1

> 求 $|\Psi_{AB}\rangle = x_1 |0_A, 0_B\rangle + x_2 |0_A, 1_B\rangle + x_3 |1_A, 0_B\rangle + x_4 |1_A, 1_B\rangle$ 的约化密度矩阵 ρ_A 和 ρ_B

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B(\rho_{AB}) \\ &= \langle 0_B|\rho_{AB}|0_B\rangle + \langle 1_B|\rho_{AB}|1_B\rangle \\ &= (|x_1|^2 + |x_2|^2)|0_A\rangle\langle 0_A| + (|x_3|^2 + |x_4|^2)|1_A\rangle\langle 1_A| + (x_1x_3^* + x_2x_4^*)|0_A\rangle\langle 1_A| + (x_3x_1^* + x_4x_2^*)|1_A\rangle\langle 0_A|\end{aligned}$$

$$\begin{aligned}\rho_B &= \mathrm{Tr}_A(\rho_{AB}) \\ &= \langle 0_A|\rho_{AB}|0_A\rangle + \langle 1_A|\rho_{AB}|1_A\rangle \\ &= (|x_1|^2 + |x_3|^2)|0_B\rangle\langle 0_B| + (|x_2|^2 + |x_4|^2)|1_B\rangle\langle 1_B| + (x_1x_2^* + x_3x_4^*)|0_B\rangle\langle 1_B| + (x_2x_1^* + x_4x_3^*)|1_B\rangle\langle 0_B|\end{aligned}$$

2

> 设双电子自旋态为：
>
> $$\rho_{AB} = \alpha|+_A^z, +_B^z\rangle\langle +_A^z, +_B^z| + (1-\alpha)|-_A^z, -_B^z\rangle\langle -_A^z, -_B^z| + \gamma|+_A^z, +_B^z\rangle\langle -_A^z, -_B^z| + \gamma^*|-_A^z, -_B^z\rangle\langle +_A^z, +_B^z|$$
>
> 求 $\hat{\sigma}_A^z$ 和 $\hat{\sigma}_A^x$ 在此态的平均值。

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B(\rho_{AB}) \\ &= \langle +_B^z|\rho_{AB}|+_B^z\rangle + \langle -_B^z|\rho_{AB}|-_B^z\rangle \\ &= \alpha|+_A^z\rangle\langle +_A^z| + (1-\alpha)|-_A^z\rangle\langle -_A^z|\end{aligned}$$

由于 $\hat{\sigma}_A^z|+_A^z\rangle = +1|+_A^z\rangle, \hat{\sigma}_A^z|-_A^z\rangle = -|-_A^z\rangle$，于是：

$$\hat{\sigma}_A^z\rho_A = \alpha|+_A^z\rangle\langle +_A^z| - (1-\alpha)|-_A^z\rangle\langle -_A^z|$$

$\hat{\sigma}_A^z$ 在 ρ_{AB} 态的平均值为：

$$\begin{aligned}\langle\hat{\sigma}_A^z\rangle &= \mathrm{Tr}_A(\hat{\sigma}_A^z\rho_A) \\ &= \langle +_A^z|\hat{\sigma}_A^z\rho_A|+_A^z\rangle + \langle -_A^z|\hat{\sigma}_A^z\rho_A|-_A^z\rangle \\ &= 2\alpha - 1\end{aligned}$$

由于：

$$\hat{\sigma}_A^x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \ \rho_A = \alpha|+_A^z\rangle\langle +_A^z| + (1-\alpha)|-_A^z\rangle\langle -_A^z| = \begin{bmatrix}\alpha & 0\\ 0 & 1-\alpha\end{bmatrix}$$

$$\hat{\sigma}_A^x\rho_A \doteq \begin{bmatrix}0 & 1-\alpha\\ \alpha & 0\end{bmatrix}$$

于是 $\hat{\sigma}_A^x$ 在 ρ_{AB} 的平均值为：

$$\langle\hat{\sigma}_A^x\rangle = \mathrm{Tr}_A(\hat{\sigma}_A^x\rho_A) = 0$$

3

> 求哈密顿量 $\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}[\hat{\sigma}_+\exp(-\mathrm{i}\omega_L t) + \mathrm{h.c.}]$ 在 $\hat{H}_0 = \frac{\hbar\omega_L}{2}\hat{\sigma}_z$ 的相互作用绘景的形式。

$$\begin{aligned}\hat{H} &= \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}[\hat{\sigma}_+\exp(-\mathrm{i}\omega_L t) + \hat{\sigma}_-\exp(\mathrm{i}\omega_L t)] \\ &= \frac{\hbar\omega_L}{2}\hat{\sigma}_z + \frac{\hbar(\omega_0-\omega_L)}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}[\hat{\sigma}_+\exp(-\mathrm{i}\omega_L t) + \hat{\sigma}_-\exp(\mathrm{i}\omega_L t)] \\ &= \hat{H}_0 + \hat{V}(t)\end{aligned}$$

其中，

$$
\hat{H}_0 = \frac{\hbar\omega_L}{2}\hat{\sigma}_z
$$

$$
\hat{V}(t) = \frac{\hbar(\omega_0-\omega_L)}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\left[\hat{\sigma}_+\exp(-\mathrm{i}\omega_L t) + \hat{\sigma}_-\exp(\mathrm{i}\omega_L t)\right]
$$

$\hat{H}_0$ 的相互作用绘景形式：

$$
\begin{aligned}
\hat{H}_{0,\mathcal{I}} &= \exp(\mathrm{i}\hat{H}_0 t/\hbar)\hat{H}_0\exp(-\mathrm{i}\hat{H}_0 t/\hbar) \\
&= \hat{H}_0 + \frac{1}{1!}\left[\mathrm{i}\hat{H}_0 t/\hbar, \hat{H}_0\right] + \cdots \\
&= \hat{H}_0 \\
&= \frac{\hbar\omega_L}{2}\hat{\sigma}_z
\end{aligned}
$$

注意到：

$$
\exp(\mathrm{i}\omega_L t\hat{\sigma}_z/2)\hat{\sigma}_z\exp(-\mathrm{i}\omega_L t\hat{\sigma}_z/2) = \hat{\sigma}_z
$$

$$
\begin{aligned}
\exp(\mathrm{i}\omega_L t\hat{\sigma}_z/2)\hat{\sigma}_+\exp(-\mathrm{i}\omega_L t\hat{\sigma}_z/2) &= \hat{\sigma}_+ + [\mathrm{i}\omega_L t\hat{\sigma}_z/2, \hat{\sigma}_+] + \frac{1}{2!}[\mathrm{i}\omega_L t\hat{\sigma}_z/2, [\mathrm{i}\omega_L t\hat{\sigma}_z/2, \hat{\sigma}_+]] + \cdots \\
&= \hat{\sigma}_+ + (\mathrm{i}\omega_L t/2)\hat{\sigma}_+ + \frac{(\mathrm{i}\omega_L t/2)^2}{2!}\hat{\sigma}_+ + \cdots \\
&= [1 + (\mathrm{i}\omega_L t/2) + \frac{(\mathrm{i}\omega_L t/2)^2}{2!} + \cdots]\hat{\sigma}_+ \\
&= \exp(\mathrm{i}\omega_L t/2)\hat{\sigma}_+
\end{aligned}
$$

同理，

$$
\begin{aligned}
\exp(\mathrm{i}\omega_L t\hat{\sigma}_z/2)\hat{\sigma}_-\exp(-\mathrm{i}\omega_L t\hat{\sigma}_z/2) &= \hat{\sigma}_- + [\mathrm{i}\omega_L t\hat{\sigma}_z/2, \hat{\sigma}_-] + \frac{1}{2!}[\mathrm{i}\omega_L t\hat{\sigma}_z/2, [\mathrm{i}\omega_L t\hat{\sigma}_z/2, \hat{\sigma}_-]] + \cdots \\
&= \hat{\sigma}_- + (-\mathrm{i}\omega_L t/2)\hat{\sigma}_- + \frac{(-\mathrm{i}\omega_L t/2)^2}{2!}\hat{\sigma}_- + \cdots \\
&= [1 + (-\mathrm{i}\omega_L t/2) + \frac{(-\mathrm{i}\omega_L t/2)^2}{2!} + \cdots]\hat{\sigma}_- \\
&= \exp(-\mathrm{i}\omega_L t/2)\hat{\sigma}_-
\end{aligned}
$$

于是可得 $\hat{V}(t)$ 的相互作用绘景形式：

$$
\begin{aligned}
\hat{V}_{\mathcal{I}}(t) &= \exp(\mathrm{i}\hat{H}_0 t/\hbar)\hat{V}(t)\exp(-\mathrm{i}\hat{H}_0 t/\hbar) \\
&= \exp(\mathrm{i}\omega_L t\hat{\sigma}_z/2)\hat{V}(t)\exp(-\mathrm{i}\omega_L t\hat{\sigma}_z/2) \\
&= \exp(\mathrm{i}\omega_L t\hat{\sigma}_z/2)\left\{\frac{\hbar(\omega_0-\omega_L)}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\left[\hat{\sigma}_+\exp(-\mathrm{i}\omega_L t) + \hat{\sigma}_-\exp(\mathrm{i}\omega_L t)\right]\right\}\exp(-\mathrm{i}\omega_L t\hat{\sigma}_z/2) \\
&= \frac{\hbar(\omega_0-\omega_L)}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\left[\exp(\mathrm{i}\omega_L t/2)\hat{\sigma}_+\exp(-\mathrm{i}\omega_L t) + \exp(-\mathrm{i}\omega_L t/2)\hat{\sigma}_-\exp(\mathrm{i}\omega_L t)\right] \\
&= \frac{\hbar(\omega_0-\omega_L)}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\left[\exp(-\mathrm{i}\omega_L t/2)\hat{\sigma}_+ + \exp(\mathrm{i}\omega_L t/2)\hat{\sigma}_-\right] \\
&= \frac{\hbar(\omega_0-\omega_L)}{2}\hat{\sigma}_z + \hbar\Omega\left[\cos(\omega_L t/2)\hat{\sigma}_x + \sin(\omega_L t/2)\hat{\sigma}_y\right]
\end{aligned}
$$

综上，

$$
\begin{aligned}
H_{\mathcal{I}} &= H_{0,\mathcal{I}} + V_{\mathcal{I}}(t) \\
&= \frac{\hbar\omega_L}{2}\hat{\sigma}_z + \frac{\hbar(\omega_0-\omega_L)}{2}\hat{\sigma}_z + \hbar\Omega\left[\cos(\omega_L t/2)\hat{\sigma}_x + \sin(\omega_L t/2)\hat{\sigma}_y\right] \\
&= \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\Omega\left[\cos(\omega_L t/2)\hat{\sigma}_x + \sin(\omega_L t/2)\hat{\sigma}_y\right]
\end{aligned}
$$

4

> 求哈密顿量 $\hat{H}=\dfrac{\hbar\omega_0}{2}\hat{\sigma}_z+\hbar\omega\hat{a}^\dagger\hat{a}+\hbar g\hat{\sigma}_x\left(\hat{a}+\hat{a}^\dagger\right)$ 在 $\hat{H}_0=\dfrac{\hbar\omega_0}{2}\hat{\sigma}_z+\hbar\omega\hat{a}\hat{a}^\dagger$ 的相互作用绘景的形式。

$$\hat{V}=\hbar g\hat{\sigma}_x\left(\hat{a}+\hat{a}^\dagger\right)$$

$$\hat{H}_{0,\mathcal{I}}=\hat{H}_0=\frac{\hbar\omega_0}{2}\hat{\sigma}_z+\hbar\omega\hat{a}\hat{a}^\dagger$$

由于:

$$\left[\frac{\omega_0}{2}\hat{\sigma}_z,\omega\hat{a}\hat{a}^\dagger\right]=0$$

于是:

$$\begin{aligned}\exp(\mathrm{i}\hat{H}_0t/\hbar)&=\exp\left[\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z+\omega\hat{a}\hat{a}^\dagger\right)\right]\\&=\exp\left[\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z\right)\right]\exp\left[\mathrm{i}t\left(\omega\hat{a}\hat{a}^\dagger\right)\right]\end{aligned}$$

于是:

$$\begin{aligned}\frac{\hat{V}_{\mathcal{I}}(t)}{\hbar g}&=\exp(\mathrm{i}\hat{H}_0t/\hbar)\frac{\hat{V}}{\hbar g}\exp(-\mathrm{i}\hat{H}_0t/\hbar)\\&=\exp\left[\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z+\omega\hat{a}\hat{a}^\dagger\right)\right]\left[\hat{\sigma}_x\left(\hat{a}+\hat{a}^\dagger\right)\right]\exp\left[-\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z+\omega\hat{a}\hat{a}^\dagger\right)\right]\\&=\exp\left[\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z\right)\right]\exp\left[\mathrm{i}t\left(\omega\hat{a}\hat{a}^\dagger\right)\right]\left[\hat{\sigma}_x\left(\hat{a}+\hat{a}^\dagger\right)\right]\exp\left[-\mathrm{i}t\left(\omega\hat{a}\hat{a}^\dagger\right)\right]\exp\left[-\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z\right)\right]\\&=\exp\left[\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z\right)\right]\hat{\sigma}_x\exp\left[-\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z\right)\right]\exp\left[\mathrm{i}t\left(\omega\hat{a}\hat{a}^\dagger\right)\right]\left(\hat{a}+\hat{a}^\dagger\right)\exp\left[-\mathrm{i}t\left(\omega\hat{a}\hat{a}^\dagger\right)\right]\\&=\frac{1}{0!}\hat{\sigma}_x+\frac{1}{1!}\left[\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z\right),\hat{\sigma}_x\right]+\cdots+\frac{1}{0!}\left(\hat{a}+\hat{a}^\dagger\right)+\frac{1}{1!}\left[\mathrm{i}t\left(\omega\hat{a}\hat{a}^\dagger\right),\left(\hat{a}+\hat{a}^\dagger\right)\right]+\cdots\\&=\frac{1}{0!}\hat{\sigma}_x+\frac{1}{1!}\left(-t\omega_0\hat{\sigma}_y\right)+\frac{1}{2!}\left[\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z\right),\left(-t\omega_0\hat{\sigma}_y\right)\right]+\cdots+\frac{1}{0!}\left(\hat{a}+\hat{a}^\dagger\right)+\frac{1}{1!}\mathrm{i}t\omega\left(\hat{a}^\dagger-\hat{a}\right)+\frac{1}{2!}\left[\mathrm{i}t\left(\omega\hat{a}\hat{a}^\dagger\right),\mathrm{i}t\omega\left(\hat{a}^\dagger-\hat{a}\right)\right]+\cdots\\&=\frac{1}{0!}\hat{\sigma}_x+\frac{1}{1!}\left(-t\omega_0\hat{\sigma}_y\right)+\frac{1}{2!}\left(-t^2\omega_0^2\hat{\sigma}_x\right)+\frac{1}{3!}\left[\mathrm{i}t\left(\frac{\omega_0}{2}\hat{\sigma}_z\right),\left(-t^2\omega_0^2\hat{\sigma}_x\right)\right]+\cdots+\frac{1}{0!}\left(\hat{a}+\hat{a}^\dagger\right)+\frac{1}{1!}\mathrm{i}t\omega\left(\hat{a}^\dagger-\hat{a}\right)+\frac{1}{2!}\mathrm{i}^2\omega^2t^2\left(\hat{a}+\hat{a}^\dagger\right)+\\&=\frac{1}{0!}\hat{\sigma}_x+\frac{1}{1!}\left(-t\omega_0\hat{\sigma}_y\right)+\frac{1}{2!}\left(-t^2\omega_0^2\hat{\sigma}_x\right)+\frac{1}{3!}t^3\omega_0^3\hat{\sigma}_y+\cdots+\frac{1}{0!}\left(\hat{a}+\hat{a}^\dagger\right)+\frac{1}{1!}\mathrm{i}t\omega\left(\hat{a}^\dagger-\hat{a}\right)+\frac{1}{2!}\mathrm{i}^2\omega^2t^2\left(\hat{a}+\hat{a}^\dagger\right)+\frac{1}{3!}\mathrm{i}^3\omega^3t^3\left(\hat{a}^\dagger-\hat{a}\right)\\&=\left[\frac{1}{0!}+\frac{1}{2!}\left(-t^2\omega_0^2\right)+\cdots\right]\hat{\sigma}_x+\left[\frac{1}{1!}\left(-t\omega_0\right)+\frac{1}{3!}\left(t^3\omega_0^3\right)+\cdots\right]\hat{\sigma}_y+\left[\frac{1}{0!}+\frac{1}{2!}\left(-\omega^2t^2\right)+\cdots\right]\left(\hat{a}+\hat{a}^\dagger\right)+\left[\frac{1}{1!}\mathrm{i}\omega t+\frac{1}{3!}\left(-\mathrm{i}\omega^3t^3\right)\right.\\&=\cos\left(\omega_0t\right)\hat{\sigma}_x-\sin\left(\omega_0t\right)\hat{\sigma}_y+\cos\left(\omega t\right)\left(\hat{a}+\hat{a}^\dagger\right)+\mathrm{i}\sin\left(\omega t\right)\left(\hat{a}^\dagger-\hat{a}\right)\\&=\cos\left(\omega_0t\right)\hat{\sigma}_x-\sin\left(\omega_0t\right)\hat{\sigma}_y+\mathrm{e}^{-\mathrm{i}\omega t}\hat{a}+\mathrm{e}^{\mathrm{i}\omega t}\hat{a}^\dagger\end{aligned}$$

于是:

$$\begin{aligned}\hat{H}_{\mathcal{I}}&=\hat{H}_{0,\mathcal{I}}+\hat{V}_{\mathcal{I}}(t)\\&=\frac{\hbar\omega_0}{2}\hat{\sigma}_z+\hbar\omega\hat{a}\hat{a}^\dagger+\hbar g\left[\cos\left(\omega_0t\right)\hat{\sigma}_x-\sin\left(\omega_0t\right)\hat{\sigma}_y+\mathrm{e}^{-\mathrm{i}\omega t}\hat{a}+\mathrm{e}^{\mathrm{i}\omega t}\hat{a}^\dagger\right]\end{aligned}$$

5

> 设仅存两种中微子 ν_μ 和 ν_τ，其哈密顿量为 $\hat{H}=\sum\limits_{j=1}^{2}E_j\left|\nu_j\right\rangle\left\langle\nu_j\right|$，其中 $E_j=\sqrt{c^2p^2+m_j^2c^4}$。已知两种中微子的状态可表示为 $\left|\nu_\mu\right\rangle=\cos\theta\left|\nu_1\right\rangle+\sin\theta\left|\nu_2\right\rangle$ 和 $\left|\nu_\tau\right\rangle=-\sin\theta\left|\nu_1\right\rangle+\cos\theta\left|\nu_2\right\rangle$，设 $t=0$ 时刻体系产生一个 ν_μ，求 t 时刻探测到 ν_τ 的概率。

采用薛定谔绘景，态矢随时间演化，哈密顿量不随时间演化。

时间演化算符为:

$$\hat{U}(t,0)=\exp(-\mathrm{i}\hat{H}t/\hbar)$$

初态:

$$|\psi(0)\rangle = |\nu_\mu\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

演化到 t 时刻的态矢：

$$\begin{aligned}
|\psi(t)\rangle &= \hat{U}(t,0)\,|\psi(0)\rangle \\
&= \exp(-\mathrm{i}\hat{H}t/\hbar)\,(\cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle) \\
&= \exp\left(-\mathrm{i}E_1 t/\hbar\right)\cos\theta\,|\nu_1\rangle + \exp\left(-\mathrm{i}E_2 t/\hbar\right)\sin\theta\,|\nu_2\rangle
\end{aligned}$$

t 时刻探测到 $|\nu_\tau\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$ 的概率：

$$\begin{aligned}
P &= |\langle\nu_\tau|\psi(t)\rangle|^2 \\
&= |(-\sin\theta\,\langle\nu_1| + \cos\theta\,\langle\nu_2|)\left(\exp\left(-\mathrm{i}E_1 t/\hbar\right)\cos\theta\,|\nu_1\rangle + \exp\left(-\mathrm{i}E_2 t/\hbar\right)\sin\theta\,|\nu_2\rangle\right)|^2 \\
&= \sin^2\theta\cos^2\theta\left[2 - 2\cos\left(\frac{E_1 - E_2}{\hbar}t\right)\right] \\
&= \sin^2(2\theta)\sin^2\left(\frac{E_1 - E_2}{2\hbar}t\right)
\end{aligned}$$

6

> 体系哈密顿量 $\hat{H}$ 不含时且具有非简并本征值 $\hbar\nu_n$ 和本征态 $|\nu_n\rangle$，物理量 $\hat{A}$ 的本征解 $\hat{A}\,|a_m\rangle = a_m\,|a_m\rangle$。设 $|\Psi(0)\rangle = |\nu_1\rangle$，此时测 $\hat{A}$ 得 a_m 的概率和总平均值为多少？；若测得 a_m，经 t 时间后再重复测量，再次得到 a_m 的概率为多少？

$|\Psi(0)\rangle = |\nu_1\rangle$，测 $\hat{A}$ 得 a_m 的概率：

$$\begin{aligned}
P &= |\,\langle a_m|\Psi(0)\rangle\,|^2 \\
&= |\,\langle a_m|\nu_1\rangle\,|^2
\end{aligned}$$

总平均值：

$$\begin{aligned}
\bar{A} &= \langle\Psi(0)|\hat{A}|\Psi(0)\rangle \\
&= \langle\nu_1|\hat{A}|\nu_1\rangle
\end{aligned}$$

若测得 a_m，则 $\Psi(0) = |\nu_1\rangle$ 塌缩到 $\Psi(0) = |a_m\rangle$，经时间 t 后，由于哈密顿量不含时，于是态矢演化到：

$$\begin{aligned}
|\Psi(t)\rangle &= \hat{U}(t,0)\,|\Psi(0)\rangle \\
&= \exp\left(-\mathrm{i}\hat{H}t/\hbar\right)|a_m\rangle \\
&= \sum_n \langle\nu_n|a_m\rangle \exp\left(-\mathrm{i}\hat{H}t/\hbar\right)|\nu_n\rangle \\
&= \sum_n c_{nm}\exp(-\mathrm{i}\nu_n t)\,|\nu_n\rangle\,,\quad c_{nm} = \langle\nu_n|a_m\rangle
\end{aligned}$$

再次对 $\hat{A}$ 进行测量，再次测得 a_m 的概率为：

$$\begin{aligned}
P' &= |\,\langle a_m|\Psi(t)\rangle\,|^2 \\
&= \left|\sum_n c_{nm}\,\langle a_m|\nu_n\rangle\exp(-\mathrm{i}\nu_n t)\right|^2 \\
&= \left|\sum_n |c_{nm}|^2\exp(-\mathrm{i}\nu_n t)\right|^2 \\
&= \sum_{n,l}|c_{nm}|^2|c_{lm}|^2\exp\left[\mathrm{i}\left(\nu_l - \nu_n\right)t\right]
\end{aligned}$$

7

> 求状态 $|\Psi\rangle = \cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}\,|-\rangle$ 的 Bloch 矢量 $\vec{r}$

σ_z 表象下，

$$|\Psi\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$$

$$\rho = |\Psi\rangle\langle\Psi| = \begin{bmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}$$

$$\begin{aligned} \vec{r} &= \mathrm{Tr}\left(\rho\vec{\sigma}\right) \\ &= \mathrm{Tr}\left(\begin{bmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{\mathrm{e}}_x + \begin{bmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}\vec{\mathrm{e}}_y + \begin{bmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\vec{\mathrm{e}}_z\right) \\ &= \sin\theta\vec{\mathrm{e}}_x + \cos\theta\vec{\mathrm{e}}_z \end{aligned}$$

8

> 设某二能级系统的哈密顿量为 $\hat{H} = \hbar\vec{\omega}\cdot\hat{\vec{\sigma}} = \hbar\sum_{j=x,y,z}\omega_j\hat{\sigma}_j$，求其 Bloch 矢量 $\vec{r}(t)$ 满足的动力学方程。

薛定谔绘景下密度矩阵的动力学方程为：

$$\mathrm{i}\hbar\frac{\mathrm{d}\rho}{\mathrm{d}t} = [\hat{H}, \rho]$$

在 $\hat{\sigma}_z$ 表象下，二能级系统的密度矩阵 ρ 可写为：

$$\rho = \frac{1}{2}\left(I + \vec{r}(t)\cdot\vec{\sigma}\right)$$

计算对易关系（采用爱因斯坦求和约定）：

$$\begin{aligned} [\hat{H}, \rho] &= \left[\hbar\omega_i\hat{\sigma}_i, \frac{1}{2}\left(I + \vec{r}(t)\cdot\hat{\vec{\sigma}}\right)\right] \\ &= \frac{\hbar}{2}\left[\omega_i\hat{\sigma}_i, r_j(t)\hat{\sigma}_j\right] \\ &= \frac{\hbar}{2}\omega_i r_j(t)\left[\hat{\sigma}_i, \hat{\sigma}_j\right] \\ &= \frac{\hbar}{2}\omega_i r_j(t)\cdot 2\mathrm{i}\varepsilon_{ijk}\hat{\sigma}_k \\ &= \mathrm{i}\hbar\varepsilon_{ijk}\omega_i r_j(t)\hat{\sigma}_k \\ &= \mathrm{i}\hbar\left\{\omega_x\left[y(t)\hat{\sigma}_z - z(t)\hat{\sigma}_y\right] + \omega_y\left[z(t)\hat{\omega}_x - x(t)\hat{\sigma}_z\right] + \omega_z\left[x(t)\hat{\sigma}_y - y(t)\hat{\sigma}_x\right]\right\} \end{aligned}$$

代入密度矩阵动力学方程可得 Bloch 矢量 $\vec{r}(t) = x(t)\vec{\mathrm{e}}_x + y(t)\vec{\mathrm{e}}_y + z(t)\vec{\mathrm{e}}_z$ 动力学方程：

$$\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t}\cdot\vec{\sigma} = 2\left\{\omega_x\left[y(t)\hat{\sigma}_z - z(t)\hat{\sigma}_y\right] + \omega_y\left[z(t)\hat{\omega}_x - x(t)\hat{\sigma}_z\right] + \omega_z\left[x(t)\hat{\sigma}_y - y(t)\hat{\sigma}_x\right]\right\}$$

9

> 求 $\hat{S}\hat{J}_z\hat{S}^\dagger$，其中 $\hat{S} = \exp\left(-\mathrm{i}\phi\hat{J}_z/\hbar\right)\exp\left(-\mathrm{i}\theta\hat{J}_y/\hbar\right)$

$$
\begin{aligned}
\exp\left(-\mathrm{i}\theta\hat{J}_y/\hbar\right)\hat{J}_z\exp\left(\mathrm{i}\theta\hat{J}_y/\hbar\right) &= \hat{J}_z+\frac{1}{1!}\left[-\mathrm{i}\theta\hat{J}_y/\hbar,\hat{J}_z\right]+\cdots \\
&= \hat{J}_z+\frac{1}{1!}\left(\theta\hat{J}_x\right)+\frac{1}{2!}\left[-\mathrm{i}\theta\hat{J}_y/\hbar,\theta\hat{J}_x\right]+\cdots \\
&= \hat{J}_z+\frac{1}{1!}\left(\theta\hat{J}_x\right)+\frac{1}{2!}\left(-\theta^2\hat{J}_z\right)+\frac{1}{3!}\left[-\mathrm{i}\theta\hat{J}_y/\hbar,-\theta^2\hat{J}_z\right]+\cdots \\
&= \hat{J}_z+\frac{1}{1!}\left(\theta\hat{J}_x\right)+\frac{1}{2!}\left(-\theta^2\hat{J}_z\right)+\frac{1}{3!}\left(-\theta^3\hat{J}_x\right)+\frac{1}{4!}\left[-\mathrm{i}\theta\hat{J}_y/\hbar,-\theta^3\hat{J}_x\right]+\cdots \\
&= \hat{J}_z+\frac{1}{1!}\left(\theta\hat{J}_x\right)+\frac{1}{2!}\left(-\theta^2\hat{J}_z\right)+\frac{1}{3!}\left(-\theta^3\hat{J}_x\right)+\frac{1}{4!}\left(\theta^4\hat{J}_z\right)+\cdots \\
&= \left[1+\frac{-\theta^2}{2!}+\frac{\theta^4}{4!}+\cdots\right]\hat{J}_z+\left[\frac{\theta}{1!}+\frac{-\theta^3}{3!}+\cdots\right]\hat{J}_x \\
&= \cos\theta\hat{J}_z+\sin\theta\hat{J}_x
\end{aligned}
$$

$$
\exp\left(-\mathrm{i}\phi\hat{J}_z/\hbar\right)\hat{J}_z\exp\left(\mathrm{i}\phi\hat{J}_z/\hbar\right)=\hat{J}_z
$$

$$
\begin{aligned}
\exp\left(-\mathrm{i}\phi\hat{J}_z/\hbar\right)\hat{J}_x\exp\left(\mathrm{i}\phi\hat{J}_z/\hbar\right) &= \hat{J}_x+\frac{1}{1!}\left[-\mathrm{i}\phi\hat{J}_z/\hbar,\hat{J}_x\right]+\cdots \\
&= \hat{J}_x+\frac{1}{1!}\left(\phi\hat{J}_y\right)+\frac{1}{2!}\left[-\mathrm{i}\phi\hat{J}_z/\hbar,\phi\hat{J}_y\right]+\cdots \\
&= \hat{J}_x+\frac{1}{1!}\left(\phi\hat{J}_y\right)+\frac{1}{2!}\left(-\phi^2\hat{J}_x\right)+\frac{1}{3!}\left[-\mathrm{i}\phi\hat{J}_z/\hbar,-\phi^2\hat{J}_x\right]+\cdots \\
&= \hat{J}_x+\frac{1}{1!}\left(\phi\hat{J}_y\right)+\frac{1}{2!}\left(-\phi^2\hat{J}_x\right)+\frac{1}{3!}\left(-\phi^3\hat{J}_y\right)+\cdots \\
&= \left[1+\frac{1}{2!}\left(-\phi^2\right)+\cdots\right]\hat{J}_x+\left[\frac{1}{1!}\left(\phi\right)+\frac{1}{3!}\left(-\phi^3\right)+\cdots\right]\hat{J}_y \\
&= \cos\phi\hat{J}_x+\sin\phi\hat{J}_y
\end{aligned}
$$

于是:

$$
\begin{aligned}
\hat{S}\hat{J}_z\hat{S}^\dagger &= \exp\left(-\mathrm{i}\phi\hat{J}_z/\hbar\right)\exp\left(-\mathrm{i}\theta\hat{J}_y/\hbar\right)\hat{J}_z\exp\left(\mathrm{i}\theta\hat{J}_y/\hbar\right)\exp\left(\mathrm{i}\phi\hat{J}_z/\hbar\right) \\
&= \exp\left(-\mathrm{i}\phi\hat{J}_z/\hbar\right)\left(\cos\theta\hat{J}_z+\sin\theta\hat{J}_x\right)\exp\left(\mathrm{i}\phi\hat{J}_z/\hbar\right) \\
&= \cos\theta\left(\hat{J}_z\right)+\sin\theta\left(\cos\phi\hat{J}_x+\sin\phi\hat{J}_y\right) \\
&= \sin\theta\cos\phi\hat{J}_x+\sin\theta\sin\phi\hat{J}_z+\cos\theta\hat{J}_z
\end{aligned}
$$

10

> 在 Ramsey 谱学中，需要测量如下二能级系统哈密顿量中的频率 $\mathit{\Delta}$: $\hat{H}=-\mathit{\Delta}\hat{\sigma}_z$。为此制备系统初态 $|\Psi(0)\rangle=\frac{1}{\sqrt{2}}\left(|+_z\rangle+|-_z\rangle\right)$，并让它在 $\hat{H}$ 支配下演化固定时间 T，然后测量 $\hat{\sigma}_x$，求测得 $+_x$ 的概率，从中解出 $\mathit{\Delta}$；如果重复该实验 N 次，计算得到 n 次 $+_x$ 的概率。

$\hat{\sigma}_z$ 表象下，

$$
|\Psi(0)\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},\ \ \hat{H}=-\mathit{\Delta}\begin{bmatrix}1&0\\0&-1\end{bmatrix}
$$

让 $|\Psi(0)\rangle$ 在 $\hat{H}$ 支配下演化时间 T，态矢演化到:

$$
\begin{aligned}
|\Psi(T)\rangle &= \exp(-\mathrm{i}\hat{H}T/\hbar)\,|\Psi(0)\rangle \\
&= \exp\left(\mathrm{i}T\mathit{\Delta}\hat{\sigma}_z/\hbar\right)\frac{1}{\sqrt{2}}\left(|+_z\rangle+|-_z\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left[\exp\left(\mathrm{i}T\mathit{\Delta}/\hbar\right)|+_z\rangle+\exp\left(-\mathrm{i}T\mathit{\Delta}/\hbar\right)|-_z\rangle\right]
\end{aligned}
$$

$\hat{\sigma}_z$ 表象下，

$$
|+_z\rangle=\begin{bmatrix}1\\0\end{bmatrix},\ \ |-_z\rangle=\begin{bmatrix}0\\1\end{bmatrix},\ \ |+_x\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}
$$

$$\langle +_x|+_z\rangle = \frac{1}{\sqrt{2}},\ \langle +_x|-_z\rangle = \frac{1}{\sqrt{2}}$$

对 $\hat{\sigma}_x$ 测量，测得 $+_x$ 的概率为：

$$\begin{aligned} P &= |\langle +_x|\Psi(T)\rangle|^2 \\ &= \left|\frac{1}{\sqrt{2}}\left[\exp\left(\mathrm{i}T\mathit{\Delta}/\hbar\right)\langle +_x|+_z\rangle + \exp\left(-\mathrm{i}T\mathit{\Delta}/\hbar\right)\langle +_x|-_z\rangle\right]\right|^2 \\ &= \cos^2\left(\frac{T\mathit{\Delta}}{\hbar}\right) \end{aligned}$$

将该实验重复 N 次，n 次得到 $+_x$ 的概率为：

$$\begin{aligned} P' &= \mathrm{C}_N^n P^n (1-P)^{N-n} \\ &= \frac{N!}{n!(N-n)!}\cos^{2n}\left(\frac{T\mathit{\Delta}}{\hbar}\right)\sin^{2(N-n)}\left(\frac{T\mathit{\Delta}}{\hbar}\right) \end{aligned}$$

11

> 求量子谐振子降算符 $\hat{a}$ 的本征解 $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$

$\hat{a}$ 的本征方程：

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

设 $|\alpha\rangle$ 可展为：

$$|\alpha\rangle = \sum_n C_n\,|n\rangle$$

代入 $\hat{a}$ 的本征方程得：

$$\sum_{n=1}^{\infty} C_n\sqrt{n}\,|n-1\rangle = \sum_{m=0}^{\infty}\alpha C_m\,|m\rangle$$

对比得：

$$\frac{C_n}{C_{n-1}} = \frac{\alpha}{\sqrt{n}}$$

于是：

$$C_n = \frac{C_n}{C_{n-1}}\frac{C_{n-1}}{C_{n-2}}\cdots\frac{C_2}{C_1}\frac{C_1}{C_0}C_0 = \frac{\alpha^n}{\sqrt{n!}}C_0$$

由归一化条件 $\langle\alpha|\alpha\rangle = 1$ 可得：

$$|C_0|^2\sum_{n=0}^{\infty}\frac{\left|\alpha^2\right|^n}{n!} = 1$$

即：

$$|C_0|^2\mathrm{e}^{|\alpha|^2} = 1$$

实数解为：

$$C_0 = \mathrm{e}^{-|\alpha|^2/2}$$

综上，α 可表达为：

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$$

12

> 证明量子谐振子降算符 $\hat{a}$ 的本征态 $|\alpha\rangle$ 可以写为 $|\alpha\rangle=\hat{D}(\alpha)\,|0\rangle$，其中 $\hat{D}(\alpha)=\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})$

注意到 $\hat{D}^\dagger(\alpha)$ 是幺正的：

$$\begin{aligned}\hat{D}^\dagger(\alpha)\hat{D}(\alpha)&=\exp\left(\alpha^*\hat{a}-\alpha\hat{a}^\dagger\right)\exp\left(\alpha\hat{a}^\dagger-\alpha^*\hat{a}\right)\\&=\exp\left(\left[\alpha^*\hat{a}-\alpha\hat{a}^\dagger\right]+\left[\alpha\hat{a}^\dagger-\alpha^*\hat{a}\right]+\frac{1}{2}\left[\alpha^*\hat{a}-\alpha\hat{a}^\dagger,\alpha\hat{a}^\dagger-\alpha^*\hat{a}\right]+\cdots\right)\\&=\hat{I}\end{aligned}$$

$$\begin{aligned}\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)&=\exp\left(\alpha^*\hat{a}-\alpha\hat{a}^\dagger\right)\hat{a}\exp\left(\alpha\hat{a}^\dagger-\alpha^*\hat{a}\right)\\&=\hat{a}+\frac{1}{1!}\left[\alpha^*\hat{a}-\alpha\hat{a}^\dagger,\hat{a}\right]+\cdots\\&=\hat{a}+\frac{1}{1!}\alpha+\frac{1}{2!}\left[\alpha^*\hat{a}-\alpha\hat{a}^\dagger,\alpha\right]+\cdots\\&=\hat{a}+\alpha\end{aligned}$$

两边左乘 $\hat{D}(\alpha)$ 得：

$$\hat{a}\hat{D}(\alpha)=\hat{D}(\alpha)\left(\hat{a}+\alpha\right)$$

同时作用在真空态上：

$$\hat{a}\hat{D}(\alpha)\,|0\rangle=\hat{D}(\alpha)\left(\hat{a}+\alpha\right)|0\rangle=\alpha\hat{D}(\alpha)\,|0\rangle$$

即：

$$\hat{a}\left(\hat{D}(\alpha)\,|0\rangle\right)=\alpha\left(\hat{D}(\alpha)\,|0\rangle\right)$$

与 $\hat{a}$ 的本征方程 $\hat{a}\,|\alpha\rangle=\alpha\,|\alpha\rangle$ 比较，结合归一化条件，得：

$$\hat{D}(\alpha)\,|0\rangle=|\alpha\rangle$$

13

> 求 $\hat{S}^\dagger\hat{a}\hat{S}$，其中 $\hat{S}=\exp\left[\frac{1}{2}\left(\xi^*\hat{a}^2-\xi\hat{a}^{\dagger2}\right)\right]$ 与 $\xi=r\mathrm{e}^{\mathrm{i}\theta}$

$$
\begin{aligned}
\hat{S}^{\dagger}\hat{a}\hat{S} &= \exp\left[\frac{1}{2}\left(\xi\hat{a}^{\dagger 2}-\xi^*\hat{a}^2\right)\right]\hat{a}\exp\left[\frac{1}{2}\left(\xi^*\hat{a}^2-\xi\hat{a}^{\dagger 2}\right)\right] \\
&= \frac{1}{0!}\hat{a}+\frac{1}{1!}\left[\frac{1}{2}\left(\xi\hat{a}^{\dagger 2}-\xi^*\hat{a}^2\right),\hat{a}\right]+\cdots \\
&= \frac{1}{0!}\hat{a}+\frac{1}{1!}\left(-\xi\hat{a}^{\dagger}\right)+\frac{1}{2!}\left[\frac{1}{2}\left(\xi\hat{a}^{\dagger 2}-\xi^*\hat{a}^2\right),-\xi\hat{a}^{\dagger}\right]+\cdots \\
&= \frac{1}{0!}\hat{a}+\frac{1}{1!}\left(-\xi\hat{a}^{\dagger}\right)+\frac{1}{2!}\left(\xi^*\xi\hat{a}\right)+\frac{1}{3!}\left[\frac{1}{2}\left(\xi\hat{a}^{\dagger 2}-\xi^*\hat{a}^2\right),\xi^*\xi\hat{a}\right]+\cdots \\
&= \frac{1}{0!}\hat{a}+\frac{1}{1!}\left(-\xi\hat{a}^{\dagger}\right)+\frac{1}{2!}\left(\xi^*\xi\hat{a}\right)+\frac{1}{3!}\left(-\xi\xi^*\xi\hat{a}^{\dagger}\right)+\frac{1}{4!}\left[\frac{1}{2}\left(\xi\hat{a}^{\dagger 2}-\xi^*\hat{a}^2\right),-\xi\xi^*\xi\hat{a}^{\dagger}\right]+\cdots \\
&= \frac{1}{0!}\hat{a}+\frac{1}{1!}\left(-\xi\hat{a}^{\dagger}\right)+\frac{1}{2!}\left(\xi^*\xi\hat{a}\right)+\frac{1}{3!}\left(-\xi\xi^*\xi\hat{a}^{\dagger}\right)+\frac{1}{4!}\left(\xi^*\xi\xi^*\xi\hat{a}\right)+\frac{1}{5!}\left[\frac{1}{2}\left(\xi\hat{a}^{\dagger 2}-\xi^*\hat{a}^2\right),\xi^*\xi\xi^*\xi\hat{a}\right]+\cdots \\
&= \frac{1}{0!}\hat{a}+\frac{1}{1!}\left(-\xi\hat{a}^{\dagger}\right)+\frac{1}{2!}\left(\xi^*\xi\hat{a}\right)+\frac{1}{3!}\left(-\xi\xi^*\xi\hat{a}^{\dagger}\right)+\frac{1}{4!}\left(\xi^*\xi\xi^*\xi\hat{a}\right)+\frac{1}{5!}\left(-\xi\xi^*\xi\xi^*\xi\hat{a}^{\dagger}\right)+\cdots \\
&= \left[\frac{1}{0!}|\xi|^0+\frac{1}{2!}|\xi|^2+\frac{1}{4!}|\xi|^4+\cdots\right]\hat{a}+\left[\frac{1}{1!}|\xi|^0+\frac{1}{3!}|\xi|^2+\frac{1}{5!}|\xi|^4+\cdots\right]\left(-\xi\hat{a}^{\dagger}\right) \\
&= \left[\frac{1}{0!}|\xi|^0+\frac{1}{2!}|\xi|^2+\frac{1}{4!}|\xi|^4+\cdots\right]\hat{a}+\left[\frac{1}{1!}|\xi|^1+\frac{1}{3!}|\xi|^3+\frac{1}{5!}|\xi|^5+\cdots\right]\left(-\frac{\xi}{|\xi|}\hat{a}^{\dagger}\right) \\
&= \left(\cosh|\xi|\right)\hat{a}+\left(\sinh|\xi|\right)\left(-\frac{\xi}{|\xi|}\hat{a}^{\dagger}\right) \\
&= \left(\cosh r\right)\hat{a}-\left(\sinh r\right)\mathrm{e}^{\mathrm{i}\theta}\hat{a}^{\dagger}
\end{aligned}
$$

14

> 用 Peres-Horodecki 判据判断如下态是纠缠态的条件，其中 $0\leqslant\lambda\leqslant1$，$|\Psi_{\pm}\rangle=\dfrac{|01\rangle\pm|10\rangle}{\sqrt{2}},|\Phi_{\pm}\rangle=\dfrac{|00\rangle\pm|11\rangle}{\sqrt{2}}$

> $\rho_1=\lambda|\Phi_+\rangle\langle\Phi_+|+(1-\lambda)|\Psi_+\rangle\langle\Psi_+|$

> $\rho_2=(1-\lambda)|\Psi_-\rangle\langle\Psi_-|+\lambda|11\rangle\langle11|$

> $\rho_3=\lambda|\Psi_-\rangle\langle\Psi_-|+\dfrac{1-\lambda}{3}(|\Psi_+\rangle\langle\Psi_+|+|\Phi_+\rangle\langle\Phi_+|+|\Phi_-\rangle\langle\Phi_-|)$

以 $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ 为基，

$$
|\Psi_+\rangle\langle\Psi_+|=\frac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{bmatrix},\quad |\Psi_-\rangle\langle\Psi_-|=\frac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{bmatrix},
$$

$$
|\Phi_+\rangle\langle\Phi_+|=\frac{1}{2}\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix},\quad |\Phi_-\rangle\langle\Phi_-|=\frac{1}{2}\begin{bmatrix}1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&1\end{bmatrix},
$$

$$
\begin{aligned}
\rho_1&=\lambda|\Phi_+\rangle\langle\Phi_+|+(1-\lambda)|\Psi_+\rangle\langle\Psi_+| \\
&=\begin{bmatrix}\frac{\lambda}{2}&0&0&\frac{\lambda}{2}\\0&\frac{1-\lambda}{2}&\frac{1-\lambda}{2}&0\\0&\frac{1-\lambda}{2}&\frac{1-\lambda}{2}&0\\\frac{\lambda}{2}&0&0&\frac{\lambda}{2}\end{bmatrix}
\end{aligned}
$$

部分转置：

$$
\rho_1^{T_B}=\begin{bmatrix}\frac{\lambda}{2}&0&0&\frac{1-\lambda}{2}\\0&\frac{1-\lambda}{2}&\frac{\lambda}{2}&0\\0&\frac{\lambda}{2}&\frac{1-\lambda}{2}&0\\\frac{1-\lambda}{2}&0&0&\frac{\lambda}{2}\end{bmatrix}
$$

利用 Mathematica，可以解得 $\rho_1^{T_B}$ 的本征值为：

$$\lambda'_{1,2,3,4}=\frac{1}{2},\frac{1}{2},\frac{1-2\lambda}{2},\frac{-1+2\lambda}{2}$$

当 $\lambda=\frac{1}{2}$，ρ_1 不是纠缠态；当 $0\leqslant\lambda\leqslant 1,\lambda\neq\frac{1}{2}$，$\rho_1$ 是纠缠态。

$$\begin{aligned}\rho_2&=(1-\lambda)\left|\Psi_-\right\rangle\left\langle\Psi_-\right|+\lambda\left|11\right\rangle\left\langle 11\right|\\&=\begin{bmatrix}0&0&0&0\\0&\frac{1-\lambda}{2}&\frac{\lambda-1}{2}&0\\0&\frac{\lambda-1}{2}&\frac{1-\lambda}{2}&0\\0&0&0&\lambda\end{bmatrix}\end{aligned}$$

部分转置：

$$\rho_2^{T_B}=\begin{bmatrix}0&0&0&\frac{\lambda-1}{2}\\0&\frac{1-\lambda}{2}&0&0\\0&0&\frac{1-\lambda}{2}&0\\\frac{\lambda-1}{2}&0&0&\lambda\end{bmatrix}$$

利用 Mathematica，可以解得 $\rho_2^{T_B}$ 的特征值为：

$$\lambda'_{1,2,3,4}=\frac{1-\lambda}{2},\frac{1-\lambda}{2},\frac{\lambda-\sqrt{1-2\lambda+2\lambda^2}}{2},\frac{\lambda+\sqrt{1-2\lambda+2\lambda^2}}{2}$$

当 $0\leqslant\lambda\leqslant 1$，四个本征值 $\lambda'_{1,2,3,4}\geqslant 0$，因此 ρ_2 不是纠缠态。

$$\begin{aligned}\rho_3&=\lambda\left|\Psi_-\right\rangle\left\langle\Psi_-\right|+\frac{1-\lambda}{3}(\left|\Psi_+\right\rangle\left\langle\Psi_+\right|+\left|\Phi_+\right\rangle\left\langle\Phi_+\right|+\left|\Phi_-\right\rangle\left\langle\Phi_-\right|)\\&=\begin{bmatrix}\frac{1-\lambda}{3}&0&0&0\\0&\frac{1+2\lambda}{6}&\frac{1-4\lambda}{6}&0\\0&\frac{1-4\lambda}{6}&\frac{1+2\lambda}{6}&0\\0&0&0&\frac{1-\lambda}{3}\end{bmatrix}\end{aligned}$$

部分转置 $\rho_3^{T_B}$ 为：

$$\rho_3^{T_B}=\begin{bmatrix}\frac{1-\lambda}{3}&0&0&\frac{1-4\lambda}{6}\\0&\frac{1+2\lambda}{6}&0&0\\0&0&\frac{1+2\lambda}{6}&0\\\frac{1-4\lambda}{6}&0&0&\frac{1-\lambda}{3}\end{bmatrix}$$

利用 Mathematica，可以解得本征值为：

$$\lambda'_{1,2,3,4}=\frac{1-2\lambda}{2},\frac{1+2\lambda}{6},\frac{1+2\lambda}{6},\frac{1+2\lambda}{6}$$

当 $0\leqslant\lambda\leqslant\frac{1}{2}$，$\rho_3$ 不是纠缠态；当 $\frac{1}{2}<\lambda\leqslant 1$，$\rho_3$ 是纠缠态。

15

> 定义量子谐振子的两个正交分量 $\hat{X}_1=\dfrac{\hat{a}+\hat{a}^\dagger}{2}$ 和 $\hat{X}_2=\dfrac{\hat{a}-\hat{a}^\dagger}{2\mathrm{i}}$，求在降算符本征态 $|\alpha\rangle$ 的 $\Delta\hat{X}_\theta$，其中 $\hat{X}_\theta=\cos\theta\hat{X}_1+\sin\theta\hat{X}_2$

$$\begin{aligned}\hat{X}_\theta&=\cos\theta\hat{X}_1+\sin\theta\hat{X}_2\\&=\cos\theta\frac{\hat{a}+\hat{a}^\dagger}{2}+\sin\theta\frac{\hat{a}-\hat{a}^\dagger}{2\mathrm{i}}\\&=\left(\frac{\cos\theta}{2}+\frac{\sin\theta}{2\mathrm{i}}\right)\hat{a}+\left(\frac{\cos\theta}{2}-\frac{\sin\theta}{2\mathrm{i}}\right)\hat{a}^\dagger\\&=\frac{\cos\theta-\mathrm{i}\sin\theta}{2}\hat{a}+\frac{\cos\theta+\mathrm{i}\sin\theta}{2}\hat{a}^\dagger\\&=\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\hat{a}+\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\hat{a}^\dagger\end{aligned}$$

$$
\begin{aligned}
\hat{X}_\theta^2 &= \left(\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\hat{a} + \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\hat{a}^\dagger\right)^2 \\
&= \frac{1}{4}\mathrm{e}^{-\mathrm{i}2\theta}\hat{a}^2 + \frac{1}{4}\mathrm{e}^{\mathrm{i}2\theta}\hat{a}^{\dagger 2} + \frac{1}{4}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)
\end{aligned}
$$

$$
\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \quad \left\langle\alpha\right|\hat{a}^\dagger = \alpha^*\left\langle\alpha\right|, \quad \left[\hat{a}, \hat{a}^\dagger\right] = 1 \Longrightarrow \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1
$$

在 $\left|\alpha\right\rangle$ 态下 $\hat{X}_\theta$ 的平均值:

$$
\begin{aligned}
\left\langle\hat{X}_\theta\right\rangle &= \left\langle\alpha\middle|\hat{X}_\theta\middle|\alpha\right\rangle \\
&= \left\langle\alpha\middle|\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\hat{a} + \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\hat{a}^\dagger\middle|\alpha\right\rangle \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\alpha + \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\alpha^*
\end{aligned}
$$

在 $\left|\alpha\right\rangle$ 态下 $\hat{X}_\theta^2$ 的平均值:

$$
\begin{aligned}
\left\langle\hat{X}_\theta^2\right\rangle &= \left\langle\alpha\middle|\frac{1}{4}\mathrm{e}^{-\mathrm{i}2\theta}\hat{a}^2 + \frac{1}{4}\mathrm{e}^{\mathrm{i}2\theta}\hat{a}^{\dagger 2} + \frac{1}{4}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)\middle|\alpha\right\rangle \\
&= \left\langle\alpha\middle|\frac{1}{4}\mathrm{e}^{-\mathrm{i}2\theta}\hat{a}^2 + \frac{1}{4}\mathrm{e}^{\mathrm{i}2\theta}\hat{a}^{\dagger 2} + \frac{1}{4}\left(\hat{a}^\dagger\hat{a} + 1 + \hat{a}^\dagger\hat{a}\right)\middle|\alpha\right\rangle \\
&= \frac{1}{4}\mathrm{e}^{-\mathrm{i}2\theta}\alpha^2 + \frac{1}{4}\mathrm{e}^{\mathrm{i}2\theta}\alpha^{*2} + \frac{1}{2}|\alpha|^2 + \frac{1}{4}
\end{aligned}
$$

在 $\left|\alpha\right\rangle$ 态下 $\hat{X}_\theta$ 的不确定度:

$$
\begin{aligned}
\Delta\hat{X}_\theta &= \sqrt{\left\langle\hat{X}_\theta^2\right\rangle - \left\langle\hat{X}_\theta\right\rangle^2} \\
&= \sqrt{\frac{1}{4}\mathrm{e}^{-\mathrm{i}2\theta}\alpha^2 + \frac{1}{4}\mathrm{e}^{\mathrm{i}2\theta}\alpha^{*2} + \frac{1}{2}|\alpha|^2 + \frac{1}{4} - \left(\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\alpha + \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\alpha^*\right)^2} \\
&= \frac{1}{2}
\end{aligned}
$$

16

> 求 $\left|\Psi\right\rangle = \sum_{i,j=0}^{1} a_{ij}\left|ij\right\rangle$ 的 von Neumann 熵，其中 $\sum_{i,j=0}^{1}\left|a_{ij}\right|^2 = 1$

以 $\{\left|00\right\rangle, \left|01\right\rangle, \left|10\right\rangle, \left|11\right\rangle\}$ 为基，密度矩阵可写为:

$$
\begin{aligned}
\rho &= \left|\Psi\right\rangle\left\langle\Psi\right| \\
&= \begin{bmatrix}
\left|a_{00}\right|^2 & a_{00}a_{01}^* & a_{00}a_{10}^* & a_{00}a_{11}^* \\
a_{01}a_{00}^* & \left|a_{01}\right|^2 & a_{01}a_{10}^* & a_{01}a_{11}^* \\
a_{10}a_{00}^* & a_{10}a_{01}^* & \left|a_{10}\right|^2 & a_{10}a_{11}^* \\
a_{11}a_{00}^* & a_{11}a_{01}^* & a_{11}a_{10}^* & \left|a_{11}\right|^2
\end{bmatrix}
\end{aligned}
$$

利用 Mathematica，可以解得其本征值分别为:

$$
\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = \left|a_{00}\right|^2 + \left|a_{01}\right|^2 + \left|a_{10}\right|^2 + \left|a_{11}\right|^2 = 1
$$

因此 ρ 可化为:

$$
\rho = PDP^{-1}
$$

其中,

$$
D = \mathrm{diag}\left(0, 0, 0, 1\right)
$$

von Neumann 熵 为:

$$
\begin{aligned}
S &= -\mathrm{Tr}\left(\rho\ln\rho\right)\\
&= -\mathrm{Tr}\left(PDP^{-1}P\ln DP^{-1}\right)\\
&= -\mathrm{Tr}\left(PD\ln DP^{-1}\right)\\
&= -\mathrm{Tr}\left(D\ln DP^{-1}P\right)\\
&= -\mathrm{Tr}\left(D\ln D\right)\\
&= -\mathrm{Tr}\left[\mathrm{diag}\left(0,0,0,1\right)\mathrm{diag}\left(-\infty,-\infty,-\infty,0\right)\right]\\
&= -\mathrm{Tr}\left[\mathrm{diag}\left(0,0,0,0\right)\right]\\
&= 0
\end{aligned}
$$

17

> 求以下状态的 Concurrence：(1) Bell 态：$|\Psi_+\rangle = \dfrac{|01\rangle+|10\rangle}{\sqrt{2}}$；(2) Werner 态：$\rho_{\mathrm{W}} = p\,|\Psi_+\rangle\langle\Psi_+| + (1-p)\dfrac{I_{4\times4}}{4}$，其中 $I_{4\times4}$ 为四维单位矩阵。

（1）

$\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ 基矢下，

$$
\rho_{AB} = |\Psi_+\rangle\langle\Psi_+| \quad = \frac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{bmatrix}
$$

$$
\begin{aligned}
\sigma_y\otimes\sigma_y &= \begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix}\otimes\sigma_y\\
&= \begin{bmatrix}0\sigma_y&-\mathrm{i}\sigma_y\\\mathrm{i}\sigma_y&0\sigma_y\end{bmatrix}\\
&= \begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix}
\end{aligned}
$$

利用 Mathematica 计算可得，$\rho_{AB}\left(\sigma_y\otimes\sigma_y\rho_{AB}^*\sigma_y\otimes\sigma_y\right)$ 的本征值为：

$$
\lambda'_{1,2,3,4} = 1,0,0,0
$$

本征值的平方根为：

$$
\lambda_{1,2,3,4} = \sqrt{\lambda'_{1,2,3,4}} = 1,0,0,0
$$

Concurrence 为：

$$
C = \max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\} = 1
$$

（2）

$\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ 基矢下，

$$
\begin{aligned}
\rho_{\mathrm{W}} &= p\,|\Psi_+\rangle\langle\Psi_+| + (1-p)\frac{I_{4\times4}}{4}\\
&= \begin{bmatrix}\frac{1-p}{4}&0&0&0\\0&\frac{1+p}{4}&\frac{p}{2}&0\\0&\frac{p}{2}&\frac{1+p}{4}&0\\0&0&0&\frac{1-p}{4}\end{bmatrix}
\end{aligned}
$$

利用 Mathematica 计算可得，$\rho_{\mathrm{W}}\left(\sigma_y\otimes\sigma_y\rho_{\mathrm{W}}^*\sigma_y\otimes\sigma_y\right)$ 的本征值为：

$$
\lambda'_{1,2,3,4} = \frac{(3p+1)^2}{16},\frac{(p-1)^2}{16},\frac{(p-1)^2}{16},\frac{(p-1)^2}{16}
$$

本征值的平方根为：

$$
\begin{aligned}
\lambda_{1,2,3,4} = \sqrt{\lambda'_{1,2,3,4}} = \frac{3p+1}{4}, \frac{1-p}{4}, \frac{1-p}{4}, \frac{1-p}{4} \\
C &= \max\left\{0, \frac{3p+1}{4} - 3\times\frac{1-p}{4}\right\} \\
&= \max\left\{0, \frac{3p-1}{2}\right\} \\
&= \begin{cases} 0 & ,\ 0\leqslant p\leqslant \frac{1}{3} \\ \frac{3p-1}{2} & ,\ \frac{1}{3}<p\leqslant 1 \end{cases}
\end{aligned}
$$

18

> 两种电子自旋处于 $\left|\Psi^{AB}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_z^A -_z^B\right\rangle - \left|-_z^A +_z^B\right\rangle\right)$

> （1）先后测量 $\hat{S}_z^A$ 和 $\hat{S}_z^B$，测值和概率为多少？

先测量 $\hat{S}_z^A$，测得 $+\hbar/2$ 的概率为 $1/2$，测得 $-\hbar/2$ 的概率为 $1/2$。

若测得 $\hat{S}_z^A$ 测值为 $+\hbar/2$ 后对 $\hat{S}_z^B$ 进行测量，测得 $\hat{S}_z^B$ 测值 $-\hbar/2$ 的概率为 1

若测得 $\hat{S}_z^A$ 测值为 $-\hbar/2$ 后对 $\hat{S}_z^B$ 进行测量，测得 $\hat{S}_z^B$ 测值 $+\hbar/2$ 的概率为 1

> （2）先后测量 $\hat{S}_x^A$ 和 $\hat{S}_x^B$，测值和概率为多少？

由于：

$$
\left|+_z^A\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_x^A\right\rangle + \left|-_x^A\right\rangle\right),\ \left|-_z^A\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_x^A\right\rangle - \left|-_x^A\right\rangle\right)
$$

$$
\left|+_z^B\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_x^B\right\rangle + \left|-_x^B\right\rangle\right),\ \left|-_z^B\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_x^B\right\rangle - \left|-_x^B\right\rangle\right)
$$

于是：

$$
\begin{aligned}
\left|\Psi^{AB}\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|+_z^A -_z^B\right\rangle - \left|-_z^A +_z^B\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\left|+_x^A\right\rangle + \left|-_x^A\right\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(\left|+_x^B\right\rangle - \left|-_x^B\right\rangle\right) - \frac{1}{\sqrt{2}}\left(\left|+_x^A\right\rangle - \left|-_x^A\right\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(\left|+_x^B\right\rangle + \left|-_x^B\right\rangle\right)\right] \\
&= \frac{1}{\sqrt{2}}\left(\left|-_x^A +_x^B\right\rangle - \left|+_x^A -_x^B\right\rangle\right)
\end{aligned}
$$

先测量 $\hat{S}_x^A$，测得 $+\hbar/2$ 的概率为 $1/2$，测得 $-\hbar/2$ 的概率为 $1/2$。

若测得 $\hat{S}_x^A$ 测值为 $+\hbar/2$ 后对 $\hat{S}_x^B$ 进行测量，测得 $\hat{S}_x^B$ 测值 $-\hbar/2$ 的概率为 1

若测得 $\hat{S}_x^A$ 测值为 $-\hbar/2$ 后对 $\hat{S}_x^B$ 进行测量，测得 $\hat{S}_x^B$ 测值 $+\hbar/2$ 的概率为 1

> （3）先后测量 $\hat{S}_n^A$ 和 $\hat{S}_n^B$，测值和概率为多少？其中，$\hat{S}_n = \vec{n}\cdot\hat{\vec{S}}, \vec{n} = \sin\theta\cos\varphi\vec{e}_x + \sin\theta\sin\varphi\vec{e}_y + \cos\theta\vec{e}_z$

在 $\hat{\sigma}_z$ 表象下，

$$
\begin{aligned}
\hat{\sigma}_n &= \vec{n}\cdot\hat{\vec{\sigma}} \\
&= \sin\theta\cos\varphi\hat{\sigma}_x + \sin\theta\sin\varphi\hat{\sigma}_y + \cos\theta\hat{\sigma}_z \\
&= \begin{bmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{bmatrix}
\end{aligned}
$$

$\hat{\sigma}_n$ 的两个本征态 $\left|+_n\right\rangle$ 和 $\left|-_n\right\rangle$ 在 $\hat{\sigma}_z$ 表象下的表示为：

$$
\left|+_n\right\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{bmatrix},\ \left|-_n\right\rangle = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\varphi} \end{bmatrix}
$$

即：

$$\begin{cases}|+_n\rangle=\cos\dfrac{\theta}{2}|+_z\rangle+\sin\dfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}|-_z\rangle\\|-_n\rangle=\sin\dfrac{\theta}{2}|+_z\rangle-\cos\dfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}|-_z\rangle\end{cases}\Longrightarrow\begin{cases}|+_z\rangle=\cos\dfrac{\theta}{2}|+_n\rangle+\sin\dfrac{\theta}{2}|-_n\rangle\\|-_z\rangle=\sin\dfrac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}|+_n\rangle-\cos\dfrac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}|-_n\rangle\end{cases}$$

于是：

$$\begin{aligned}\left|\Psi^{AB}\right\rangle&=\frac{1}{\sqrt{2}}\left(\left|+_z^A-_z^B\right\rangle-\left|-_z^A+_z^B\right\rangle\right)\\&=\frac{1}{\sqrt{2}}\left[\left(\cos\frac{\theta}{2}\left|+_n^A\right\rangle+\sin\frac{\theta}{2}\left|-_n^A\right\rangle\right)\otimes\left(\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}\left|+_n^B\right\rangle-\cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}\left|-_n^B\right\rangle\right)-\left(\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}\left|+_n^A\right\rangle-\cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}\left|-_n^A\right\rangle\right)\otimes\left(\cos\frac{\theta}{2}\left|+_n^B\right.\right.\\&=\frac{\mathrm{e}^{-\mathrm{i}\varphi}}{\sqrt{2}}\left(\left|-_n^A+_n^B\right\rangle-\left|+_n^A-_n^B\right\rangle\right)\end{aligned}$$

先测量 $\hat{S}_n^A$，测得 $+\hbar/2$ 的概率为 $1/2$，测得 $-\hbar/2$ 的概率为 $1/2$。

若测得 $\hat{S}_n^A$ 测值为 $+\hbar/2$ 后对 $\hat{S}_n^B$ 进行测量，测得 $\hat{S}_n^B$ 测值 $-\hbar/2$ 的概率为 1

若测得 $\hat{S}_n^A$ 测值为 $-\hbar/2$ 后对 $\hat{S}_n^B$ 进行测量，测得 $\hat{S}_n^B$ 测值 $+\hbar/2$ 的概率为 1

19

> 求证三粒子自旋态 $|W\rangle=\dfrac{1}{\sqrt{3}}(|+_z-_z-_z\rangle+|-_z+_z-_z\rangle+|-_z-_z+_z\rangle)$ 是总自旋算符平方及其第三分量的共同本征态 $\left|\frac{3}{2},-\frac{1}{2}\right\rangle$

$|+_z-_z-_z\rangle\equiv|a\rangle\,,|-_z+_z-_z\rangle\equiv|b\rangle\,,|-_z-_z+_z\rangle\equiv|c\rangle$

由于：

$$\begin{aligned}\hat{S}_z|W\rangle&=\left(\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}\right)\frac{1}{\sqrt{3}}(|a\rangle+|b\rangle+|c\rangle)\\&=\frac{1}{\sqrt{3}}\cdot\frac{\hbar}{2}(|a\rangle-|b\rangle-|c\rangle-|a\rangle+|b\rangle-|c\rangle-|a\rangle-|b\rangle+|c\rangle)\\&=\frac{-\hbar}{2\sqrt{3}}(|a\rangle+|b\rangle+|c\rangle)\\&=-\frac{\hbar}{2}|W\rangle\end{aligned}$$

$$\begin{aligned}\hat{S}^2&=\left(\hat{\vec{S}}_1+\hat{\vec{S}}_2+\hat{\vec{S}}_3\right)\cdot\left(\hat{\vec{S}}_1+\hat{\vec{S}}_2+\hat{\vec{S}}_3\right)\\&=\hat{S}_1^2+\hat{S}_2^2+\hat{S}_3^2+2\left(\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2+\hat{\vec{S}}_2\cdot\hat{\vec{S}}_3+\hat{\vec{S}}_3\cdot\hat{\vec{S}}_1\right)\end{aligned}$$

$$\begin{aligned}\hat{S}_1^2|W\rangle&=\frac{1}{\sqrt{3}}\hat{S}_1^2(|+_z-_z-_z\rangle+|-_z+_z-_z\rangle+|-_z-_z+_z\rangle)\\&=\frac{3\hbar^2}{4}\frac{1}{\sqrt{3}}(|+_z-_z-_z\rangle+|-_z+_z-_z\rangle+|-_z-_z+_z\rangle)\\&=\frac{3\hbar^2}{4}|W\rangle\end{aligned}$$

$$\begin{aligned}\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2&=\hat{S}_{1x}\hat{S}_{2x}+\hat{S}_{1y}\hat{S}_{2y}+\hat{S}_{1z}\hat{S}_{2z}\\&=\frac{\hat{S}_{1+}+\hat{S}_{1-}}{2}\frac{\hat{S}_{2+}+\hat{S}_{2-}}{2}+\frac{\hat{S}_{1+}-\hat{S}_{1-}}{2\mathrm{i}}\frac{\hat{S}_{2+}-\hat{S}_{2-}}{2\mathrm{i}}+\hat{S}_{1z}\hat{S}_{2z}\\&=\frac{1}{2}\left(\hat{S}_{1+}\hat{S}_{2-}+\hat{S}_{1-}\hat{S}_{2+}\right)+\hat{S}_{1z}\hat{S}_{2z}\end{aligned}$$

$$\begin{aligned}\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2\left|W\right\rangle &= \left(\frac{1}{2}\left(\hat{S}_{1+}\hat{S}_{2-}+\hat{S}_{1-}\hat{S}_{2+}\right)+\hat{S}_{1z}\hat{S}_{2z}\right)\frac{1}{\sqrt{3}}(\left|+_z-_z-_z\right\rangle+\left|-_z+_z-_z\right\rangle+\left|-_z-_z+_z\right\rangle)\\ &= \frac{1}{\sqrt{3}}\left[\frac{1}{2}\left(\hbar^2\left|+_z-_z-_z\right\rangle+\hbar^2\left|-_z+_z-_z\right\rangle\right)+\frac{\hbar^2}{4}\left(-\left|+_z-_z-_z\right\rangle-\left|-_z+_z-_z\right\rangle+\left|-_z-_z+_z\right\rangle\right)\right]\\ &= \frac{\hbar^2}{4\sqrt{3}}\left(\left|a\right\rangle+\left|b\right\rangle+\left|c\right\rangle\right)\\ &= \frac{\hbar^2}{4}\left|W\right\rangle\end{aligned}$$

$$\begin{aligned}\left(\hat{S}_1^2+2\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2\right)\left|W\right\rangle &= \frac{3\hbar^2}{4}\left|W\right\rangle+2\cdot\frac{\hbar^2}{4}\left|W\right\rangle\\ &= \frac{5\hbar^2}{4}\left|W\right\rangle\end{aligned}$$

由轮换对称性可知：

$$\left(\hat{S}_1^2+2\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2\right)\left|W\right\rangle=\left(\hat{S}_2^2+2\hat{\vec{S}}_2\cdot\hat{\vec{S}}_3\right)\left|W\right\rangle=\left(\hat{S}_3^2+2\hat{\vec{S}}_3\cdot\hat{\vec{S}}_1\right)\left|W\right\rangle=\frac{5\hbar^2}{4}\left|W\right\rangle$$

$$\begin{aligned}\hat{S}^2\left|W\right\rangle &= \left[\hat{S}_1^2+\hat{S}_2^2+\hat{S}_3^2+2\left(\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2+\hat{\vec{S}}_2\cdot\hat{\vec{S}}_3+\hat{\vec{S}}_3\cdot\hat{\vec{S}}_1\right)\right]\left|W\right\rangle\\ &= \frac{15\hbar^2}{4}\left|W\right\rangle\end{aligned}$$

$$\boxed{\hat{S}_z\left|W\right\rangle=-\frac{\hbar}{2}\left|W\right\rangle=-\frac{1}{2}\hbar\left|W\right\rangle,\ \ \hat{S}^2\left|W\right\rangle=\frac{15\hbar^2}{4}=\frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2\left|W\right\rangle}$$

综上，$\left|W\right\rangle$ 是总自旋算符平方 $\hat{S}^2$ 及其第三分量 $\hat{S}_z$ 的共同本征态 $\left|\frac{3}{2},-\frac{1}{2}\right\rangle$

20

> 考虑一个处于 $\left|+_z\right\rangle$ 的粒子，执行 N 次关于算符 $\hat{\sigma}_k=\vec{n}_k\cdot\hat{\vec{\sigma}}$ 的测量，其中 $\vec{n}_k=\sin\frac{k\pi}{2N}\vec{e}_x+\cos\frac{k\pi}{2N}\vec{e}_z(k=1,2,\cdots,N)$，求：

> （1）全部测量结果都是 $+1$ 的概率。当 $N\to\infty$ 时出现什么？

在 $\hat{\sigma}_z$ 表象下，

$$\hat{\sigma}_k=\vec{n}_k\cdot\hat{\vec{\sigma}}=\sin\frac{k\pi}{2N}\hat{\sigma}_x+\cos\frac{k\pi}{2N}\hat{\sigma}_z=\begin{bmatrix}\cos\frac{k\pi}{2N} & \sin\frac{k\pi}{2N}\\ \sin\frac{k\pi}{2N} & -\cos\frac{k\pi}{2N}\end{bmatrix}$$

$\hat{\sigma}_k$ 的本征方程：

$$\hat{\sigma}_k\left|\lambda\right\rangle=\lambda\left|\lambda\right\rangle$$

解得：

$$\lambda=\pm1$$

当 $\lambda=+1$，对应的本征态记为 $\left|+_k\right\rangle$，在 $\hat{\sigma}_z$ 表象下，

$$\left|+_k\right\rangle=\begin{bmatrix}\cos\frac{k\pi}{4N}\\ \sin\frac{k\pi}{4N}\end{bmatrix}$$

当 $\lambda=-1$，对应的本征态记为 $\left|-_k\right\rangle$，在 $\hat{\sigma}_z$ 表象下，

$$\left|-_k\right\rangle=\begin{bmatrix}\sin\frac{k\pi}{4N}\\ -\cos\frac{k\pi}{4N}\end{bmatrix}$$

第一次测量 $\hat{\sigma}_1$，$k=1$，测量前系统处于 $\left|+_z\right\rangle$，测得 $+1$ 的概率为：

$$\begin{aligned}P_1 &= \left|\left\langle+_z\middle|+_1\right\rangle\right|^2\\ &= \cos^2\frac{\pi}{4N}\end{aligned}$$

第一次测量 $\hat{\sigma}_1$ 得到 $+1$ 后系统塌缩到 $|+_1\rangle$，第二次测量 $\hat{\sigma}_2$，$k=2$，显然，第二次测量测得 $+1$ 的概率与第一次相同，即：

$$P_2 = P_1 = \cos^2\frac{\pi}{4N}$$

以此类推，

$$P_3 = P_4 = \cdots = P_n = \cos^2\frac{\pi}{4N}$$

全部测量结果都是 $+1$ 的概率为：

$$P = P_1P_2\cdots P_N = \cos^{2N}\frac{\pi}{4N}$$

当 $N\to\infty$，

$$P = \lim_{N\to\infty}\cos^{2N}\frac{\pi}{4N} = 1$$

> （2）若初态为 $|-_z\rangle$，全部测量结果都是 $+1$ 的概率。当 $N\to\infty$ 时 出现什么？

第一次测量 $\hat{\sigma}_1$ ，$k=1$，测量前系统处于 $|-_z\rangle$，测得 $+1$ 的概率为：

$$\begin{aligned} P_1' &= |\langle -_z \mid +_1\rangle|^2 \\ &= \sin^2\frac{\pi}{4N} \end{aligned}$$

第一次测量 $\hat{\sigma}_1$ 得到 $+1$ 后系统塌缩到 $|+_1\rangle$，从第二次测量开始的结果应与（1）中一致，即：

$$P_2' = P_1 = \cos^2\frac{\pi}{4N}$$

以此类推，

$$P_3' = P_4' = \cdots = P_N' = \cos^2\frac{\pi}{4N}$$

全部测量结果都是 $+1$ 的概率为：

$$P' = P_1'P_2'\cdots P_N' = \sin^2\frac{\pi}{4N}\cos^{2N-2}\frac{\pi}{4N}$$

当 $N\to\infty$，

$$P' = \lim_{N\to\infty}\sin^2\frac{\pi}{4N}\cos^{2N-2}\frac{\pi}{4N} = 0$$