Online Appendix to "Dynamic Coordination with Informational Externalities"

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OA.1 Omitted Proofs for Section 3

OA.1.1 Proof of Lemma 6

I show the first inequality, which is equivalent to $f^0(z)(F^1(\hat{z})-F^1(z)) > f^1(z)(F^0(\hat{z})-F^0(z))$ if and only if $\hat{z} > z$. The proof of the second inequality is analogous.

First note that if $\hat{z} = z$, then $f^0(z)(F^1(\hat{z}) - F^1(z)) = 0 = f^1(z)(F^0(\hat{z}) - F^0(z))$. For a fixed z, $f^0(z)(F^1(\hat{z}) - F^1(z))$ is increasing in $\hat{z}$ with derivative $f^0(z)f^1(\hat{z})$; $f^1(z)(F^0(\hat{z}) - F^0(z))$ is increasing in $\hat{z}$ with derivative $f^1(z)f^0(\hat{z})$. By MLRP, $\hat{z} > z$ if and only if $f^1(z)f^0(\hat{z}) < f^0(z)f^1(\hat{z})$, which implies $f^0(z)(F^1(\hat{z}) - F^1(z)) > f^1(z)(F^0(\hat{z}) - F^0(z))$ for $\hat{z} > z$.

OA.1.2 Proof of Lemma 7

Leader x's expected payoff from stopping at t^* is

$$\begin{aligned}\mathcal{L}(x,t^*) =& \lim_{\varepsilon\to 0}\left(q_L(x)\int_0^{t^*-\varepsilon} e^{-r\tau}\mathrm{d}G_F^1(\tau)H - (1-q_L(x))\int_0^{t^*-\varepsilon} e^{-r\tau}\mathrm{d}G_F^0(\tau)L\right).\\ &+\lim_{\varepsilon\to 0}\left(q_L(x)(G_F^1(t^*)-G_F^1(t^*-\varepsilon)) + (1-q_L(x))(G_F^0(t^*)-G_F^0(t^*-\varepsilon)\right)\cdot 0\\ =& \lim_{\varepsilon\to 0} q_L(x)\left(\int_0^{t^*-\varepsilon} e^{-r\tau}\mathrm{d}G_F^1(\tau)H + \int_0^{t^*-\varepsilon} e^{-r\tau}\mathrm{d}G_F^0(\tau)L\right) - \int_0^{t^*-\varepsilon} e^{-r\tau}\mathrm{d}G_F^0(\tau)L.\end{aligned}$$

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Follower y's expected payoff from stopping at t^* is

$$\mathcal{F}(y,t^*) =e^{-rt^*}\Bigg(\left(q_F(y)\left((1-G_L^1(t^*))H+(1-G_L^0(t^*))L\right)-(1-G_L^0(t^*))L\right) \\ +\lim_{\varepsilon\to 0}\left(q_F(y)(1-G_L^1(t^*-\varepsilon))+(1-q_F(y))(1-G_L^0(t^*-\varepsilon))\right)(-c)\Bigg).$$

I show the leader's expected payoff is supermodular and the follower's is submodular.

Denote $\Delta\mathcal{L}(x,t,t')=\mathcal{L}(x,t')-\mathcal{L}(x,t)$. For $t'>t$ and $x'>x$,

$$\begin{aligned}&\Delta\mathcal{L}(x',t,t')-\Delta\mathcal{L}(x,t,t')\\ =&\lim_{\varepsilon\to 0}(q_L(x')-q_L(x))\left(\int_{t-\varepsilon}^{t'-\varepsilon}e^{-r\tau}\mathrm{d}G_F^1(\tau)H+\int_{t-\varepsilon}^{t'-\varepsilon}e^{-r\tau}\mathrm{d}G_F^0(\tau)L\right).\end{aligned}$$

By MLRP, $q_L(x')-q_L(x)>0$. For $t'>t$, $G_F^\theta(t')\geq G_F^\theta(t)$. So $\Delta\mathcal{L}(x',t,t')-\Delta\mathcal{L}(x,t,t')>0$. Therefore, $\mathcal{L}(x,t)$ is supermodular in (x,t). By Topkis's theorem, $\sigma_L(x)=\arg\max_{t\geq 0}\mathcal{L}(x,t)$ is non-decreasing in x.

Denote $\Delta\mathcal{F}(y,t,t')=\mathcal{F}(y,t')-\mathcal{F}(y,t)$. For $t'>t$ and $y'>y$,

$$\begin{aligned}&\Delta\mathcal{F}(y',t,t')-\Delta\mathcal{F}(y,t,t')\\ =&(q_F(y')-q_F(y))\left(e^{-rt'}(1-G_L^1(t'))-e^{-rt}(1-G_L^1(t))\right)H\\ &-(q_F(y')-q_F(y))\left(e^{-rt}(1-G_L^0(t))-e^{-rt'}(1-G_L^0(t'))\right)L\\ &-\lim_{\varepsilon\to 0}c\Bigg(e^{-r(t'-\varepsilon)}(q_F(y')-q_F(y))\left((1-G_L^1(t'-\varepsilon))+(1-G_L^0(t'-\varepsilon))\right)\\ &\qquad+e^{-r(t-\varepsilon)}(q_F(y')-q_F(y))\left((1-G_L^1(t-\varepsilon))+(1-G_L^0(t-\varepsilon))\right)\Bigg).\end{aligned}$$

By MLRP, $q_F(y')-q_F(y)>0$. For $t'>t$, $e^{-rt'}(1-G_L^\theta(t'))<e^{-rt}(1-G_L^\theta(t'))\leq e^{-rt}(1-G_L^\theta(t))$. So $\Delta\mathcal{F}(y',t,t')-\Delta\mathcal{F}(y,t,t')<0$. Therefore, $\mathcal{F}(y,t)$ is submodular in (y,t). By Topkis's theorem, $\sigma_F(y)=\arg\max_{t\geq 0}\mathcal{F}(y,t)$ is non-increasing in y.

Follower's equilibrium distribution of stopping time is non-atomic at T

Suppose there is an atom at $t=T$ in the follower's equilibrium distribution of stopping time. By a similar argument to Claim 1, there exists a unique $\bar{x}\in(x(0),1)$ such that $B(\bar{x},T)=0$. So for all $x\in(\bar{x},1)$, $B(x,T)>0$. By a similar argument to Claim 2,

$\sup_{t\in(T-\varepsilon,T+\varepsilon)} \mathcal{L}(x,t) = \mathcal{L}(x,T) + A(x,T)$ with $A(x,T) > 0$. Because the probability of the follower investing at any $t \geq T$ is zero, the change in leader x's expected payoff at and after T is zero. This implies that the leader's expected payoff from disinvesting at any $t \geq T$ is constant at $\mathcal{L}(x,T) + A(x,T)$. Thus, the leader is indifferent between disinvesting at any $t \in (T, \infty]$. By the (indifference) tie-breaking rule, x disinvests at the infimum of the set $(T, \infty]$, which is T. However, if x disinvests at T, he gets $\mathcal{L}(x,T) < \mathcal{L}(x,T) + A(x,T)$. Thus, for all $x \in (\bar{x}, 1)$, x does not have a best response.

Leader's equilibrium distribution of stopping time is non-atomic at T

Suppose there is an atom at T. By the definition of T, it must be that all remaining leader types disinvest. That is, for $\theta = 0, 1$, $\Pr(\sigma_L(x) > T|\sigma_L(x) > 0, \theta) = 0$. So $\Pr(\sigma_L(x) \geq T|\sigma_L(x) > 0, \theta) = \Pr(\sigma_L(x) = T|\sigma_L(x) > 0, \theta)$, which means

$$\mathcal{F}_{-}(y,T) = e^{-rT} \left(A(y,T) - C(y,T)\right),$$

$$\mathcal{F}(y,T) = \mathcal{F}_{-}(y,T) - e^{-rT}A(y,T) = -e^{-rT}C(y,T) < 0,$$

$\mathcal{F}_{+}(y,T) < 0$, and $\mathcal{F}(y,t) < 0$ for all $t > T$. $A(y,T) - C(y,T)$ is strictly increasing in y and there exists a unique $\bar{y} \in (0, y(0))$ such that $A(\bar{y},T) - C(\bar{y},T) = 0$.

Case (i) Suppose $\mathcal{F}(\bar{y},t)$ attains a maximum at $t^* \in [0,T]$. Because $\sigma_F(y)$ is non-increasing, if $\bar{y}$ invests at t^*, then all $y > \bar{y}$ will invest (at or) before t^* and all $y < \bar{y}$ will invest (at or) after t^*. So the only types who might invest in $(t^*, T]$ are $y < \bar{y}$. For $y < \bar{y}$, $\mathcal{F}_{-}(y,T) = A(y,T) - C(y,T) < 0$ and $\mathcal{F}(y,t) < 0$ for all $t \geq T$, so if investing at $t^{**} \in (t^*, T]$ is optimal, it must be that $\mathcal{F}(y, t^{**}) > 0$ and $t^{**} < T$. So there exist a $\delta > 0$ small such that $t^{**} \notin [T-\delta, T]$. This implies there does not exist a y such that $\sigma_F(y) \in [T-\delta, T]$. The leader is indifferent between stopping at $T-\delta$ and T. By the (indifference) tie-breaking rule, the leader stops at $T-\delta$, which contradicts the hypothesis that there is a mass of leader types stopping at T.

Case (ii) Suppose $\mathcal{F}(\bar{y},t)$ does not attain a maximum in $[0,T]$. Fix $y > \bar{y}$. Then $A(y,T) - C(y,T) > 0$ so $\mathcal{F}_{-}(y,T) > 0 > \mathcal{F}(y,T)$. There are two sub-cases.

First, all $\mathcal{F}(y,t)$ with $y > \bar{y}$ up to sets of measure zero attains a maximum in $[0,T]$. If $\mathcal{F}$ attains a maximum at some $t^* \in [0,T]$, it must be that $\mathcal{F}(y,t^*) \geq \mathcal{F}_{-}(y,T) > 0$ and $t^* < T$. So there exist a $\delta > 0$ small such that $t^* \notin [T-\delta, T]$, a contradiction.

Second, there exist a positive measure of types $y > \bar{y}$ such that $\mathcal{F}(y,t)$ does not attain a maximum in $[0,T]$. $\mathcal{F}_{-}(y,T)$ is the supremum of $\mathcal{F}(y,t)$ for all t. However, this supremum cannot be achieved because $\mathcal{F}(y,T) = -e^{-rT}C(y,T) < 0 < \mathcal{F}_{-}(y,T)$. So a positive measure of follower types does not have a best response, a contradiction.

Combining case (i) and case (ii), there cannot exist an atom in the leader's equilibrium distribution of stopping time at T.

OA.2 Omitted Proofs for Section 4

OA.2.1 Proof of Lemma 11

First, I establish a useful equality: it follows directly from (6) that for all $\mu \in (0,1)$,

$$\frac{f^0(\mu)}{\hat{f}^0(\mu)} = \frac{f^1(\mu)}{\hat{f}^1(\mu)}. \tag{OA.1}$$

Define the survival rate conditional on θ, and the probability rate conditional on θ as

$$Q^\theta(\mu) := \frac{1-F^\theta(\mu)}{1-\hat{F}^\theta(\mu)},\ P^\theta(\mu) := \frac{F^\theta(\mu)}{\hat{F}^\theta(\mu)}.$$

By (OA.1), showing $h_F(\mu) > h_{\hat{F}}(\mu)$ is equivalent to showing Corollary 3

$$\frac{1-F^1(\mu)}{1-\hat{F}^1(\mu)} < \frac{1-F^0(\mu)}{1-\hat{F}^0(\mu)}, \tag{OA.2}$$

that is, $Q^1(\mu) < Q^0(\mu)$. Figure OA.1 provides an illustration of this inequality.

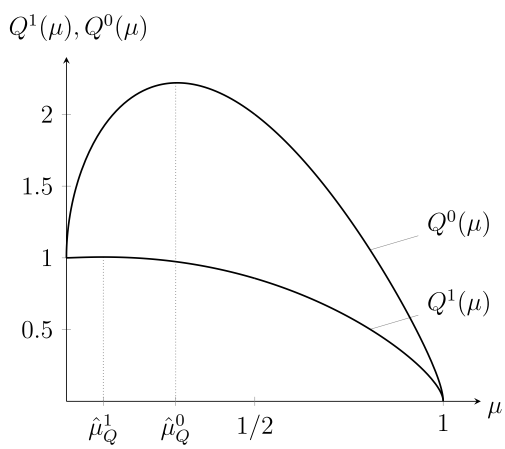

Figure OA.1: Illustration for (OA.2) with F and $\hat{F}$ induced by signals distributed according to $Beta(1+\theta, 1+(1-\theta))$ and $Beta(1+2\theta, 1+2(1-\theta))$ respectively.

Consider the slope of $Q^\theta(\mu)$:

$$\frac{\mathrm{d}Q^1}{\mathrm{d}\mu} = \frac{f^1(\mu)}{1-\hat{F}^1(\mu)}\left(\frac{1-F^1(\mu)}{1-\hat{F}^1(\mu)} - \frac{f^1(\mu)}{\hat{f}^1(\mu)}\right), \frac{\mathrm{d}Q^0}{\mathrm{d}\mu} = \frac{f^0(\mu)}{1-\hat{F}^0(\mu)}\left(\frac{1-F^0(\mu)}{1-\hat{F}^0(\mu)} - \frac{f^0(\mu)}{\hat{f}^0(\mu)}\right).$$

By Proposition 2 in Hopkins and Kornienko (2007), if $f^\theta(\mu)/\hat{f}^\theta(\mu)$ is unimodal with maximum at $1/2$, then $Q^\theta(\mu)$ is unimodal with maximum achieved at $\hat{\mu}_Q^\theta < 1/2$.

First, consider $\mu \geq \max\{\hat{\mu}_Q^1, \hat{\mu}_Q^0\}$. Both $Q^1(\mu)$ and $Q^0(\mu)$ are decreasing. Suppose the contrary $Q^0(\mu) \leq Q^1(\mu)$. By (OA.1),

$$\frac{1-F^0(\mu)}{1-\hat{F}^0(\mu)} - \frac{f^0(\mu)}{\hat{f}^0(\mu)} \leq \frac{1-F^1(\mu)}{1-\hat{F}^1(\mu)} - \frac{f^1(\mu)}{\hat{f}^1(\mu)} < 0,$$

By MLRP, $\hat{f}^0(\mu)/(1-\hat{F}^0(\mu)) > \hat{f}^1(\mu)/(1-\hat{F}^1(\mu))$, so $\mathrm{d}Q^0/\mathrm{d}\mu < \mathrm{d}Q^1/\mathrm{d}\mu < 0$. This means that if $Q^0(\mu) \leq Q^1(\mu)$, then $Q^0(\mu)$ must decrease faster than $Q^1(\mu)$. This is a contradiction because $\lim_{\mu\to 1} Q^1(\mu) = \lim_{\mu \to 1} Q^0(\mu)$ as

$$\lim_{\mu\to 1}\frac{1-F^1(\mu)}{1-\hat{F}^1(\mu)} = \lim_{\mu\to 1}\frac{f^1(\mu)}{\hat{f}^1(\mu)} = \lim_{\mu\to 1}\frac{f^0(\mu)}{\hat{f}^0(\mu)} = \lim_{\mu\to 1}\frac{1-F^0(\mu)}{1-\hat{F}^0(\mu)}.$$

Because $\hat{\mu}_Q^1, \hat{\mu}_Q^0 < 1/2$ and $f^1(\mu)/\hat{f}^1(\mu)$ is increasing for $\mu \leq 1/2$, $\hat{\mu}_Q^1 < \hat{\mu}_Q^0$.

Next, consider $\mu \in (\hat{\mu}_Q^1, \hat{\mu}_Q^0)$. Q^0 is increasing and Q^1 is decreasing, which means $Q^1(\mu) < f^1(\mu)/\hat{f}^1(\mu) = f^0(\mu)/\hat{f}^0(\mu) < Q^0(\mu)$.

Lastly, consider $\mu \leq \hat{\mu}_Q^1$. $\lim_{\mu\to 0} Q^1(\mu) = \lim_{\mu\to 0} Q^0(\mu) = 1$ and $\lim_{\mu\to 0} \mathrm{d}Q^1/\mathrm{d}\mu < \lim_{\mu\to 0} \mathrm{d}Q^0/\mathrm{d}\mu$. Suppose there exists $\tilde{\mu}$ such that $Q^1(\tilde{\mu}) = Q^0(\tilde{\mu})$. Then at $\tilde{\mu}$, $\mathrm{d}Q^1/\mathrm{d}\mu < \mathrm{d}Q^0/\mathrm{d}\mu$. This is a contradiction. Because $Q^1(\mu)$ and $Q^0(\mu)$ are increasing, they start at the same value, and $Q^0(\mu)$ increases faster than $Q^1(\mu)$ at the beginning. So if they were to cross, $Q^1(\mu)$ must cross $Q^0(\mu)$ from below which means it must be $\mathrm{d}Q^1/\mathrm{d}\mu > \mathrm{d}Q^0/\mathrm{d}\mu$. Therefore $Q^1(\mu) < Q^0(\mu)$ for all $\mu \leq \hat{\mu}_Q^1$.

OA.2.2 Proof of Lemma 12

$F \succ_{\mathrm{ULR}} \hat{F}$ implies $f/\hat{f}$ is unimodal. The likelihood ratio of F and $(1-\lambda)F + \lambda\hat{F}$ is $f/((1-\lambda)f + \lambda\hat{f})$ and the likelihood ratio of $(1-\lambda)F+\lambda\hat{F}$ and $\hat{F}$ is $((1-\lambda)f+\lambda\hat{f})/\hat{f}$. Both are unimodal as implied by that $f/\hat{f}$ is unimodal.

$F \succ_{\mathrm{ULR}} \hat{F}$ implies the mean of F is greater than or equal to the mean of $\hat{F}$. So the mean of F is greater than or equal to the mean of $(1-\lambda)F + \lambda\hat{F}$, which is greater than or equal to the mean of $\hat{F}$. The result follows.

OA.2.3 Proof of Claim 8

The proof is mostly algebraic. The idea is to write $\mathcal{V}$ as a function of h so the condition reduces to terms that are easier to compare. For conciseness, I omit the argument of the functions from now on. After some rearranging, $\mathcal{V}$ can be written as

$$\mathcal{V} = \underbrace{q\left(1 - \frac{1-\mu}{\mu}\right)}_{=:b} \underbrace{-q\left(1 - \frac{1-\mu}{\mu}\right)\left(\frac{1-\mu}{\mu}\frac{1-F^1}{F^1}\right)}_{=:a} h.$$

So $\mathcal{V} = ah + b$. Let the subscript of the function denote the partial derivative the function is taken with respect to. Then

$$\frac{\partial h/\partial \lambda}{\partial h/\partial \mu} - \frac{\partial \mathcal{V}/\partial \lambda}{\partial \mathcal{V}/\partial \mu} = \frac{h_\lambda}{h_\mu}\left(\frac{a_\mu h + b_\mu}{\partial \mathcal{V}/\partial \mu}\right) - \frac{a_\lambda h + b_\lambda}{\partial \mathcal{V}/\partial \mu}.$$

Because $\partial \mathcal{V}/\partial \mu > 0$ (and also $a_\mu h + b_\mu > 0$), to show the above expression is negative, it is equivalent to showing

$$\frac{h_\lambda}{h_\mu} < \frac{a_\lambda h + b_\lambda}{a_\mu h + b_\mu}.$$

This follows from the following chain of inequality: for all $\mu \geq 1/2$,

$$\frac{h_\lambda}{h_\mu} < \frac{q_\lambda}{q_\mu} < \frac{a_\lambda h + b_\lambda}{a_\mu h + b_\mu}.$$

I now prove this chain of inequality holds.

For the first inequality $h_\lambda/h_\mu < q_\lambda/q_\mu$, write q as a function of h,

$$q = \frac{1}{1 + \underbrace{\frac{1-\mu}{\mu}\frac{1}{F^1}}_{=:c} \underbrace{-\left(\frac{1-\mu}{\mu}\right)^2 \frac{1-F^1}{V^1}}_{=:d} h}.$$

That is, $q = 1/(1 + c + dh)$. Then

$$\frac{h_\lambda}{h_\mu} - \frac{q_\lambda}{q_\mu} = \frac{h_\lambda}{h_\mu}\left(1 - \frac{h_\mu d}{q_\mu}\right) - \frac{c_\lambda + d_\lambda h}{q_\mu}.$$

Because $h_\lambda < 0$, $h_\mu > 0$, $q_\mu > 0$, $d < 0$, so $1 - h_\mu d/q_\mu > 0$. So the first term is negative. For the second term, by definition, $c = ((1-\mu)/\mu)(1/F^1)$, and $d = -c(1-\mu)/\mu + ((1-\mu)/\mu)^2$. Because $(1-F^0)/(1-F^1) < 1$, and $c_\lambda > 0$, so $d_\lambda h = -c_\lambda(1-F^0)/(1-F^1) > -c_\lambda$, which implies $(c_\lambda + d_\lambda h)/q_\mu > 0$.

For the second inequality $q_\lambda/q_\mu < (a_\lambda h + b_\lambda)/(a_\mu h + b_\mu)$, the right-hand side is

$$\frac{q_\lambda \overbrace{\left(2-\frac{1}{\mu}\right)\left(1-\frac{1-F^0}{F^1}\right)}^{=:\alpha} \overbrace{-\left(\frac{1-F^1}{F^1}\right)_\lambda \frac{1-\mu}{\mu} bh}^{=:\beta}}{q_\mu \underbrace{\left(2-\frac{1}{\mu}\right)\left(1-\frac{1-F^0}{F^1}\right)}_{=:\alpha} + \underbrace{\left(2-\frac{1}{\mu}\right)_\mu q\left(1-\frac{1-F^0}{F^1}\right) - \left(\frac{1-\mu}{\mu}\frac{1-F^1}{F^1}\right)_\mu bh}_{=:\eta}},$$

so

$$\frac{q_\lambda}{q_\mu} - \frac{a_\lambda h + b_\lambda}{a_\mu h + b_\mu} = \frac{q_\lambda}{q_\mu}\frac{\eta}{q_\mu \alpha + \eta} - \frac{\beta}{q_\mu \alpha + \eta}.$$

I want to show this is negative. Because $q_\mu \alpha + \eta > 0$, this is equivalent to showing

$$\frac{q_\mu}{q_\lambda} - \frac{\eta}{\beta} > 0.$$

Writing out all the terms, the left-hand side is equal to

$$\underbrace{\frac{\left(\frac{1-\mu}{\mu}\right)_\mu}{\frac{1-\mu}{\mu}\left(\frac{F^0}{F^1}\right)_\lambda \left(\frac{1-F^1}{F^1}\right)_\lambda \left(2-\frac{1}{\mu}\right)h}}_{<0}$$
$$\cdot \underbrace{\left(\frac{F^0}{F^1}\left(\frac{1-F^1}{F^1}\right)_\lambda \left(2-\frac{1}{\mu}\right)h - \left(\frac{F^0}{F^1}\right)_\lambda \left(1-\frac{1-F^0}{F^1}\right) - \left(\frac{F^0}{F^1}\right)_\lambda \frac{1-F^1}{F^1}\left(2-\frac{1}{\mu}\right)h\right)}_{<0}$$
$$+\underbrace{\frac{\frac{1-\mu}{\mu}}{\frac{1-\mu}{\mu}\left(\frac{F^0}{F^1}\right)_\lambda \left(\frac{1-F^1}{F^1}\right)_\lambda \left(2-\frac{1}{\mu}\right)h}}_{>0}$$
$$\cdot \underbrace{\left(\left(\frac{F^0}{F^1}\right)_\mu \left(\frac{1-F^1}{F^1}\right)_\lambda \left(2-\frac{1}{\mu}\right)h - \left(\frac{F^0}{F^1}\right)_\lambda \left(\frac{1-F^1}{F^1}\right)_\mu \left(2-\frac{1}{\mu}\right)h\right)}_{>0}.$$

The inequalities of each of the four terms follow from $(F^0/F^1)_\lambda > 0$ and $((1 - F^1)/F^1)_\lambda > 0$ (both are implied by the ULR order), MLRP, IHRP and symmetry.

OA.3 Omitted Proofs for Section 5

OA.3.1 Proof of Claim 10

(i), (ii), and (iii) of the claim directly follow from taking the partial derivative and applying MLRP. I prove (iv). Rewrite $W_0(x, y)$,

$$\begin{aligned} W_0(x,y) = & \Pr(s_{-i} \in [y,x) | s_i = x, s_{-i} \leq x) \\ & \cdot \left[\Pr(\theta = 1 | s_i = x, s_{-i} \in [y,x]) H - \Pr(\theta = 0 | s_i = x, s_{-i} \in [y,x]) L \right]. \end{aligned}$$

In what follows, I show both $\Pr(\theta = 1|s_i = x, s_{-i} \in [y, x])$ and $\Pr(s_{-i} \in [y, x)|s_i = x, s_{-i} \leq x)$ are increasing in x for $x > y$. Given these two probabilities are increasing in x, $W_0(x, y)$ is increasing in x if $W_0(x, y) > 0$.

First, it follows from MLRP and Lemma 6 that for $x > y$,

$$\Pr(\theta = 1|s_i = x, s_{-i} \in [y,x]) = \frac{\rho_0 f^1(x)(F^1(x) - F^1(y))}{\rho_0 f^1(x)(F^1(x) - F^1(y)) + (1 - \rho_0) f^0(x)(F^0(x) - F^0(y))}$$

is increasing in x. I show $\Pr(s_{-i} \in [y, x)|s_i = x, s_{-i} \leq x)$ is increasing in x for $x > y$. By the law of total probability,

$$\begin{aligned} & \Pr(s_{-i} \in [y,x) | s_i = x, s_{-i} \leq x) \\ & = \Pr(s_{-i} \in [y,x) | s_i = x, s_{-i} \leq x, \theta = 1) \Pr(\theta = 1 | s_i = x, s_{-i} \leq x) \\ & \quad + \Pr(s_{-i} \in [y,x) | s_i = x, s_{-i} \leq x, \theta = 0) \Pr(\theta = 0 | s_i = x, s_{-i} \leq x), \end{aligned} \tag{OA.3}$$

where for $\theta = 0, 1$,

$$\Pr(s_{-i} \in [y,x) | s_i = x, s_{-i} \leq x, \theta) = \frac{\Pr(s_{-i} \in [y,x) | \theta)}{\Pr(s_{-i} \leq x | \theta)} = 1 - \frac{F^\theta(y)}{F^\theta(x)}.$$

Denote $q(x) = \Pr(\theta = 1|s_i = x, s_{-i} \leq x)$, (OA.3) can be written as

$$\Pr(s_{-i} \in [y,x) | x, s_{-i} \leq x) = \left(1 - \frac{F^1(y)}{F^1(x)}\right) q(x) + \left(1 - \frac{F^0(y)}{F^0(x)}\right)(1 - q(x)).$$

The partial derivative with respect to x is equal to

$$\left(\frac{F^1(y)f^1(x)}{F^1(x)^2}q(x)+\frac{F^0(y)f^0(x)}{F^0(x)^2}(1-q(x))\right)+q'(x)\left(1-\frac{F^1(y)}{F^1(x)}-1+\frac{F^0(y)}{F^0(x)}\right).$$

The first term is positive. MLRP implies $q(x)$ is increasing in x and $F^1(x)/F^0(x) > F^1(y)/F^0(y)$ for $x > y$, so the second term is also positive.

OA.3.2 Proof of Theorem 2

Equilibrium conditions

Leader-follower continuation game. Introducing a flow cost for the leader does not affect the follower's incentive. Same as the no-flow-cost case, the follower's first-order condition implies $x'(t) = \phi(x(t), y(t))$, where

$$\phi(x,y) := -r\left(\frac{\rho_0 f^1(y)(1-F^1(x))(H-c)-(1-\rho_0)f^0(y)(1-F^0(x))(L+c)}{\rho_0 f^1(y)f^1(x)(H-c)-(1-\rho_0)f^0(y)f^0(x)(L+c)}\right).$$

For leader of type x, same as before, denote his belief at the beginning of the leader-follower continuation game by $q_L(x) = \Pr(\theta = 1|x, s_F < y(0))$. His expected payoff from disinvesting at t is

$$\begin{aligned}\mathcal{L}(x,t) =& q_L(x)\\ &\cdot\left(\int_0^t -y'(\tau)\frac{f^1(y(\tau))}{F^1(y(0))}\left(e^{-r\tau}H-\int_0^\tau e^{-r\tilde{\tau}}\eta \mathrm{d}\tilde{\tau}\right)\mathrm{d}\tau-\frac{F^1(y(t))}{F^1(y(0))}\int_0^t e^{-r\tilde{\tau}}\eta \mathrm{d}\tilde{\tau}\right)\\ &-(1-q_L(x))\\ &\cdot\left(\int_0^t -y'(\tau)\frac{f^0(y(\tau))}{F^0(y(0))}\left(e^{-r\tau}L+\int_0^\tau e^{-r\tilde{\tau}}\eta \mathrm{d}\tilde{\tau}\right)\mathrm{d}\tau+\frac{F^0(y(t))}{F^0(y(0))}\int_0^t e^{-r\tilde{\tau}}\eta \mathrm{d}\tilde{\tau}\right).\end{aligned}$$

The first-order condition implies $y'(t) = \psi(x(t), y(t))$, where

$$\psi(x,y) := -\eta\left(\frac{\rho_0 f^1(x)F^1(y)+(1-\rho_0)f^0(x)F^0(y)}{\rho_0 f^1(x)f^1(y)H-(1-\rho_0)f^0(x)f^0(y)L}\right).$$

Initial conditions. With strictly monotonic strategies, the flow cost does not affect the initial conditions. So the same as the no-flow cost case, $y(0) < z = x(0)$ and z's indifference condition implies $W_0(x(0), y(0)) = c$, where

$$W_0(x,y) := \frac{\rho_0 f^1(x)(F^1(x) - F^1(y))H}{\rho_0 f^1(x)F^1(x) + (1-\rho_0)f^0(x)F^0(x)} - \frac{(1-\rho_0)f^0(x)(F^0(x) - F^0(y))L}{\rho_0 f^1(x)F^1(x) + (1-\rho_0)f^0(x)F^0(x)}.$$

Optimality

To show optimality, one needs to show (i) $\mathcal{F}(y,t)$ is single-peaked in t, (ii) $\mathcal{L}(x,t)$ is single-peaked in t, and (iii) all types above z invest and all types below do not. (i) is the same as the no-flow-cost case. The following lemma establishes (ii) holds. Given (i) and (ii), the proof of (iii) is the same as the no-flow-cost case.

Lemma OA.1. *For a fixed x, $\mathcal{L}(x,t)$ is single-peaked in t.*

Proof. The proof is analogous to the proof of the follower's expected payoff being single-peaked in the no-flow-cost case (Lemma 8). To simplify notation, define

$$M(x,t) := \frac{q_L(x)}{F^1(y(0))}(-y'(t))f^1(y(t))H - \frac{1-q_L(x)}{F^0(y(0))}(-y'(t))f^0(y(t))L$$

$$N(x,t) := \left(\frac{q_L(x)}{F^1(y(0))}F^1(y(t)) + \frac{1-q_L(x)}{F^0(y(0))}F^0(y(t))\right)\eta.$$

In words, $e^{-rt}M(x,t)\mathrm{d}t$ is type x's marginal benefit from waiting for $\mathrm{d}t$ before disinvesting and $e^{-rt}N(x,t)\mathrm{d}t$ is the marginal cost. Let the subscript i denote the partial derivative with respect to the i-th argument. The first-order condition of $\mathcal{L}$ implies $M(x(t),t) = N(x(t),t)$. Because strategies are strictly monotone and everywhere differentiable, at each t, there exists one and only one type whose first-order condition is satisfied at t. Denote the type whose first-order condition is satisfied at t^* by x^*, that is, $M(x^*,t^*) = N(x^*,t^*)$. Suppose x^* mimics the behavior of type $\hat{x}$ by stopping at $\hat{t}$. Because $M(x,t)$ is differentiable in x, by the fundamental theorem of calculus,

$$M(x^*,\hat{t}) = M(\hat{x},\hat{t}) + \int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x = N(\hat{x},\hat{t}) + \int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x,$$

where $M_1(x,\hat{t}) = \mathrm{d}M(x,\hat{t})/\mathrm{d}x$. The second equality follows from $\hat{x}$'s first-order condition $M(\hat{x},\hat{t}) = N(\hat{x},\hat{t})$. By MLRP, $q_L(x)$ is decreasing in x and because $y'(t) < 0$, so $M_1(x,\hat{t}) > 0$. Thus, if $\hat{x} < x^*$, then

$$M(x^*,\hat{t}) = N(\hat{x},\hat{t}) + \int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x > N(\hat{x},\hat{t}) > N(x^*,\hat{t}),$$

where the first inequality follows from $\int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x > 0$, and the second inequality follows from that N is decreasing in x because of MLRP and $y(t) < y(0)$. Similarly, if $\hat{x} > x^*$, then $\int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x < 0$, so

$$M(x^*,\hat{t}) = N(\hat{x},\hat{t}) + \int_{\hat{x}}^{x^*} M_1(x,\hat{t})\mathrm{d}x < N(\hat{x},\hat{t}) < N(x^*,\hat{t}).$$

$x(t)$ is increasing, so $\hat{x} < (>) x^*$ is equivalent to $\hat{t} < (>) t^*$. The above argument shows $M(x^*,\hat{t}) - N(x^*,\hat{t}) > 0$ for all $\hat{t} < t^*$ and $M(x^*,\hat{t}) - N(x^*,\hat{t}) < 0$ for all $\hat{t} > t^*$. $\square$

Existence

In any dynamic equilibrium in strictly monotonic and differentiable strategies,
(i) by optimality, players must get strictly positive payoff;
(ii) strategies are strictly monotone: $x'(t) > 0$ and $y'(t) < 0$ for all $t \geq 0$;
(iii) strategies are differentiable for all $t \geq 0$ and $x(t), y(t) \in (0,1)$.

(i) In the leader-follower game, for the leader, disinvesting at $t = 0$ generates payoff 0 for any types of the leader, that is, $\mathcal{L}(x,0) = 0$ for all $x \geq x(0)$. By Lemma OA.1, $\mathcal{L}(x,t)$ is single-peaked in t, so by optimality, if a type optimally disinvests at $t > 0$, he must expect to get a strictly higher payoff than disinvesting at $t = 0$. That is, $\mathcal{L}(x(t),t) > \mathcal{L}(x(t),0) = 0$ for all $x(t) > x(0)$. For the follower, $\mathcal{F}(y(t),t) > 0$ if and only if

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(y(t))}{f^0(y(t))}\frac{1-F^1(x(t))}{1-F^0(x(t))} > \frac{L+c}{H-c}. \tag{OA.4}$$

I now show players' expected payoff at the beginning of the game is positive. Because

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(x(0))}{f^0(x(0))}\frac{1-F^1(x(0))}{1-F^0(x(0))} > \frac{\rho_0}{1-\rho_0}\frac{f^1(y(0))}{f^0(y(0))}\frac{1-F^1(x(0))}{1-F^0(x(0))} > \frac{L+c}{H-c},$$

where the first inequality follows from $x(0) > y(0)$, and the second inequality follows from evaluating (OA.4) at $t = 0$. This implies z's ex ante expected payoff is strictly positive. By MLRP, all types above z receive strictly positive payoffs. Types below z do not invest at the beginning of the game so their payoff is at least 0.

(ii) $y'(t) < 0$ if and only if

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(y(t))}{f^0(y(t))}\frac{f^1(x(t))}{f^0(x(t))} > \frac{L}{H}. \tag{OA.5}$$

Given (OA.4), $x'(t) > 0$ if and only if

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(y(t))}{f^0(y(t))}\frac{f^1(x(t))}{f^0(x(t))} < \frac{L+c}{H-c}. \tag{OA.6}$$

(iii) Because $\phi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$ are autonomous first-order differential equations and are continuous for all (x, y) such that $\phi(x, y) > 0$ and $\psi(x, y) < 0$, and $x(t)$ and $y(t)$ are bounded, so as $t \to \infty$,

$$x'(t) \to 0 \text{ and } y'(t) \to 0.$$

$x'(t) = 0$ and $y'(t) = 0$ if and only if $x(t) = 1$ and $y(t) = 0$. So $\phi(x(t), y(t)) \to 0$ and $\psi(x(t), y(t)) \to 0$ if and only if $x(t) \to 1$ and $y(t) \to 0$.

Define $\mathcal{D} \subset (0, 1)^2$ as

$$\mathcal{D} := \{(x, y) : \text{(OA.4), (OA.5) and (OA.6) hold}\},$$

and define $\mathcal{D}_0 \subset (0, 1)^2$ as

$$\mathcal{D}_0 := \mathcal{D} \cap \{(x, y) : x > y \text{ and } V(x, y) = c\}.$$

In words, if a solution $(x(t), y(t))$ to the differential system (9) is an equilibrium, then it must be that $(x(t), y(t)) \in \mathcal{D}$ for all $t \geq 0$ with initial values $(x(0), y(0)) \in \mathcal{D}_0$.

For the purpose of this proof, it is helpful to consider the (x, y)-plane and the differential equation

$$y'(x) = \Upsilon(x, y) := \frac{\psi(x, y)}{\phi(x, y)},\ \forall (x, y) \in \mathcal{D}. \tag{OA.7}$$

By definition, $\Upsilon(x, y)$ is continuous in (x, y) for all $(x, y) \in \mathcal{D}$. An equilibrium is a solution $y(x)$ to the differential equation (OA.7) in $\mathcal{D}$ with $y(x) < x$ that goes through a point in $\mathcal{D}_0$ and converges to 0 as x goes to 1. Showing an equilibrium exists and is unique is equivalent to showing such solution exists and is unique. Lemma OA.2 shows there exists a trajectory in $\mathcal{D}$ that converges to 0 as x goes to 1. Under certain parametric restrictions (OA.14), the solution is also unique, and Lemma OA.3 shows this (unique) trajectory goes through one and only one point in $\mathcal{D}_0$ for $y(x) < x$.

Figure OA.2 illustrates the unique equilibrium trajectory (red arrowed curve) which goes through exactly one point in $\mathcal{D}_0$ and converges to the point $(1, 0)$. All other solution trajectories (black arrowed curves) will diverge to the boundaries of $\mathcal{D}$. Figure OA.2 also displays annotations that facilitate the rest of the proof.

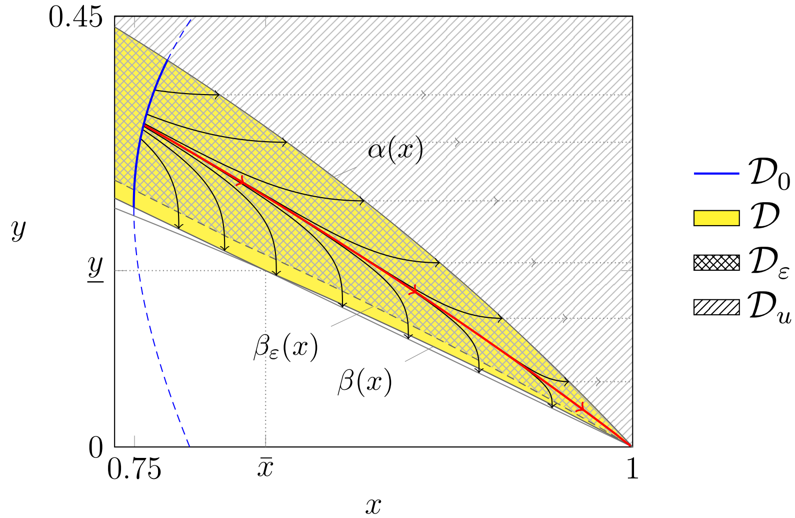

Figure OA.2: Equilibrium trajectory (red arrowed curve) and sample trajectories (non-equilibrium, black arrowed curves) to the differential system (9) for $\rho_0 = 1/2, H = L = 1, r = 1/5, c = 0.38, \eta = 1/20$ and posterior beliefs distributed according to $Beta(1+\theta, 1+(1-\theta))$.

Lemma OA.2. *For any feasible parameters, there exists a solution $y(x)$ to the differential equation (OA.7) in $\mathcal{D}$ with $y(x) \to 0$ as $x \to 1$.*

Proof. Consider the boundaries of $\mathcal{D}$. For any fixed $x \in (0, 1)$, let $\beta_F(x)$ be such that

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(\beta_F(x))}{f^0(\beta_F(x))}\frac{1-F^1(x)}{1-F^0(x)} = \frac{L+c}{H-c}, \tag{OA.8}$$

$\beta_f(x)$ be such that

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(\beta_f(x))}{f^0(\beta_f(x))}\frac{f^1(x)}{f^0(x)} = \frac{L}{H}, \tag{OA.9}$$

and $\alpha(x)$ be such that

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(\alpha(x))}{f^0(\alpha(x))}\frac{f^1(x)}{f^0(x)} = \frac{L+c}{H-c}. \tag{OA.10}$$

Finally, define

$$\beta(x) := \max_{x\in(0,1)}\{\beta_F(x), \beta_f(x)\}.$$

By IHRP, $\beta_f(x)$ and $\beta_F(x)$ intersect at most once for $x \in (0,1)$.

Claim OA.1. (i) $\mathcal{D}$ is non-empty. (ii) $(1,0) \in \mathrm{cl}(\mathcal{D})$ and $(0,1) \in \mathrm{cl}(\mathcal{D})$.

Proof of Claim OA.1. (i) Fix an $x \in (0,1)$. By MLRP, the left-hand side of (OA.8) evaluated at any $(x', y') > (x, \beta_F(x))$ is strictly higher than $(L+c)/(H-c)$, the left-hand side of (OA.9) evaluated at any $(x', y') > (x, \beta_f(x))$ is strictly higher than L/H, and the left-hand side of (OA.10) evaluated at any $(x', y') < (x, \alpha(x))$ is strictly lower than $(L+c)/(H-c)$. $\alpha(x) > \beta(x)$ for all $x \in (0,1)$. So $\mathcal{D}$ is non-empty.

(ii) Fix $x \in (0,1)$. Consider (OA.8). Take the limit of both sides as $x \to 1$. The right-hand side is constant at $(L+c)/(H-c)$. On the left-hand side, because

$$\lim_{x\to 1}\frac{1-F^1(x)}{1-F^0(x)} = \lim_{x\to 1}\frac{f^1(x)}{f^0(x)} = \infty,$$

it must be $f^1(\beta_F(x))/f^0(\beta_F(x)) \to 0$, which means $\beta_F(x) \to 0$. The same argument applies for equations (OA.9) and (OA.10). This implies $(1,0) \in \mathrm{cl}(\mathcal{D})$. An analogous argument shows $(0,1) \in \mathrm{cl}(\mathcal{D})$. $\square$

By definition, for all $(x,y) \in \mathcal{D}$, $\psi(x,y) < 0$ and $\phi(x,y) > 0$, so $\Upsilon(x,y) < 0$. Define $\mathcal{D}_u \in (0,1)^2$ (the subscript u stands for "upper") as

$$\mathcal{D}_u := \left\{(x,y) : \frac{\rho_0}{1-\rho_0}\frac{f^1(y)}{f^0(y)}\frac{f^1(x)}{f^0(x)} \geq \frac{L+c}{H-c}\right\}.$$

In words, $\mathcal{D}_u$ is the set of points in the (x,y)-plane that are equal to or above $\alpha(x)$. By definition and the continuity of the distribution functions, $\mathcal{D} \cup \mathcal{D}_u$ is connected. For any fixed $x \in (0,1)$, as $y \to \alpha(x)$, $\Upsilon(x,y) \to 0$. Let $\Upsilon(x,y) = 0$ for all $(x,y) \in \mathcal{D}_u$. Then $\Upsilon(x,y)$ is continuous in (x,y) for all $(x,y) \in \mathcal{D}\cup\mathcal{D}_u$. Apply the implicit function theorem to (OA.10), MLRP implies for all feasible parameters and $x \in (0,1)$,

$$\alpha'(x) < 0 = \Upsilon(x, \alpha(x)).$$

This means $\alpha(x)$ is a strong lower fence (or lower solution, see Hubbard and West, 1991, Section 1.3, or Teschl, 2012, Section 1.5) for the differential equation

$$y'(x) = \Upsilon(x,y) = \begin{cases} \psi(x,y)/\phi(x,y) & (x,y) \in \mathcal{D} \\ 0 & (x,y) \in \mathcal{D}_u \end{cases}. \tag{OA.11}$$

Consider an ε-variation of $\beta_F(x)$ and $\beta_f(x)$. Let $\beta_{F,\varepsilon}(x)$ be such that

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(\beta_{F,\varepsilon}(x))}{f^0(\beta_{F,\varepsilon}(x))}\frac{1-F^1(x)}{1-F^0(x)} = \frac{L+c}{H-c} + \varepsilon, \tag{OA.12}$$

and $\beta_{f,\varepsilon}(x)$ be such that

$$\frac{\rho_0}{1-\rho_0}\frac{f^1(\beta_{f,\varepsilon}(x))}{f^0(\beta_{f,\varepsilon}(x))}\frac{f^1(x)}{f^0(x)} = \frac{L}{H} + \varepsilon. \tag{OA.13}$$

Let

$$\beta_\varepsilon(x) := \max_{x\in(0,1)} \{\beta_{F,\varepsilon}(x), \beta_{f,\varepsilon}(x)\}.$$

Define

$$\mathcal{D}_\varepsilon := \{(x,y) : x \in (0,1) \text{ and } \beta_\varepsilon(x) \le y < \alpha(x)\}.$$

By MLRP, for all $x \in (0,1)$, $\beta_{F,\varepsilon}(x) < \alpha(x)$. For all $\varepsilon < (L+c)/(H-c) - L/H$, $\beta_{f,\varepsilon}(x) < \alpha(x)$. By the same argument as Claim OA.1, $\mathcal{D}_\varepsilon$ is non-empty, and the points $(1,0)$ and $(0,1)$ are in the closure of $\mathcal{D}_\varepsilon$. Moreover, $\mathcal{D}_\varepsilon \cup \mathcal{D}_u$ is connected and $\Upsilon(x,y)$ is continuous in (x,y) for all $(x,y) \in \mathcal{D}_\varepsilon \cup \mathcal{D}_u$.

Apply the implicit function theorem to (OA.12) and (OA.13), MLRP implies that for all feasible parameters and any $\varepsilon > 0$, $\beta'_{F,\varepsilon}(x)$ and $\beta'_{f,\varepsilon}(x)$ are both finite and negative. Therefore $\beta'_\varepsilon(x) > -\infty$ for all $x \in (0,1)$.

Claim OA.2. There exists $\hat{\varepsilon} > 0$ such that $\Upsilon(x, \beta_{\hat{\varepsilon}}(x)) < \beta'_{\hat{\varepsilon}}(x)$ for all x.

Proof of Claim OA.2. For all $x \in (0,1)$, by definition, as $\varepsilon \to 0$, $\beta_\varepsilon(x) \to \beta(x)$, which implies $\Upsilon(x, \beta_\varepsilon(x)) \to -\infty$. So for any x, there exists $\varepsilon(x) > 0$ (ε might depend on x) such that for all $\varepsilon < \varepsilon(x)$, $\Upsilon(x, \beta_\varepsilon(x)) < \beta'_\varepsilon(x)$. Let $\hat{\varepsilon} = \inf_{x\in(0,1)} \varepsilon(x)$.

It remains to show that $\hat{\varepsilon} > 0$. Suppose $\hat{\varepsilon} = 0$. Then there exists a sequence ε_n with $\varepsilon_n \to 0$ such that for each ε_n there exists x_n such that $\Upsilon(x_n, \beta_{\varepsilon_n}(x_n)) \geq \beta'(x_n)$. This is a contradiction because for all x_n, $\beta'(x_n) > -\infty$ but as $\varepsilon_n \to 0$, $\Upsilon(x_n, \beta_{\varepsilon_n}(x_n)) \to -\infty$. $\square$

This means $\beta_\varepsilon(x)$ is a strong upper fence (or upper solution) for the differential equation (OA.11). Therefore, in $\mathcal{D}_\varepsilon \cup \mathcal{D}_u$, there exists a solution $y(x)$ to the differential equation (OA.7) with $\beta_\varepsilon(x) \leq y(x) \leq \alpha(x)$ for all $x \in (0, 1)$ (see Hubbard and West, 1991, Theorem 1.4.4, or Teschl, 2012, Lemma 1.2).

The above argument establishes there exists a solution in $\mathcal{D}_\varepsilon \cup \mathcal{D}_u$. It remains to show that the solution is within $\mathcal{D}_\varepsilon$ (and thus within $\mathcal{D}$), not in $\mathcal{D}_u$. This boils down to showing that solutions in $\mathcal{D}_u$ do not converge to 0 as $x \to 1$. This follows from the definition that $y'(x) = 0$ for all $(x, y) \in \mathcal{D}_u$. So for any $(x, y(x)) \in \mathcal{D}_u$ that solves the differential equation (OA.11), $y(x) > 0$ for all x. $\square$

Uniqueness

Assumption. *Assume the following condition holds:*

$$\forall (x, y) \in \mathcal{D},\ \partial\Upsilon(x, y)/\partial y \geq 0. \tag{OA.14}$$

The uniqueness of a global condition can be established if the primitives satisfy the above condition. It can be numerically verified that (OA.14) is satisfied if f^θ is induced by signals distributed according to the Beta distributions or the Normal distributions. Moreover, by definition, as $x \to 1$, $\alpha(x) \to 0$ and $\beta_\varepsilon(x) \to 0$, so

$$\lim_{x \to 1} |\alpha(x) - \beta_\varepsilon(x)| = 0. \tag{OA.15}$$

Conditions (OA.14) and (OA.15) imply the solution is unique in $\mathcal{D}_\varepsilon$ (see Hubbard and West, 1991, Theorem 1.4.5, or Teschl, 2012, Section 1.5).

The above argument establishes the unique solution is in $\mathcal{D}_\varepsilon$. It remains to show this solution is unique in $\mathcal{D}$. Because $\mathcal{D} = \mathcal{D}_\varepsilon \cup \{(x, y) : x \in (0, 1) \text{ and } \beta(x) < y < \beta_\varepsilon(x)\}$, it boils down to showing there does not exist a solution that converges to 0 as $x \to 1$ in the set $\{(x, y) : x \in (0, 1) \text{ and } \beta(x) < y < \beta_\varepsilon(x)\}$. For all $y(x)$ such that $\beta(x) < y(x) < \beta_\varepsilon(x)$, $y'(x) \to -\infty$, which implies for all $x \in (0, 1)$, $y(x) \to \beta(x) > 0$.

Denote this unique solution by $\hat{y}(x)$. I prove there exists a unique set of initial values that satisfy this unique solution. This is summarized in the following lemma.

Lemma OA.3. *There exists a unique* $(x_0, y_0) \in \mathcal{D}_0$ *such that* $y_0 = \hat{y}(x_0)$.

Proof. To simplify notation, define

$$\ell(x, y) := \frac{\rho_0}{1-\rho_0} \frac{f^1(y)}{f^0(y)} \frac{f^1(x)}{f^0(x)}.$$

Recall that $\mathcal{D}_0$ is the set of points $(x, y) \in \mathcal{D}$ that satisfies the equation $W_0(x, y) = c$. Solve $W_0(x, y) = c$ for y in terms of x and denote the solution by $y_{W_0}(x)$. By Claim 10 (iii) and (iv), $y_{W_0}(x)$ is increasing and continuous in x for all x such that $y_{W_0}(x) < x$.

By a change of variable, Lemma OA.2 shows $\hat{y}(x)$ also converges to 1 as $x \to 0$. So $\hat{y}(x)$ is a strictly decreasing function that converges to 1 as $x \to 0$ and converges to 0 as $x \to 1$, and satisfies $\ell(x, \hat{y}(x)) \in (L/H, (L+c)/(H-c))$ for all $x \in (0, 1)$. So points in $\mathcal{D}_0$ constitute a strictly increasing and continuous function that starts at a point below $\hat{y}(x)$, and ends at a point above $\hat{y}(x)$. The result follows. $\square$

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