写出协变形式的电荷守恒定律,并由 4 维电磁势写出协变形式的达朗贝尔方程,推导电磁场张量 $F_{\mu\nu}$ 的各分量,并利用 $F_{\mu\nu}$ 将真空中的麦克斯韦方程组写成协变形式。

协变形式的电荷守恒定律:

$$\partial_{\mu}J_{\mu}=0$$

协变形式的达朗贝尔方程:

$$\Box A_{\mu} = -\mu_0 J_{\mu}$$

推导电磁场张量 $F_{\mu\nu}$ 的各分量:

$$\begin{split} A_{\mu} &= \left(\vec{A}, \frac{\mathrm{i}}{c} \varphi \right), \ \, \vec{B} = \nabla \times \vec{A}, \ \, \vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \\ F_{\mu\nu} &= \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \\ F_{11} &= F_{22} = F_{33} = F_{44} = 0 \\ F_{12} &= \partial_{1} A_{2} - \partial_{2} A_{1} = (\nabla \times \vec{A})_{3} = B_{3}, \ \, F_{21} = -F_{12} = -B_{3} \\ F_{13} &= \partial_{1} A_{3} - \partial_{3} A_{1} = -(\nabla \times \vec{A})_{2} = -B_{2}, \ \, F_{31} = -F_{13} = B_{2} \\ F_{14} &= \partial_{1} A_{4} - \partial_{4} A_{1} = \frac{\mathrm{i}}{c} \partial_{1} \varphi - \frac{1}{\mathrm{i}c} \frac{\partial A_{1}}{\partial t} = \frac{1}{\mathrm{i}c} \left(-\partial_{1} \varphi - \frac{\partial A_{1}}{\partial t} \right) = \frac{1}{\mathrm{i}c} E_{1} = -\frac{\mathrm{i}}{c} E_{1}, \ \, F_{41} = -F_{14} = \frac{\mathrm{i}}{c} E_{1} \\ F_{23} &= \partial_{2} A_{3} - \partial_{3} A_{2} = (\nabla \times \vec{A})_{1} = B_{1}, \ \, F_{32} = -F_{23} = -B_{1} \\ F_{24} &= \partial_{2} A_{4} - \partial_{4} A_{2} = \frac{\mathrm{i}}{c} \partial_{2} \varphi - \frac{1}{\mathrm{i}c} \frac{\partial A_{2}}{\partial t} = \frac{1}{\mathrm{i}c} \left(-\partial_{2} \varphi - \frac{\partial A_{2}}{\partial t} \right) = -\frac{\mathrm{i}}{c} E_{2}, \ \, F_{42} = -F_{24} = \frac{\mathrm{i}}{c} E_{2} \\ F_{34} &= \partial_{3} A_{4} - \partial_{4} A_{3} = \frac{\mathrm{i}}{c} \partial_{3} \varphi - \frac{1}{\mathrm{i}c} \frac{\partial A_{3}}{\partial t} = \frac{1}{\mathrm{i}c} \left(-\partial_{3} \varphi - \frac{\partial A_{3}}{\partial t} \right) = -\frac{\mathrm{i}}{c} E_{3}, \ \, F_{43} = -F_{34} = \frac{\mathrm{i}}{c} E_{3} \end{split}$$

综上,

$$F_{\mu
u} = egin{bmatrix} 0 & B_3 & -B_2 & -rac{\mathrm{i}}{c}E_1 \ -B_3 & 0 & B_1 & -rac{\mathrm{i}}{c}E_2 \ B_2 & -B_1 & 0 & -rac{\mathrm{i}}{c}E_3 \ rac{\mathrm{i}}{c}E_1 & rac{\mathrm{i}}{c}E_2 & rac{\mathrm{i}}{c}E_3 & 0 \end{bmatrix}$$

真空中麦克斯韦方程组协变形式:

$$\partial_
u F_{\mu
u} = \mu_0 J_\mu$$

2

由 4 维动量推出相对论中能量、动量和质量间的关系式,并推导相对论力学方程,讨论各分量方程的意义。

一方面, 四维动量 p_{μ} 是四维矢量, 其与自己的内积是不变量:

$$egin{aligned} p_{\mu}p_{\mu} &= m_0 U_{\mu} \cdot m_0 U_{\mu} = m_0 rac{\mathrm{d}x_{\mu}}{\mathrm{d} au} \cdot m_0 rac{\mathrm{d}x_{\mu}}{\mathrm{d} au} = m_0^2 \gamma^2 rac{\mathrm{d}x_{\mu}}{\mathrm{d}t} rac{\mathrm{d}x_{\mu}}{\mathrm{d}t} \ &= m_0^2 \gamma^2 \left(v^2 - c^2\right) = m_0^2 rac{1}{1 - v^2/c^2} (v^2 - c^2) = m_0^2 rac{c^2}{c^2 - v^2} (v^2 - c^2) \ &= -m_0^2 c^2 \end{aligned}$$

另一方面,

$$p_{\mu}=\left(ec{p},rac{\mathrm{i}W}{c}
ight)$$

于是:

$$p^2 - rac{W^2}{c^2} = -m_0^2 c^2$$

即:

$$W = \sqrt{p^2c^2 + m_0^2c^4}$$

一方面,四维力可表达为:

$$k_{\mu} = rac{\mathrm{d}p_{\mu}}{\mathrm{d} au} = \gamma rac{\mathrm{d}}{\mathrm{d}t} \left(ec{p}, rac{\mathrm{i}W}{c}
ight) = \left(\gamma rac{\mathrm{d}ec{p}}{\mathrm{d}t}, \gamma rac{\mathrm{i}}{c} rac{\mathrm{d}W}{\mathrm{d}t}
ight)$$

另一方面,

$$k_{\mu} = \left(\gamma ec{F}, rac{\mathrm{i} \gamma}{c} ec{F} \cdot ec{v}
ight)$$

对比可得分量方程:

$$egin{cases} ec{F} = rac{\mathrm{d}ec{p}}{\mathrm{d}t} \ ec{F} \cdot ec{v} = rac{\mathrm{d}W}{\mathrm{d}t} \end{cases}$$

上面第一条方程就是牛顿第二定律,第二条方程表示外力做功导致系统能量变化,也就是能量守恒。

写出协变形式的洛伦兹力公式(分别写出点电荷以及力密度形式的协变洛伦兹力公式)。

点电荷形式协变洛伦兹力公式:

$$K_{\mu} = q F_{\mu\nu} U_{\nu}$$

力密度形式协变洛伦兹力公式:

$$f_{\mu}=F_{\mu
u}J_{
u}$$

4

由电磁场能动张量表达式 $T_{\mu\lambda}=rac{1}{\mu_0}\left(F_{\mu\nu}F_{\nu\lambda}+rac{1}{4}\delta_{\mu\lambda}F_{\nu\tau}F_{\nu\tau}
ight)$,计算能动张量的各分量,并说明与 3 维电磁场能量密度、能流密度、动量密度、动量流密度之间的关系。

$$F_{\nu\tau}F_{\nu\tau} = 2\vec{B}^2 - \frac{2\vec{E}^2}{c^2}$$

$$T_{11} = \frac{1}{\mu_0} \left(F_{1\nu}F_{\nu 1} + \frac{1}{4}\delta_{11}F_{\nu\tau}F_{\nu\tau} \right) = \frac{1}{2} \left[\frac{1}{\mu_0} \left(B_1^2 - B_2^2 - B_3^2 \right) + \varepsilon_0 (E_1^2 - E_2^2 - E_3^2) \right]$$

$$T_{12} = \frac{1}{\mu_0} \left(F_{1\nu}F_{\nu 2} + \frac{1}{4}\delta_{12}F_{\nu\tau}F_{\nu\tau} \right) = \frac{1}{\mu_0}B_1B_2 + \varepsilon_0E_1E_2$$

$$T_{21} = T_{12} = \frac{1}{\mu_0}B_1B_2 + \varepsilon_0E_1E_2$$

$$T_{13} = \frac{1}{\mu_0} \left(F_{1\nu}F_{\nu 3} + \frac{1}{4}\delta_{13}F_{\nu\tau}F_{\nu\tau} \right) = \frac{1}{\mu_0}B_1B_3 + \varepsilon_0E_1E_3$$

$$T_{31} = T_{13} = \frac{1}{\mu_0}B_1B_3 + \varepsilon_0E_1E_3$$

$$T_{14} = \frac{1}{\mu_0} \left(F_{1\nu}F_{\nu 4} + \frac{1}{4}\delta_{14}F_{\nu\tau}F_{\nu\tau} \right) = \frac{i}{\mu_0c} \left(B_2E_3 - B_3E_2 \right)$$

$$T_{41} = T_{14} = \frac{i}{\mu_0c} \left(B_2E_3 - B_3E_2 \right)$$

$$T_{22} = \frac{1}{\mu_0} \left(F_{2\nu}F_{\nu 2} + \frac{1}{4}\delta_{22}F_{\nu\tau}F_{\nu\tau} \right) = \frac{1}{2} \left[\frac{1}{\mu_0} \left(-B_1^2 + B_2^2 - B_3^2 \right) + \varepsilon_0 \left(-E_1^2 + E_2^2 - E_3^2 \right) \right]$$

$$T_{23} = \frac{1}{\mu_0} \left(F_{2\nu}F_{\nu 3} + \frac{1}{4}\delta_{23}F_{\nu\tau}F_{\nu\tau} \right) = \frac{1}{\mu_0}B_2B_3 + \varepsilon E_2E_3$$

$$T_{32} = T_{23} = \frac{1}{\mu_0} B_2 B_3 + \varepsilon E_2 E_3$$

$$T_{24} = \frac{1}{\mu_0} \left(F_{2\nu} F_{\nu 4} + \frac{1}{4} \delta_{24} F_{\nu \tau} F_{\nu \tau} \right) = \frac{\mathrm{i}}{\mu_0 c} \left(B_3 E_1 - B_1 E_3 \right)$$

$$T_{42} = T_{24} = \frac{\mathrm{i}}{\mu_0 c} \left(B_3 E_1 - B_1 E_3 \right)$$

$$T_{33} = \frac{1}{\mu_0} \left(F_{3\nu} F_{\nu 3} + \frac{1}{4} \delta_{33} F_{\nu \tau} F_{\nu \tau} \right) = \frac{1}{2} \left[\frac{1}{\mu_0} \left(-B_1^2 - B_2^2 + B_3^2 \right) + \varepsilon_0 \left(-E_1^2 - E_2^2 + E_3^2 \right) \right]$$

$$T_{34} = \frac{1}{\mu_0} \left(F_{3\nu} F_{\nu 4} + \frac{1}{4} \delta_{34} F_{\nu \tau} F_{\nu \tau} \right) = \frac{\mathrm{i}}{\mu_0 c} \left(B_1 E_2 - B_2 E_1 \right)$$

$$T_{43} = T_{34} = \frac{\mathrm{i}}{\mu_0 c} \left(B_1 E_2 - B_2 E_1 \right)$$

$$T_{44} = \frac{1}{\mu_0} \left(F_{4\nu} F_{\nu 4} + \frac{1}{4} \delta_{44} F_{\nu \tau} F_{\nu \tau} \right) = \frac{1}{2} \left(\frac{1}{\mu_0} B^2 + \varepsilon_0 E^2 \right)$$

与3维电磁场能量密度、能流密度、动量密度、动量流密度之间的关系:

能量密度(真空中):

$$w=rac{1}{2}\left(ec{E}\cdotec{D}+ec{H}\cdotec{B}
ight)=rac{1}{2}\left(arepsilon_0E+rac{1}{\mu_0}B^2
ight) \ T_{44}=w$$

能流密度(真空中):

$$ec{S} = ec{E} imes ec{H} = rac{1}{\mu_0} ec{E} imes ec{B}$$
 $T_{14} = T_{41} = rac{\mathrm{i}}{\mu_0 c} \left(B_2 E_3 - B_3 E_2
ight) = -rac{\mathrm{i}}{c} S_1$
 $T_{24} = T_{42} = rac{\mathrm{i}}{\mu_0 c} \left(B_3 E_1 - B_1 E_3
ight) = -rac{\mathrm{i}}{c} S_2$
 $T_{34} = T_{43} = rac{\mathrm{i}}{\mu_0 c} \left(B_1 E_2 - B_2 E_1
ight) = -rac{\mathrm{i}}{c} S_3$

动量密度(真空中):

$$ec{g} = ec{D} imes ec{B} = arepsilon_0 ec{E} imes ec{B} \ T_{14} = T_{41} = rac{\mathrm{i}}{\mu_0 c} \left(B_2 E_3 - B_3 E_2
ight) = -\mathrm{i} c g_1 \ T_{24} = T_{42} = rac{\mathrm{i}}{\mu_0 c} \left(B_3 E_1 - B_1 E_3
ight) = -\mathrm{i} c g_2 \ T_{24} = T_{42} = rac{\mathrm{i}}{\mu_0 c} \left(B_3 E_1 - B_1 E_3
ight) = -\mathrm{i} c g_2 \ T_{24} = T_{42} = rac{\mathrm{i}}{\mu_0 c} \left(B_3 E_1 - B_1 E_3
ight) = -\mathrm{i} c g_2 \ T_{24} = T_$$

$$T_{34} = T_{43} = rac{\mathrm{i}}{\mu_0 c} \left(B_1 E_2 - B_2 E_1
ight) = -\mathrm{i} c g_3$$

动量流密度(真空中):

$$\stackrel{\leftrightarrow}{T} = -(ec{D}ec{E} + ec{B}ec{H}) + rac{1}{2}\stackrel{\leftrightarrow}{I}(ec{E}\cdotec{D} + ec{B}\cdotec{H}) = -\left(arepsilon_0ec{E}ec{E} + rac{1}{\mu_0}ec{B}ec{B}
ight) + rac{1}{2}\stackrel{\leftrightarrow}{I}\left(arepsilon_0E^2 + rac{1}{\mu_0}B^2
ight) \ T_{ij} = -\stackrel{\leftrightarrow}{T}, \ \ i,j = 1,2,3$$

综上, 电磁场能动张量与能量密度、能流密度、动量密度、动量流密度的关系为:

$$T_{\mu
u} = egin{bmatrix} -\overset{\leftrightarrow}{T}_{3 imes 3} & -\mathrm{i} cec{g}_{3 imes 1} \ -\mathrm{i} cec{g}_{1 imes 3}^{\,\mathrm{T}} & w_{1 imes 1} \end{bmatrix} = egin{bmatrix} -\overset{\leftrightarrow}{T}_{3 imes 3} & -rac{\mathrm{i}}{c}ec{S}_{3 imes 1} \ -rac{\mathrm{i}}{c}ec{S}_{1 imes 3}^{\,\mathrm{T}} & w_{1 imes 1} \end{bmatrix}$$

5

分别写出自由电磁场与一般电磁场的拉格朗日密度,并变分得到场方程。

对于电磁场, 拉格朗日密度是场量及其一阶导的函数:

$$\mathscr{L} = \mathscr{L}\left(arphi_{\sigma}(x_{
u}), \partial_{\mu}arphi_{\sigma}(x_{
u})
ight)$$

最小作用量原理:

$$\delta\int\mathscr{L}\left(arphi_{\sigma},\partial_{\mu}arphi_{\sigma}
ight)\mathrm{d}\Omega=0$$

拉格朗日密度的变分为:

$$\begin{split} \delta \mathcal{L} &= \mathcal{L}[\varphi_{\sigma} + \delta \varphi_{\sigma}, \partial_{\mu} \varphi_{\sigma} + \delta(\partial_{\mu} \varphi_{\sigma})] - \mathcal{L}[\varphi_{\sigma}, \partial_{\mu} \varphi_{\sigma}] \\ &= \frac{\partial \mathcal{L}}{\partial \varphi_{\sigma}} \delta \varphi_{\sigma} + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi_{\sigma})} \delta(\partial_{\mu} \varphi_{\sigma}) \\ &= \frac{\partial \mathcal{L}}{\partial \varphi_{\sigma}} \delta \varphi_{\sigma} - \frac{\partial}{\partial x_{\mu}} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi_{\sigma})} \right) \delta \varphi_{\sigma} + \frac{\partial}{\partial x_{\mu}} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi_{\sigma})} \delta \varphi_{\sigma} \right) \end{split}$$

由于:

$$\int\limits_{\Omega} rac{\partial}{\partial x_{\mu}} \left(rac{\partial \mathscr{L}}{\partial (\partial_{\mu} arphi_{\sigma})} \delta arphi_{\sigma}
ight) \mathrm{d}\Omega = \int\limits_{\partial \Omega} rac{\partial \mathscr{L}}{\partial (\partial_{\mu} arphi_{\sigma})} \delta arphi_{\sigma} \mathrm{d}\Omega_{\mu} = 0$$

代入最小最用量原理,得:

$$\int \left[rac{\partial}{\partial x_{\mu}}\left(rac{\partial\mathscr{L}}{\partial(\partial_{\mu}arphi_{\sigma})}
ight) - rac{\partial\mathscr{L}}{\partialarphi_{\sigma}}
ight]\deltaarphi_{\sigma}\mathrm{d}\Omega = 0$$

由 $\delta\varphi_{\sigma}$ 的任意性,得到场方程:

$$\frac{\partial}{\partial x_{\mu}} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi_{\sigma})} \right) - \frac{\partial \mathcal{L}}{\partial \varphi_{\sigma}} = 0$$

对于电磁场,

$$A_
u(x_\mu) = \left(ec{A}(x_\mu), rac{\mathrm{i}}{c} arphi(x_\mu)
ight)$$

电磁场的拉格朗日方程为:

$$\frac{\partial \mathscr{L}}{\partial A_{\nu}} - \frac{\partial}{\partial x_{\mu}} \left(\frac{\partial \mathscr{L}}{\partial (\partial_{\mu} A_{\nu})} \right) = 0$$

自由电磁场的拉格朗日密度:

$$\mathscr{L}_0(\partial_\mu A_\mu) = -rac{1}{4\mu_0} F_{\mu
u} F_{\mu
u}$$

代入电磁场的拉格朗日方程,得:

$$rac{\partial (F_{lphaeta}F_{lphaeta})}{\partial A_
u} - rac{\partial}{\partial x_\mu}rac{\partial (F_{lphaeta}F_{lphaeta})}{\partial (\partial_\mu A_
u)} = 0$$

即:

$$\partial_{\mu}F_{\mu
u}=0$$

一般电磁场的拉格朗日密度:

$$\mathscr{L}(A_{\mu},\partial_{\mu}A_{
u})=-rac{1}{4\mu_{0}}F_{\mu
u}F_{\mu
u}+J_{\mu}A_{\mu}$$

代入电磁场的拉格朗日方程,得:

$$\partial_{\mu}F_{\mu
u}=-\mu_{0}J_{
u}$$