Online Appendix to "Optimal Disclosure Windows"

Beixi Zhou*

April 19, 2023

OA.1 Omitted Proofs for Section 3

OA.1.1 Proof of Lemma 17

Recall that by definition,

$$q(w^*, w^*, \rho) = \frac{\rho}{\rho + e^{-\lambda w^*}(1-\rho)}$$

and

$$\frac{\partial}{\partial \rho} q(w, w^*, \rho) = \frac{e^{-\lambda w^*}}{\left(\rho + e^{-\lambda w^*}(1-\rho)\right)^2} e^{-(r+\lambda)(w^*-w)} > 0.$$

First, $\partial \hat{V}/\partial \rho > 0$:

$$\frac{\partial}{\partial \rho}\hat{V}(w^*, \rho) = e^{-rw^*}e^{-\lambda w^*}e^{-\beta w^*}\frac{\partial}{\partial \rho}q(w^*, w^*, \rho) + \int_0^{w^*} e^{-rs}e^{-\lambda s}\beta e^{-\beta s}\frac{\partial}{\partial \rho}q(s, w^*, \rho)\mathrm{d}s,$$

which is positive because as is shown above, for all $s \leq w^*$,

$$\frac{\partial}{\partial \rho}q(s, w^*, \rho) > 0.$$

The same argument shows $\partial \hat{U}_1/\partial \rho > 0$.

Second, $\partial \hat{U}/\partial \rho > 0$: by (25),

$$\frac{\partial}{\partial \rho}\hat{U}(w^*, \rho) = \hat{U}_1(w^*, \rho) - \hat{V}(w^*, \rho) + (1-\rho)\frac{\partial}{\partial \rho}\hat{V}(w^*, \rho) + \rho\frac{\partial}{\partial \rho}\hat{U}_1(w^*, \rho).$$

*Department of Economics, Boston University, bzhou@bu.edu.

Because $\hat{U}(w^*, \rho) = (1-\rho)\hat{V}(w^*, \rho) + \rho\hat{U}_1(w^*, \rho) > \hat{V}(w^*, \rho)$, so $\hat{U}_1(w^*, \rho) > \hat{V}(w^*, \rho)$. Therefore, $\partial\hat{U}/\partial\rho > 0$.

Next, $\partial\hat{V}/\partial w^* < 0$:

$$\begin{aligned}\frac{\partial}{\partial w^*}\hat{V}(w^*, \rho) =& e^{-rw^*}e^{-\lambda w^*}e^{-\beta w^*}\left(\frac{\partial}{\partial w^*}q(w^*, w^*, \rho) - r - (r+\lambda)q(w^*, w^*, \rho)\right) \\ &+ \int_0^{w^*} e^{-rs}e^{-\lambda s}\beta e^{-\beta s}\frac{\partial}{\partial w^*}q(s, w^*, \rho)\mathrm{d}s.\end{aligned}$$

To show this is negative, it suffices to show $\partial q(s, w^*, \rho)/\partial w^* < 0$ for all $s \leq w^*$. This partial derivative is given by

$$\begin{aligned}\frac{\partial}{\partial w^*}q(s, w^*, \rho) =& -(r+\lambda)e^{-(r+\lambda)(w^*-s)}\left(\frac{\rho}{\rho + e^{-\lambda w^*}(1-\rho)} + \frac{r}{r+\lambda}\right) \\ &+ e^{-(r+\lambda)(w^*-s)}\lambda(1 - q(w^*, w^*, \rho))q(w^*, w^*, \rho).\end{aligned}$$

After some rearranging,

$$\frac{\partial}{\partial w^*}q(s, w^*, \rho) = e^{-(r+\lambda)(w^*-s)}\left(-\lambda q(w^*, w^*, \rho)^2 - rq(w^*, w^*, \rho) - r\right) < 0.$$

The result follows. Note that $\partial q(s, w^*, \rho)/\partial w^* < 0$ also implies $\partial\hat{U}_1(w^*, \rho)/\partial w^* < 0$.

Finally, $\partial\hat{U}/\partial w^* < 0$: by (25),

$$\frac{\partial}{\partial w^*}\hat{U}(w^*, \rho) = (1-\rho)\frac{\partial}{\partial w^*}\hat{V}(w^*, \rho) + \rho\frac{\partial}{\partial w^*}\hat{U}_1(w^*, \rho) < 0$$

because, as is shown above, $\partial\hat{V}(w^*, \rho)/\partial w^* < 0$ and $\partial\hat{U}_1(w^*, \rho)/\partial w^* < 0$.

OA.1.2 Proof of Claim 3

by definition,

$$\frac{\partial\hat{V}}{\partial\rho} = \frac{e^{-rw^*}}{\left(1 + \left(e^{\lambda w^*} - 1\right)\rho\right)^2} \text{ and } \frac{\partial\hat{U}_1}{\partial\rho} = \frac{\beta - \lambda e^{-(\beta-\lambda)w^*}}{\beta - \lambda}\frac{\partial\hat{V}}{\partial\rho}, \tag{OA.1}$$

where $\left(\beta - \lambda e^{-(\beta-\lambda)w^*}\right)/(\beta - \lambda) > 1$.

For the first inequality, $\rho'(t) - r(1 + \rho(t)) \leq 0$, $\partial\hat{V}/\partial\rho > 0$, and $\partial\hat{U}_1/\partial\rho > 0$

implies

$$\rho'(t)\left((1-\rho(t))\frac{\partial \hat{V}}{\partial \rho}(w^*(t),\rho(t))+\rho(t)\frac{\partial \hat{U}_1}{\partial \rho}(w^*(t),\rho(t))\right)$$
$$\leq r(1+\rho(t))\left((1-\rho(t))\frac{\partial \hat{V}}{\partial \rho}(w^*(t),\rho(t))+\rho(t)\frac{\partial \hat{U}_1}{\partial \rho}(w^*(t),\rho(t))\right).$$

So it suffices to show

$$r(1+\rho(t))\left((1-\rho(t))\frac{\partial \hat{V}}{\partial \rho}(w^*(t),\rho(t))+\rho(t)\frac{\partial \hat{U}_1}{\partial \rho}(w^*(t),\rho(t))\right) < r\hat{U}(w^*(t),\rho(t)).$$

I show the above inequality holds for all ρ and all w^*. Plug in the expression for $\partial \hat{V}/\partial \rho$ and $\partial \hat{U}_1/\partial \rho$,

$$(1+\rho(t))\frac{\partial \hat{V}}{\partial \rho}(w^*(t),\rho(t))\left((1-\rho(t))+\rho(t)\frac{\beta-\lambda e^{-(\beta-\lambda)w^*}}{\beta-\lambda}\right) < \hat{U}(w^*(t),\rho(t)).$$

After some rearranging, the inequality becomes

$$\frac{1+\rho}{\left(1+\left(e^{\lambda w^*}-1\right)\rho\right)^2}\left((1-\rho)+\rho\frac{\beta-\lambda e^{-(\beta-\lambda)w^*}}{\beta-\lambda}\right) < e^{rw^*}\hat{U}(w^*,\rho).$$

The left-hand side is independent of r. For the right-hand side, note that $\hat{U}(w^*,\rho)$ also depends on r and $e^{rw^*}\hat{U}(w^*,\rho)=(1-\rho)e^{rw^*}\hat{V}(w^*,\rho)+\rho e^{rw^*}\hat{U}_1(w^*,\rho)$, where

$$\begin{aligned} e^{rw^*}\hat{V}(w^*,\rho) =& e^{-\lambda w^*}e^{-\beta w^*}(1+q(w^*,w^*,\rho)) \\ &+\int_0^{w^*} e^{r(w^*-s)}\lambda e^{-\lambda s}e^{-\beta s}\mathrm{d}s+\int_0^{w^*} e^{r(w^*-s)}e^{-\lambda s}\beta e^{-\beta s}(1+q(s,w^*,\rho))\mathrm{d}s, \end{aligned}$$

which is increasing in r. So it suffices to show this inequality holds when the right-hand side is evaluated at $r=0$. That is,

$$\frac{(1+\rho)\left((1-\rho)+\rho\frac{\beta-\lambda e^{-(\beta-\lambda)w^*}}{\beta-\lambda}\right)}{\left(1+\left(e^{\lambda w^*}-1\right)\rho\right)^2} < \frac{\beta+\beta e^{\lambda w^*}\rho-\lambda\left(1+\rho\left(e^{\lambda w^*}-\rho+e^{(\lambda-\beta)w^*}\rho\right)\right)}{(\beta-\lambda)(1+(e^{\lambda w^*}-1)\rho)}.$$

Evaluating at $w^*=0$, the left-hand side is equal to the right-hand side and is equal to $1+\rho$. So to prove this inequality, it suffices to show the left-hand side is decreasing in w^* and the right-hand side is increasing in w^*.

Take the derivative of the right-hand side with respect to w^*, $\lambda\rho\left(e^{\lambda w^*} + e^{(\lambda-\beta)w^*}\rho\right) > 0$. Take the derivative of the left-hand side with respect to w^*,

$$\frac{e^{(\lambda-\beta)w^*}\lambda\rho(1+\rho)}{\left(1+\left(e^{\lambda w^*}-1\right)\rho\right)^2}\frac{\beta-\lambda-e^{\beta w^*}\left(\beta+\lambda(\rho-1)\right)-\beta\rho+\beta e^{\lambda w^*}\rho+\lambda\rho}{\beta-\lambda}.$$

Show this is negative is equivalent to showing the second term is negative. After some simplifying, the goal is to show

$$\frac{\lambda\left(1-e^{\beta w^*}\right)-\beta\left(1-e^{\lambda w^*}\right)}{\beta-\lambda} < 0.$$

Note that this term is equal to 0 at $w^* = 0$. Its derivative with respect to w^* is $\beta\lambda\left(e^{\lambda w^*}-e^{\beta w^*}\right)/(\beta-\lambda) < 0$. So the inequality holds.

The second inequality follows a similar argument. Analogously, $\rho'(t)-r(1+\rho(t)) \leq 0$ and $\partial\hat{V}/\partial\rho > 0$ implies

$$\rho'(t)\frac{\partial\hat{V}}{\partial\rho}(w^*(t),\rho(t)) \leq r(1+\rho(t))\frac{\partial\hat{V}}{\partial\rho}(w^*(t),\rho(t)).$$

So it suffices to show

$$(1+\rho(t))\frac{\partial\hat{V}}{\partial\rho}(w^*(t),\rho(t)) < \hat{V}(w^*(t),\rho(t)).$$

I show the above inequality holds for all ρ and all w^*. Plug in the expression for $\partial\hat{V}(w^*,\rho)/\partial\rho$ using (OA.1), the inequality becomes

$$\frac{1+\rho}{\left(1+\left(e^{\lambda w^*}-1\right)\rho\right)^2} < e^{rw^*}\hat{V}(w^*,\rho).$$

Note that the left-hand side is independent of r and the right-hand side is increasing in r (shown in the first part of the proof). So it suffices to show this inequality holds when the right-hand side is evaluated at $r = 0$. That is,

$$1+\rho < \left(1+e^{\lambda w^*}\rho\right)\left(1+\left(e^{\lambda w^*}-1\right)\rho\right).$$

The right-hand side increases in w^* and equals $1+\rho$ at $w^* = 0$. The inequality holds.