提示: 用电势叠加原理可快速出答案

$$egin{align} U_3 &= rac{1}{4\piarepsilon_0}rac{Q_a+Q_b}{r} \ U_2 &= rac{1}{4\piarepsilon_0}igg(rac{Q_a}{r}+rac{Q_b}{R_b}igg) \ U_1 &= rac{1}{4\piarepsilon_0}igg(rac{Q_a}{R_a}+rac{Q_b}{R_b}igg) \ \end{split}$$

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由对称性可知, 球心处的电场强度只有沿 z 轴负方向的分量

 $heta\sim heta+\mathrm{d} heta, arphi\sim arphi+\mathrm{d}arphi$ 小范围内的面元 $\mathrm{d}S$ 所带面电荷在球心处产生的电场强度大小:

$$\begin{split} \mathrm{d}E &= \frac{1}{4\pi\varepsilon_0} \frac{\sigma \mathrm{d}S}{R^2} \\ &= \frac{1}{4\pi\varepsilon_0 R^2} P\cos\theta \cdot R \mathrm{d}\theta \cdot R\sin\theta \mathrm{d}\varphi \\ &= \frac{1}{4\pi\varepsilon_0} P\cos\theta\sin\theta \mathrm{d}\theta \mathrm{d}\varphi \end{split}$$

积分可得球心处电场强度大小 E_O :

$$egin{aligned} E_O &= \int \mathrm{d}E \cos \theta \ &= rac{P}{4\piarepsilon_0} \int_{arphi=0}^{arphi=2\pi} \mathrm{d}arphi \int_{ heta=0}^{ heta=\pi/2} \sin heta \cos^2 heta \mathrm{d} heta \ &= -rac{P}{2arepsilon_0} \int_{ heta=0}^{ heta=\pi/2} \cos^2 heta \mathrm{d}(\cos heta) \ &= -rac{P}{2arepsilon_0} \cdot rac{\cos^3 heta}{3} igg|_{ heta=0}^{ heta=\pi/2} \ &= rac{P}{6arepsilon_0} \end{aligned}$$

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(1)

$$\left\{egin{aligned} (\sigma_1+\sigma_2)S &= Q_A\ (\sigma_3+\sigma_4)S &= 0\ (\sigma_5+\sigma_6)S &= Q_C \end{aligned}
ight.$$

每个导体内部电场为零:

$$\begin{cases} E_A = 0 \\ E_B = 0 \\ E_C = 0 \end{cases}$$

将每个导体板看成两个无限大均匀带电平面,选择水平向右为正方向,上面三个条件可化为:

$$\begin{cases} \frac{\sigma_1}{2\varepsilon_0} - \frac{\sigma_2}{2\varepsilon_0} - \frac{\sigma_3}{2\varepsilon_0} - \frac{\sigma_4}{2\varepsilon_0} - \frac{\sigma_5}{2\varepsilon_0} - \frac{\sigma_6}{2\varepsilon_0} = 0\\ \frac{\sigma_1}{2\varepsilon_0} + \frac{\sigma_2}{2\varepsilon_0} + \frac{\sigma_3}{2\varepsilon_0} - \frac{\sigma_4}{2\varepsilon_0} - \frac{\sigma_5}{2\varepsilon_0} - \frac{\sigma_6}{2\varepsilon_0} = 0\\ \frac{\sigma_1}{2\varepsilon_0} + \frac{\sigma_2}{2\varepsilon_0} + \frac{\sigma_3}{2\varepsilon_0} + \frac{\sigma_4}{2\varepsilon_0} + \frac{\sigma_5}{2\varepsilon_0} - \frac{\sigma_6}{2\varepsilon_0} = 0 \end{cases}$$

最终解得:

$$egin{aligned} \sigma_1 &= rac{Q_A + Q_C}{2S} \ \sigma_2 &= rac{Q_A - Q_C}{2S} \ \sigma_3 &= rac{Q_C - Q_A}{2S} \ \sigma_4 &= rac{Q_A - Q_C}{2S} \ \sigma_5 &= rac{Q_C - Q_A}{2S} \ \sigma_6 &= rac{Q_A + Q_C}{2S} \end{aligned}$$

(2)

A, B 间场强:

$$E_{AB}\Delta S = rac{1}{arepsilon_0}\Delta S \sigma_2 \Longrightarrow E_{AB} = rac{Q_A - Q_C}{2arepsilon_0 S}$$

A, B 间电势:

$$U_{AB} = E_{AB} \cdot d_1 = rac{d_1(Q_A - Q_C)}{2arepsilon_0 S}$$

B, C 间场强:

$$E_{BC}\Delta S = rac{1}{arepsilon_0}\Delta S \sigma_4 \Longrightarrow E_{BC} = rac{Q_A - Q_C}{2arepsilon_0 S}$$

B, C 间电势:

$$U_{BC} = E_{BC} \cdot d_2 = rac{d_2(Q_A - Q_C)}{2arepsilon_0 S}$$

(3)

合上电键后 A 板和 C 板间可能存在电荷转移,但总量不变。设稳定后 A 板带电量 Q_A' ,C 板带电量 Q_C' ,则:

$$Q_A' + Q_C' = Q_A + Q_C$$
 $U_{AB}' = rac{d_1(Q_A' - Q_C')}{2arepsilon_0 S}$ $U_{BC}' = rac{d_2(Q_A' - Q_C')}{2arepsilon_0 S}$

一方面:

$$U_{AC}' = U_{AB}' + U_{BC}' = rac{d_1(Q_A' - Q_C')}{2arepsilon_0 S} + rac{d_2(Q_A' - Q_C')}{2arepsilon_0 S}$$

另一方面:

$$U'_{AC} = U_0$$

$$\begin{cases} Q'_A + Q'_C = Q_A + Q_C \\ \frac{d_1(Q'_A - Q'_C)}{2\varepsilon_0 S} + \frac{d_2(Q'_A - Q'_C)}{2\varepsilon_0 S} = U_0 \end{cases}$$

解得:

$$\left\{ egin{aligned} Q_A' &= rac{Q_A + Q_C}{2} + rac{arepsilon_0 S U_0}{d_1 + d_2} \ Q_C' &= rac{Q_A + Q_C}{2} - rac{arepsilon_0 S U_0}{d_1 + d_2} \end{aligned}
ight.$$

其实没必要解出 Q_A',Q_C' ,需要的只是:

$$\begin{cases} Q_A' + Q_C' = Q_A + Q_C \\ \frac{Q_A' - Q_C'}{2S} = \frac{\varepsilon_0 U_0}{d_1 + d_2} \end{cases}$$

于是:

$$\begin{cases} \sigma_1' = \frac{Q_A' + Q_C'}{2S} = \frac{Q_A + Q_C}{2S} \\ \sigma_2' = \frac{Q_A' - Q_C'}{2S} = \frac{\varepsilon_0 U_0}{d_1 + d_2} \\ \sigma_3' = \frac{Q_C' - Q_A'}{2S} = -\frac{\varepsilon_0 U_0}{d_1 + d_2} \\ \sigma_4' = \frac{Q_A' - Q_C'}{2S} = \frac{\varepsilon_0 U_0}{d_1 + d_2} \\ \sigma_5' = \frac{Q_C' - Q_A'}{2S} = -\frac{\varepsilon_0 U_0}{d_1 + d_2} \\ \sigma_6' = \frac{Q_A' + Q_C'}{2S} = \frac{Q_A + Q_C}{2S} \end{cases}$$

$$U_{AB}' = \frac{d_1(Q_A' - Q_C')}{2\varepsilon_0 S} = \frac{d_1}{d_1 + d_2} U_0$$

$$U_{BC}' = \frac{d_2(Q_A' - Q_C')}{2\varepsilon_0 S} = \frac{d_2}{d_1 + d_2} U_0$$

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(1)

$$\oint\limits_{\partial V} ec{D} \cdot \mathrm{d}ec{S} = Q_0$$

其中, Q_0 是 V 内总自由电荷

选取竖直向下为正方向,

$$D_1 \Delta S = \sigma_0 \Delta S \Longrightarrow D_1 = \sigma_0$$

"好"介质:

$$egin{aligned} ec{D} &\equiv arepsilon_0 ec{E} + ec{P} \ &= arepsilon_0 ec{E} + \chi_e arepsilon_0 ec{E} \ &= arepsilon_0 (1 + \chi_e) ec{E} \ &= arepsilon_0 arepsilon ec{E} \end{aligned}$$

于是:

$$E_1 = \frac{D_1}{\varepsilon_0 \varepsilon_1} = \frac{\sigma_0}{\varepsilon_0 \varepsilon_1}$$

注意, D_1 是一维矢量,其方向由正负给出。上面的式子的正负号要服从同一正方向规定。

$$D_2(-\Delta S) = \Delta S(-\sigma_0) \Longrightarrow D_2 = \sigma_0$$

$$E_2 = rac{D_2}{arepsilon_0 arepsilon_2} \ = rac{\sigma_0}{arepsilon_0 arepsilon_2}$$

(2)

"好"介质:

 $ec{P} = \chi_e arepsilon_0 ec{E} \ = (arepsilon - 1) arepsilon_0 ec{E}$

于是:

$$P_1 = (arepsilon_1 - 1)arepsilon_0 E_1 \ = rac{\sigma_0(arepsilon_1 - 1)}{arepsilon_1}$$

$$P_2 = (arepsilon_2 - arepsilon_0) E_2 \ = rac{\sigma_0 (arepsilon_2 - 1)}{arepsilon_2}$$

(3)

$$egin{aligned} U_{AB} &= E_1 d_1 + E_2 d_2 \ &= rac{\sigma_0 d_1}{arepsilon_0 arepsilon_1} + rac{\sigma_0 d_2}{arepsilon_0 arepsilon_2} \ &= rac{\sigma_0 (d_1 arepsilon_2 + d_2 arepsilon_1)}{arepsilon_0 arepsilon_1 arepsilon_2} \end{aligned}$$

(4)

$$\begin{aligned} \sigma_1' &= P_{1n} \\ &= -P_1 \\ &= -\frac{\sigma_0(\varepsilon_1 - 1)}{\varepsilon_1} \end{aligned}$$

$$\sigma_2' = P_{2n}$$

$$= P_2$$

$$= \frac{\sigma_0(\varepsilon_2 - 1)}{\varepsilon_2}$$

两介质交界面处:

$$\oint \int _{\partial V} ec{P} \cdot \mathrm{d}ec{S} = -Q_{\mathrm{p}}$$

其中, Q_{p} 是 ∂V 内总束缚电荷

$$P_1(-\Delta S) + P_2(\Delta S) = -\sigma' \Delta S$$

解得:

$$egin{aligned} \sigma' &= P_1 - P_2 \ &= rac{\sigma_0(arepsilon_1 - 1)}{arepsilon_1} - rac{\sigma_0(arepsilon_2 - 1)}{arepsilon_2} \ &= rac{arepsilon_1 - arepsilon_2}{arepsilon_1 arepsilon_2} \sigma_0 \end{aligned}$$

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$$\begin{cases} I_1 + I_2 = I_3 \\ -E_1 + R_1 I_1 + E_2 - R_2 I_2 = 0 \\ -E_2 + R_2 I_2 + R_3 I_3 = 0 \end{cases}$$

解得:

$$\left\{egin{aligned} I_1 = 6 \; \mathrm{A} \ I_2 = -3 \; \mathrm{A} \ I_3 = 3 \; \mathrm{A} \end{aligned}
ight.$$

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(1)

电流连续方程积分形式:

$$\iint\limits_{\partial V} \vec{j} \cdot \mathrm{d}\vec{S} = -\frac{\mathrm{d}Q}{\mathrm{d}t}$$

其中, Q 是 ∂V 内总电荷

电流连续方程微分形式:

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

(2)

欧姆定律微分形式:

$$ec{j}=\sigmaec{E}$$

其中, σ 是电导率

(3)

$$egin{aligned} & egin{aligned} \oint \int \vec{E} \cdot \mathrm{d} \vec{S} &= rac{Q}{arepsilon_0} \ &= rac{1}{arepsilon_0} \iiint
ho \mathrm{d} V \end{aligned}$$

高斯公式:

$$\iint\limits_{\partial V} \vec{E} \cdot \mathrm{d}\vec{S} = \iiint\limits_{V} (\nabla \cdot \vec{E}) \mathrm{d}V$$

对比得静电场高斯定理的微分形式:

$$abla \cdot ec{E} = rac{
ho}{arepsilon_0}$$

(4)

$$\begin{cases} \vec{j} = \sigma \vec{E} \\ \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0 \Longrightarrow \nabla \cdot \vec{E} = -\frac{1}{\sigma} \frac{\partial \rho}{\partial t} \end{cases}$$
$$\begin{cases} \nabla \cdot \vec{E} = -\frac{1}{\sigma} \frac{\partial \rho}{\partial t} \\ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \end{cases} \Longrightarrow \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon_0} \rho = 0$$

考虑某一确定场点处的体电荷密度:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\sigma}{\varepsilon_0}\rho = 0$$

解得:

$$ho = C e^{-rac{\sigma}{arepsilon_0}t}$$

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安培环路定理:

$$\oint\limits_{\partial S}ec{B}\cdot\mathrm{d}ec{l}=\mu_0\sum I_0$$
 $2B\Delta L=\mu_0 j\Delta L\Longrightarrow B=rac{\mu j}{2}$

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(1)

负电荷导电

(2)

电子所受洛伦兹力和电场力平衡:

$$evB = erac{U_{AA'}}{b} \Longrightarrow vB = rac{U_{AA'}}{b}$$

其中, v 是电子速度

电流微观表达式:

$$I \equiv rac{\Delta Q}{\Delta t}$$

$$= rac{enSv\Delta t}{\Delta t}$$

$$= enSv$$

$$= enabv$$

其中, n 是单位体积内参与导电的电子数。

$$egin{cases} vB = rac{U_{AA'}}{b} \ I = enabv \end{cases}$$

解得:

$$n = rac{IB}{eaU_{AA'}} \ = 2.9 imes 10^{14} \ {
m cm}^{-3}$$