Exercise 3

Deadline: 10.11.2014

The goal of this exercise ist to derive the closed-form expressions for the optimal parameters $\bar{w}$ and $\bar{b}$ in LDA, given some training set.

1 LDA - Optimal Parameters (20 points)

Remember that in Linear discriminant analysis the decision boundary is given by a $d-1$ dimensional hyperplane (where d is the dimension of the featurespace) that we parametrize via

$$w^T x + b = 0. \tag{1}$$

w is a normal vector of the hyperplane and b a vector that fixes the position of the hyperplane. The decision rule in a classification task with two classes for a query point x is then

$$\hat{y} = \begin{cases} 1, & \text{if } w^T x + b > 0 \\ 0, & \text{if } w^T x + b < 0 \end{cases} \tag{2}$$

In the training phase we are given a set of datapoints $\{x_i\}_{i\in 1,\dots,N}$ with $x_i \in \mathbb{R}^d$ and their respective labels $\{y_i\}_{i\in 1,\dots,N}$ with $y_i \in [0,1]$. We assume that the training set is balanced (same amount of examples for each class) and therefore we can define the response z_i in the following way:

$$z_i = \begin{cases} -1, & \text{if } y_i = 0 \\ 1, & \text{if } y_i = 1 \end{cases} \tag{3}$$

The optimal parameter $\bar{w}$ and $\bar{b}$ are now the ones that minimize following expression:

$$\bar{w}, \bar{b} = \operatorname{argmin}_{w,b} \sum_{i=1}^{N} \left(w^T x_i + b - z_i \right)^2 \tag{4}$$

From the lecture we know that

$$\left(N S_w - \frac{N_0 N_1}{N} S_B \right) \bar{w} = \frac{N}{2} \left(m_1 - m_0 \right) \tag{5}$$

determines $\bar{w}$. (The $\frac{1}{2}$ on the RHS is due to the convention in eq. 3). We saw how this could be transformed to

$$\bar{w} \propto S_w^{-1} (m_1 - m_0) \tag{6}$$

Our goal is to derive eq. 5 from 4. Therefore compute $\bar{b}$ from

$$\frac{\partial}{\partial b} \sum_{i=1}^{N} \left(w^T x_i + b - z_i \right)^2 = 0\ . \tag{7}$$

Use this when reshuffling

$$\frac{\partial}{\partial w}\sum_{i=1}^{N}\left(w^T x_i + b - z_i\right)^2 = 0 \; . \tag{8}$$

to end up with eq. 5.

You need to remember the definitions of the between class covariance S_B:

$$S_B = (m_1 - m_0)(m_1 - m_0)^T \tag{9}$$

and the within class covariance S_W

$$S_W = \frac{1}{N}\sum_{i=1}^{N}(x_i - m_{y_i})(x_i - m_{y_i})^T \; . \tag{10}$$

where

$$m_0 = \frac{1}{N_0}\sum_{i:\; y_i=0} x_i \tag{11}$$

$$m_1 = \frac{1}{N_1}\sum_{i:\; y_i=1} x_i \tag{12}$$

$$m = \frac{1}{N}\sum_{i}^{N} x_i \tag{13}$$

$$N_0 = N_1 = \frac{N}{2} \tag{14}$$

During your calculations you may find the following relation for general vectors a, b and c useful:

$$a \cdot (b^T \cdot c) = (a \cdot c^T) \cdot b \tag{15}$$

Regulations

Either hand in your solution as a .pdf or hand in a readable, handwritten solution. Send the .pdf to `niko.krasowski@hci.iwr.uni-heidelberg.de`. Either way, please make sure that you hand in your solution before the deadline. You may hand in the exercises in teams of maximally three people, which must be clearly named on the solution sheet (one email is sufficient). Discussions between different teams about the exercises are encouraged, but the solutions must not be copied verbatim. Please respect particularly this rule, otherwise we cannot give you a passing grade. Solutions are due by email at the beginning of the next exercise. For each exercise there will be maximally 20 points assigned. If you have 50% or more points in the end of the semester you will be allowed to take the exam.