9

(a)

$$
\begin{aligned}
L &= \frac{1}{2}mv^2 - e\phi + \frac{e}{c}\vec{A}\cdot\vec{v} \\
\frac{\partial L}{\partial x} &= -e\frac{\partial \phi}{\partial x} + \frac{e}{c}\frac{\partial}{\partial x}(\vec{A}\cdot\vec{v}) \\
&= eE_x + e\frac{\partial A_x}{\partial t} + \frac{e}{c}\frac{\partial(\vec{A}\cdot\vec{v})}{\partial x} \\
\frac{\partial L}{\partial \dot{x}} &= mv_x + \frac{e}{c}A_x
\end{aligned}
$$

该拉式量关于 x 坐标的欧拉-拉格朗日方程为：

$$
m\dot{v}_x + \frac{e}{c}\dot{A}_x - eE_x - e\frac{\partial A_x}{\partial t} - \frac{e}{c}\frac{\partial(\vec{A}\cdot\vec{v})}{\partial x} = 0
$$

(2)

该拉式量对应的欧拉-拉格朗日方程的矢量表达式为：

$$
m\dot{\vec{v}} + \frac{e}{c}\dot{\vec{A}} - e\vec{E} - e\frac{\partial \vec{A}}{\partial t} - \frac{e}{c}\nabla(\vec{A}\cdot\vec{v}) = \vec{0}
$$

(3)

$$
\begin{cases}
\phi' = \phi + \dfrac{1}{c}\dfrac{\partial f}{\partial t} \\
\vec{A}' = \vec{A} - \nabla f
\end{cases}
\tag{1}
$$

$$
\vec{E}' = -\nabla\phi' - \frac{\partial \vec{A}'}{\partial t} = \vec{E} - \frac{1}{c}\nabla\frac{\partial f}{\partial t} + \frac{\partial(\nabla f)}{\partial t}
\tag{2}
$$

变换后的系统满足：

$$
m\dot{\vec{v}} + \frac{e}{c}\dot{\vec{A}}' - e\vec{E}' - e\frac{\partial \vec{A}'}{\partial t} - \frac{e}{c}\nabla(\vec{A}'\cdot\vec{v}) = \vec{0}
$$

将关系 $(1)(2)$ 代入得：

$$
m\dot{\vec{v}} + \frac{e}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\vec{A} - \nabla f) - e(\vec{E} - \frac{1}{c}\nabla\frac{\partial f}{\partial t} + \frac{\partial(\nabla f)}{\partial t}) - e\frac{\partial(\vec{A} - \nabla f)}{\partial t} - \frac{e}{c}\nabla[(\vec{A} - \nabla f)\cdot\vec{v}] = \vec{0}
$$

即：

$$
m\dot{\vec{v}} + \frac{e}{c}\dot{\vec{A}} - e\vec{E} - e\frac{\partial \vec{A}}{\partial t} - \frac{e}{c}(\vec{v}\cdot\nabla)\vec{A} + \left[-\frac{e}{c}\frac{\mathrm{d}}{\mathrm{d}t}\nabla f + \frac{e}{c}\nabla\frac{\partial f}{\partial t} - e\frac{\partial(\nabla f)}{\partial t} + e\frac{\partial(\nabla f)}{\partial t} + \frac{e}{c}\nabla(\vec{v}\cdot\nabla f) \right] = \vec{0}
$$

注意到：

$$
\begin{aligned}
-\frac{e}{c}\frac{\mathrm{d}}{\mathrm{d}t}\nabla f + \frac{e}{c}\nabla\frac{\partial f}{\partial t} - e\frac{\partial(\nabla f)}{\partial t} + e\frac{\partial(\nabla f)}{\partial t} + \frac{e}{c}\nabla(\vec{v}\cdot\nabla f) &= -\frac{e}{c}\frac{\mathrm{d}}{\mathrm{d}t}\nabla f + \frac{e}{c}\nabla\frac{\partial f}{\partial t} + \frac{e}{c}\nabla(\vec{v}\cdot\nabla f) \\
&= \frac{e}{c}\left[-\frac{\partial \nabla f}{\partial \vec{r}}\vec{v} - \frac{\partial \nabla f}{\partial t} + \nabla\frac{\partial f}{\partial t} + \nabla(\vec{v}\cdot\nabla f) \right] \\
&= \frac{e}{c}\left[-\nabla(\vec{v}\cdot\nabla f) - \nabla\frac{\partial f}{\partial t} + \nabla\frac{\partial f}{\partial t} + \nabla(\vec{v}\cdot\nabla f) \right] \\
&= \vec{0}
\end{aligned}
$$

这就是说，新的拉式量给出相同的运动方程

10

(a)

以固定点 O 为原点，过 O 竖直向下为 x 轴正方向，过 O 水平向右为 y 轴正方向建系

选取图中角度 φ 和质点与 O 的距离 r 为广义坐标

$$\vec{r}=r\cos\varphi\vec{e}_x+r\sin\varphi\vec{e}_y$$

$$\dot{\vec{r}}=(\dot{r}\cos\varphi-r\dot{\varphi}\sin\varphi)\vec{e}_x+(\dot{r}\sin\varphi+r\dot{\varphi}\cos\varphi)\vec{e}_y$$

$$T=\frac{1}{2}m\dot{\vec{r}}^2=\frac{1}{2}m(\dot{r}^2+r^2\dot{\varphi}^2)$$

选取原点 O 所在水平面为零势能面，计算系统势能：

$$V=\frac{1}{2}k(r-l)^2-mgr\cos\varphi$$

于是得到系统的拉格朗日量：

$$L=T-V=\frac{1}{2}m(\dot{r}^2+r^2\dot{\varphi}^2)-\frac{1}{2}k(r-l)^2+mgr\cos\varphi$$

(b)

欧拉-拉格朗日方程给出：

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{r}}-\frac{\partial L}{\partial r}=0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}}-\frac{\partial L}{\partial\varphi}=0$$

将 L 代入，得：

$$m\ddot{r}-mr\dot{\varphi}^2+k(r-l)-mg\cos\varphi=0$$

$$mr^2\ddot{\varphi}+2mr\dot{r}\dot{\varphi}+mgr\sin\varphi=0$$

即：

$$\ddot{r}-r\dot{\varphi}^2+\frac{k}{m}(r-l)-g\cos\varphi=0$$

$$\ddot{\varphi}+\frac{2\dot{r}}{r}\dot{\varphi}+\frac{g}{r}\sin\varphi=0$$

(c)

当对平衡位置的角位移和径向位移都很小，$\cos\varphi\approx 1,\sin\varphi\approx\varphi,\dot{\varphi}\approx 0$，方程组简化为：

$$\ddot{r}+\frac{k}{m}r=\frac{kl}{m}+g \tag{1}$$

$$\ddot{\varphi}+\frac{g}{r}\varphi=0 \tag{2}$$

方程 (1) 的通解为：

$$r=C_1\cos\sqrt{\frac{k}{m}}t+C_2\sin\sqrt{\frac{k}{m}}t+l+\frac{mg}{k}$$

代入 (2)，解得：

$$\varphi=C_3\cos\sqrt{\frac{g}{r}}t+C_4\sin\sqrt{\frac{g}{r}}t$$

11

位力定理给出，当时间趋于无穷时：

$$\langle T\rangle=-\frac{1}{2}\langle\sum_i\vec{F}_i\cdot\vec{r}_i\rangle$$

设气体装于体积为 V 容器中，气体压强为 p，气体内部粒子之间碰撞导致的 $\sum_i\vec{F}_i\cdot\vec{r}_i$ 为零，于是只用考虑气体粒子与容器壁的碰撞：

$$
\begin{aligned}
\langle\sum_i \vec{F}_i\cdot\vec{r}_i\rangle &= \langle-\oiint_{\partial V} p\vec{r}\cdot\mathrm{d}\vec{S}\rangle \\
&= \langle -p\iiint_V \nabla\vec{r}\mathrm{d}V\rangle \\
&= -3pV
\end{aligned}
$$

理想气体的平均动能为 $\bar{T}=\frac{3}{2}Nk_BT$，于是 $\langle T\rangle=\bar{T}=\frac{3}{2}Nk_BT$，代入位力定理，得：

$$\frac{3}{2}Nk_BT=-\frac{1}{2}\cdot(-3pV)$$

于是：

$$pV=Nk_BT$$

这就是理想气体状态方程

12

作用量为：

$$S\equiv\int_{t_1}^{t_2}L(t,q,\dot{q},\ddot{q},\cdots,q^{(k)})\mathrm{d}t$$

设 $\delta q^{(i)}(t)$ 满足 $\delta q^{(i)}(t_1)=\delta q^{(i)}(t_2)=0,(i=0,1,2,\cdots,k)$

则 S 的变分为：

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2}\sum_{i=0}^{k}\frac{\partial L}{\partial q^{(i)}}\delta q^{(i)}\mathrm{d}t \\
&= \sum_{i=0}^{k}\int_{t_1}^{t_2}\frac{\partial L}{\partial q^{(i)}}\delta q^{(i)}\mathrm{d}t
\end{aligned}
$$

注意到，对于 $i=1,2,\cdots,k$：

$$
\begin{aligned}
\int_{t_1}^{t_2}\frac{\partial L}{\partial q^{(i)}}\delta q^{(i)}\mathrm{d}t &= \int_{t_1}^{t_2}\frac{\partial L}{\partial q^{(i)}}\delta[\frac{\mathrm{d}}{\mathrm{d}t}q^{(i-1)}]\mathrm{d}t \\
&= \int_{t_1}^{t_2}\frac{\partial L}{\partial q^{(i)}}\frac{\mathrm{d}}{\mathrm{d}t}\delta q^{(i-1)}\mathrm{d}t \\
(\text{分部积分}) &= \frac{\partial L}{\partial q^{(i)}}\delta q^{(i-1)}\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\delta q^{(i-1)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^{(i)}}\mathrm{d}t \\
&= -\int_{t_1}^{t_2}\delta q^{(i-1)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^{(i)}}\mathrm{d}t \\
&= -\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^{(i)}}\delta\frac{\mathrm{d}}{\mathrm{d}t}q^{(i-2)}\mathrm{d}t \\
&= -\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^{(i)}}\frac{\mathrm{d}}{\mathrm{d}t}\delta q^{(i-2)}\mathrm{d}t \\
(\text{分部积分}) &= (-1)^2\frac{\partial L}{\partial q^{i}}\delta q^{(i-2)}\Big|_{t_1}^{t_2}+(-1)^2\int_{t_1}^{t_2}\delta q^{(i-2)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial q^{(i)}}\mathrm{d}t \\
&= (-1)^2\int_{t_1}^{t_2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial q^{(i)}}\delta q^{(i-2)}\mathrm{d}t \\
&= \cdots \\
&= (-1)^i\int_{t_1}^{t_2}\frac{\mathrm{d}^i}{\mathrm{d}t^i}\frac{\partial L}{\partial q^{(i)}}\delta q^{(0)}\mathrm{d}t
\end{aligned}
$$

于是：

$$
\begin{aligned}
\delta S &= \sum_{i=0}^{k}\int_{t_1}^{t_2}\frac{\partial L}{\partial q^{(i)}}\delta q^{(i)}\mathrm{d}t \\
&= \sum_{i=0}^{k}(-1)^i\int_{t_1}^{t_2}\frac{\mathrm{d}^i}{\mathrm{d}t^i}\frac{\partial L}{\partial q^{(i)}}\delta q^{(0)}\mathrm{d}t \\
&= \int_{t_1}^{t_2}\sum_{i=0}^{k}(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}t^i}\frac{\partial L}{\partial q^{(i)}}\delta q^{(0)}\mathrm{d}t \\
&= \int_{t_1}^{t_2}\left(\sum_{i=0}^{k}(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}t^i}\frac{\partial L}{\partial q^{(i)}}\right)\delta q^{(0)}\mathrm{d}t
\end{aligned}
$$

系统的真实运动路径应使 S 取极小值，这就要求 $\delta S = 0$，于是得到：

$$
\sum_{i=0}^{k}(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}t^i}\frac{\partial L}{\partial q^{(i)}} = 0
$$

即：

$$
\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}} - \cdots + (-1)^k\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{\partial L}{\partial q^{(k)}} = 0
$$