推导 $\hat{H}=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+rac{1}{2}
ight)$,完成腔内单模电磁场的量子化。

腔内电场满足波动方程:

$$abla^2 ec{E} - rac{1}{c^2} rac{\partial^2 ec{E}}{\partial t^2} = ec{0}$$

考虑沿z轴传播的x方向偏振的线偏振光,波动方程化为:

$$\frac{\partial^2 E_x(z,t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial E_x(z,t)}{\partial t^2} = 0$$

设 $E_x(z,t)=E_x(z)q(t)$, 分离变量得:

$$rac{\mathrm{d}^2 E_x(z)}{\mathrm{d}z^2} + k^2 E_x(z) = 0$$

$$rac{\mathrm{d}^2q(t)}{\mathrm{d}t^2}+\omega^2q(t)=0, \ \ \omega=ck$$

可以解得通解:

$$E_x(z) = A\sin(kz + \alpha)$$

$$q(t) = c_1 \mathrm{e}^{-\mathrm{i}\omega t} + c_2 \mathrm{e}^{\mathrm{i}\omega t}$$

边界条件:

$$E_x(0) = E_x(L) = 0$$

得到:

$$\alpha = 0, \ k = \frac{n\pi}{L}$$

因此:

$$ec{E} = ec{\mathrm{e}}_x A q(t) \sin(kz), \;\; k = rac{n\pi}{L}$$

由 $abla imes ec{B} = \mu_0 arepsilon_0 rac{\partial ec{E}}{\partial t}$ 可得:

$$ec{B} = ec{\mathrm{e}}_y \left(rac{\mu_0 arepsilon_0}{k}
ight) A \dot{q}(t) \cos(kz)$$

取基模 n=1, 电磁场的能量为:

$$\begin{split} H &= \frac{1}{2} \int \mathrm{d}^{3}\vec{r} \left[\varepsilon_{0} \left| E_{x}(z,t) \right|^{2} + \frac{1}{\mu_{0}} \left| B_{y}(z,t) \right|^{2} \right] \\ &= \frac{\varepsilon_{0} V A^{2}}{4} \left[q^{2}(t) + \frac{\dot{q}^{2}(t)}{c^{2}k^{2}} \right] \\ &= \frac{\varepsilon_{0} V A^{2}}{2m\omega^{2}} \left[\frac{m\omega^{2}q^{2}(t)}{2} + \frac{p^{2}(t)}{2m} \right] \\ &= \frac{p^{2}(t)}{2m} + \frac{m\omega^{2}q^{2}(t)}{2}, \ \ A = \left(\frac{2m\omega^{2}}{V\varepsilon_{0}} \right)^{1/2} \end{split}$$

$$q
ightarrow \hat{q}, p
ightarrow \hat{p}, [\hat{q}(t), \hat{p}(t')] = \mathrm{i}\hbar \delta_{t,t'}$$

$$\hat{H}=rac{\hat{p}^2(t)}{2m}+rac{m\omega^2\hat{q}^2(t)}{2}$$

$$\hat{a}\mathrm{e}^{-\mathrm{i}\omega t}=rac{1}{\sqrt{2m\hbar\omega}}\left[m\omega\hat{q}(t)+\mathrm{i}\hat{p}(t)
ight]$$

则:

$$\hat{H}=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+rac{1}{2}
ight)$$
 $\hat{ec{E}}(z,t)=ec{e}_{x}\mathcal{E}\left(\hat{a}\mathrm{e}^{-\mathrm{i}\omega t}+\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\omega t}
ight)\sin(kz)$

2-2

推导
$$\hat{H}=\sum_{ec{k}s}\hbar\omega_k\left(\hat{a}_{ec{k}s}^\dagger\hat{a}_{ec{k}s}+rac{1}{2}
ight)$$
 ,完成自由空间多模电磁场的量子化。

光场由矢势确定:

$$ec{E}(ec{r},t) = -rac{\partial ec{A}(ec{r},t)}{\partial t} \ ec{B}(ec{r},t) =
abla imes ec{A}(ec{r},t)$$

麦克斯韦方程:

$$abla imes ec{H} = ec{J} + rac{\partial ec{D}}{\partial t}$$

自由空间中 $ec{J}=ec{0},ec{D}=arepsilon_0ec{E},ec{H}=ec{B}/\mu_0$,代入上式得:

$$abla imes ec{B} = arepsilon_0 \mu_0 rac{\partial ec{E}}{\partial t} = rac{1}{c^2} rac{\partial ec{E}}{\partial t}$$

将上式可化为关于矢势 \vec{A} 的方程:

$$abla imes (
abla imes ec{A}) = -rac{1}{c^2} rac{\partial^2 ec{A}}{\partial t^2}$$

利用公式 $abla imes (
abla imes \vec{A}) =
abla (
abla imes \vec{A}) -
abla^2 \vec{A}$,并采用库伦规范 $abla imes \vec{A} = 0$ 可得:

$$abla^2 ec{A}(ec{r},t) - rac{1}{c^2} rac{\partial^2 ec{A}(ec{r},t)}{\partial t^2} = 0$$

设 $\vec{A}(\vec{r},t) = \vec{A}(\vec{r})A(t)$,分离变量可得:

$$\left\{egin{aligned}
abla^2ec{A}(ec{r})+k^2ec{A}(ec{r})&=ec{0}\ \ddot{A}(t)+\omega^2A(t)&=0\ \omega^2&=c^2k^2 \end{aligned}
ight.$$

把自由空间看作边长为 L 的立方腔,可得通解:

$$ec{A}(ec{r},t) = \sum_{ec{k}\,s} ec{\mathrm{e}}_{ec{k}s} \left[A_{ec{k}s} \mathrm{e}^{\mathrm{i}(ec{k}\cdotec{r}-\omega_k t)} + A_{ec{k}s}^* \mathrm{e}^{-\mathrm{i}(ec{k}\cdotec{r}-\omega_k t)}
ight]$$

采用周期性边界条件可得:

$$ec{k}=rac{2\pi}{L}(m_x,m_y,m_z),\;\;m_x,m_y,m_z$$
为整数

由于 $\nabla\cdot\vec{A}=0$,因此 $\vec{k}\cdot\vec{\mathbf{e}}_{\vec{k}s}=0$,于是电磁波为横波,有两个独立的偏振方向,s=1,2电场:

$$\begin{split} \vec{E}(\vec{r},t) &= -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} \\ &= -\frac{\partial}{\partial t} \left\{ \sum_{\vec{k},s} \vec{\mathbf{e}}_{\vec{k}s} \left[A_{\vec{k}s} \mathbf{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} + A_{\vec{k}s}^* \mathbf{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right] \right\} \\ &= \mathrm{i} \sum_{\vec{k},s} \omega_k \vec{\mathbf{e}}_{\vec{k}s} \left[A_{\vec{k}s} \mathbf{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} - A_{\vec{k}s}^* \mathbf{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right] \end{split}$$

结合 $abla imes (arphi ec{A}) = (
abla arphi) imes ec{A} + arphi (
abla imes ec{A})$,可得磁场:

$$\begin{split} \vec{B}(\vec{r},t) &= \nabla \times \vec{A}(\vec{r},t) \\ &= \nabla \times \left\{ \sum_{\vec{k},s} \vec{\mathbf{e}}_{\vec{k}s} \left[A_{\vec{k}s} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} + A_{\vec{k}s}^* \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right] \right\} \\ &= \sum_{\vec{k},s} \left[\left(A_{\vec{k}s} \nabla \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right) \times \vec{\mathbf{e}}_{\vec{k}s} + \left(A_{\vec{k}s}^* \nabla \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right) \times \vec{\mathbf{e}}_{\vec{k}s} \right] \\ &= \sum_{\vec{k},s} \left[\left(\mathrm{i} A_{\vec{k}s} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right) \vec{k} \times \vec{\mathbf{e}}_{\vec{k}s} - \left(\mathrm{i} A_{\vec{k}s}^* \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right) \vec{k} \times \vec{\mathbf{e}}_{\vec{k}s} \right] \\ &= \mathrm{i} \sum_{\vec{k},s} \left(\vec{k} \times \vec{\mathbf{e}}_{\vec{k}s} \right) \left[A_{\vec{k}s} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} - A_{\vec{k}s}^* \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right] \\ &= \frac{\mathrm{i}}{c} \sum_{\vec{k},s} \omega_k \left(\hat{k} \times \vec{\mathbf{e}}_{\vec{k}s} \right) \left[A_{\vec{k}s} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} - A_{\vec{k}s}^* \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega_k t)} \right] \end{split}$$

哈密顿量为:

$$egin{align} H &= rac{1}{2} \int\limits_V \left(arepsilon_0 ec{E} \cdot ec{E} + rac{1}{\mu_0} ec{B} \cdot ec{B}
ight) \mathrm{d}V \ &= 2 arepsilon_0 V \sum_{ec{k} \ s} \omega_k^2 A_{ec{k} s}^* A_{ec{k} s} \end{aligned}$$

引入 $p_{\vec{k}s}, q_{\vec{k}s}$ 使得:

$$A_{ec{k}s} = rac{1}{2\omega_k \left(arepsilon_0 V
ight)^{1/2}} \left[\omega_k q_{ec{k}s} + \mathrm{i} p_{ec{k}s}
ight]$$

则:

$$H=rac{1}{2}\sum_{ec{k}\;s}\left(p_{ec{k}s}^2+\omega_kq_{ec{k}s}^2
ight)$$

量子化:

$$\left[\hat{q}_{ec{k}s},\hat{p}_{ec{k}'s'}
ight]=\mathrm{i}\hbar\delta_{ec{k}ec{k}'}\delta_{ss'}$$

令:

$$\hat{a}_{ec{k}s} = rac{1}{\left(2\hbar\omega_k
ight)^{1/2}}\left(\omega_k\hat{q}_{ec{k}s} + \mathrm{i}\hat{p}_{ec{k}s}
ight)$$

则:

$$\hat{H} = \sum_{ec{k},s} \hbar \omega_k \left(\hat{a}_{ec{k}s}^\dagger \hat{a}_{ec{k}s} + rac{1}{2}
ight) \ \hat{ec{E}}(ec{r},t) = \mathrm{i} \sum_{ec{k},s} \left(rac{\hbar \omega_k}{2 arepsilon_0 V}
ight)^{1/2} ec{\mathrm{e}}_{ec{k}s} \left[\hat{a}_{ec{k}s} \mathrm{e}^{\mathrm{i}(ec{k}\cdot ec{r} - \omega_k t)} - \hat{a}_{ec{k}s}^\dagger \mathrm{e}^{-\mathrm{i}(ec{k}\cdot ec{r} - \omega_k t)}
ight]$$

$$\hat{\vec{B}}(\vec{r},t) = \frac{\mathrm{i}}{c} \sum_{\vec{k}.s} \left(\frac{\hbar \omega_k}{2\varepsilon_0 V} \right)^{1/2} \left(\hat{k} \times \vec{\mathrm{e}}_{\vec{k}s} \right) \left[\hat{a}_{\vec{k}s} \mathrm{e}^{\mathrm{i}(\vec{k} \cdot \vec{r} - \omega_k t)} - \hat{a}_{\vec{k}s}^{\dagger} \mathrm{e}^{-\mathrm{i}(\vec{k} \cdot \vec{r} - \omega_k t)} \right]$$

求证,对单模自由光场 $\hat{H}=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+1/2\right)$,其本征态满足: (1) $|n\rangle=\frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}}|0\rangle$ (2) $\hat{U}(t)=\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)=\sum_{n}\exp\left(-\frac{\mathrm{i}}{\hbar}E_{n}t\right)|n\rangle\left\langle n|$,其中 $E_{n}=\hbar\omega\left(n+1/2\right)$

(1)

$$|\hat{a}^{\dagger}|n
angle = \sqrt{n+1}\,|n+1
angle \Longrightarrow |n+1
angle = rac{\hat{a}^{\dagger}}{\sqrt{n+1}}\,|n
angle$$

于是:

$$\begin{split} |n\rangle &= \frac{\hat{a}^{\dagger}}{\sqrt{n}} \, |n-1\rangle \\ &= \frac{\left(\hat{a}^{\dagger}\right)^2}{\sqrt{n}\sqrt{n-1}} \, |n-2\rangle \\ &= \vdots \\ &= \frac{\left(\hat{a}^{\dagger}\right)^n}{\sqrt{n!}} \, |0\rangle \end{split}$$

(2)

$$\hat{H}\ket{n}=E_{n}\ket{n}=\hbar\omega\left(n+1/2
ight)\ket{n}$$

完备性:

$$egin{aligned} \left| n
ight
angle \left\langle n
ight| &= I \ \hat{U}(t) = \exp\left(-rac{\mathrm{i}}{\hbar}\hat{H}t
ight) \ &= \exp\left(-rac{\mathrm{i}}{\hbar}\hat{H}t
ight) \sum_{n} \left| n
ight
angle \left\langle n
ight| \ &= \sum_{n} \exp\left(-rac{\mathrm{i}}{\hbar}\hat{H}t
ight) \left| n
ight
angle \left\langle n
ight| \ &= \sum_{n} \exp\left(-rac{\mathrm{i}}{\hbar}E_{n}t
ight) \left| n
ight
angle \left\langle n
ight| \end{aligned}$$

2-4

推导方程:
$$ho = rac{\mathrm{e}^{-eta \hat{H}}}{Z} = \sum_{n=0}^{\infty} rac{ar{n}^n}{\left(1 + ar{n}
ight)^{n+1}} \ket{n}ra{n}, \;\; Z = \mathrm{Tr}\left(\mathrm{e}^{-eta \hat{H}}\right)$$
 $eta = rac{1}{k_{\mathrm{B}}T}, \;\; \hat{H} = \hbar\omega(\hat{a}^{\dagger}\hat{a} + 1/2)$

热场的密度矩阵:

$$ho = rac{\exp\left(-\hat{H}/k_{
m B}T
ight)}{{
m Tr}\left[\exp\left(-\hat{H}/k_{
m B}T
ight)
ight]}, \;\; \hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + 1/2
ight)$$
 $m Tr}\left[\exp\left(-\hat{H}/k_{
m B}T
ight)
ight] = \sum_{n=0}^{\infty}\left\langle n\left|\exp\left(-\hat{H}/k_{
m B}T
ight)\left|n
ight
angle$
 $= \sum_{n=0}^{\infty}\exp\left(-E_n/k_{
m B}T
ight)$
 $\equiv Z$

其中 $Z\equiv\sum_{n=0}^{\infty}\exp\left(-E_{n}/k_{\mathrm{B}}T
ight)$ 称为配分函数。

$$\begin{split} Z &\equiv \sum_{n=0}^{\infty} \exp\left(-E_n/k_{\rm B}T\right) \\ &= \exp\left(-\hbar\omega/2k_{\rm B}T\right) \sum_{n=0}^{\infty} \exp\left(-\hbar\omega n/k_{\rm B}T\right) \\ &= \exp\left(-\hbar\omega/2k_{\rm B}T\right) \cdot \frac{1}{1 - \exp\left(-\hbar\omega/k_{\rm B}T\right)} \\ &= \frac{\exp\left(-\hbar\omega/2k_{\rm B}T\right)}{1 - \exp\left(-\hbar\omega/k_{\rm B}T\right)} \end{split}$$

设:

$$ho = \sum_{n=0}^{\infty} P_n \ket{n} ra{n}$$

则:

$$P_n = \langle n \, | \,
ho \, | \, n
angle$$

$$= \frac{1}{Z} \exp \left(-E_n/k_{
m B}T
ight)$$

注意到:

$$\sum_{n=0}^{\infty} n e^{-nx} = -\sum_{n=0}^{\infty} \frac{\partial}{\partial x} (e^{-nx})$$
$$= -\frac{d}{dx} \sum_{n=0}^{\infty} e^{-nx}$$
$$= \frac{e^{-x}}{(1 - e^{-x})^2}$$

热场的平均光子数:

$$egin{aligned} ar{n} &= \langle \hat{n}
angle \ &= \operatorname{Tr} \left(\hat{n}
ho
ight) \ &= \sum_{n=0}^{\infty} \langle n \, | \, \hat{n}
ho \, | \, n
angle \ &= \sum_{n=0}^{\infty} n P_n \ &= \frac{\exp \left(-\hbar \omega / 2 k_{\mathrm{B}} T
ight)}{Z} \sum_{n=0}^{\infty} n \exp \left(-\hbar \omega n / k_{\mathrm{B}} T
ight) \ &= \frac{1}{\exp \left(\hbar \omega / k_{\mathrm{B}} T
ight) - 1} \end{aligned}$$

因此:

$$\exp\left(-\hbar\omega/k_{
m B}T
ight)=rac{ar{n}}{ar{n}+1}$$

于是:

$$egin{aligned}
ho &= \sum_{n=0}^{\infty} P_n \ket{n}ra{n} \ &= \sum_{n=0}^{\infty} rac{1}{Z} \exp\left(-E_n/k_{
m B}T
ight)\ket{n}ra{n} \ &= \sum_{n=0}^{\infty} rac{\left(rac{ar{n}}{ar{n}+1}
ight)^{1/2}\left(rac{ar{n}}{ar{n}+1}
ight)^n}{\left(rac{ar{n}}{ar{n}+1}
ight)^{1/2}/\left(1-rac{ar{n}}{ar{n}+1}
ight)}\ket{n}ra{n} \ &= \sum_{n=0}^{\infty} rac{ar{n}^n}{\left(1+ar{n}
ight)^{n+1}}\ket{n}ra{n} \ &|n
angle \ \end{aligned}$$

2-5

推导黑体辐射公式: $\bar{U}(\omega)=\hbar\omega \bar{n} \rho(\omega)=rac{\hbar\omega^3}{\pi^2c^3}rac{1}{\mathrm{e}^{eta\hbar\omega}-1}$

由 $ec{k}=rac{2\pi}{L}(m_xec{\mathbf{e}}_x+m_yec{\mathbf{e}}_y+m_zec{\mathbf{e}}_z),\;\;m_x,m_y,m_z\in\mathbb{N}$ 可得:

动量空间中 $k \sim k + \mathrm{d}k$ 动量范围内的电磁模式数:

$$rac{2 imes 4\pi k^2\mathrm{d}k}{\left(2\pi/L
ight)^3}=rac{Vk^2}{\pi^2}\mathrm{d}k$$

由色散关系 $\omega_k=ck, \mathrm{d}\omega_k=c\mathrm{d}k$, $\omega_k\sim\omega_k+\mathrm{d}\omega_k$ 圆频率范围内的电磁模式数:

$$\frac{V\omega_k^2}{\pi^2c^3}\mathrm{d}\omega_k$$

模密度(单位体积内 ω 附近单位圆频率区间内的电磁模式数) :

$$egin{aligned}
ho(\omega) &= rac{\omega^2}{\pi^2 c^3} \ ar{U}(\omega) &= \hbar \omega ar{n}
ho(\omega) \ &= rac{\hbar \omega}{\exp\left(\hbar \omega/k_{
m B} T
ight) - 1} \cdot rac{\omega^2}{\pi^2 c^3} \ &= rac{\hbar \omega^3}{\pi^2 c^3} rac{1}{e^{eta \hbar \omega} - 1} \end{aligned}$$

2-6

】 对相干态,求证: (1) $\hat{a}^{\dagger}\ket{lpha}ra{lpha}=\left(lpha^*+\partial_{lpha}
ight)\ket{lpha}ra{lpha}$ (2) $\ket{lpha}ra{lpha}=\left(lpha+\partial_{lpha^*}
ight)\ket{lpha}ra{lpha}$

(1)

$$\left|\alpha\right\rangle\left\langle \alpha\right|=\mathrm{e}^{-lphalpha^{*}}\sum_{n,m=0}^{\infty}rac{lpha^{n}}{\sqrt{n!}}rac{lpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m
ight|$$

一方面:

$$\begin{split} \hat{a}^{\dagger} \left| \alpha \right\rangle \left\langle \alpha \right| &= \hat{a}^{\dagger} \left(\mathrm{e}^{-\alpha \alpha^{*}} \sum_{n,m=0}^{\infty} \frac{\alpha^{n}}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \left| n \right\rangle \left\langle m \right| \right) \\ &= \mathrm{e}^{-\alpha \alpha^{*}} \sum_{n,m=0}^{\infty} \frac{\alpha^{n}}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \hat{a}^{\dagger} \left| n \right\rangle \left\langle m \right| \\ &= \mathrm{e}^{-\alpha \alpha^{*}} \sum_{n,m=0}^{\infty} \frac{\alpha^{n}}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \sqrt{n+1} \left| n+1 \right\rangle \left\langle m \right| \\ &= \mathrm{e}^{-\alpha \alpha^{*}} \sum_{n,m=0}^{\infty} \frac{\alpha^{n}}{\sqrt{(n+1)!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n+1) \left| n+1 \right\rangle \left\langle m \right| \\ &= \mathrm{e}^{-\alpha \alpha^{*}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{n-1}}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n) \left| n \right\rangle \left\langle m \right| \end{split}$$

另一方面:

$$\begin{split} \left(\alpha^{*}+\partial_{\alpha}\right)\left|\alpha\right\rangle\left\langle\alpha\right| &=\left(\alpha^{*}+\partial_{\alpha}\right)\left(\mathrm{e}^{-\alpha\alpha^{*}}\sum_{n,m=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right) \\ &=\alpha^{*}\left(\mathrm{e}^{-\alpha\alpha^{*}}\sum_{n,m=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right)-\alpha^{*}\left(\mathrm{e}^{-\alpha\alpha^{*}}\sum_{n,m=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right)+\left(\mathrm{e}^{-\alpha\alpha^{*}}\sum_{n,m=0}^{\infty}\frac{n\alpha^{n-1}}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right) \\ &=\mathrm{e}^{-\alpha\alpha^{*}}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{n-1}}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}(n)\left|n\right\rangle\left\langle m\right| \\ &=\mathrm{e}^{-\alpha\alpha^{*}}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{n-1}}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}(n)\left|n\right\rangle\left\langle m\right| \end{split}$$

对比可得:

$$\hat{a}^{\dagger}\ket{lpha}ra{lpha}=(lpha^*+\partial_lpha)\ket{lpha}ra{lpha}$$

(2)

$$\ket{lpha}ra{lpha}=\mathrm{e}^{-lphalpha^*}\sum_{n.m=0}^{\infty}rac{lpha^n}{\sqrt{n!}}rac{lpha^{*m}}{\sqrt{m!}}\ket{n}ra{m!}$$

产生算符作用于 Fock 态:

$$\hat{a}^{\dagger}\ket{m}=\sqrt{m+1}\ket{m+1}$$

取厄米共轭得:

$$\langle m | \, \hat{a} = \sqrt{m+1} \, \langle m+1 |$$

一方面:

$$\begin{split} \left|\alpha\right\rangle\left\langle\alpha\right|\hat{a} &= \left(\mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\right)\hat{a} \\ &= \mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\left|n\right\rangle\left\langle m\right|\hat{a} \\ &= \mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\sqrt{m+1}\left|n\right\rangle\left\langle m+1\right| \\ &= \mathrm{e}^{-\alpha\alpha^*}\sum_{n,m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{(m+1)!}}(m+1)\left|n\right\rangle\left\langle m+1\right| \\ &= \mathrm{e}^{-\alpha\alpha^*}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*(m-1)}}{\sqrt{m!}}(m)\left|n\right\rangle\left\langle m\right| \end{split}$$

另一方面:

$$\begin{split} \left(\alpha + \partial_{\alpha^*}\right) \left|\alpha\right\rangle \left\langle\alpha\right| &= \left(\alpha + \partial_{\alpha^*}\right) \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \left|n\right\rangle \left\langle m\right|\right) \\ &= \alpha \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \left|n\right\rangle \left\langle m\right|\right) - \alpha \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} \left|n\right\rangle \left\langle m\right|\right) + \left(\mathrm{e}^{-\alpha\alpha^*} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{m\alpha^{*(m-1)}}{\sqrt{m!}} \left|n\right\rangle \left\langle m\right|\right) \\ &= \mathrm{e}^{-\alpha\alpha^*} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{m\alpha^{*(m-1)}}{\sqrt{m!}} \left|n\right\rangle \left\langle m\right| \\ &= \mathrm{e}^{-\alpha\alpha^*} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{m\alpha^{*(m-1)}}{\sqrt{m!}} \left|n\right\rangle \left\langle m\right| \\ &= \mathrm{e}^{-\alpha\alpha^*} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*(m-1)}}{\sqrt{m!}} \left(m\right) \left|n\right\rangle \left\langle m\right| \end{split}$$

对比可得:

$$\ket{lpha}ra{lpha}=\left(lpha+\partial_{lpha^*}
ight)\ket{lpha}ra{lpha}$$

2-7

若单模自由光场初态为 $|\psi(0)
angle$ 为相干态,求证 $|\psi(t)
angle$ 也是相干态。

单模光场哈密顿量:

$$\hat{H}=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+rac{1}{2}
ight)=\hbar\omega\left(\hat{n}+rac{1}{2}
ight)$$

其中, $\hat{n}\equiv\hat{a}^{\dagger}\hat{a}$ 是粒子数算符。Fock 态是 \hat{n} 的本征态:

$$\hat{n}\ket{n}=n\ket{n}$$

设 $|\psi(0)\rangle = |\alpha\rangle$, 则:

$$\begin{split} |\psi(t)\rangle &= \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar} \, |\psi(0)\rangle \\ &= \mathrm{e}^{-\mathrm{i}(\hat{n}+1/2)\omega t} \, |\alpha\rangle \\ &= \mathrm{e}^{-\mathrm{i}\omega t/2} \mathrm{e}^{-\mathrm{i}\omega t\hat{n}} \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, |n\rangle \\ &= \mathrm{e}^{-\mathrm{i}\omega t/2} \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mathrm{e}^{-\mathrm{i}\omega t\hat{n}} \, |n\rangle \\ &= \mathrm{e}^{-\mathrm{i}\omega t/2} \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mathrm{e}^{-\mathrm{i}\omega tn} \, |n\rangle \\ &= \mathrm{e}^{-\mathrm{i}\omega t/2} \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\left(\alpha \mathrm{e}^{-\mathrm{i}\omega t}\right)^n}{\sqrt{n!}} \, |n\rangle \\ &= \mathrm{e}^{-\mathrm{i}\omega t/2} \mathrm{e}^{-|\alpha \mathrm{e}^{-\mathrm{i}\omega t}|^2/2} \sum_{n=0}^{\infty} \frac{\left(\alpha \mathrm{e}^{-\mathrm{i}\omega t}\right)^n}{\sqrt{n!}} \, |n\rangle \\ &= \mathrm{e}^{-\mathrm{i}\omega t/2} \, |\alpha \mathrm{e}^{-\mathrm{i}\omega t}\rangle \end{split}$$

相位因子的冗余是允许的,因此 $|\psi(t)\rangle$ 也是相干态。

2-8

$$\begin{split} |\xi,\alpha\rangle &= \hat{S}(\xi)\hat{D}(\alpha)\,|0\rangle\,,\ \, \hat{S}(\xi) = \exp\left[\frac{1}{2}\left(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger^2}\right)\right],\ \, \xi = r\mathrm{e}^{\mathrm{i}\theta} \\ \hat{Y}_1 &= \exp\left(\frac{\mathrm{i}\theta\hat{a}^\dagger\hat{a}}{2}\right)\hat{X}_1\exp\left(\frac{-\mathrm{i}\theta\hat{a}^\dagger\hat{a}}{2}\right) = \frac{1}{2}\left[\exp\left(-\frac{\mathrm{i}\theta}{2}\right)\hat{a} + \exp\left(\frac{\mathrm{i}\theta}{2}\right)\hat{a}^\dagger\right] \\ \hat{Y}_2 &= \exp\left(\frac{\mathrm{i}\theta\hat{a}^\dagger\hat{a}}{2}\right)\hat{X}_2\exp\left(\frac{-\mathrm{i}\theta\hat{a}^\dagger\hat{a}}{2}\right) = \frac{1}{2\mathrm{i}}\left[\exp\left(-\frac{\mathrm{i}\theta}{2}\right)\hat{a} - \exp\left(\frac{\mathrm{i}\theta}{2}\right)\hat{a}^\dagger\right] \\ \hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) &= \hat{a}\cosh r - \hat{a}^\dagger\mathrm{e}^{\mathrm{i}\theta}\sinh r \\ \hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) &= \left[\left(\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi)\right)^\dagger\right]^\dagger \\ &= \left(\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)\right)^\dagger \\ &= -\hat{a}\mathrm{e}^{-\mathrm{i}\theta}\sinh r + \hat{a}^\dagger\cosh r \\ \hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{a}\hat{S}(\xi) &= \hat{a}^\dagger\hat{a}\left(\cosh^2r + \sinh^2r\right) + \sinh^2r - \left(\hat{a}^{\dagger2}\mathrm{e}^{\mathrm{i}\theta} + \hat{a}^2\mathrm{e}^{-\mathrm{i}\theta}\right)\cosh r\sinh r \\ \Big\langle \xi,\alpha \mid \hat{Y}_1 \mid \xi,\alpha \Big\rangle &= \mathrm{e}^{-r}\left(\alpha\mathrm{e}^{-\mathrm{i}\theta/2} + \alpha^*\mathrm{e}^{\mathrm{i}\theta/2}\right)/2 \end{split}$$

$$\left\langle \xi, \alpha \left| \hat{Y}_{1} \right| \xi, \alpha \right\rangle = \mathrm{e}^{-r} \left(\alpha \mathrm{e}^{-\mathrm{i}\theta/2} + \alpha^{*} \mathrm{e}^{\mathrm{i}\theta/2} \right) / 2$$

$$\left\langle \xi, \alpha \left| \hat{Y}_{1}^{2} \right| \xi, \alpha \right\rangle = \mathrm{e}^{-2r} \left[1 + \left(\alpha \mathrm{e}^{-\mathrm{i}\theta/2} + \alpha^{*} \mathrm{e}^{\mathrm{i}\theta/2} \right)^{2} \right] / 4$$

$$\left\langle \xi, \alpha \left| \hat{Y}_{2} \right| \xi, \alpha \right\rangle = \mathrm{e}^{r} \left(\alpha \mathrm{e}^{-\mathrm{i}\theta/2} - \alpha^{*} \mathrm{e}^{\mathrm{i}\theta/2} \right) / (2\mathrm{i})$$

$$\left\langle \xi, \alpha \left| \hat{Y}_{2}^{2} \right| \xi, \alpha \right\rangle = \mathrm{e}^{2r} \left[1 - \left(\alpha \mathrm{e}^{-\mathrm{i}\theta/2} - \alpha^{*} \mathrm{e}^{\mathrm{i}\theta/2} \right)^{2} \right] / 4$$

于是:

$$egin{aligned} \Delta \hat{Y}_1 &= \sqrt{\left\langle \xi, lpha \left| \hat{Y}_1^2 \left| \xi, lpha \right
ight
angle - \left\langle \xi, lpha \left| \hat{Y}_1 \left| \xi, lpha
ight
angle^2} \ &= rac{\mathrm{e}^{-r}}{2} \end{aligned} \ \Delta \hat{Y}_2 &= \sqrt{\left\langle \xi, lpha \left| \hat{Y}_2^2 \left| \xi, lpha
ight
angle - \left\langle \xi, lpha \left| \hat{Y}_2 \left| \xi, lpha
ight
angle^2} \ &= rac{\mathrm{e}^r}{2} \end{aligned}$$

2-9

常证:
$$\exp\left(\xi^*\hat{a}_A\hat{a}_B - \xi\hat{a}_A^{\dagger}\hat{a}_B^{\dagger}\right) = \frac{1}{\cosh r} \exp\left(-\Omega\hat{a}_A^{\dagger}\hat{a}_B^{\dagger}\right) \exp\left[\ln\cosh r\left(\hat{a}_A^{\dagger}\hat{a}_A + \hat{a}_B^{\dagger}\hat{a}_B\right)\right] \exp\left(\Omega^*\hat{a}_A\hat{a}_B\right)$$

$$\left[\xi^*\hat{a}_A\hat{a}_B, -\xi\hat{a}_A^{\dagger}\hat{a}_B^{\dagger}\right] = -|\xi|^2 \left[\hat{a}_A\hat{a}_B, \hat{a}_A^{\dagger}\hat{a}_B^{\dagger}\right]$$

$$= -|\xi|^2 \left(\hat{a}_A\hat{a}_A^{\dagger} + \hat{a}_B^{\dagger}\hat{a}_B\right)$$

$$\left[\xi^*\hat{a}_A\hat{a}_B, \hat{a}_A\hat{a}_A^{\dagger} + \hat{a}_B^{\dagger}\hat{a}_B\right] = 0$$

$$\left[-\xi\hat{a}_A^{\dagger}\hat{a}_B^{\dagger}, \hat{a}_A\hat{a}_A^{\dagger} + \hat{a}_B^{\dagger}\hat{a}_B\right] = 0$$

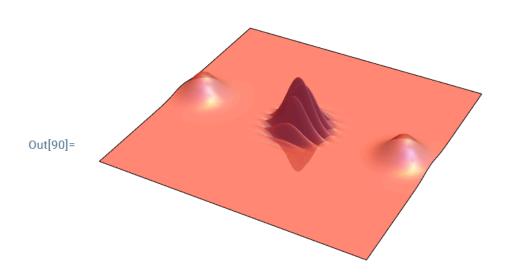
因此由 BCH 公式:

$$\begin{split} \exp\left(\xi^{*}\hat{a}_{A}\hat{a}_{B} - \xi\hat{a}_{A}^{\dagger}\hat{a}_{B}^{\dagger}\right) &= \exp\left(\xi^{*}\hat{a}_{A}\hat{a}_{B}\right) \exp\left(|\xi|^{2}\left(\hat{a}_{A}\hat{a}_{A}^{\dagger} + \hat{a}_{B}^{\dagger}\hat{a}_{B}\right)/2\right) \exp\left(-\xi\hat{a}_{A}^{\dagger}\hat{a}_{B}^{\dagger}\right) \\ &= \frac{1}{\cosh r} \exp\left(-\Omega\hat{a}_{A}^{\dagger}\hat{a}_{B}^{\dagger}\right) \exp\left[\ln\cosh r\left(\hat{a}_{A}^{\dagger}\hat{a}_{A} + \hat{a}_{B}^{\dagger}\hat{a}_{B}\right)\right] \exp\left(\Omega^{*}\hat{a}_{A}\hat{a}_{B}\right) \end{split}$$

求薛定谔猫态 $|\Psi\rangle=\frac{1}{\sqrt{N}}\left(|\alpha_0\rangle+|-\alpha_0\rangle\right)$,其中 $N=2\left(1+\mathrm{e}^{-2|\alpha_0|^2}\right)$ 的 $W(\alpha,\alpha^*)$ 函数,并以 $\mathrm{Re}[\alpha]$ 和 $\mathrm{Im}[\alpha]$ 为两个水平轴,所得的 $W(\alpha,\alpha^*)$ 函数为竖直轴画出其三维图。

 $ho = \ket{\Psi}ra{\Psi}$

$$\begin{split} &=\frac{1}{N}\left(|\alpha_{0}\rangle+|-\alpha_{0}\rangle\right)\left(\langle\alpha_{0}|+\langle-\alpha_{0}|\right) \\ &C_{\mathrm{W}}(\lambda)=\mathrm{Tr}\left(\rho\hat{D}(\lambda)\right) \\ &=\mathrm{Tr}\left[\frac{1}{N}\left(|\alpha_{0}\rangle+|-\alpha_{0}\rangle\right)\left(\langle\alpha_{0}|+\langle-\alpha_{0}|\right)\hat{D}(\lambda)\right] \\ &=\frac{1}{N}\mathrm{Tr}\left[\left(\langle\alpha_{0}|+\langle-\alpha_{0}|\right)\hat{D}(\lambda)\left(|\alpha_{0}\rangle+|-\alpha_{0}\rangle\right)\right] \\ &=\frac{1}{N}\left(\langle\alpha_{0}|+\langle-\alpha_{0}|\right)\hat{D}(\lambda)\left(|\alpha_{0}\rangle+|-\alpha_{0}\rangle\right) \\ &=\frac{1}{N}\left\{\left(\langle\alpha_{0}|+\langle-\alpha_{0}|\right)\left(\mathrm{e}^{\mathrm{i}\Im(\lambda\alpha_{0}^{*})}|\lambda+\alpha_{0}\rangle+\mathrm{e}^{-\mathrm{i}\Im(\lambda\alpha_{0}^{*})}|\lambda-\alpha_{0}\rangle\right)\right\} \\ &=\frac{1}{N}\mathrm{e}^{-|\alpha_{0}|^{2}}\mathrm{e}^{-|\lambda|^{2}/2}\left[\mathrm{e}^{-\lambda^{*}\alpha_{0}}\left(\mathrm{e}^{-|\alpha_{0}|^{2}-\alpha_{0}^{*}\lambda}+\mathrm{e}^{|\alpha_{0}|^{2}+\alpha_{0}^{*}\lambda}\right)+\mathrm{e}^{\lambda^{*}\alpha_{0}}\left(\mathrm{e}^{|\alpha_{0}|^{2}-\alpha_{0}^{*}\lambda}+\mathrm{e}^{-|\alpha_{0}|^{2}+\alpha_{0}^{*}\lambda}\right)\right] \\ &W(\alpha)&=\frac{1}{\pi^{2}}\int C_{\mathrm{W}}(\lambda)\mathrm{e}^{\lambda^{*}\alpha-\lambda\alpha^{*}}\mathrm{d}^{2}\lambda \\ &=\frac{2}{\pi}\left[\mathrm{e}^{-2|\alpha-\alpha_{0}|^{2}}+\mathrm{e}^{-2|\alpha+\alpha_{0}|^{2}}\mathrm{e}^{-2|\alpha|^{2}}\left(\mathrm{e}^{-2(\alpha_{0}\alpha^{*}-\alpha\alpha_{0}^{*})}+\mathrm{e}^{-2(-\alpha_{0}\alpha^{*}+\alpha\alpha_{0}^{*})}\right)\right] \end{split}$$



2-11

求相干压缩态 $|\Psi\rangle=\hat{S}(\xi)\hat{D}(\alpha_0)\,|0\rangle$ 的 $W(\alpha,\alpha^*)$ 函数,并以 $\mathrm{Re}[\alpha]$ 和 $\mathrm{Im}[\alpha]$ 为两个水平轴,所得的 $W(\alpha,\alpha^*)$ 函数为竖直轴画出其三维图。

$$ho = \hat{S}(\xi)\hat{D}(lpha_0)\ket{0}ra{0}\hat{D}^\dagger(lpha_0)\hat{S}^\dagger(\xi)$$

$$\begin{split} W(\alpha,\alpha^*) &= \frac{1}{\pi^2} \int \mathrm{d}^2 \lambda \mathrm{Tr} \left[\rho \hat{D}(\lambda) \right] \mathrm{e}^{-\lambda \alpha^* + \lambda^* \alpha} \\ &= \frac{1}{\pi^2} \int \mathrm{d}^2 \lambda \left\langle 0 \left| \, \hat{D}^\dagger(\alpha_0) \hat{S}^\dagger(\xi) \hat{D}(\lambda) \hat{S}(\xi) \hat{D}(\alpha_0) \, \right| \, 0 \right\rangle \mathrm{e}^{-\lambda \alpha^* + \lambda^* \alpha} \end{split}$$

利用公式:

$$\hat{D}^{\dagger}(lpha_0)\hat{D}(\lambda)\hat{D}(lpha_0) = \hat{D}(\lambda)\mathrm{e}^{\lambdalpha_0^*-\lambda^*lpha_0} \ \hat{S}^{\dagger}(\xi)\hat{D}(\lambda)\hat{S}(\xi) = \hat{D}\left(\lambda\cosh r + \lambda^*\mathrm{e}^{\mathrm{i} heta}\sinh r
ight)$$

可得:

$$\begin{split} W(\alpha,\alpha^*) &= \frac{1}{\pi^2} \int \mathrm{d}^2 \lambda \left\langle 0 \, \Big| \, \hat{D}^\dagger(\alpha_0) \hat{S}^\dagger(\xi) \hat{D}(\lambda) \hat{S}(\xi) \hat{D}(\alpha_0) \, \Big| \, 0 \right\rangle \mathrm{e}^{-\lambda \alpha^* + \lambda^* \alpha} \\ &= \frac{1}{\pi^2} \int \mathrm{d}^2 \lambda \left\langle 0 \, \Big| \, \hat{D} \left(\lambda \cosh r + \lambda^* \mathrm{e}^{\mathrm{i}\theta} \sinh r \right) \, \Big| \, 0 \right\rangle \mathrm{e}^{-\lambda (\alpha^* - \alpha_0^*) + \lambda^* (\alpha - \alpha_0)} \end{split}$$

作变量代换:

$$\chi = \lambda \cosh r + \lambda^* e^{i\theta} \sinh r$$

积分可化为:

$$\begin{split} W(\alpha,\alpha^*) &= \frac{1}{\pi^2} \int \mathrm{d}^2 \lambda \left\langle 0 \, \Big| \, \hat{D} \left(\lambda \cosh r + \lambda^* \mathrm{e}^{\mathrm{i} \theta} \sinh r \right) \, \Big| \, 0 \right\rangle \mathrm{e}^{-\lambda (\alpha^* - \alpha_0^*) + \lambda^* (\alpha - \alpha_0)} \\ &= \frac{1}{\pi^2} \int \mathrm{d}^2 \chi \left\langle 0 \, \Big| \, \hat{D}(\chi) \, \Big| \, 0 \right\rangle \mathrm{e}^{-\chi A^* + \chi^* A} \\ &= \frac{2}{\pi} \mathrm{e}^{-2|A|^2} \\ &= \frac{2}{\pi} \mathrm{e}^{-2\left|(\alpha^* - \alpha_0^*) \mathrm{e}^{\mathrm{i} \theta} \sinh r + (\alpha - \alpha_0) \cosh r \right|^2} \end{split}$$

其中,

$$A\equiv (lpha^*-lpha_0^*)\,\mathrm{e}^{\mathrm{i} heta}\sinh r+(lpha-lpha_0)\cosh r$$

2-12

从方程 $\hat{H}=\frac{1}{2m}\left[\hat{\vec{p}}-\frac{e}{c}\vec{A}(\vec{r},t)\right]^2+eU(\vec{r})+V(r)$ 出发,(1)推导偶极近似、二能级近似和旋波近似哈密顿量 $\hat{H}(t)=\frac{\hbar\omega_0}{2}\hat{\sigma}_z+\frac{\hbar\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}_++\mathrm{H.c.}\right)$ (2)求二能级原子的布居反转的 Rabi 振荡,并用 Mathematica 软件画出结果。

$$egin{aligned} \hat{H} &= rac{1}{2m} \left[\hat{ec{p}} - rac{e}{c} ec{A}(ec{r},t)
ight]^2 + eU(ec{r}) + V(r) \ \hat{\mathcal{U}} &\equiv \exp \left(rac{\mathrm{i}e}{\hbar c} \int_{ec{r}_0}^{ec{r}} ec{A}(ec{r}',t) \cdot \mathrm{d}ec{r}'
ight) \ \hat{ ilde{H}} &\equiv \hat{\mathcal{U}}^\dagger \hat{H} \hat{\mathcal{U}} - \mathrm{i}\hbar \hat{\mathcal{U}}^\dagger \hat{\mathcal{U}} \ &= rac{\hat{ec{p}}^2}{2m} + eU(ec{r}) + V(r) - rac{e}{c} \int_{ec{r}_0}^{ec{r}} ec{E}(ec{r}',t) \cdot \mathrm{d}ec{r}' \end{aligned}$$

偶极近似:

$$ec{A}(ec{r},t)pproxec{A}(ec{r}_0,t)$$

二能级近似:

$$\hat{ec{p}}^2/2m+V(r)pprox E_e\ket{e}ra{e}+E_g\ket{g}ra{g}=rac{\hbar\omega_0}{2}\hat{\sigma}_z, \ \ \hbar\omega_0=E_e-E_g, \hat{\sigma_z}=\ket{e}ra{e}-\ket{g}ra{g}$$

$$egin{aligned} \hat{H} pprox rac{\hbar \omega_0}{2} \hat{\sigma_z} - rac{e}{c} ec{r} \cdot ec{E}(t) &= rac{\hbar \omega_0}{2} \hat{\sigma_z} - rac{e}{c} ec{1}_r \cdot ec{E}(t) \left(r_{ge} \hat{\sigma}_+ + ext{H.c.}
ight) \ ec{E}(t) &= ec{1}_E E_0 \cos(\omega t) \ \\ \hat{H} &= rac{\hbar \omega_0}{2} \hat{\sigma_z} - rac{e}{c} ec{1}_r \cdot ec{E}(t) \left(r_{ge} \hat{\sigma}_+ + ext{H.c.}
ight) \ &= rac{\hbar \omega_0}{2} \hat{\sigma_z} + rac{\hbar \Omega}{2} \left(ext{e}^{ ext{i}\omega t} + ext{e}^{- ext{i}\omega t}
ight) \left(ext{e}^{ ext{i}\phi} \hat{\sigma}_+ + ext{H.c.}
ight) \end{aligned}$$

其中,

$$r_{ge} = \left| r_{ge}
ight| \mathrm{e}^{\mathrm{i}\phi}, \ \ \Omega = -rac{e ec{1}_r \cdot ec{1}_E E_0 \left| r_{ge}
ight|}{\hbar c}$$

定义:

$$\hat{H}_0 = rac{\hbar \omega_0}{2}\hat{\sigma}_z, ~~\hat{H}_I = \hat{H} - \hat{H}_0$$

相互作用绘景:

$$\hat{H}_{I}^{(I)}(t)=rac{\hbar\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{e}^{-\mathrm{i}\omega t}
ight)\left(\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\omega_{0}t}\hat{\sigma}_{+}+\mathrm{H.c.}
ight)$$

旋波近似,忽略快速振荡项 $\mathrm{e}^{\pm\mathrm{i}(\omega_0+\omega)t}$,

$$egin{aligned} \hat{H}_{I}^{(I)}(t) &= rac{\hbar\Omega}{2} \left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}
ight) \left(\mathrm{e}^{\mathrm{i}\phi} \mathrm{e}^{\mathrm{i}\omega_0 t} \hat{\sigma}_+ + \mathrm{H.c.}
ight) \ &pprox rac{ar{\Omega}}{2} \left[\mathrm{e}^{\mathrm{i}\phi} \mathrm{e}^{\mathrm{i}\omega_0 t} \hat{\sigma}_+ + \mathrm{H.c.}
ight] \end{aligned}$$

回到薛定谔绘景:

$$\hat{H}=rac{\hbar\omega_0}{2}\hat{\sigma}_z+rac{\hbar\Omega}{2}\left(\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\omega t}\hat{\sigma}_++\mathrm{H.c}
ight)$$

设t时刻体系的状态可展开为:

$$\ket{\Psi(t)} = c_g(t) \mathrm{e}^{\mathrm{i}\omega t/2} \ket{g} + c_e(t) \mathrm{e}^{-\mathrm{i}\omega_0 t/2} \ket{e}$$

由薛定谔方程可得:

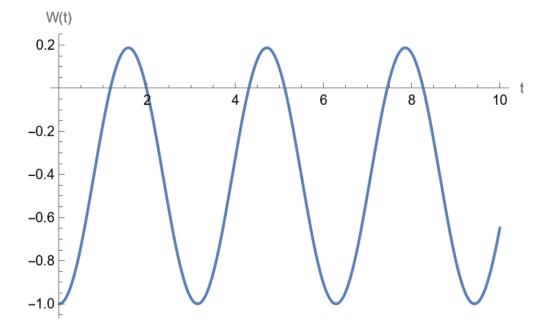
$$\mathrm{i} \dot{c}_e(t) = rac{\Omega}{2} c_g(t) \mathrm{e}^{-\mathrm{i} \varDelta t}, \ \ \mathrm{i} \dot{c}_g(t) = rac{\Omega}{2} c_e(t) \mathrm{e}^{\mathrm{i} \varDelta t}$$

可以解得:

$$\begin{split} c_e(t) &= \mathrm{e}^{-\mathrm{i} \varDelta t/2} \left[c_e(0) \left(\cos \frac{\mu t}{2} + \frac{\mathrm{i} \varDelta}{\mu} \sin \frac{\mu t}{2} \right) - \mathrm{i} c_g(0) \frac{\Omega}{\mu} \sin \frac{\mu t}{2} \right] \\ c_g(t) &= \mathrm{e}^{\mathrm{i} \varDelta t/2} \left[c_g(0) \left(\cos \frac{\mu t}{2} - \frac{\mathrm{i} \varDelta}{\mu} \sin \frac{\mu t}{2} \right) - \mathrm{i} c_e(0) \frac{\Omega}{\mu} \sin \frac{\mu t}{2} \right] \end{split}$$

选择初始条件 $c_g(0) = 1, c_e(0) = 1$, 布居反转为:

$$egin{aligned} W(t) &= \langle \hat{\sigma}_z
angle \ &= rac{\Omega^2 - \Delta^2}{\mu^2} \sin^2 rac{\mu t}{2} - \cos^2 rac{\mu t}{2} \end{aligned}$$



在光与物质相互作用的全量子描述中,光的初态为相干态时,(1)求二能级原子布居反转的动力学演化;(2)用 Mathematica 画出你的结果;(3)与问题 12 得到的 Rabi 振荡相对比,体会量子光-物质相互作用的效应。

设体系状态可展为:

$$| ilde{\Psi}_{n+1}(t)
angle = c_{n+1,g}(t)\mathrm{e}^{-\mathrm{i}[(n+1)\omega-\omega_0/2]t}\,|n+1,g
angle + c_{n,e}(t)\mathrm{e}^{-\mathrm{i}[n\omega+\omega_0/2]}\,|n,e
angle$$

代入薛定谔方程可得:

$$\mathrm{i}\dot{c}_{n+1,g}(t)=g\sqrt{n+1}\mathrm{e}^{-\mathrm{i}\Delta t}c_{n,e}(t)$$

$$\mathrm{i}\dot{c}_{n,e}(t)=g\sqrt{n+1}\mathrm{e}^{\mathrm{i}\Delta t}c_{n+1,g}(t)$$

解得:

$$c_{n+1,g}(t) = \mathrm{e}^{-\mathrm{i} \varDelta t/2} \left[c_{n+1,g}(0) \left(\cos \dfrac{\Omega_n(\varDelta)t}{2} + \mathrm{i} \dfrac{\varDelta \sin \dfrac{\Omega_n(\varDelta)t}{2}}{\Omega_n(\varDelta)}
ight) + c_{n,e}(0) \dfrac{-2\mathrm{i} g \sqrt{n+1} \sin \dfrac{\Omega_n(\varDelta t)}{2}}{\Omega_n(\varDelta)}
ight] \ c_{n,e}(t) = \mathrm{e}^{\mathrm{i} \varDelta t/2} \left[c_{n+1,g}(0) \dfrac{-2\mathrm{i} g \sqrt{n+1} \sin \dfrac{\Omega_n(\varDelta t)}{2}}{\Omega_n(\varDelta)} + c_{n,e}(0) \left(\cos \dfrac{\Omega_n(\varDelta)t}{2} - \mathrm{i} \dfrac{\varDelta \sin \dfrac{\Omega_n(\varDelta)t}{2}}{\Omega_n(\varDelta)}
ight)
ight]$$

布居反转:

$$W(t) = \langle \Psi_{n+1}(t) \, | \, \hat{\sigma}_z \, | \, \Psi_{n+1}(t)
angle = |c_{n,e}(t)|^2 - |c_{n+1,g}(t)|^2$$

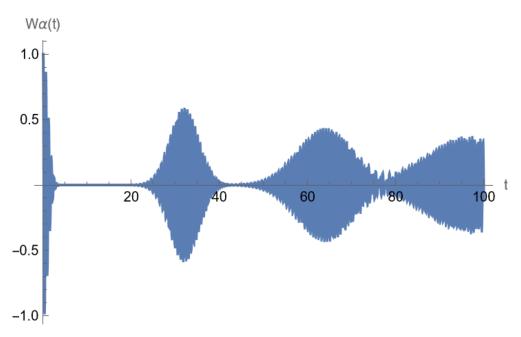
当初态为相干态,即:

$$|\Psi(0)
angle = |lpha,e
angle = \mathrm{e}^{-|lpha|^2/2} \sum_{n=0}^{\infty} rac{lpha^n}{\sqrt{n!}} \ket{n,e}$$

$$\Psi(t)=\mathrm{e}^{-|lpha|^2/2}\sum_{n=0}^{\infty}rac{lpha^n}{\sqrt{n!}}\ket{ ilde{\Psi}_{n+1}(t)}$$

布居反转:

$$egin{align*} W_lpha(t) &= \mathrm{e}^{-|lpha|^2} \sum_{n=0}^\infty rac{|lpha|^{2n} \, W_n(t)}{n!} \ &= \mathrm{e}^{-|lpha|^2} \sum_{n=0}^\infty rac{|lpha|^{2n}}{n!} \left[rac{\Delta^2}{\Omega_n^2(\Delta)} + rac{4g^2(n+1)}{\Omega_n^2(\Delta)} \cos\left[\Omega_n(\Delta)t
ight]
ight], \;\; \Omega_n(\Delta) &= \sqrt{\Delta^2 + 4g^2(n+1)} \end{split}$$



在大 ω_0/ω 极限下研究量子 Rabi 模型,并用 Mathematica 画出你的结果(1)数值求解其本征能量随 g/ω 的变化;(2)数值求解基态能量 $E_g, \frac{\mathrm{d}E_g}{\mathrm{d}g}, \frac{\mathrm{d}^2E_g}{\mathrm{d}g^2}$ 随 g/ω 的变化,揭示在 $g=\sqrt{\omega\omega_0}/2$ 处发生量子相变。

量子 Rabi 模型哈密顿量:

$$\hat{H} = \omega \hat{a}^{\dagger} \hat{a} + rac{\omega_0}{2} \hat{\sigma}_z - g \left(\hat{a} + \hat{a}^{\dagger}
ight) \hat{\sigma}_x$$

其中, ω 是光场圆频率, ω_0 是跃迁圆频率, g 是耦合强度。

当:

$$\frac{2g}{\sqrt{\omega\omega_0}} = 1$$

时,发生量子相变,即:

$$g=rac{\sqrt{\omega\omega_0}}{2}$$

