Sum and Mixture of Gaussians

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Proposition 1. ***(Sum of Gaussians)*** *Let X and Y be independent random variables that are normally distributed, then their sum is also normally distributed. i.e., if*

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2) \tag{1}$$
$$Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2) \tag{2}$$
$$Z = X + Y, \tag{3}$$

then

$$Z \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2) \tag{4}$$

Proof. (using moment generating function) The moment generating function of Z is given by:

$$M_Z(s) = \mathbb{E}(\exp(sZ)) = \mathbb{E}(\exp(s(X+Y))) = \mathbb{E}(\exp(sX))\mathbb{E}(\exp(sY)) = M_X(s)M_Y(s). \tag{5}$$

As the sum of two independent random variables X and Y is just the product of the two separate characteristic functions. The moment generating function of the normal distribution is $M_X(s) = \exp(s\mu_X + s^2\sigma_X^2/2)$. We have:

$$M_Z(s) = \exp(s\mu_X + s^2\sigma_X^2/2)\exp(s\mu_Y + s^2\sigma_Y^2/2) \tag{6}$$
$$= \exp(s(\mu_X + \mu_Y) + s^2(\sigma_X^2/2 + \sigma_Y^2/2)) \tag{7}$$

This is the moment generating function of the normal distribution with the mean $\mu_X + \mu_Y$ and the variance $\sigma_X^2 + \sigma_Y^2$. $\square$

Definition 1. ***(Mixture of Gaussian)*** *Let Z be a categorical distribution, sampling process for mixture of two Gaussian is given by:*

$$Z \sim \textit{Categorical}(\pi_1, \pi_2) \tag{8}$$
$$X|Z = 0 \sim \mathcal{N}(\mu_1, \sigma_1^2) \tag{9}$$
$$X|Z = 1 \sim \mathcal{N}(\mu_2, \sigma_2^2), \tag{10}$$

where $\pi_1 + \pi_2 = 1$, $\pi_1 \geq 0$ and $\pi_2 \geq 0$. The pdf can be represented as:

$$p(x) = \pi_1\mathcal{N}(x|\mu_1, \sigma_1^2) + \pi_2\mathcal{N}(x|\mu_2, \sigma_2^2) \tag{11}$$

Proposition 2. ***(Expectation and Variance)***

$$\mathbb{E}(X) = \pi_1\mu_1 + \pi_2\mu_2 \tag{12}$$

$$\mathbb{E}(X^2) = \int x^2 p(x)dx = \pi_1(\sigma_1^2 + \mu_1^2) + \pi_2(\sigma_2^2 + \mu_2^2) \tag{13}$$
$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \tag{14}$$