Optimal Disclosure Windows*

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Abstract

I study a dynamic disclosure game in which an agent controls the time window over which information flows to the decision maker, but does not control the content of that information. In equilibrium, the agent has incentives to delay the start of disclosure to continue to learn privately for some time. This delay exacerbates the information asymmetry between the agent and the decision maker as the agent is learning while the decision maker is not. The length of the disclosure window is determined by the degree of information asymmetry at the beginning of the window, with longer windows associated with greater information asymmetry. As a result, the delay in the start of disclosure requires a longer disclosure window.

Keywords: information disclosure, strategic timing, dynamic signaling.

JEL Codes: C73, D82, D83

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1 Introduction

I study a novel dynamic model of verifiable information disclosure. I consider a disclosure environment in which an agent, who observes a flow of information over time, controls the window of time over which this information is disclosed to a decision maker, but does not control the content of the information being disclosed. The agent chooses when to open this disclosure window and how long to keep it open, taking into account that while disclosure is in progress, he cannot prevent unfavorable information from coming to light.

This disclosure environment is common in many corporate and financial settings. For example, consider a pharmaceutical company (the agent) seeking drug approvals from the Food and Drug Administration (FDA, the decision maker). The pharmaceutical company first files for patents to establish proprietary, and privately gathers information about the drug through discovery and preclinical research. It then registers the drug with the FDA to start clinical trials. At this point, the company must report all results from clinical trials. The company then chooses a time to conclude clinical trials, after which it submits a drug application to the FDA, who then decides the extent (treatment conditions, age groups, etc) to which they approve the drug.[1] Similarly, consider a private company (the agent) trying to raise capital through an Initial Public Offering (IPO). The company is initially privately owned and learns about its potential market value through operations. It announces an IPO when it decides to go public and performs due diligence, where it undergoes investigations into its financials and performances. The company then chooses an initial offering date, at which time the market (the decision maker) evaluates and prices the company.

This disclosure environment prevents the agent from cherry-picking which information to disclose. Instead, the agent (partially) controls the information disclosure by choosing when to open and close the disclosure window, based on what he knows and what has been revealed. As the examples above show, sometimes information transparency during a disclosure window is a legal or institutional requirement, and the timing of disclosure is a crucial strategic component in determining the final outcome. The voluntary disclosure literature has largely focused on "*what* to disclose," while the study of "*when* to disclose" has received relatively little attention.

[1] Details about the FDA drug approval process can be found at https://www.fda.gov/patients/learn-about-drug-and-device-approvals/drug-development-process.

My paper contributes to this literature by studying both *when* and *how long* to disclose. I explore the strategic timing of both the beginning and the end of disclosing an information process, whilst capturing the idea that the information arriving during this window of time must be disclosed in full. I characterize the agent's optimal time to start disclosure and the optimal duration to keep it open. I derive a key insight that shows delays in the start of disclosure lead to longer disclosure windows.

Specifically, in my model, an agent and a decision maker engage in a dynamic disclosure game that takes place in continuous time with an infinite horizon. The underlying state of the world is either good or bad, initially unknown to both the agent and the decision maker. Over time, the agent privately receives a flow of information about the state. The information takes the form of signals that arrive at random times and deliver conclusive news that the state is bad. The agent chooses a time to start disclosing this information process to the decision maker. While the information process unfolds, he chooses a time to stop disclosing. To capture the idea that the beginning and end of a disclosure process are often restricted by exogenous factors such as financing, I assume the agent can start disclosing at any time only after he gets an opportunity to do so, and disclosure might exogenously terminate at any time after it starts. The decision maker observes the times at which disclosure begins and ends and the signal arrivals (or lack thereof) in between, but not the underlying reasons for why disclosure doesn't begin or why it ends. Given this information, the decision maker takes an action at the end of disclosure. While the decision maker prefers an action that matches the state, the agent prefers a higher action regardless of the state.

This setup poses some analytical challenges. First, this game is effectively a dynamic signaling game with an agent whose (private) information changes over time. Second, the agent choosing both when to start and when to stop disclosing results in two stopping problems that are intertwined. Both issues lead to an obscured inference problem for the decision maker. The standard exponential bandit framework with conclusive bad news helps keep the problem tractable: at each point in time, the agent can be either of only two types: *informed* if he has observed a conclusive bad signal, or *uninformed* if he has not. Although the agent's belief still changes over time if he is uninformed, the agent's type can only change from being uninformed to informed. Together with the fact that learning is common knowledge, the decision maker knows the agent's belief evolution at each point in time.[2]

The equilibrium strategies are as follows. The decision maker takes an action that is equal to her posterior belief that $\theta = 1$ at the time disclosure stops. If the agent is patient, then there exists a fixed period of time at the beginning of the game where both types of agent delay the start of disclosure in order to privately learn about the state. After having done so, the agent starts disclosing as soon as he gets the opportunity. If the agent is impatient, he starts disclosing as soon as he gets an opportunity right from the beginning of the game. Once disclosure has started, the uninformed agent has stronger incentives to keep disclosure open and let information flow to the decision maker. In particular, along the history in which no signal arrives and disclosure does not terminate exogenously, the uninformed agent keeps the disclosure window open until such a time that the decision maker is convinced that the agent is uninformed. While the uninformed agent strictly prefers to keep the disclosure window open until this time, the informed agent is indifferent and randomizes over waiting times within this disclosure window.

The equilibrium captures a novel interaction between the duration of the disclosure window and the time at which it starts: a later start of disclosure leads to a longer disclosure window. The duration of the disclosure window is determined by the degree of information asymmetry between the agent and the decision maker at the beginning of the disclosure window. At the beginning of the game, the agent and the decision maker have the same information about the state. Prior to the start of disclosure, the agent learns about the state privately: he either becomes more optimistic that the state is good if he remains uninformed, or receives a signal and learns that the state is bad. Because the two types of agent adopt the same strategy in choosing the start of disclosure, the decision maker cannot infer anything about the state from the agent's behavior and thus is not learning about the state while the agent continues to do so. A longer delay till the start of disclosure exacerbates this information asymmetry between the agent and the decision maker. The now-more-optimistic uninformed agent needs to keep the disclosure window open longer to reduce the difference between his belief that the state is good and that of the decision maker.

At the beginning of the game where the agent chooses when to start disclosure, the agent faces a tradeoff between private learning and discounting. If a signal arrives, the

[2]In other words, "time 0" is common knowledge. The decision maker does not know what the agent has learned but does know how much the agent has learned.

agent would rather observe that signal in private and learn the state without having to share this information with the decision maker. However, delay is costly because of discounting and because a delay in the start of disclosure leads to an even longer delay in the continuation game. Thus, the expected time until the payoff realizes is longer. The agent finds it optimal to delay to learn only if he is sufficient patient.

In the continuation stopping game while the disclosure window is open, the agent faces a tradeoff between inducing a more favorable action and higher risks. In the conclusive bad news environment, "no news is good news": the decision maker (and the agent) become more optimistic that the state is good in the absence of signals. However, a longer disclosure time exposes the agent to higher risks because a signal is more likely to arrive during a larger time interval. This tradeoff leads to delayed stopping by the more optimistic (uninformed) agent as he regards the risk lower than the informed agent does. In equilibrium, the uninformed agent keeps the disclosure window open in the hope that both the absence of signals for a certain length of time, and the fact that he is willing to let information flow for that long, convince the decision maker that he is the uninformed agent; hence the decision maker should be as optimistic as he is.

Related Literature

My paper contributes to the literature on voluntary disclosure of verifiable information pioneered by Grossman and Hart (1980), Grossman (1981), and Milgrom (1981). These earlier works establish an unraveling result: under certain conditions, all types of agents (or senders) disclose their information and the decision maker (receiver) learns the agent's type. Subsequent work in this literature has shown that the unraveling result fails if disclosure is costly (see Jovanovic, 1982 and Verrecchia, 1983), or if the decision maker is uncertain about the agent's information endowment, that is, whether the agent has information or not (see Dye, 1985 and Jung and Kwon, 1988). I adopt the approach proposed in Dye (1985) and Jung and Kwon (1988).

As mentioned, most of the existing literature has focused on studying "what to disclose" while very few consider "when to disclose."[3] In my paper, the nature of disclosure is multi-dimensional: not only does the content of the disclosure matter, the timing of disclosure also plays a key role in determining the equilibrium outcome.

[3] This was pointed out in, for example, Guttman, Kremer, and Skrzypacz (2014) and Hirst, Koonce, and Venkataraman (2008).

Some examples that explore the timing of disclosure include Acharya, DeMarzo, and Kremer (2011) and Guttman, Kremer, and Skrzypacz (2014), both of which are dynamic versions of Dye (1985).

Specifically, Acharya et al. (2011) consider a model in which the agent gets one piece of private information and his timing decision to disclose this information is constrained by the arrival of some public information. Guttman et al. (2014) study a two-signal, two-period model where the agent chooses which period to disclose and what signal to disclose. The content of disclosure is completely flexible in Guttman et al. (2014). If the agent waits until the second period to disclose, there is a chance that he might get an additional signal in the second period, and can then choose which signal to disclose. In my model, the agent cannot pick and choose the information he discloses. In fact, the agent's inability to control the information content is one of the key considerations of the agent's timing decisions.

My model features a disclosure environment different from the papers mentioned above. When the agent starts disclosure, he does not disclose a piece of evidence that can be immediately verified. Instead, "disclosure" starts a learning process for the decision maker. In this regard, the most closely related paper is Gratton, Holden, and Kolotilin (2018). Over a finite time horizon, a perfectly informed agent receives a signal process at some random time, and then chooses a time after getting this information to start disclosing this process to the decision maker. Their model highlights a "credibility vs. scrutiny" tradeoff, where the agent signals that his type is good by starting disclosure early, but is exposed to greater scrutiny. The continuation game in my model, although studies the agent's decision to stop disclosure not start, features a similar tradeoff. There are a few key differences between their model and mine. I consider a model where the agent is uninformed of the state and has the same information as the decision maker at the beginning of the game, but gradually learns about the state over time. More importantly, the agent controls not only when to start disclosure, but also when to stop. I focus on understanding the interactions between these two timing decisions, and the fact that the agent's (private) information evolves over time is an important determinant of this interaction.

On a more technical level, my model results in a dynamic signaling game with changing types. I consider exponential learning with conclusive news (as studied in Keller, Rady, and Cripps, 2005), which keeps the complications from changing types at bay. Thomas (2019) studies an experimentation problem with reputation concerns where the effect of changing type is more prominent. In addition, I adopt an equilibrium refinement that is in the same spirit as the divinity refinement. Halac and Kremer (2020) also adopts this refinement and applies it to an infinite horizon continuous-time game.

Structure of the paper

I introduce the model in Section 2. In Section 3, I characterize a class of equilibria of the game, and discuss the equilibrium properties and dynamics. Next, in Section 4, I study the duration of the disclosure window, and show that delays in the start of disclosure leads to longer disclosure windows. Lastly, in Section 5, I present a benchmark model where the agent knows the state from the start. I then study an extension that characterizes the decision maker's optimal duration of a disclosure window, and show that the decision maker's incentive to keep the disclosure window open contrasts the agent's in the main model. All proofs are relegated to the Appendix.

2 Model

Time is continuous and the horizon is infinite. There are two players, an agent (he) and a decision maker (she). There is an unknown state of the world $\theta \in \{0, 1\}$, where $\theta = 1$ indicates the state is good and $\theta = 0$ indicates the state is bad. The players share a common prior that $\theta = 1$, denoted by $\mu \in (0, 1)$.

Information

Over time, the agent receives private signals that have Poisson arrival rate λ^θ, where $\lambda^1 = 0$ and $\lambda^0 = \lambda > 0$. In other words, signals are conclusive that $\theta = 0$. This means the agent's posterior belief that $\theta = 1$ increases in the absence of signals. The agent can receive multiple signals over time, but because signals are conclusive, the agent's belief that $\theta = 1$ does not change upon receiving additional signals.

Agent's (private) belief/types. Throughout the game, the agent updates his belief about the state through the realization of the signal process. In particular, let t_s denote the arrival time of the first signal. Define the agent's posterior belief that $\theta = 1$ conditional on no signal arriving by t as $\rho(t) := \Pr(\theta = 1|t_s > t)$. By Bayes' rule,

$$\rho(t) = \frac{\mu}{\mu + e^{-\lambda t}(1-\mu)}. \tag{1}$$

This belief $\rho(t)$ is strictly increasing in t. If a signal arrives at t_s, the agent's belief drops down to 0 for all $t \geq t_s$. The agent's belief remains his private information—even though the evolution of the signal process will be publicly observable once the agent starts disclosing, the signal arrivals (or lack thereof) prior to the start of disclosure remains private.

At any t, the agent either has observed a signal and has belief 0 or he has not observed a signal and has belief $\rho(t)$. I say that the agent is *informed* at t if he has observed a signal at or before t, or *uninformed* at t if he has not.

Actions and payoffs

Over time, the agent privately receives an opportunity to start disclosing that has Poisson arrival rate $\alpha > 0$. This process is independent of the state and the signal process. The agent can start disclosing at any time after the arrival of the opportunity.[4] Suppose the opportunity arrives at time t_o. If the agent starts disclosing at time $t_{\text{start}} \geq t_o$, he commits to disclosing all signal arrivals (or lack thereof) for $t > t_{\text{start}}$. The agent cannot disclose the realization of the signal process for $t \leq t_{\text{start}}$. The decision maker observes t_{start}, the time at which the agent starts disclosing, but not the time at which the opportunity arrives, nor can the agent disclose this information.

Once disclosure starts at t_{start}, the agent chooses a time $t_{\text{stop}} \geq t_{\text{start}}$ to stop disclosing. Also, starting at time t_{start}, an exogenous termination arrives with Poisson arrival rate $\beta > 0$. This process is independent of the state and the signal process. As soon as a termination occurs, the game ends. Suppose a termination occurs at time t_{term}. The decision maker observes the time at which disclosure ends, $\min\{t_{\text{stop}}, t_{\text{term}}\}$, but does not observe whether the end of disclosure is the agent's choice or exogenous.

The decision maker takes an action $a \in \mathbb{R}$ at and only at the time disclosure ends. The decision maker's realized payoff is $1 - (a-\theta)^2$ and the agent's realized payoff is $\kappa + a$, where $\kappa > 0$ is constant.[5] Both players collect payoffs at the time disclosure

[4] I assume there is only one such opportunity. Multiple opportunity arrivals do not matter as long as the agent can start disclosing when the first one arrives.

[5] The constant term $\kappa > 0$ in the agent's payoff indicates delaying is costly: if there exists a time $\bar{t}$ such that the decision maker takes a fixed action for any $t \geq \bar{t}$, the agent strictly prefers stopping disclosure at the earliest time $\bar{t}$. The decision maker's payoff has an analogous constant.

stops, and discount future payoffs at a common discount rate $r > 0$.

Discussion of Assumptions. The two Poisson arrival processes, opportunity to start disclosing and exogenous termination, capture the fact that the timing of the disclosure window is sometimes restricted by exogenous factors unrelated to the underlying nature of the state. Take the drug approval process for example. The drug company needs to get the relevant paperwork ready, sort out administrative issues, and get financing in order before they can file for clinical trials. Similarly for the end of disclosure, clinical trials might be terminated unexpectedly because of fundings falling through, or a change in the company's leadership where the new board has no interest in developing the drug and wants to wrap up testings as soon as possible.[6]

From a modeling perspective, these two processes provide the noise that prevents the decision maker from getting all of the agent's information upon an observable action, and thus prevents unraveling.[7] The decision maker observes the times at which disclosure starts and stops, but not the underlying reasons. To be specific, if disclosure has not started, the decision maker does not know whether it is because the agent chooses not to start, or because the agent does not have an opportunity to do so. Similarly, if disclosure stops, the decision maker does not know whether it is because the agent chooses to stop, or because disclosure was exogenously terminated.

Strategies and solution concept

The strategy of the agent describes when to start and when to stop disclosing. As is standard, strategies can be described in terms of probability distribution functions. I focus on strategies that satisfy a Markov restriction, and introduce the formal definition in the contest of Markov strategies.

For a mixed starting strategy, let $\Phi(t|t_o, t_s)$ denote the probability that the agent starts disclosing before or at time t given that the opportunity arrives at $t_o < t$ and the first signal arrives at $t_s < t$. Let $\Phi(t|t_o, \varnothing)$ denote the probability that the agent starts disclosure by time t given that the opportunity arrives at $t_o < t$ and no signal has arrived by time t. That is, for each t_o and each t_s, $\Phi(t|t_o, t_s)$ measures, for each t,

This constant is strategically irrelevant and is normalized to 1.

[6] See Sica, 2002 or Williams, Tse, DiPiazza, and Zarin, 2015.

[7] Similar assumptions can be found in, for example, Ekmekci, Gorno, Maestri, Sun, and Wei (2022). They study a game between a principal and an agent, and assume the principal gets stochastic opportunities to terminate her relationship with the agent.

the probability to start disclosing by time t; for each t_o, $\Phi(t|t_o, \varnothing)$ measures, for each t, the probability to start disclosing by time t. Note that defining mixed strategy in this way imposes an implicit Markov restriction that specifies that the stopping decision depends only on the time of the first signal arrival, even if the agent may have observed more than one. I impose an additional Markov restriction that specifies

$$\Phi(t|t_o, t_s') = \Phi(t|t_o, t_s'') \text{ for all } t_s', t_s'' < t_o.$$

In words, if the agent is informed by the time he gets the opportunity to disclose, his starting decision does not depend on the time at which he becomes informed.

I impose a similar set of Markov restrictions on the stopping strategy. At the time disclosure starts, denoted by t_{start}, recall that the agent's private history contains the signal arrivals prior to the start of disclosure, and the time of his opportunity arrival. I impose a Markov restriction that specifies that, instead of the entire private history, the agent's stopping strategy only depends on whether he is informed or uninformed prior to the start of disclosure. Because the game ends when an exogenous termination occurs, a stopping strategy only needs to specify the agent's actions in the absence of exogenous termination.

I reason in terms of the waiting time since t_{start}. That is, if the agent stops at $w \geq 0$, it translates to stopping at the calendar time $t_{\text{start}} + w$. Within $[0, w]$, signal arrivals are public. I impose a Markov restriction that specifies that the agent's stopping strategy depends only on the arrival time of the first public signal. Formally, let $w_s \geq 0$ denote arrival of time of the first public signal. Let $G_I(w|t_{\text{start}}, w_s)$ denote the probability that an informed agent stops disclosure by time w, given disclosure started at t_{start} and the first public signal arrives at $w_s < w$. Let $G_I(w|t_{\text{start}}, \varnothing)$ denote the probability that an informed agent stops disclosing by time w, given disclosure started at t_{start} and no public signal has arrived by w. The uninformed agent's probabilities $G_U(w|t_{\text{start}}, w_s)$ and $G_U(w|t_{\text{start}}, \varnothing)$ are defined in the same way. I impose an additional Markov restriction that specifies

$$G_I(w|t_{\text{start}}, \hat{w}) = G_U(w|t_{\text{start}}, \hat{w}).$$

In words, if a public signal arrives at $\hat{w}$, the informed agent and the uninformed agent adopt the same stopping strategies at and after $\hat{w}$.[8]

[8]In equilibrium, both types of agent stop disclosing immediately as soon as a public signal arrives.

The decision maker takes an action when and only when disclosure ends. Denote this time by t_{end}. Note that disclosure ends at the minimum of the time the agent stops and the time when termination occurs. Denote the set of public history at time t_{end} by $\mathcal{H}^{\text{pub}}_{t_{\text{end}}}$ and an element of it $h^{\text{pub}}_{t_{\text{end}}}$. Then $h^{\text{pub}}_{t_{\text{end}}}$ consists of the time at which disclosure starts t_{start}, the time at which disclosure stops t_{end}, and the evolution of the signal process in $[t_{\text{start}}, t_{\text{end}}]$. The decision maker's strategy maps a public history to a real number.

I focus on the set of perfect Bayesian equilibria in which the agent adopts Markov strategies defined above (referred to as equilibrium hereinafter).

To rule out equilibria that arise due to unreasonable off-path beliefs, I use an equilibrium refinement that is in the spirit of the divinity refinement introduced by Banks and Sobel (1987).[9] I discuss how divinity plays a role in more details when I characterize the equilibrium.

3 Equilibrium Analysis

In this section, I characterize a class of equilibria of this game. I begin by characterizing the decision maker's equilibrium strategy, and devote the rest of the section to characterizing the agent's.

3.1 Decision Maker's Equilibrium Strategy

The decision maker takes an action at and only at the time disclosure ends, t_{end}.[10] As mentioned, a public history at t_{end}, denoted by $h^{\text{pub}}_{t_{\text{end}}}$, contains the time at which disclosure starts, the time at which disclosure ends, and signal arrivals (or lack thereof) in between. Given any public history $h^{\text{pub}}_{t_{\text{end}}}$, the decision maker's maximization problem is

$$\max_{a\in\mathbb{R}} \mathbb{E}_\sigma \left[e^{-rt_{\text{end}}} \left(\kappa - (a-\theta)^2 \right) | h^{\text{pub}}_{t_{\text{end}}} \right],$$

where the expectation is taken over the agent's strategies, denoted by σ. This implies that in equilibrium, the decision maker's action is equal to her expectation of the

[9]With a slight abuse of terminology, I refer to this refinement as divinity. The divinity refinement introduced by Banks and Sobel (1987) cannot be applied directly to a continuous time infinite horizon setting.

[10]In particular, the decision maker does not/cannot take an action if disclosure never ends.

state given the public history. Because the state is either 0 or 1, this expectation is equal to her posterior belief that $\theta = 1$. The following lemma summarizes this result.

Lemma 1. *In any equilibrium, at the time disclosure ends t_{end}, given any public history $h^{pub}_{t_{end}}$, the decision maker's action is*

$$a = \mathbb{E}_\sigma[\theta | h^{pub}_{t_{end}}] = \Pr(\theta = 1 | h^{pub}_{t_{end}}).$$

3.2 Agent's Equilibrium Strategy

The rest of Section 3 is dedicated to characterizing the agent's equilibrium strategies. The game has two stages: the initial *starting game* where the agent chooses a time to start disclosing, and the continuation *stopping game* where disclosure has started and the agent chooses a time to stop disclosing. Both games are of some substance. To keep the analysis in perspective, I first provide a characterization of the equilibrium and discuss the intuition in Section 3.2. I then analyze the equilibrium in detail and discuss some of the interesting dynamics and properties in Section 3.3 and Section 3.4.

Equilibrium characterization

Theorem 1 below characterizes a class of equilibria of this game. This class of equilibria features the two types of agent following the same starting strategy and different stopping strategies. Specifically, at the beginning of the game, conditional on having the opportunity, both types of agent delay starting until some time $\tau^* \geq 0$.[11] At and after τ^*, both types of agent start disclosure as soon as an opportunity arrives. Once disclosure starts, the uninformed agent waits for a certain amount of time w^* (where w^* depends on the time disclosure starts) to stop disclosure, while the informed agent randomizes over waiting times in $[0, w^*]$.

With a slight abuse of notation, given that disclosure starts at t_{start}, let $G_I(w)$ and $G_U(w)$ denote respectively the informed and uninformed agent's probability of stopping by w given no signal in $[0, w]$ (and no exogenous termination by w). That is, $G_I(w) = G_I(w|t_{\text{start}}, w_s > w)$ and $G_U(w) = G_U(w|t_{\text{start}}, w_s > w)$. The equilibrium can be stated as follows.

[11]Depending on the parameters, τ^* can be 0 which means the agent does not delay.

Theorem 1.A. *The following strategy is an equilibrium of the starting game. There exists $\tau^* \geq 0$ such that neither type of agent starts disclosing for $t < \tau^*$, and both types of agent start disclosing as soon as an opportunity arrives for $t \geq \tau^*$. That is, for all t_s,*

$$\Phi(t|t_o, t_s) = \begin{cases} 0 & t < \tau^* \\ 1 & t \geq \max\{\tau^*, t_o\} \end{cases}.$$

The decision maker's belief that $\theta = 1$ if disclosure starts is 0 for $t < \tau^*$ (starting disclosure is off-path), and is μ for $t \geq \tau^*$ (starting disclosure is on-path).

Theorem 1.B. *Suppose disclosure starts at $t \geq \tau^*$. There exists a unique divine equilibrium which takes the following form: there exists a waiting time $w^* \geq 0$ (where w^* depends on t) such that*

(i) if a signal arrives at $w \in [0, w^*]$, the agent stops disclosing immediately at w;

(ii) if no signal arrives in $[0, w^*]$, the uninformed agent stops disclosing at w^* with probability 1:

$$G_U(w) = \begin{cases} 0 & w \in [0, w^*) \\ 1 & w \geq w^* \end{cases}. \tag{2}$$

The informed agent randomizes over waiting times in $[0, w^*]$. His stopping probability $G_I(w)$ is the unique solution to the following boundary value problem:[12] *for all $w \in [0, w^*]$,*

$$G''_I(w) = \mathscr{G}(G_I(w), G'_I(w), w)$$

[12]Given that disclosure starts at t, denote the uninformed agent's belief that $\theta = 1$ at t is $\rho(t)$, the decision maker's belief is μ, $\mathscr{G}$ is given by

$$\begin{aligned} G''_I(w) = &\mathscr{G}(G_I(w), G'_I(w), w) \\ := &\beta G'_I(w) - \frac{\beta\rho(t)}{\rho(t) - \mu}\left(r(1-\mu) + e^{\lambda w}\mu\left(r\kappa\left(\frac{\mu + e^{-\lambda w}(1-\mu)}{\mu}\right)^2 + r + \lambda\right)\right) \\ &+ r\left(1 + 2\kappa\left(\frac{\mu + e^{-\lambda w}(1-\mu)}{\mu}\right)\right)(\beta G_I(w) - G'_I(w)) \\ &- \frac{r\kappa e^{-\lambda w}(\rho(t) - \mu)}{\beta\rho(t)\mu}(\beta G_I(w) - G'_I(w))^2. \end{aligned}$$

with boundary conditions

$$G_I(0) = 0, G_I(w^*) = 1, \textit{ and } G_I'(w^*) = 0.$$

Theorem 1.A characterizes the agent's strategies in the starting game. The agent either delays starting disclosure ($\tau^* > 0$), or immediately starts disclosing ($\tau^* = 0$) as soon as an opportunity arrives. The parametric restriction that defines these two cases comes down to how discounting compares to the arrival rate of signals. Loosely speaking, because the decision maker only takes the action when disclosure ends, delaying the start of disclosure is costly due to discounting. The benefit, on the other hand, is the agent's ability to learn about the state in private: if a (conclusive bad) signal arrives, the agent learns that the state is bad without having to share this information with the decision maker. Delaying the start of disclosure is beneficial if discounting is small comparing to the arrival rate of signals.

Theorem 1.B characterizes the agent's strategies in the stopping game. While disclosure is open, the longer the decision maker has not seen a signal, the more optimistic she becomes. However, the risk of a signal arriving also increases. This tradeoff leads to delayed stopping by the more optimistic (uninformed) agent.

The equilibrium dynamics are best illustrated in terms of belief evolution the decision maker. Figure 1 plots an example of the decision maker's (on-path) belief conditional on no signal arrivals in an equilibrium with initial delay till time $\tau^* > 0$.

Because signals are conclusive that the state is 0, the uninformed agent's posterior belief that state is 1 increases over time in the absence of signals. The decision maker's belief that $\theta = 1$ evolves according to the arrowed path. In this equilibrium, the two types of agent adopt the same starting strategy. The observation that disclosure does not start before τ^* or starts after τ^* is uninformative about the agent's type and therefore uninformative about the state. So the decision maker's belief that $\theta = 1$ is equal to her prior.

Suppose disclosure starts at time t_{start}. Along the history in which a signal does not arrive, disclosure lasts for a (maximum) duration of w^*.[13] The shaded region from time t_{start} to $t_{\text{start}} + w^*$ is the *disclosure window* and w^* is the *length*, or the *duration*, of this disclosure window. During the disclosure window, not only is the evolution of

[13]More precisely, the agent can choose to stop disclosing prematurely before w^*, or disclosure can terminate exogenously before w^*.

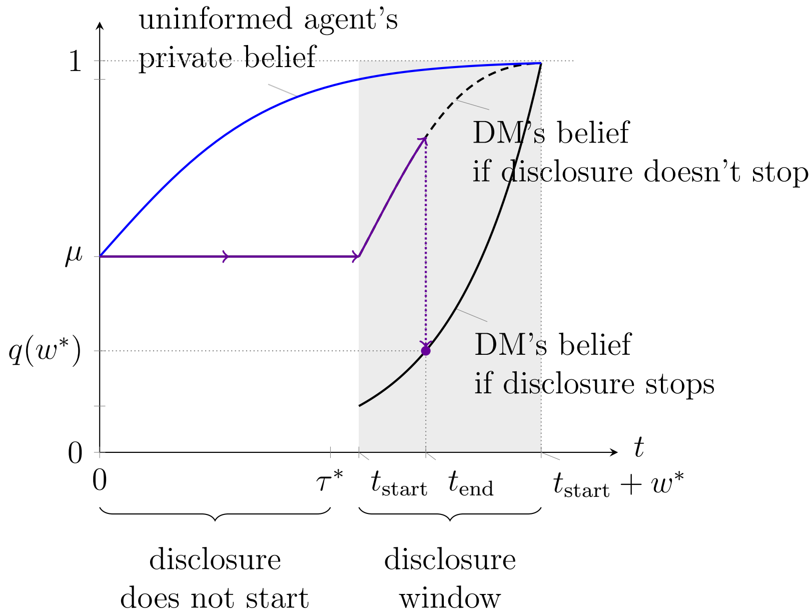

Figure 1: Agent's and decision maker's belief evolutions in an equilibrium with initial delay for $\mu = 0.5, r = 0.01, \lambda = 5, \alpha = 1, \beta = 0.5$.

the signal process informative of the state, but also the stopping of disclosure is bad news about the agent's private information prior to disclosure. In particular, only the informed agent intentionally stops within this window. Thus, "not stopping" is good news: the decision maker's belief if disclosure does not stop increases over time, and drops down if disclosure stops. The decision maker's posterior belief given the entire public history at the end of disclosure t_{end} is $q(w^*)$, as illustrated in the figure.[14]

I now analyze the equilibrium in the two stage games respectively. To better understand the dynamics of this game, it is more intuitive to start with the continuation stopping game given a starting time, and then work backwards to solve the initial starting game.

3.3 Stopping Game

The stopping game is a game of incomplete information in which at the beginning of the stopping game, the agent is either informed or uninformed. Conceptually, the stopping game can be parameterized by the uninformed agent's (private) belief that $\theta = 1$, denoted by ρ, and the decision maker's belief that $\theta = 1$ conditional on

[14]Recall that disclosure ends either when the agent stops or when exogenous termination occurs.

disclosure starting, denoted by η.[15] In this game, the decision maker cannot be more optimistic about the state than the uninformed agent. That is, $\rho \geq \eta$. I characterize the equilibrium conditional on exogenous termination not occurring.[16]

In the stopping game, there are two types of (public) histories: a history along which there is a (conclusive bad) signal arrival and one where there is not. When a signal arrives at some time, it overrides information transmitted through all other channels. By Lemma 1, the decision maker's action is 0 at any stopping time after a signal. Because of discounting, the agent strictly prefers stopping immediately at the time of the signal arrival. This is formalized in part (i) of Theorem 1.B.

Along the history with no signal arrivals, the decision maker has no way of learning the state with certainty and the agent signals his type through the stopping time of disclosure. Because of the possibility of exogenous termination, the informed agent can disguise himself as an uninformed agent whose disclosure process has been exogenously terminated. This gives the uninformed agent incentives to delay stopping in an attempt to differentiate himself from the informed agent.

To formalize this result, I first derive the decision maker's belief when disclosure stops. Because this is also the action the decision maker takes, it is a key element in determining the agent's equilibrium stopping strategies.

Decision maker's (public) belief. While the disclosure window is open, the decision maker updates her belief that $\theta = 1$ through the evolution of the signal process and the observation of disclosure stopping. Let $q(w)$ denote the decision maker's posterior belief that $\theta = 1$ if disclosure stops at waiting time w. Given disclosure has started, the public history at w consists of the event that no signal arrives in $[0, w]$,[17] and that disclosure stops at w. Let w_s denote the arrival time of the first public signal. Then

$$q(w) := \Pr(\theta = 1 | w_s > w, w_{\text{stop}} = w).$$

To derive this belief, let $F^\theta(w)$ denote the probability that disclosure stops by w conditional on state θ, and $f^\theta(w)$ its density whenever differentiable. Conditional on

[15]I characterize the equilibrium for any $\rho \geq \eta$. Both ρ and η are determined by the equilibrium starting time t_{start} in the initial starting game. Specifically, $\rho = \rho(t_{\text{start}})$ where $\rho(\cdot)$ is defined in (1). The decision maker's belief is $\eta = \eta(t_{\text{start}})$. (Given the equilibrium starting strategy characterized in Theorem 1.A, $\eta = \eta(t_{\text{start}}) = \mu$ for all t_{start}.)

[16]Recall that the game ends when exogenous termination occurs.

[17]Recall that $w \geq 0$ is the waiting time after disclosure starts, not the calendar time.

$\theta = 1$, the agent can only be uninformed. Disclosure stops when either the agent chooses to stop or termination occurs. That is,

$$f^1(w) = G_U'(w)e^{-\beta w} + \beta e^{-\beta w}(1 - G_U(w)). \tag{3}$$

To derive the rate at which disclosure stops at w conditional on $\theta = 0$, it is useful to first define the probability that the agent is informed at the beginning of the continuation game conditional on $\theta = 0$. Denote this probability by γ. Given ρ and η, γ is given by $\eta = \rho(1 - \gamma(1 - \eta))$. By the same logic as the $\theta = 1$ case, conditional on $\theta = 0$,

$$f^0(w) = (1 - \gamma)f^1(w) + \gamma \left[G_I'(w)e^{-\beta w} + \beta e^{-\beta w}(1 - G_I(w))\right]. \tag{4}$$

By Bayes' rule,

$$q(w) = \frac{f^1(w)\eta}{f^1(w)\eta + e^{-\lambda w} f^0(w)(1 - \eta)}. \tag{5}$$

Agent's expected payoff. Given $q(w)$, the informed agent's expected payoff from waiting w to stop disclosing consists of three cases. The first case is when a signal arrives before a termination occurs during the waiting time. The agent stops immediately and the decision maker takes action 0. The second case is when a termination occurs at some time s before a signal arrives. The decision maker takes action $q(s)$. The third case is when no signal arrives and no termination occurs. The decision maker takes action $q(w)$ at w.[18] Denote by $V(w)$ the informed agent's expected payoff from waiting w, and $U(w)$ the uninformed agent's.

$$V(w) = \int_0^w e^{-rs}\lambda e^{-\lambda s}e^{-\beta s}\kappa \mathrm{d}s + \int_0^w e^{-rs}e^{-\lambda s}\beta e^{-\beta s}(\kappa + q(s))\mathrm{d}s + e^{-rw}e^{-\lambda w}e^{-\beta w}(\kappa + q(w)).$$

Conditional on $\theta = 0$, the uninformed agent's expected payoff is the same as the informed. The uninformed agent's expected payoff from waiting w is

$$U(w) = (1 - \rho)V(w) + \rho \left(\int_0^w e^{-rs}\beta e^{-\beta s}(\kappa + q(s))\mathrm{d}s + e^{-rw}e^{-\beta w}(\kappa + q(w)) \right).$$

[18]The event that both a signal and termination arrive in a small interval $[w, w + \mathrm{d}w)$ is of order $(\mathrm{d}w)^2$ and can be neglected.

It can be verified that if $V(w)$ is (weakly) increasing in w, then $U(w)$ is strictly increasing in w. Intuitively, conditional on $\theta = 0$, the longer disclosure stays open, the more likely it is for a signal to arrive. The uninformed agent is more optimistic, so if the informed agent is (weakly) willing to wait, the uninformed agent must strictly prefer doing so.

Exploiting this property, I show existence of an equilibrium of the following form. Over an interval of waiting time $[0, w^*]$, the informed agent randomizes over waiting times in $[0, w^*]$, and the uninformed agent strictly prefers waiting till w^*. Both types stop with probability 1 by w^*. I then show that this is the unique equilibrium of the continuation game that survives the divinity refinement.

3.3.1 Equilibrium Dynamics

The most intuitive way to understand the equilibrium dynamics is through the decision maker's belief evolution. In equilibrium, the informed agent's randomization over waiting times feeds into the decision maker's belief evolution which in turn keeps the informed agent's expected payoff from stopping constant over time. Specifically, $V'(w) = 0$ implies

$$q'(w) = r\kappa + (r + \lambda)q(w). \tag{6}$$

While the disclosure window is open, the informed agent discounts at an exponential rate, and the signal arrival times are exponentially distributed. The informed agent's indifference condition (6) captures that the decision maker's belief when disclosure stops needs to counteract these two forces and increase exponentially.[19]

In this game, the decision maker's belief that $\theta = 1$ is bounded above by the uninformed agent's at any point in time. This suggests $q(w)$ cannot exponentially increase indefinitely and disclosure must stop before this belief exceeds the uninformed agent's. The following lemma takes this observation one step further and shows that if disclosure stops at the end of the disclosure window, the decision maker infers that the agent must be uninformed, and therefore, her posterior belief that $\theta = 1$ must equal the uninformed agent's.

[19]Specifically, the solution to the first-order differential equation (6) is $q(w) = me^{(r+\lambda)w} - \kappa r/(r+\lambda)$ with constant $m > 0$ to be determined in equilibrium.

Lemma 2. *If the informed agent's strategy is atomless with support* $[0, w^*]$*, then*

$$q(w^*) = \frac{\rho}{\rho + e^{-\lambda w^*}(1-\rho)}. \tag{7}$$

Moreover, $w^* < \infty$*.*

Intuitively, at the end of the disclosure window, both types of agent stop disclosing with probability 1. Moreover, the informed agent stops continuously throughout. Therefore, the mass of informed agent who has not stopped at this point is zero, and stopping can only come from the uninformed agent. As will be discussed later, this structure where the informed agent stops continuously throughout the entire duration of the disclosure window is the unique equilibrium that survives the divinity refinement.

To further understand the equilibrium dynamics, consider the decision maker's belief before disclosure stops. Let $q^C(w)$ denote the decision maker's belief given this history. That is,

$$q^C(w) := \Pr(\theta = 1 | w_s > w, w_{\text{stop}} > w).$$

Because only the informed agent voluntarily stops during the disclosure window, $q^C(w)$ is increasing in w and $q^C(w) > q(w)$ for all $w < w^*$.

In summary, from the decision maker's perspective, stopping prematurely before the disclosure window duration w^* is bad news—whenever disclosure stops, the decision maker's belief drops from $q^C(w)$ to $q(w)$. However, later premature stopping is good news—a longer duration without a signal arrival induces a higher belief that the state is good, and later stopping implies there's a higher chance that the agent is uninformed to begin with. Figure 2 illustrates the dynamics by plotting the decision maker's beliefs as functions of waiting time w.

Boundary value problem. While the evolution of the decision maker's belief $q(w)$ illustrates the equilibrium dynamics, it does not pin down the equilibrium strategies $G_I(w)$—some properties of the agent's strategies get lost during their translation to beliefs. In the strategy space, $G_I(w)$ is continuous at any $w \in [0, w^*]$, in particular, at $w = 0$, so $G_I(0) = 0$. The informed agent stops with probability 1 by w^* means $G_I(w^*) = 1$. Together with Lemma 2, they imply $G_I'(w^*) = 0$. Moreover, equation (6) defines a second-order differential equation in G_I. All together, they form the boundary value problem stated in part (ii) of Theorem 1.B. By a shooting argument,

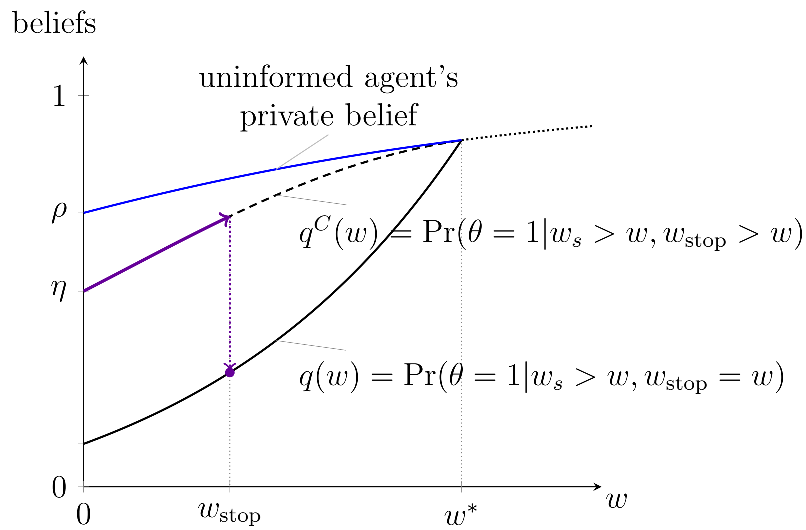

Figure 2: The decision maker's beliefs for $\rho = 0.7$, $\eta = \mu = 0.5, \lambda = 3, r = 0.5, \alpha = 1, \beta = 0.5$, and $\kappa = 1$.

this boundary value problem has a unique solution. Figure 3 plots this solution, $G_I(w)$, as well as the uninformed stopping strategy, $G_U(w)$.

3.3.2 Divinity and Equilibrium Uniqueness

Given a fixed equilibrium, divinity specifies that the decision maker's off-path belief should assign zero weight to the agent being the type that has less incentive to deviate. In this game, continuing disclosure after w^* is off the equilibrium path. Intuitively, for any given belief of the decision maker, the uninformed agent always has a stronger incentive to keep disclosure open. Therefore, the decision maker's off-path belief should assign zero weight to the agent being informed.

This means that the uninformed agent can always "prove" that he is uninformed by deviating to this off-path play. If the uninformed agent were ever to stop on the equilibrium path, it must be that whenever the uninformed agent chooses to stop, the decision maker's belief upon stopping is that the agent is uninformed. In turn, because posterior belief is a martingale, stopping at any $w \geq w^*$ induces the same expected posterior belief. Because of discounting, the uninformed agent stops at w^*.

The above argument shows that in a divine equilibrium, the decision maker's belief that $\theta = 1$ at w^* must be equal to the uninformed agent's. In addition, the decision maker's belief that $\theta = 1$ must also be continuous at w^*: any jump in belief at w^*

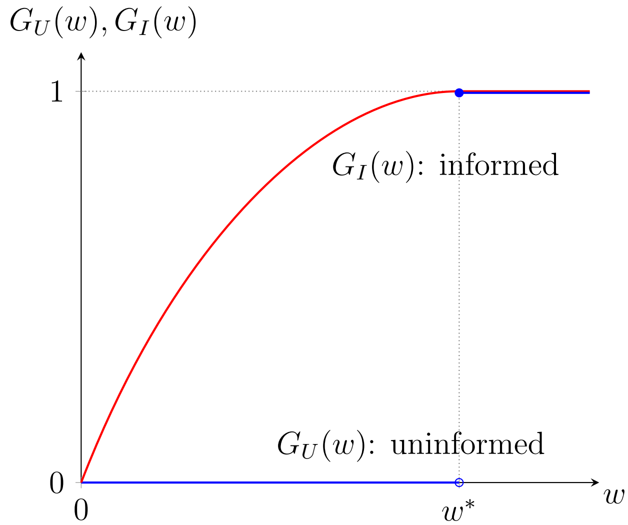

Figure 3: Agent's equilibrium strategies $G_U(w)$ and $G_I(w)$ for $\rho = 0.7, \eta = 0.5, \lambda = 3, r = 0.5, \alpha = 1,$ and $\beta = 0.5$.

will result in a profitable deviation to stopping at w^* for the informed agent. These conditions pin down the boundary conditions in the boundary value problem stated in Theorem 1.B (ii), giving rise to equilibrium uniqueness. Without divinity, one can construct equilibria where disclosure stops after any amount of waiting time by imposing a punishing belief that deters the agent from keeping the disclosure window open after said waiting time.[20]

3.3.3 Role of Information Asymmetry

Because the agent might be informed prior to starting disclosure, at the time disclosure starts, the decision maker's belief that $\theta = 1$ is lower than the uninformed agent's. As it turns out, it is precisely this information asymmetry that enables information transmission in the stopping game.

To see this, suppose the agent is uninformed and the decision maker knows that. Because the posterior belief is a martingale, stopping at any time would induce the same expected action. Because of discounting, the agent will stop disclosing immediately and the decision maker takes an action that is equal to the common belief. Hence, the possibility that the agent is informed forces the uninformed agent to keep

[20] More precisely, with a punishing belief, one can construct equilibria that "ends early" by having a mass of informed agent stopping at the end of the disclosure window.

the disclosure window open in the hope that the lack of signals corroborates that he is uninformed, enabling information transmission. The following lemma formalizes this result.

Proposition 1. *$w^* > 0$ if and only if $\rho > \eta$.*

In some sense, the agent does not care about what he knows but what the decision maker knows. The uninformed agent is more optimistic than the decision maker, and his incentive to delay stopping and let information flow is not due to learning for himself. Instead, he delays so that the decision maker can learn and become as optimistic as he is.

Comparative statics. As analyzed above, the duration of the disclosure window is determined by how long it takes to eliminate the information asymmetry between the agent and the decision maker at the start of disclosure. It is intuitive that the larger this information asymmetry is, the longer it takes to weed it out. This suggests that the duration of the disclosure window should increase in the magnitude of information asymmetry. The following result formalizes this intuition.

Lemma 3. *The equilibrium waiting time w^* is increasing in ρ and decreasing in η.*

Although intuitive, technically w^* is part of the solution to a boundary value problem and is pinned down in equilibrium through the informed agent's randomization, so proving Lemma 3 is not straightforward. Nevertheless, this result is crucial in analyzing the starting game in Section 3.4, and deriving the main insight of the paper in Section 4. The analysis so far has demonstrated that the degree of information asymmetry at the beginning of the disclosure window impacts the duration of the disclosure window. Section 4 concerns how this degree of information asymmetry is affected by the time disclosure starts. On that note, I now turn to analyzing the agent's optimal time to start disclosure.

3.4 Starting Game

In the initial starting game, the agent chooses a time to start disclosure, knowing that this starting time induces a duration for which he will keep the disclosure window open, given by the (unique) equilibrium in the continuation stopping game.

Unlike the stopping game where any signal arrival is public information, the agent's learning is private prior to the start of disclosure. On the one hand, if the agent delays the start of disclosure and a signal arrives during this time, the agent is able to conceal this information from the decision maker, and behave optimally as an informed agent onward. Had the agent not delayed, the decision maker would have seen this signal and taken action 0. On the other hand, delay in the start of disclosure translates to a delay in the expected time till payoff realizes, which incurs a cost due to discounting.[21]

If the agent is impatient relative to the value of information, he starts disclosing as soon as he gets the opportunity. Otherwise, he delays to privately learn for a certain amount of time before starting. I provide sufficient conditions under which starting immediately is an equilibrium, and sufficient conditions under which delaying is an equilibrium whenever starting immediately is not.[22]

To simply the exposition, I impose the following assumption. This assumption plays no role in the stopping game; it simplifies the analysis in the starting game and delivers a clean intuition.

Assumption. *Assume the (common) prior that $\theta = 1$ is $\mu \geq 1/2$.*

3.4.1 Immediate Disclosure Equilibrium

The following proposition provides sufficient conditions under which it is an equilibrium for both types of the agent to start disclosing as soon as he gets an opportunity. I refer to this equilibrium as the *immediate disclosure equilibrium.*

Proposition 2. *If*

$$\frac{r}{\lambda} \geq \frac{(1-\mu)\mu}{\kappa+\mu}, \tag{8}$$

then immediate disclosure is an equilibrium.

[21] More precisely, suppose the decision maker's belief does not change no matter when disclosure starts. Prior to the start of disclosure, the (uninformed) agent becomes optimistic. Thus, the uninformed agent's belief that $\theta = 1$ and the decision maker's belief grows divergent. As implied by Lemma 3, the (uninformed) agent needs to keep the disclosure window open longer in the continuation game. Therefore, a delay in the start of disclosure results in an even longer disclosure window, and hence a longer expected time until payoff realizes.

[22] I focus on the incentive of an agent who has the opportunity to disclose at time 0. An agent who got the opportunity at later times has the same incentive. Details of this argument are in Proof of Theorem 1.A in the Appendix.

To interpret condition (8), consider a scenario in which whenever disclosure starts, it stops immediately and the decision maker believes that the agent is uninformed. This is the best case scenario for both types of agent—the decision maker takes the highest possible action and the agent does not need to engage in any waiting to induce it.[23]

Condition (8) can be rearranged as $\lambda(1-\mu)\mu \leq r(\kappa+\mu)$. The left-hand side, $\lambda(1-\mu)\mu$, is the marginal benefit from delaying in this best case scenario. At time 0, with an expected rate of $\lambda(1-\mu)$, a signal arrives. By delaying for an infinitesimal amount of time, the agent learns that the state is $\theta = 0$ but the decision maker does not. The agent then immediately stops and gets payoff $\kappa + \mu$. If he had not delayed and started disclosing right away, the decision maker would have seen the signal and taken action 0, the agent then gets payoff $\kappa + 0$. The gain is therefore μ. The right-hand-side, $r(\kappa + \mu)$, is the marginal cost from delaying. Because there is no waiting in the continuation game, the cost purely comes from discounting.

Therefore, condition (8) implies that, in the best case scenario where the agent does not need to wait in the continuation game, discounting (r) is so large relative to information (λ) that the agent does not want to delay starting at any point in time.[24] Thus, in the case where the agent does have to wait in the continuation game and have to wait even longer if he starts later, the agent would not want to delay.

3.4.2 Delayed Equilibrium

In a delayed equilibrium, neither type of agent starts disclosure until some $\tau^* > 0$. Then both start immediately at and after τ^*. That is, if an agent already has the opportunity before τ^*, they start disclosing immediately at τ^*, otherwise they start disclosing as soon as they receive the opportunity. The decision maker's belief that $\theta = 1$ is 0 if disclosure starts off-path before τ^*.

The key is to characterize τ^*. By the above argument, if immediate disclosure is not an equilibrium, then there must exist a point in time where the value of private learning is large relative to discounting. By assumption $\mu \geq 1/2$, the value of private

[23]The highest possible action is subject to the decision maker's optimality: the decision maker's action is equal to her posterior belief that $\theta = 1$. In this best case scenario, this belief is equal to the uninformed agent's belief which is the highest possible belief any player can have.

[24]More precisely, condition (8) implies that the agent does not want delay at $t = 0$. For $t > 0$, a similar condition can be obtained by replacing μ on the right-hand side with the uninformed agent's belief $\rho(t)$. Condition (8) implies the same inequality holds with $\rho(t)$ for all $t > 0$.

learning diminishes as time goes by, and will eventually be overwhelmed by discounting. This means that if an equilibrium prescribes immediate disclosure after some time τ^*, this τ^* needs to be large enough until the discounting effect dominates.

On the other hand, τ^* cannot be too large. Otherwise the (informed) agent would rather deviate to starting disclosure right away and getting action 0 (as the decision maker's off-path belief is 0), than waiting till τ^* for the chance of inducing a higher action. The following proposition provides sufficient conditions for the existence of a τ^* that strike a balance between these two forces.

Proposition 3. *Fix parameters such that immediate disclosure is not an equilibrium. There exists $0 < \overline{\kappa} \leq (\lambda/r)(1-\mu)\mu - \mu$ such that for all $\kappa \leq \overline{\kappa}$, there exists $\underline{\tau}^*(\kappa) > 0$ such that it is an equilibrium for neither type of agent to start disclosing for $t < \underline{\tau}^*(\kappa)$, and both types of agent to start disclosing as soon as an opportunity arrives for $t \geq \underline{\tau}^*(\kappa)$. The decision maker's belief that $\theta = 1$ if disclosure starts off-path at $t < \underline{\tau}^*(\kappa)$ is* 0.

In words, the key to sustain this delayed equilibrium is the additive constant κ in the agent's payoff being small. Intuitively, in the stopping equilibrium analyzed in Section 3.3, because of exogenous termination, it is always possible that stopping is due to an uninformed agent getting terminated. So in the absence of signals, the decision maker's belief when disclosure stops at any point in time is strictly positive. This means if the agent follows the equilibrium path and waits till τ^* to start disclosure, his on-path expected action is strictly positive. Thus, if κ is small, the agent (both types, the informed in particular) would prefer waiting to induce a strictly positive action, than getting the zero action right away.

3.4.3 Divinity and Equilibrium Multiplicity

The immediate disclosure equilibrium survives the divinity refinement trivially because there are no off-path actions. For the delayed equilibrium, starting disclosure before τ^* is off the equilibrium path and, as mentioned, the decision maker's off-path belief that $\theta = 1$ is zero. Is this punishing off-path belief consistent with the divinity refinement?

As discussed in the stopping game in Section 3.3, the divinity refinement specifies that the decision maker's off-path belief should assign zero weight to the type of agent that has less incentive to deviate. The zero off-path belief assigns zero weight to the uninformed agent: if disclosure starts before τ^*, the decision maker thinks that it is the informed agent who has deviated, and thus believes that $\theta = 0$. For this to be consistent with the divinity refinement, at each $t < \tau^*$, for any decision maker's belief $\eta \in [0, \rho(t)]$ such that the uninformed agent finds it (at least weakly) profitable to deviate to starting at t, the informed agent must find it strictly profitable to deviate. In other words, the set of decision maker's beliefs that induce a profitable deviation for the uninformed agent must be a proper subset of that for the informed agent.

Whether this condition is satisfied in equilibrium depends on the parameters. Figure 4 plots two numerical examples, both of which satisfy the sufficient condition in Proposition 3 and thus are equilibria. The picture on the left is an example where the equilibrium is divine, the one on the right is not. The red (solid) area plots the set of decision maker's belief η such that the informed agent finds it profitable to deviate, and the blue (patterned) area is where the uninformed agent finds profitable to deviate. To satisfy divinity, the blue set needs to be entirely contained in the red.

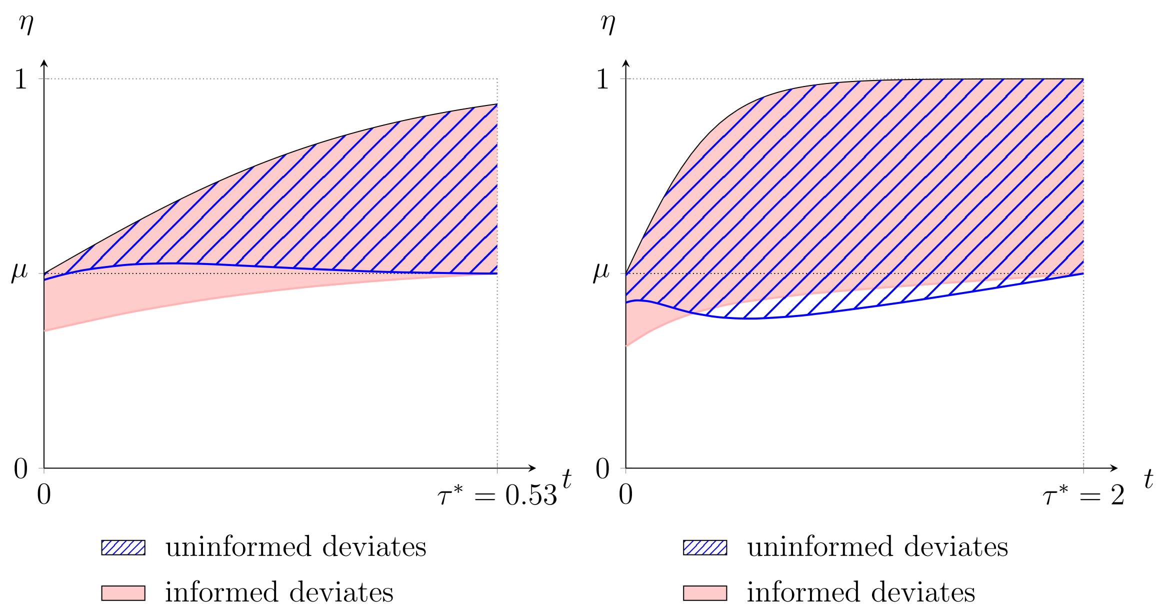

Figure 4: Numerical examples where a delayed equilibrium is divine (left panel, $\tau^* = 0.53$) and is not (right panel, $\tau^* = 2$). Parameters are $\mu = 0.5, r = 0.01, \lambda = 5, \alpha = 1, \beta = 0.5$, and $\kappa = 1$.

In general, the initial starting game will have multiple equilibria. Within the class of pooling equilibria characterized here, there typically exists a continuum of τ^*'s that can be sustained as an equilibrium and survives the divinity refinement. The numerical examples above show that the divinity refinement can indeed eliminate some equilibria, but in general, it cannot reduce the equilibrium set to a singleton.

4 Duration of Disclosure Windows

The equilibrium features a novel dynamic between the start disclosing time and the stop disclosing time: delay in the start of disclosure leads to a longer disclosure window. Proposition 4 formalizes this result and Figure 5 illustrates.

Proposition 4. *The agent's waiting time $w^*(t)$ is increasing in the disclosure starting time t. Moreover, there exists $\overline{w}^* < \infty$ such that $\lim_{t\to\infty} w^*(t) = \overline{w}^*$.*

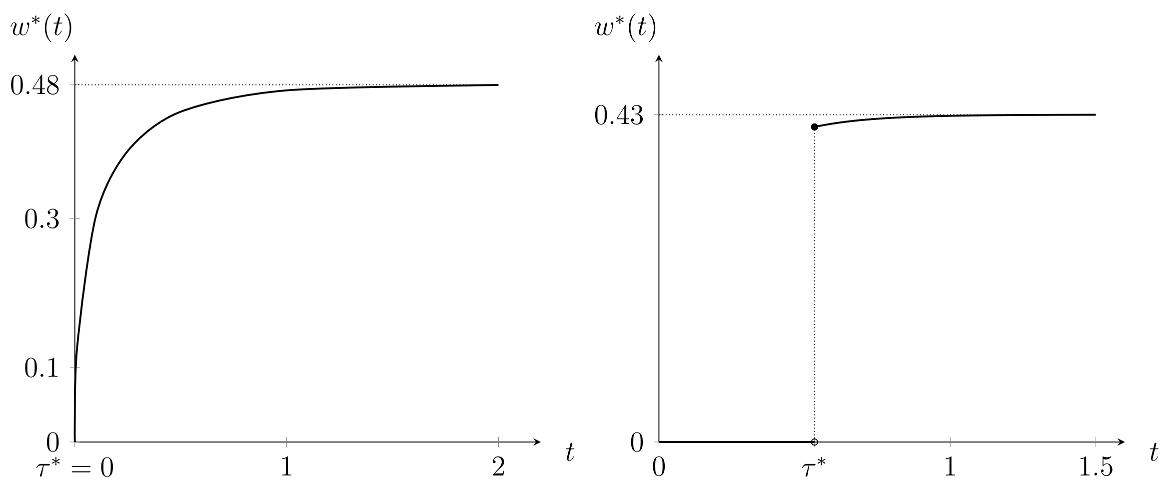

Figure 5: Agent's waiting time $w^*(t)$ as a function of the disclosure starting time t. Left panel: $\tau^* = 0$; $\mu = 0.5, r = 0.5, \lambda = 3, \alpha = 1, \beta = 0.5$, and $\kappa = 1$. Right panel: $\tau^* = 0.53$; $\mu = 0.5, r = 0.01, \lambda = 5, \alpha = 1, \beta = 0.5$, and $\kappa = 1$.

The agent and the decision maker start off with the same information about the state, and become increasingly asymmetrically informed while the start of disclosure is delayed. During this delay, the agent privately learns about the state through the information process. He either becomes more optimistic in the absence of signals or becomes informed whenever a signal arrives. The decision maker on the other hand, does not get any information about the state: she does not observe the information process, nor can she infer anything from the agent's behavior because both types of the agent adopt the same starting strategy. So on the equilibrium path, the decision maker's belief that $\theta = 1$ stays the same as her prior.[25] The longer the start of disclosure is delayed, the more divergent the agent's and the decision maker's beliefs become. The duration of the disclosure window is determined by the amount of time it takes to eliminate this information asymmetry through disclosure. The uninformed agent keeps the disclosure window open longer to reduce the now-larger information asymmetry between him and the decision maker.

With time, the agent becomes more and more certain of the state. If disclosure starts late in time, the continuation game approaches to a benchmark case in which the agent knows whether the state is 0 or 1. Let $\overline{w}^*$ denote the optimal waiting time in this benchmark case. The waiting time $w^*(t)$ then converges to $\overline{w}^*$ as t increases. In the next section, I present this benchmark model and characterize $\overline{w}^*$.

5 Benchmarks and Extensions

5.1 Perfectly Informed Agent

As a benchmark, consider the scenario in which the agent is informed of whether the state is $\theta = 0$ or $\theta = 1$ at time 0. As the start of disclosure increases, the game approximates this benchmark case where the agent is perfectly informed. As mentioned in Proposition 4, the waiting time in the continuation stopping game is bounded above by $\overline{w}^*$, which is the waiting time in this benchmark. Proposition 5 formalizes this argument (the characterization of $\overline{w}^*$ is related to the Appendix).

Proposition 5. *In the continuation game, there exists $0 < \overline{w}^* < \infty$ such that the informed agent randomizes over stopping times in $[0, \overline{w}^*]$ and uninformed agent stops with probability 1 at $\overline{w}^*$. (The equilibrium is similarly characterized as in Theorem 1.B).*

5.2 Decision-Maker-Optimal Disclosure Duration

The disclosure environment features the agent controlling both the start and the stop of the disclosure window. The resulting disclosure duration is what's optimal for the

[25] While the decision maker's belief about the state does not change, her belief about the agent's type does change. The probability that the agent is informed increases as time goes by, but if the agent remains uninformed, his belief also increases. The decision maker's belief that $\theta = 1$ takes expectations of these two beliefs and stays constant at the prior on average.

agent. What is the optimal disclosure duration for the decision maker?

In what follows, I study a model in which the decision maker chooses the stopping time of disclosure in addition to the action taken at the end of disclosure. This model highlights the fact that the (uninformed) agent's incentive to keep disclosure open differs from the decision maker's. Same as the main model, the decision maker can stop and therefore take an action if and only if disclosure has started at some time prior. The rest of the setup remains the same.[26]

In this model, the decision maker's problem is Markov in her belief. She either stops when she sees a signal and thus learns the state is 0, or she waits until her belief reaches a threshold $\overline{\eta}$ that is independent of the time at which disclosure started. As before, denote the decision maker's belief at the beginning of the continuation game by η. The following proposition formalizes this intuition.

Proposition 6. *There exists $\Delta > 0$ such that*
(i) if $r/\lambda \geq \Delta$, the decision maker stops immediately for all $\eta \in (0,1)$;
(ii) if $r/\lambda < \Delta$, there exists $\underline{\eta}$ and $\overline{\eta}$ with $0 < \underline{\eta} < \overline{\eta} < 1$ such that if $\eta < \underline{\eta}$ or $\eta \geq \overline{\eta}$, the decision maker stops immediately. If $\eta \in [\underline{\eta}, \overline{\eta})$, the decision maker either stops at the first time her posterior belief is equal to 0 or the first time her posterior belief is equal to $\overline{\eta}$.

Figure 6 plots the decision maker's optimal waiting time that waits as a function of her initial belief η (Case (ii) of Proposition 6). Recall that in the conclusive bad news setting, the decision maker's belief in the absence of signals increasing over time. So the closer the decision maker's initial belief is to the target belief $\overline{\eta}$, the less time it takes for her belief to increase to $\overline{\eta}$.

The decision maker's and the agent's incentives to delay in the stopping game differ. The decision maker wants to take an action that matches the state. Her incentive to delay stopping is driven by learning: the later she stops, the more she learns about the state and is thus able to take a more informative action. This is in contrast to the (uninformed) agent's incentive to delay stopping in the main model. The agent wants the decision maker to take a high action regardless of the state. Although the uninformed agent wants the decision maker to learn, he cares asymmetrically about

[26] In particular, I maintain the assumption that disclosure can exogenously terminate at some random time after it starts. This assumption plays no role other than to keep the model comparable with the main model in the paper.

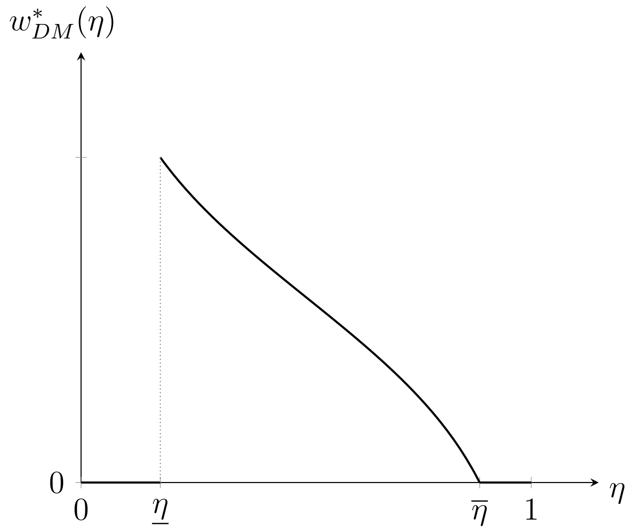

Figure 6: Decision maker's optimal waiting time for $r = 0.5, \lambda = 5$, and $\beta = 0.5$.

the direction in which the decision maker learns. The uninformed agent cares about learning only to the extent of getting the decision maker to believe the way he does.

A Appendix

A.1 Proofs for Section 3

A.1.1 Proof of Lemma 1

Given any public history h_T^{pub} at T, denote $\Pr(\theta = 1|h_T^{\text{pub}}) = q(T)$. The decision maker's problem is

$$\max_{a \in \mathbb{R}} e^{-rT} \left[\left(1 - (a-0)^2\right)(1 - q(T)) + \left(1 - (a-1)^2\right) q(T) \right]$$

Setting the derivative with respect to a to 0 results in $a = q(T)$.

A.1.2 Proof of Theorem 1.B (i)

Suppose a signal arrives at $\hat{w} \geq 0$. By Bayes' rule, both the agent and the decision maker's belief is zero for all $w \geq \hat{w}$. By Lemma 1, the decision maker takes action 0 if disclosure stops at any $w \geq \hat{w}$. The agent's expected payoff from stopping at any $w \geq \hat{w}$ (from the perspective of $\hat{w}$) is $e^{-r(w-\hat{w})}\kappa$, which is maximized at $\hat{w}$.

A.1.3 Proof of Theorem 1.B (ii)

Necessary conditions

Suppose disclosure starts at t_{start}. At the beginning of the continuation stopping game, the uninformed agent's belief is $\rho(t_{\text{start}})$, where $\rho(\cdot)$ is given by (1). To simplify notation, let $\rho = \rho(t_{\text{start}})$ and denote by $p(w)$ the uninformed agent's private belief that $\theta = 1$ calculated from the perspective of t_{start},

$$p(w) := \frac{\rho}{\rho + e^{-\lambda w}(1-\rho)}.$$

In other words, $p(w) = \rho(t_{\text{start}} + w)$.

First, I establish that at a common finite time, both types of agent stops disclosing with probability one.

Lemma 4. *In any equilibrium, there exists $\overline{w} < \infty$ such that*

$$\overline{w} = \inf\{w : G_U(w) = 1\} = \inf\{w : G_I(w) = 1\}.$$

Proof of Lemma 4. The proof is organized as the following series of claims.

Claim 1. In any equilibrium, if there exists $\hat{w}$ such that $G_U(\hat{w}) = 1$, then $G_I(\hat{w}) = 1$.

Proof. Suppose $G_U(\hat{w}) = 1$ for some $\hat{w}$. If there exists $w' > \hat{w}$ such that $G_I(w') = 1$, then the decision maker's belief about the agent if disclosure stops at any $w \in (\hat{w}, w')$ is that the agent is informed—because the uninformed agent has stopped with probability 1 by time $\hat{w}$. From the perspective of time $\hat{w}$, if the informed agent stops at $\hat{w}$, his expected payoff is $\kappa + q(\hat{w})$, where $q(\hat{w}) \geq 0$. If the informed agent stops at $w > \hat{w}$, his expected payoff is

$$\kappa \left(\int_{\hat{w}}^{w} \lambda e^{-(r+\lambda+\beta)(s-\hat{w})} \mathrm{d}s + \int_{\hat{w}}^{w} \beta e^{-(r+\lambda+\beta)(s-\hat{w})} \mathrm{d}s + e^{-(r+\lambda+\beta)(w-\hat{w})} \right) < \kappa \leq \kappa + q(\hat{w}).$$

Therefore, stopping at any $w > \hat{w}$ is dominated by stopping at $\hat{w}$. This implies that the informed agent stops by time $\hat{w}$, that is, $G_I(\hat{w}) = 1$. □

Next, I establish that both types of agent stop with probability one in finite time. Intuitively, because the decision maker only takes the action when disclosure stops, so both types of agent's expected payoff from never stopping is 0. If they stop at some finite w, their payoff is bounded below by $e^{-\lambda w}\kappa > 0$. The following lemma formalizes this intuition.

Claim 2. In any equilibrium, there exists $\overline{w} < \infty$ s.t. $G_I(\overline{w}) = 1$ and $G_U(\overline{w}) = 1$.

Proof. Recall that the informed agent's expected payoff from stopping at w is $V(w)$, and the uninformed agent's expected payoff from stopping at w is $U(w)$ where

$$V(w) = \int_0^w e^{-rs}\lambda e^{-\lambda s}e^{-\beta s}\kappa \mathrm{d}s + \int_0^w e^{-rs}e^{-\lambda s}\beta e^{-\beta s}(\kappa + q(s))\mathrm{d}s + e^{-rw}e^{-\lambda w}e^{-\beta w}(\kappa + q(w))$$

and

$$U(w) = (1-\rho)V(w) + \rho\left(\int_0^w e^{-rs}\beta e^{-\beta s}(\kappa + q(s))\mathrm{d}s + e^{-rw}e^{-\beta w}(\kappa + q(w))\right).$$

Take the limit as w goes to ∞,

$$\lim_{w\to\infty} V(w) = 0 \text{ and } \lim_{w\to\infty} U(w) = 0.$$

For any fixed $\hat{w} < \infty$, $V(\hat{w})$ and $U(\hat{w})$ are minimized at $q(s) = 0$ for all $s \in [0, \hat{w}]$. Then

$$V(\hat{w}) \geq \int_0^{\hat{w}} e^{-rs}\lambda e^{-\lambda s}e^{-\beta s}\kappa \mathrm{d}s + \int_0^{\hat{w}} e^{-rs}e^{-\lambda s}\beta e^{-\beta s}\kappa \mathrm{d}s + e^{-r\hat{w}}e^{-\lambda\hat{w}}e^{-\beta\hat{w}}\kappa > 0$$

and

$$U(\hat{w}) \geq (1-\rho)V(\hat{w}) + \rho\left(e^{-r\hat{w}}e^{-\beta\hat{w}}\kappa + \int_0^{\hat{w}} e^{-rs}\beta e^{-\beta s}\kappa \mathrm{d}s\right) > 0.$$

This means that both types of agent strictly prefer stopping at $\hat{w} < \infty$ to never stopping. Define

$$\overline{w} := \inf\{w : G_U(w) = 1\}.$$

The above argument shows $\overline{w} < \infty$. By Lemma 4, $G_I(\overline{w}) = 1$. □

Claim 3. The following is true: $\overline{w} = \inf\{w : G_I(w) = 1\}$.

Proof. Suppose there exists $w' < \overline{w}$ such that $G_I(w') = 1$. Then the decision maker's belief about the agent if disclosure stops at any $w > w'$ is that the agent is uninformed—because the informed agent has stopped with probability 1 by time

w'. That is, $q(w) = p(w)$ for all $w > w'$. Consider $\tilde{w} = (w' + \overline{w})/2$. From the perspective of $\tilde{w}$, the uninformed agent's expected payoff from stopping at $w > \tilde{w}$ is

$$
\begin{aligned}
p(\tilde{w}) &\left(\int_{\tilde{w}}^{w} e^{-r(s-\tilde{w})} \beta e^{-\beta(s-\tilde{w})} (\kappa + p(s)) \mathrm{d}s + e^{-r(w-\tilde{w})} e^{-\beta(w-\tilde{w})} (\kappa + p(w)) \right) \\
&+ (1 - p(\tilde{w})) \Bigg(\int_{\tilde{w}}^{w} \lambda e^{-(r+\lambda+\beta)(s-\tilde{w})} \kappa \mathrm{d}s + \int_{\tilde{w}}^{w} \beta e^{-(r+\lambda+\beta)(s-\tilde{w})} (\kappa + p(s)) \mathrm{d}s \\
&\qquad + e^{-(r+\lambda+\beta)(w-\tilde{w})} (\kappa + p(w)) \Bigg).
\end{aligned}
$$

Take the derivative with respect to w (the function is differentiable in w),

$$
\begin{aligned}
&-(1 - p(\tilde{w})) e^{-(r+\lambda+\beta)(w-\tilde{w})} \left(r\kappa + (r + \lambda) p(w) - p'(w) \right) \\
&\qquad - p(\tilde{w}) e^{-(\lambda+\beta)(w-\tilde{w})} \left(\lambda p(w) - p'(w) \right) < 0.
\end{aligned}
$$

So the expected payoff is maximized at $w = \tilde{w}$. That is, stopping at any $w > \tilde{w}$ is dominated by stopping at $\tilde{w}$. This implies $G_U(\tilde{w}) = 1$ which contradicts the definition of $\overline{w}$. □

It is useful to observe that the decision maker's belief that $\theta = 1$ if disclosure stops at any $w \in [0, \overline{w})$ is strictly less than the uninformed agent's. That is,

$$
q(w) < p(w) \text{ for all } w \in [0, \overline{w}).
$$

Intuitively, before $\overline{w}$, there is a strictly positive probability that the agent is informed. Regardless what strategy the agent adopts, there is a strictly positive probability that stopping is due to the informed agent being terminated. So when disclosure stops, the decision maker puts a strictly positive weight on the agent being informed, and thus will not be as optimistic as the uninformed agent's.

The next result establishes that during this interval $[0, \overline{w}]$, for any given decision maker's belief path $q(w)$, if the informed agent (weakly) prefers stopping at a later time, the uninformed agent strictly prefers doing so.

Lemma 5. *Fix any $w_1, w_2 \in [0, \overline{w}]$ where $w_2 > w_1$. If $V(w_2) \geq V(w_1)$, then $U(w_2) > U(w_1)$.*

Proof. With some algebra,

$$
\begin{aligned}
\frac{V(w_2)-V(w_1)}{e^{-\lambda w_1}} &= \int_{w_1}^{w_2} \beta e^{-(\beta+r+\lambda)s} e^{\lambda w_1} q(s)\mathrm{d}s \\
&\quad + e^{\lambda w_1} \frac{\left(e^{-(\beta+r+\lambda)w_2} - e^{-(\beta+r+\lambda)w_1}\right)}{\beta+r+\lambda} r\kappa \\
&\quad + e^{\lambda w_1} e^{-(\beta+r)w_2} q(w_2)\left(e^{-\lambda w_2} - e^{-\lambda w_1}\right) \\
&\quad + e^{-(\beta+r)w_2} q(w_2) - e^{-(\beta+r)w_1} q(w_1)
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\frac{U(w_2)-U(w_1)}{\rho+(1-\rho)e^{-\lambda w_1}} &= \int_{w_1}^{w_2} \beta e^{-(\beta+r)s} \frac{\rho+(1-\rho)e^{-\lambda s}}{\rho+(1-\rho)e^{-\lambda w_1}} q(s)\mathrm{d}s \\
&\quad + \frac{\rho\left(e^{-(\beta+r)w_2} - e^{-(\beta+r)w_1}\right)}{\left(\rho+(1-\rho)e^{-\lambda w_1}\right)(\beta+r)} r\kappa \\
&\quad + \frac{(1-\rho)\left(e^{-(\beta+r+\lambda)w_2} - e^{-(\beta+r+\lambda)w_1}\right)}{\left(\rho+(1-\rho)e^{-\lambda w_1}\right)(\beta+r+\lambda)} r\kappa \\
&\quad + \frac{1}{\rho+(1-\rho)e^{-\lambda w_1}}(1-\rho)e^{-(\beta+r)w_2} q(w_2)\left(e^{-\lambda w_2} - e^{-\lambda w_1}\right) \\
&\quad + e^{-(\beta+r)w_2} q(w_2) - e^{-(\beta+r)w_1} q(w_1).
\end{aligned} \tag{10}
$$

It can be verified that the first four lines in (10) is less than the first three lines in (9). This implies $U(w_2) - U(w_1) > 0$ given $V(w_2) - V(w_1) \geq 0$. The proof is algebraic and is omitted. $\square$

Lemma 6 (uninformed agent's stopping strategy)**.** *In any equilibrium, $G_U(w) = 0$ for all $w \in [0, \overline{w})$ and $G_U(\overline{w}) = 1$.*

Proof. Suppose there exists $\tilde{w}$ such that $G_U(\tilde{w}) > 0$. This means at some $\hat{w} \leq \tilde{w}$, the uninformed stops with positive probability. then $U(\hat{w}) \geq U(w)$ for all $w \neq \hat{w}$. By definition, $\overline{w}$ is the first time $G_U(w)$ is equal to 1, so the uninformed agent stops with positive probability at $\overline{w}$. This means $U(\hat{w}) = U(\overline{w}) \geq U(w)$ for all $w \neq \hat{w}$ and $w \neq \overline{w}$. Then there exists a neighborhood around $\overline{w}$, $(\overline{w} - \varepsilon, \overline{w}]$ with $\varepsilon > 0$, such that $U(w) \leq U(\overline{w})$ for all $w \in (\overline{w} - \varepsilon, \overline{w}]$.

By (the contrapositive of) Lemma 5, $V(\hat{w}) < V(\overline{w})$ and $V(\hat{w}) < V(w)$ for all $w \in (\overline{w} - \varepsilon, \overline{w}]$. This means that stopping at any $w \in (\overline{w} - \varepsilon, \overline{w}]$ is dominated by stopping at $\hat{w}$, so $G(w) = G(\overline{w}) = 1$ for all $w \in (\overline{w} - \varepsilon, \overline{w}]$, which contradicts the definition that $\overline{w} = \inf\{w : G(w) = 1\}$. $\square$

In words, Lemma 6 says that the uninformed agent does not stop at any $w \in [0, \overline{w})$ and stops with probability 1 at $\overline{w}$. This implies that conditional on $\theta = 1$, the probability that disclosure stops by w for $w \in [0, \overline{w})$ is equal to the probability that exogenous termination occurs by w. That is,

$$F^1(w) = \begin{cases} 1 - e^{-\beta w} & w \in [0, \overline{w}) \\ 1 & w = \overline{w} \end{cases}. \tag{11}$$

By definition,

$$\begin{aligned} F^0(w) = 1 &- \Pr(w_{\text{stop}} > w|\theta = 0, \text{informed}) \Pr(\text{informed}|\theta = 0) \\ &- \Pr(w_{\text{stop}} > w|\theta = 0, \text{uninformed}) \Pr(\text{uninformed}|\theta = 0), \end{aligned}$$

where $\Pr(w_{\text{stop}} > w|\theta = 0, \text{uninformed}) = 1 - F^1(w)$. By definition, $\Pr(\text{informed}|\theta = 0)$ is denoted by γ, which is given by $\eta = \rho(1 - \gamma(1 - \eta))$. After some simplifying, for $w \in [0, \overline{w})$,

$$F^0(w) = 1 - e^{-\beta w} \left(1 - \gamma G_I(w)\right). \tag{12}$$

Given that the uninformed agent does not stop in $[0, \overline{w})$, if the informed agent stops with a strictly positive probability in $[0, \overline{w})$, it reveals that the agent is informed. So the informed agent must stop continuously. The following lemma formalize this intuition.

Lemma 7 (informed agent's stopping strategy)**.** *In any equilibrium, $G_I(w)$ is (i) continuous and (ii) strictly increasing in w for $w \in [0, \overline{w})$.*

Proof. **Part (i).** If $G_I(w)$ is discontinuous at some $\hat{w}$, it means the informed agent stops with positive probability at $\hat{w}$. Suppose there exists $\hat{w} \in [0, \overline{w})$ such that $G_I(w)$ is discontinuous. That is, $\lim_{\varepsilon \to 0} G_I(\hat{w} - \varepsilon) < G_I(\hat{w})$. The decision maker's belief if disclosure stops at w can be written as

$$q(w) = \left(\frac{\Pr(w_{\text{stop}} = w|\theta = 1)}{\Pr(w_{\text{stop}} = w|\theta = 0)} \frac{\eta}{1 - \eta}\right) \Bigg/ \left(\frac{\Pr(w_{\text{stop}} = w|\theta = 1)}{\Pr(w_{\text{stop}} = w|\theta = 0)} \frac{\eta}{1 - \eta} + e^{-\lambda w}\right).$$

Then at $\hat{w}$,

$$\frac{\Pr(w_{\text{stop}} = \hat{w}|\theta = 1)}{\Pr(w_{\text{stop}} = \hat{w}|\theta = 0)} = \frac{\lim_{\varepsilon\to 0} \left(F^1(\hat{w}) - F^1(\hat{w} - \varepsilon)\right)/\varepsilon}{\lim_{\varepsilon\to 0} \left(F^0(\hat{w}) - F^0(\hat{w} - \varepsilon)\right)/\varepsilon} = 0,$$

where F^1 and F^0 are defined in (11) and (12) respectively. Thus, the decision maker's belief is $q(\hat{w}) = 0$. Therefore, from the perspective of time $\hat{w}$, if the informed agent stops at $\hat{w}$, his expected payoff is κ. If the informed agent stops at $\hat{w}+\varepsilon$, the decision maker's belief is $q(\hat{w}+\varepsilon) > 0$ for all $\varepsilon > 0$. The informed agent's expected payoff is

$$\lim_{\varepsilon\to 0} \left(1 - e^{-\lambda\varepsilon}\right)\kappa + e^{-\lambda\varepsilon}e^{-r\varepsilon}\left(\kappa + q(\hat{w}+\varepsilon)\right) = \kappa + \lim_{\varepsilon\to 0} q(\hat{w}+\varepsilon) > \kappa.$$

Therefore, stopping at $\hat{w}$ is dominated by stopping at $\hat{w}+\varepsilon$, which contradicts the premise that the agent stops at $\hat{w}$ with positive probability.

Part (ii). If $G_I(w)$ is constant over some interval, it means that the informed agent does not stop during that interval. Suppose there exists an interval $[w_1, w_2]$ with $0 \le w_1 < w_2 < \overline{w}$ such that $G_I(w)$ is constant for $w \in [w_1, w_2]$. Let $\tilde{w}$ be the supremum of w_2 for which over $[w_1, w_2]$, $G_I(w)$ is constant.

By Lemma 6, the uninformed agent does not stop in $[w_1, \tilde{w}]$. So if disclosure stops in $[w_1, \tilde{w}]$, it can only be due to exogenous termination, which is uninformative of the state. The decision maker's belief if disclosure stops at any $w \in [w_1, \tilde{w}]$ is

$$q(w) = \frac{\eta}{\eta + e^{-\lambda w}\left(1 - \gamma G_I(w_1)\right)(1-\eta)},$$

which means $q'(w) = \lambda(1 - q(w))q(w)$. Given $q(w)$ is differentiable in w for $w \in (w_1, \tilde{w})$, the informed agent's expected payoff from stopping at $w \in [w_1, \tilde{w}]$ is differentiable for $w \in (w_1, \tilde{w})$ with derivative proportional to

$$q'(w) - r\kappa - (\lambda + r)\, q(w) < 0,$$

which means the expected payoff $V(\tilde{w}) < V(w_1)$. Because $V(w)$ is continuous in w for a neighborhood around $\tilde{w}$, $w \in (\tilde{w}, \tilde{w}+\varepsilon)$, $\lim_{\varepsilon\to 0} V(\tilde{w}+\varepsilon) = V(\tilde{w}) < V(w_1)$. So stopping at any $w \in (\tilde{w}, \tilde{w}+\varepsilon)$ is dominated by stopping at w_1, which means $G_I(w)$ is constant over the interval $[\tilde{w}, \tilde{w}+\varepsilon]$. This contradicts the definition of $\tilde{w}$. □

Lemma 8. *In any equilibrium, $G_I(w)$ is twice differentiable in w for $w \in (0, \overline{w})$.*

Proof. First, by the Lebesgue's theorem (see, for example, Royden and Fitzpatrick, 1988), because $G_I(w)$ is monotone so it is almost everywhere differentiable for $w \in (0, \overline{w})$. By (12), $F^0(w)$ is a differentiable function of $G_I(w)$ and is also almost everywhere differentiable. Denote the derivative of $F^0(w)$, whenever exists, by $f^0(w)$, and the derivative of $F^1(w)$ is $f^1(w) = \beta e^{-\beta w}$. Then the decision maker's belief that $\theta = 1$ is

$$q(w) = \frac{\beta e^{-\beta w}\eta}{\beta e^{-\beta w}\eta + e^{-\lambda w} f^0(w)(1-\eta)}. \tag{13}$$

Lemma 7 says that in any equilibrium, the informed agent must be indifferent with respect to waiting times in $[0, \overline{w})$. This implies that the informed agent's expected payoff from stopping at w is constant in $[0, \overline{w})$. Because $V(w)$ is an everywhere differentiable function of $q(w)$, so $q(w)$ is everywhere differentiable in w. This implies $f^0(w)$, and thus $F^0(w)$, are everywhere differentiable in w. The result follows and (13) holds true for all $w \in (0, \overline{w})$. □

Take the derivative of $V(w)$ with respect to w and setting it to zero, we get $q'(w) = r\kappa + (\lambda + r)\, q(w)$. Plugging in (13), one obtains a second-order differentiable equation in $G_I(w)$, $G_I''(w) = \mathscr{G}(G_I(w), G_I'(w), w)$, where $\mathscr{G}$ is given by[27]

$$\begin{aligned}
&\mathscr{G}(G_I(w), G_I'(w), w) \\
=&\beta G_I'(w) - \frac{\beta\rho}{\rho-\eta}\left(r(1-\eta) + e^{\lambda w}\eta\left(r\kappa\left(\frac{\eta + e^{-\lambda w}(1-\eta)}{\eta}\right)^2 + r + \lambda\right)\right) \\
&+ r\left(1 + 2\kappa\left(\frac{\eta + e^{-\lambda w}(1-\eta)}{\eta}\right)\right)\left(\beta G_I(w) - G_I'(w)\right) \\
&- \frac{r\kappa e^{-\lambda w}(\rho-\eta)}{\beta\rho\eta}\left(\beta G_I(w) - G_I'(w)\right)^2.
\end{aligned} \tag{14}$$

Divinity. By Lemma 4, both types of agent stop by $\overline{w} < \infty$. This means that stopping at $w > \overline{w}$ is off the equilibrium path. First, I show that if an equilibrium survives the divinity refinement, the decision maker's belief that $\theta = 1$ if disclosure stops off-path at $w > \overline{w}$ is equal to the uninformed agent's belief that $\theta = 1$, $p(w)$. Next, I show that the informed agent's probability of stopping must be continuous for all $w \in [0, \overline{w}]$.

[27] Recall that given that disclosure starts at t and given the agent's equilibrium starting strategies, $\rho = \rho(t)$ and $\eta = \mu$, which is the same $\mathscr{G}$ given in footnote 12.

Lemma 9. *In a divine equilibrium, the decision maker's belief that $\theta = 1$ if disclosure stops at $w > \overline{w}$ is $q(w) = p(w)$.*

Proof. From the perspective of time $\overline{w}$, the agent's expected payoff is given by $V(w)$ and $U(w)$ where both the uninformed agent's and the decision maker's belief that $\theta = 1$ at $\overline{w}$ is given by $p(\overline{w})$. It follows from Lemma 5 that for any decision maker's belief $q(w)$ for $w > \overline{w}$, if $V(w)$ increases in w, then $U(w)$ increases in w. That is, if the informed agent finds it optimal to deviate from stopping at $\overline{w}$ to stopping at $w > \overline{w}$, the uninformed agent also finds it optimal to deviate. As mentioned in the paper, the divinity refinement prescribes that the decision maker's off-path belief should assign zero weight to the type of agent that has less incentive to deviate. The result follows. $\square$

Lemma 10. *In a divine equilibrium, $G_I(w)$ is continuous at $w = \overline{w}$.*

Proof. By Lemma 9, from the perspective of time $\overline{w}$, if the uninformed agent stops at $\overline{w}$, his expected payoff is

$$\kappa + q(\overline{w}).$$

By Lemma 9, $\lim_{\varepsilon \to 0} q(\overline{w} + \varepsilon) = p(\overline{w})$, and if the uninformed agent stops at $\overline{w} + \varepsilon$ for $\varepsilon > 0$ small, his expected payoff is

$$\lim_{\varepsilon \to 0} (1 - p(\overline{w})) \left(1 - e^{-\lambda\varepsilon}\right) \kappa + \left((1 - p(\overline{w}))\, e^{-\lambda\varepsilon} + p(\overline{w})\right) e^{-r\varepsilon} \left(\kappa + q(\overline{w} + \varepsilon)\right) = \kappa + p(\overline{w}).$$

Recall that by Lemma 9, in equilibrium, the uninformed agent stops with probability 1 at $\overline{w}$. This implies $\kappa + q(\overline{w}) = \kappa + p(\overline{w})$: because if $q(\overline{w}) < p(\overline{w})$, the uninformed agent finds it profitable to deviate to stopping at $\overline{w} + \varepsilon$, violating Lemma 9. At $\overline{w}$,

$$\Pr(w_{\text{stop}} = \overline{w} | \theta = 1) = 1$$

and

$$\Pr(w_{\text{stop}} = \overline{w} | \theta = 0) = (1 - \gamma) \cdot 1 + \gamma(G_I(\overline{w}) - G_I(\overline{w}^-)),$$

where $G_I(\overline{w}^-) = \lim_{\varepsilon \to 0} G_I(\overline{w} - \varepsilon)$ is the left limit of G_I at $\overline{w}$. Recall that γ is the probability that the agent is informed at the beginning of the continuation game conditional on $\theta = 0$, and is given by $\eta = \rho(1 - \gamma(1 - \eta))$. Therefore,

$$q(\overline{w}) = \frac{\eta}{\eta + e^{-\lambda\overline{w}} \left((1 - \gamma) + \gamma(G_I(\overline{w}) - G_I(\overline{w}^-))\right)(1 - \eta)}.$$

Thus, $q(\overline{w}) = p(\overline{w})$ if and only if $G_I(\overline{w}) = G_I(\overline{w}^-)$. $\square$

Boundary conditions. By Lemma 7 (i), $G_I(0) = 0$. By Lemma 4, $G_I(\overline{w}) = 1$. By Lemma 10, $G_I(w)$ is continuous for all $w \in [0, \overline{w}]$ and $q(\overline{w}) = p(\overline{w})$, which implies

$$\lim_{\varepsilon \to 0} q(\overline{w} - \varepsilon) = q(\overline{w}) = p(\overline{w}).$$

Writing out $p(\overline{w}) = q(\overline{w})$,

$$\frac{\rho}{\rho + e^{-\lambda \overline{w}}(1-\rho)} = \frac{\beta\eta}{\beta\eta + e^{-\lambda \overline{w}}\left((1-\gamma)\beta + \gamma\left(G_I'(\overline{w}) + \beta(1 - G_I(\overline{w}))\right)\right)(1-\eta)}.$$

Because $G_I(\overline{w}) = 1$, the above equality implies

$$G_I'(\overline{w}) = 0.$$

To sum up, the boundary conditions are therefore given by $G_I(0) = 0$, $G_I(\overline{w}) = 1$, and $G_I'(\overline{w}) = 0$.

Therefore, in any equilibrium, for all $w \in [0, \overline{w}]$,

$$f^1(w) = \beta e^{-\beta w}$$

and

$$f^0(w) = (1-\gamma)\beta e^{-\beta w} + \gamma\left(G_I'(w)e^{-\beta w} + \beta e^{-\beta w}(1 - G_I(w))\right).$$

Substituting these expressions for $f^1(w)$ and $f^0(w)$ into (13), the decision maker's belief that $\theta = 1$ when disclosure stops at $w \in [0, \overline{w}]$ can be written as

$$q(w) = \frac{\beta\eta}{\beta\eta + e^{-\lambda w}\left((1-\gamma)\beta + \gamma\left(G_I'(w) + \beta(1 - G_I(w))\right)\right)(1-\eta)}. \tag{15}$$

Existence and uniqueness

The following theorem establishes existence and uniqueness of a solution to the desired boundary value problem. Moreover, this unique solution to the boundary value problem is an equilibrium.

Theorem 2. *There exists a unique solution, w^* and $G_I^*(w)$, to the following boundary* *value problem: for all* $w \in [0, w^*]$,

$$G_I''(w) = \mathscr{G}(G_I(w), G_I'(w), w), \quad G_I(0) = 0, G_I(w^*) = 1, \textit{ and } G_I'(w^*) = 0, \quad \text{(BVP)}$$

where $\mathscr{G}$ *is given by (14).*

To prove this, I first present a useful lemma that establishes equivalence between two boundary value problems (or initial value problems).

Lemma 11. *There exists* $0 < \overline{w}_{\max} < \infty$ *such that for any* $\overline{w} \in (0, \overline{w}_{\max})$*, there exists a unique solution to the initial value problem (BVP-q): for all* $w \in [0, \overline{w}]$,

$$q'(w) = r\kappa + (r+\lambda)q(w), \quad q(\overline{w}) = p(\overline{w}) \quad \text{(BVP-q)}$$

and a unique solution to the initial value problem (BVP-1): for all $w \in [0, \overline{w}]$,

$$G_I''(w) = \mathscr{G}(G_I(w), G_I'(w), w), \quad G_I(\overline{w}) = 1 \textit{ and } G_I'(\overline{w}) = 0. \quad \text{(BVP-1)}$$

Moreover, given the solution to (BVP-1), the corresponding $q(w)$ *given by (15) solves (BVP-q) and vice versa.*

Proof. First, I show these two initial value problems are equivalent.

Equation (15) links $G_I(w)$ with $q(w)$. In equilibrium, the informed agent's indifference condition $V'(w) = 0$ reduces to $q'(w) = r\kappa + (r+\lambda)q(w)$, which is the differential equation in (BVP-q). By (15), this differential equation is equivalent to the differential equation in (BVP-1). Moreover, by (15), the boundary conditions $G_I(w^*) = 1$ and $G_I'(w^*) = 0$ imply $q(w^*) = p(w^*)$.

Next, I derive conditions under which each initial value problem admits a unique solution.

By the Picard-Lindelöf Theorem (see, for example, Teschl, 2012, Theorem 2.2), for any $\overline{w} > 0$, (BVP-q) has a unique solution. In fact, the solution admits a closed form. To emphasize this solution depends on $\overline{w}$, denote it by $\overline{q}(w; \overline{w})$, where

$$\overline{q}(w; \overline{w}) = \left(p(\overline{w}) + \frac{\kappa\lambda}{r+\lambda}\right) e^{-(r+\lambda)(\overline{w}-w)} - \frac{\kappa\lambda}{r+\lambda}. \tag{16}$$

For (BVP-1), because $G_I(w)$ is twice differentiable for $w \in (0, \overline{w})$, so $G'(w) < \infty$ and $G(w) < \infty$ for all $w \in (0, \overline{w})$. By (15), this means that there does not exists

$w \in (0, \overline{w})$ such that $q(w) = 0$. If (BVP-1) admits a solution, $\overline{w}$ must be such that $q(w) > 0$ for all $w \in [0, \overline{w}]$. This means $\overline{q}(w; \overline{w}) > 0$ for all $w \in [0, \overline{w}]$. I now derive conditions under which $\overline{q}(w; \overline{w}) > 0$ for all $w \in [0, \overline{w}]$.

By (16), $\overline{q}(w; \overline{w})$ is strictly increasing in w and $\overline{q}(\overline{w}; \overline{w}) = p(\overline{w}) > 0$, so $q(w; \overline{w}) > 0$ for all w as long as $q(0; \overline{w}) > 0$.

Claim 4. There exists a unique $\overline{w}_{\max} > 0$ such that $\overline{q}(0; \overline{w}) > 0$ if and only if $\overline{w} < \overline{w}_{\max}$.

Proof of the claim. Plugging in $w = 0$ into (16),

$$\overline{q}(0; \overline{w}) = \left(p(\overline{w}) + \frac{\kappa\lambda}{r + \lambda} \right) e^{-(r+\lambda)\overline{w}} - \frac{\kappa\lambda}{r + \lambda}.$$

Then

$$\lim_{\overline{w} \to 0} \overline{q}(0; \overline{w}) = \overline{q}(0; 0) = \rho > 0$$

and

$$\lim_{\overline{w} \to \infty} \overline{q}(0; \overline{w}) = -\frac{\kappa\lambda}{r + \lambda} < 0.$$

It can be readily verified that the derivative of $\overline{q}(0; \overline{w})$ with respect to $\overline{w}$ is strictly negative. Because $\overline{q}(0; \overline{w})$ is continuous in $\overline{w}$, by the intermediate value theorem, there exists a unique $0 < \overline{w}_{\max} < \infty$ such that $\overline{q}(0; \overline{w}_{\max}) = 0$, $\overline{q}(0; \overline{w}) > 0$ for all $\overline{w} < \overline{w}_{\max}$, and $\overline{q}(0; \overline{w}) < 0$ for all $\overline{w} > \overline{w}_{\max}$. □

Therefore, by the Picard-Lindelöf Theorem, for all $\overline{w} < \overline{w}_{\max}$, there exists a unique solution to (BVP-1). □

Lemma 11 is useful because (BVP-q) has a closed-form solution. By exploiting the relationship between $G_I(w)$ and $q(w)$ using Lemma 9, one obtains properties of $G_I(w)$ that are otherwise harder to derive.

Lemma 12. *Let $\overline{G}_I(w; \overline{w})$ be the (unique) solution to (BVP-1) for some $\overline{w} \in (0, \overline{w}_{\max})$. There exists a unique $w^* \in (0, \overline{w}_{\max})$ such that $\overline{G}_I(0; w^*) = 0$.*

Proof. By Lemma 11, I derive properties of the (unique) solution to (BVP-1) through (16) via (15). Equate (16) with (15),

$$G_I'(w; \overline{w}) = \beta G_I(w; \overline{w}) - \frac{\beta\rho}{\rho - \eta} \left(1 - \eta + e^{\lambda w} \eta \left(1 - \frac{1}{\overline{q}(w; \overline{w})} \right) \right). \tag{17}$$

Note that this condition is in itself a differential equation in $G_I(w;\overline{w})$ and the solution to this differential equation with $G_I(\overline{w};\overline{w}) = 1$ is the solution to (BVP-1) for a fixed $\overline{w}$. More specifically, the initial value problem, (BVP-2),

$$G_I'(w;\overline{w}) = \beta G_I(w;\overline{w}) - \frac{\beta\rho}{\rho-\eta}\left(1-\eta+e^{\lambda w}\eta\left(1-\frac{1}{\overline{q}(w;\overline{w})}\right)\right), \; G_I(\overline{w};\overline{w}) = 1 \tag{BVP-2}$$

is equivalent to (BVP-q) and thus is equivalent to (BVP-1). Define

$$H(w;\overline{w}) := \frac{\beta\rho}{\rho-\eta}\left(1-\eta+e^{\lambda w}\eta\left(1-\frac{1}{\overline{q}(w;\overline{w})}\right)\right). \tag{18}$$

The differential equation (17) becomes

$$G_I'(w;\overline{w}) = \beta G_I(w;\overline{w}) - H(w;\overline{w}),$$

which has a closed-form solution, denoted by

$$\overline{G}_I(w;\overline{w}) = ce^{\beta w} - e^{\beta w}\int_0^w e^{-\beta s}H(s;\overline{w})\mathrm{d}s,$$

where $c \in \mathbb{R}$ is an integration constant to be determined. By the boundary condition $G_I(\overline{w};\overline{w}) = 1$,

$$1 = ce^{\beta\overline{w}} - e^{\beta\overline{w}}\int_0^{\overline{w}} e^{-\beta s}H(s;\overline{w})\mathrm{d}s \implies c = e^{-\beta\overline{w}} + \int_0^{\overline{w}} e^{-\beta s}H(s;\overline{w})\mathrm{d}s.$$

Therefore,

$$\overline{G}_I(w;\overline{w}) = \left(e^{-\beta\overline{w}} + \int_0^{\overline{w}} e^{-\beta s}H(s;\overline{w})\mathrm{d}s\right)e^{\beta w} - e^{\beta w}\int_0^w e^{-\beta s}H(s;\overline{w})\mathrm{d}s. \tag{19}$$

By definition, for all $\overline{w} \in (0,\overline{w}_{\max})$ and all $w < \overline{w}$, $H(w;\overline{w}) < \infty$. Set $w = 0$,

$$\overline{G}_I(0;\overline{w}) = e^{-\beta\overline{w}} + \int_0^{\overline{w}} e^{-\beta s}H(s;\overline{w})\mathrm{d}s.$$

The goal is to show there exists a unique $w^* \in (0,\overline{w}_{\max})$ such that $\overline{G}_I(0;w^*) = 0$. The proof uses the intermediate value theorem.

Take the derivative $\overline{G}_I(0;\overline{w})$ with respect to $\overline{w}$,

$$\frac{\mathrm{d}}{\mathrm{d}\overline{w}}\overline{G}_I(0;\overline{w}) = -\beta e^{-\beta\overline{w}} + e^{-\beta\overline{w}}H(\overline{w};\overline{w}) + \int_0^{\overline{w}} e^{-\beta s}\frac{\partial}{\partial\overline{w}}H(s;\overline{w})\mathrm{d}s < 0. \tag{20}$$

This derivative is negative because $H(\overline{w};\overline{w}) = \beta$; and because $\partial\overline{q}(s;\overline{w})/\partial\overline{w} < 0$ for $s \leq \overline{w}$, $\partial H(s;\overline{w})/\partial\overline{w} < 0$.

Let $\overline{w} \downarrow 0$. Then

$$\lim_{\overline{w}\downarrow 0}\overline{G}_I(0;\overline{w}) = \lim_{\overline{w}\downarrow 0} e^{-\beta\overline{w}} + \lim_{\overline{w}\downarrow 0}\int_0^{\overline{w}} e^{-\beta s}H(s;\overline{w})\mathrm{d}s = 1.$$

Let $\overline{w} \uparrow \overline{w}_{\max}$. Then

$$\lim_{\overline{w}\uparrow\overline{w}_{\max}}\overline{G}_I(0;\overline{w}) = e^{-\beta\overline{w}_{\max}} + \lim_{\overline{w}\uparrow\overline{w}_{\max}}\int_0^{\overline{w}} e^{-\beta s}H(s;\overline{w})\mathrm{d}s.$$

Claim 5. The following is true:

$$\lim_{\overline{w}\uparrow\overline{w}_{\max}}\int_0^{\overline{w}} e^{-\beta s}H(s;\overline{w})\mathrm{d}s = -\infty.$$

Proof of the claim. Substituting in the definition of $H(s;\overline{w})$, given by (18),

$$\begin{aligned}\lim_{\overline{w}\uparrow\overline{w}_{\max}}\int_0^{\overline{w}} e^{-\beta s}H(s;\overline{w})\mathrm{d}s = &\lim_{\overline{w}\uparrow\overline{w}_{\max}}\int_0^{\overline{w}} e^{-\beta s}\frac{\beta\rho}{\rho-\eta}\left(1-\eta+e^{\lambda s}\eta\right)\mathrm{d}s\\ &- \lim_{\overline{w}\uparrow\overline{w}_{\max}}\frac{\beta\rho\eta}{\rho-\eta}\int_0^{\overline{w}}\frac{e^{-\beta s}e^{\lambda s}}{\overline{q}(s;\overline{w})}\mathrm{d}s.\end{aligned}$$

The first limit is finite. It suffices to show the second limit is infinite. Because $\overline{q}(s;\overline{w}) > 0$ for all $s \in (0,\overline{w})$,

$$\lim_{\overline{w}\uparrow\overline{w}_{\max}}\int_0^{\overline{w}}\frac{e^{-\beta s}e^{\lambda s}}{\overline{q}(s;\overline{w})}\mathrm{d}s \geq \lim_{\overline{w}\uparrow\overline{w}_{\max}}\int_0^{\overline{w}}\frac{e^{-\beta\overline{w}}}{\overline{q}(s;\overline{w})}\mathrm{d}s.$$

Because $\lim_{\overline{w}\uparrow\overline{w}_{\max}} e^{-\beta\overline{w}} = e^{-\beta\overline{w}_{\max}}$ is finite, it suffices to show

$$\lim_{\overline{w}\uparrow\overline{w}_{\max}}\int_0^{\overline{w}}\frac{1}{\overline{q}(s;\overline{w})}\mathrm{d}s = \infty.$$

Substituting in the definition of $\overline{q}(s;\overline{w})$ given by (16), this integral has a closed-form solution and

$$\lim_{\overline{w}\uparrow\overline{w}_{\max}} \int_0^{\overline{w}} \frac{1}{\overline{q}(s;\overline{w})} \mathrm{d}s = \lim_{\overline{w}\uparrow\overline{w}_{\max}} \frac{1}{r\kappa} \ln\left(\frac{\rho(r+\lambda)}{\rho(r+\lambda) - r\kappa\left(e^{(r+\lambda)\overline{w}}-1\right)\left(\rho+e^{-\lambda\overline{w}}(1-\rho)\right)}\right).$$

Recall that $\overline{w}_{\max}$ is given by $\overline{q}(0;\overline{w}_{\max}) = 0$ (and $\overline{q}(0;\overline{w}_{\max}) > 0$ for $\overline{w} < \overline{w}_{\max}$). It follows directly from rearranging the equation $\overline{q}(0;\overline{w}_{\max}) = 0$ that as $\overline{w}$ increasing to $\overline{w}_{\max}$, the denominator above goes to zero from above. That is,

$$\lim_{\overline{w}\uparrow\overline{w}_{\max}} \rho(r+\lambda) - r\kappa\left(e^{(r+\lambda)\overline{w}}-1\right)\left((1-\rho)\,e^{-\lambda\overline{w}}+\rho\right) = 0^+.$$

The result follows. $\square$

It follows from the claim that $\lim_{\overline{w}\uparrow\overline{w}_{\max}} \overline{G}_I(0;\overline{w}) = -\infty$. By the intermediate value theorem, there exists a unique $w^* \in (0,\overline{w}_{\max})$ such that $\overline{G}_I(0;w^*) = 0$. $\square$

Lemma 13. *Denote $G_I^*(w) = \overline{G}_I(w;w^*)$. Then the pair w^* and $G_I^*(w)$ is the unique solution to (BVP).*

Proof. Note that (BVP) subsumes (BVP-1) with the additional boundary condition $G_I(0) = 0$. By Lemma 12, $G_I^*(w)$ is the solution to (BVP-1) such that $G_I^*(0) = 0$. The result follows. $\square$

To sum up, by Lemma 12, w^* is unique. By Lemma 11 and Lemma 13, the pair w^* and $G_I^*(w)$ solves (BVP). The theorem follows.

Sufficient conditions

To establish sufficiency, one needs to show that the (unique) solution to the boundary value problem (BVP) is a proper probability distribution and satisfies the equilibrium conditions. Because the solution must satisfy the boundary conditions, $G_I(0) = 0$ and $G_I(\overline{w}) = 1$, to show the solution is a proper probability distribution function, it remains to show the solution is strictly increasing. This is formalized in the following lemma.

Lemma 14. *Let the pair w^* and $G_I^*(w)$ be the solution to the boundary value problem (BVP). Then $G_I^*(w)$ is strictly increasing in w for $w \in [0,w^*]$.*

Proof. By Lemma 11, I consider the solution to (BVP-q). Take the derivative of (17) with respect to w on both sides,

$$\beta G_I^{*\prime}(w) = G_I^{*\prime\prime}(w) + \frac{\beta\rho}{\rho-\eta}\eta e^{\lambda w}\left(\lambda\left(1-\frac{1}{q^*(w)}\right)+\frac{q^{*\prime}(w)}{q^*(w)^2}\right).$$

Because at $w = w^*$, $G_I^{*\prime}(w^*) = 0$ and $q^*(w^*) = p(w^*)$, moreover, $q^{*\prime}(w) = r\kappa + (r+\lambda)q^*(w)$, so

$$G_I^{*\prime\prime}(w^*) = -\frac{\beta\rho}{\rho-\eta}\eta e^{\lambda w^*}\left(\frac{\lambda p(w^*)^2 + r\kappa + rp(w^*)}{p(w^*)^2}\right) < 0,$$

where the inequality follows from $\rho > \eta$. Suppose there exists $\hat{w} < w^*$ such that $G_I^{*\prime}(\hat{w}) = 0$. Then

$$G_I^{*\prime\prime}(\hat{w}) = -\frac{\beta\rho}{\rho-\eta}\eta e^{\lambda\hat{w}}\left(\lambda\left(1-\frac{1}{q^*(\hat{w})}\right)+\frac{r\kappa+(r+\lambda)q^*(\hat{w})}{q^*(\hat{w})^2}\right). \tag{21}$$

As is shown in the proof of Lemma 11, $q(\hat{w}) > 0$, so $r\kappa+(r+\lambda)q(\hat{w}) > \lambda\left(1-q^*(\hat{w})\right)q^*(\hat{w})$. This implies the term in the parenthesis of (21) is positive. It then follows from $\rho > \eta$ that $G_I^{*\prime\prime}(\hat{w}) < 0$.

This says that if there exists $\hat{w} < w^*$ such that $G_I^{*\prime}(\hat{w}) = 0$, then it must be $G_I^{*\prime\prime}(\hat{w}) < 0$. This is a contradiction because as shown, $G_I^{*\prime}(w)$ decreases to 0 at w^*, so if there exists a point in time before w^* at which $G_I^{*\prime}(w)$ is equal zero, it must be either increasing to zero from below or tangent to zero. □

A.1.4 Proof of Lemma 2

The proof is subsumed by the proof of Lemma 9 and Lemma 10 above.

A.1.5 Proof of Lemma 3

I first prove the result for η.

I first show that for a fixed $\overline{w}$, the solution to (BVP-2) is increasing in η. Let $\overline{G}_I(w;\overline{w};\eta)$ denote the solution to (BVP-2) given $\overline{w}$ and η. For any two η_1 and η_2 where $\eta_1 < \eta_2$, the boundary condition in (BVP-2) says $\overline{G}_I(w;\overline{w};\eta_1) = \overline{G}_I(w;\overline{w};\eta_2) =$

1. Now I show the differential equation (17) in (BVP-2),

$$G_I'(w;\overline{w};\eta) = \beta G_I(w;\overline{w};\eta) - \frac{\beta\rho}{\rho-\eta}\left(1-\eta+e^{\lambda w}\eta\left(1-\frac{1}{\overline{q}(w;\overline{w})}\right)\right)$$

is increasing in η for a fixed $G_I(w;\overline{w};\eta)$ for all $w<\overline{w}$. The derivative of the right-hand side of the differential equation with respect to η is equal to

$$-\frac{\beta\rho}{(\rho-\eta)^2}\left(1-\rho\left(1+e^{\lambda w}\left(\frac{1}{\overline{q}(w;\overline{w})}-1\right)\right)\right)>0,$$

where the inequality follows from plugging in the definition of $\overline{q}(w;\overline{w})$ given by (16). By a standard comparison argument (see Teschl, 2012, Theorem 1.3), the solution to (BVP-2), $\overline{G}_I(w;\overline{w};\eta)$, is decreasing pointwise in η for $[0,\overline{w}]$.

Let $w^*(\eta_1)$ be the (unique) solution such that $\overline{G}_I(w;w^*(\eta_1);\eta_1)=0$. The above argument implies $\overline{G}_I(w;w^*(\eta_1);\eta_2)<0$. Let $w^*(\eta_2)$ be the (unique) solution such that $\overline{G}_I(w;w^*(\eta_2);\eta_2)=0$. Recall that by (20), for a fixed η, $\partial\overline{G}_I(0;\overline{w};\eta)/\partial\overline{w}<0$. So it must be that $w^*(\eta_2)<w^*(\eta_1)$.

The proof for ρ is analogous. Only that it needs to be shown that the differential equation (17) in (BVP-2) is decreasing in ρ. Below is the proof.

By definition (18), differential equation (17) can be written as,

$$G_I'(w;\overline{w};\rho) = \beta G_I(w;\overline{w};\rho) - H(w;\overline{w};\rho).$$

Showing this differential equation is decreasing in ρ for a fixed $G_I(w;\overline{w};\rho)$ is equivalent to showing $H(w;\overline{w};\rho)$ is increasing in ρ for a fixed $G_I(w;\overline{w};\rho)$. Take the derivative of H with respect to w. Because $\overline{q}(w;\overline{w})$ is a solution to (BVP-q), so $\partial\overline{q}(w;\overline{w};\rho)/\partial w = r\kappa+(r+\lambda)\overline{q}(w;\overline{w};\rho)$, then

$$\frac{\partial}{\partial w}H(w;\overline{w};\rho) = e^{\lambda w}\frac{\beta\rho\eta}{\rho-\eta}\left(\lambda+\frac{r\kappa}{\overline{q}(w;\overline{w};\rho)^2}+\frac{r}{\overline{q}(w;\overline{w};\rho)}\right).$$

Because $\overline{q}(w;\overline{w};\rho)$ is increasing in ρ and $\beta\rho\eta/(\rho-\eta)$ is decreasing in ρ, so $\partial H(w;\overline{w};\rho)/\partial w<0$. Because $H(\overline{w};\overline{w};\rho)=\beta$ for all ρ, by a standard comparison argument (see Teschl, 2012, Theorem 1.3), $H(w;\overline{w};\rho)$ is increasing pointwise in ρ. The result follows.

A.1.6 Proof of Proposition 1

By definition, $w^* \geq 0$ and $\rho \geq \eta$. Proving Proposition 1 is equivalent to showing $w^* = 0$ if and only if $\rho = \eta$. Fix $\rho = \eta$. The uninformed agent's expected payoff from waiting $w \geq 0$ to disclose is

$$\begin{aligned}
U(w) =& e^{-\beta w}\left(\rho + e^{-\lambda w}(1-\rho)\right) e^{-rw}\left(\kappa + \frac{\rho}{\rho + e^{-\lambda w}(1-\rho)}\right) \\
&+ e^{-\beta w}(1-\rho)\int_0^w \lambda e^{-\lambda s} e^{-rs}\kappa \mathrm{d}s \\
&+ \int_0^w \beta e^{-\beta\tau}\left(\rho + e^{-\lambda\tau}(1-\rho)\right) e^{-r\tau}\left(\kappa + \frac{\rho}{\rho + e^{-\lambda\tau}(1-\rho)}\right)\mathrm{d}\tau \\
&+ \int_0^w \beta e^{-\beta\tau}\left((1-\rho)\int_0^\tau \lambda e^{-\lambda s}e^{-rs}\kappa \mathrm{d}s\right)\mathrm{d}\tau.
\end{aligned}$$

It can be readily verified that $U(w)$ is strictly decreasing in w for all $w \geq 0$ and therefore maximized at $w = 0$. The same calculation and conclusion apply to the informed agent's expected payoff $V(w)$.

A.1.7 Proof of Theorem 1.A

The conditions under which $\tau^* \geq 0$ exists are stated in Proposition 2 and Proposition 3. I establish some preliminary expressions and notations below; the proofs of the theorem is subsumed by Proof of Proposition 2 and Proof of Proposition 3.

Preliminaries

Suppose disclosure starts at t. Recall that the continuation stopping game can be parameterized by the uninformed agent's belief that $\theta = 1$ at the decision maker's belief that $\theta = 1$ at the beginning of the continuation game. With a slight abuse of notation, denote the equilibrium waiting time in the continuation stopping game by

$$w^*(t) = w^*(\rho(t), \eta(t)).$$

It is convenient to define the decision maker's belief that $\theta = 1$ when disclosure stops at w explicitly as a function of the disclosure starting time t. With a slight abuse of notation, denote this function by the same letter as before, q. Then for $w \leq w^*(t)$,

$$q(w, w^*(t), \rho(t)) = q(w, t) = \left(\frac{\rho(t)}{\rho(t) + e^{-\lambda w^*(t)}(1 - \rho(t))} + \frac{r}{r + \lambda} \right) e^{-(r+\lambda)(w^*(t) - w)} - \frac{r}{r + \lambda}.$$

For $w > w^*(t)$, $q(w, t) = \rho(t)$ (off-path belief).

In equilibrium, $V(w)$ is maximized at $w = w^*$. Define the informed agent's equilibrium payoff in this continuation game as $\hat{V}(w^*, \rho)$,

$$\begin{aligned} \hat{V}(w^*, \rho) :=& e^{-rw^*} e^{-\lambda w^*} e^{-\beta w^*} (1 + q(w^*, w^*, \rho)) \\ &+ \int_0^{w^*} e^{-rs} \lambda e^{-\lambda s} e^{-\beta s} \mathrm{d}s + \int_0^{w^*} e^{-rs} e^{-\lambda s} \beta e^{-\beta s} (1 + q(s, w^*, \rho)) \mathrm{d}s. \end{aligned} \tag{22}$$

Similarly, define the uninformed agent's equilibrium payoff of this continuation game as $\hat{U}(w^*, \rho)$, that is,

$$\hat{U}(w^*, \rho) := (1 - \rho) \hat{V}(w^*, \rho) + \rho \hat{U}_1(w^*, \rho), \tag{23}$$

where

$$\hat{U}_1(w^*, \rho) = e^{-rw^*} e^{-\beta w^*} (1 + q(w^*, w^*, \rho)) + \int_0^{w^*} e^{-rs} \beta e^{-\beta s} (1 + q(s, w^*, \rho)) \mathrm{d}s.$$

Define the informed agent's equilibrium payoff as a function of starting time t as

$$V^*(t) := \hat{V}(w^*(t), \rho(t)) = \hat{V}(0, \rho(t)), \tag{24}$$

where the equality follows from the informed agent being indifferent over $w \in [0, w^*(t)]$ in the continuation game. Define the uninformed agent's equilibrium payoff as a function of starting time t as

$$U^*(t) := \hat{U}(w^*(t), \rho(t)). \tag{25}$$

The following lemma establishes some useful properties of $\hat{V}$ and $\hat{U}$. The proof is mostly algebraic and is relegated to the Online Appendix.

Lemma 15. *For all feasible parameters,*

$$\frac{\partial \hat{V}}{\partial \rho} > 0, \frac{\partial \hat{U}}{\partial \rho} > 0, \frac{\partial \hat{V}}{\partial w^*} < 0, \text{ and } \frac{\partial \hat{U}}{\partial w^*} < 0.$$

Expected payoffs in the starting game

Suppose $\eta(t) = \mu$ for all $t \geq 0$. Therefore, $w^*(t) = w^*(\rho(t), \mu)$. By Lemma 3, w^* is increasing in ρ which is increasing in t. By the Lebesgue's Theorem, $w^*(t)$ is almost everywhere differentiable.[28] Suppose the agent is uninformed at $t = 0$ and has the opportunity to disclose at $t = 0$. His expected payoff from waiting till t to start disclosing if he remains uninformed at t and starting immediately if he becomes informed is

$$Y(t) = (1-\mu) \int_0^t \lambda e^{-\lambda s} e^{-rs} V^*(s) \mathrm{d}s + \left((1-\mu)e^{-\lambda t} + \mu\right) e^{-rt} U^*(t).$$

The derivative of Y with respect to t is proportional to $y(t)$ where $y(t)$ is defined as

$$y(t) := \lambda(1-\rho(t))(V^*(t) - U^*(t)) + U^{*\prime}(t) - rU^*(t),$$

whenever differentiable. After some simplifying,

$$y(t) = \rho'(t) \left((1-\rho(t)) \frac{\partial \hat{V}}{\partial \rho} + \rho(t) \frac{\partial \hat{U}_1}{\partial \rho} \right) + \left((1-\rho(t)) \frac{\partial \hat{V}}{\partial w^*} + \rho \frac{\partial \hat{U}_1}{\partial w^*} \right) w^{*\prime}(t) - rU^*(t). \tag{26}$$

Similarly, define the informed agent's expected payoff from starting at t as

$$Z(t) = e^{-rt} V^*(t).$$

The derivative is proportional to

$$z(t) = V^{*\prime}(t) - rV^*(t),$$

whenever differentiable.

Suppose the agent got opportunity at time $t_0 > 0$. Let $Y(t|t_0)$ denote the uninformed agent's expected payoff from starting disclosure at $t > t_0$ and $Z(t|t_0)$ the

[28] Differentiability helps simplify notations but plays no role in the proofs.

informed. The following result establishes that the agent's intertemporal incentive is independent of the time at which he got the opportunity. Therefore, it is sufficient to focus the analysis on the agent who got the opportunity to start disclosing at time 0.

Lemma 16. *If $Y(t)$ is increasing (decreasing), then $Y(t|t_0)$ is increasing (decreasing). Similarly, if $Z(t)$ is increasing (decreasing), then $Z(t|t_0)$ is increasing (decreasing).*

Proof. I prove the case for the uninformed agent, the informed agent follows from the same algebra. By definition, $Y(t|t_0)$ is given by

$$\begin{aligned} Y(t|t_0) =&(1-\rho(t_0))\int_{t_0}^{t} \lambda e^{-\lambda(s-t_0)}e^{-r(s-t_0)}V^*(s)\mathrm{d}s \\ &+\left((1-\rho(t_0))e^{-\lambda(t-t_0)}+\rho(t_0)\right)e^{-r(t-t_0)}U^*(t). \end{aligned}$$

Take the derivative with respect to t,

$$\begin{aligned} Y'(t|t_0) =&e^{-r(t-t_0)}\frac{e^{r(t-t_0)}\rho(t)}{(1-\rho(t_0))e^{-\lambda(t-t_0)}+\rho(t_0)} \\ &\cdot\left(\rho'(t)\left(V^*(t)-U^*(t)\right)+\rho(t)\left(U^{*\prime}(t)-rU^*(t)\right)\right). \end{aligned}$$

where the first line is strictly positive and the second line is proportional to $y(t)$. The result follows. $\square$

A.1.8 Proof of Proposition 2

Immediate disclosure is an equilibrium if and only if $Y(t)$ is decreasing in t for all $t \geq 0$ and $Z(t)$ is decreasing in t for all $t \geq 0$.

I derive sufficient conditions under which immediate disclosure is an equilibrium if the waiting time in the continuation game is zero for all disclosure starting time. I then show that under these conditions, immediate disclosure is an equilibrium if the waiting time in the continuation game is the equilibrium waiting time.

Intuitively, if the agent (informed and uninformed) is impatient enough that he does not want to delay starting when waiting time is zero, he does not want to delay starting when waiting time is longer.

Recall that $U^*(t) = \hat{U}(w^*(t), \rho(t))$ and $V^*(t) = \hat{V}(w^*(t), \rho(t))$. Note that $w^*(t) =$

0 for all $t \geq 0$ implies that $q(0,t) = \rho(t)$, therefore,

$$\hat{U}(w^*(t), \rho(t)) = \hat{V}(w^*(t), \rho(t)) = 1 + q(0,t) = 1 + \rho(t).$$

Let $Y_0(t)$ denote the uninformed agent's expected payoff from waiting till t if remains uninformed and starting immediately if becomes informed. Then

$$Y_0(t) = (1-\mu)\int_0^t \lambda e^{-\lambda s} e^{-rs}\left(1+\rho(s)\right) \mathrm{d}s + \left((1-\mu)e^{-\lambda t} + \mu\right) e^{-rt}\left(1+\rho(t)\right).$$

The derivative is proportional to

$$y_0(t) = \rho'(t) - r(1+\rho(t)).$$

Therefore, $Y_0(t)$ is decreasing for all $t \geq 0$ if and only if $y_0(t) \leq 0$ for all $t \geq 0$. This is true if (i) $y_0'(t) \leq 0$ for all $t \geq 0$ and (ii) $y_0(0) \leq 0$. For condition (i), take the derivative of $y_0(t)$ with respect to t, $y_0'(t) \leq 0$ for all t if and only if $r/\lambda \geq 1 - 2\rho(t)$ for all t. Because $\rho(t)$ is increasing in t, it must be that

$$\frac{r}{\lambda} \geq 1 - 2\mu.$$

By assumption $\mu \geq 1/2$, this condition is always satisfied. For condition (ii), set $y_0(0) \leq 0$, then $\lambda(1-\mu)\mu \leq r(1+\mu)$, that is,

$$\frac{r}{\lambda} \geq \frac{(1-\mu)\mu}{1+\mu}. \tag{27}$$

Therefore, $Y_0(t)$ is decreasing in t for all $t \geq 0$ if (8) holds. For the informed agent, $Z_0(t) = e^{-rt}V^*(t) = e^{-rt}(1+\rho(t))$. The same condition (8) implies $Z_0(t)$ is decreasing in t for all $t \geq 0$. The following result shows if $Y_0(t)$ and $Z_0(t)$ are decreasing in t for all $t \geq 0$, $Y(t)$ and $Z(t)$ are decreasing in t for all $t \geq 0$.

Lemma 17. *If $y_0(t) \leq 0$, then $y(t) \leq 0$ and $z(t) \leq 0$.*

Proof. Let $\tilde{y}(w,\rho)$ be

$$\tilde{y}(w,\rho) = \lambda(1-\rho)\rho\left((1-\rho)\frac{\partial \hat{V}(w,\rho)}{\partial \rho} + \rho\frac{\partial \hat{U}_1(w,\rho)}{\partial \rho}\right) - r\hat{U}(w,\rho).$$

Evaluate $\tilde{y}(w, \rho)$ at the optimal $w^*(t)$ and $\rho(t)$ and denote the resulting $\tilde{y}(w^*(t), \rho(t))$ by $\tilde{y}^*(t)$, that is,

$$\begin{aligned}\tilde{y}^*(t) &:= \tilde{y}(w^*(t), \rho(t)) \\ &= \rho'(t)\left((1-\rho(t))\frac{\partial \hat{V}}{\partial \rho}(w^*(t), \rho(t)) + \rho(t)\frac{\partial \hat{U}_1}{\partial \rho}(w^*(t), \rho(t))\right) - r\hat{U}(w^*(t), \rho(t)).\end{aligned}$$

By Lemma 15, $\partial \hat{V}/\partial w^* < 0$, $\partial \hat{U}_1/\partial w^* < 0$; by Lemma 3, w^* is increasing in ρ, which is increasing in t, so $w^*(t)$ is increasing in t. Therefore, $y(t) < \tilde{y}^*(t)$ for all $t \geq 0$. So it suffices to show $\tilde{y}^*(t) < 0$ if $y_0(t) \leq 0$. Recall that $y_0(t) \leq 0$ if and only if $\rho'(t) - r(1 + \rho(t)) \leq 0$. So it reduces to showing the following claim.

Claim 6. Inequality $\rho'(t) - r(1 + \rho(t)) \leq 0$ implies

$$\rho'(t)\left((1-\rho(t))\frac{\partial \hat{V}}{\partial \rho}(w^*(t), \rho(t)) + \rho(t)\frac{\partial \hat{U}_1}{\partial \rho}(w^*(t), \rho(t))\right) - r\hat{U}(w^*(t), \rho(t)) < 0$$

and

$$\rho'(t)\frac{\partial \hat{V}}{\partial \rho}(w^*(t), \rho(t)) - r\hat{V}(w^*(t), \rho(t)) < 0.$$

The proof of the claim is mostly algebraic and is relegated to the Online Appendix. It follows from Claim 6 that $Y_0(t)$ decreasing implies $Y(t)$ decreasing (first inequality), and that $Z_0(t)$ decreasing implies $Z(t)$ decreasing (second inequality). □

A.1.9 Proof of Proposition 3

For a fixed κ, if the proposed strategies under $\underline{\tau}^*(\kappa)$ are (part of) an equilibrium, the following must be true. First, the uninformed agent's expected payoff from starting at t, $Y(t)$, and the informed agent's expected payoff from starting at t, $Z(t)$, must be decreasing for all $t \geq \underline{\tau}^*(\kappa)$. Second, neither the uninformed nor the informed agent wants to deviate to starting before $\underline{\tau}^*(\kappa)$. The goal is to derive sufficient conditions under which there exists $\underline{\tau}^*(\kappa)$ such that both statements hold.

Lemma 18. *Fix parameters such that immediate disclosure is not an equilibrium. There exists $\underline{\tau}^*(\kappa) > 0$ such that $Y(t)$ and $Z(t)$ are both decreasing in t for all $t \geq \underline{\tau}^*(\kappa)$.*

Proof. If immediate disclosure is not an equilibrium, by contrapositive of Proposition 2, condition (8) does not hold. That is, Fix κ such that

$$\kappa < (\lambda/r)(1-\mu)\mu - \mu. \tag{28}$$

In this case, $y_0(t)$ is still (strictly) decreasing in t for all $t \geq 0$: the condition that implies $y_0(t)$ is decreasing is $r/\lambda \geq 1-2\mu$, which always holds by assumption $\mu \geq 1/2$. However, if (8) does not hold, $y_0(0) > 0$. Because $y_0(t)$ is continuous in t, and $\lim_{t\to\infty} y_0(t) = \lim_{t\to\infty} \rho'(t) - r(\kappa + \rho(t)) = -r(\kappa+1) < 0$, there exists a $\tau_0(\kappa)$ where $0 < \tau_0(\kappa) < \infty$ such that $y_0(\tau_0(\kappa)) = 0$ and $y_0(t) < 0$ for all $t > \tau_0(\kappa)$. Moreover, $\tau_0(\kappa)$ is unique because $y_0(t)$ is strictly decreasing in t. We have shown that $y_0(t) \leq 0$ implies $y(t) < 0$. So $y(t) < 0$ for all $t \geq \tau_0(\kappa)$.

If immediate disclosure is not an equilibrium, there must exists some $t \geq 0$ such that $y(t) > 0$. Let $\underline{\tau}^*(\kappa) := \sup\{t : y(t) \geq 0\}$. By the argument above, $\underline{\tau}^*(\kappa) \leq \tau_0(\kappa)$. □

Next, I derive conditions on κ such that neither the uninformed nor the informed agent wants to deviate to starting before $\underline{\tau}^*(\kappa)$.

Lemma 19. *There exists* $0 < \overline{\kappa} \leq (\lambda/r)(1-\mu)\mu - \mu$ *such that for all* $\kappa < \overline{\kappa}$,

$$e^{-r\underline{\tau}^*(\kappa)} V^*(\underline{\tau}^*(\kappa)) \geq \kappa.$$

Proof. Recall that the informed agent is indifferent with respect to waiting times in $[0, w^*]$ in the continuation game. So for any starting time τ, $V^*(\tau) = \kappa + q(0,\tau)$. So if the informed agent does not want to deviate to starting before some τ, it must be that $e^{-r\tau}(\kappa + q(0,\tau)) \geq \kappa$, which is equivalent to $q(0,\tau) \geq \kappa\left(e^{r\tau} - 1\right)$. The goal is to show

$$q(0, \underline{\tau}^*(\kappa)) \geq \kappa\left(e^{r\underline{\tau}^*(\kappa)} - 1\right). \tag{29}$$

Recall the definition of $\tau_0(\kappa)$ from the proof of the previous lemma: $\tau_0(\kappa)$ is such that $y_0(\tau_0(\kappa)) = 0$, that is

$$\lambda(1 - \rho(\tau_0(\kappa)))\rho(\tau_0(\kappa)) - r(\kappa + \rho(\tau_0(\kappa))) = 0.$$

Consider $\kappa = 0$. Because $\rho(\cdot) > \mu > 0$, the above equality implies $\tau_0(0) < \infty$. Then $\underline{\tau}^*(0) \leq \tau_0(0) < \infty$. Because $\kappa\left(e^{r\underline{\tau}^*(\kappa)} - 1\right)$ is continuous in κ, as $\kappa \to 0$,

$\kappa\left(e^{r\underline{\tau}^*(\kappa)}-1\right) \to 0$. Because in equilibrium, stopping disclosure is always on the equilibrium path and with strictly positive probability, the stopping comes from an uninformed agent who got terminated, so $q(0,\underline{\tau}^*(\kappa)) > 0$ for all $\kappa \geq 0$. Therefore, the inequality (29) holds strictly at $\kappa = 0$. Because both $q(0,\underline{\tau}^*(\kappa))$ and the right-hand side of (29) are continuous in κ, there exists a neighborhood of 0 such that $q(0,\underline{\tau}^*(\kappa)) > \kappa\left(e^{r\underline{\tau}^*(\kappa)}-1\right)$.

Consider the solutions to $q(0,\underline{\tau}^*(\kappa)) = \kappa\left(e^{r\underline{\tau}^*(\kappa)}-1\right)$. If there exists solutions to this equation, define $\overline{\kappa}$ to be the smallest solution, so for all $\kappa < \overline{\kappa}$, $q(0,\underline{\tau}^*(\kappa)) > \kappa\left(e^{r\underline{\tau}^*(\kappa)}-1\right)$. If there does not exist a solution, this means for all $\kappa \in [0,(\lambda/r)(1-\mu)\mu-\mu)$, $q(0,\underline{\tau}^*(\kappa)) > \kappa\left(e^{r\underline{\tau}^*(\kappa)}-1\right)$. In this case, define $\overline{\kappa} = (\lambda/r)(1-\mu)\mu-\mu$. The result follows. $\square$

A.2 Proofs for Section 4

A.2.1 Proof of Proposition 4

Because both types of the agent adopt the same starting strategy, the decision maker's belief that $\theta = 1$ if disclosure starts at any time t (that is on the equilibrium path) is μ. By Lemma 3, w^* is increasing in ρ, which is increasing in τ, which means w^* is increasing in τ.

As the disclosure time τ increases, the uninformed agent's belief that $\theta = 1$ at the beginning of the continuation stopping game is $\rho = \lim_{\tau\to\infty}\rho(\tau) = 1$; the decision maker's belief that $\theta = 1$ is $\eta = \mu$. One can obtain the upper bound $\overline{w}$ by solving the boundary value problem (BVP) evaluated at $\rho = 1$.

A.3 Proofs for Section 5

A.3.1 Proof of Proposition 5

The proof is the same as the proof of Theorem 1.B (ii) as is thus omitted.

A.3.2 Proof of Proposition 6

Let $q_D(w)$ denote the decision maker's belief that $\theta = 1$ conditional on no signals arriving in $[0, w]$. By Bayes' rule,

$$q_D(w) = \frac{\eta}{\eta + e^{-\lambda w}(1-\eta)}. \tag{30}$$

Given this belief $q_D(w)$, from the perspective of the beginning of the continuation stopping game, denote the decision maker's expected payoff from stopping at w by $D(w)$, then

$$
\begin{aligned}
D(w) =&(1-\eta)\int_0^w e^{-rs}\lambda e^{-\lambda s}e^{-\beta s}\mathrm{d}s \\
&+(1-\eta)\int_0^w e^{-rs}e^{-\lambda s}\beta e^{-\beta s}\left(1-(q_D(s)-0)^2\right)\mathrm{d}s \\
&+(1-\eta)e^{-rw}e^{-\lambda w}e^{-\beta w}\left(1-(q_D(w)-0)^2\right) \\
&+\eta\int_0^w e^{-rs}\beta e^{-\beta s}\left(1-(q_D(s)-1)^2\right)\mathrm{d}s \\
&+\eta e^{-rw}e^{-\beta w}\left(\left(1-(q_D(w)-1)^2\right)\right).
\end{aligned} \tag{31}
$$

Take the derivative of $D(w)$ with respect to w. After simplifying,

$$
D'(w)\propto \lambda(1-q_D(w))q_D(w)^2 - r\left(1-(1-q_D(w))q_D(w)\right). \tag{32}
$$

Define

$$
R(w;\eta) := \left(\frac{e^{-\lambda w}(1-\eta)+\eta}{e^{-\lambda w}(1-\eta)} + \left(\frac{\eta+e^{-\lambda w}(1-\eta)}{\eta}\right)^2\right)^{-1}. \tag{33}
$$

Rearrange the right-hand side of (32) and substitute (30) for $q_D(w)$, $D'(w)\le 0$ if and only if $R(w;\eta)\le r/\lambda$.

Claim 7. For any w and η, $\partial R(w;\eta)/\partial w$ and $\partial R(w;\eta)/\partial\eta$ have the same sign.

Proof. To facilitate the proof, note that $q_D(w)$ depends on both w and η. With a slight abuse of notation, denote (30) by $q_D(w;\eta)$. Then $R(w;\eta)$, defined in (34), can be written as

$$
R(w;\eta) = \left(\frac{1}{1-q_D(w;\eta)} + \frac{1}{q_D(w;\eta)^2}\right)^{-1}. \tag{34}
$$

Then

$$
\frac{\partial R(w;\eta)}{\partial w} = \frac{\partial R(w;\eta)}{\partial q_D}\frac{\partial q_D(w;\eta)}{\partial w} \text{ and } \frac{\partial R(w;\eta)}{\partial \eta} = \frac{\partial R(w;\eta)}{\partial q_D}\frac{\partial q_D(w;\eta)}{\partial \eta}.
$$

By (30), $\partial q_D(w;\eta)/\partial w>0$ and $\partial q_D(w;\eta)/\partial\eta>0$. The result follows. ☐

Claim 8. $R(w;\eta)$ has the following properties.
(i) For all $0<\eta<1$, $R(w;\eta)$ is single-peaked in w: there exists a unique $w_R^*(\eta)=\arg\max_w R(w;\eta)$ with $w_R^*(\eta)\geq 0$ such that $R(w;\eta)$ is strictly increasing in w for $w\leq w_R^*(\eta)$ and strictly decreasing in w for $w>w_R^*(\eta)$.
(ii) There exists a unique $0<\eta_R<1$ such that

(a) for all $\eta<\eta_R$, $w_R^*(\eta)>0$, $w_R^{*\prime}(\eta)<0$, and $R(w_R^*(\eta);\eta)$ is constant in η;

(b) for all $\eta\geq\eta_R$, $w_R^*(\eta)=0$ and $R(w_R^*(\eta);\eta)=R(0;\eta)$ is decreasing in η;

(c) Moreover, $w_R^*(\eta)$ is continuous for all $0<\eta<1$. That is, $\lim_{\eta\to\eta_R} w_R^*(\eta)\to 0$.

(iii) For all $0<\eta<1$, $\lim_{w\to\infty} R(w;\eta)\to 0$.

Proof. These properties follow directly from operating on (33).

Specifically, for (ii) (a) and (c), if $w_R^*(\eta)$ is interior, namely, $w_R^*(\eta)>0$, $w_R^*(\eta)$ has a closed form which can be obtained by solving $\partial R(w;\eta)/\partial w=0$. It follows from the functional form of $w_R^*(\eta)$ that $w_R^{*\prime}(\eta)<0$ and $\lim_{\eta\to\eta_R} w_R^*(\eta)\to 0$. By definition,

$$\frac{\mathrm{d}R(w_R^*(\eta);\eta)}{\mathrm{d}\eta}=\frac{\partial R(w_R^*(\eta);\eta)}{\partial\eta}+\frac{\partial R(w_R^*(\eta);\eta)}{\partial w}\frac{\mathrm{d}w_R^*(\eta)}{\mathrm{d}\eta}.$$

The second term on the right-hand side is zero because $\partial R(w_R^*(\eta);\eta)/\partial w=0$. By Claim 7, the first term $\partial R(w_R^*(\eta);\eta)/\partial\eta=0$. Therefore, $\mathrm{d}R(w_R^*(\eta);\eta)/\mathrm{d}\eta=0$, which means $R(w_R^*(\eta);\eta)$ is constant in η.

For (ii) (c), it follows from the definition (33) that $R(0;\eta)=\left(\frac{1}{1-\eta}+\frac{1}{\eta^2}\right)^{-1}$ and is decreasing in η. ☐

Part (i). It follows from Claim 8 (ii) that $R(0;\eta_R)=\max_{\eta\in(0,1)} R(w_R^*(\eta);\eta)$. Denote this maximum value by

$$\Delta:=R(0;\eta_R).$$

Then for all $w\geq 0$ and $0<\eta<1$, $R(w;\eta)\leq R(0;\eta_R)=\Delta$. This means if $\Delta\leq r/\lambda$, $R(w;\eta)\leq\Delta$, which means $D'(w)<0$. (In fact, for any parameters, Δ is a constant number is approximately 0.1916.)

Part (ii). Fix $r/\lambda<\Delta$. The shape of $D(w)$ depends on η. There are three cases.

Case 1. By Claim 8, for all $\eta\geq\eta_R$, $R(w;\eta)$ is decreasing in w for all $w\geq 0$ and is thus maximized at $w=0$. Because $R(0;\eta)$ is decreasing in η, there exists $\overline{\eta}>\eta_R$

such that $R(0;\overline{\eta}) = r/\lambda$ and $R(0;\eta) < r/\lambda$ (and therefore $R(w;\eta) < r/\lambda$) for all $\eta > \overline{\eta}$. This implies $D'(w) \le 0$ for all $w \ge 0$ and is thus maximized at $w = 0$ for all $\eta \ge \overline{\eta}$.

Case 2. By Claim 8, for all $\eta < \eta_R$, $R(w;\eta)$ is single-peaked in w and is maximized at $w_R^*(\eta) > 0$. Because $R(0;\eta)$ is decreasing in η, there exists $\tilde{\eta} < \eta_R$ such that there exists a unique $w_{DM}^*(\eta)$ where $R(w_{DM}^*(\eta);\eta) = r/\lambda$, $R(w;\eta) < r/\lambda$ for $w < w_{DM}^*(\eta)$, and $R(w;\eta) > r/\lambda$ for $w > w_{DM}^*(\eta)$. This implies $D(w)$ single-peaked in w is uniquely maximized at $w = w_{DM}^*(\eta) > 0$ for $\eta \in [\tilde{\eta}, \overline{\eta})$.

Case 3. For all $\eta < \tilde{\eta}$, $R(0;\eta) < r/\lambda$ and $R(w;\eta)$ intersects with the line r/λ twice. Denote these two intersections by $w_L(\eta)$ and $w_H(\eta)$ with $w_L(\eta) < w_H(\eta)$. That is, $R(w_L(\eta);\eta) = R(w_H(\eta);\eta) = r/\lambda$.

Moreover, $R(w;\eta) < r/\lambda$ for $w < w_L(\eta)$ and $w \ge w_H(\eta)$, and $R(w;\eta) > r/\lambda$ for $w \in [w_L(\eta), w_H(\eta))$. This means that $D(w)$ decreases for $w \in [0, w_L(\eta))$, increases for $w \in [w_L(\eta), w_H(\eta))$, and decreases for $w \ge w_H(\eta)$. Therefore, for each $\eta < \tilde{\eta}$, $D(w)$ has two local maxima, one at $w = 0$ and one at $w = w_H(\eta)$. To determine which is the global maximizer, one can obtain $D(w_H(\eta))$ by plugging in the closed form expression for $w_H(\eta)$, and compare it with $D(0) = 1 - (1-\eta)\eta$. It can be shown that there exists $\underline{\eta} < \tilde{\eta}$ such that $D(w_H(\eta)) < 1 - (1-\eta)\eta$ for all $\eta < \underline{\eta}$, and $D(w_H(\eta)) \ge 1 - (1-\eta)\eta$ for all $\eta \in [\underline{\eta}, \tilde{\eta})$. Then the optimal waiting time $w_{DM}^*(\eta) = 0$ for $\eta < \underline{\eta}$ and $w_{DM}^*(\eta) = w_H(\eta)$ for $\eta \ge \underline{\eta}$.

This concludes the proof. As an illustration, Figure 7 plots the function $D(w)$ for several values of η. The resulting optimal waiting time $w_{DM}^*(\eta)$ is given by Figure 6.

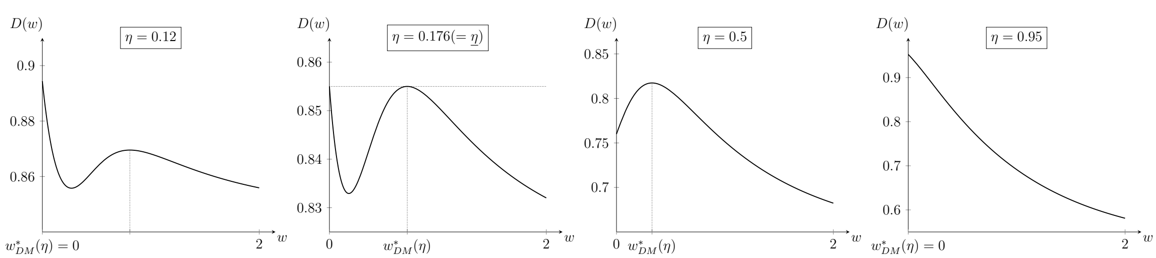

Figure 7: Decision maker's expected payoff from stopping at w $D(w)$ for $r = 0.5, \lambda = 5$, and $\beta = 0.5$. From left to right, the values of η are respectively $\eta = 0.12, 0.176, 0.5, 0.95$.

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