求解单位圆内的二维亥姆霍兹方程：

$$(\nabla^2+k^2)u=0$$

解析解

极坐标系下，二维亥姆霍兹方程化为：

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+k^2u=0$$

设 $u(\rho,\varphi)=R(\rho)\Phi(\varphi)$，分离变量得：

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+m^2\Phi=0$$

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho})+(k^2-\frac{m^2}{\rho^2})R=0$$

形式解为：

$$\Phi^{(m)}(\varphi)=A_m\cos(m\theta)+B_m\sin(m\theta)$$

$$R^{(m)}(\rho)=C_m\mathrm{J}_m(k\rho)+D_m\mathrm{N}_m(k\rho)$$

$R(0)$ 是有限的，于是：

$$R^{(m)}(\rho)=C_m\mathrm{J}_m(k\rho)$$

第一类齐次边界条件

$$u\Big|_{\rho=1}=0$$

由此可得本征值：

$$k_n^{(m)}=x_n^{(m)}$$

其中，$x_n^{(m)}$ 是 m 阶贝塞尔函数 J_m 的第 n 个零点

本征振动模式为：

$$R_n^{(m)}(\rho)=\mathrm{J}_m(x_n^{(m)}\rho)$$

第二类齐次边界条件

$$\left.\frac{\partial u}{\partial n}\right|_{\rho=1}=0$$

由此可得本征值：

$$k_n^{(m)}=y_n^{(m)}$$

其中，$y_n^{(m)}$ 是 m 阶贝塞尔函数的导数 J_m' 的第 n 个零点

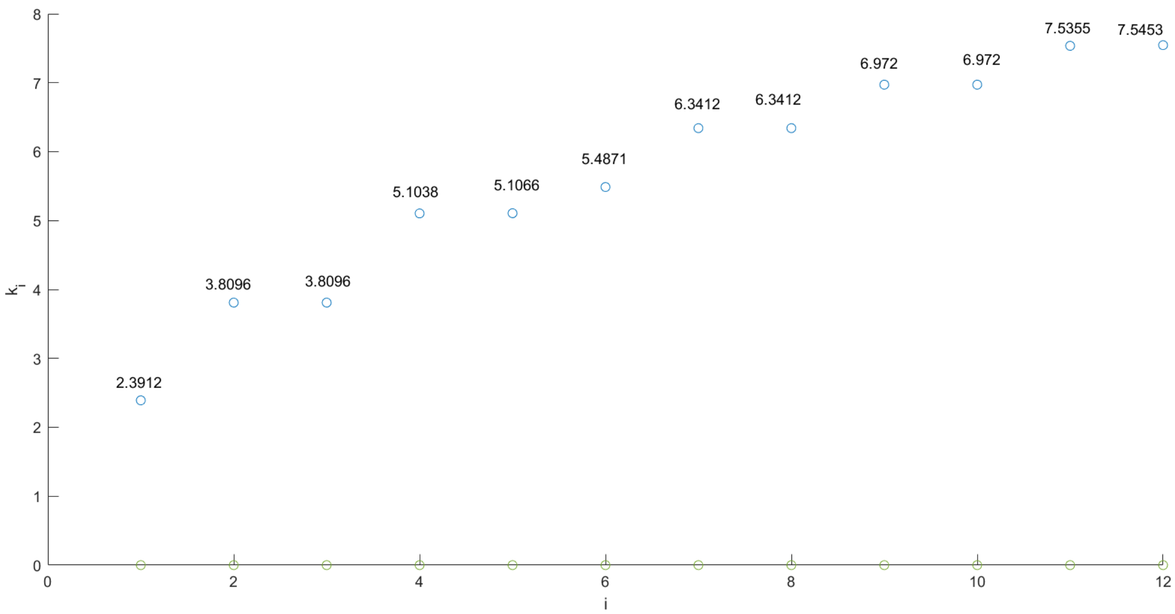
本征振动模式为：

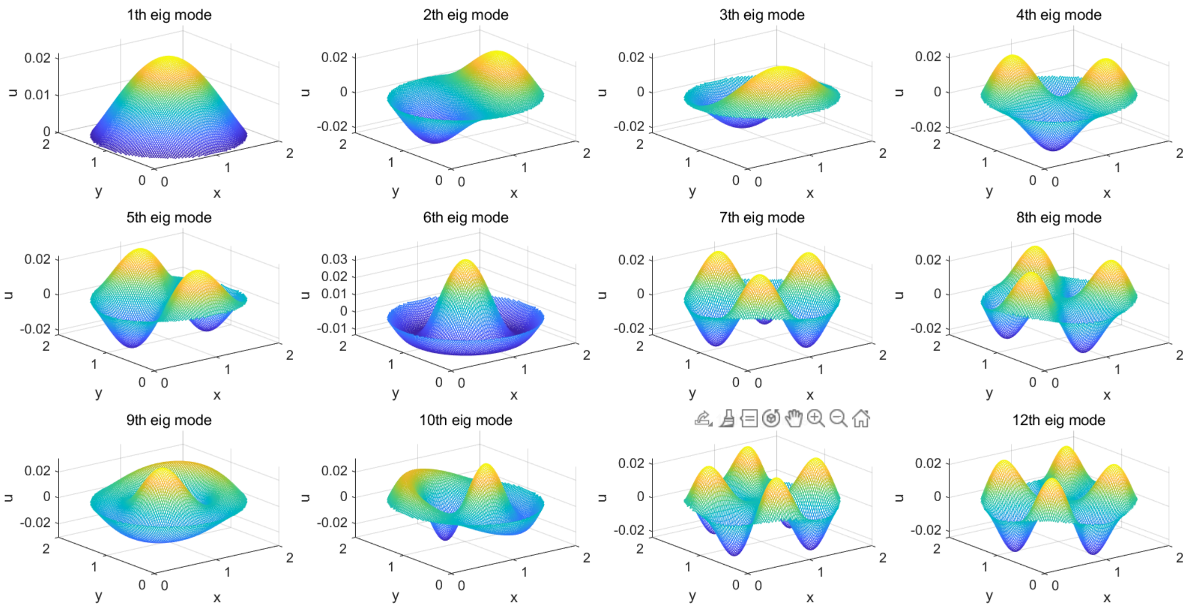
$$R_n^{(m)}(\rho)=\mathrm{J}_m(y_n^{(m)}\rho)$$

数值解

第一类齐次边界条件

绘图如下：




1th eig mode
2th eig mode
3th eig mode
4th eig mode
5th eig mode
6th eig mode
7th eig mode
8th eig mode
9th eig mode
10th eig mode
12th eig mode

matlab 代码如下:

```
N = 101;
h = 2 / (N-1);
cnt = 0;
cnt_to_n = zeros(N*N);
n_to_cnt = zeros(N*N);

X = [0];
Y = [0];

for n = 1:N*N
    i = mod(n-1, N)+1;
    j = floor(n/N)+1;
    if ( (i-1)*h - 1 )^2 + ( (j-1)*h - 1 )^2 < 1
        cnt = cnt + 1;
        cnt_to_n(cnt) = n;
        n_to_cnt(n) = cnt;
        X(cnt) = (i-1)*h;
        Y(cnt) = (j-1)*h;
    end
end

M = zeros(cnt, cnt);

for k = 1:cnt
    M(k, k) = -4;
    n = cnt_to_n(k);
    i = mod(n-1, N) + 1;
    j = floor(n/N) + 1;

    if i > 1 && n_to_cnt(n-1) > 0
        M(k, n_to_cnt(n-1) ) = 1;
    end

    if  i<N && n_to_cnt(n+1) > 0
        M(k, n_to_cnt(n+1) ) = 1;
    end

    if j>1 && n_to_cnt(n-N) > 0
        M(k, n_to_cnt(n-N) ) = 1;
    end

    if j < N && n_to_cnt(n+N) > 0
        M(k, n_to_cnt(n+N) ) = 1;
    end

end
```

```
M = -M./(h^2);

[V, D] = eig(M);

x = 1:12;
y = zeros(12);

for i = 1:12
    y(i) = sqrt(D(i, i));
end

figure(1);
scatter(x, y);
xlabel("i");
ylabel("k_i");

for i =1:12
    text(i,y(i), num2str(y(i)));
end

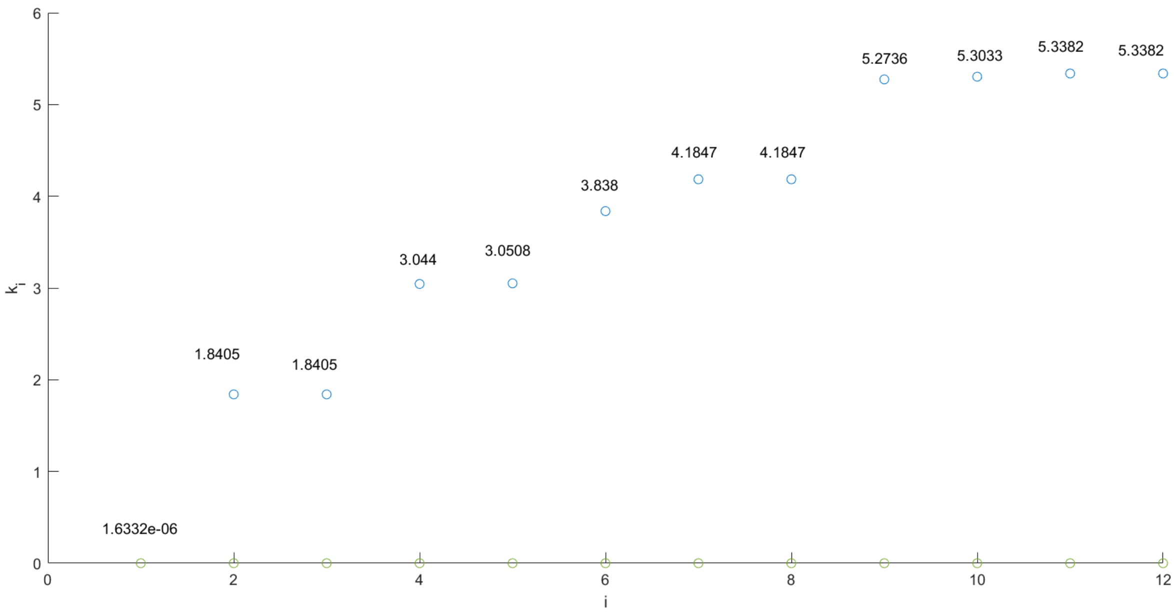
[xq, yq] = meshgrid(0:0.02:2, 0:0.02:2);

figure(2);

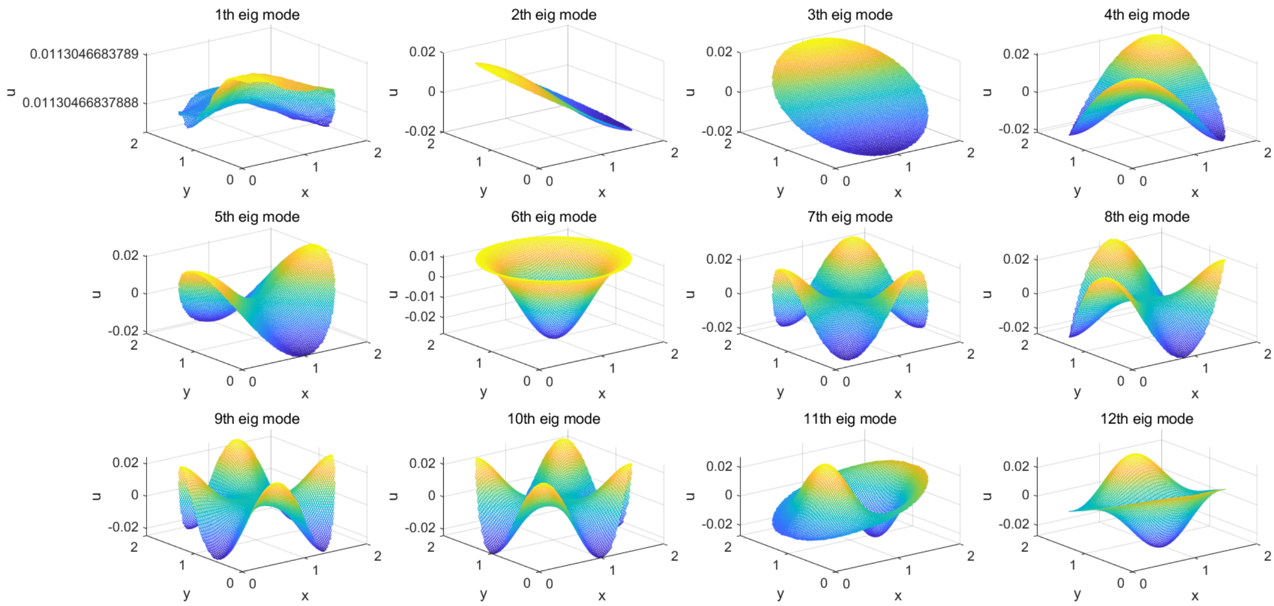
for k = 1:12
    subplot(3,4,k);
    Z = V(:, k);
    zq = griddata(X,Y,Z,xq,yq);
    mesh(xq,yq,zq);
    title([num2str(k), 'th eig mode']);
    xlabel('x');
    ylabel('y');
    zlabel('u');
end
```

第二类齐次边界条件

绘图如下:



1.6332e-06
1.8405
1.8405
3.044
3.0508
3.838
4.1847
4.1847
5.2736
5.3033
5.3382
5.3382
k_i
i



1th eig mode
2th eig mode
3th eig mode
4th eig mode
5th eig mode
6th eig mode
7th eig mode
8th eig mode
9th eig mode
10th eig mode
11th eig mode
12th eig mode
0.0113046683789
0.01130466837888

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cnt = 0;
cnt_to_n = zeros(N*N);
n_to_cnt = zeros(N*N);

X = [0];
Y = [0];

for n = 1:N*N
    i = mod(n-1, N)+1;
    j = floor(n/N)+1;
    if ( (i-1)*h - 1 )^2 + ( (j-1)*h - 1 )^2 < 1
        cnt = cnt + 1;
        cnt_to_n(cnt) = n;
        n_to_cnt(n) = cnt;
        X(cnt) = (i-1)*h;
        Y(cnt) = (j-1)*h;
    end
end

M = zeros(cnt, cnt);

for k = 1:cnt
    M(k, k) = -4;
    n = cnt_to_n(k);
    i = mod(n-1, N) + 1;
    j = floor(n/N) + 1;

    if i > 1 && n_to_cnt(n-1) > 0
        M(k, n_to_cnt(n-1) ) = 1;
    else
        M(k, k) = M(k, k) + 1;
    end

    if  i<N && n_to_cnt(n+1) > 0
        M(k, n_to_cnt(n+1) ) = 1;
    else
        M(k, k) = M(k, k) + 1;
    end

    if j>1 && n_to_cnt(n-N) > 0
        M(k, n_to_cnt(n-N) ) = 1;
    else
        M(k, k) = M(k, k) + 1;
    end
```

```
    if j < N && n_to_cnt(n+N) > 0
        M(k, n_to_cnt(n+N) ) = 1;
    else
        M(k, k) = M(k, k) + 1;
    end

end

M = -M./(h^2);

[V, D] = eig(M);

x = 1:12;
y = zeros(12);

for i = 1:12
    y(i) = sqrt(D(i, i));
end

figure(1);
scatter(x, y);
xlabel("i");
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for i =1:12
    text(i,y(i), num2str(y(i)));
end

[xq, yq] = meshgrid(0:0.02:2, 0:0.02:2);

figure(2);

for k = 1:12
    subplot(3,4,k);
    Z = V(:, k);
    zq = griddata(X,Y,Z,xq,yq);
    mesh(xq,yq,zq);
    title([num2str(k), 'th eig mode']);
    xlabel('x');
    ylabel('y');
    zlabel('u');
end
```