}(r)}{\mathrm{d} r} &= -rac{l^2}{mr^3} + rac{3}{r^4} \end{aligned}$$

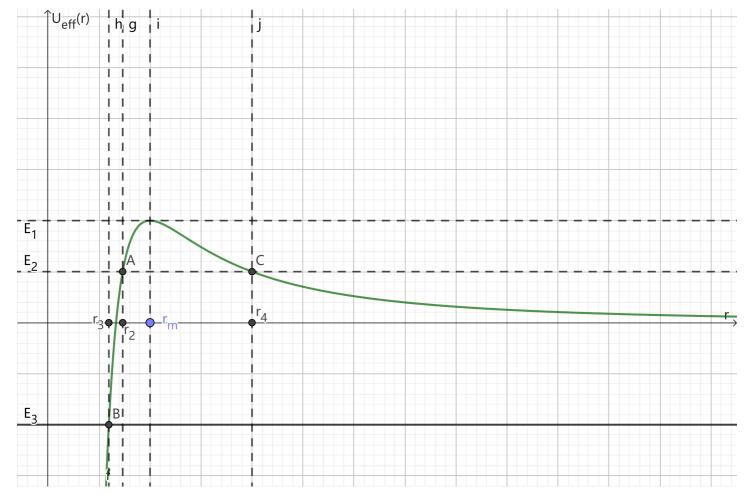
令 $U'_{\mathrm{eff}}(r_m)=0$, 得:

$$egin{aligned} r_m &= rac{3m}{l^2} > 0 \ & U_{ ext{eff}}''(r_m) &= rac{1}{r_m^4} \cdot (rac{-l^2}{m}) < 0 \end{aligned}$$

于是 r_m 是极小值点

$$\lim_{r o 0^+} U_{
m eff}(r) = 0$$
 $\lim_{r o +\infty} U_{
m eff}(r) = -\infty$

$U_{\rm eff}(r)$ 关于 r 的函数关系图大致如下:



图中,
$$E_1 = U_{ ext{eff}}(r_m), 0 < E_2 < U_{ ext{eff}}(r_m), E_3 \leqslant 0$$

当能量 $E>E_1=U_{
m eff}(r_m)$ 时,质点可在全空间运动,轨道不闭合

当 $E=E_1$ 时,质点绕力心做圆周运动,轨道为圆形

当
$$0 < E = E_2 < U_{ ext{eff}}(r_m)$$
 时,质点在 $0 < r < r_2$ 和 $r > r_4$ 区域内运动

当 $E = E_3 \leqslant 0$ 时,质点在 $0 < r < r_3$ 的区域内运动

体系的拉氏量为:

$$L=rac{m(\dot{r}^2+r^2\dot{ heta}^2)}{2}+rac{1}{r^3}$$

关于广义坐标 r 的 E-L 方程为:

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{3}{r^4} = 0 \tag{1}$$

体系角动量守恒, 而:

$$egin{aligned} l &\equiv |ec{r} imes ec{p}| \ &= m|ec{r} imes ec{v}| \ &= m|rec{e}_r imes (\dot{r}ec{e}_r + r\dot{ heta}ec{e}_ heta)| \ &= mr^2\dot{ heta} \end{aligned}$$

于是:

$$\dot{ heta}=rac{l}{mr^2}$$

将上式代入(1),消去 $\dot{\theta}$,得:

$$m\ddot{r}=rac{l^2}{mr^3}-rac{3}{r^4}$$

注意到:

$$\ddot{r} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}t}$$

$$= \frac{\mathrm{d}r}{\mathrm{d}t} \frac{\mathrm{d}\dot{r}}{\mathrm{d}r}$$

$$= \dot{r} \frac{\mathrm{d}\dot{r}}{\mathrm{d}r}$$

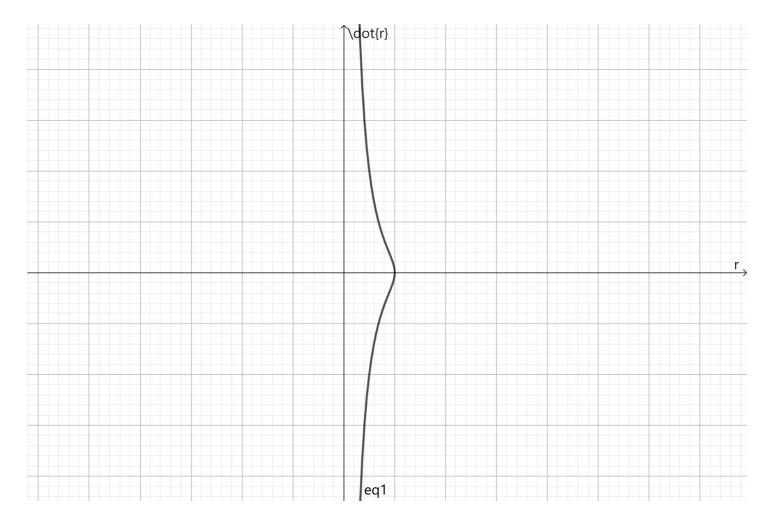
于是得到:

$$m\dot{r}\mathrm{d}\dot{r}=(rac{l^2}{mr^3}-rac{3}{r^4})\mathrm{d}r$$

积分得:

$$rac{m\dot{r}^2}{2} = rac{1}{r^3} - rac{l^2}{2mr^2} + C$$

在 (r,\dot{r}) 空间的相流大致如下:



2

对于 $V(r)=-rac{k}{r^n}$,

$$egin{aligned} U_{ ext{eff}}(r) &= rac{l^2}{2mr^2} - rac{k}{r^n} \ rac{\mathrm{d}U_{ ext{eff}}(r)}{\mathrm{d}r} &= rac{-l^2}{mr^3} + knrac{1}{r^{n+1}} \end{aligned}$$

存在圆轨道要求存在 r_m 使得:

$$\left.rac{\mathrm{d}U_{\mathrm{eff}}(r)}{\mathrm{d}r}
ight|_{r=r_m}=0$$

得到:

$$r_m^{2-n} = \frac{l^2}{mkn}$$

$$\frac{d^2 U_{\text{eff}}(r)}{dr^2} = \frac{3l^2}{mr^4} - kn(n+1)\frac{1}{r^{n+2}}$$
(1)

存在稳定圆轨道要求:

$$\left.rac{\mathrm{d}^2 U_{\mathrm{eff}}(r)}{\mathrm{d}r^2}
ight|_{r=r_m}>0$$

结合 (1), 得到:

对于 $V = \alpha r^m$,

$$egin{aligned} U_{ ext{eff}}(r) &= rac{l^2}{2mr^2} + lpha r^m \ &rac{\mathrm{d} U_{ ext{eff}}(r)}{\mathrm{d} r} &= rac{-l^2}{mr^3} + lpha mr^{m-1} \end{aligned}$$

存在圆轨道要求存在 r_k 使得:

$$\left. rac{\mathrm{d} U_{\mathrm{eff}}(r)}{\mathrm{d} r} \right|_{r=r_{b}} = 0$$

得到:

$$r_k^{2+m} = \frac{l^2}{\alpha m^2}$$

$$\frac{d^2 U_{\text{eff}}(r)}{dr^2} = \frac{3l^2}{mr^4} + \alpha m(m-1)r^{m-2}$$
(2)

存在稳定圆轨道要求:

$$\left.rac{\mathrm{d}^2 U_{\mathrm{eff}}(r)}{\mathrm{d}r^2}
ight|_{r=r_k}>0$$

结合 (2), 得到:

$$m > -2$$

3

(a)

轨道方程为:

$$heta = \int_{r_0}^r rac{l}{r^2\sqrt{2m[E-V(r)]-rac{l^2}{r^2}}}\mathrm{d}r + heta_0$$

当 $V(r) = -\frac{k}{r}$, 令 $u = \frac{1}{r}$, 得:

$$\theta = \theta_0 - \int \frac{\mathrm{d}u}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}}$$

利用积分公式:

$$\int \frac{\mathrm{d}x}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \arccos \frac{-(\beta + 2\gamma x)}{\sqrt{\beta^2 - \alpha \gamma}}$$

得:

$$heta = heta_0 - rccos rac{rac{l^2 u}{mk} - 1}{\sqrt{1 + rac{2El^2}{mk^2}}}$$

再将 $u=\frac{1}{r}$ 代回得:

$$r=rac{rac{l^2}{mk}}{1+\sqrt{1+rac{2El^2}{mk^2}\cos(heta- heta_0)}}$$

令 $p=rac{l^2}{mk}, e=\sqrt{1+rac{2El^2}{mk^2}}$,r, heta 的关系可改写为:

$$r = rac{p}{1 + e\cos(heta - heta_0)}$$

 $\cos(\theta - \theta_0) \in [-1, 1]$, 于是:

$$r_{ ext{min}} = rac{p}{1+e}, \;\; r_{ ext{max}} = rac{p}{1-e}$$

于是:

$$\begin{split} \Delta\theta &= 2 \int_{r_{\min}}^{r_{\max}} \frac{l \mathrm{d}r/r^2}{\sqrt{2m(E - V_{\mathrm{eff}}(r))}} \\ &= -2 \int_{u = \frac{1 - e}{p}}^{u = \frac{1 - e}{p}} \frac{\mathrm{d}u}{\sqrt{\frac{2mE}{l^2} + \frac{2mk}{l^2}u - u^2}} \\ &= 2 \arccos \frac{\frac{l^2u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}} \bigg|_{u = \frac{1 - e}{p}}^{u = \frac{1 - e}{p}} \\ &= 2 \cdot (\arccos(-1) - \arccos 1) \\ &= 2\pi \end{split}$$

(b)

 $\diamondsuit u = rac{1}{r}, \mathrm{d} u = -rac{1}{r^2} \mathrm{d} r$, $\diamondsuit x = u^2, \mathrm{d} x = 2u \mathrm{d} u$

$$\begin{split} \mathrm{d}\theta &= \frac{l \mathrm{d}r/r^2}{\sqrt{2m(E - V_{\mathrm{eff}}(r))}} \\ &= \frac{\frac{l^2}{r^2} \mathrm{d}r}{\sqrt{2m(E - \frac{l^2}{2mr^2} - \frac{1}{2}kr^2)}} \\ &= \frac{-l \mathrm{d}u}{\sqrt{2m(E - \frac{l^2}{2m}u^2 - \frac{k}{2}\frac{1}{u^2})}} \\ &= \frac{-\frac{l}{2}\mathrm{d}(u^2)}{\sqrt{2m(Eu^2 - \frac{l^2}{2m}u^4 - \frac{k}{2})}} \\ &= \frac{-l}{2\sqrt{2m}} \frac{\mathrm{d}x}{\sqrt{-\frac{l^2}{2m}x^2 + Ex - \frac{k}{2}}} \end{split}$$

积分得:

$$egin{aligned} heta - heta_0 &= rac{-l}{2\sqrt{2m}} \int_{x_0}^x rac{\mathrm{d}x}{\sqrt{-rac{l^2}{2m}x^2 + Ex - rac{k}{2}}} \ &= -rac{1}{2} rccos rac{rac{l^2}{m}x - E}{\sqrt{E^2 - rac{kl^2}{m}}} \ &= -rac{1}{2} rccos rac{rac{l^2}{mr^2} - E}{\sqrt{E^2 - rac{kl^2}{m}}} \end{aligned}$$

于是:

$$\cos[2(heta- heta_0)] = rac{rac{l^2}{mr^2}-E}{\sqrt{E^2-rac{kl^2}{m}}}$$

得到:

$$egin{aligned} r_{ ext{min}} &= rac{l}{\sqrt{m}} \cdot rac{1}{\sqrt{E + \sqrt{E^2 - rac{kl^2}{m}}}} \ & \ r_{ ext{max}} &= rac{l}{\sqrt{m}} \cdot rac{1}{\sqrt{E - \sqrt{E^2 - rac{kl^2}{m}}}} \end{aligned}$$

于是:

于是:

$$\Delta \theta = 2 \cdot (\theta_{\text{max}} - \theta_{\text{min}})$$

$$= 2 \cdot [(\theta_0 + \frac{\pi}{2}) - \theta_0]$$

$$= \pi$$

(c)

当 $E o 0^-$,有:

$$egin{aligned} r_{ ext{max}} = +\infty \ V_{ ext{eff}}(r_{ ext{min}}) &= rac{l^2}{2mr_{ ext{min}}^2} - kr_{ ext{min}}^{-eta} = 0 \Longrightarrow r_{ ext{min}} = (rac{l^2}{2mk})^{rac{1}{2-eta}} \ \Delta heta_0 &\equiv \lim_{E o 0^-} \Delta heta \ &= \lim_{E o 0^-} 2 \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{l ext{d} r/r^2}{\sqrt{2m(E - V_{ ext{eff}}(r))}} \ &= \lim_{E o 0^-} rac{2}{\sqrt{2m}} \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{l ext{d} r/r^2}{\sqrt{E - rac{l^2}{2mr^2} + kr^{-eta}}} \ &= rac{2l}{\sqrt{2m}} \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{ ext{d} r}{\sqrt{-rac{l^2}{2mr^2} + kr^{-eta}}} \ &= rac{2l}{\sqrt{2m}} \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{ ext{d} r}{\sqrt{-rac{l^2}{2mr^2} + kr^{-eta}}} \end{aligned}$$

令:

$$u=\sqrt{-rac{l^2}{2m}+kr^{2-eta}}$$

$$egin{split} r &= (rac{u^2}{k} + rac{l^2}{2mk})^{rac{1}{2-eta}} \ &\ln r = rac{1}{2-eta} [\ln(u^2 + rac{l^2}{2m}) - \ln k] \ &rac{\mathrm{d}r}{r} = \mathrm{d}\ln r \ &= rac{1}{2-eta} rac{2u\mathrm{d}u}{u^2 + rac{l^2}{2m}} \end{split}$$

代回积分表达式:

$$\begin{split} \Delta\theta_0 &= \frac{2l}{\sqrt{2m}} \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{r\sqrt{-\frac{l^2}{2m} + kr^{2-\beta}}} \\ &= \frac{2l}{\sqrt{2m}} \cdot \frac{2}{2-\beta} \int_0^{+\infty} \frac{\mathrm{d}u}{u^2 + \frac{l^2}{2m}} \\ &= \frac{2l}{\sqrt{2m}} \cdot \frac{2}{2-\beta} \cdot \frac{2m}{l^2} \cdot \frac{l}{\sqrt{2m}} \int_0^{+\infty} \frac{\mathrm{d}(\frac{\sqrt{2m}}{l}u)}{1 + (\frac{\sqrt{2m}}{l}u)^2} \\ &= \frac{2l}{\sqrt{2m}} \cdot \frac{2}{2-\beta} \cdot \frac{2m}{l^2} \cdot \frac{l}{\sqrt{2m}} \arctan \frac{\sqrt{2m}}{l} u \Big|_0^{+\infty} \\ &= \frac{2l}{\sqrt{2m}} \cdot \frac{2}{2-\beta} \cdot \frac{2m}{l^2} \cdot \frac{l}{\sqrt{2m}} \cdot \frac{\pi}{2} \\ &= \frac{2\pi}{2-\beta} \end{split}$$

4

(a)

体系能量守恒:

$$E=rac{m\dot{r}^2}{2}+rac{l^2}{2mr^2}-rac{k}{r}$$

得到:

$$\mathrm{d}t = rac{\mathrm{d}r}{\sqrt{rac{2}{m}[E + rac{k}{r} - rac{l^2}{2mr^2}]}}$$

积分得:

$$egin{aligned} rac{ au}{2} &= \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{ ext{d}r}{\sqrt{rac{2}{m}[E+rac{k}{r}-rac{l^2}{2mr^2}]}} \ &= \sqrt{rac{m}{2}} \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{ ext{d}r}{\sqrt{E-rac{l^2}{2mr^2}+rac{k}{r}}} \end{aligned}$$

于是:

$$au = \sqrt{2m} \int_{r_{
m min}}^{r_{
m max}} rac{{
m d}r}{\sqrt{E-rac{l^2}{2mr^2}+rac{k}{r}}}$$

(b)

半长轴 a 为:

$$a\equivrac{r_{
m max}+r_{
m min}}{2}=-rac{k}{2E}$$
 $e\equiv\sqrt{1+rac{2El^2}{mk^2}}=\sqrt{1-rac{l^2}{mka}}$

令 $r=a(1-e\cos\psi)$,当 $r=r_{\max},\psi=0$;当 $r=r_{\min},\psi=\pi$,于是:

$$egin{aligned} au &= \sqrt{2m} \int_{r_{ ext{min}}}^{r_{ ext{max}}} rac{ ext{d}r}{\sqrt{E - rac{l^2}{2mr^2} + rac{k}{r}}} \ &= \sqrt{2m} \int_0^{\pi} rac{ae \sin\psi ext{d}\psi}{\sqrt{-rac{k}{2a} - rac{(1 - e^2)k}{2a(1 - e\cos\psi)^2} + rac{k}{a(1 - e\cos\psi)}}} \ &= \sqrt{2m} \cdot \sqrt{rac{2}{k}} a^{rac{3}{2}} \cdot \int_0^{\pi} (1 - e\cos\psi) ext{d}\psi \ &= 2\pi a^{rac{3}{2}} \sqrt{rac{m}{k}} \end{aligned}$$

5

$$ec{J} = ec{r} imes ec{p} \Longrightarrow J_j = arepsilon_{lmj} x_l p_m \ ec{A} = ec{p} imes ec{J} - mk ec{e}_r \Longrightarrow A_a = arepsilon_{ija} p_i J_j - mk rac{x_a}{r} \ = arepsilon_{ija} arepsilon_{lmj} p_i p_m x_l - mk rac{x_a}{r} \ = arepsilon_{jia} arepsilon_{jml} p_i p_m x_l - mk rac{x_a}{r} \ = (\delta_{im} \delta_{al} - \delta_{il} \delta_{am}) p_i p_m x_l - mk rac{x_a}{r} \ = p^2 x_a - p_l x_l p_a - mk rac{x_a}{r} \ \end{cases}$$

于是:

$$\{A_a,H\}=\{p^2x_a-p_lx_lp_a-mkrac{x_a}{r},rac{p^2}{2m}-rac{k}{r}\}$$

注意到:

$$\{p^2x_a,rac{p^2}{2m}\}=rac{p_ap^2}{m} \ \{p^2x_a,rac{k}{r}\}=2krac{p_bx_bx_a}{r^3} \$$

$$\{p_{l}x_{l}p_{a},rac{p^{2}}{2m}\}=rac{p_{a}p^{2}}{m}$$
 $\{p_{l}x_{l}p_{a},rac{k}{r}\}=rac{kx_{l}p_{l}x_{a}}{r^{3}}+rac{kp_{a}}{r}$ $\{mkrac{x_{a}}{r},rac{p^{2}}{2m}\}=-rac{kx_{b}p_{b}x_{a}}{r^{3}}+rac{kp_{a}}{r}$ $\{mkrac{x_{a}}{r},rac{k}{r}\}=0$

于是:

$$\begin{aligned}
\{A_a, H\} &= \{p^2 x_a - p_l x_l p_a - m k \frac{x_a}{r}, \frac{p^2}{2m} - \frac{k}{r}\} \\
&= \{p^2 x_a, \frac{p^2}{2m}\} - \{p^2 x_a, \frac{k}{r}\} - \{p_l x_l p_a, \frac{p^2}{2m}\} + \{p_l x_l p_a, \frac{k}{r}\} - \{m k \frac{x_a}{r}, \frac{p^2}{2m}\} + \{m k \frac{x_a}{r}, \frac{k}{r}\} \\
&= 0
\end{aligned}$$

于是 \vec{A} 是个守恒量

6

$$r = ae^{b\theta} \Longrightarrow \theta = \frac{1}{b} \ln \frac{r}{a} \Longrightarrow \frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{1}{br}$$
 (1)

而轨道方程的微分形式为:

$$rac{\mathrm{d} heta}{\mathrm{d}r} = rac{rac{l}{r^2}}{\sqrt{2m[E-U(r)]-rac{l^2}{r^2}}}$$

在有心力 $F=-rac{k}{r^3}$ 作用下,势能为:

$$U(r)=\int_{r}^{+\infty}-rac{k}{r^{3}}\mathrm{d}r=-rac{k}{2r^{2}}$$

代入轨道方程的微分形式,得:

$$rac{\mathrm{d} heta}{\mathrm{d}r} = rac{rac{l}{r^2}}{\sqrt{2mE + rac{mk}{r^2} - rac{l^2}{r^2}}}$$

与 (1) 对比得, E=0, 进一步得到:

$$rac{\mathrm{d} heta}{\mathrm{d}r} = rac{l}{r\sqrt{mk-l^2}}$$

再与(1)对比,得:

$$\frac{l}{\sqrt{mk-l^2}} = \frac{1}{b}$$

解得:

$$l=\sqrt{rac{mk}{b^2+1}}$$

综上,

$$\begin{cases} E = 0 \\ l = \sqrt{\frac{mk}{b^2 + 1}} \end{cases}$$

7

(a)

由对称性可知,散射角 Θ 与入射方向极角 χ_0 的关系为:

$$\Theta=\pi-2\chi_0$$

轨道方程的微分形式为:

$$d\theta = \frac{\frac{l}{r^2}dr}{\sqrt{2m[E - V(r)] - \frac{l^2}{r^2}}}$$
(1)

整个运动过程机械能守恒,而在无穷远处势能为零,于是:

$$E = \frac{1}{2}mv_{\infty}^2 \tag{2}$$

在无穷远处,设 \vec{r} 与 \vec{p} 的夹角为 ϕ ,有:

$$|\vec{l}| = |\vec{r} \times \vec{p}|$$

$$= |\frac{b}{\sin \phi} m v_{\infty} \sin \phi|$$

$$= mbv_{\infty}$$
(3)

(2)(3) 代入(1), 消去E, l得:

$$\mathrm{d} heta = rac{rac{b}{r^2} \mathrm{d} r}{\sqrt{1 - rac{b^2}{r^2} - rac{2V(r)}{mv_\infty^2}}}$$

积分得:

$$\chi_0 = \int_{r_{
m min}}^{+\infty} rac{rac{b}{r^2} \mathrm{d}r}{\sqrt{1 - rac{b^2}{r^2} - rac{2V(r)}{mv_{\infty}^2}}}$$

(b)

$$\chi_0 = \int_R^{+\infty} rac{b \mathrm{d}r/r^2}{\sqrt{1 - rac{b^2}{r^2}}}
onumber \ = \int_R^{+\infty} rac{b \mathrm{d}r}{r\sqrt{r^2 - b^2}}$$

 $\Rightarrow u = \sqrt{r^2 - b^2}$

$$egin{aligned} r &= \sqrt{u^2 + b^2} \ \ln r &= rac{1}{2} \ln(u^2 + b^2) \ &rac{\mathrm{d} r}{r} &= \mathrm{d} (\ln r) \ &= rac{u \mathrm{d} u}{u^2 + b^2} \end{aligned}$$

代回积分式,得:

$$\chi_0 = \int_{r=R}^{r=+\infty} \frac{b dr}{r \sqrt{r^2 - b^2}}$$

$$= \int_{u=\sqrt{R^2 - b^2}}^{u=+\infty} \frac{b du}{u^2 + b^2}$$

$$= \int_{u=\sqrt{R^2 - b^2}}^{u=+\infty} \frac{d(\frac{u}{b})}{1 + (\frac{u}{b})^2}$$

$$= \arctan \frac{u}{b} \Big|_{u=\sqrt{R^2 - b^2}}^{u=+\infty}$$

$$= \frac{\pi}{2} - \arctan \frac{\sqrt{R^2 - b^2}}{b}$$

又 $\Theta=\pi-2\chi_0$,与上式联立,消去 χ_0 得:

$$\tan\frac{\Theta}{2} = \frac{\sqrt{R^2 - b^2}}{b}$$

即:

$$b = \sqrt{rac{R^2}{ an^2rac{\Theta}{2}+1}} \ = rac{R}{\sqrt{rac{\sin^2rac{\Theta}{2}}{\cos^2rac{\Theta}{2}}+1}} \ = R\cosrac{\Theta}{2}$$

(c)

$$\sigma = \frac{b}{\sin \Theta} \left| \frac{\mathrm{d}b}{\mathrm{d}\Theta} \right|$$
$$= \frac{R^2}{4}$$

(d)

$$\sigma = \frac{R^2}{4}$$

$$\begin{cases} b = R\cos\frac{\Theta}{2} \\ b \in [0, R] \end{cases} \Longrightarrow \Theta \in [0, \pi]$$

根据对称性,入射钢球的出射方向可以是空间中任意一个方向,即:

$$\int \mathrm{d}\Omega = 4\pi$$

于是总散射截面为:

$$S = \int \sigma \mathrm{d}\Omega = 4\pi \cdot rac{R^2}{4} = \pi R^2$$