Dynamic Coordination with Informational Externalities*

Beixi Zhou†

April 16, 2021

Abstract

I study a two-player continuous-time dynamic coordination game with observational learning. Each player has one opportunity to make a reversible investment with an uncertain return that is realized only when both players invest. Each player learns about the potential return by observing a private signal and the actions of the other player. In equilibrium, players' roles as leader and follower are endogenously determined. Information aggregates in a single burst initially, then gradually through delayed investment and disinvestment over time. More precise signals lead to faster coordination conditional on initial disagreement, but might also increase the probability of initial disagreement.

Keywords: dynamic coordination, observational learning, real option.

JEL Codes: C73, D82, D83

*I am indebted to Bart Lipman and Chiara Margaria for their invaluable guidance and support throughout this project. I thank Christophe Chamley, Juan Ortner, and Allen Vong for detailed comments and discussions, and Cuimin Ba, Joyee Deb, Tan Gan, Marina Halac, Dilip Mookherjee, Colin Stewart, and audiences at Boston University, the 30th Stony Brook International Conference on Game Theory, the 12th World Congress of the Econometric Society, and the 2020 Young Economist Symposium for comments and suggestions.

†Department of Economics, Boston University, bzhou@bu.edu.

1 Introduction

Two phenomena often arise in the diffusion of innovative technologies. The first one is observational learning: players learn about the profitability of a new technology by observing the investment behavior of other players. Second, technology adoption often exhibits strategic complementarities among investors. Investors with insufficient funds to initiate a project rely on other investors to fill the gap.

The objective of this paper is to study observational learning in the context of complementary investments with reversible investment decisions. In such situations, coordination among investors might make an investment profitable when it would not have been so if undertaken by only one investor, and an investor can back out of an investment at any point in time should he become pessimistic about its prospects.

Specifically, I study the interplay of observational learning and coordination in a two-player timing game of investment in continuous time. Each player is endowed with one opportunity to invest in a risky project. The decision to invest is reversible, but a player cannot reinvest after having disinvested. The return of the investment is ex ante unknown. Each player receives a private signal about the return only at the outset, and decides when to invest and when to disinvest after having invested. Actions are public; thus each player learns about the return not only from the private signal, but also by observing the investment decisions of the other player. The players pay a (lump-sum) investment cost at the time of their investment, which is not recoverable should they decide to ever disinvest. The (lump-sum) return of the project is only realized at the time when both players invest.

My main findings are as follows. First, I characterize the symmetric equilibria in which players use threshold strategies and there is a positive probability of investment at more than one instant in time.[1] I show uniqueness within this class of equilibria. In this unique equilibrium, players first decide whether to invest at the start of the game. This initial investment decision endogenously determines the players' roles as the leader (if a player invests) or the follower (if a player doesn't invest). The continuation game unfolds as a timing game, in which the leader chooses when to disinvest and the follower chooses when to invest. The duration a player is willing to wait signals his private information to the other player. Information is revealed in a single burst at the beginning of the game, and then aggregates continuously and

[1]This class of equilibria is of special interest. I also characterize other symmetric equilibria.

gradually over time through delayed disinvestment and investment decisions.

Next, I investigate how changes in the informativeness of the signal distributions affect the learning dynamics and the equilibrium outcome. In particular, does having a more precise signal increase the probability that the players coordinate initially (either both investing or both not investing), and do players coordinate faster if they cannot reach an initial agreement? I find that, while a more precise information structure always increases the speed of the observational learning process conditional on initial disagreement, its effect on the probability of initial agreement is ambiguous. Specifically, when information gets arbitrarily precise, the probability of initial agreement converges to 1; but when precision is low, there exist parameters where an increase in precision leads to a decrease in the probability of initial agreement. While better information enables players to make more informed decisions, it can also increase the variance in the two players' beliefs, which leads to more disagreement. The effect of the increase in precision on efficiency, defined as the sum of the players' ex ante equilibrium payoffs, is positive in the limit: when information gets arbitrarily precise, efficiency approaches the first-best level.

Related literature

My model combines a dynamic coordination game (Gale, 1995) and an observational learning game with endogenous timing (Chamley and Gale, 1994, or Murto and Välimäki, 2013) and reversible actions.

As noted, my model features strategic delay. Although strategic delay is a common feature in coordination games or observational learning games with endogenous timing, the novelty is the way in which delay acts as a signal and facilitates information transmission. In the dynamic coordination literature, delay arises due to strategic uncertainty. Players want to coordinate on the same action but are unsure of what others will do. In a seminal paper, Gale (1995) studies delay in an N-player dynamic coordination game where each player decides whether to make an irreversible investment, the return of which is increasing in the number of investors. In the observational learning with endogenous timing literature, delay arises due to informational externalities. Players "wait and see" what others do in order to learn from their actions. In a seminal paper, Chamley and Gale (1994) study an N-player investment timing game with a pure informational externality. Each player observes a private signal and the actions of the others, and his payoff only depends on his own action and the state. Murto and Välimäki (2013) generalize Chamley and Gale (1994) by modeling uncertainty over when to invest, not just whether to invest. The equilibria in Murto and Välimäki (2013) feature alternating phases between investment waves (sudden bursts of information) and waiting phases (gradual revelation of information). In my model, a burst of information occurs only at time 0, and afterward, the information revelation can only be gradual.

Turning to observational learning models with payoff externalities, there is a small theoretical literature that incorporates payoff externalities into a standard sequential social learning model as studied by Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992). Early work includes Dasgupta (2000), who studies a sequential social learning model where the return of the investment is realized if and only if all players invest. Although Dasgupta's model has both coordination and informational externalities, the exogenous order of actions in the standard sequential social learning framework leaves open the possibilities for enriched learning behavior if players are allowed to choose the timing of their actions. An experimental study by Brindisi, Çelen, and Hyndman (2014) investigates the effect of endogenous timing in a dynamic coordination setting with observable actions. In their model, investment is irreversible, thus the decision to invest eliminates strategic uncertainty once and for all, and all actions occur at time 0. With reversible investment and different modeling choices, my model allows for richer learning dynamics in a more general setting.

An important assumption of my model is the reversibility of investment decisions, in contrast to the irreversible investments assumed in much of the literature. The usual formulation of irreversible investment, as in Dixit and Pindyck (1994), is that investment expenditures are sunk costs. The investment in my model is also "irreversible" in this sense as the cost of investment is nonrecoverable. On the other hand, it is reversible in the sense that a player can shut down or abandon an investment without bearing its return. Abandonment of this kind is a common practice in investment (e.g., McDonald and Siegel, 1985, Bar-Ilan and Strange, 1996), yet seems understudied in the theoretical literature. In fact, in this paper, this reversibility is one of the key ingredients that give delay its informational value. It makes it possible that after players endogenously determine their roles in the game (leader or follower), they can subsequently signal their optimism or pessimism through delayed actions.[2] Few other papers assume reversible actions in dynamic settings. Kováč and Steiner

[2] I discuss the case of irreversible investments in Section 5.2.

(2013) study the role of reversibility of actions in a two-period coordination game with unobservable actions. Klein and Wagner (2019) analyze a strategic experimentation problem with private information where players can invest and disinvest at any time.

2 Model

Time is continuous and the horizon is infinite, $t \geq 0$. There are two players $i = 1, 2$. Each player chooses when (if ever) to invest in a risky project, and when (if ever) to disinvest. The return of the investment depends on an unknown state of the world $\theta \in \{0, 1\}$, where $\theta = 1$ indicates the project is good, while $\theta = 0$ indicates the project is bad. The common prior belief of $\theta = 1$ is $\rho_0 \in (0, 1)$.

Information. At the outset, player i receives a private signal $s_i \in S = (0, 1)$ about the state θ, which is conditionally independent across players. Conditional on state θ, the distribution of s is denoted $F^\theta(s)$ with a continuously differentiable density function $f^\theta(s) = \mathrm{d}F^\theta(s)/\mathrm{d}s$ that is strictly positive on $(0, 1)$. The two distributions F^0 and F^1 are mutually absolutely continuous with common support $[0, 1]$, which ensures there does not exist a signal that perfectly reveals the state.

The distributions are assumed to satisfy the strict monotone likelihood ratio property (MLRP) on $(0, 1)$ and have an unbounded likelihood. That is, the likelihood ratio $f^0(s)/f^1(s)$ is strictly decreasing on $(0, 1)$ with $\lim_{s \to 0} f^0(s)/f^1(s) = \infty$ and $\lim_{s \to 1} f^0(s)/f^1(s) = 0$. MLRP guarantees that a higher signal induces a higher posterior belief of the good state for any prior; unbounded likelihood guarantees that the distribution of the induced posterior beliefs has full support over $[0, 1]$. Without loss of generality, I assume $\Pr(\theta = 1|s) = s$ for all $s \in (0, 1)$. By definition, the likelihood ratio is proportional to the posterior belief of the bad state over the good state, $f^0(s)/f^1(s) = ((1 - s)/s)(\rho_0/(1 - \rho_0))$, and strict MLRP is always satisfied.

Players cannot communicate. They do not observe each other's private signal, but observe each other's actions (investing, disinvesting, and not investing).

Actions and payoffs. Each player chooses when, if ever, to invest, and when, if ever, to disinvest after having invested. A player cannot reinvest after having disinvested. If player i invests at t_i, he pays a lump-sum (sunk) cost $c > 0$ at t_i. As soon as both players invest, the project generates a lump-sum return R^θ to each of the players. If the state is good, the return is $R^1 = H > c$, and if the state is bad, the return is negative: $R^0 = -L < 0$.[3] If only one player invests, the payoff from the investment is 0. The payoff from never investing is 0. The (payoff) tie-breaking rule in the case of a simultaneous move, that is, a player invests at the same instant the other player disinvests, is that the return R^θ is not realized. Players discount payoffs at a common discount rate $r > 0$.

As an example, suppose player 1 invests at t_1 and player 2 invests at $t_2 \geq t_1$. The realized payoff of player 1 and player 2 are $u_1 = e^{-rt_2}R^\theta + e^{-rt_1}(-c)$ and $u_2 = e^{-rt_2}R^\theta + e^{-rt_2}(-c)$. Suppose player 1 disinvests at $\tau_1 \in (t_1, t_2]$. Then $u_1 = e^{-rt_1}(-c)$ and $u_2 = e^{-rt_2}(-c)$.[4]

Strategies. The usual challenge of defining strategies in continuous-time models with observable actions arises. A player updates his belief in response to the other player's actions, and may want to react to his updated belief without delay. To overcome this challenge, I follow the definition of strategy in Murto and Välimäki (2013) and model the dynamic game as a multi-stage stopping game as follows.

The game consists of at most two stages. Stage 0 starts at time 0. In this initial stage, the public history is that nobody has invested. Denote this history by $h^0 \in \mathcal{H}^0 := \{\varnothing\}$. Given this history and the private signal s_i, player $i \in \{1, 2\}$ chooses a time to invest $\sigma_i(s_i, \varnothing) \geq 0$. The initial stage ends at $t^0 := \inf_i \sigma_i(s_i, \varnothing)$.

As soon as stage 0 ends, the game immediately moves to stage 1. The public history after the initial stage and at the beginning of the next stage consists of t^0 and the identity of the player(s) who invested at t^0, that is, $\{(\{1, 2\}, t^0), (\{1\}, t^0), (\{2\}, t^0)\}$. The history $(\{1, 2\}, t^0)$ is terminal: if both players invest simultaneously, the return of the project is realized at t^0. The histories $(\{1\}, t^0)$ and $(\{2\}, t^0)$ are non-terminal. If only one player invests, the player who invested is called the leader and the player who did not invest is called the follower; the subsequent continuation game is called a *leader-follower continuation game*. Define the set of non-terminal histories as $\mathcal{H}^1 := \{(\{1\}, t^0), (\{2\}, t^0)\}$. Given $h^1 \in \mathcal{H}^1$ and the private signal s_i, the leader decides if and when to disinvest and the follower decides if and when to invest, each conditional on the other player not moving first, $\sigma_i(s_i, h^1) \geq t^0$. The leader-follower

[3]The assumption $R^0 < 0$ is essential to the analysis. It ensures that a player will ever have an incentive to disinvest after having invested.

[4]Player 2 still pays the investment cost if he were to invest after player 1 has disinvested — although in equilibrium it is suboptimal to do so and thus does not happen.

continuation game ends either after the follower invests in which case payoff realizes, or after the leader disinvests, in which case it is a dominant strategy for the follower to never invest afterward. I treat these histories as if they were terminal.

Denote the set of all non-terminal public histories by $\mathcal{H} := \mathcal{H}^0 \cup \mathcal{H}^1 = \{\varnothing\} \cup \{(\{1\}, t^0), (\{2\}, t^0)\}$. A pure strategy for player i for stage $k \in \{0, 1\}$ is a function

$$\sigma_i^k : S \times \mathcal{H}^k \to [t^{k-1}, \infty]$$

that maps the private signal and a non-terminal history to a time to switch actions (either switch from not investing to investing or from investing to not investing), conditional on the other player $-i$ not switching before that time. Define player i's strategy on the set of all non-terminal histories $\sigma_i : S \times \mathcal{H} \to [0, \infty]$ as

$$\sigma_i(s_i, h) = \sigma_i^k(s_i, h) \text{ whenever } h \in \mathcal{H}^k.$$

Equilibrium concept. I analyze symmetric perfect Bayesian equilibria in which players use monotonic strategies, defined as follows.

Definition 1. A strategy σ_i is monotonic if $\sigma_i(s_i, \varnothing)$ is decreasing in s_i.

In words, in the history that nobody has invested, a player chooses to invest earlier if his signal is higher. An equilibrium is monotonic if players use monotonic strategies.

In the case that a player is indifferent between two investment (or disinvestment) times t' and t'' with $t' < t''$, I restrict attention to strategies that satisfy the following (indifference) tie-breaking rule. The strategies prescribe investment at t'' to a player who has not invested, and disinvestment at t' to a player who has invested.[5]

A perfect Bayesian equilibrium is a pair of strategies and a system of beliefs for each player such that each player's strategy maximizes his expected payoff and beliefs are updated via Bayes' rule at any history reached with positive probability.

Off the equilibrium path, I assume each player's belief about the deviator's type is the lowest possible type, so neither player invests after any observable deviations. That is, upon observing a deviation by player $-i$, player i never invests if he has not invested, and disinvests immediately if he has invested. This assumption is without

[5]This tie-breaking rule is innocuous. It assumes players do not invest or stay in the investment if they are indifferent between in and out. The same equilibrium can be obtained if indifference is consistently broken in the opposite fashion instead (some of the proofs will take a different approach).

loss of generality in the sense that if a (perfect Bayesian) equilibrium outcome can be sustained with a different off-path belief, it can be sustained with this off-path belief.

3 Equilibrium Analysis

3.1 Multiplicity of Equilibria

As is common in coordination games, this game admits multiple equilibria. The first type of multiplicity comes from a continuum of "starting times." An equilibrium is said to start at $\hat{t} \geq 0$ if there is a positive probability of investment by either player at $\hat{t}$ and zero probability for all $t < \hat{t}$. For any equilibrium that starts at time 0, one can construct another equilibrium by postponing all actions to a later time. Such equilibria are considered equivalent up to starting time and are inefficient due to discounting. Consequently, I focus on equilibria that start at time 0.

Within the set of equilibria that start at time 0, there are three types of equilibria. One equilibrium that always exists is one in which neither player invests for all $t \geq 0$. I call this a *no-investment equilibrium*. Another equilibrium prescribes investment only at time 0: there is a positive probability of investment at $t = 0$, and zero probability for all $t > 0$. I call this a *myopic equilibrium*. In the myopic equilibrium, player i invests at time 0 if and only if his type s_i is above (or equal to) threshold s^*, where

$$\Pr(s_{-i} \geq s^*, \theta = 1 | s_i = s^*)H - \Pr(s_{-i} \geq s^*, \theta = 0 | s_i = s^*)L - c = 0.$$

Unless both players invest at time 0, in which case the payoff realizes, the player who invested at time 0 disinvests immediately, and the player who did not never invests.[6]

A more interesting type of equilibrium is one in which there is a positive probability of investment at more than one point in time. I call this a *dynamic equilibrium*. I focus on characterizing monotonic symmetric dynamic equilibria.

[6]The no-investment equilibrium and the myopic equilibrium constitute the set of (pure strategy) equilibria of a one-shot game at $t = 0$ with the following payoff matrix:

	Invest	Not invest
Invest	$R^\theta - c, R^\theta - c$	$-c, 0$
Not invest	$0, -c$	$0, 0$

By MLRP, the equilibrium of this one-shot game must be monotonic. The threshold type s^* is indifferent between investing and not investing, all types above it invest, and all types below do not.

3.2 Monotonic Symmetric Dynamic Equilibrium

The structure of a monotonic symmetric dynamic equilibrium is as follows. The initial stage starts at $t = 0$. In the initial stage, a player invests if his type is above threshold z and does not invest if below. After players take an action in the initial stage, there are three possible continuation games. First, if both players invest, payoff realizes. Second, if neither player invests, neither ever invests afterward.[7] Third, if only one player invests, the game immediately moves to the leader-follower continuation game. In this continuation game, players play according to a sequence of time-dependent thresholds: at each $t \geq 0$, the leader stays invested if his type is above threshold $x(t)$ and the follower does not invest if his type is below threshold $y(t)$.

The rest of the section is organized as follows. First, I discuss the players' incentives at time 0 and characterize the initial investment thresholds $(z, x(0), y(0))$ in Proposition 1. Next, I characterize the continuation thresholds $x(t)$ and $y(t)$ for all t in Proposition 2 and discuss their properties. Lastly, in Theorem 1, I present sufficient conditions for existence and uniqueness of such an equilibrium.

3.2.1 Incentives at time 0

Consider the leader-follower continuation game.[8] At time 0, upon observing an investment in the initial stage, in the second stage, the follower invests if his type is above $y(0)$, and the leader stays invested if his type is above $x(0)$. The following result characterizes the investment thresholds $(z, x(0), y(0))$ in these two stages.

Proposition 1. *In any monotonic symmetric dynamic equilibrium, the initial values $(z, x(0), y(0)) \in (0, 1)^3$ must satisfy*

$$y(0) < z = x(0), \tag{1}$$

$$\frac{\rho_0 f^1(z)(F^1(z) - F^1(y(0)))H}{\rho_0 f^1(z)F^1(z) + (1 - \rho_0)f^0(z)F^0(z)} - \frac{(1 - \rho_0)f^0(z)(F^0(z) - F^0(y(0)))L}{\rho_0 f^1(z)F^1(z) + (1 - \rho_0)f^0(z)F^0(z)} = c, \tag{2}$$

$$\rho_0 f^1(x(0))f^1(y(0))H - (1 - \rho_0)f^0(x(0))f^0(y(0))L = 0. \tag{3}$$

[7] This is shown in Lemma 9 in the Appendix. In short, Lemma 9 states that in any monotonic symmetric dynamic equilibrium, neither player invests after no initial investment.

[8] As mentioned, if both players invest, payoff realizes; if neither player invests, neither invests afterward. It only remains to analyze the leader-follower continuation game.

Condition (1) describes the equilibrium behavior at the beginning of the leader-follower continuation game. The leader stays invested for sure ($x(0) = z$), and there is a strictly positive probability that the follower follows suit ($y(0) < z$).

Intuitively, a player pays the investment cost upfront if he becomes the leader. A player is willing to do so only if he expects this action to induce a high enough probability of investment by the other player. This suggests there has to be a mass of follower types following suit, namely, $y(0) < z$. In turn, these higher follower types, upon seeing the good news that the leader has invested, now find investment profitable and invest without further delay. The leader, anticipating a mass of follower types following suit, will want to stay invested and wait for the (possibility of) investment by the follower, rather than to disinvest right away. This suggests $x(0) = z$.

Equation (2) follows from type z's indifference between investing and not investing in the initial stage at time 0 given his continuation strategies.[9] Equation (3) follows from type $x(0)$'s optimality condition in the leader-follower continuation game, which is analyzed in the following section.

3.2.2 Leader-follower continuation game

In this continuation game, each player solves an optimal stopping problem. If the leader stops, he disinvests, while if the follower stops, he invests. I derive conditions that any monotonic symmetric dynamic equilibrium must satisfy.

With a slight abuse of notation, denote player i's pure strategy in this continuation game by $\sigma_i : S_i \to [0, \infty]$, which maps player i's type to a stopping time conditional on the other player not having stopped. Let $i \in \{L, F\}$, where L denotes the leader and F the follower.[10] Let $G_i(t)$ denote the probability that i stops no later than t according to σ_i, and $G_i^\theta(t)$ denote this probability conditional on state θ. Define T_L as the earliest time by which all types of leader decide to disinvest, and T_F as the earliest time by which all types of follower decide to not invest ever after. That is, $T_i := \inf\{t > 0 : G_i(t) = \lim_{\tau\to\infty} G_i(\tau)\}$.

Because of coordination, if at some point in time, a player chooses to not invest ever after, the other player will do the same. Lemma 1 formalizes this intuition.

Lemma 1. *In any monotonic symmetric dynamic equilibrium, $T := T_L = T_F \le \infty$.*

[9] z is indifferent in the sense that his expected payoff from not investing at time 0 is equal to the supremum of his expected payoff over all continuation strategies that prescribe investing at time 0.

[10] Conditional on reaching the leader-follower continuation game, $S_L = [x(0), 1)$ and $S_F = (0, y(0)]$.

In other words, if the players cannot reach an agreement (either both investing or both not investing) by T, the probability of investment by either player after T is zero. In what follows, I determine players' equilibrium behavior for $t \in [0, T]$.

The following properties of the strategies shed lights on how information aggregates in equilibrium. Lemma 2 establishes there does not exist a time at which a mass of types stop. This means information is revealed gradually, never in bursts.

Lemma 2. *In any monotonic symmetric dynamic equilibrium, σ_L is strictly increasing and σ_F is strictly decreasing.*

Lemma 2 in its weak form follows from a revealed-preference argument. By MLRP, it is not surprising that a higher type of leader (lower type of follower) prefers to wait longer because he is more optimistic (pessimistic). However, strict monotonicity in the context of coordination is not as straightforward. σ_i constant at $\hat{t}$ over an interval means all types in this interval stop at $\hat{t}$. Intuitively, if a player is getting a continuous flow of information about the other player, he will want to respond continuously. If there is a mass of types wanting to stop at a particular time, this must be a response to a mass of types stopping by the other player. This cannot happen in equilibrium because by the (payoff) tie-breaking rule, it is a strictly dominated strategy for the follower to invest at the same time the leader disinvests.[11]

Lemma 3 establishes there does not exist a time interval over which neither player stops for sure. This means information is revealed continuously, never with pauses.

Lemma 3. *In any monotonic symmetric dynamic equilibrium, σ_L and σ_F are continuous.*

A pause is detrimental to payoff due to discounting and has no informational value. Because a player's belief is the same at the beginning and the end of the pause, were he willing to stop at the end of the pause, he would deviate to doing so at the beginning of it.[12]

By Lemma 2 and Lemma 3, the inverse mappings of σ_L and σ_F, defined as $x(t) := \sigma_L^{-1}(t)$ and $y(t) := \sigma_F^{-1}(t)$, are continuous, and strictly increasing and decreasing in t

[11] Recall the (payoff) tie-breaking rule says that the return of the investment is not realized when the follower invests at the same time the leader disinvests.

[12] To be precise, the follower strictly prefers investing at the beginning of the pause due to discounting. The leader is indifferent between disinvesting at any time during the pause. The leader will deviate to disinvesting the beginning of the pause given the (indifference) tie-breaking rule.

respectively. I refer $x(t)$ and $y(t)$ as the equilibrium (inverse) strategies of the leader and the follower. In words, leader of type $x(t^*)$ and follower of type $y(t^*)$ optimally stop at t^*. If the leader has not stopped by t^*, the follower knows the leader's type must be higher than $x(t^*)$. If the follower has not stopped by t^*, the leader knows the follower's type must be lower than $y(t^*)$. Given the follower's conditional distributions of stopping time $G_F^\theta(t)$, leader of type x's expected payoff from stopping at t is

$$\begin{aligned}\mathcal{L}(x,t) =& \Pr(\theta=1|s_L=x, s_F \le y(0)) \int_0^t e^{-r\tau} \mathrm{d}G_F^1(\tau) H \\ &- \Pr(\theta=0|s_L=x, s_F \le y(0)) \int_0^t e^{-r\tau} \mathrm{d}G_F^0(\tau) L,\end{aligned}$$

and given $G_L^\theta(t)$, follower of type y's expected payoff from stopping at t is

$$\begin{aligned}\mathcal{F}(y,t) =& e^{-rt}\big(\Pr(\theta=1|s_F=y, s_L > x(0))(1-G_L^1(t))(H-c) \\ &- \Pr(\theta=0|s_F=y, s_L > x(0))(1-G_L^0(t))(L+c)\big).\end{aligned}$$

In equilibrium, a pair of (inverse) strategies $(x(t), y(t))$ must satisfy $\mathcal{L}(x(t),t) \ge \mathcal{L}(x(t),t')$ and $\mathcal{F}(y(t),t) \ge \mathcal{F}(y(t),t')$ for all t and all $t' \ne t$. The following proposition characterizes the equilibrium (inverse) strategies $x(t)$ and $y(t)$.

Proposition 2. *In any monotonic symmetric dynamic equilibrium, $x(t)$ and $y(t)$ are differentiable functions that satisfy*

$$\rho_0 f^1(x(t)) f^1(y(t)) H - (1-\rho_0) f^0(x(t)) f^0(y(t)) L = 0, \tag{4}$$

$$x'(t) = r\left(\frac{(H-c)L}{(L+H)c}\frac{1-F^1(x(t))}{f^1(x(t))} - \frac{(L+c)H}{(L+H)c}\frac{1-F^0(x(t))}{f^0(x(t))}\right). \tag{5}$$

Equation (4) follows from leader $x(t)$'s first-order condition. If the follower does not invest in $[t, t+\mathrm{d}t)$, the leader's marginal cost from waiting is 0. If the follower invests in $[t, t+\mathrm{d}t)$, the return realizes, which would not have happened had the leader not waited. So the leader's marginal benefit from waiting is

$$\begin{aligned}&-y'(t)\Pr(s_F=y(t)|s_L=x(t), s_F \le y(t)) \\ &\quad \cdot \big[\Pr(\theta=1|s_L=x(t), s_F=y(t))H - \Pr(\theta=0|s_L=x(t), s_F=y(t))L\big].\end{aligned}$$

This marginal benefit must be equal to the marginal cost which is 0. The first line is the probability that the follower invests in $[t, t + \mathrm{d}t)$, which is strictly positive. Therefore, the second line must be 0. This means if the follower invests in $[t, t + \mathrm{d}t)$, the leader learns that the follower's type is equal to $y(t)$, and the expected return of the project given the two signals $x(t)$ and $y(t)$ must be 0. In other words, the leader stays invested as long as his private signal (belief), conditional on the follower investing immediately, maps to a weakly positive expected payoff.

Equation (5) is derived from follower $y(t)$'s first-order condition. If the leader stays invested in $[t, t + \mathrm{d}t)$, the follower incurs a loss by waiting due to discounting:

$$r\big[\Pr(\theta = 1|s_F = y(t), s_L \geq x(t))H - \Pr(\theta = 0|s_F = y(t), s_L \geq x(t))L - c\big].$$

This term is strictly positive for otherwise the follower would not want to invest at t. If the leader disinvests in $[t, t + \mathrm{d}t)$, the follower's marginal benefit from waiting is

$$\begin{aligned}
&- x'(t)\Pr(s_L = x(t)|s_F = y(t), s_L \geq x(t)) \\
&\qquad \cdot \big[\Pr(\theta = 1|s_L = x(t), s_F = y(t))H - \Pr(\theta = 0|s_L = x(t), s_F = y(t))L - c\big].
\end{aligned}$$

The first term is the probability that the leader disinvests in $[t, t+\mathrm{d}t)$, which is strictly positive. The second line is the follower's expected payoff from investing, which is negative. If the leader disinvests in $[t, t + \mathrm{d}t)$, the follower learns that the leader's type is $x(t)$. As noted, the leader's first-order condition (4) implies that the expected return given $x(t)$ and $y(t)$ is 0. However, in addition to getting this zero return, the follower also pays the investment cost $c > 0$. Therefore, by waiting, the follower saves himself from an unprofitable investment.

It follows from the autonomous differential equation (5) and Lemma 2 that $T = \infty$. This means a leader with a posterior belief arbitrarily close to 1 never disinvests and a follower with a posterior belief arbitrarily close to 0 never invests.

Lemma 4. *In any monotonic symmetric dynamic equilibrium, $T = \infty$.*

3.2.3 Existence and uniqueness

Proposition 1 and Proposition 2 establish the necessary conditions for equilibrium. I now derive sufficient conditions for the existence and uniqueness of such an equilibrium. To this end, I impose the following two assumptions on the signal distributions.

Define the hazard ratio $h(s)$ and the failure ratio $k(s)$ as

$$h(s) := \frac{1 - F^0(s)}{1 - F^1(s)} \frac{f^1(s)}{f^0(s)} \text{ and } k(s) := \frac{F^0(s)}{F^1(s)} \frac{f^1(s)}{f^0(s)}.$$

Assumption. (F^0, F^1) satisfies the following properties:
(i) Increasing hazard ratio property (IHRP): $h(s)$ *is strictly increasing in* s.
(ii) Increasing failure ratio property (IFRP): $k(s)$ *is strictly increasing in* s.

In words, IHRP and IFRP state that higher signals are better news conditional on truncations.[13] Many of the commonly used signal distributions satisfy IHRP and IFRP. Examples include Beta distributions and Normal distributions.[14]

IHRP is not necessary and is assumed only for convenience: it enables a clean statement of the parametric restriction for the existence of the equilibrium. IFRP ensures uniqueness of the initial values and further uniqueness of the equilibrium.

Theorem 1. *There exists $\bar{c} > 0$ such that a monotonic symmetric dynamic equilibrium exists if and only if $c < \bar{c}$. Moreover, this equilibrium is unique.*

Learning dynamics. Figure 1 plots the equilibrium (inverse) strategies and illustrates the equilibrium dynamics. At $t = 0$, information is aggregated in a single burst. A player with a type higher than z invests right away. Upon observing this investment, a player with a type in $[y(0), z]$ follows suit. For $t > 0$, information is aggregated gradually and continuously through delayed investment and disinvestment. At each t, the leader disinvests if his type is $x(t)$ and stays invested if above $x(t)$. The follower invests if his type is $y(t)$ and does not invest if below $y(t)$.

Evolution of beliefs. I discuss the evolution of the players' actual posterior beliefs and the public belief in the leader-follower continuation game. A player's actual posterior belief is his posterior belief given his private signal and the public information

[13] Kalashnikov and Rachev (1985) introduce these two concepts in statistics, and Herrera and Hörner (2012) derive their properties in a standard sequential social learning model (à la Banerjee, 1992 and Bikhchandani et al., 1992).

[14] IHRP and IFRP are assumptions on the signal distributions. Because signals are taken to be posterior beliefs, IHRP and IFRP are also imposed on the posterior belief distributions. If the identification between signals and posterior beliefs is dropped, it can be readily verified that if the signal distributions satisfy IHRP, IFRP, and MLRP, the induced posterior belief distributions also satisfy IHRP and IFRP (in addition to MLRP).

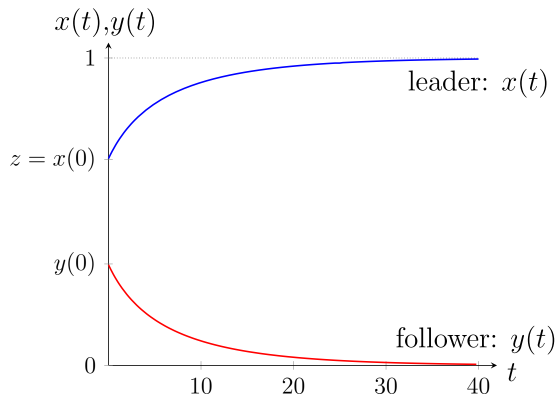

Figure 1: Equilibrium (inverse) strategies for $\rho_0 = 1/2, H = L = 1, r = 1/5, c = 1/5$ and posterior beliefs distributed according to $Beta(1 + \theta, 1 + (1 - \theta))$.

that the other player has not stopped. The public belief, denoted by $\rho(t)$, is the posterior belief given only the observable actions, $\rho(t) := \Pr(\theta = 1 | s_L \geq x(t), s_F < y(t))$.

A direct implication of the monotonicity of the equilibrium (inverse) strategies is that a player's actual posterior belief is monotone: the leader's posterior belief decreases over time and the follower's increases. The evolution of the public belief, however, is not as clear. The evolution of this belief is driven by two opposing forces: the leader staying in is good news but the follower staying out is bad. Which effect dominates depends on the relative rate at which the leader and the follower stop, as well as the relative likelihood of having a leader and a follower wait that long. Although the exact interplay of these two forces is difficult to pin down generally, one effect never overwhelms the other. The public belief is bounded away from 0 and 1, and eventually settles at an interior value ρ^* (ρ^* is defined in the Appendix).

Lemma 5. *$\rho(t) \in (0, 1)$ for all $t \geq 0$. As $t \to \infty$, $\rho(t) \to \rho^*$ with $\rho^* \in (0, 1)$.*

3.2.4 Takeaways

I conclude the equilibrium analysis by highlighting three features of the equilibrium.

First, the players' roles as the leader and follower are determined by their actions at time 0, when the players are ex ante symmetric. After time 0, the leader-follower continuation game is one with asymmetric players and this asymmetry is endogenous.

Second, the learning dynamic features bursts of information at the start of the game, and gradual revelation of information in the continuation game. In the continuation game, the duration of the leader's and the follower's waiting times respectively signal their optimism and pessimism. For each instant the leader stays invested, the follower becomes more optimistic; for each instant the follower stays out, the leader becomes more pessimistic. Moreover, the investment and disinvestment times fully reveal the types of the leader and the follower.

Lastly, the equilibrium dynamics are driven by a novel feature that the leader's incentive to disinvest solely comes from his fear of implementing a bad project and thus realizing a negative return — there is no flow cost of investment,[15] and when the leader disinvests, he does not recover the investment cost, nor get any return.

4 Role of Information Precision

In this section, I study how changes in the precision (or informativeness) of the players' private signals affect the equilibrium dynamics and outcomes. For tractability, I focus on a *symmetric environment*.[16] In a symmetric environment, the return H from investing in state $\theta = 1$ is equal to the loss L from investing in $\theta = 0$ and is normalized to 1, so $H = L = 1$. The prior belief about $\theta = 1$ is $\rho_0 = 1/2$. The conditional distributions F^0 and F^1 are symmetric about the prior 1/2: $F^1(\mu) = 1 - F^0(1 - \mu)$. In other words, the probability of having a posterior belief μ in state $\theta = 1$ is equal to the probability of having a posterior belief $1 - \mu$ in $\theta = 0$.

Definition of precision

Recall that without loss of generality, the players' types are taken to be their posterior beliefs given their private signals. I define precision in terms of the posterior belief distributions. By definition, any pair of conditional posterior belief distributions must satisfy the consistency condition:

$$\mu = \frac{\rho_0 f^1(\mu)}{\rho_0 f^1(\mu) + (1 - \rho_0) f^0(\mu)}. \tag{6}$$

A common way to rank distributions according to their precision is the mean-preserving spread of posteriors. I adopt a stronger notion, the unimodal likelihood ratio (ULR)

[15] One can show an equilibrium in a model where the leader pays a flow cost of investment converges in strategy to the equilibrium characterized in this section as the flow cost decreases to zero.

[16] I expect the results in this section hold for asymmetric environments up to some extent.

order,[17] which has more structure than mean-preserving spread: the posterior belief about the good state conditional on left (right) truncation is higher (lower) under a more precise distribution. The standard definition of the ULR order is as follows.

Definition 2. A function $\ell(\mu)$ is *unimodal* around $\tilde{\mu}$ if $\ell(\mu)$ is strictly increasing for $\mu < \tilde{\mu}$ and strictly decreasing for $\mu > \tilde{\mu}$.

Definition 3 (Hopkins and Kornienko, 2007, Definition 2)**.** For two distributions F and $\hat{F}$ with density f and $\hat{f}$ respectively and common support, F dominates $\hat{F}$ in the *unimodal likelihood ratio (ULR) order*, written as $F \succ_{\text{ULR}} \hat{F}$, if the likelihood ratio $f(\mu)/\hat{f}(\mu)$ is unimodal and the mean of F is (weakly) higher than the mean of $\hat{F}$.

I define *more precise than* by adapting the ULR order, which ranks two distributions, for two pairs of distributions. To set notation, define the pair of conditional posterior belief distributions as $\mathbf{F} := (F^0, F^1)$, and the ex ante posterior belief distribution of $\mathbf{F}$ as $F(\mu) := \rho_0 F^1(\mu) + (1 - \rho_0)F^0(\mu)$.

Definition 4. For two pairs of conditional posterior belief distributions $\mathbf{F}$ and $\hat{\mathbf{F}}$, $\hat{\mathbf{F}}$ is *more precise than* $\mathbf{F}$ if (i) the ex ante posterior belief distribution F dominates $\hat{F}$ in the unimodal likelihood ratio order, $F \succ_{\text{ULR}} \hat{F}$, and (ii) the mean of $\hat{F}^1$ is higher than the mean of F^1.[18]

This definition is better interpreted in the symmetric environment. Condition (ii) is intuitive: conditional on the state, a more precise pair of distributions should be on average "more accurate" than a less precise pair. Condition (i) implies that the likelihood ratio of F and $\hat{F}$ is unimodal and symmetric about the prior 1/2. Because the mean of F and $\hat{F}$ are both equal to the prior, condition (i) further implies $\hat{F}$ is a mean-preserving spread of F (see Hopkins and Kornienko, 2007, Proposition 1).

In line with the interpretation of the mean-preserving spread, $\hat{\mathbf{F}}$ is more precise than $\mathbf{F}$ if the ex ante posterior belief distribution $\hat{F}$ is more dispersed than F. It is less likely to get a posterior belief closer to the prior when information is more precise. This is captured by the likelihood ratio being unimodal around the prior.

Many commonly used (signal) distributions, such as Beta distributions with different variances and Normal distributions with different variances, induce posterior belief distributions that satisfy the ULR order.

[17] Ramos, Ollero, and Sordo (2000) first introduce the ULR order and Hopkins and Kornienko (2007) summarizes some of its properties.

[18] Because the mean of any ex ante posterior belief distribution is equal to the prior, condition (ii) is equivalent to the mean of $\hat{F}^0$ being lower than the mean of F^0.

4.1 Speed of Learning

I show that in the leader-follower continuation game, the equilibrium (inverse) strategies $x(t)$ and $y(t)$ are ordered pointwise for different levels of precisions.

Proposition 3. *Take two pairs of conditional posterior belief distributions* $\mathbf{F}$ *and* $\hat{\mathbf{F}}$ *with which a monotonic symmetric dynamic equilibrium exists. Suppose* $\hat{\mathbf{F}}$ *is more precise than* $\mathbf{F}$*. In the leader-follower continuation game, the type of leader (follower) who disinvests (invests) at any* $t \geq 0$ *is higher (lower) with* $\hat{\mathbf{F}}$*.*

Figure 2 illustrates this result by plotting the leader and the follower's equilibrium (inverse) strategies $x(t)$ and $y(t)$ as a function of t, for posterior beliefs that are induced by signals distributed according to the Beta distributions $Beta(1+\gamma\theta, 1+\gamma(1-\theta))$ for $\gamma > 0$. It is readily verified that the higher γ is, the higher the precision of the induced posterior belief distributions.

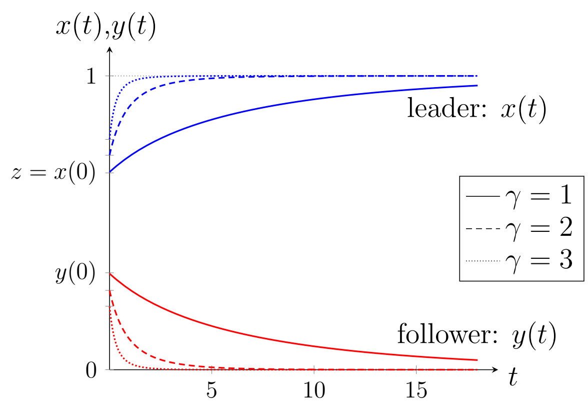

Figure 2: Equilibrium (inverse) strategies in a symmetric environment with $r = 1/5, c = 1/5$, and posterior beliefs induced by signals distributed according to $Beta(1+\gamma\theta, 1+\gamma(1-\theta))$.

The proposition can be restated as follows. For a player with any given private belief, if the precision is higher, he stops earlier. The leader and the follower reach an agreement (either both investing or both not investing) at the minimum of their stopping times. This means players reach an agreement faster.

Corollary 1. *Fix a pair of types for the leader and the follower. The time at which the leader and the follower reach an agreement is lower with* $\hat{\mathbf{F}}$*.*

In other words, higher precision leads to faster coordination. This result might seem intuitive, but it is not obvious. Suppose the leader's strategy $x(t)$ does not change with precision. If the leader has stayed invested up to t, the follower knows the leader's type must be above $x(t)$. The ULR order implies knowing the leader's type is above $x(t)$ is more optimistic news if information is more precise. Thus, if the follower waits the same amount of time before investing under a more precise distribution, his belief would be higher by the end of his waiting time, and thus he would deviate to investing earlier. The leader in turn would want to disinvest earlier. This suggests players reach an agreement faster if information is more precise.

4.2 Probability of Initial Coordination

I say that the players reach an initial agreement if either both players invest or neither player invests at $t = 0$. This includes three events: both players invest initially, neither invests initially, or one invests initially and the other follows suit. The previous section establishes that if information is more precise, conditional on no initial agreement, the resolution of disagreement is faster in the continuation game. How does the probability of initial agreement change with precision?

I begin by setting the notation for precision. Consider a set of distributions $\{\mathbf{F}_\gamma\}_{\gamma \geq 0}$ ordered by precision and indexed by a parameter $\gamma \geq 0$, where a higher γ indicates a higher precision. I refer this index γ as the precision of $\mathbf{F}$. Assume this indexed set of distributions satisfies the following property: as $\gamma \to 0$, $\mathbf{F}_\gamma$ converges (pointwise) to the (pair of) uninformative distributions, and as $\gamma \to \infty$, $\mathbf{F}_\gamma$ converges (pointwise) to the (pair of) perfectly informative distributions.[19] Denote the probability of initial agreement in the dynamic equilibrium by $\Pi_\gamma(x_\gamma(0))$, where $x_\gamma(0)$ is the initial value under $\mathbf{F}_\gamma$.

As before, consider the Beta distributions $\{Beta(1 + \gamma\theta, 1 + \gamma(1 - \theta))\}_{\gamma>0}$ for an illustration. Figure 3 plots the probability of initial agreement as a function of precision γ for two different levels of investment costs. It illustrates that there exists $\underline{\gamma}$ such that the dynamic equilibrium does not exist for $\gamma \leq \underline{\gamma}$. For $\gamma > \underline{\gamma}$, as γ increases, depending on the value of c, the probability of initial agreement either first decreases and then increases to 1, or monotonically increases to 1.

This example shows there exist parameters such that the probability of initial

[19] The indexing is arbitrary. The results hold for any set of distributions that satisfies this property.

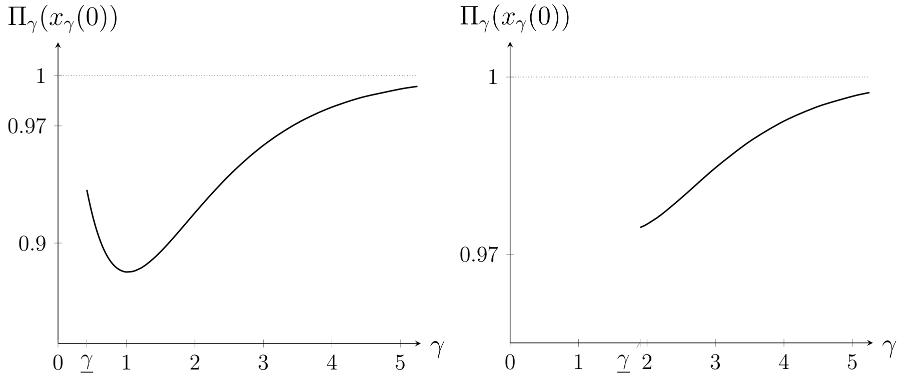

Figure 3: Probability of initial agreement in a symmetric environment with $c = 1/5$ (left panel) and $c = 3/5$ (right panel), and posterior beliefs induced by signals distributed according to $Beta(1 + \gamma\theta, 1 + \gamma(1 - \theta))$.

agreement is U-shaped in precision. Intuitively, the probability of coordination at the beginning of the game is governed by the difference in players' posterior beliefs upon observing the private signal. If the signals are uninformative, the players' posterior beliefs are the same as their prior. If the signals are perfectly informative, players update their belief to either 0 or 1. In both cases, the players' beliefs are the same as one another, so they would take the same action.[20] For signals that are partially informative, as precision increases, while they make players more informed, they also create variances in beliefs and might lead to a higher probability of disagreement.

The following proposition formalizes this intuition in the limit. The dynamic equilibrium exists only when precision is sufficiently high, and the probability of initial agreement in the dynamic equilibrium converges to 1 as precision gets arbitrarily high.

Proposition 4. *There exists $\underline{\gamma}$ such that a dynamic equilibrium exists if and only if $\gamma > \underline{\gamma}$. As $\gamma \to \infty$, the probability of initial agreement converges to* 1.

4.3 Ex ante Efficiency

A natural question is how ex ante efficiency, defined as the sum of the two players' ex ante equilibrium payoffs, changes as precision increases. In what follows, I derive a limiting result: the ex ante efficiency of the dynamic equilibrium approaches the full-efficiency payoff as precision gets arbitrarily high.

[20] In the symmetric environment, if the signals are uninformative, the expected payoff from investing given the prior is $-c < 0$. So neither player invests.

As information gets arbitrarily precise, by Proposition 4, the probability of initial agreement converges to 1; by Proposition 3, players reach an agreement faster if they had disagreed initially. Both players learn the state almost surely upon receiving the signals and are very likely to invest in the good state and not invest in the bad state right at time 0, which is the full-efficiency outcome, with payoff denoted by $\mathcal{E}^* = 1 - c$.

Proposition 5. *As $\gamma \to \infty$, the ex ante efficiency of the dynamic equilibrium converges to the full-efficiency payoff $\mathcal{E}^*$ from below.*

Numerical examples suggest the ex ante efficiency is increasing in precision. The limiting result Proposition 5, albeit seemingly intuitive, does not hold for all equilibria because of coordination. In particular, the no-investment equilibrium always generates 0 payoff, and all equilibria that do not start at time 0 will always fall short of the full-efficiency payoff regardless of how precise information gets.

5 Benchmarks

5.1 Constrained Efficiency

Consider a social planner who does not observe the signals or the true state θ and who seeks to maximize the sum of the expected payoff of the two players. The optimal mechanism, which is also incentive compatible, is straightforward. Each player truthfully reports his signal to the social planner; given the signals s_i and s_j, the social planner recommends investing at $t = 0$ to both players if the expected return from investing is higher than the cost, $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L \geq c$, and not investing for any $t \geq 0$ otherwise.

This constrained efficient outcome features efficient information aggregation, no delay, and no coordination failure. How is the eventual outcome of the equilibrium (whether the two players eventually invest or not[21]) different from the constrained efficient outcome? To illustrate, Figure 4 plots the eventual investment region (the complement is the no-investment region) in the type space. In the area above the blue curve, players eventually both invest in equilibrium. In the area above the red curve, players eventually both invest in the constrained efficient outcome.

[21] To be specific, in equilibrium, the outcome "eventual investment" is the events that both players invest initially and the follower invests before the leader disinvests. "Eventual no-investment" is the events that neither player invests initially and the leader disinvests before the follower invests.

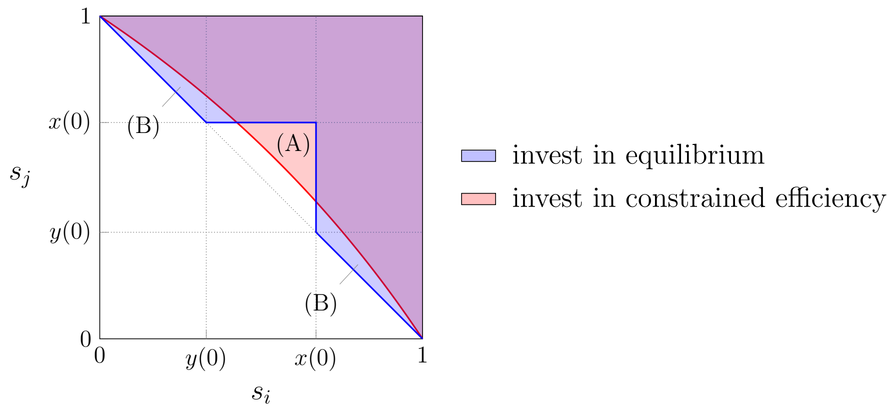

Figure 4: Investment regions in a symmetric environment with $c = 1/5, r = 1/5$ and posterior beliefs induced by $Beta(1 + \theta, 1 + (1 - \theta))$.

The eventual outcome of the equilibrium differs from the constrained efficient outcome in two ways. The first one is under-investment due to lack of information aggregation. In area (A), players invest in the constrained efficient outcome, but do not in equilibrium. Players are optimistic, but not optimistic enough without information about the other player. The second case is over-investment due to inefficient information aggregation. In area (B), the follower eventually invests in equilibrium but does not in the constrained efficient outcome. In equilibrium, the follower knows the leader's type is above a certain value, which makes him more optimistic than knowing it is equal to that value as in the constrained efficient outcome. This leads to over-investment by some lower types of the follower.

The eventual outcome of the equilibrium is consistent with the constrained efficient outcome if the posterior belief given the two signals is low enough, in which case eventually neither player invests, or high enough, in which case eventually both players invest. Proposition 6 provides sufficient conditions for this consistency.[22]

Proposition 6. *Fix a dynamic equilibrium. The eventual outcome of this equilibrium coincides with the constrained efficient outcome if* (s_i, s_j) *is such that*
(i) $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L < 0$*, or*
(ii) $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L > \kappa$ *with* $\kappa > c$.[23]

[22]Even if the equilibrium outcome is eventually consistent with the constrained efficient outcome, there are still inefficiencies due to delay and the leader paying the sunk investment cost.

[23]$\kappa := \Pr(\theta = 1|x(0), x(0))H - \Pr(\theta = 0|x(0), x(0))L$. This implies if both players' types are high enough that they invest initially in equilibrium, they invest in the constrained efficient outcome.

5.2 Irreversible Investments

The reversibility of investment is crucial for generating the dynamics in the continuation game. If investment is irreversible, all actions occur at time 0. A player invests initially if his type is above a threshold x_{ir}, and the other player follows suit if his type is above a threshold y_{ir} with $y_{\text{ir}} < x_{\text{ir}}$. Information revelation is coarse and occurs in a single burst at time 0.[24] (x_{ir} and y_{ir} are characterized in the Appendix.)

Compared to the investing behavior at time 0 in the dynamic equilibrium, the inability to exit from a potentially unprofitable investment deters some lower types of the leader from investing initially. In turn, an initial investment is better news and encourages more follower types to follow suit. Proposition 7 formalizes this intuition.

Proposition 7. *If $(z, (x(t), y(t))_{t\geq 0})$ is the equilibrium with reversible investment and (x_{ir}, y_{ir}) is an equilibrium with irreversible investment, then $x_{ir} > x(0)$ and $y_{ir} < y(0)$.*

I conclude with the remark that the equilibrium with irreversible investment might sometimes be more efficient than the dynamic equilibrium with reversible investment. In particular, players benefit from observational learning generated by reversible investments when the information is valuable to learn. If the precision of information is low, the benefit of commitment from irreversible investments outweighs. Figure 5 illustrates this intuition by plotting the ratio of the ex ante efficiency in the dynamic equilibrium to the irreversible investment equilibrium.

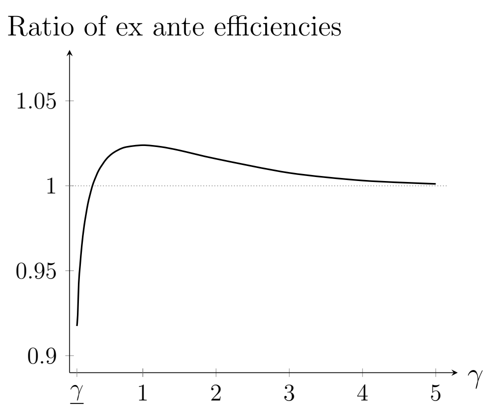

Figure 5: Ratio of ex ante efficiencies in a symmetric environment with $c = 1/20, r = 1/5$ and posterior beliefs induced by signals distributed as $Beta(1+\gamma\theta, 1+\gamma(1-\theta))$.

[24]One can show there exists a perfect Bayesian equilibrium in a model with no discounting that has the same outcome as this irreversible investment equilibrium.

A Appendix

A.1 Proofs for Section 3

A.1.1 Preliminaries

I first establish a useful implication of MLRP, which says knowing the signal is equal to z is worse news than knowing the signal is in an interval above z, and is better news than knowing the signal is in an interval below z. The proof follows directly from applying MLRP and is omitted.

Lemma 6. *$\hat{z} > z$ if and only if*

$$\frac{f^0(z)}{f^1(z)} > \frac{F^0(\hat{z}) - F^0(z)}{F^1(\hat{z}) - F^1(z)} > \frac{f^0(\hat{z})}{f^1(\hat{z})}.$$

The two tie-breaking rules will be referenced frequently in this section. Recall that the (payoff) tie-breaking rule says if the leader and the follower stop simultaneously, the leader gets 0 and the follower gets $-c$. The (indifference) tie-breaking rule says if the leader is indifferent between stopping at t' and at $t'' > t'$, he stops at t'.

A.1.2 Proof of Lemma 1

By the (payoff) tie-breaking rule, the follower's payoff from investing at any $t \geq T_L$ is $-c < 0$. So for any follower y, either $\sigma_F(y) < T_L$ or $\sigma_F(y) = \infty$. $T_F = \inf_t \{t : G_F(t) = G_F(T_L)\} \leq T_L$. Given T_F, the leader is indifferent between disinvesting at any $t \geq T_F$. By the (indifference) tie-breaking rule, leader disinvests at T_F. $T_L = \inf_t\{t : G_L(t) = G_L(T_F)\} \leq T_F$.

A.1.3 Proof of Lemma 2

Lemma 7. *$\sigma_L(x)$ is non-decreasing and $\sigma_F(y)$ is non-increasing.*

The proof of Lemma 7 follows from a standard revealed preference argument and is omitted. By MLRP, the leader of type x's expected payoff $\mathcal{L}(x, t)$ is supermodular in (x, t) and the follower of type y's expected payoff $\mathcal{F}(x, t)$ is submodular. Weak monotonicity follows from the Topkis's theorem.

I prove the σ_i's are strictly monotone. This is equivalent to proving i's equilibrium distribution of stopping time is non-atomic at any $t \in (0, T]$ for $i \in \{L, F\}$. There cannot be an atom at $t = 0$ because by construction, an atom at $t = 0$ is not well-defined in the leader-follower continuation game with $s_L \geq x(0)$ and $s_F \leq y(0)$.[25]

Leader's equilibrium distribution of stopping time is non-atomic

The idea is, if there is an atom at $\hat{t}$ in the leader's distribution of stopping time, either there is a mass of follower types that don't have a best response, or there does not exist a follower who invests in $[\hat{t} - \delta, \hat{t}]$. By the (indifference) tie-breaking rule, the leader stops at $\hat{t} - \delta$, which contradicts a mass of leader stopping at $\hat{t}$.

Suppose there is an atom at $\hat{t} \in (0, T]$ in the leader's equilibrium distribution of stopping time. Denote the follower of type y's belief at the beginning of the leader-follower game by $q_F(y) := \Pr(\theta = 1 | y, \sigma_L(x) \geq 0)$. Let $A(y, \hat{t})$ denote the jump in y's expected payoff from investing at $\hat{t}$ and $C(y, \hat{t})$ the expected cost from investing at $\hat{t}$,

$$
\begin{aligned}
A(y,t) :=& q_F(y)\Pr(\sigma_L(x) = t | \sigma_L(x) \geq 0, \theta = 1)H \\
& - (1 - q_F(y))\Pr(\sigma_L(x) = t | \sigma_L(x) \geq 0, \theta = 0)L; \\
C(y,t) :=& \big(q_F(y)\Pr(\sigma_L(x) = t | \sigma_L(x) \geq 0, \theta = 1) \\
& + (1 - q_F(y))\Pr(\sigma_L(x) = t | \sigma_L(x) \geq 0, \theta = 0)\big)c.
\end{aligned}
$$

Follower y's expected payoffs from investing at $\hat{t} - \varepsilon$, at $\hat{t}$, and at $\hat{t} + \varepsilon$ are respectively[26]

$$
\begin{aligned}
\mathcal{F}_-(y, \hat{t}) :=& \lim_{\varepsilon \to 0} \mathcal{F}(y, \hat{t} - \varepsilon) = e^{-r\hat{t}}\big(q_F(y)\Pr(\sigma_L(x) \geq \hat{t} | \sigma_L(x) \geq 0, \theta = 1)(H - c) \\
& - (1 - q_F(y))\Pr(\sigma_L(x) \geq \hat{t} | \sigma_L(x) \geq 0, \theta = 0)(L + c)\big), \\
\mathcal{F}(y, \hat{t}) =& \mathcal{F}_-(y, \hat{t}) - e^{-r\hat{t}}A(y, \hat{t}), \\
\mathcal{F}_+(y, \hat{t}) :=& \lim_{\varepsilon \to 0} \mathcal{F}(y, \hat{t} + \varepsilon) = \mathcal{F}(y, \hat{t}) + e^{-r\hat{t}}C(y, \hat{t}) = \mathcal{F}_-(y, \hat{t}) - e^{-r\hat{t}}(A(y, \hat{t}) - C(y, \hat{t})).
\end{aligned}
$$

By strict MLRP and that the f^θ's are continuous, $A(y, \hat{t}) - C(y, \hat{t})$ is strictly increasing and continuous in y. Because the likelihood ratio is unbounded, $\lim_{y \to 0} A(y, \hat{t}) - C(y, \hat{t}) < 0$. Because $y(0)$ optimally stops at $t = 0$, so $\mathcal{F}(y(0), 0) > 0$, together with Lemma 6 and $\sigma_L(x)$ non-decreasing, this implies $A(y(0), \hat{t}) - C(y(0), \hat{t}) > 0$. So there exists a unique $\hat{y} \in (0, y(0))$ such that $A(\hat{y}, \hat{t}) - C(\hat{y}, \hat{t}) = 0$, $A(y, \hat{t}) - C(y, \hat{t}) < 0$ for

[25] The time 0 here is not the calendar time 0 and is thus not the proper beginning of a stage. So a mass of types stopping at $t = 0$ of this continuation game is not a well-defined strategy.

[26] If $\hat{t} = T$, for $\theta = 0, 1$, $\Pr(\sigma_L(x) > T | \sigma_L(x) \geq 0, \theta) = 0$. So $\mathcal{F}_-(y, T) = e^{-rT}(A(y, T) - C(y, T))$, $\mathcal{F}_+(y, T) < 0$, and $\mathcal{F}(y, t) < 0$ for all $t \geq T$. The rest of the argument follows.

all $y \in (0, \hat{y})$, and $A(y, \hat{t}) - C(y, \hat{t}) > 0$ for all $y \in (\hat{y}, y(0))$.

Claim 1. For all y such that $A(y, \hat{t}) - C(y, \hat{t}) \geq 0$, $\sup_{t > \hat{t}} \mathcal{F}(y, t) = \mathcal{F}_+(y, \hat{t})$.

Proof. Let y be $A(y, \hat{t}) - C(y, \hat{t}) \geq 0$. y's expected payoff from investing at $t > \hat{t}$ is

$$\begin{aligned}
\mathcal{F}(y,t) =& e^{-rt}\big(q_F(y)\Pr(\sigma_L(x) > \hat{t} | \sigma_L(x) \geq 0, \theta = 1)(H - c) \\
&\qquad - (1 - q_F(y))\Pr(\sigma_L(x) > \hat{t} | \sigma_L(x) \geq 0, \theta = 0)(L + c)\big) \\
&- e^{-rt}\big(q_F(y)\Pr(\hat{t} < \sigma_L(x) < t | \sigma_L(x) \geq 0, \theta = 1)(H - c) \\
&\qquad - (1 - q_F(y))\Pr(\hat{t} < \sigma_L(x) < t | \sigma_L(x) \geq 0, \theta = 0)(L + c)\big) \\
&- e^{-rt} A(y, t).
\end{aligned}$$

$\mathcal{F}_+(y, \hat{t})$ can be written as

$$\begin{aligned}
\mathcal{F}_+(y,\hat{t}) =& e^{-r\hat{t}}\big(q_F(y)\Pr(\sigma_L(x) > \hat{t} | \sigma_L(x) \geq 0, \theta = 1)(H - c) \\
&\qquad - (1 - q_F(y))\Pr(\sigma_L(x) > \hat{t} | \sigma_L(x) \geq 0, \theta = 0)(L + c)\big).
\end{aligned}$$

$\sigma_L(x)$ non-decreasing and $A(y, \hat{t}) - C(y, \hat{t}) \geq 0$ imply $\mathcal{F}_+(y, \hat{t}) \geq 0$. $e^{-r\hat{t}} > e^{-rt}$ so the first term in $\mathcal{F}(y, t)$ is less than $\mathcal{F}_+(y, \hat{t})$. The second term in $\mathcal{F}(y, t)$ is negative by Lemma 6. For the third term, $A(y, t) = 0$ if there is no atom at t. Otherwise, $A(y, \hat{t}) \geq C(y, \hat{t}) > 0$ implies $A(y, t) > 0$ for $t > \hat{t}$. □

For type $\hat{y}$, $\mathcal{F}(\hat{y}, \hat{t}) < \mathcal{F}_-(\hat{y}, \hat{t}) = \mathcal{F}_+(\hat{y}, \hat{t})$. By Claim 1, $\mathcal{F}_+(\hat{y}, \hat{t})$ is the supremum over t for $t > \hat{t}$, so $\mathcal{F}(\hat{y}, t)$ cannot attain a maximum at $t > \hat{t}$. Therefore, either (i) $\mathcal{F}(\hat{y}, t)$ attains a maximum in $[0, \hat{t}]$, or (ii) $\mathcal{F}(\hat{y}, t)$ does not attain a maximum in $[0, \hat{t}]$.

Case (i) By definition, $C(y, \hat{t}) > 0$ for all y, so $\mathcal{F}_+(y, \hat{t}) > \mathcal{F}(y, \hat{t})$ for all y. If $\mathcal{F}(\hat{y}, t)$ attains a maximum at some $t^* \in [0, \hat{t}]$, it must be that $t^* < \hat{t}$. Because $\sigma_F(y)$ is non-increasing, if $\hat{y}$ invests at t^*, then all $y > \hat{y}$ will invest (at or) before t^* and all $y < \hat{y}$ will invest (at or) after t^*. So the only types who might invest in $(t^*, \hat{t}]$ are $y < \hat{y}$. For $y < \hat{y}$, $A(y, \hat{t}) - C(y, \hat{t}) < 0$, which implies $\mathcal{F}_-(y, \hat{t}) < \mathcal{F}_+(y, \hat{t})$. So if investing at some $t^{**} \in (t^*, \hat{t}]$ is optimal, it must be that $\mathcal{F}(y, t^{**}) \geq \mathcal{F}_+(y, \hat{t}) > \mathcal{F}_-(y, \hat{t})$ if $\mathcal{F}_+(y, \hat{t}) \geq 0$, or $\mathcal{F}(y, t^{**}) \geq 0 > \mathcal{F}_-(y, \hat{t})$ if $\mathcal{F}_+(y, \hat{t}) < 0$. Either way, there exists a $\delta > 0$ small such that $t^{**} \notin [\hat{t} - \delta, \hat{t}]$. This implies there does not exist a y such that $\sigma_F(y) \in [\hat{t} - \delta, \hat{t}]$. This means the leader is indifferent between stopping at $\hat{t} - \delta$ and $\hat{t}$. By the (indifference) tie-breaking rule, the leader stops at $\hat{t} - \delta$, which contradicts the hypothesis that there is a mass of leader types stopping at $\hat{t}$.

Case (ii) Suppose $\mathcal{F}(\hat{y}, t)$ doesn't attain a maximum in $[0, \hat{t}]$. Consider the incentive of the types that are higher than $\hat{y}$. Fix $y > \hat{y}$. Then $A(y, \hat{t}) - C(y, \hat{t}) > 0$, which implies $\mathcal{F}_-(y, \hat{t}) > \mathcal{F}_+(y, \hat{t}) > 0$ and $\mathcal{F}_-(y, \hat{t}) > \mathcal{F}(y, \hat{t})$. Claim 1 states that $\mathcal{F}_+(y, \hat{t})$ is the supremum over t for all $t > \hat{t}$. There are two sub-cases.

First, all $\mathcal{F}(y, t)$ with $y > \hat{y}$ up to sets of measure zero attain a maximum in $[0, \hat{t}]$. If $\mathcal{F}(y, t)$ attains a maximum at $t^* \in [0, \hat{t}]$, it must be that $\mathcal{F}(y, t^*) \geq \mathcal{F}_-(y, \hat{t}) > 0$ and $t^* < \hat{t}$. So there exists a $\delta > 0$ small such that $t^* \notin [\hat{t} - \delta, \hat{t}]$, which is a contradiction by the same argument as case (i). Second, there exists a positive measure of follower types y with $y > \hat{y}$ such that $\mathcal{F}(y, t)$ does not attain a maximum in $[0, \hat{t}]$. Then $\mathcal{F}_-(y, \hat{t})$ is the supremum of $\mathcal{F}(y, t)$ for all t. However, this supremum cannot be achieved: because $A(y, \hat{t}) > C(y, \hat{t}) > 0$, $\mathcal{F}(y, \hat{t}) = \mathcal{F}_-(y, \hat{t}) - e^{-r\hat{t}} A(y, \hat{t}) < \mathcal{F}_-(y, \hat{t})$.

Follower's equilibrium distribution of stopping time is non-atomic

Leader of type x's expected payoff from stopping at any t can be written as

$$\mathcal{L}(x, t) = \int_0^t e^{-r\tau} \big(\mathrm{d} \Pr(\theta = 1, \sigma_F(y) < \tau | x, \sigma_F(y) \geq 0) H \\ - \mathrm{d} \Pr(\theta = 0, \sigma_F(y) < \tau | x, \sigma_F(y) \geq 0) L \big).$$

For a fixed x, by the (first) fundamental theorem of calculus, $\mathcal{L}(x, t)$ is differentiable at every t where the integrand is continuous. Its derivative with respect to t, whenever exists, is given by the integrand evaluated at t, which can be written as $e^{-rt} \lim_{\varepsilon \to 0} (\Pr(\sigma_F(y) \in (t - \varepsilon, t] | x, \sigma_F(y) \geq 0) / \varepsilon) \cdot B(x, t)$, where

$$B(x, t) := \Pr(\theta = 1 | x, \sigma_F(y) = t) H - \Pr(\theta = 0 | x, \sigma_F(y) = t) L.$$

In words, $B(x, t)$ is leader x's expected return conditional on the follower investing at t. Thus, the derivative of $\mathcal{L}(x, t)$ at t is proportional to $B(x, t)$ when $\lim_{\varepsilon \to 0} \Pr(\sigma_F(y) \in (t - \varepsilon, t] | x, \sigma_F(y) \geq 0) / \varepsilon > 0$, and is equal to zero when this probability is zero.

Suppose there is an atom at $\hat{t} \in (0, T]$ in the follower's equilibrium distribution of stopping time. That is, $\sigma_F(y) = \hat{t}$ for all $y \in [y', y'']$ with $0 < y' < y'' < y(0)$. x's expected payoff from stopping at $\hat{t}$ is $\mathcal{L}(x, \hat{t})$, and from stopping at $\hat{t} + \varepsilon$ is

$$\lim_{\varepsilon \to 0} \mathcal{L}(x, \hat{t} + \varepsilon) = \mathcal{L}(x, \hat{t}) + e^{-r\hat{t}} \left(\Pr(\sigma_F(y) = \hat{t} | x, y < y(0)) \cdot B(x, \hat{t}) \right),$$

where $\Pr(\sigma_F(y) = \hat{t}|x, y < y(0)) = \Pr(y \in [y', y'']|x, y < y(0)) > 0$ and

$$B(x, \hat{t}) = \Pr(\theta = 1|x, y \in [y', y''])H - \Pr(\theta = 0|x, y \in [y', y''])L.$$

Define the left limit of $B(x, \hat{t})$ as $B_-(x, \hat{t})$,

$$B_-(x, \hat{t}) := \lim_{t \to \hat{t}-} B(x, t) = \Pr(\theta = 1|x, y = y'')H - \Pr(\theta = 0|x, y = y'')L.$$

I first show there is a positive measure of leader types that satisfy $B_-(x, \hat{t}) > 0 > B(x, \hat{t})$. By Lemma 6, for all x, $B_-(x, \hat{t}) > B(x, \hat{t})$. By strict MLRP and the f^θ's are continuous, $B(x, \hat{t})$ is strictly increasing and continuous in x. Because the likelihood ratio is unbounded, $\lim_{x \to 1} B(x, \hat{t}) > 0$. Because $x(0)$ optimally stops at $t = 0$, $B(x(0), 0) = 0$, together with Lemma 6 and $\sigma_F(y)$ non-increasing, $B(x(0), \hat{t}) < 0$. So there exists a unique $\hat{x} \in (x(0), 1)$ such that $B(\hat{x}, \hat{t}) = 0$. Similarly, there exists a unique $\hat{x}_- \in (x(0), 1)$ such that $B_-(\hat{x}_-, \hat{t}) = 0$. $\hat{x}_- < \hat{x}$ by MLRP. So for all $x \in (\hat{x}_-, \hat{x})$, $B_-(x, \hat{t}) > 0 > B(x, \hat{t})$.

Next, I show for all $x \in (\hat{x}_-, \hat{x})$, x stops at $\hat{t}$. Because $\sigma_F(y)$ is non-increasing, $B(x, t)$ is non-increasing in t. Then $B(x, t) \geq B_-(x, \hat{t}) > 0$ for all $t < \hat{t}$ and $B(x, t) \leq B(x, \hat{t}) < 0$ for all $t > \hat{t}$. This implies x's expected payoff from stopping at t is increasing in t for $t < \hat{t}$ and decreasing for $t > \hat{t}$. If x has a best response, it can only be at $\hat{t}$. $B(x, \hat{t}) < 0$ implies $\lim_{\varepsilon \to 0} \mathcal{L}(x, \hat{t} + \varepsilon) < \mathcal{L}(x, \hat{t})$. So $\sup_t \mathcal{L}(x, t) = \mathcal{L}(x, \hat{t})$.

Thus, there exists a positive measure of leader types who stop at $\hat{t}$, which contradicts the leader's distribution of stopping time is non-atomic.

A.1.4 Proof of Lemma 3

Recall that $G_i(t) = \Pr(\sigma_i(s_i) \leq t|\sigma_i(s_i) \geq 0)$ for $i \in \{L, F\}$ and $G_i^\theta(t) = \Pr(\sigma_i(s_i) \leq t|\sigma_i(s_i) \geq 0, \theta)$ for $\theta \in \{0, 1\}$. To simplify notation, let $\mathcal{F}(y, t) = e^{-rt}\mathcal{G}(y, t)$ where $\mathcal{G}(y, t) := q_F(y)(1 - G_L^1(t))(H - c) - (1 - q_F(y))(1 - G_L^0(t))(L + c)$.

I first show if G_i is constant on $[t', t'']$, G_{-i} is constant on $[t', t'']$ for $0 < t' < t'' < T$.

Suppose $G_L(t)$, thus $G_L^\theta(t)$, is constant on $[t', t'']$. So $\mathcal{G}(y, t)$ is constant on $[t', t'']$. For $t \in [t', t'']$, fix y such that $\mathcal{G}(y, t) > 0$, then $\mathcal{F}(y, t') > \mathcal{F}(y, t) > 0$ for all $t \in (t', t'']$, so investing at any $t \in (t', t'']$ is dominated by investing at t'. If $\mathcal{G}(y, t) < 0$, investing at any $t \in [t', t'']$ is dominated by not investing. By strict MLRP, the set of y's that satisfy $\mathcal{G}(y, t) = 0$ is of measure zero. Thus, there does not exist a positive measure of follower types that invest in $(t', t'']$, which means $G_F(t)$ is constant on $[t', t'']$.

Suppose $G_F(t)$ is constant on $[t', t'']$. By the (indifference) tie-breaking rule, for any leader x such that disinvesting at any $t \in [t', t'']$ is optimal, x disinvests at t'. So no leader will disinvest in $(t', t'']$, which means $G_L(t)$ is constant on $[t', t'']$.

Next, I show that there does not exist an interval $[t', t'']$ with $0 < t' < t'' < T$ such that both G_F and G_L are constant. Suppose the contrary and let $\bar{t}$ be the supremum of t'' for which over $[t', t'']$, G_L and G_F are constant.

Fix y such that $\mathcal{G}(y,t) > 0$, so $\mathcal{F}(y,t') > \mathcal{F}(y,t)$ for all $t \in (t', \bar{t}]$. In particular, $\mathcal{F}(y,t') > \mathcal{F}(y,\bar{t})$. By Lemma 2, $\mathcal{F}(y,t)$ is continuous in t, so for $\delta > 0$, $\lim_{\delta \to 0} \mathcal{F}(y, \bar{t} + \delta) = \mathcal{F}(y,\bar{t}) < \mathcal{F}(y,t')$. So investing at any $t \in [\bar{t}, \bar{t}+\delta]$ is dominated by investing at t'. If $\mathcal{G}(y,t) < 0$, $\mathcal{F}(y,t') < \mathcal{F}(y,\bar{t}) < 0$ for all $t \in (t', \bar{t}]$. $\lim_{\delta \to 0} \mathcal{F}(y, \bar{t}+\delta) = \mathcal{F}(y,\bar{t}) < 0$, so investing at $t \in [\bar{t}, \bar{t}+\delta]$ is dominated by not investing. There does not exist a positive measure of follower types that invest in $[\bar{t}, \bar{t}+\delta]$. This means G_F is constant on $[t', \bar{t}+\delta]$, so G_L is also constant on $[t', \bar{t}+\delta]$. This contradicts the definition of $\bar{t}$.

A.1.5 Proof of Proposition 2

Differentiability

First, consider the leader's incentive. Recall the proof of Lemma 2. By Lemma 2 and Lemma 3, for any fixed x, x's expected payoff $\mathcal{L}(x,t)$ is differentiable everywhere in $t \in (0,T)$. The first-order condition $B(x(t),t) = 0$ implies

$$\frac{f^1(x(t))}{f^0(x(t))} \frac{f^1(y(t))}{f^0(y(t))} = \frac{L}{H} \frac{1-\rho_0}{\rho_0}. \tag{7}$$

Next, consider the follower's expected payoff. As before, let $\mathcal{F}(y,t) = e^{-rt}\mathcal{G}(y,t)$. By Lemma 2 and Lemma 3, $\mathcal{G}(y,t)$ can be written as

$$\mathcal{G}(y,t) = q_F(y) \frac{1 - F^1(x(t))}{1 - F^1(x(0))}(H - c) - (1 - q_F(y)) \frac{1 - F^0(x(t))}{1 - F^0(x(0))}(L + c) \tag{8}$$

with

$$q_F(y) = \frac{\rho_0 f^1(y)(1 - F^1(x(0)))}{\rho_0 f^1(y)(1 - F^1(x(0))) + (1-\rho_0) f^0(y)(1 - F^0(x(0)))}. \tag{9}$$

Fix $t^* \in (0,T)$. Let $\{t_n\}_{n=1}^{\infty}$ be a decreasing sequence such that $t_n \geq t^*$ for all n and $t_n \to t^*$ as $n \to \infty$. Denote the type of follower who optimally stops at t^* by $y^* = y(t^*)$. By optimality, $\mathcal{F}(y^*, t_n) \leq \mathcal{F}(y^*, t^*)$ for all t_n, which means $\mathcal{G}(y^*, t_n) \leq$

$e^{r(t_n-t^*)}\mathcal{G}(y^*,t^*)$. Because e^{-rt} is Lipschitz-continuous with Lipschitz constant r, so $e^{r(t_n-t^*)} \leq re^{rt_n}(t_n-t^*)+1$. Thus, $\mathcal{G}(y^*,t_n)-\mathcal{G}(y^*,t^*) \leq re^{rt_n}(t_n-t^*)\mathcal{G}(y^*,t^*)$. I now show $\mathcal{G}(y^*,t_n)-\mathcal{G}(y^*,t^*) \geq 0$. Writing out $\mathcal{G}(y^*,t_n)-\mathcal{G}(y^*,t^*)$ using the definition of $\mathcal{G}$ given by (8), $\mathcal{G}(y^*,t_n)-\mathcal{G}(y^*,t^*) \geq 0$ if and only if

$$\frac{F^1(x(t_n))-F^1(x(t^*))}{F^0(x(t_n))-F^0(x(t^*))}\frac{f^1(y^*)}{f^0(y^*)} \leq \frac{L+c}{H-c}\frac{1-\rho_0}{\rho_0}.$$

Take the limit of the left-hand side as $t_n \to t^*$,

$$\lim_{t_n\to t^*}\frac{F^1(x(t_n))-F^1(x(t^*))}{F^0(x(t_n))-F^0(x(t^*))}\frac{f^1(y^*)}{f^0(y^*)} = \frac{f^1(x(t^*))}{f^0(x(t^*))}\frac{f^1(y^*)}{f^0(y^*)} = \frac{L}{H}\frac{1-\rho_0}{\rho_0} < \frac{L+c}{H-c}\frac{1-\rho_0}{\rho_0},$$

where the first equality follows from $x(t)$ continuous and the second equality is (7). Thus, there exists N such that for all $n \geq N$, $\mathcal{G}(y^*,t_n)-\mathcal{G}(y^*,t^*) \geq 0$. Therefore,

$$0 \leq \mathcal{G}(y^*,t_n)-\mathcal{G}(y^*,t^*) \leq re^{rt_n}(t_n-t^*)\mathcal{G}(y^*,t^*).$$

Because $re^{rt_n}\mathcal{G}(y^*,t^*)$ is finite, $\mathcal{G}(y^*,t)$ is locally Lipschitz-continuous and thus differentiable almost everywhere in a neighborhood around t^*. Because the F^θ's are (twice) continuously differentiable, for a fixed y, $\mathcal{G}(y,t)$ is a continuously differentiable function of $x(t)$, which implies $x(t)$ is differentiable almost everywhere in a neighborhood around t^*. The same argument holds for every $t^* \in (0,T)$, so $x(t)$ is differentiable almost everywhere in $t \in (0,T)$. By (7), $y(t)$ can be written as a continuously differentiable function of $x(t)$, so $y(t)$ is also differentiable almost everywhere.

Equilibrium conditions (4) and (5) follow from differentiating $\mathcal{L}(x,t)$ and $\mathcal{F}(x,t)$ with respect to t, and setting the resulting derivative to zero whenever it exists, where

$$\mathcal{L}(x,t) = q_L(x)\int_0^t -e^{-r\tau}y'(\tau)\frac{f^1(y(\tau))}{F^1(y(0))}\mathrm{d}\tau H - (1-q_L(x))\int_0^t -e^{-r\tau}y'(\tau)\frac{f^0(y(\tau))}{F^0(y(0))}\mathrm{d}\tau L$$

$$\mathcal{F}(y,t) = e^{-rt}\left(q_F(y)\frac{1-F^1(x(t))}{1-F^1(x(0))}(H-c) - (1-q_F(y))\frac{1-F^0(x(t))}{1-F^0(x(0))}(L+c)\right)$$

with $q_F(y)$ given by (9) and $q_L(x) = \Pr(\theta=1|x, y \leq y(0))$ is given by

$$q_L(x) = \frac{\rho_0 f^1(x)F^1(y(0))}{\rho_0 f^1(x)F^1(y(0)) + (1-\rho_0)f^0(x)F^0(y(0))}.$$

By the (second) fundamental theorem of calculus, (5) can be written as

$$x(t) = x(0) + \int_0^t r\left(\frac{(H-c)L}{(L+H)c}\frac{1-F^1(x(\tau))}{f^1(x(\tau))} - \frac{(L+c)H}{(L+H)c}\frac{1-F^0(x(\tau))}{f^0(x(\tau))}\right)d\tau.$$

By Lemma 2, $x(t)$ is continuous for all $t \in [0,T]$ and so is the integrand. By the (first) fundamental theorem of calculus, $x(t)$ is differentiable everywhere in $t \in (0,T)$. So $y(t)$ is also differentiable everywhere.

Optimality

Lemma 8. *Fix x, $\mathcal{L}(x,t)$ is single-peaked in t. Fix y, $\mathcal{F}(y,t)$ is single-peaked in t.*

Proof. Let the subscript i of a function denote the derivative with respect to the function's i-th argument. The derivative of $\mathcal{L}(x,t)$ with respect to t is

$$\mathcal{L}_2(x,t) = -y'(t)e^{-rt}\left(q_L(x)\frac{f^1(y(t))}{F^1(y(0))}H - (1-q_L(x))\frac{f^0(y(t))}{F^0(y(0))}L\right).$$

By strict MLRP and $y(t)$ strictly decreasing, there exists a unique t^* such that $\mathcal{L}_2(x,t^*) = 0$, $\mathcal{L}_2(x,t) > 0$ for $t < t^*$, and $\mathcal{L}_2(x,t) < 0$ for $t > t^*$.

For the follower, the first-order condition implies $\mathcal{G}_2(y(t),t) = r\mathcal{G}(y(t),t)$. Because strategies are everywhere differentiable, at each t, there is one and only one type whose first-order condition is satisfied. Denote the type whose first-order condition is satisfied at t^* by y^*, that is, $\mathcal{G}_2(y^*,t^*) = r\mathcal{G}(y^*,t^*)$. Suppose y^* mimics type $\hat{y}$ by stopping at $\hat{t}$. By the (second) fundamental theorem of calculus,

$$\mathcal{G}_2(y^*,\hat{t}) = \mathcal{G}_2(\hat{y},\hat{t}) + \int_{\hat{y}}^{y^*}\mathcal{G}_{21}(y,\hat{t})\mathrm{d}y = r\mathcal{G}(\hat{y},\hat{t}) + \int_{\hat{y}}^{y^*}\mathcal{G}_{21}(y,\hat{t})\mathrm{d}y,$$

where $\mathcal{G}_{21}(y,\hat{t}) = \mathrm{d}\mathcal{G}_2(y,\hat{t})/\mathrm{d}y$. The second equality follows from $\hat{y}$'s first-order condition $\mathcal{G}_2(\hat{y},\hat{t}) = r\mathcal{G}(\hat{y},\hat{t})$. By MLRP, $\mathcal{G}_{21}(y,\hat{t}) < 0$. Thus, if $\hat{y} < y^*$,

$$\mathcal{G}_2(y^*,\hat{t}) = r\mathcal{G}(\hat{y},\hat{t}) + \int_{\hat{y}}^{y^*}\mathcal{G}_{21}(y,\hat{t})\mathrm{d}y < r\mathcal{G}(\hat{y},\hat{t}) < r\mathcal{G}(y^*,\hat{t}),$$

where the first inequality follows from $\int_{\hat{y}}^{y^*}\mathcal{G}_{21}(y,\hat{t})\mathrm{d}y < 0$ and the second inequality follows from MLRP. Similarly, if $\hat{y} > y^*$, then $\mathcal{G}_2(y^*,\hat{t}) > r\mathcal{G}(y^*,\hat{t})$. Because $y(t)$ is decreasing, $\hat{y} < (>) y^*$ if and only if $\hat{t} > (<) t^*$. The result follows. □

A.1.6 Proof of Lemma 4

Define the right-hand side of (5) as $\phi(\cdot)$ and rewrite (5) as $x'(t) = \phi(x(t))$. Let $\bar{x} := \min\{x : \phi(x) = 0\}$. Because the support of f^1 and f^0 is $[0,1]$ which is bounded, as $x \to 1$, $(1 - F^1(x))/f^1(x) \to 0$ and $(1 - F^0(x))/f^0(x) \to 0$, so $\phi(x) \to 0$. Thus, $\bar{x} \leq 1$. The differential equation $x'(t) = \phi(x(t))$ is autonomous and $\phi(x)$ is continuous for all $x \in (0,1)$. Given an initial value $x^*(0)$, the solution to $x'(t) = \phi(x(t))$, denoted by $x^*(t)$, is either constant or monotone. So if $\phi(x^*(0)) > 0$, then $\phi(x^*(t)) > 0$ for all t and $\phi(x^*(t)) \to 0$ as $x^*(t) \to \bar{x}$ (Teschl, 2012, Lemma 1.1). This implies $x^*(t) < \bar{x} \leq 1$ for all t. Let $x(t)$ denote the equilibrium (inverse) strategy. If $x(t)$ is a solution to $x'(t) = \phi(x(t))$ for all t, then $x(t) < \bar{x} \leq 1$ for all t. Therefore, if there exists $T < \infty$ such that $x(T) = 1$, $x(t)$ must be discontinuous at T. This contradicts Lemma 2.

A.1.7 Proof of Proposition 1

Equilibrium conditions

Equation (3) is given by evaluating equation (4) at $t = 0$. I first derive condition (1).

By definition, $y(0) \leq z \leq x(0)$. I first show $y(0) < z$ if and only if $x(0) = z$.

Suppose $y(0) < z$. Suppose $y(0)$ waits a small amount of time $\mathrm{d}t$ before investing. If the leader stays invested in $[0, \mathrm{d}t)$, $y(0)$ invests at $\mathrm{d}t$ and gets $\lim_{\mathrm{d}t \to 0} \mathcal{F}(y(0), \mathrm{d}t) = \mathcal{F}(y(0), 0)$. If the leader disinvests in $[0, \mathrm{d}t)$, $y(0)$ never invests and gets 0. $y(0)$'s expected payoff from waiting is

$$\Pr(s_L \geq x(0) | y(0), s_L > z)\mathcal{F}(y(0), 0) + \Pr(s_L \in (z, x(0)) | y(0), s_L > z)(0). \tag{10}$$

Suppose $y(0)$ invests. If the leader stays invested, $y(0)$ gets $\Pr(\theta = 1 | y(0), s_L \geq x(0))H - \Pr(\theta = 0 | y(0), s_L \geq x(0))L - c$, which is equal to $\mathcal{F}(y(0), 0)$. If the leader disinvests, $y(0)$ gets $-c$. So his expected payoff from investing is

$$\Pr(s_L \geq x(0) | y(0), s_L > z)\mathcal{F}(y(0), 0) + \Pr(s_L \in (z, x(0)) | y(0), s_L > z)(-c). \tag{11}$$

By optimality, (11)$-$(10)$= 0$, which holds if and only if $x(0) = z$.

Next, I show there does not exist a dynamic equilibrium with $x(0) > z$. Suppose $x(0) > z$. From step 1, $y(0) = z$. Consider z's incentive in the initial stage. Suppose z invests in the initial stage. If the other player invests, payoff realizes. If the other player does not invest, z disinvests immediately (because $x(0) > z$). z's expected payoff from investing in the initial stage is

$$\Pr(\theta = 1, s_{-i} > z|z)H - \Pr(\theta = 0, s_{-i} > z|z)L - c. \tag{12}$$

Suppose z does not invest in the initial stage. If the other player does not invest, by optimality, z gets at least 0. If the other player invests, z invests at $\mathrm{d}t$ (because $y(0) = z$). z's expected payoff from not investing in the initial stage is at least

$$\Pr(\theta = 1, s_{-i} > x(0)|z)(H - c) - \Pr(\theta = 0, s_{-i} > x(0)|z)(L + c). \tag{13}$$

Players' incentive in the leader-follower continuation game doesn't change. For $x(0) > z = y(0)$, (4) becomes $\rho_0 f^1(x(0))f^1(z)H = (1-\rho_0)f^0(x(0))f^0(z)L$. Together with MLRP, (13)>(12). This is a contradiction as z is not indifferent in the initial stage.

Equation (2) is given by type z's indifference between investing and not investing in the initial stage given the continuation strategies.

Suppose z invests in the initial stage. If the other player invests, payoff realizes. If the other player does not invest, the game moves to the leader-follower continuation game where z is the leader. z stays invested at the beginning of the leader-follower continuation game at which if the follower follows suit, payoff realizes, otherwise z disinvests at $\mathrm{d}t$ and gets $\lim_{\mathrm{d}t\to 0}\mathcal{L}(z, \mathrm{d}t) = 0$.[27] z's expected payoff from investing is

$$\begin{aligned} &\Pr(s_{-i} > z|z)\left[\Pr(\theta = 1|z, s_{-i} > z)H - \Pr(\theta = 0|z, s_{-i} > z)L - c\right] \\ &+ \Pr(s_{-i} \le z|z)\big(\Pr(s_{-i} \in [y(0), z]|z, s_{-i} \le z) \\ &\qquad \cdot \left[\Pr(\theta = 1|z, s_{-i} \in [y(0), z])H - \Pr(\theta = 0|z, s_{-i} \in [y(0), z])L\right] - c\big). \end{aligned} \tag{14}$$

Suppose z does not invest in the initial stage. If the other player does not invest, the game moves to a continuation game in which neither player has invested. Call this a *no-investment continuation game*. Denote z's payoff in the no-investment continuation game by $U(z)$. By optimality, $U(z) \ge 0$. If the other player invests, the game moves to the leader-follower continuation game where z is the follower. At the beginning of the leader-follower continuation game, by (1), z invests and the leader

[27] Technically, "disinvesting at $\mathrm{d}t$" is not a well-defined best response. This does not compromise the analysis as it arises only on a zero-measure set. The threshold z is well-defined as it is shown below that all types above z strictly prefer investing and all types below strictly prefer not investing.

stays invested with probability 1. z's expected payoff from not investing is

$$\begin{aligned}&\Pr(s_{-i} > z|z)\left[\Pr(\theta = 1|z, s_{-i} > z)H - \Pr(\theta = 0|z, s_{-i} > z)L - c\right]\\ &+ \Pr(s_{-i} \leq z|z)U(z).\end{aligned} \tag{15}$$

I now show that players never invest after no initial investments. So $U(z) = 0$.

Lemma 9. *In any monotonic symmetric dynamic equilibrium, $\sigma_i(s_i, \varnothing) \in \{0, \infty\}$.*

Proof. If there exists an equilibrium where players invest in the no-investment continuation game, z, being the highest type, must find it optimal to invest. The best z can do in this continuation game is to signal he is the highest type. Suppose he can do that. Given i's type is equal to z, by strict MLRP, there exists a unique $\underline{y}$ that is indifferent between investing and not investing, types in $(\underline{y}, z)$ invest, and types below $\underline{y}$ never invest. Let $\overline{U}(z)$ denote z's expected payoff in this best case scenario,

$$\overline{U}(z) = \Pr(\theta = 1, s_{-i} \in [\underline{y}, z]|z, s_{-i} \leq z)H - \Pr(\theta = 0, s_{-i} \in [\underline{y}, z]|z, s_{-i} \leq z)L - c.$$

$\underline{y}$'s indifference condition is $\Pr(\theta = 1|\underline{y}, z)H - \Pr(\theta = 0|\underline{y}, z)L = c$. By (1) and (4), $\Pr(\theta = 1|y(0), z)H - \Pr(\theta = 0|y(0), z)L = 0$. So $y(0) < \underline{y}$ by MLRP. By Lemma 6,

$$\begin{aligned}&\Pr(\theta = 1, s_{-i} \in [y(0), z]|z, s_{-i} \leq z)H - \Pr(\theta = 0, s_{-i} \in [y(0), z]|z, s_{-i} \leq z)L - c\\ >&\Pr(\theta = 1, s_{-i} \in [\underline{y}, z]|z, s_{-i} \leq z)H - \Pr(\theta = 0, s_{-i} \in [\underline{y}, z]|z, s_{-i} \leq z)L - c = \overline{U}(z).\end{aligned}$$

This means z's expected payoff from investing in the initial stage is strictly higher than not investing. This is a contradiction as z is not indifferent. □

In equilibrium, (15)=(14). With $U(z) = 0$, this reduces to

$$\begin{aligned}c =& \Pr(s_{-i} \in [y(0), z]|s_i = z, s_{-i} \leq z)\\ &\cdot\left[\Pr(\theta = 1|s_i = z, s_{-i} \in [y(0), z])H - \Pr(\theta = 0|s_i = z, s_{-i} \in [y(0), z])L\right].\end{aligned} \tag{16}$$

(2) follows from writing out these probability terms. By the law of total probability,

$$\begin{aligned}&\Pr(s_{-i} \in [y(0), z]|s_i = z, s_{-i} \leq z)\\ =&\Pr(s_{-i} \in [y(0), z]|s_i = z, s_{-i} \leq z, \theta = 1)\Pr(\theta = 1|s_i = z, s_{-i} \leq z)\\ &+ \Pr(s_{-i} \in [y(0), z]|s_i = z, s_{-i} \leq z, \theta = 0)\Pr(\theta = 0|s_i = z, s_{-i} \leq z).\end{aligned}$$

Because signals are conditionally independent, for $\theta = 0, 1$,

$$\begin{aligned}
\Pr(s_{-i} \in [y(0), z] | s_i = z, s_{-i} \leq z, \theta) =& 1 - F^\theta(y(0))/F^\theta(z), \\
q(z) := \Pr(\theta = 1 | s_i = z, s_{-i} \leq z) =& \frac{\rho_0 f^1(z) F^1(z)}{\rho_0 f^1(z) F^1(z) + (1-\rho_0) f^0(z) F^1(z)}, \\
\Pr(\theta = 1 | s_i = z, s_{-i} \in [y(0), z]) =& \frac{\rho_0 f^1(z)(F^1(z) - F^1(y(0)))}{\begin{pmatrix} \rho_0 f^1(z)(F^1(z) - F^1(y(0))) \\ +(1-\rho_0) f^0(z)(F^0(z) - F^0(y(0))) \end{pmatrix}}.
\end{aligned}$$

Optimality

By (16) and Lemma 6, z's expected payoff from investing is strictly positive. I verify that types above z strictly prefer investing at time 0. The argument for types below z is analogous and thus omitted. Fix a type $x > z$. The only difference between x's and z's strategies is x disinvests at $\sigma_L(x) > 0$ in the leader-follower continuation game. The difference in x's payoff between investing and not investing at time 0 is

$$\begin{aligned}
\Pr(s_{-i} \leq z | x) \big(& \Pr(s_{-i} \in [y(0), z] | x, s_{-i} \leq z) \\
& \cdot [\Pr(\theta = 1 | x, s_{-i} \in [y(0), z]) H - \Pr(\theta = 0 | x, s_{-i} \in [y(0), z]) L] - c \\
& + \Pr(s_{-i} < y(0) | x, s_{-i} \leq z) \mathcal{L}(x, \sigma_L(x)) \big).
\end{aligned}$$

To show this is positive, first, by optimality and Lemma 8, for all $x > z$, the last line $\mathcal{L}(x, \sigma_L(x)) > \lim_{\mathrm{d}t \to 0} \mathcal{L}(x, \mathrm{d}t) = 0$. It then suffices to establish the first two lines is positive, which follows from z's indifference condition (16) and MLRP.

A.1.8 Proof of Theorem 1

Lemma 10. *There exists a unique set of solution $(z, x(0), y(0))$ to the system of equations (1), (2), and (3).*

Proof. From (3), one can write $y(0)$ as a function of z, denoted by $y_0(z)$. $y_0(z)$ is decreasing in z by MLRP. The initial condition (16) can be written as $\mathcal{V}(z) = c$ where

$$\mathcal{V}(z) := q(z) \left(1 - \frac{F^1(y_0(z))}{F^1(z)}\right) H - (1 - q(z)) \left(1 - \frac{F^0(y_0(z))}{F^0(z)}\right) L.$$

$\mathcal{V}(z)$ is continuous in z. As $z \to y(0)$, $\mathcal{V}(z) \to 0 < c$. As $z \to 1$, because the likelihood ratio is unbounded, $y_0(z) \to 0$ and $q(z) \to 1$. So $\lim_{z\to 1} \mathcal{V}(z) = H > c$. By strict MLRP and IFRP, $\mathcal{V}(z)$ is strictly increasing in z. The result follows. $\square$

Therefore, $(z, x(0), y(0))$ is unique. Because the f^θ's are continuously differentiable, by the Picard–Lindelöf theorem, given $(x(0), y(0))$, there exists a unique solution $(x(t), y(t))$ to the differential system (4) and (5). This solution is an equilibrium if and only if $x(t)$ is strictly increasing and $y(t)$ is strictly decreasing. Equation (4) and MLRP imply it suffices to establish conditions under which $x(t)$ is strictly increasing.

Claim 2. Let $x(t)$ be the solution to the differential equation (5) with initial value $x(0)$. $x(t)$ is strictly increasing for $t \geq 0$ if and only if

$$h(x(0)) < \frac{(H-c)L}{(L+c)H}. \tag{17}$$

Proof. Because $x'(t) = \phi(x(t))$ is a first-order autonomous differential equation where $\phi(\cdot)$ is continuous, the solution $x(t)$ to this differential equation with initial value $x(0)$ is either constant or monotone. $x(t)$ is strictly increasing if and only if $\phi(x(0)) > 0$ (Teschl, 2012, Lemma 1.1), which reduces to (17).[28] $\square$

Claim 3. There exists a unique $\bar{c} \in (0, H)$ such that

$$h(\mathcal{V}^{-1}(\bar{c})) = \frac{(H-\bar{c})L}{(L+\bar{c})H}. \tag{18}$$

Proof. The right-hand side of (18) is continuous and strictly decreasing in c. For all $c \in (0, H)$, $(H-c)L/((L+c)H)$ takes values in $(0,1)$ and converges to 1 as $c \to 0$ and to 0 as $c \to H$. For the left-hand side of (18), from the proof of Lemma 10, for all $z > \underline{z}$, where $\underline{z}$ solves $\rho_0 f^1(\underline{z}) f^1(\underline{z}) H - (1-\rho_0) f^0(\underline{z}) f^0(\underline{z}) L = 0$, $\mathcal{V}(z)$ is increasing in z. This implies $\mathcal{V}^{-1}(c)$ is increasing in c with $\mathcal{V}^{-1}(c) \to \underline{z}$ as $c \to 0$ and $\mathcal{V}^{-1}(c) \to 1$ as $c \to H$. By IHRP, $h(\mathcal{V}^{-1}(c))$ is strictly increasing in c, and by MLRP, $h(\mathcal{V}^{-1}(c))$ converges to some constant $\underline{h} \geq 0$ as $c \to 0$ and converges to $\bar{h} \leq 1$ as $c \to H$. The result then follows from the intermediate value theorem. $\square$

It follows from the proof of Claim 3 that $c < \bar{c}$ if and only if $h(\mathcal{V}^{-1}(c)) < (H - c)L/((L+c)H)$ where by definition $x(0) = \mathcal{V}^{-1}(c)$. The result follows.

[28] It is worth noting that condition (17) coincides with the condition that $y(0)$, the lowest type who invests at $t = 0$, gets positive payoff from investing at $t = 0$.

A.1.9 Proof of Lemma 5

By definition $\rho(t) := \Pr(\theta = 1 | s_L \geq x(t), s_F < y(t))$, which can be written as

$$\rho(t) = 1 \Bigg/ \left(1 + \frac{1-\rho_0}{\rho_0} \frac{F^0(y(t))}{F^1(y(t))} \frac{1-F^0(x(t))}{1-F^1(x(t))}\right).$$

For all $t \geq 0$, $x(t), y(t) \in (0,1)$ which implies $\rho(t) \in (0,1)$.

Because the likelihood ratio is unbounded, (4) implies as $t \to \infty$, either $x(t)$ and $y(t)$ converge to interior values, or to 1 and 0 respectively. $x(t)$ solves the autonomous differential equation (5). So (i) if there exists $\bar{x} < 1$ such that $h(\bar{x}) = L(H-c)/(H(L+c))$, then $x(t) \to \bar{x} < 1$ and $y(t) \to \underline{y} > 0$ where $\bar{x}$ and $\underline{y}$ satisfy (4); (ii) otherwise $x(t) \to 1$ and $y(t) \to 0$ (Teschl, 2012, Lemma 1.1). Note that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1-\rho_0}{\rho_0} \frac{F^0(y(t))}{F^1(y(t))} \frac{1-F^0(x(t))}{1-F^1(x(t))} \\
=& \lim_{t\to\infty} \frac{1-\rho_0}{\rho_0} \frac{f^0(x(t))}{f^1(x(t))} \frac{f^0(y(t))}{f^1(y(t))} \cdot \underbrace{\frac{F^0(y(t))}{F^1(y(t))} \frac{f^1(y(t))}{f^0(y(t))}}_{=k(y(t))} \cdot \underbrace{\frac{1-F^0(x(t))}{1-F^1(x(t))} \frac{f^1(x(t))}{f^0(x(t))}}_{=h(x(t))}, \qquad (19)
\end{aligned}$$

where $k(\cdot)$ is the failure ratio and $h(\cdot)$ is the hazard ratio. By MLRP, $k(\cdot) > 1$ and $h(\cdot) < 1$. IFRP and IHRP say $k(\cdot)$ and $h(\cdot)$ are both strictly increasing. So in case (i) where $x(t) \to \bar{x} < 1$ and $y(t) \to \underline{y} > 0$, $\lim_{t\to\infty} k(y(t))h(x(t))$ is positive and finite. In case (ii), as $y(t) \to 0$, $k(y(t))$ converges a finite number (larger than 1); as $x(t) \to 1$, $h(x(t))$ converges to a positive number (smaller than 1). Thus, $\lim_{t\to\infty} k(y(t))h(x(t))$ is positive and finite. So (19) is equal to $(H/L)\lim_{t\to\infty} k(y(t))h(x(t))$, where H/L, by (4), is the limit of the first term in (19). Therefore,

$$\lim_{t\to\infty} \rho(t) = \rho^* := 1 \Big/ \left(1 + (H/L) \lim_{t\to\infty} k(y(t))h(x(t))\right) \in (0,1).$$

A.2 Proofs for Section 4

A.2.1 Preliminaries

I first derive some useful implications of the precision definition and the ULR order. I derive these results in the symmetric environment, but they can be generalized to general environments.

Lemma 11. *Suppose $\hat{\mathbf{F}}$ is more precise than $\mathbf{F}$. For all $\mu \in (0,1)$, the hazard ratio of*

$\mathbf{F}$ *is higher than the hazard ratio of* $\hat{\mathbf{F}}$*:* $h(\mu) > \hat{h}(\mu)$*. In the symmetric environment, equivalently, the failure ratio of* $\mathbf{F}$ *is lower than the failure ratio of* $\hat{\mathbf{F}}$*:* $k(\mu) < \hat{k}(\mu)$*.*

Proof. Define $Q^\theta(\mu) := (1 - F^\theta(\mu))/(1 - \hat{F}^\theta(\mu))$. It follows directly from (6) that $h(\mu) > \hat{h}(\mu)$ if and only if $Q^1(\mu) < Q^0(\mu)$. Moreover, by (6), for all $\mu \in (0, 1)$,

$$\frac{f^0(\mu)}{\hat{f}^0(\mu)} = \frac{f^1(\mu)}{\hat{f}^1(\mu)} = \frac{f^0(\mu) + f^1(\mu)}{\hat{f}^0(\mu) + \hat{f}^1(\mu)}. \tag{20}$$

Because $F \succ_{\text{ULR}} \hat{F}$, all three ratios in (20) are unimodal and symmetric about 1/2. Then $Q^\theta(\mu)$ is unimodal with maximum achieved at $\hat{\mu}_Q^\theta < 1/2$ (Hopkins and Kornienko, 2007, Proposition 2). Moreover, $\lim_{\mu\to 0} Q^1(\mu) = \lim_{\mu\to 0} Q^0(\mu) = 1$ and

$$\lim_{\mu\to 1} Q^1(\mu) = \lim_{\mu\to 1} \frac{1 - F^1(\mu)}{1 - \hat{F}^1(\mu)} = \lim_{\mu\to 1} \frac{f^1(\mu)}{\hat{f}^1(\mu)} = \lim_{\mu\to 1} \frac{f^0(\mu)}{\hat{f}^0(\mu)} = \lim_{\mu\to 1} \frac{1 - F^0(\mu)}{1 - \hat{F}^0(\mu)} = \lim_{\mu\to 1} Q^0(\mu).$$

The proof concerns comparing the derivatives of Q^1 and Q^0, which are given by

$$\frac{\mathrm{d}Q^1}{\mathrm{d}\mu} = \frac{f^1(\mu)}{1 - \hat{F}^1(\mu)} \left(Q^1(\mu) - \frac{f^1(\mu)}{\hat{f}^1(\mu)} \right) \text{ and } \frac{\mathrm{d}Q^0}{\mathrm{d}\mu} = \frac{f^0(\mu)}{1 - \hat{F}^0(\mu)} \left(Q^0(\mu) - \frac{f^0(\mu)}{\hat{f}^0(\mu)} \right).$$

By MLRP and (20), $f^1(\mu)/(1 - \hat{F}^1(\mu)) < f^0(\mu)/(1 - \hat{F}^0(\mu))$ for all μ.

Consider $\mu \geq \max\{\hat{\mu}_Q^1, \hat{\mu}_Q^0\}$, then both $Q^1(\mu)$ and $Q^0(\mu)$ are decreasing. Suppose there exists $\tilde{\mu}$ such that $Q^0(\tilde{\mu}) \leq Q^1(\tilde{\mu})$. Then at $\tilde{\mu}$, $\mathrm{d}Q^0/\mathrm{d}\mu < \mathrm{d}Q^1/\mathrm{d}\mu < 0$. This is a contradiction because $\lim_{\mu\to 1} Q^1(\mu) = \lim_{\mu\to 1} Q^0(\mu)$.

At $\mu = \max\{\hat{\mu}_Q^1, \hat{\mu}_Q^0\}$, one of $\mathrm{d}Q^1/\mathrm{d}\mu$ and $\mathrm{d}Q^0/\mathrm{d}\mu$ is zero and the other is strictly negative. As is shown above, $Q^1(\mu) < Q^0(\mu)$, so it must be that $\mathrm{d}Q^1/\mathrm{d}\mu < 0$ and $\mathrm{d}Q^0/\mathrm{d}\mu = 0$. This implies $\hat{\mu}_Q^1 < \hat{\mu}_Q^0$.

Consider $\mu \in (\hat{\mu}_Q^1, \hat{\mu}_Q^0)$, then Q^1 is decreasing and Q^0 is increasing. $\mathrm{d}Q^1/\mathrm{d}\mu < 0$ and $\mathrm{d}Q^0/\mathrm{d}\mu > 0$ implies $Q^1(\mu) < f^1(\mu)/\hat{f}^1(\mu) = f^0(\mu)/\hat{f}^0(\mu) < Q^0(\mu)$.

Consider $\mu \leq \hat{\mu}_Q^1$, then both $Q^1(\mu)$ and $Q^0(\mu)$ are increasing. Suppose there exists $\tilde{\mu}$ such that $Q^0(\tilde{\mu}) \leq Q^1(\tilde{\mu})$. Then at $\tilde{\mu}$, $0 < \mathrm{d}Q^1/\mathrm{d}\mu < \mathrm{d}Q^0/\mathrm{d}\mu$. This is a contradiction because $\lim_{\mu\to 0} Q^1(\mu) = \lim_{\mu\to 0} Q^0(\mu)$. □

Lemma 12. *Suppose* $\hat{\mathbf{F}}$ *is more precise than* $\mathbf{F}$*.* $h^0(\mu) > \hat{h}^0(\mu)$ *and* $F^1(\mu) > \hat{F}^1(\mu)$ *for* $\mu \geq 1/2$*.*

Proof. Let $h^\theta(\mu) = f^\theta(\mu)/(1 - F^\theta(\mu))$ denote the hazard rate conditional on θ. The posterior distribution conditional on $\theta = 0$ satisfies the definition of the ULR order: $F^0(\mu) \succ_{\text{ULR}} \hat{F}^0(\mu)$. Then $h^0(\mu) > \hat{h}^0(\mu)$ for $\mu \geq 1/2$ (Hopkins and Kornienko, 2007, Corollary 1). The ULR order implies the ex ante distribution $\hat{F}$ is a mean-preserving spread of F (Hopkins and Kornienko, 2007, Proposition 1), so $F^1(\mu) + F^0(\mu) > \hat{F}^1(\mu) + \hat{F}^0(\mu)$ for $\mu \geq 1/2$. It then follows from Lemma 11 that $F^1(\mu) > \hat{F}^1(\mu)$. $\square$

Lemma 13. *Suppose $F \succ_{ULR} \hat{F}$, for any $\lambda \in (0, 1)$, $F \succ_{ULR} ((1-\lambda)F + \lambda\hat{F}) \succ_{ULR} \hat{F}$.*

Proof. For any two distributions $F \succ_{\text{ULR}} \hat{F}$, $f/\hat{f}$ is unimodal. The likelihood ratio of F and $(1-\lambda)F + \lambda\hat{F}$ is $f/((1-\lambda)f + \lambda\hat{f})$ and the likelihood ratio of $(1-\lambda)F + \lambda\hat{F}$ and $\hat{F}$ is $((1-\lambda)f + \lambda\hat{f})/\hat{f}$. Both are unimodal as implied by that $f/\hat{f}$ is unimodal.

$F \succ_{\text{ULR}} \hat{F}$ implies the mean of F is (weakly) higher than the mean of $\hat{F}$. So the mean of F is (weakly) higher than the mean of $(1-\lambda)F + \lambda\hat{F}$, which is (weakly) greater than the mean of $\hat{F}$. The result follows. $\square$

Let $\mathbf{F}_0 = (F_0^0, F_0^1)$ denote the uninformative distribution and $\mathbf{F}_\infty = (F_\infty^0, F_\infty^1)$ denote the perfectly informative distribution. $\mathbf{F}_0$ and $\mathbf{F}_\infty$ are defined as follows.

$$F_0^0(\mu) = F_0^1(\mu) = \begin{cases} 0 & \mu < 1/2 \\ 1 & \mu \geq 1/2 \end{cases}; \; F_\infty^0(\mu) = 1 \forall \mu \geq 0 \text{ and } F_\infty^1(\mu) = \mathbf{1}_{\{1\}}(\mu). \tag{21}$$

In the symmetric environment, for any precision, (6) says $f^1(\mu)/f^0(\mu) = \mu/(1-\mu)$ and (4) reduces to $x(t) = 1 - y(t)$ for all $t \geq 0$. Because $x(0) \geq y(0)$, $x(0) \geq 1/2$. Therefore, in the following proofs, it suffices to analyze $x(t)$ with $x(0) \geq 1/2$.

A.2.2 Proof of Proposition 3

Initial value. Denote the initial value by $x_\gamma(0)$ for distribution $\mathbf{F}_\gamma$. To show the initial value is increasing in precision, rewrite the initial condition (2) with the precision index γ, $\mathcal{V}_\gamma(x_\gamma(0)) = c$. In the symmetric environment,

$$\mathcal{V}_\gamma(\mu) = \underbrace{\frac{f_\gamma^1(\mu)F_\gamma^1(\mu)}{f_\gamma^1(\mu)F_\gamma^1(\mu) + f_\gamma^0(\mu)F_\gamma^0(\mu)}}_{=:q_\gamma(\mu)} \underbrace{\left(1 - \frac{1 - F_\gamma^0(\mu)}{F_\gamma^1(\mu)}\right)}_{=:1-R_\gamma(\mu)} \left(1 - \frac{1-\mu}{\mu}\right). \tag{22}$$

From the proof of Lemma 10, $\mathcal{V}_\gamma(\mu)$ is increasing in μ for $\mu \geq 1/2$ for a fixed γ. So showing $x_\gamma(0)$ is increasing in γ is equivalent to showing $\mathcal{V}_\gamma(\mu)$ is decreasing in γ for a fixed μ. By Lemma 11, $q_\gamma(\mu)$ is decreasing in γ. I show $R_\gamma(\mu)$ is increasing in γ.

Fix two pairs of posterior belief distributions $\mathbf{F}_{\gamma_1}$ and $\mathbf{F}_{\gamma_2}$ with $\gamma_1 < \gamma_2$. Let $Q^0(\mu) = (1 - F^0_{\gamma_1}(\mu))/(1 - F^0_{\gamma_2}(\mu))$ and $P^1(\mu) = F^1_{\gamma_1}(\mu)/F^1_{\gamma_2}(\mu)$. Showing $R_{\gamma_2}(\mu) > R_{\gamma_1}(\mu)$ for $\mu \geq 1/2$ is equivalent to showing $Q^0(\mu) < P^1(\mu)$ for $\mu \geq 1/2$.

By (20), $f^1(\mu)/\hat{f}^1(\mu)$ and $f^0(\mu)/\hat{f}^0(\mu)$ are unimodal and symmetric around 1/2. Then $P^1(\mu)$ is unimodal with a maximum at $\hat{\mu}^1_P \geq 1/2$ and $Q^0(\mu)$ is unimodal with a maximum at $\hat{\mu}^0_Q \leq 1/2$ (Hopkins and Kornienko, 2007, Proposition 2). In the symmetric environment, $Q^0(\mu) = P^1(1-\mu)$ for all μ. In particular, $Q^0(1/2) = P^1(1/2)$. So for all $\mu \in [1/2, \hat{\mu}^1_P)$, $Q^0(\mu)$ is decreasing and $P^1(\mu)$ is increasing, so $Q^0(\mu) < P^1(\mu)$. For all $\mu \geq \hat{\mu}^1_P$, $Q^0(\mu) < f^0_{\gamma_1}(\mu)/f^0_{\gamma_2}(\mu) = f^1_{\gamma_1}(\mu)/f^1_{\gamma_2}(\mu) < P^1(\mu)$ (Hopkins and Kornienko, 2007, Corollary 1).

Differential equation. Rewrite (5) with the precision index γ, $x'_\gamma(t) = \phi_\gamma(x_\gamma(t)) = K^1/h^1_\gamma(x_\gamma(t)) - K^0/h^0_\gamma(x_\gamma(t))$, where $K^1 = r(1-c)/2c$ and $K^0 = r(1+c)/2c$. To show $x'_\gamma(t)$ is increasing pointwise in precision, first, by Lemma 12, $h^0_{\gamma_1}(\mu) > h^0_{\gamma_2}(\mu)$. Then by Lemma 11, $h^1_{\gamma_1}(\mu)/h^1_{\gamma_2}(\mu) > h^0_{\gamma_1}(\mu)/h^0_{\gamma_2}(\mu) > 1$. $\phi_\gamma(\mu) > 0$ implies $K^1/h^1_\gamma(\mu) > K^0/h^0_\gamma(\mu)$ for $\gamma = \gamma_1, \gamma_2$. Then

$$\phi_{\gamma_2}(\mu) - \phi_{\gamma_1}(\mu) = \frac{K^1}{h^1_{\gamma_1}(\mu)}\left(\frac{h^1_{\gamma_1}(\mu)}{h^1_{\gamma_2}(\mu)} - 1\right) - \frac{K^0}{h^0_{\gamma_1}(\mu)}\left(\frac{h^0_{\gamma_1}(\mu)}{h^0_{\gamma_2}(\mu)} - 1\right) > 0.$$

The result follows from a standard comparison argument (Teschl, 2012, Theorem 1.3).

A.2.3 Proof of Proposition 4

Existence. Step 1. There exists a (unique) initial value $x_\gamma(0)$ for all $\gamma > 0$.

From the proof of Lemma 10, $\mathcal{V}_\gamma(\mu)$, defined in (22), is continuous and increasing in μ for $\mu \geq 1/2$. $\mathcal{V}_\gamma(\mu) \to 0$ as $\mu \to 1/2$ and $\mathcal{V}_\gamma(\mu) \to 1$ as $\mu \to 1$. The uninformative ($\gamma = 0$) and perfectly informative ($\gamma = \infty$) distributions are defined in (21).

Take $\gamma \to 0$. For any $\mu \geq 1/2$, $F^1_\gamma(\mu)/F^0_\gamma(\mu) \to 1$, so $q_\gamma(\mu) \to \mu$. For any $\mu > 1/2$, $(1 - F^0_\gamma(\mu))/F^1_\gamma(\mu) \to 0$ so $\mathcal{V}_\gamma(\mu) \to 2\mu - 1$. At $\mu = 1/2$, $\mathcal{V}_\gamma(\mu) = 0$. Therefore, for all $\mu \geq 1/2$, $\mathcal{V}_\gamma(\mu) \to 2\mu - 1$. Take $\gamma \to \infty$. For all $\mu < 1$, $F^1_\gamma(\mu)/F^0_\gamma(\mu) \to 0$ so $q_\gamma(\mu) \to 0$ and $\mathcal{V}_\gamma(\mu) \to 0$. At $\mu = 1$, $\mathcal{V}_\gamma(\mu) = 1$. So $\mathcal{V}_\gamma(\mu) \to \mathbf{1}_{\{1\}}(\mu)$.

Thus, for all $\gamma \in (0, \infty)$, $0 < \mathcal{V}_\gamma(\mu) < 2\mu - 1$ for a fixed $\mu \in (1/2, 1)$. By the intermediate value theorem, there exists a unique $x_\gamma(0)$ such that $\mathcal{V}_\gamma(x_\gamma(0)) = c$. Moreover, $\lim_{\gamma \to 0} x_\gamma(0) = (c+1)/2$ and $\lim_{\gamma \to \infty} x_\gamma(0) \to 1$.

Step 2. A dynamic equilibrium exists when γ is sufficiently large.

Let $x_\gamma(t)$ be the solution to $x'_\gamma(t) = \phi_\gamma(x_\gamma(t))$ with initial value $x_\gamma(0)$. A dynamic equilibrium exists at γ if $x_\gamma(t)$ is strictly increasing in t. I show $x_\gamma(t)$ cannot be decreasing or constant in t when $\gamma \to \infty$. The result follows as $x_\gamma(t)$ is either monotone or constant (Teschl, 2012, Lemma 1.1). The equation $\phi_\gamma(x) = 0$ has at least one at most two solutions and one of them must be 1 (proof of Lemma 4). The other solution is obtained by setting $h_\gamma(\bar{x}) = (1-c)/(1+c)$ if such $\bar{x}$ exists. Take $\gamma \to \infty$. $h_\gamma(x) \to 0$ for all $x < 1$, so if there exists $\bar{x}$ such that $h_\gamma(\bar{x}) = (1-c)/(1+c) > 0$, $\bar{x}$ can only be 1. Because $x_\gamma(0) < 1$, if $x_\gamma(t)$ is decreasing in t, $x_\gamma(t)$ must be decreasing to 0 (Teschl, 2012, Lemma 1.1). However, as $x \to 0$, $h_\gamma(x) \to 0$ which means $\phi_\gamma(x) > 0$, a contradiction. If $x_\gamma(t)$ is constant, the above argument shows the solution to $\phi_\gamma(x) = 0$ can only be 1, so $x_\gamma(t) = 1$ for all $t \geq 0$. This contradicts $x_\gamma(0) < 1$.

Step 3. Fix two pairs of posterior belief distributions $\mathbf{F}_{\gamma_1}$ and $\mathbf{F}_{\gamma_2}$ with $\gamma_1 < \gamma_2$. If a dynamic equilibrium exists with $\mathbf{F}_{\gamma_1}$, a dynamic equilibrium exists with $\mathbf{F}_{\gamma_2}$.

By Claim 2, $x_\gamma(t)$ is strictly increasing in t if and only if $h_\gamma(x_\gamma(0)) < (1-c)/(1+c)$. The result follows from $h_{\gamma_2}(x_{\gamma_2}(0)) < h_{\gamma_1}(x_{\gamma_1}(0))$, which is shown below.

Define the mixture distribution with weight $\lambda \in (0,1)$ as $\mathbf{F}_\lambda = (F^0_\lambda, F^1_\lambda)$, where $F^\theta_\lambda = (1-\lambda)F^\theta_{\gamma_1} + \lambda F^\theta_{\gamma_2}$ and $F_\lambda = (1-\lambda)F_{\gamma_1} + \lambda F_{\gamma_2}$. It can be verified that $\mathbf{F}_\lambda$ defines a pair of posterior belief distributions. By Lemma 13, $F_{\gamma_1} \succ_{\text{ULR}} F_\lambda \succ_{\text{ULR}} F_{\gamma_2}$. So $\mathbf{F}_\lambda$ is more precise than $\mathbf{F}_{\gamma_1}$ and less precise than $\mathbf{F}_{\gamma_2}$. Denote the hazard ratio of $\mathbf{F}_\lambda$ by $h_\lambda(\mu)$ and the initial value by $x_\lambda(0)$. Take the derivative of $h_\lambda(x_\lambda(0))$ with respect to λ,

$$\frac{\mathrm{d}h_\lambda(x_\lambda(0))}{\mathrm{d}\lambda} = \frac{\partial h}{\partial \mu}\frac{\mathrm{d}x_\lambda(0)}{\mathrm{d}\lambda} + \frac{\partial h}{\partial \lambda} = \frac{\partial h}{\partial \mu}\left(-\frac{\partial \mathcal{V}/\partial \lambda}{\partial \mathcal{V}/\partial \mu} + \frac{\partial h/\partial \lambda}{\partial h/\partial \mu}\right).$$

By IHRP, $\partial h/\partial \mu > 0$. The result follows from the following claim.

Claim 4. For all $\mu \geq 1/2$, $-\frac{\partial \mathcal{V}/\partial \lambda}{\partial \mathcal{V}/\partial \mu} + \frac{\partial h/\partial \lambda}{\partial h/\partial \mu} < 0$.

Step 4. To conclude the proof, let $\underline{\gamma}$ be the highest γ such that a dynamic equilibrium does not exist. That is, $\underline{\gamma} := \sup_\gamma\{\gamma : h_\gamma(x_\gamma(0)) \geq (1-c)/(1+c)\}$. By step 3, a dynamic equilibrium exists if and only if $\gamma > \underline{\gamma}$.

Proof of Claim 4. The proof is mostly algebraic. For conciseness, I omit the argument of the functions. After some rearranging, $\mathcal{V}$ can be written in terms of h,

$$\mathcal{V} = \underbrace{q\left(1 - \frac{1-\mu}{\mu}\right)}_{=:b}\underbrace{-q\left(1 - \frac{1-\mu}{\mu}\right)\left(\frac{1-\mu}{\mu}\frac{1-F^1}{F^1}\right)}_{=:a}h.$$

That is, $\mathcal{V} = ah + b$. Let the superscript denote the (partial) derivative. Then $h^\lambda/h^\mu - \mathcal{V}^\lambda/\mathcal{V}^\mu = (h^\lambda/h^\mu)(a^\mu h + b^\mu)/\mathcal{V}^\mu - (a^\lambda h + b^\lambda)/\mathcal{V}^\mu$. Because $\mathcal{V}^\mu > 0$, $a^\mu h + b^\mu > -ah^\mu > 0$, showing Claim 4 is equivalent to showing $h^\lambda/h^\mu < (a^\lambda h + b^\lambda)/(a^\mu h + b^\mu)$. I prove the following chain of inequality: for all $\mu \geq 1/2$, $h^\lambda/h^\mu < q^\lambda/q^\mu < (a^\lambda h + b^\lambda)/(a^\mu h + b^\mu)$.

For the first inequality $h^\lambda/h^\mu < q^\lambda/q^\mu$, let $q = 1/(1 + m + dh)$ where

$$q = 1 \Big/ \left(1 + \underbrace{\frac{1-\mu}{\mu}\frac{1}{F^1}}_{=:m} \underbrace{- \left(\frac{1-\mu}{\mu}\right)^2 \frac{1-F^1}{F^1}}_{=:d} h\right).$$

It reduces to showing $h^\lambda/h^\mu - q^\lambda/q^\mu = (h^\lambda/h^\mu)(1 - h^\mu d/q^\mu) - (m^\lambda + d^\lambda h)/q^\mu < 0$. $h^\lambda < 0$ (Lemma 11), $h^\mu > 0$, $q^\mu > 0$, and $d < 0$, so $(h^\lambda/h^\mu)(1 - h^\mu d/q^\mu) < 0$. Note that $d = -m(1-\mu)/\mu + ((1-\mu)/\mu)^2$. Because $(1-F^0)/(1-F^1) < 1$ (MLRP) and $m^\lambda > 0$ (Lemma 12), $d^\lambda h = -m^\lambda(1-F^0)/(1-F^1) > -m^\lambda$, so $(m^\lambda + d^\lambda h)/q^\mu > 0$.

For the second inequality $q^\lambda/q^\mu < (a^\lambda h + b^\lambda)/(a^\mu h + b^\mu)$, the right-hand side is

$$\frac{q^\lambda \overbrace{\left(2 - \frac{1}{\mu}\right)\left(1 - \frac{1-F^0}{F^1}\right)}^{=:\alpha} \overbrace{- \left(\frac{1-F^1}{F^1}\right)^\lambda \frac{1-\mu}{\mu} bh}^{=:\beta}}{q^\mu \underbrace{\left(2 - \frac{1}{\mu}\right)\left(1 - \frac{1-F^0}{F^1}\right)}_{=\alpha} + \underbrace{\left(2 - \frac{1}{\mu}\right)^\mu q \left(1 - \frac{1-F^0}{F^1}\right) - \left(\frac{1-\mu}{\mu}\frac{1-F^1}{F^1}\right)^\mu bh}_{=:\eta}}.$$

It reduces to showing $q^\lambda/q^\mu - (a^\lambda h + b^\lambda)/(a^\mu h + b^\mu) = (q^\lambda/q^\mu)\eta/(q^\mu\alpha + \eta) - \beta/(q^\mu\alpha + \eta) < 0$. Because $q^\mu\alpha + \eta > 0$, it is equivalent to $q^\mu/q^\lambda - \eta/\beta > 0$. Writing out all the terms, this inequality follows from Lemma 11, Lemma 12, MLRP, IHRP, and symmetry. $\square$

Convergence. The probability of initial agreement is $\Pi_\gamma(x_\gamma(0))$ where $\Pi_\gamma(\mu) = 1 - 2(1 - F_\gamma^1(\mu))(1 - F_\gamma^0(\mu))$. The following claim establishes the result.

Claim 5. As $\gamma \to \infty$, $F_\gamma^0(x_\gamma(0)) \to 1$ and $F_\gamma^1(x_\gamma(0)) \to 0$.

Proof. Take $\gamma \to \infty$. By the Dini's theorem, $F_\gamma^0(\mu) \to 1$ uniformly for all $\mu \geq 1/2$. $x_\gamma(0) \to 1$ (step 1 above). So $F_\gamma^0(x_\gamma(0)) \to 1$. Rearrange $\mathcal{V}_\gamma(x_\gamma(0)) = c$,

$$F_\gamma^1(x_\gamma(0))\left(1 - \frac{x_\gamma(0)}{2x_\gamma(0) - 1}c\right) = F_\gamma^0(x_\gamma(0))\left(\frac{1 - x_\gamma(0)}{2x_\gamma(0) - 1}c - 1\right) + 1.$$

Take the limit of both sides as $\gamma \to \infty$. Because $F^0_\gamma(x_\gamma(0)) \to 1$ and $x_\gamma(0) \to 1$, $\lim_{\gamma\to\infty} F^1_\gamma(x_\gamma(0))\,(1-c) = 0$. Because $1-c>0$, it must be that $F^1_\gamma(x_\gamma(0)) \to 0$. $\square$

A.2.4 Proof of Proposition 5

To facilitates the proof, Figure 6 illustrates the partition of the type space using the equilibrium (inverse) strategies. In area (i), both players invest initially. In area (ii), one player invests initially, the other player follows suit. In area (iii), the follower invests before the leader disinvests. In area (iv), the leader disinvests before the follower invests. Players do not invest in the white area.

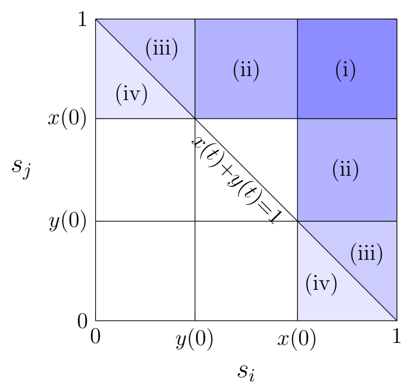

Figure 6: Partition of the type space in a symmetric environment with $c = 1/5, r = 1/5$, and posterior beliefs induced by $Beta(1+\theta, 1+(1-\theta))$.

Let $\mathcal{E}_\gamma(x_\gamma(0))$ denote the ex ante efficiency in a dynamic equilibrium with distribution $\mathbf{F}_\gamma$, where $\mathcal{E}_\gamma(\mu)$ is given by

$$\begin{aligned}
\mathcal{E}_\gamma(\mu) =& (1-F^1_\gamma(\mu))^2(1-c) - (1-F^0_\gamma(\mu))^2(1+c) && (\mathcal{E}_{\text{i}})\\
&+ 2(1-F^1_\gamma(\mu))(F^1_\gamma(\mu) - F^1_\gamma(1-\mu))(1-c) && (\mathcal{E}_{\text{ii-1}})\\
&- 2(1-F^0_\gamma(\mu))(F^0_\gamma(\mu) - F^0_\gamma(1-\mu))(1+c) && (-\mathcal{E}_{\text{ii-0}})\\
&+ \left((1-F^1_\gamma(\mu))F^1_\gamma(1-\mu) + (1-F^0_\gamma(\mu))F^0_\gamma(1-\mu)\right)(-c) && (\mathcal{E}_{\text{LF-c}})\\
&+ \int_0^\infty \left(\int_0^\tau -e^{-rt}(2-c) f^1_\gamma(y_\gamma(t)) y'_\gamma(t)\mathrm{d}t\right) f^1_\gamma(x_\gamma(\tau))x'_\gamma(\tau)\mathrm{d}\tau && (\mathcal{E}_{\text{LF-1}})\\
&- \int_0^\infty \left(\int_0^\tau -e^{-rt}(2+c) f^0_\gamma(y_\gamma(t)) y'_\gamma(t)\mathrm{d}t\right) f^0_\gamma(x_\gamma(\tau))x'_\gamma(\tau)\mathrm{d}\tau && (-\mathcal{E}_{\text{LF-0}})
\end{aligned}$$

where by symmetry, $F^1_\gamma(1-\mu) = 1 - F^0_\gamma(\mu)$ and $F^0_\gamma(1-\mu) = 1-F^1_\gamma(\mu)$. Take $\gamma \to \infty$. By Claim 5, $\mathcal{E}_{\text{i}} + \mathcal{E}_{\text{ii-1}} - \mathcal{E}_{\text{ii-0}} \to 1-c$ and $\mathcal{E}_{\text{LF-c}} \to 0$. Because $\mathcal{E}_{\text{LF-1}}$ and $\mathcal{E}_{\text{LF-0}}$ are both positive and decreasing in r (keeping $x_\gamma(t)$ and $y_\gamma(t)$ fixed), $\mathcal{E}_{\text{LF-1}} - \mathcal{E}_{\text{LF-0}}$ is bounded above by $\mathcal{E}_{\text{LF-1}}$ evaluated at $r = 0$ and bounded below by $-\mathcal{E}_{\text{LF-0}}$ evaluated at $r = 0$:

$$\mathcal{E}_{\text{LF-1}} - \mathcal{E}_{\text{LF-0}} \leq (2-c)\int_{x_\gamma(0)}^{1} \left(F_\gamma^0(x) - F_\gamma^0(x_\gamma(0))\right) f_\gamma^1(x)\mathrm{d}x,$$
$$\mathcal{E}_{\text{LF-1}} - \mathcal{E}_{\text{LF-0}} \geq -(2+c)\int_{x_\gamma(0)}^{1} \left(F_\gamma^1(x) - F_\gamma^1(x_\gamma(0))\right) f_\gamma^0(x)\mathrm{d}x.$$

Because the integrands are finite and $x_\gamma(0) \to 1$, both the upper and the lower bounds converge to zero. So $\mathcal{E}_{\text{LF-1}} - \mathcal{E}_{\text{LF-0}} \to 0$. The full-efficiency payoff is given by $\mathcal{E}^* = 2\Pr(\theta = 1)(1-c) = 1-c$. So $\mathcal{E}_\gamma(x_\gamma(0)) \to \mathcal{E}^*$.

A.3 Proofs for Section 5

A.3.1 Proof of Proposition 6

No investment region. Let (s_i, s_j) be $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L < 0$. Because $c > 0$, eventually neither player invests in the constrained outcome.

To show eventually neither player invests in equilibrium, fix $s_i = \hat{s}_i$. If $\hat{s}_i > x(0)$ or $\hat{s}_i < y(0)$, players eventually do not invest if s_j is such that $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L < 0$, which is given by the assumption. If $\hat{s}_i \in [y(0), x(0)]$, players eventually do not invest if $s_j \leq x(0)$. By MLRP, $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L$ is increasing in s_i and s_j. Recall the leader's first-order condition $\Pr(\theta = 1|x(0), y(0))H - \Pr(\theta = 0|x(0), y(0))L = 0$. For all $\hat{s}_i \geq y(0)$, $\Pr(\theta = 1|\hat{s}_i, x(0))H - \Pr(\theta = 0|\hat{s}_i, x(0))L \geq 0$. So if $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L < 0$, it must be $s_j < x(0)$.

Investment region. Define $\kappa := \Pr(\theta = 1|x(0), x(0))H - \Pr(\theta = 0|x(0), x(0))L$. To show eventually both players invest in the constrained outcome, it suffices to show $\kappa > c$. The initial condition (16) implies $\Pr(\theta = 1|s_i = x(0), s_j \in [y(0), x(0)])H - \Pr(\theta = 0|s_i = x(0), s_j \in [y(0), x(0)])L > c$. By Lemma 6, $\Pr(\theta = 1|s_i = x(0), s_j = x(0)) > \Pr(\theta = 1|s_i = x(0), s_j \in [y(0), x(0)])$. The result follows.

To show eventually both players invest in equilibrium, fix $s_i = \hat{s}_i$. If $\hat{s}_i > x(0)$ or $\hat{s}_i < y(0)$, players eventually invest if s_j is such that $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L > 0$, which is given by the assumption $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L > \kappa$ and $\kappa > c > 0$ as shown above. Suppose $\hat{s}_i \in [y(0), x(0)]$. Both players eventually invest if $s_j > x(0)$. By MLRP, For $\hat{s}_i \leq x(0)$, $\Pr(\theta = 1|\hat{s}_i, x(0)) < \Pr(\theta =$

$1|x(0), x(0))$. So if $\Pr(\theta = 1|\hat{s}_i, s_j) > \Pr(\theta = 1|x(0), x(0))$, it must be $s_j > x(0)$.

A.3.2 Proof of Proposition 7

Characterizing x_{ir} and y_{ir}. I maintain the assumption that players use symmetric monotonic strategies. A similar argument to Lemma 9 shows players do not invest after no initial investment. By MLRP, there exists a threshold x_{ir} in the first stage of time 0 such that a player invests in the first stage if and only if his type is above x_{ir}, and a threshold y_{ir} in the second stage of time 0 such that a player invests if and only if his type is above y_{ir}. x_{ir}'s indifference condition is the same as the initial threshold z's indifference condition in the dynamic equilibrium, which is $W_0(x_{\text{ir}}, y_{\text{ir}}) = c$, where

$$W_0(x,y) := \frac{\rho_0 f^1(x)(F^1(x) - F^1(y))H}{\rho_0 f^1(x)F^1(x) + (1-\rho_0)f^0(x)F^0(x)} - \frac{(1-\rho_0)f^0(x)(F^0(x) - F^0(y))L}{\rho_0 f^1(x)F^1(x) + (1-\rho_0)f^0(x)F^0(x)}.$$

Given the other player's type is above x_{ir}, type y_{ir} is indifferent between investing right away and never investing. His indifference condition is $W_1(x_{\text{ir}}, y_{\text{ir}}) = c$, where

$$\begin{aligned} W_1(x,y) :=& \frac{\rho_0 f^1(y)(1-F^1(x))H}{\rho_0 f^1(y)(1-F^1(x)) + (1-\rho_0)f^0(y)(1-F^0(x))} \\ &- \frac{(1-\rho_0)f^0(y)(1-F^0(x))L}{\rho_0 f^1(y)(1-F^1(x)) + (1-\rho_0)f^0(y)(1-F^0(x))}. \end{aligned}$$

Comparing to $x(0)$ and $y(0)$. Recall that the set of initial values $(x(0), y(0))$ of the dynamic equilibrium solves $W_0(x,y) = c$ and $W^*(x,y) = 0$, where

$$W^*(x,y) := \rho_0 f^1(y) f^1(x) H - (1-\rho_0) f^0(y) f^0(x) L.$$

Figure 7 illustrates $x_{\text{ir}} > x(0)$ and $y_{\text{ir}} < y(0)$. $W^*(x,y) = 0$ and $W_1(x,y) = c$ defines decreasing curves. $W_0(x,y) = c$ defines a decreasing curve below $W^*(x,y) = 0$ and an increasing one above $W^*(x,y) = 0$. Claim 6 formalizes this observation.

Claim 6. For all $(x,y) \in (0,1)^2$, the following holds:
(i) $\partial W^*(x,y)/\partial x > 0$ and $\partial W^*(x,y)/\partial y > 0$;
(ii) $\partial W_1(x,y)/\partial x > 0$ and $\partial W_1(x,y)/\partial y > 0$;
(iii) $\partial W_0(x,y)/\partial y \geq 0$ if and only if $W^*(x,y) \leq 0$;
(iv) For $x > y$ and $W_0(x,y) > 0$, $\partial W_0(x,y)/\partial x > 0$.

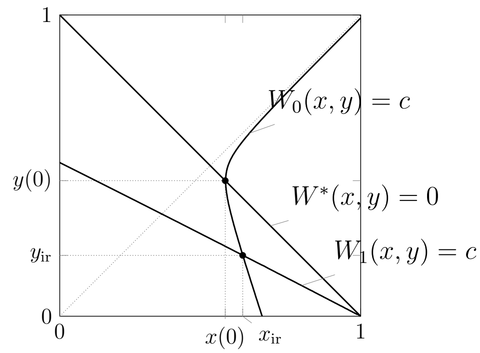

Figure 7: $(x(0), y(0))$ and $(x_{\text{ir}}, y_{\text{ir}})$ in a symmetric environment with $c = 1/50$, $r = 1/5$, and posterior beliefs distributed according to $Beta(1 + \theta, 1 + (1 - \theta))$.

Because $W^*(x(0), y(0)) = 0$ and $W_0(x(0), y(0)) = c$, by Claim 6 (i), (iii), and (iv), if $W_0(x_{\text{ir}}, y_{\text{ir}}) = c$, either $x_{\text{ir}} > x(0)$ and $y_{\text{ir}} < y(0)$, or $x_{\text{ir}} > x(0)$ and $y_{\text{ir}} > y(0)$. Recall the analysis of the follower's first-order condition in Section 3. $y(0)$'s marginal cost is strictly positive, which implies $W_1(x(0), y(0)) > c$. By Claim 6 (ii), if $x_{\text{ir}} > x(0)$ and $y_{\text{ir}} > y(0)$, $W_1(x_{\text{ir}}, y_{\text{ir}}) > W_1(x(0), y(0)) > c$, which contradicts the equilibrium condition $W_1(x_{\text{ir}}, y_{\text{ir}}) = c$. Therefore, it must be $x_{\text{ir}} > x(0)$ and $y_{\text{ir}} < y(0)$.

Proof of Claim 6. (i), (ii), and (iii) follow directly from MLRP. For (iv), fix $x > y$ and $W_0(x, y) > 0$ (both are necessary conditions for equilibrium). Note that

$$\begin{aligned} W_0(x, y) = &\Pr(s_{-i} \in [y, x] | s_i = x, s_{-i} \leq x) \\ &\cdot [\Pr(\theta = 1 | s_i = x, s_{-i} \in [y, x]) H - \Pr(\theta = 0 | s_i = x, s_{-i} \in [y, x]) L] . \end{aligned}$$

By MLRP and Lemma 6, $\Pr(\theta = 1 | s_i = x, s_{-i} \in [y, x])$ is increasing in x. Let $q(x) = \Pr(\theta = 1 | s_i = x, s_{-i} \leq x)$. So $\Pr(s_{-i} \in [y, x) | s_i = x, s_{-i} \leq x) = (1 - F^1(y)/F^1(x))\, q(x) + (1 - F^0(y)/F^0(x))\, (1 - q(x))$, which is increasing in x by MLRP. $\square$

References

Banerjee, Abhijit V. (1992) "A Simple Model of Herd Behavior," *The Quarterly Journal of Economics*, 107, 797–817.

Bar-Ilan, Avner and William C. Strange (1996) "Investment Lags," *The American Economic Review*, 86, 610–622.

Bikhchandani, Sushil, David Hirshleifer, and Ivo Welch (1992) "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades," *Journal of Political Economy*, 100, 992–1026.

Brindisi, Francesco, Boğaçhan Çelen, and Kyle Hyndman (2014) "The Effect of Endogenous Timing on Coordination under Asymmetric Information: An Experimental Study," *Games and Economic Behavior*, 86, 264–281.

Chamley, Christophe and Douglas Gale (1994) "Information Revelation and Strategic Delay in a Model of Investment," *Econometrica*, 1065–1085.

Dasgupta, Amil (2000) "Social Learning with Payoff Complementarities."

Dixit, Avinash K. and Robert S. Pindyck (1994) *Investment under uncertainty*, Princeton, N.J: Princeton University Press.

Gale, Douglas (1995) "Dynamic Coordination Games," *Economic theory*, 5, 1–18.

Herrera, Helios and Johannes Hörner (2012) "A Necessary and Sufficient Condition for Information Cascades."

Hopkins, Ed and Tatiana Kornienko (2007) "Cross and Double Cross: Comparative Statics in First Price and All Pay Auctions," *The B.E. Journal of Theoretical Economics*, 7.

Kalashnikov, V. V. and S. T. Rachev (1985) "Characterization Problems in Queueing and Their Stability," *Advances in Applied Probability*, 17, 868–886.

Klein, Nicolas and Peter Wagner (2019) "Strategic Investment and Learning with Private Information."

Kováč, Eugen and Jakub Steiner (2013) "Reversibility in Dynamic Coordination Problems," *Games and Economic Behavior*, 77, 298–320.

McDonald, Robert L. and Daniel R. Siegel (1985) "Investment and the Valuation of Firms When There is an Option to Shut Down," *International Economic Review*, 26, 331.

Murto, Pauli and Juuso Välimäki (2013) "Delay and Information Aggregation in Stopping Games with Private Information," *Journal of Economic Theory*, 148, 2404–2435.

Ramos, Hector M., Jorge Ollero, and Miguel A. Sordo (2000) "A Sufficient Condition for Generalized Lorenz Order," *Journal of Economic Theory*, 90, 286–292.

Teschl, Gerald (2012) *Ordinary Differential Equations and Dynamical Systems*, Providence, R.I: American Mathematical Society.