Dynamic Coordination with Informational Externalities*

Beixi Zhou†

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Abstract

I study a two-player continuous-time dynamic coordination game with observational learning. Each player has one opportunity to make a reversible investment with an uncertain return that is realized only when both players invest. Each player learns about the potential return by observing a private signal and the actions of the other player. In equilibrium, players' roles as leader and follower are endogenously determined. Information aggregates in a single burst initially, then gradually through delayed investment and disinvestment over time. More precise signals lead to faster coordination conditional on initial disagreement, but might also increase the probability of initial disagreement.

Keywords: dynamic coordination, observational learning, real option.

JEL Codes: C73, D82, D83

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†Department of Economics, Boston University, bzhou@bu.edu.

1 Introduction

Two phenomena often arise in the diffusion of innovative technologies. The first one is observational learning: players learn about the profitability of a new technology by observing the investment behavior of other players. Second, technology adoption often exhibits strategic complementarities among investors. Investors with insufficient funds to initiate a project rely on other investors to fill the gap.

The objective of this paper is to study observational learning in the context of complementary investments with reversible investment decisions. In such situations, coordination among investors might make an investment profitable when it would not have been so if undertaken by only one investor, and an investor can back out of an investment at any point in time should he become pessimistic about its prospects.

Specifically, I study the interplay of observational learning and coordination in a two-player timing game of investment in continuous time. Each player is endowed with one opportunity to invest in a risky project. The decision to invest is reversible, but a player cannot reinvest after having disinvested. The return of the investment is ex ante unknown. Each player receives a private signal about the return only at the outset, and decides when to invest and when to disinvest after having invested. Actions are public; thus each player learns about the return not only from the private signal, but also by observing the investment decisions of the other player. The players pay a (lump-sum) investment cost at the time of their investment, which is not recoverable should they decide to ever disinvest. The (lump-sum) return of the project is only realized at the time when both players invest.

My main findings are as follows. First, I characterize the symmetric equilibria in which players use threshold strategies and there is a positive probability of investment at more than one instant in time.[1] I show uniqueness within this class of equilibria. In this unique equilibrium, players first decide whether to invest at the start of the game. This initial investment decision endogenously determines the players' roles as the leader (if a player invests) or the follower (if a player doesn't invest). The continuation game unfolds as a timing game, in which the leader chooses when to disinvest and the follower chooses when to invest. The duration a player is willing to wait signals his private information to the other player. Information is revealed in

[1]This class of equilibria is of special interest. I also characterize and discuss other symmetric equilibria of the game.

a single burst at the beginning of the game, and then aggregates continuously and gradually over time through delayed disinvestment and investment decisions.

Next, I investigate how changes in the informativeness of the signal distributions affect the learning dynamics and the equilibrium outcome. In particular, does having a more precise signal increase the probability that the players coordinate initially (either both investing or both not investing), and do players coordinate faster if they cannot reach an initial agreement? I find that, while a more precise information structure always increases the speed of the observational learning process conditional on initial disagreement, its effect on the probability of initial agreement is ambiguous. Specifically, when information gets arbitrarily precise, the probability of initial agreement converges to 1; but when precision is low, there exist parameters where an increase in precision leads to a decrease in the probability of initial agreement. While better information enables players to make more informed decisions, it can also increase the variance in the two players' beliefs, which leads to more disagreement. The effect of the increase in precision on efficiency, defined as the sum of the players' ex ante equilibrium payoffs, is positive in the limit: when information gets arbitrarily precise, efficiency approaches the first-best level.

Related literature

My model combines a dynamic coordination game (Gale, 1995) and an observational learning game with endogenous timing (Chamley and Gale, 1994, or Murto and Välimäki, 2013) and reversible actions.

As noted, my model features strategic delay. Although strategic delay is a common feature in coordination games or observational learning games with endogenous timing, the novelty is the way in which delay acts as a signal and facilitates information transmission. In the dynamic coordination literature, delay arises due to strategic uncertainty. Players want to coordinate on the same action but are unsure of what others will do. In a seminal paper, Gale (1995) studies delay in an N-player dynamic coordination game where each player decides whether to make an irreversible investment, the return of which is increasing in the number of investors. In the observational learning with endogenous timing literature, delay arises due to informational externalities. Players "wait and see" what others do in order to learn from their actions. In a seminal paper, Chamley and Gale (1994) study an N-player investment timing game with a pure informational externality. Each player observes a private signal and the actions of the others, and his payoff only depends on his own action and the state. Murto and Välimäki (2013) generalize Chamley and Gale (1994) by modeling uncertainty over when to invest, not just whether to invest. The equilibria in Murto and Välimäki (2013) feature alternating phases between investment waves (sudden bursts of information) and waiting phases (gradual revelation of information). In my model, a burst of information occurs only at time 0, and afterward, the information revelation can only be gradual.

Turning to observational learning models with payoff externalities, there is a small theoretical literature that incorporates payoff externalities into a standard sequential social learning model as studied by Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992). Early work includes Dasgupta (2000), who studies a sequential social learning model where the return of the investment is realized if and only if all players invest. Although Dasgupta's model has both coordination and informational externalities, the exogenous order of actions in the standard sequential social learning framework leaves open the possibilities for enriched learning behavior if players are allowed to choose the timing of their actions. An experimental study by Brindisi, Çelen, and Hyndman (2014) investigates the effect of endogenous timing in a dynamic coordination setting with observable actions. In their model, investment is irreversible, thus the decision to invest eliminates strategic uncertainty once and for all, and all actions occur at time 0. With reversible investment and different modeling choices, my model allows for richer learning dynamics in a more general setting.

An important assumption of my model is the reversibility of investment decisions, in contrast to the irreversible investments assumed in much of the literature. The usual formulation of irreversible investment, as in Dixit and Pindyck (1994), is that investment expenditures are sunk costs. The investment in my model is also "irreversible" in this sense as the cost of investment is nonrecoverable. On the other hand, it is reversible in the sense that a player can shut down or abandon an investment without bearing its return. Abandonment of this kind is a common practice in investment (e.g., McDonald and Siegel, 1985, Bar-Ilan and Strange, 1996), yet seems understudied in the theoretical literature. In fact, in this paper, this reversibility is one of the key ingredients that give delay its informational value. It makes it possible that after players endogenously determine their roles in the game (leader or follower), they can subsequently signal their optimism or pessimism through delayed actions.[2]

[2] I discuss the case of irreversible investments in Section 5.2.

Few other papers assume reversible actions in dynamic settings. Kováč and Steiner (2013) study the role of reversibility of actions in a two-period coordination game with unobservable actions. Klein and Wagner (2019) analyze a strategic experimentation problem with private information where players can invest and disinvest at any time.

2 Model

Time is continuous and the horizon is infinite, $t \geq 0$. There are two players $i = 1, 2$. Each player chooses when (if ever) to invest in a risky project, and when (if ever) to disinvest. The return of the investment depends on an unknown state of the world $\theta \in \{0, 1\}$, where $\theta = 1$ indicates the project is good, while $\theta = 0$ indicates the project is bad. The common prior belief of $\theta = 1$ is $\rho_0 \in (0, 1)$.

Information. At the outset, player i receives a private signal $s_i \in S = (0, 1)$ about the state θ, which is conditionally independent across players. Conditional on state θ, the distribution of s is denoted $F^\theta(s)$ with a continuously differentiable density function $f^\theta(s) = \mathrm{d}F^\theta(s)/\mathrm{d}s$ that is strictly positive on $(0, 1)$. The two distributions F^0 and F^1 are mutually absolutely continuous with common support $[0, 1]$, which ensures there does not exist a signal that perfectly reveals the state.

The distributions are assumed to satisfy the strict monotone likelihood ratio property (MLRP) on $(0, 1)$ and have an unbounded likelihood. That is, the likelihood ratio $l(s) := f^0(s)/f^1(s)$ is strictly decreasing on $(0, 1)$ with $\lim_{s \to 0} l(s) = \infty$ and $\lim_{s \to 1} l(s) = 0$. MLRP guarantees that a higher signal induces a higher posterior belief of the good state for any prior; unbounded likelihood guarantees that the distribution of the induced posterior beliefs has full support over $[0, 1]$. Without loss of generality, I assume $\Pr(\theta = 1|s) = s$ for all $s \in (0, 1)$. By definition, the likelihood ratio is proportional to the posterior belief of the bad state over the good state, namely, $((1 - \rho_0)/\rho_0)l(s) = (1 - s)/s$, and strict MLRP is always satisfied.

Players cannot communicate. They do not observe each other's private signal, but observe each other's investment and disinvestment decisions.

Actions and payoffs. Each player chooses when, if ever, to invest, and when, if ever, to disinvest after having invested. A player cannot reinvest after having disinvested. If player i invests at t_i, he pays a lump-sum (sunk) cost $c > 0$ at t_i. As soon as both players invest, the project generates a lump-sum return R^θ to each of the players. If the state is good, the return is $R^1 = H > c$, and if the state is bad, the return is negative: $R^0 = -L < 0$.[3] If only one player invests, the payoff from the investment is 0. The payoff from never investing is 0. The (payoff) tie-breaking rule in the case of a simultaneous move, that is, a player invests at the same instant the other player disinvests, is that the return R^θ is not realized. Players discount payoffs at a common discount rate $r > 0$.

As an example, suppose player 1 invests at t_1 and player 2 invests at $t_2 \geq t_1$. The realized payoff of player 1 and player 2, denoted by u_1 and u_2, are

$$u_1 = e^{-rt_2} R^\theta + e^{-rt_1}(-c), \quad u_2 = e^{-rt_2} R^\theta + e^{-rt_2}(-c).$$

Suppose player 1 disinvests at $\tau_1 \in (t_1, t_2]$. Then[4]

$$u_1 = e^{-rt_1}(-c), \quad u_2 = e^{-rt_2}(-c).$$

Strategies. The usual challenge of defining strategies in continuous-time models with observable actions arises. A player updates his belief in response to the other player's actions, and may want to react to his updated belief without delay. To overcome this challenge, I follow the definition of strategy in Murto and Välimäki (2013) and model the dynamic game as a multi-stage stopping game as follows.

The game consists of at most two stages. Stage 0 starts at time 0. In this initial stage, the public history is that nobody has invested. Denote this history by $h^0 \in \mathcal{H}^0 := \{\varnothing\}$. Given this history and the private signal s_i, player $i \in \{1, 2\}$ chooses a time to invest $\sigma_i(s_i, \varnothing) \geq 0$. The initial stage ends at $t^0 := \inf_i \sigma_i(s_i, \varnothing)$.

As soon as stage 0 ends, the game immediately moves to stage 1. The public history after the initial stage and at the beginning of the next stage consists of t^0 and the identity of the player(s) who invested at t^0, that is, $\{(\{1,2\}, t^0), (\{1\}, t^0), (\{2\}, t^0)\}$. The history $(\{1,2\}, t^0)$ is terminal: if both players invest simultaneously, the return of the project is realized at t^0. The histories $(\{1\}, t^0)$ and $(\{2\}, t^0)$ are non-terminal. If only one player invests, the player who invested is called the leader and

[3] The assumption $R^0 < 0$ is essential to the analysis. It ensures that a player will ever have an incentive to disinvest after having invested.

[4] Player 2 still pays the investment cost if he were to invest after player 1 has disinvested — although this case never happens in equilibrium.

the player who did not invest is called the follower; the subsequent continuation game is called a *leader-follower continuation game*. Define the set of non-terminal histories as $\mathcal{H}^1 := \{(\{1\}, t^0), (\{2\}, t^0)\}$. Given $h^1 \in \mathcal{H}^1$ and the private signal s_i, the leader decides if and when to disinvest and the follower decides if and when to invest, each conditional on the other player not moving first, $\sigma_i(s_i, h^1) \geq t^0$. The leader-follower continuation game ends either after the follower invests in which case payoff realizes, or after the leader disinvests, in which case it is a dominant strategy for the follower to never invest afterward. I treat these histories as if they were terminal.

Denote the set of all non-terminal public histories by $\mathcal{H} := \mathcal{H}^0 \cup \mathcal{H}^1 = \{\varnothing\} \cup \{(\{1\}, t^0), (\{2\}, t^0)\}$. A pure strategy for player i for stage $k \in \{0, 1\}$ is a function

$$\sigma_i^k : S \times \mathcal{H}^k \to [t^{k-1}, \infty]$$

that maps the private signal and a non-terminal history to a time to switch actions (either switch from not investing to investing or from investing to not investing), conditional on the other player $-i$ not switching before that time. Define player i's strategy on the set of all non-terminal histories $\sigma_i : S \times \mathcal{H} \to [0, \infty]$ as

$$\sigma_i(s_i, h) = \sigma_i^k(s_i, h) \text{ whenever } h \in \mathcal{H}^k.$$

Equilibrium concept. I analyze symmetric perfect Bayesian equilibria in which players use monotonic strategies, defined as follows.

Definition 1. A strategy σ_i is monotonic if $\sigma_i(s_i, \varnothing)$ is decreasing in s_i.

In words, in the history that nobody has invested, a player chooses to invest earlier if his signal is higher. An equilibrium is monotonic if players use monotonic strategies.

In the case that a player is indifferent between two investment (or disinvestment) times t' and t'' with $t' < t''$, I restrict attention to strategies that satisfy the following (indifference) tie-breaking rule. The strategies prescribe investment at t'' to a player who has not invested, and disinvestment at t' to a player who has invested.[5]

A perfect Bayesian equilibrium is a pair of strategies and a system of beliefs for each player such that each player's strategy maximizes his expected payoff and beliefs are updated via Bayes' rule at any history reached with positive probability.

[5]This tie-breaking rule is innocuous. It assumes players do not invest or stay in the investment if they are indifferent between in and out. The same equilibrium can be obtained if indifference is consistently broken in the opposite fashion instead (some of the proofs will take a different approach).

Off the equilibrium path, I assume each player's belief about the deviator's type is the lowest possible type, so neither player invests after any observable deviations. That is, upon observing a deviation by player $-i$, player i never invests if he has not invested, and disinvests immediately if he has invested. This assumption is without loss of generality in the sense that if a (perfect Bayesian) equilibrium outcome can be sustained with a different off-path belief, it can be sustained with this off-path belief.

3 Equilibrium Analysis

3.1 Multiplicity of Equilibria

As is common in coordination games, this game admits multiple equilibria. The first type of multiplicity comes from a continuum of "starting times." An equilibrium is said to start at $\hat{t} \geq 0$ if there is a positive probability of investment by either player at $\hat{t}$ and zero probability for all $t < \hat{t}$. For any equilibrium that starts at time 0, one can construct another equilibrium by postponing all actions to a later time. Such equilibria are considered equivalent up to starting time and are inefficient due to discounting. Consequently, I focus on equilibria that start at time 0.

Within the set of equilibria that start at time 0, there are three types of equilibria. One equilibrium that always exists is one in which neither player invests for all $t \geq 0$. I call this a *no-investment equilibrium.* Another equilibrium prescribes investment only at time 0: there is a positive probability of investment at $t = 0$, and zero probability for all $t > 0$. I call this a *myopic equilibrium.* In the myopic equilibrium, player i invests at time 0 if and only if his type s_i is above (or equal to) threshold s^*, where

$$\Pr(s_{-i} \geq s^*, \theta = 1 | s_i = s^*)H - \Pr(s_{-i} \geq s^*, \theta = 0 | s_i = s^*)L - c = 0.$$

Unless both players invest at time 0, in which case the payoff realizes, the player who invested at time 0 disinvests immediately, and the player who did not never invests.[6]

[6]The no-investment equilibrium and the myopic equilibrium constitute the set of (pure strategy) equilibria of a one-shot game at $t = 0$ with the following payoff matrix:

	Invest	Not invest
Invest	$R^\theta - c, R^\theta - c$	$-c, 0$
Not invest	$0, -c$	$0, 0$

By MLRP, the equilibrium of this one-shot game must be monotonic. The threshold type s^* is

A more interesting type of equilibrium is one in which there is a positive probability of investment at more than one point in time. I call this a *dynamic equilibrium*. I focus on characterizing monotonic symmetric dynamic equilibria.

3.2 Monotonic Symmetric Dynamic Equilibrium

The overall structure of a monotonic symmetric dynamic equilibrium is as follows. The initial stage starts at $t = 0$. In the initial stage, a player invests if his type is above a threshold and does not invest if below. Denote this threshold by z. After players take an action in the initial stage, there are three possible continuation games. First, if both players invest, then the payoff realizes. Second, if neither player invests, then neither player ever invests afterward.[7] Third, if only one player invests, the game immediately moves to the leader-follower continuation game. In this continuation game, players play according to a sequence of time-dependent thresholds: at each $t \geq 0$, the leader stays invested if his type is above the threshold $x(t)$ and the follower does not invest if his type is below the threshold $y(t)$.

The rest of the section is organized as follows. First, I discuss the players' incentives at time 0 and characterize the initial investment thresholds $(z, x(0), y(0))$ in Proposition 1. Next, I characterize the continuation thresholds $x(t)$ and $y(t)$ for all t in Proposition 2 and discuss important properties of these thresholds along the way. Lastly, in Theorem 1, I present sufficient conditions for existence and uniqueness of such an equilibrium.

3.2.1 Incentives at time 0

Consider the leader-follower continuation game.[8] At time 0, upon observing an investment in the initial stage, in the second stage, the follower invests if his type is above $y(0)$, and the leader stays invested if his type is above $x(0)$. The following result characterizes the investment threshold z in the initial stage, and the initial thresholds $x(0)$ and $y(0)$ in the leader-follower continuation game.

indifferent between investing and not investing, all types above it invest, and all types below do not.

[7]This is shown in Lemma 9 in the Appendix. In short, Lemma 9 states that in any monotonic symmetric dynamic equilibrium, neither player invests after no initial investment.

[8]As mentioned, if both players invest, payoff realizes; if neither player invests, neither invests afterward. It only remains to analyze the leader-follower continuation game.

Proposition 1. *In any monotonic symmetric dynamic equilibrium, the initial values* $(z, x(0), y(0)) \in (0,1)^3$ *must satisfy*

$$y(0) < z = x(0), \tag{1}$$

$$\frac{\rho_0 f^1(z)(F^1(z) - F^1(y(0)))H}{\rho_0 f^1(z)F^1(z) + (1-\rho_0)f^0(z)F^0(z)} - \frac{(1-\rho_0)f^0(z)(F^0(z) - F^0(y(0)))L}{\rho_0 f^1(z)F^1(z) + (1-\rho_0)f^0(z)F^0(z)} = c, \tag{2}$$

$$\rho_0 f^1(x(0))f^1(y(0))H - (1-\rho_0)f^0(x(0))f^0(y(0))L = 0. \tag{3}$$

Condition (1) describes the equilibrium behavior at the beginning of the leader-follower continuation game. The leader stays invested for sure ($x(0) = z$), and there is a strictly positive probability that the follower follows suit ($y(0) < z$).

Intuitively, a player pays the investment cost upfront if he becomes the leader. A player is willing to do so only if he expects this action to induce a high enough probability of investment by the other player. This suggests there has to be a mass of follower types following suit, namely, $y(0) < z$. In turn, these higher follower types, upon seeing the good news that the leader has invested, now find investment profitable and invest without further delay. The leader, anticipating a mass of follower types following suit, will want to stay invested and wait for the (possibility of) investment by the follower, rather than to disinvest right away and thus waste the investment cost. This suggests $x(0) = z$.

Equation (2) follows from type z's indifference between investing and not investing in the initial stage at time 0 given his continuation strategies.[9] Equation (3) follows from type $x(0)$'s optimality condition in the leader-follower continuation game, which is analyzed in the following section.

3.2.2 Leader-follower continuation game

In this continuation game, each player solves an optimal stopping problem. If the leader stops, he disinvests, while if the follower stops, he invests. I derive conditions that any monotonic symmetric dynamic equilibrium must satisfy.

[9] To be precise, type z is indifferent in the sense that his expected payoff from not investing at time 0 is equal to the supremum of his expected payoff over all continuation strategies that prescribe investing at time 0.

With a slight abuse of notation, denote player i's pure strategy by $\sigma_i : S_i \to [0, \infty]$, which maps player i's type to a stopping time conditional on the other player not having stopped. Let $i \in \{L, F\}$, where L denotes the leader and F the follower.[10] Let $G_i(t)$ denote the probability that i stops no later than t according to σ_i, and $G_i^\theta(t)$ denote this probability conditional on θ. Define T_L as the earliest time by which all types of leader decide to disinvest, and T_F as the earliest time by which all types of follower decide to not invest ever after:

$$T_i := \inf\left\{t \geq 0 : G_i(t) = \lim_{\tau \to \infty} G_i(\tau)\right\}, i \in \{L, F\}.$$

Because of coordination, if at some point in time, a player decides to not invest for the rest of the game, the other player will do the same. The following result formalizes this intuition.

Lemma 1. *In any monotonic symmetric dynamic equilibrium, $T := T_L = T_F \leq \infty$.*

T can be interpreted as an equilibrium deadline: if the players cannot reach an agreement (either both investing or both not investing) by T, the probability of investment by either player after T is zero. In what follows, I determine players' equilibrium behavior before T.

The following properties of the strategies shed light on how information aggregates in equilibrium. Lemma 2 establishes there does not exist a time at which a mass of types stop. This means information is revealed gradually, never in bursts.

Lemma 2. *In any monotonic symmetric dynamic equilibrium, σ_L is strictly increasing and σ_F is strictly decreasing.*

Lemma 2 in its weak form follows from a revealed-preference argument. By MLRP, it is not surprising that a higher type of leader (lower type of follower) prefers to wait longer because he is more optimistic (pessimistic). However, strict monotonicity in the context of coordination is not as straightforward. σ_i constant at $\hat{t}$ over an interval means all types in this interval stop at $\hat{t}$. Intuitively, if a player is getting a continuous flow of information about the other player, he will want to respond continuously. If there is a mass of types wanting to stop at a particular time, this must be a response to a mass of types stopping by the other player. This cannot happen in equilibrium

[10]Conditional on reaching the leader-follower continuation game, $S_L = (x(0), 1)$ and $S_F = (0, y(0)]$.

because by the (payoff) tie-breaking rule, it is a strictly dominated strategy for the follower to stop at the same time the leader stops.[11]

Lemma 3 establishes there does not exist a time interval over which neither player stops for sure. This means information is revealed continuously, never with pauses.

Lemma 3. *In any monotonic symmetric dynamic equilibrium, σ_L and σ_F are continuous.*

A pause is detrimental to payoff due to discounting and has no informational value. Because a player's belief is the same at the beginning and the end of the pause, were he willing to stop at the end of the pause, he would deviate to doing so at the beginning of it.[12]

By Lemma 2 and Lemma 3, the inverse mappings of σ_L and σ_F, defined as $x(t) := \sigma_L^{-1}(t)$ and $y(t) := \sigma_F^{-1}(t)$, are continuous, and strictly increasing and decreasing in t respectively. I refer $x(t)$ and $y(t)$ as the equilibrium (inverse) strategies of the leader and the follower. The interpretation of $x(t)$ and $y(t)$ is that leader of type $x(\hat{t})$ optimally stops at $\hat{t}$ and follower of type $y(\hat{t})$ optimally stops at $\hat{t}$. If the leader has not stopped by $\hat{t}$, the follower knows the leader's type must be greater than $x(\hat{t})$. If the follower has not stopped by $\hat{t}$, the leader knows the follower's type must be less than $y(\hat{t})$. Given the follower's conditional distributions of stopping time $G_F^\theta(t)$, leader of type x's expected payoff from stopping at t is

$$\begin{aligned}\mathcal{L}(x,t) =& \Pr(\theta = 1|s_L = x, s_F < y(0)) \int_0^t e^{-r\tau} \mathrm{d}G_F^1(\tau) H \\ &- \Pr(\theta = 0|s_L = x, s_F < y(0)) \int_0^t e^{-r\tau} \mathrm{d}G_F^0(\tau) L,\end{aligned}$$

and given $G_L^\theta(t)$, follower of type y's expected payoff from stopping at t is

$$\begin{aligned}\mathcal{F}(y,t) =& e^{-rt}\Big(\Pr(\theta = 1|s_F = y, s_L > x(0))(1 - G_L^1(t))(H - c) \\ &- \Pr(\theta = 0|s_F = y, s_L > x(0))(1 - G_L^0(t))(L + c)\Big).\end{aligned}$$

[11]Recall the (payoff) tie-breaking rule says that the return of the investment is not realized when the follower invests at the same time the leader disinvests.

[12]To be precise, the follower strictly prefers investing at the beginning of the pause due to discounting. The leader is indifferent between disinvesting at any time during the pause. The leader will deviate to disinvesting the beginning of the pause given the (indifference) tie-breaking rule.

In equilibrium, a pair of (inverse) strategies $(x(t), y(t))$ must satisfy $\mathcal{L}(x(t), t) \geq \mathcal{L}(x(t), t')$ and $\mathcal{F}(y(t), t) \geq \mathcal{F}(y(t), t')$ for all t and all $t' \neq t$. The following proposition characterizes the equilibrium (inverse) strategies $x(t)$ and $y(t)$.

Proposition 2. *In any monotonic symmetric dynamic equilibrium, $x(t)$ and $y(t)$ are differentiable functions that satisfy*

$$\rho_0 f^1(x(t)) f^1(y(t)) H - (1-\rho_0) f^0(x(t)) f^0(y(t)) L = 0, \tag{4}$$

$$x'(t) = r\left(\frac{(H-c)L}{(L+H)c}\frac{1-F^1(x(t))}{f^1(x(t))} - \frac{(L+c)H}{(L+H)c}\frac{1-F^0(x(t))}{f^0(x(t))}\right). \tag{5}$$

Equation (4) follows from leader $x(t)$'s first-order condition. If the follower does not invest in $[t, t+\mathrm{d}t)$, the leader's marginal cost from waiting is 0. If the follower invests in $[t, t+\mathrm{d}t)$, the return realizes, which would not have happened had the leader not waited. So the leader's marginal benefit from waiting is

$$\begin{aligned} &- y'(t) \Pr(s_F = y(t) | s_L = x(t), s_F \leq y(t)) \\ &\qquad \cdot \big[\Pr(\theta = 1 | x(t), y(t)) H - \Pr(\theta = 0 | x(t), y(t)) L \big]. \end{aligned}$$

This marginal benefit must be equal to the marginal cost which is 0. The first line is the probability that the follower invests in $[t, t+\mathrm{d}t)$, which is strictly positive. Therefore, the second line must be 0. This means if the follower invests in $[t, t+\mathrm{d}t)$, the leader learns that the follower's type is equal to $y(t)$, and the expected return of the project given the two signals $x(t)$ and $y(t)$ must be 0. In other words, the leader stays invested as long as his private signal (belief), conditional on the follower investing immediately, maps to a weakly positive expected payoff.

Equation (5) is derived from follower $y(t)$'s first-order condition. If the leader does not disinvest in $[t, t+\mathrm{d}t)$, the follower incurs a loss by waiting due to discounting. Thus, his marginal cost from waiting is

$$r\big[\Pr(\theta = 1 | s_F = y(t), s_L \geq x(t)) H - \Pr(\theta = 0 | s_F = y(t), s_L \geq x(t)) L - c\big].$$

This term is strictly positive for otherwise the follower would not want to invest at t.

If the leader disinvests in $[t, t + \mathrm{d}t)$, the follower's marginal benefit from waiting is

$$
\begin{aligned}
&- x'(t) \Pr(s_L = x(t) | s_F = y(t), s_L \geq x(t)) \\
&\qquad\qquad \cdot \big[\Pr(\theta = 1 | x(t), y(t)) H - \Pr(\theta = 0 | x(t), y(t)) L - c \big].
\end{aligned}
$$

The first line is the probability that the leader disinvests in $[t, t+\mathrm{d}t)$, which is strictly positive. The second line is the follower's expected payoff from investing, which is negative. If the leader disinvests in $[t, t + \mathrm{d}t)$, the follower learns that the leader's type is $x(t)$. As noted, the leader's first-order condition (4) implies that the expected return given $x(t)$ and $y(t)$ is zero. However, in addition to getting this zero return, the follower also pays the investment cost $c > 0$. Therefore, by waiting, the follower saves himself from a non-profitable investment.

It follows from the autonomous differential equation (5) and Lemma 2 that $T = \infty$. This means a leader with a posterior belief arbitrarily close to 1 never disinvests and a follower with a posterior belief arbitrarily close to 0 never invests.

Lemma 4. *In any monotonic symmetric dynamic equilibrium, $T = \infty$.*

3.2.3 Existence and uniqueness

Proposition 1 and Proposition 2 establish the necessary conditions for equilibrium. I now derive sufficient conditions for the existence and uniqueness of such an equilibrium. To this end, I impose the following two assumptions on the signal distributions.

Define the hazard ratio $h(s)$ and the failure ratio $k(s)$ as

$$
h(s) := \frac{1 - F^0(s)}{1 - F^1(s)} \frac{f^1(s)}{f^0(s)}, \quad k(s) := \frac{F^0(s)}{F^1(s)} \frac{f^1(s)}{f^0(s)}.
$$

Assumption. (F^0, F^1) satisfies the following properties:
(i) Increasing hazard ratio property (IHRP): *$h(s)$ is strictly increasing in s.*
(ii) Increasing failure ratio property (IFRP): *$k(s)$ is strictly increasing in s.*

In words, IHRP and IFRP state that higher signals are better news conditional on truncations.[13] IHRP (IFRP), under MLRP, is equivalent to "updating monotonically

[13] Kalashnikov and Rachev (1985) introduce these two concepts in statistics, and Herrera and Hörner (2012) derive their properties in a standard sequential social learning model (à la Banerjee, 1992 and Bikhchandani et al., 1992).

after good (bad) news": upon observing an investment (no investment), the public posterior likelihood ratio is increasing in the prior likelihood ratio. Many of the commonly used signal distributions satisfy IHRP and IFRP. Examples include Beta distributions and Normal distributions.[14]

IHRP is not necessary and is assumed only for convenience: it enables a clean statement of the parametric restriction for the existence of the equilibrium. IFRP ensures uniqueness of the initial values and further uniqueness of the equilibrium.

Theorem 1. *There exists $\bar{c} > 0$ such that a monotonic symmetric dynamic equilibrium exists if and only if $c < \bar{c}$. Moreover, this equilibrium is unique.*

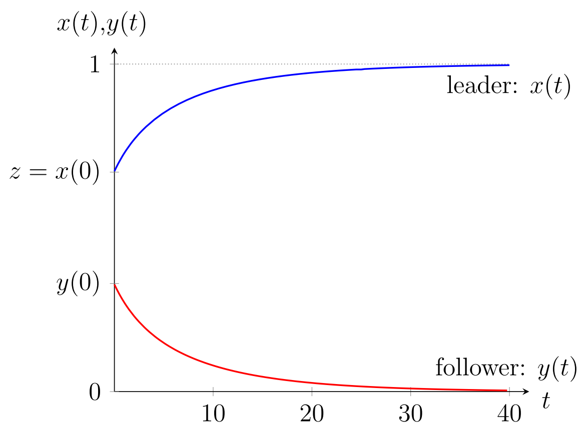

Figure 1: Equilibrium (inverse) strategies for $\rho_0 = 1/2, H = L = 1, r = 1/5, c = 1/5$ and posterior beliefs distributed according to $Beta(1+\theta, 1+(1-\theta))$.

Learning dynamics. Figure 1 plots the equilibrium (inverse) strategies and illustrates the equilibrium dynamics. At $t = 0$, information is aggregated in a single burst. A player with a type higher than z invests right away. Upon observing this investment, a player with a type in $[y(0), z]$ follows suit. For $t > 0$, information is aggregated gradually and continuously through delayed investment and disinvestment. At each t, the leader disinvests if his type is $x(t)$ and stays invested if above $x(t)$. The follower invests if his type is $y(t)$ and does not invest if below $y(t)$.

[14]IHRP and IFRP are assumptions on the signal distributions. Because signals are taken to be posterior beliefs, IHRP and IFRP are also imposed on the posterior belief distributions. If the identification between signals and posterior beliefs is dropped, it can be readily verified that if the signal distributions satisfy IHRP, IFRP, and MLRP, the induced posterior belief distributions also satisfy IHRP and IFRP (in addition to MLRP).

Evolution of beliefs. I discuss the evolution of the players' actual posterior beliefs and the public belief in the leader-follower continuation game. A player's actual posterior belief is his posterior belief given his private signal and the public information that the other player has not stopped. The public belief, denoted by $\rho(t)$, is the posterior belief given only the observable actions, $\rho(t) := \Pr(\theta = 1 | s_L \geq x(t), s_F < y(t))$.

A direct implication of the monotonicity of the equilibrium (inverse) strategies is that a player's actual posterior belief is monotone: the leader's posterior belief decreases over time and the follower's increases. The evolution of the public belief, however, is not as clear. The evolution of this belief is driven by two opposing forces: the leader staying in is good news but the follower staying out is bad. Which effect dominates depends on the relative rate at which the leader and the follower stop, as well as the relative likelihood of having a leader and a follower wait that long. Although the exact interplay of these two forces is difficult to pin down generally, one effect never overwhelms the other. The public belief is bounded away from 0 and 1, and eventually settles at an interior value ρ^* (ρ^* is defined in the Appendix).

Lemma 5. *$\rho(t) \in (0,1)$ for all $t \geq 0$. As $t \to \infty$, $\rho(t) \to \rho^*$ with $\rho^* \in (0,1)$.*

3.2.4 Takeaways

I conclude the equilibrium analysis by highlighting three features of the equilibrium.

First, the players' roles as the leader and follower are determined by their actions at time 0, when the players are ex ante symmetric. After time 0, the leader-follower continuation game is one with asymmetric players and this asymmetry is endogenous.

Second, the learning dynamic features bursts of information at the start of the game, and gradual revelation of information in the continuation game. In the continuation game, the duration of the leader's and the follower's waiting times respectively signal their optimism and pessimism. For each instant the leader stays invested, the follower becomes more optimistic; for each instant the follower stays out, the leader becomes more pessimistic. Moreover, the investment and disinvestment times fully reveal the types of the leader and the follower.

Lastly, the equilibrium dynamics are driven by a novel feature that the leader's incentive to disinvest solely comes from his fear of implementing a bad project and thus realizing a negative return — when the leader disinvests, he does not recover the investment cost, nor does he get any payoff from the project.

4 Role of Information Precision

In this section, I study the role of information precision in determining the equilibrium dynamics and outcomes.

4.1 Preliminaries

Symmetric environment

For tractability, I focus on a *symmetric environment*.[15] In a symmetric environment, the prior belief about $\theta = 1$ is $\rho_0 = 1/2$. The return H from investing in state $\theta = 1$ is equal to the loss L from investing in $\theta = 0$ and is normalized to 1, so $H = L = 1$. Moreover, the conditional distributions F^0 and F^1 are symmetric about the prior 1/2:

$$F^1(\mu) = 1 - F^0(1 - \mu).$$

In other words, the probability of having a posterior belief μ in state $\theta = 1$ is equal to the probability of having a posterior belief $1 - \mu$ in state $\theta = 0$.

Definition of precision

I define precision in terms of the posterior belief distributions. By definition, any pair of conditional posterior belief distributions must satisfy the consistency condition:

$$\mu = \frac{\rho_0 f^1(\mu)}{\rho_0 f^1(\mu) + (1 - \rho_0) f^0(\mu)}. \tag{6}$$

Recall that without loss of generality, the players' types are taken to be their posterior beliefs upon observing their private signals.

A common way to rank distributions according to their precision (or informativeness) is the notion of mean-preserving spread of posteriors. In this paper, however, it is useful to adopt a stronger notion: the unimodal likelihood ratio (ULR) order,[16] which has more structure than mean-preserving spread: the posterior belief about the good state conditional on left (right) truncation is higher (lower) under a more precise distribution. The standard definition of the ULR order is as follows.

[15] I expect the results in this section hold for asymmetric environments up to some extent.

[16] Ramos, Ollero, and Sordo (2000) first introduce the ULR order and Hopkins and Kornienko (2007) develop some of its properties.

Definition 2. A function $\ell(\mu)$ is *unimodal* around $\tilde{\mu}$ if $\ell(\mu)$ is strictly increasing for $\mu < \tilde{\mu}$ and strictly decreasing for $\mu > \tilde{\mu}$.

Definition 3 (Definition 2 in Hopkins and Kornienko, 2007)**.** For two distributions F and $\hat{F}$ (with density functions f and $\hat{f}$ respectively), F dominates $\hat{F}$ in the *unimodal likelihood ratio (ULR) order*, written as $F \succ_{\text{ULR}} \hat{F}$, if the likelihood ratio $f(\mu)/\hat{f}(\mu)$ is unimodal and the mean of F is greater than or equal to the mean of $\hat{F}$.

I define *more precise than* by adapting the ULR order, which ranks two distributions, for two pairs of distributions. To set notation, define the pair of conditional posterior belief distributions as $\mathbf{F} := (F^0, F^1)$, and the ex ante posterior belief distribution of $\mathbf{F}$ as $F(\mu) := \rho_0 F^1(\mu) + (1-\rho_0)F^0(\mu)$ with density $f(\mu) = \rho_0 f^1(\mu) + (1-\rho_0) f^0(\mu)$.

Definition 4. For two pairs of posterior belief distributions $\mathbf{F}$ and $\hat{\mathbf{F}}$, $\hat{\mathbf{F}}$ is *more precise than* $\mathbf{F}$ if
(i) the ex ante posterior belief distribution F dominates $\hat{F}$ in the unimodal likelihood ratio order $F \succ_{\text{ULR}} \hat{F}$;
(ii) the mean of $\hat{F}^1$ is higher than the mean of F^1.[17]

This definition is better interpreted in the symmetric environment. Condition (ii) is intuitive: conditional on the state, a more precise pair of distributions should be on average "more accurate" than a less precise pair. Condition (i) implies that the likelihood ratio of F and $\hat{F}$,

$$\frac{f(\mu)}{\hat{f}(\mu)} = \frac{f^1(\mu) + f^0(\mu)}{\hat{f}^1(\mu) + \hat{f}^0(\mu)},$$

is unimodal and symmetric about the prior. Because the mean of F and $\hat{F}$ are both equal to the prior, condition (i) further implies $\hat{F}$ is a mean-preserving spread of F.[18]

In line with the interpretation of the mean-preserving spread, $\hat{\mathbf{F}}$ is more precise than $\mathbf{F}$ if the ex ante posterior belief distribution $\hat{F}$ is more dispersed than F. It is less likely to get a posterior belief closer to the prior when information is more precise. This is captured by the likelihood ratio being unimodal around the prior.

[17]Because the mean of any ex ante posterior belief distribution is equal to the prior, this condition is equivalent to the mean of $\hat{F}^0$ being lower than the mean of F^0.
[18]Hopkins and Kornienko (2007), Proposition 1.

Many commonly used (signal) distributions, such as Beta distributions with different variances and Normal distributions with different variances, induce posterior belief distributions that satisfy the ULR order.

4.2 Speed of Learning

I show that in the leader-follower continuation game, the equilibrium (inverse) strategies $x(t)$ and $y(t)$ are ordered pointwise for different levels of precisions.

As an illustration, suppose the posterior beliefs are induced by signals distributed according to the Beta distributions $Beta(1+\gamma\theta, 1+\gamma(1-\theta))$ for $\gamma > 0$. It is readily verified that the higher γ is, the higher the precision of the induced posterior belief distribution is. Figure 2 plots the leader and the follower's equilibrium (inverse) strategies $x(t)$ and $y(t)$. If the precision is higher, the type of leader (follower) who disinvests (invests) at any $t \geq 0$ is higher (lower).

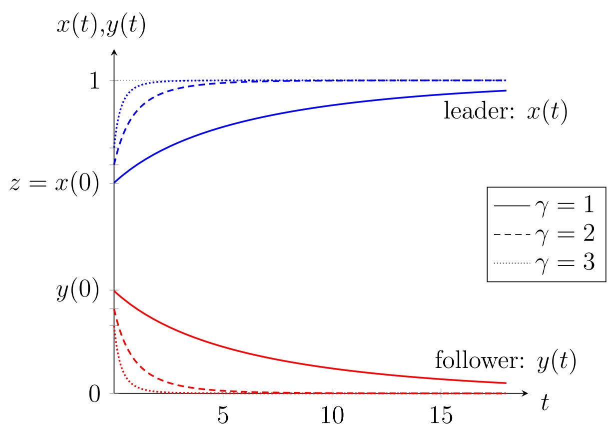

Figure 2: Equilibrium (inverse) strategies in a symmetric environment with $r = 1/5, c = 1/5$ and posterior beliefs induced by signals distributed according to $Beta(1+\gamma\theta, 1+\gamma(1-\theta))$.

Proposition 3. *Take two distributions* $\hat{\mathbf{F}}$ *and* $\mathbf{F}$ *with which a monotonic symmetric dynamic equilibrium exists. Suppose* $\hat{\mathbf{F}}$ *is more precise than* $\mathbf{F}$*. In the leader-follower continuation game, the threshold type who disinvests (invests) at* t *is higher (lower) for all* $t \geq 0$ *under* $\hat{\mathbf{F}}$*.*

The proposition can be restated as follows. For a player with any given private belief, if the precision is higher, he stops earlier. The leader and the follower reach an agreement (either both investing or both not investing) at the minimum of their stopping times. This means the players reach an agreement faster.

Corollary 1. *Fix a pair of types for the leader and the follower. The time at which the leader and the follower reach an agreement is lower under* $\hat{\mathbf{F}}$.

In other words, higher precision leads to faster coordination. This result might seem intuitive, but it is not obvious. Suppose the leader's strategy $x(t)$ does not change with precision. If the leader has stayed invested up to t, the follower knows the leader's type must be above $x(t)$. The ULR order implies knowing the leader's type is above $x(t)$ is more optimistic news if information is more precise. Thus, if the follower waits the same amount of time before investing under a more precise distribution, his belief would be higher by the end of his waiting time, and thus he would deviate to investing earlier. The leader in turn would want to disinvest earlier. This suggests players will reach an agreement faster if information is more precise.

4.3 Probability of Initial Coordination

I say that the players reach an initial agreement if either both players invest or neither player invests at $t = 0$. This includes three events: both players invest initially, neither invests initially, or one invests initially and the other follows suit. The previous section establishes that as precision increases, conditional on no initial agreement, the resolution of disagreement is faster in the continuation game. How does the probability of initial agreement change with precision?

I begin by setting the notation for precision. Consider a set of distributions $\{\mathbf{F}_\gamma\}_{\gamma\geq 0}$ ordered by precision and indexed by a parameter $\gamma \geq 0$, where a higher γ indicates a higher precision. I refer this index γ as the precision of $\mathbf{F}$. Assume this indexed set of distributions satisfies the following property: as $\gamma \to 0$, $\mathbf{F}_\gamma$ converges (pointwise) to the (pair of) uninformative distributions and as $\gamma \to \infty$, $\mathbf{F}_\gamma$ converges (pointwise) to the (pair of) perfectly informative distributions.[19] Denote the probability of initial agreement in the dynamic equilibrium by $\Pi_\gamma(x_\gamma(0))$, where $x_\gamma(0)$ is the initial value under $\mathbf{F}_\gamma$.

As before, consider the Beta distributions $\{Beta(1 + \gamma\theta, 1 + \gamma(1 - \theta))\}_{\gamma>0}$ for an illustration. Figure 3 plots the probability of initial agreement in the dynamic

[19] The indexing is arbitrary. The results in this section hold for any set of distributions that satisfies this property.

equilibrium as a function of precision γ for two different values of the investment cost c. As a reference, it also plots the probability of initial agreement in the no-investment and the myopic equilibrium, which is constant at 1.[20]

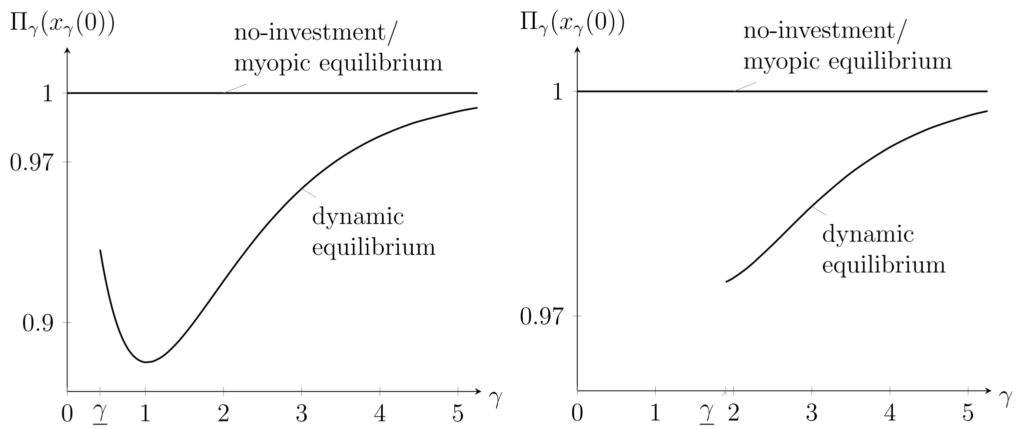

Figure 3: Probability of initial agreement in a symmetric environment with $c = 1/5$ (left panel) and $c = 3/5$ (right panel), and posterior beliefs induced by signals distributed according to $Beta(1 + \gamma\theta, 1 + \gamma(1 - \theta))$.

Figure 3 illustrates that there exists $\underline{\gamma}$ such that the dynamic equilibrium does not exist for $\gamma \leq \underline{\gamma}$. For $\gamma > \underline{\gamma}$, as γ increases, depending on the value of the investment cost, the probability of initial agreement either first decreases and then increases to 1, or monotonically increases to 1.

This example shows there exist parameters such that the probability of initial agreement in the dynamic equilibrium is U-shaped in precision. Intuitively, the probability of coordination at the beginning of the game is governed by the difference in players' posterior beliefs upon observing the private signal. At the one extreme, if the signals are uninformative, the players' posterior beliefs are the same as their prior. At the other extreme, if the signals are perfectly informative, players update their belief to either 0 or 1. In both cases, the players' beliefs are the same as one another, so they would take the same action right at the start.[21] For signals that are partially

[20]Recall that in the no-investment equilibrium, players never invest. In the myopic equilibrium, players invest with positive probability only at $t = 0$. These two equilibria generate probability 1 of initial agreement by construction. While the no-investment equilibrium always exists, the myopic equilibrium exists only when precision is high enough.

[21]In the symmetric environment, if the signals are uninformative, the expected payoff from investing given the prior is negative. The players coordinate on the strictly dominant strategy "not investing" with probability 1.

informative, as precision increases, while they make players more informed, they also create variances in beliefs and might lead to a higher probability of disagreement.

The following proposition formalizes this intuition in the limit. The dynamic equilibrium exists only when the precision of the distribution is sufficiently high, and the probability of initial agreement in the dynamic equilibrium converges to 1 as precision gets arbitrarily high.

Proposition 4. *There exists $\underline{\gamma} \in (0, \infty)$ such that a dynamic equilibrium exists if and only if $\gamma > \underline{\gamma}$. As $\gamma \to \infty$, the probability of initial agreement in the dynamic equilibrium $\Pi_\gamma(x_\gamma(0))$ converges to 1 from below.*

Although the above discussion concerns only the dynamic equilibrium, initial agreement is an observable behavior in all equilibria. In this spirit, I also discuss the probability of initial agreement as a property of the entire set of equilibria. For each γ, the probability of initial agreement can take on either one or two values: $\Pi_\gamma(x_\gamma(0))$ and 1 if a dynamic equilibrium exists under $\mathbf{F}_\gamma$, or 1 otherwise. While the maximum probability of initial agreement is always 1, the minimum probability of initial agreement, defined as $\underline{\Pi}_\gamma := \min \{\Pi_\gamma(x_\gamma(0)), 1\}$ for each γ, is non-monotone in precision. This observation is summarized in the following corollary.

Corollary 2. *The minimum probability of initial agreement is non-monotone in precision: $\underline{\Pi}_\gamma = 1$ for $\gamma \leq \underline{\gamma}$, $\underline{\Pi}_\gamma < 1$ for $\gamma > \underline{\gamma}$, and $\underline{\Pi}_\gamma \to 1$ as $\gamma \to \infty$.*

4.4 Ex ante Efficiency

A natural question is how ex ante efficiency, defined as the sum of the two players' ex ante equilibrium payoffs, changes as precision increases. In what follows, I derive a limiting result: the ex ante efficiency of the dynamic equilibrium approaches the full-efficiency payoff as precision gets arbitrarily high.

As information gets arbitrarily precise, by Proposition 4, the probability of initial agreement converges to 1; by Proposition 3, players reach an agreement faster if they had disagreed initially. Both players learn the state almost surely upon receiving the signals and are very likely to invest in the good state and not invest in the bad state right at time 0, which is the full-efficiency outcome, denoted by $\mathcal{E}^*$.[22] The following proposition formalizes this result.

[22]Formally, $\mathcal{E}^* = 2\Pr(\theta = 1)(1 - c) = 1 - c$.

Proposition 5. *As $\gamma \to \infty$, the ex ante efficiency of the dynamic equilibrium converges to the full-efficiency payoff $\mathcal{E}^*$ from below.*

In Figure 9 in the Appendix, I use a numerical example to illustrate that the ex ante efficiency is increasing in precision for all $\gamma > \underline{\gamma}$. However, to formally prove this monotonicity result is difficult. The limiting result Proposition 5, albeit seemingly intuitive, does not hold for all equilibria because of coordination. In particular, the no-investment equilibrium always generates 0 payoff, and all equilibria that do not start at time 0, by construction, will always fall short of the full-efficiency payoff regardless of how precise information gets.[23]

5 Benchmarks and Extensions

5.1 Constrained Efficiency

Consider a social planner who does not observe the signals or the true state θ and who seeks to maximize the sum of the expected payoff of the two players. The optimal mechanism, which is also incentive compatible, is straightforward. Each player truthfully reports his signal to the social planner; given the signals s_i and s_j, the social planner recommends investing at $t = 0$ to both players if the expected return from investing is higher than the cost,

$$\Pr(\theta = 1 | s_i, s_j)H - \Pr(\theta = 0 | s_i, s_j)L \geq c, \tag{7}$$

and not investing for any $t \geq 0$ otherwise.

This constrained efficient outcome features efficient information aggregation, no delay, and no coordination failure: players either both invest right away or never invest. How is the eventual outcome of the equilibrium (whether the two players eventually invest or not) different from the constrained efficient outcome?[24] To illustrate, Figure 4 plots the regions for eventual investment (and eventual no-investment) in the type space. If the pair of types (s_i, s_j) is in the blue region (area above the blue

[23] Recall that an equilibrium is said to start at $\hat{t} \geq 0$ if there is a positive probability of investment by either player at $\hat{t}$ and zero probability for all $t < \hat{t}$.

[24] To be more specific, in equilibrium, the outcome "eventual investment" is the events that (i) both players invest initially and (ii) the follower invests before the leader disinvests. The outcome "eventual no-investment" is the events that (i) neither player invests initially and (ii) the leader disinvests before the follower invests.

curve), players eventually both invest in equilibrium. If (s_i, s_j) is in the red region (area above the red curve), players eventually both invest in the constrained efficient outcome. (The complement is the no-investment region.)

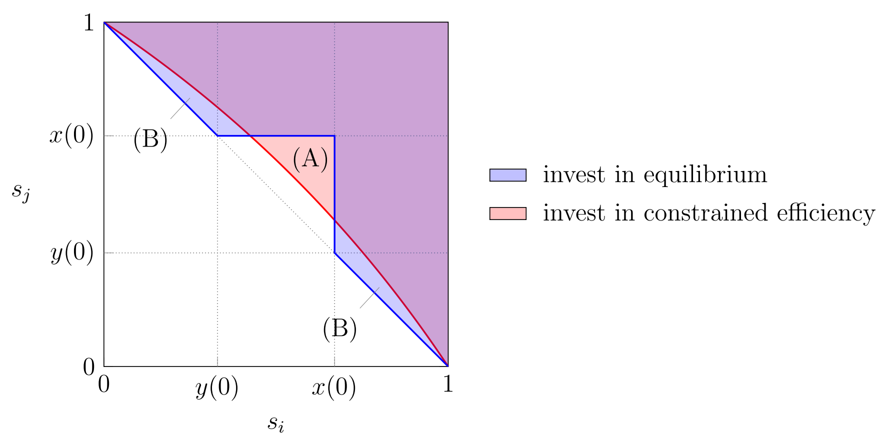

Figure 4: Investment regions in equilibrium and in the constrained efficient outcome in a symmetric environment with $c = 1/5, r = 1/5$ and posterior beliefs induced by $Beta(1 + \theta, 1 + (1 - \theta))$.

The eventual outcome of the equilibrium differs from the constrained efficient outcome in two ways. The first one is under-investment due to the lack of information transmission. In area (A), both players invest in the constrained efficient outcome, but neither invests in equilibrium. Players are optimistic, but not optimistic enough without information about the other player. The second case is over-investment due to inefficient information aggregation. In area (B), the follower eventually invests in equilibrium which he would not have done in the constrained efficient outcome. In equilibrium, the follower knows the leader's type is above a certain value, which is better news than knowing it is equal to that value as in the constrained efficient outcome. This leads to over-investment by some lower types of the follower.

The eventual outcome of the equilibrium is consistent with the constrained efficient outcome if the posterior belief given the combination of the two signals is low enough, in which case eventually neither player invests, or high enough, in which case both players invest. The following result formalizes this observation and provides sufficient conditions for this consistency.[25]

[25] If the equilibrium outcome is eventually consistent with the constrained efficient outcome, there is still an inefficiency due to delay and the leader paying the sunk investment cost.

Proposition 6. *Fix a dynamic equilibrium. The eventual outcome of this equilibrium coincides with the constrained efficient outcome if* (s_i, s_j) *is such that*
(i) $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L < 0$*, or*
(ii) $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L > \kappa$ *with* $\kappa > c$*.*[26]

5.2 Irreversible Investments

The reversibility of investment is crucial for generating the dynamics in the leader-follower continuation game. If investment is irreversible, all actions occur at time 0. The player who invests first eliminates strategic uncertainty for the other player, and the second mover's action reduces to whether or not to follow suit. A player invests initially if his type is above a threshold x_{ir}, and the other player follows suit if his type is above a threshold y_{ir} with $y_{\text{ir}} \leq x_{\text{ir}}$. Information revelation is coarse and occurs in a single burst at time 0. (x_{ir} and y_{ir} are characterized in the Appendix.)[27]

Compared to the investing behavior at time 0 in the dynamic equilibrium, the inability to back out of a potentially unprofitable investment might deter some lower types of the leader from investing initially. In turn, an initial investment is now better news and encourages more follower types to follow suit. The following result formalizes this intuition.

Proposition 7. *Fix parameters such that there exists an equilibrium in the model with irreversible investment and a dynamic equilibrium in the model with reversible investment. If* (x_{ir}, y_{ir}) *is an equilibrium with irreversible investment and* $(z, (x(t), y(t))_{t\geq 0})$ *is the equilibrium with reversible investment, then* $x_{ir} > x(0)$ *and* $y_{ir} < y(0)$*.*

I conclude this section with the remark that the equilibrium in the irreversible investment model might sometimes be more efficient than the dynamic equilibrium in the reversible investment model. Players benefit from observational learning generated by reversible investments when the information is valuable to learn. If the precision of information is low, the benefit of commitment from irreversible investments outweighs. Figure 5 illustrates this observation by plotting the ratio of the ex ante efficiency in the dynamic equilibrium $\mathcal{E}_\gamma(x_\gamma(0))$ to that in the irreversible investment equilibrium $\mathcal{E}^{\text{ir}}_\gamma(x^{\text{ir}}_\gamma, y^{\text{ir}}_\gamma)$. ($\mathcal{E}^{\text{ir}}_\gamma(x^{\text{ir}}_\gamma, y^{\text{ir}}_\gamma)$ is given in the Appendix.)

[26] $\kappa := \Pr(\theta = 1|x(0), x(0))H - \Pr(\theta = 0|x(0), x(0))L$. This implies if both players' types are high enough that they invest initially in equilibrium, they invest in the constrained efficient outcome.

[27] One can show there exists a perfect Bayesian equilibrium in a model with no discounting that has the same outcome as this irreversible investment equilibrium.

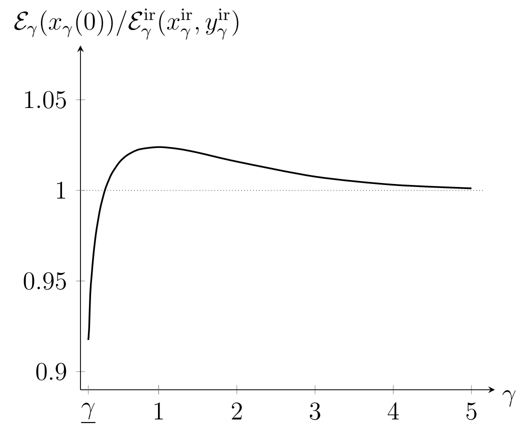

Figure 5: Ratio of the ex ante efficiency in the dynamic equilibrium to the irreversible investment equilibrium in a symmetric environment with $c = 1/20, r = 1/5$ and posterior beliefs induced by signals distributed according to $Beta(1+\gamma\theta, 1+\gamma(1-\theta))$.

5.3 Flow Cost of Investment

In this section, I characterize the equilibrium in a setting where the leader incurs a flow cost of investment $\eta > 0$ while staying invested. I call this a *flow-cost equilibrium.* I also illustrate by example that this equilibrium converges in strategies to the equilibrium characterized in Theorem 1, referred to as the *no-flow-cost equilibrium.*

To obtain a direct comparison with the no-flow-cost equilibrium, I focus on strictly monotonic and differentiable strategies and parameters such that the no-flow-cost equilibrium exists. The following theorem characterizes the flow-cost equilibrium.

Theorem 2. *Fix a flow cost of investment $\eta > 0$ for the leader. There exists a monotonic symmetric dynamic equilibrium where the inverse strategies $(z, (x(t), y(t))_{t\geq 0}) \in (0,1) \times (0,1)^\infty \times (0,1)^\infty$ are such that*
(i) the initial values z, $x(0)$, and $y(0)$ satisfy

$$y(0) < z = x(0), \; W_0(x(0), y(0)) = c; \tag{8}$$

(ii) for all $t \geq 0$, $x(t)$ is a strictly increasing differentiable function and $y(t)$ is a strictly decreasing differentiable function that solves the system of differential equations

$$x'(t) = \phi(x(t), y(t)), \; y'(t) = \psi(x(t), y(t)). \tag{9}$$

(W_0, ϕ, and ψ are defined in the Online Appendix.)

A general property of this equilibrium is $x'(t) \to 0$ and $y'(t) \to 0$ as $t \to \infty$. The initial values $x(0)$ and $y(0)$ must be that there exists a solution to the differential system (9) that satisfies this property. This is a "global boundary condition" and is similar to the one characterized in Bobtcheff, Bolte, and Mariotti (2017). Under some condition on the primitives (specified in the Online Appendix), this global solution is unique. For the rest of the analysis, assume this condition for uniqueness holds.

The effect of a positive flow cost is most prominent in the symmetric environment. Without a flow cost, it is costless for both the leader and the follower to stay in the game. This feature is necessary for the symmetry of the two players' equilibrium strategies, namely, $x(t) = 1 - y(t)$. A flow cost breaks this symmetry. While staying in the leader-follower continuation game is still costless for the follower, it is now costly for the leader. Figure 6 plots the unique flow-cost equilibrium in a symmetric environment. The dotted curve below $x(t)$ is $1 - y(t)$, which is auxiliary to bring out the now-existing asymmetry in the leader's and the follower's strategies.

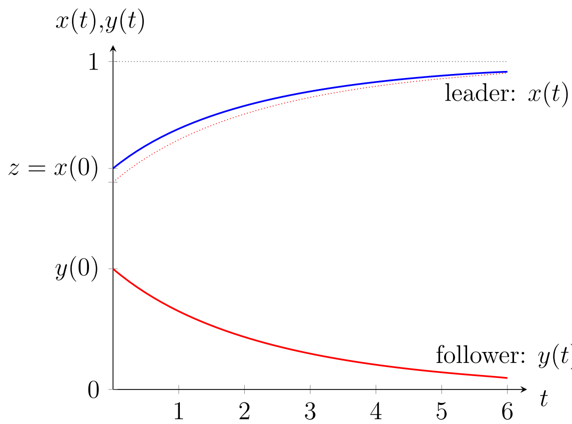

Figure 6: Flow-cost equilibrium (inverse) strategies in a symmetric environment with $c = 1/5$, $r = 1/5$, and $\eta = 1/20$ and posterior beliefs distributed according to $Beta(1 + \theta, 1 + (1 - \theta))$.

Convergence to the no-flow-cost equilibrium. To compare with the no-flow-cost equilibrium, I denote the (unique) flow-cost equilibrium (inverse) strategies with $\eta > 0$ by $x(t, \eta)$ and $y(t, \eta)$, and the (unique) no-flow-cost equilibrium by $x(t, 0)$ and $y(t, 0)$. In what follows, I verify by example that the equilibrium (inverse) strategies $x(t, \eta)$ and $y(t, \eta)$ converge to $x(t, 0)$ and $y(t, 0)$ as η converges to 0 for $t \geq 0$.

This convergence might not be as straightforward as one's intuition suggests. The flow-cost equilibrium has the property that for any c, the strategies $x(t, \eta)$ and $y(t, \eta)$ converge to the boundary points 1 and 0 as $t \to \infty$ for any $\eta > 0$. However, this is not always the case for the no-flow-cost equilibrium. In particular, when c is relatively high, $x(t, 0)$ and $y(t, 0)$ converge to interior points $\bar{x} < 1$ and $\underline{y} > 0$ as $t \to \infty$. In a numerical example with Beta distributions, albeit seemingly contradictory, the no-flow-cost equilibrium (inverse) strategies $x(t, \eta)$ and $y(t, \eta)$ converge to the flow-cost equilibrium $x(t, 0)$ and $y(t, 0)$ as the flow cost η decreases to 0.

The key observation is, the values that $x(t, \eta)$ and $y(t, \eta)$ converge to depend on the order in which the limits of t and η are taken. Take the leader's strategy $x(t, \eta)$ for example.[28] Although for any $\eta > 0$, as $t \to \infty$, $x(t, \eta) \to 1$ while $x(t, 0)$ may converge to $\bar{x} < 1$, $x(t, \eta) \to x(t, 0)$ pointwise as $\eta \to 0$ for any $t \geq 0$. The latter suggests the flow-cost equilibrium converges in strategies to the no-flow-cost equilibrium as the flow cost decreases to zero.

A Appendix

A.1 Proofs for Section 3

A.1.1 Preliminaries

I first establish a useful implication of MLRP.

Lemma 6. *$\hat{z} > z$ if and only if*

$$\frac{f^0(z)}{f^1(z)} > \frac{F^0(\hat{z}) - F^0(z)}{F^1(\hat{z}) - F^1(z)} > \frac{f^0(\hat{z})}{f^1(\hat{z})}.$$

The proof follows from MLRP and is relegated to the Online Appendix. The intuition for this result is similar to MLRP: knowing the signal is equal to z is worse news than knowing the signal is in an interval above z, and is better news than knowing the signal is in an interval below $\hat{z}$.

The two tie-breaking rules will be referred to frequently throughout the proofs in this section. The (payoff) tie-breaking rule says if the leader and the follower stop simultaneously, the leader gets 0 and the follower gets $-c$. The (indifference) tie-breaking rule says if the leader is indifferent between disinvesting at t' and at $t'' > t'$,

[28] The same argument applies to the follower's strategy $y(t, \eta)$.

he disinvests at the earlier time t'.

A.1.2 Proof of Lemma 1

By the (payoff) tie-breaking rule, the follower's payoff from investing at any $t \geq T_L$ is $-c < 0$. So for any follower y, either $\sigma_F(y) < T_L$ or $\sigma_F(y) = \infty$. $T_F = \inf_t \{t : G_F(t) = G_F(T_L)\} \leq T_L$. Given T_F, the leader is indifferent between disinvesting at any $t \geq T_F$. By the (indifference) tie-breaking rule, leader disinvests at T_F. $T_L = \inf_t\{t : G_L(t) = G_L(T_F)\} \leq T_F$.

A.1.3 Proof of Lemma 2

To simplify notation, denote the leader of type x's posterior belief about the state $\theta = 1$ at the beginning of the leader-follower continuation game by

$$q_L(x) = \Pr(\theta = 1|x, s_F < y(0)),$$

and the follower of type y's posterior belief by

$$q_F(y) = \Pr(\theta = 1|y, s_L > x(0)).$$

Denote the distribution of i's stopping time conditional on θ for $i \in \{L, F\}$ by

$$G_i^\theta(t) = \Pr(\sigma_i(s_i) \leq t|\sigma_i(s_i) > 0, \theta).$$

Lemma 7. *$\sigma_L(x)$ is non-decreasing and $\sigma_F(y)$ is non-increasing.*

The proof of Lemma 7 follows from a standard revealed-preference argument and is in the Online Appendix. In what follows, I prove the σ_i's are strictly monotone. This is equivalent to proving i's equilibrium distribution of stopping time is non-atomic for $i \in \{L, F\}$. I show that there does not exist an atom at any $t \in (0, T)$. The proof for atoms at $t = T$ is analogous and is relegated to the Online Appendix.

Follower's equilibrium distribution of stopping time is non-atomic

Fix the follower's distribution of stopping time and consider the leader's best response. Denote the leader's type by x. Suppose x disinvests at $t + \varepsilon$ for $\varepsilon > 0$ small. If the follower does not invest in $[t, t+\varepsilon)$, x gets 0. If the follower invests in $[t, t+\varepsilon)$, x gets $B(x,t)$, where

$$B(x,t) := \Pr(\theta = 1|x, \sigma_F(y) = t)H - \Pr(\theta = 0|x, \sigma_F(y) = t)L.$$

So x's expected payoff from disinvesting at $t+\varepsilon$ is $\lim_{\varepsilon\to 0}\Pr(\sigma_F(y) \in [t, t+\varepsilon)|x, \sigma_F(y) > 0)\cdot B(x,t)$. The first term is the probability that the follower invests in $[t, t+\varepsilon)$, which is positive.[29] Therefore, $B(x,t)$ is proportional to the change in x's expected payoff from disinvesting at t and at $t+\varepsilon$. In other words, x's expected payoff from disinvesting at t is increasing at t if $B(x,t) > 0$ and decreasing if $B(x,t) < 0$. By Lemma 7, $\sigma_F(y)$ is non-increasing, so $B(x,t)$ is non-increasing in t.

Suppose there is an atom at $\hat{t}$ in the follower's equilibrium distribution of stopping time. That is, $\sigma_F(y) = \hat{t}$ for all $y \in [y', y'']$ with $0 < y' < y'' < y(0)$. Then

$$B(x,\hat{t}) = \Pr(\theta = 1|x, y \in [y', y''])H - \Pr(\theta = 0|x, y \in [y', y''])L.$$

Denote the left limit of $B(x,\hat{t})$ by $B_-(x,\hat{t})$ and the right limit by $B_+(x,\hat{t})$,

$$B_-(x,\hat{t}) := \lim_{t\to\hat{t}-} B(x,t) = \Pr(\theta = 1|x, y = y'')H - \Pr(\theta = 0|x, y = y'')L,$$
$$B_+(x,\hat{t}) := \lim_{t\to\hat{t}+} B(x,t) = \Pr(\theta = 1|x, y = y')H - \Pr(\theta = 0|x, y = y')L.$$

By strict MLRP and Lemma 6, for all x,

$$B_-(x,\hat{t}) > B(x,\hat{t}) > B_+(x,\hat{t}).$$

Claim 1. There is a positive measure of leader types satisfying $B(x,\hat{t}) > 0 > B_+(x,\hat{t})$.

Proof of Claim 1. By strict MLRP and the continuity of the density functions, functions $B_-(x,\hat{t}), B(x,\hat{t})$ and $B_+(x,\hat{t})$ are continuous and strictly increasing in x. Because the likelihood ratio is unbounded, $\lim_{s\to 1} B(s,\hat{t}) > 0$; by Lemma 6, $B(x(0),\hat{t}) < 0$. So there exists a unique $\hat{x} \in (x(0), 1)$ such that $B(\hat{x},\hat{t}) = 0$. Similarly, there exists a unique $\hat{x}_+ \in (x(0), 1)$ such that $B_+(\hat{x}_+,\hat{t}) = 0$. $\hat{x}_+ > \hat{x}$ by MLRP. So for all $x \in (\hat{x}, \hat{x}_+)$, $B(x,\hat{t}) > 0 > B_+(x,\hat{t})$. □

[29] If this probability is equal to 0, that is, if there does not exist a follower investing in $[t, t+\varepsilon)$, the leader would be indifferent between stopping at t and $t+\varepsilon$, by the (indifference) tie-breaking rule, the leader stops at t, which violates the hypothesis that the leader stops at $t+\varepsilon$.

Claim 2. For all x satisfying $B(x, \hat{t}) > 0 > B_+(x, \hat{t})$, x does not have a best response.

Proof of Claim 2. Fix $x \in (\hat{x}, \hat{x}_+)$. Because $B(x,t)$ is non-increasing in t, $B(x,t) \geq B_-(x,\hat{t}) > B(x,\hat{t}) > 0$ for all $t < \hat{t}$ and $0 > B_+(x,\hat{t}) \geq B(x,t)$ for all $t > \hat{t}$. So x's expected payoff from disinvesting at t is increasing for $t < \hat{t}$ and decreasing for $t > \hat{t}$. So x will not disinvest at $t < \hat{t}$ or $t > \hat{t}$. If x has a best response, it can only be at $\hat{t}$. x's expected payoff from disinvesting at $\hat{t}$ is

$$
\begin{aligned}
\mathcal{L}(x,\hat{t}) =& q_L(x) \int_0^{\hat{t}} e^{-r\tau} \mathrm{d}\Pr(\sigma_F(y) < \tau | \sigma_F(y) > 0, \theta = 1) H \\
& - (1 - q_L(x)) \int_0^{\hat{t}} e^{-r\tau} \mathrm{d}\Pr(\sigma_F(y) < \tau | \sigma_F(y) > 0, \theta = 0) L.
\end{aligned}
$$

Denote $A(x,\hat{t})$ the size of the jump in the leader's expected payoff at $\hat{t}$:

$$
\begin{aligned}
A(x,\hat{t}) := e^{-r\hat{t}} \Big(& q_L(x) \Pr(\sigma_F(y) = \hat{t} | \sigma_F(y) > 0, \theta = 1) H \\
& - (1 - q_L(x)) \Pr(\sigma_F(y) = \hat{t} | \sigma_F(y) > 0, \theta = 0) L \Big).
\end{aligned}
$$

Then x's expected payoff from disinvesting at $\hat{t} + \varepsilon$ for $\varepsilon > 0$ small is

$$
\lim_{\varepsilon \to 0} \mathcal{L}(x, \hat{t} + \varepsilon) = \mathcal{L}(x,\hat{t}) + A(x,\hat{t}).
$$

$B(x,\hat{t}) > 0$ implies $A(x,\hat{t}) > 0$. So

$$
\sup_{t \in (\hat{t}-\varepsilon, \hat{t}+\varepsilon)} \mathcal{L}(x,t) = \mathcal{L}(x,\hat{t}) + A(x,\hat{t}).
$$

However, this supremum is not achieved: by the (payoff) tie-breaking rule, if x disinvests at $\hat{t}$, he gets $\mathcal{L}(x,\hat{t}) < \mathcal{L}(x,\hat{t}) + A(x,\hat{t})$. $\square$

Combining Claim 1 and Claim 2, there exists a positive measure of leader types whose best response is not well-defined, a contradiction.

Leader's equilibrium distribution of stopping time is non-atomic

The idea of the proof is as follows. If there is an atom at $\hat{t} \in (0, T)$ in the leader's stopping time, then either there exist follower types who don't have a best response, which is a contradiction, or there does not exist a follower type who invests in $[\hat{t}-\delta,\hat{t}]$. By the (indifference) tie-breaking rule, the leader stops at $\hat{t}-\delta$, which contradicts the hypothesis that a mass of leader stops at $\hat{t}$.

Define the size of the jump in follower y's expected payoff at t as

$$\begin{aligned} A(y,t) :=& q_F(y)\Pr(\sigma_L(x)=t|\sigma_L(x)>0,\theta=1)H \\ &- (1-q_F(y))\Pr(\sigma_L(x)=t|\sigma_L(x)>0,\theta=0)L \end{aligned}$$

and the expected cost from investing at t as

$$\begin{aligned} C(y,t) :=& c\bigg(q_F(y)\Pr(\sigma_L(x)=t|\sigma_L(x)>0,\theta=1) \\ &+ (1-q_F(y))\Pr(\sigma_L(x)=t|\sigma_L(x)>0,\theta=0)\bigg). \end{aligned}$$

Suppose there is an atom at $\hat{t}\in(0,T)$ in the leader's equilibrium distribution of stopping time. It must be that only a subset of the remaining leader types disinvest, otherwise it contradicts the definition of T. Follower y's expected payoff from investing at $\hat{t}-\varepsilon$ for $\varepsilon>0$ small is

$$\begin{aligned} \mathcal{F}_-(y,\hat{t}) :=& \lim_{\varepsilon\to 0}\mathcal{F}(y,\hat{t}-\varepsilon) \\ =& e^{-r\hat{t}}\bigg(q_F(y)\Pr(\sigma_L(x)\geq\hat{t}|\sigma_L(x)>0,\theta=1)(H-c) \\ &- (1-q_F(y))\Pr(\sigma_L(x)\geq\hat{t}|\sigma_L(x)>0,\theta=0)(L+c)\bigg), \end{aligned}$$

follower y's expected payoff from investing at $\hat{t}$ is

$$\mathcal{F}(y,\hat{t}) = \mathcal{F}_-(y,\hat{t}) - e^{-r\hat{t}}A(y,\hat{t}),$$

and follower y's expected payoff from investing at $\hat{t}+\varepsilon$ for $\varepsilon>0$ small is

$$\begin{aligned} \mathcal{F}_+(y,\hat{t}) := \lim_{\varepsilon\to 0}\mathcal{F}(y,\hat{t}+\varepsilon) &= \mathcal{F}(y,\hat{t}) + e^{-r\hat{t}}C(y,\hat{t}) \\ &= \mathcal{F}_-(y,\hat{t}) - e^{-r\hat{t}}(A(y,\hat{t})-C(y,\hat{t})). \end{aligned}$$

Note that by definition, $C(y,\hat{t})>0$ for all y, so $\mathcal{F}_+(y,\hat{t})>\mathcal{F}(y,\hat{t})$ for all y.

Claim 3. For all y such that $A(y,\hat{t}) - C(y,\hat{t}) \geq 0$, $\sup_{t>\hat{t}} \mathcal{F}(y,t) = \mathcal{F}_+(y,\hat{t})$.

Proof of Claim 3. The proof follows from MLRP and $\sigma_L(x)$ non-decreasing. Fix a y such that $A(y,\hat{t}) - C(y,\hat{t}) \geq 0$. y's expected payoff from investing at $t > \hat{t}$ is

$$\begin{aligned}
\mathcal{F}(y,t) =& e^{-rt}\Big(q_F(y)\Pr(\sigma_L(x) > \hat{t}|\sigma_L(x) > 0, \theta = 1)(H - c) \\
&\qquad - (1 - q_F(y))\Pr(\sigma_L(x) > \hat{t}|\sigma_L(x) > 0, \theta = 0)(L + c)\Big) \\
&- e^{-rt}\Big(q_F(y)\Pr(\hat{t} < \sigma_L(x) < t|\sigma_L(x) > 0, \theta = 1)(H - c) \\
&\qquad - (1 - q_F(y))\Pr(\hat{t} < \sigma_L(x) < t|\sigma_L(x) > 0, \theta = 0)(L + c)\Big) \\
&- e^{-rt}A(y,t).
\end{aligned}$$

$\mathcal{F}_+(y,\hat{t})$ can be written as

$$\begin{aligned}
\mathcal{F}_+(y,\hat{t}) =& e^{-r\hat{t}}\Big(q_F(y)\Pr(\sigma_L(x) > \hat{t}|\sigma_L(x) > 0, \theta = 1)(H - c) \\
&\qquad - (1 - q_F(y))\Pr(\sigma_L(x) > \hat{t}|\sigma_L(x) > 0, \theta = 0)(L + c)\Big).
\end{aligned}$$

MLRP and $A(y,\hat{t}) - C(y,\hat{t}) \geq 0$ imply $\mathcal{F}_+(y,\hat{t}) \geq 0$. Moreover, $e^{-r\hat{t}} > e^{-rt}$ so the first term in $\mathcal{F}(y,t)$ is less than $\mathcal{F}_+(y,\hat{t})$. The second term in $\mathcal{F}(y,t)$ is negative by Lemma 6 and $\sigma_L(x)$ non-decreasing. The third term in $\mathcal{F}(y,t)$ is the loss in payoff if there is an atom at t. If there is no atom at t, $A(y,t) = 0$. Otherwise, $A(y,\hat{t}) \geq C(y,\hat{t}) > 0$ and MLRP imply $A(y,t) > 0$ for $t > \hat{t}$. □

Analogous to the proof of Claim 1, $A(y,\hat{t}) - C(y,\hat{t})$ is strictly increasing in y and there exists a unique $\hat{y} \in (0, y(0))$ such that $A(\hat{y},\hat{t}) - C(\hat{y},\hat{t}) = 0$. Then $\mathcal{F}(\hat{y},\hat{t}) < \mathcal{F}_-(\hat{y},\hat{t}) = \mathcal{F}_+(\hat{y},\hat{t})$. By Claim 3, $\mathcal{F}_+(\hat{y},\hat{t})$ is the supremum over t for $t > \hat{t}$. In other words, $\mathcal{F}(\hat{y},t)$ cannot attain a maximum for $t > \hat{t}$. There are two cases: (i) $\mathcal{F}(\hat{y},t)$ attains a maximum in $[0,\hat{t}]$, and (ii) $\mathcal{F}(\hat{y},t)$ does not attain a maximum in $[0,\hat{t}]$.

Case (i) If $\mathcal{F}(\hat{y},t)$ attains a maximum at some $t^* \in [0,\hat{t}]$, it must be that $t^* < \hat{t}$. Because $\sigma_F(y)$ is non-increasing, if $\hat{y}$ invests at t^*, then all $y > \hat{y}$ will invest (at or) before t^* and all $y < \hat{y}$ will invest (at or) after t^*. So the only types who might invest in $(t^*,\hat{t}]$ are $y < \hat{y}$. For $y < \hat{y}$, $A(y,\hat{t}) - C(y,\hat{t}) < 0$, which implies $\mathcal{F}_-(y,\hat{t}) < \mathcal{F}_+(y,\hat{t})$.

So if investing at some $t^{**} \in (t^*, \hat{t}]$ is optimal, it must be that $\mathcal{F}(y, t^{**}) \geq \mathcal{F}_+(y, \hat{t}) > \mathcal{F}_-(y, \hat{t})$ if $\mathcal{F}_+(y, \hat{t}) \geq 0$, or $\mathcal{F}(y, t^{**}) \geq 0 > \mathcal{F}_-(y, \hat{t})$ if $\mathcal{F}_+(y, \hat{t}) < 0$. Either way, there exists a $\delta > 0$ small such that $t^{**} \notin [\hat{t} - \delta, \hat{t}]$. This implies there does not exist a y such that $\sigma_F(y) \in [\hat{t} - \delta, \hat{t}]$. This means the leader is indifferent between stopping at $\hat{t} - \delta$ and $\hat{t}$. By the (indifference) tie-breaking rule, the leader stops at $\hat{t} - \delta$, which contradicts the hypothesis that there is a mass of leader types stopping at $\hat{t}$.

Case (ii) Suppose $\mathcal{F}(\hat{y}, t)$ doesn't attain a maximum in $[0, \hat{t}]$. Consider the incentive of the types that are higher than $\hat{y}$. Fix $y > \hat{y}$. Then $A(y, \hat{t}) - C(y, \hat{t}) > 0$, which implies $\mathcal{F}_-(y, \hat{t}) > \mathcal{F}_+(y, \hat{t}) > 0$ and $\mathcal{F}_-(y, \hat{t}) > \mathcal{F}(y, \hat{t})$. Claim 3 states that $\mathcal{F}_+(y, \hat{t})$ is the supremum over t for all $t > \hat{t}$. There are two sub-cases.

First, all $\mathcal{F}(y, t)$ with $y > \hat{y}$ up to sets of measure zero attain a maximum in $[0, \hat{t}]$. If $\mathcal{F}(y, t)$ attains a maximum at some $t^* \in [0, \hat{t}]$, it must be that $\mathcal{F}(y, t^*) \geq \mathcal{F}_-(y, \hat{t}) > 0$ and $t^* < \hat{t}$. So there exists a $\delta > 0$ small such that $t^* \notin [\hat{t} - \delta, \hat{t}]$, which is a contradiction by the same argument as case (i).

Second, there exists a positive measure of follower types y with $y > \hat{y}$ such that $\mathcal{F}(y, t)$ does not attain a maximum in $[0, \hat{t}]$. Then $\mathcal{F}_-(y, \hat{t})$ is the supremum of $\mathcal{F}(y, t)$ for all t. However, this supremum cannot be achieved: $\mathcal{F}(y, \hat{t}) = \mathcal{F}_-(y, \hat{t}) - e^{-r\hat{t}} A(y, \hat{t}) < \mathcal{F}_-(y, \hat{t})$ as $A(y, \hat{t}) > C(y, \hat{t}) > 0$. So a positive measure of follower types does not have a best response, a contradiction.

Combining case (i) and (ii), there cannot exist an atom in the leader's equilibrium distribution of stopping time at any $t \in (0, T)$.

A.1.4 Proof of Lemma 3

To simplify notation, let $\mathcal{F}(y, t) = e^{-rt} \mathcal{G}(y, t)$ where

$$\mathcal{G}(y, t) := q_F(y)(1 - G_L^1(t))(H - c) - (1 - q_F(y))(1 - G_L^0(t))(L + c). \tag{10}$$

First, I show if G_i is constant on $[t', t'']$, G_{-i} is also constant on $[t', t'']$ for $0 < t' < t'' \leq T$.

Suppose $G_L(t)$ is constant on $[t', t'']$. Then $\mathcal{G}(y, t)$ is constant on $[t', t'']$. For $t \in [t', t'']$, if y is such that $\mathcal{G}(y, t) > 0$, then $\mathcal{F}(y, t') > \mathcal{F}(y, t) > 0$ for all $t \in (t', t'']$, so investing at any $t \in [t', t'']$ is dominated by investing at t'. If y is such that $\mathcal{G}(y, t) < 0$, then $\mathcal{F}(y, t') < \mathcal{F}(y, t) < \mathcal{F}(y, t'') < 0$, so investing at any $t \in [t', t'']$ is dominated by not investing. Therefore, there does not exist a positive measure of follower types that invests in $(t', t'']$,[30] which means $G_F(t)$ is constant on $[t', t'']$.

Suppose $G_F(t)$ is constant on $[t', t'']$. By the (indifference) tie-breaking rule, for any leader x such that disinvesting at any $t \in [t', t'']$ is optimal, x disinvests at t'. So no x will disinvest in $(t', t'']$, which means $G_L(t)$ is constant on $[t', t'']$.

Next, I show that there does not exist an interval $[t', t'']$ with $0 < t' < t'' \leq T$ such that both G_F and G_L are constant. Suppose the contrary and let $\bar{t} \leq T$ be the supremum of t'' for which over $[t', t'']$, G_L and G_F are constant.

Fix a y such that $\mathcal{G}(y, t) > 0$, so $\mathcal{F}(y, t') > \mathcal{F}(y, t)$ for all $t \in (t', \bar{t}]$. In particular, $\mathcal{F}(y, t') > \mathcal{F}(y, \bar{t})$. By Lemma 2, $\mathcal{F}(y, t)$ is continuous in t, so for $\delta > 0$, $\lim_{\delta \to 0} \mathcal{F}(y, \bar{t} + \delta) = \mathcal{F}(y, \bar{t}) < \mathcal{F}(y, t')$. So y will not invest in $[\bar{t}, \bar{t} + \delta]$.

Fix a y such that $\mathcal{G}(y, t) < 0$, so $\mathcal{F}(y, t') < \mathcal{F}(y, \bar{t}) < 0$ for all $t \in (t', \bar{t}]$. Similarly, $\lim_{\delta \to 0} \mathcal{F}(y, \bar{t} + \delta) = \mathcal{F}(y, \bar{t}) < 0$, so investing at $\bar{t} + \delta$ is dominated by not investing. So y will not invest in $[\bar{t}, \bar{t} + \delta]$.

Therefore, there does not exist a positive measure of y that invests in $[\bar{t}, \bar{t} + \delta]$. This means G_F is constant on $[t', \bar{t} + \delta]$, so G_L is also constant on $[t', \bar{t} + \delta]$. This contradicts the definition of $\bar{t}$.

A.1.5 Proof of Proposition 2

Differentiability

The main idea is to show (some of) the Dini derivatives of $x(t)$ and $y(t)$ with respect to t are bounded. Lemma 2 shows $x(t)$ and $y(t)$ are continuous in t, which then implies $x(t)$ are $y(t)$ are Lipschitz continuous in t (Giorgi and Komlósi, 1992, Theorem 1.16).

Write $G_i^\theta(t)$ in terms of $x(t)$ and $y(t)$,

$$G_L^\theta(t) = F_L^\theta(x(t)) := \Pr(s_L < x(t) | s_L > x(0), \theta) = \frac{F^\theta(x(t)) - F^\theta(x(0))}{1 - F^\theta(x(0))},$$

$$G_F^\theta(t) = 1 - F_F^\theta(y(t)) := \Pr(s_F > y(t) | s_F < y(0), \theta) = \frac{F^\theta(y(0)) - F^\theta(y(t))}{F^\theta(y(0))}.$$

Same as before, let $\mathcal{F}(y, t) = e^{-rt}\mathcal{G}(y, t)$ where $\mathcal{G}(y, t)$ is given by (10).

Fix $t^* \in (0, T)$. Let $\{t_n\}_{n=1}^\infty$ be a decreasing sequence such that $t_n > t^*$ for all n and $t_n \to t^*$ as $n \to \infty$. Let $y^* = y(t^*)$ denote the type who optimally stops at t^*.

[30] By strict MLRP, the set of y's that satisfy $\mathcal{G}(y, t) = 0$ is of measure zero.

By optimality, $\mathcal{F}(y^*, t_n) \leq \mathcal{F}(y^*, t^*)$ for all t_n. This means

$$e^{-rt_n}\mathcal{G}(y^*, t_n) \leq e^{-rt^*}\mathcal{G}(y^*, t^*) \implies \mathcal{G}(y^*, t_n) \leq e^{r(t_n - t^*)}\mathcal{G}(y^*, t^*).$$

Because e^{-rt} is Lipschitz with Lipschitz constant r, so $e^{r(t_n-t^*)} - e^{r\cdot 0} \leq r(t_n - t^* - 0)$, which is equivalent to $e^{r(t_n-t^*)} \leq r(t_n - t^*) + 1$, then

$$\mathcal{G}(y^*, t_n) - \mathcal{G}(y^*, t^*) \leq r(t_n - t^*)\mathcal{G}(y^*, t^*).$$

Claim 4. For n large enough, $\mathcal{G}(y^*, t_n) > \mathcal{G}(y^*, t^*)$.[31]

Suppose Claim 4 established. Then

$$0 < \mathcal{G}(y^*, t_n) - \mathcal{G}(y^*, t^*) \leq r(t_n - t^*)\mathcal{G}(y^*, t^*). \tag{11}$$

Divide through by $t_n - t^*$ and take the limit superior as $t_n \downarrow t^*$,

$$0 < D^+\mathcal{G}(y^*, t^*) := \limsup_{t_n \to t^*+} \frac{\mathcal{G}(y^*, t_n) - \mathcal{G}(y^*, t^*)}{t_n - t^*} \leq r\mathcal{G}(y^*, t^*), \tag{12}$$

where $D^+\mathcal{G}(y^*, t^*)$ denotes the upper right Dini derivative of $\mathcal{G}$ at t^*. Because $\mathcal{G}(y^*, t^*)$ is finite, so the upper right Dini derivative is bounded. Moreover, by Lemma 2 and Lemma 3, $\mathcal{G}$ is continuous in t. Thus $\mathcal{G}$ is Lipschitz-continuous in t (Giorgi and Komlósi, 1992, Theorem 1.16).

Writing out $\mathcal{G}(y^*, t_n) - \mathcal{G}(y^*, t^*)$,

$$\begin{aligned}\mathcal{G}(y^*, t_n) - \mathcal{G}(y^*, t^*) =& q_F(y^*)\left(F_L^1(x(t^*)) - F_L^1(x(t_n))\right)(H - c)\\ &- (1 - q_F(y^*))\left(F_L^0(x(t^*)) - F_L^0(x(t_n))\right)(L + c).\end{aligned}$$

[31] The intuition is as follows. $\mathcal{G}$ is the follower's undiscounted payoff. That is, $\mathcal{G}$ is $\mathcal{F}$ with $r = 0$. Fix the type of the follower. If an impatient follower with payoff function $\mathcal{F}$ waits till t^* to invest, then a patient follower with payoff function $\mathcal{G}$ must be willing to wait longer than t^*. Thus $\mathcal{G}$ is maximized at some $t^{**} > t^*$. So when $\mathcal{F}$ starts to decrease after t^*, $\mathcal{G}$ would still be increasing at t^*.

Then

$$
\begin{aligned}
D^+\mathcal{G}(y^*,t^*) =& \limsup_{t_n\to t^*+} \frac{\mathcal{G}(y^*,t_n)-\mathcal{G}(y^*,t^*)}{t_n-t^*} \\
=& q_F(y^*) \limsup_{t_n\to t^*+} \frac{F_L^1(x(t^*))-F_L^1(x(t_n))}{t_n-t^*}(H-c) \\
&- (1-q_F(y^*)) \liminf_{t_n\to t^*+} \frac{F_L^0(x(t^*))-F_L^0(x(t_n))}{t_n-t^*}(L+c) \\
=& - q_F(y^*) \liminf_{t_n\to t^*+} \frac{F_L^1(x(t_n))-F_L^1(x(t^*))}{t_n-t^*}(H-c) \\
&+ (1-q_F(y^*)) \limsup_{t_n\to t^*+} \frac{F_L^0(x(t_n))-F_L^0(x(t^*))}{t_n-t^*}(L+c) \\
=& - q_F(y^*)D_+F_L^1(x(t^*))(H-c) + (1-q_F(y^*))D^+F_L^0(x(t^*))(L+c),
\end{aligned}
$$

where $D_+F_L^1(x(t^*))$ is the lower right Dini derivative of $F_L^1(x(\cdot))$ at t^*, and $D^+F_L^0(x(t^*))$ is the upper right Dini derivative of $F_L^1(x(\cdot))$ at t^*. So (12) becomes

$$
0 < (1-q_F(y^*))D^+F_L^0(x(t^*)) - q_F(y^*)D_+F_L^1(x(t^*))(H-c) \le r\mathcal{G}(y^*,t^*). \tag{13}
$$

Similarly, take the limit inferior of (11),

$$
0 < (1-q_F(y^*))D_+F_L^0(x(t^*))(L+c) - q_F(y^*)D^+F_L^1(x(t^*))(H-c) \le r\mathcal{G}(y^*,t^*). \tag{14}
$$

Combine (13) and (14),

$$
\begin{aligned}
0 <& (1-q_F(y^*))D_+F_L^0(x(t^*))(L+c) - q_F(y^*)D^+F_L^1(x(t^*))(H-c) \\
\le& (1-q_F(y^*))D^+F_L^0(x(t^*))(L+c) - q_F(y^*)D^+F_L^1(x(t^*))(H-c) \\
\le& (1-q_F(y^*))D^+F_L^0(x(t^*))(L+c) - q_F(y^*)D_+F_L^1(x(t^*))(H-c) \\
\le& r\mathcal{G}(y^*,t^*).
\end{aligned}
$$

Specifically, the second line is

$$
0 < (1-q_F(y^*))D^+F_L^0(x(t^*))(L+c) - q_F(y^*)D^+F_L^1(x(t^*))(H-c) \le r\mathcal{G}(y^*,t^*). \tag{15}
$$

Consider the leader's expected payoff. Let $x^* = x(t^*)$ denote the type of the leader who optimally stops at t^*. By optimality, $\mathcal{L}(x^*, t_n) \leq \mathcal{L}(x^*, t^*)$ for all t_n. This implies

$$q_L(x^*) \int_{t^*}^{t_n} e^{-r\tau} \mathrm{d}(1 - F_F^1(y(\tau)))H - (1 - q_L(x^*)) \int_{t^*}^{t_n} e^{-r\tau} \mathrm{d}(1 - F_F^0(y(\tau)))L \leq 0.$$

Because

$$\int_{t^*}^{t_n} e^{-r\tau} \mathrm{d}(1 - F_F^1(y(\tau))) \geq e^{-rt_n} \int_{t^*}^{t_n} \mathrm{d}(1 - F_F^1(y(\tau))) = e^{-rt_n}(F_F^1(y(t^*)) - F_F^1(y(t_n)))$$

$$\int_{t^*}^{t_n} e^{-r\tau} \mathrm{d}(1 - F_F^0(y(\tau))) \leq e^{-rt^*} \int_{t^*}^{t_n} \mathrm{d}(1 - F_F^0(y(\tau))) = e^{-rt^*}(F_F^0(y(t^*)) - F_F^0(y(t_n))),$$

so

$$q_L(x^*)e^{-rt_n}(F_F^1(y(t^*)) - F_F^1(y(t_n)))H - (1 - q_L(x^*))e^{-rt^*}(F_F^0(y(t^*)) - F_F^0(y(t_n)))L \leq 0,$$

which implies (by Lemma 3, the denominator is strictly positive)

$$\frac{q_L(x^*)\,(F_F^1(y(t^*)) - F_F^1(y(t_n)))\,H}{(1 - q_L(x^*))\,(F_F^0(y(t^*)) - F_F^0(y(t_n)))\,L} \leq e^{r(t_n - t^*)}.$$

Because $e^{r(t_n - t^*)} \leq r(t_n - t^*) + 1$,

$$\begin{aligned} &q_L(x^*)\left(F_F^1(y(t^*)) - F_F^1(y(t_n))\right)H - (1 - q_L(x^*))\left(F_F^0(y(t^*)) - F_F^0(y(t_n))\right)L \\ \leq &r(t_n - t^*)(1 - q_L(x^*))\left(F_F^0(y(t^*)) - F_F^0(y(t_n))\right)L. \end{aligned} \tag{16}$$

Denote $\mathcal{U}(x,t) = q_L(x^*)(1 - F_F^1(y(t)))H - (1 - q_L(x^*))(1 - F_F^0(y(t)))L$. The following claim holds.

Claim 5. For n large enough, $\mathcal{U}(x^*, t_n) > \mathcal{U}(x^*, t^*)$.[32]

Write out $\mathcal{U}(x^*, t_n) - \mathcal{U}(x^*, t^*)$, by Claim 5, $\mathcal{U}(x^*, t_n) - \mathcal{U}(x^*, t^*) > 0$,

$$\begin{aligned} \mathcal{U}(x^*, t_n) - \mathcal{U}(x^*, t^*) = &q_L(x^*)\left(F_F^1(y(t^*)) - F_F^1(y(t_n))\right)H \\ &- (1 - q_L(x^*))\left(F_F^0(y(t^*)) - F_F^0(y(t_n))\right)L > 0. \end{aligned} \tag{17}$$

Combine (16) and (17). Divide through by $t_n - t^*$ and take the limit superior as

[32]The proof of Claim 5 is omitted as it is the same as Claim 4. The intuition is also the same: $\mathcal{U}$ is the leader's undiscounted payoff. If $\mathcal{L}$ is maximized at t^*, then $\mathcal{U}$ is maximized at some $t^{**} > t^*$.

$t_n \downarrow t^*$,

$$\begin{aligned}0 <& q_L(x^*)D_+F_F^1(y(t^*))H - (1-q_L(x^*))D^+F_F^0(y(t^*))L \\ &\leq \lim_{t_n \to t^*+} r(1-q_L(x^*))\left(F_F^0(y(t^*)) - F_F^0(y(t_n))\right)L.\end{aligned}$$

By Lemma 2, $F_F^0(y(t))$ is continuous in t, so the right-hand side is equal to 0. Then

$$q_L(x^*)D_+F_F^1(y(t^*))H - (1-q_L(x^*))D^+F_F^0(y(t^*))L = 0. \tag{18}$$

Similarly, take the limit inferior as $t_n \downarrow t^*$,

$$q_L(x^*)D^+F_F^1(y(t^*))H - (1-q_L(x^*))D_+F_F^0(y(t^*))L = 0. \tag{19}$$

Combine (18) and (19),

$$\begin{aligned}0 =& q_L(x^*)D_+F_F^1(y(t^*))H - (1-q_L(x^*))D^+F_F^0(y(t^*))L \\ \leq& q_L(x^*)D^+F_F^1(y(t^*))H - (1-q_L)D^+F_F^0(y(t^*))L \\ \leq& q_L(x^*)D^+F_F^1(y(t^*))H - (1-q_L)D_+F_F^0(y(t^*))L \\ =& 0.\end{aligned}$$

Specifically, the second line is

$$q_L(x^*)D^+F_F^1(y(t^*))H - (1-q_L(x^*))D^+F_F^0(y(t^*))L = 0. \tag{20}$$

Because F^θ is differentiable in y and $y(t)$ is continuous in t by Lemma 2,

$$\begin{aligned}D^+F_F^\theta(y(t^*)) =& \limsup_{t_n \to t^*+} \frac{F_F^\theta(y(t_n)) - F_F^\theta(y(t^*))}{t_n - t^*} \\ =& \limsup_{t_n \to t^*+} \frac{F_F^\theta(y(t_n)) - F_F^\theta(y(t^*))}{y(t_n) - y(t^*)} \frac{y(t_n) - y(t^*)}{t_n - t^*} \\ =& \lim_{t_n \to t^*} \frac{F_F^\theta(y(t_n)) - F_F^\theta(y(t^*))}{y(t_n) - y(t^*)} \limsup_{t_n \to t^*+} \frac{y(t_n) - y(t^*)}{t_n - t^*} \\ =& f_F^\theta(y(t^*))D^+y(t^*).\end{aligned}$$

Thus (20) can be written as

$$D^+y(t^*)\left[f^1(x(t^*))f^1(y(t^*))H - f^0(x(t^*))f^0(y(t^*))L\right] = 0.$$

By Lemma 3, $y(t)$ is strictly decreasing, so $D^+y(t^*) < 0$, so it must be that

$$f^1(x(t^*))f^1(y(t^*))H - f^0(x(t^*))f^0(y(t^*))L = 0. \tag{21}$$

Similarly, $D^+F_L^\theta(x(t^*)) = f_L^\theta(x(t^*))D^+x(t^*)$. Rewrite (15),

$$0 < D^+x(t^*)\left[(1-q_F(y^*))f_L^0(x(t^*))(L+c) - q_F(y^*)f_L^1(x(t^*))(H-c)\right] \leq r\mathcal{G}(y^*, t^*).$$

Suppose $D^+x(t^*)$ is unbounded. Because $\mathcal{G}(y^*, t^*)$ is bounded, so for the above inequalities to hold, it must be that the second term is zero. That is,

$$f^1(x(t^*))f^1(y(t^*))(H-c) - f^0(x(t^*))f^0(y(t^*))(L+c) = 0.$$

This contradicts (21). So $D^+x(t^*)$ must be bounded, thus $x(t)$ is Lipschitz.

Rewrite (21) as $l(x(t^*))l(y(t^*)) = H/L$. Take the limit superior,

$$l'(x(t^*))l(y(t^*))D^+x(t^*) + l'(y(t^*))l(x(t^*))D^+y(t^*) = 0,$$

where $l'(\cdot)$ is the derivative of the likelihood. By assumption, $l(\cdot)$ is unbounded and differentiable. The above equation implies $D^+y(t^*)$ is bounded, thus $y(t)$ is Lipschitz.

Because $x(t)$ and $y(t)$ are Lipschitz, they are absolutely continuous and thus differentiable almost everywhere. Their derivatives, where they exist, can be obtained by differentiating the leader's and the follower's expected payoffs with respected to t, and set the resulting derivative to zero whenever it exists, as implied by optimality. The problems faced by the leader and and the follower are respectively

$$\max_{t>0}\mathcal{L}(x,t) = \max_{t>0} q_L(x)\int_0^t e^{-r\tau}\mathrm{d}(1-F_F^1(\tau))H - (1-q_L(x))\int_0^t e^{-r\tau}\mathrm{d}(1-F_F^0(\tau))L,$$

$$\max_{t>0}\mathcal{F}(y,t) = \max_{t>0} e^{-rt}\left(q_F(y)(1-F_L^1(t))(H-c) - (1-q_F(y))(1-F_L^0(t))(L+c)\right).$$

The leader and follower's first-order conditions reduce to (4) and (5). As a result,

$$x(t) = x(0) + \int_0^t r\left(\frac{(H-c)L}{(L+H)c}\frac{1-F^1(x(\tau))}{f^1(x(\tau))} - \frac{(L+c)H}{(L+H)c}\frac{1-F^0(x(\tau))}{f^0(x(\tau))}\right)d\tau. \quad (22)$$

$x(\tau)$ is continuous, so the integrand of (22) is continuous in τ. By the fundamental theorem of calculus, one can differentiate equation (22) everywhere with respect to t so equation (5) holds for all $t > 0$. To conclude the proof, note that by Lemma 2, $x(t)$ is continuous at $t = 0$ and so is the integrand in equation (22). Thus $x(t)$ is differentiable at $t = 0$. Equation (4) implies $y(t)$ is also everywhere differentiable.

It remains to prove Claim 4.

Proof of Claim 4. The proof goes through the following steps.

Step 1. For any given y, $t_r := \arg\max_t e^{-rt}\mathcal{G}(y,t)$ is non-increasing in r.

First, $\arg\max_t e^{-rt}\mathcal{G}(y,t) = \arg\max_t \log\left(e^{-rt}\mathcal{G}(y,t)\right) = \arg\max_t \{-rt + \log\mathcal{G}(y,t)\}$. Define $\mathcal{H}(r,t) = -rt + \log\mathcal{G}(y,t)$ and $\Delta\mathcal{H}(r,t,t') = \mathcal{H}(r,t') - \mathcal{H}(r,t)$. Let $t < t'$ and $r < r'$,

$$\Delta\mathcal{H}(r',t,t') - \Delta\mathcal{H}(r,t,t') = (r-r')(t'-t) < 0.$$

Therefore, $\mathcal{H}$ is submodular, which means its maximizer t_r is non-increasing in r.

Step 2. $t_r < t_0$ for $r > 0$.

Step 1 implies $t_r \leq t_0$ for all $r > 0$. Suppose the contrary, $t_r = t_0 = t^*$. Let y^* (with belief $q_F(y^*) =: q^*$) be the type who optimally stops at t^*. By optimality, the expected loss from waiting for $\delta > 0$ small is equal to the expected gain from waiting,

$$\begin{aligned}&\left(e^{-rt^*} - e^{-r(t^*+\delta)}\right)\mathcal{G}(y^*,t^*)\\ &= -e^{-r(t^*+\delta)}\left(q^*(G_L^1(t^*+\delta) - G_L^1(t^*))(H-c) - (1-q^*)(G_L^0(t^*+\delta) - G_L^0(t^*))(L+c)\right).\end{aligned}$$

For $r > 0$, the left-hand side is positive for any positive δ, but it is equal to zero for any δ as long as $r = 0$. This means at t^*, this optimality condition cannot be satisfied for both $r > 0$ and $r = 0$, which contradicts the hypothesis that $t_r = t_0 = t^*$.

Step 3. For any given y (with belief $q_F(y) =: q$), $\mathcal{G}(y,t)$ is single-peaked in t.

$$\begin{aligned}\mathcal{G}(y,t+\delta) - \mathcal{G}(y,t) =& (1-q)(F_L^0(x(t+\delta))(L+c) - F_L^0(x(t)))\\ &- q(F_L^1(x(t+\delta)) - F_L^1(x(t)))(H-c).\end{aligned}$$

Because $x(t)$ is strictly increasing and continuous, $x(t+\delta) > x(t)$ for all $\delta > 0$ and $x(t+\delta) \to x(t)$ as $\delta \to 0$, therefore,

$$\lim_{x(t+\delta)\to x(t)} \frac{F^0(x(t+\delta)) - F^0(x(t))}{F^1(x(t+\delta)) - F^1(x(t))} = \frac{f^0(x(t))}{f^1(x(t))}.$$

By strict MLRP and Lemma 2, $f^0(x(t))/f^1(x(t))$ is strictly decreasing in t. So there exists a unique $\hat{t} \geq 0$ such that

$$\frac{f^0(x(t))}{f^1(x(t))} > \frac{q(H-c)}{(1-q)(L+c)} \text{ for } t < \hat{t}, \text{and } \frac{f^0(x(t))}{f^1(x(t))} < \frac{q(H-c)}{(1-q)(L+c)} \text{ for } t > \hat{t}.$$

This means for $\delta \to 0$, $\mathcal{G}(y, t+\delta) - \mathcal{G}(y,t)$ is positive for $t < \hat{t}$ and negative for $t > \hat{t}$.

Fix $r > 0$. Let $t^* = \arg\max_\tau e^{-r\tau}\mathcal{G}(y^*, \tau)$. Step 2 shows $t^* < t_0 = \arg\max_\tau \mathcal{G}(y^*, \tau)$. Because $\mathcal{G}$ is single-peaked with maximum achieved at t_0, so $t^* < t_0$ implies $\mathcal{G}(y^*, t^*) < \mathcal{G}(y^*, t_0)$. For the sequence $\{t_n\}$ such that $t_n \to t^*$ with $t_n > t^*$ for all n, there exists an N such that for $n > N$, $t_n < t_0$. Therefore, $\mathcal{G}(y^*, t^*) < \mathcal{G}(y^*, t_n)$. □

Optimality

Lemma 8. *For a fixed x, $\mathcal{L}(x,t)$ is single-peaked in t; for a fixed y, $\mathcal{F}(y,t)$ is single-peaked in t.*

Proof. For the leader, let $\mathcal{L}_i$ denote the derivative of $\mathcal{L}$ with respect to its i-th argument. Take the derivative of $\mathcal{L}$ with respect to t,

$$\mathcal{L}_2(x,t) = -y'(t)e^{-rt}\left(\frac{q_L(x)}{F^1(y(0))} f^1(y(t))H - \frac{1-q_L(x)}{F^0(y(0))} f^0(y(t))L\right).$$

By strict MLRP and $y(t)$ strictly decreasing, there exists a unique t^* such that $\mathcal{L}_2(x, t^*) = 0$. For $t < t^*$, $\mathcal{L}_2(x,t) > 0$; for $t > t^*$, $\mathcal{L}_2(x,t) < 0$.

For the follower, let $\mathcal{G}_i$ denote the derivative of $\mathcal{G}$ with respect to its i-th argument. The first-order condition of $\mathcal{F}$ implies $\mathcal{G}_2(y(t), t) = r\mathcal{G}(y(t), t)$. Because strategies are strictly monotone and everywhere differentiable, at each t, there exists one and only one type whose first-order condition is satisfied. Denote the type whose first-order condition is satisfied at t^* by y^*, that is, $\mathcal{G}_2(y^*, t^*) = r\mathcal{G}(y^*, t^*)$. Suppose y^* mimics the behavior of type $\hat{y}$ by stopping at $\hat{t}$. Because $\mathcal{G}$ is differentiable in y, by the fundamental theorem of calculus,

$$\mathcal{G}_2(y^*,\hat{t}) = \mathcal{G}_2(\hat{y},\hat{t}) + \int_{\hat{y}}^{y^*} \mathcal{G}_{21}(y,\hat{t})\mathrm{d}y = r\mathcal{G}(\hat{y},\hat{t}) + \int_{\hat{y}}^{y^*} \mathcal{G}_{21}(y,\hat{t})\mathrm{d}y,$$

where $\mathcal{G}_{21}(y,\hat{t}) = \mathrm{d}\mathcal{G}_2(y,\hat{t})/\mathrm{d}y$. The second equality follows from $\hat{y}$'s first-order condition $\mathcal{G}_2(\hat{y},\hat{t}) = r\mathcal{G}(\hat{y},\hat{t})$. Recall the definition of $\mathcal{G}(y,t)$ in (10) and

$$\mathcal{G}_2(y,t) = -q_F(y)\frac{f^1(x(t))x'(t)}{1-F^1(x(0))}(H-c) + (1-q_F(y))\frac{f^0(x(t))x'(t)}{1-F^0(x(0))}(L+c).$$

By MLRP, $q_F(y)$ is increasing in y, so $\mathcal{G}_{21}(y,\hat{t}) < 0$. Thus, if $\hat{y} < y^*$, then

$$\mathcal{G}_2(y^*,\hat{t}) = r\mathcal{G}(\hat{y},\hat{t}) + \int_{\hat{y}}^{y^*} \mathcal{G}_{21}(y,\hat{t})\mathrm{d}y < r\mathcal{G}(\hat{y},\hat{t}) < r\mathcal{G}(y^*,\hat{t}),$$

where the first inequality follows from $\int_{\hat{y}}^{y^*} \mathcal{G}_{21}(y,\hat{t})\mathrm{d}y < 0$, the second inequality follows from that $\mathcal{G}$ is increasing in y. Similarly, if $\hat{y} > y^*$, $\int_{\hat{y}}^{y^*} \mathcal{G}_{21}(y,\hat{t})\mathrm{d}y > 0$, so

$$\mathcal{G}_2(y^*,\hat{t}) = r\mathcal{G}(\hat{y},\hat{t}) + \int_{\hat{y}}^{y^*} \mathcal{G}_{21}(y,\hat{t})\mathrm{d}y > r\mathcal{G}(\hat{y},\hat{t}) > r\mathcal{G}(y^*,\hat{t}).$$

$y(t)$ is decreasing so $\hat{y} < (>) y^*$ if and only if $\hat{t} > (<) t^*$. The above argument shows $\mathcal{G}_2(y^*,\hat{t}) - r\mathcal{G}(y^*,\hat{t}) < 0$ for all $\hat{t} > t^*$ and $\mathcal{G}_2(y^*,\hat{t}) - r\mathcal{G}(y^*,\hat{t}) > 0$ for all $\hat{t} < t^*$. $\square$

A.1.6 Proof of Lemma 4

Define the right-hand side of (5) as $\phi(\cdot)$ and rewrite (5) as $x'(t) = \phi(x(t))$. Let $\bar{x} := \min\{x : \phi(\bar{x}) = 0\}$. Because the (common) support of f^1 and f^0 is $[0,1]$ which is bounded, as $x \to 1$, $(1-F^1(x))/f^1(x) \to 0$ and $(1-F^0(x))/f^0(x) \to 0$. So $\phi(x) \to 0$. Therefore, $\bar{x} \leq 1$. Because $x'(t) = \phi(x(t))$ is an autonomous first-order differential equation and $\phi(x)$ is continuous for all $x \in (0,1)$, the solution to $x'(t) = \phi(x(t))$ given the initial value, denoted by $x^*(t)$, is either constant or monotone. This means if the initial value $x^*(0)$ is such that $\phi(x^*(0)) > 0$, then $\phi(x^*(t)) > 0$ for all t, and $\phi(x^*(t)) \to 0$ as $x^*(t) \to \bar{x}$ (Teschl, 2012, Lemma 1.1). This implies $x^*(t) < \bar{x} \leq 1$ for all t. Let $x(t)$ denote the equilibrium (inverse) strategy. The above argument shows if $x(t)$ is a solution to $x'(t) = \phi(x(t))$ for all t, then $x(t) < \bar{x} \leq 1$ for all t. Therefore, if there exists $T < \infty$ such that $x(T) = 1$, $x(t)$ must be discontinuous at T. This contradicts Lemma 2.

A.1.7 Proof of Proposition 1

Equilibrium conditions

Leader-follower continuation game. Equation (3) is given by evaluating equation (4) in Proposition 2 at $t = 0$. It remains to derive (1) and (2). Condition (1) follows from the following two steps.

Step 1. $y(0) < z$ if and only if $x(0) = z$.

Suppose $y(0) < z$. If type $y(0)$ invests at time 0 of the leader-follower continuation game, his expected payoff is

$$\begin{aligned} &\Pr(s_L \geq x(0)|y(0), s_L > z) \\ &\cdot [\Pr(\theta = 1|y(0), s_L \geq x(0))H - \Pr(\theta = 0|y(0), s_L \geq x(0))L - c] \\ +&\Pr(s_L \in (z, x(0))|y(0), s_L > z)(-c). \end{aligned} \tag{23}$$

Suppose $y(0)$ waits for a small amount of time $\mathrm{d}t$ before investing. If the leader disinvests in $[0, \mathrm{d}t)$, it is a dominant strategy for $y(0)$ to never invest, so $y(0)$ gets 0; if the leader stays invested in $[0, \mathrm{d}t)$, then $y(0)$ invests at $\mathrm{d}t$ and gets payoff $\lim_{\mathrm{d}t\to 0} \mathcal{F}(y(0), \mathrm{d}t) = \mathcal{F}(y(0), 0)$. $y(0)$'s expected payoff is

$$\begin{aligned} &\Pr(s_L \geq x(0)|y(0), s_L > z) \\ &\cdot [\Pr(\theta = 1|y(0), s_L \geq x(0))H - \Pr(\theta = 0|y(0), s_L \geq x(0))L - c] \\ +&\Pr(s_L \in (z, x(0))|y(0), s_L > z)(0). \end{aligned} \tag{24}$$

$y(0)$ is a type that optimally invests at time 0. So the change in $y(0)$'s expected payoff from investing at 0 and at $\mathrm{d}t$ is zero. That is, (23)$-$(24)$= 0$, which holds if and only if $x(0) = z$.

Step 2. There does not exist an equilibrium with $x(0) > z$.

Suppose $x(0) > z$. From Step 1 (and by definition $x(0) \geq z$), $y(0) = z$. Consider z's incentive in the initial stage. Suppose z invests in the initial stage. If the other player invests, payoff realizes; if the other player does not invest, z disinvests immediately (because $x(0) > z$). z's expected payoff from investing is

$$\Pr(\theta = 1, s_{-i} > z|z)H - \Pr(\theta = 0, s_{-i} > z|z)L - c. \tag{25}$$

Suppose z does not invest in the initial stage. If the other player does not invest, by optimality, z gets at least 0. If the other player invests, z invests at dt (because $y(0) = z$). z's expected payoff from not investing is at least

$$\Pr(\theta = 1, s_{-i} > x(0)|z)(H - c) - \Pr(\theta = 0, s_{-i} > x(0)|z)(L + c). \tag{26}$$

Players' incentive in the leader-follower continuation game doesn't change. In particular, for $x(0) > z = y(0)$, (4) becomes $\rho_0 f^1(x(0)) f^1(z)H = (1 - \rho_0) f^0(x(0)) f^0(z)L$. This condition, together with MLRP, implies (26)>(25). This is a contradiction as z is not indifferent in the initial stage.

Initial stage. Equation (2) is given by type z's indifference between investing and not investing in the initial stage given the continuation strategies.

Suppose z does not invest in the initial stage. If the other player does not invest, the game moves to a continuation game in which neither player has invested. Call this a *no-investment continuation game.* Denote z's payoff in the no-investment continuation game by $U(z)$. By optimality, $U(z) \geq 0$. If the other player invests, the game moves to the leader-follower continuation game in which z is the follower. At the beginning of the leader-follower continuation game, by (1), z invests and the leader stays invested with probability 1. z's expected payoff from not investing is

$$\begin{aligned} &\Pr(s_{-i} > z|z) \left[\Pr(\theta = 1|z, s_{-i} > z)H - \Pr(\theta = 0|z, s_{-i} > z)L - c\right] \\ &+ \Pr(s_{-i} \leq z|z)U(z) \end{aligned} \tag{27}$$

Suppose z invests in the initial stage. If the other player invests, payoff realizes. If the other player does not invest, the game moves to the leader-follower continuation game in which z is the leader. z stays invested at the beginning of the leader-follower continuation game. If the follower follows suit, payoff realizes, otherwise z disinvests at $\mathrm{d}t$ and gets payoff $\lim_{\mathrm{d}t\to 0}\mathcal{L}(z,\mathrm{d}t)=0$.[33] So z's expected payoff from investing is

$$\begin{aligned}&\Pr(s_{-i}>z|z)\left[\Pr(\theta=1|z,s_{-i}>z)H-\Pr(\theta=0|z,s_{-i}>z)L-c\right]\\ +&\Pr(s_{-i}\le z|z)\\ &\cdot\Bigg(\Pr(s_{-i}\in[y(0),z]|z,s_{-i}\le z)\\ &\quad\cdot\left[\Pr(\theta=1|z,s_{-i}\in[y(0),z])H-\Pr(\theta=0|z,s_{-i}\in[y(0),z])L\right]-c\Bigg).\end{aligned}\tag{28}$$

In equilibrium, (27)=(28), which reduces to

$$\begin{aligned}U(z)=&\Pr(s_{-i}\in[y(0),z)|z,s_{-i}\le z)\\ &\cdot\left[\Pr(\theta=1|z,s_{-i}\in[y(0),z])H-\Pr(\theta=0|z,s_{-i}\in[y(0),z])L\right]-c.\end{aligned}\tag{29}$$

I now show that the players never invest after no initial investments. So $U(z)=0$. Equation (2) then follows from evaluating (29) at $U(z)=0$.

Lemma 9. *In any monotonic symmetric dynamic equilibrium, $\sigma_i(s_i,\varnothing)\in\{0,\infty\}$.*

Proof. If there exists an equilibrium where players invest in the no-investment continuation game, z, being the highest type, must find it optimal to invest. The best z can do in this continuation game is to signal he is the highest type. Suppose he can do that. Given player i's type is equal to z, by strict MLRP, there exists a unique type $\underline{y}$ that is indifferent between investing and not investing, types in $(\underline{y},z)$ invest, and types below $\underline{y}$ never invest. Let $\overline{U}(z)$ denote z's expected payoff in this case,

$$\overline{U}(z)=\Pr(\theta=1,s_{-i}\in[\underline{y},z]|z,s_{-i}\le z)H-\Pr(\theta=0,s_{-i}\in[\underline{y},z]|z,s_{-i}\le z)L-c.$$

$\underline{y}$'s indifference condition is

$$\Pr(\theta=1|\underline{y},z)H-\Pr(\theta=0|\underline{y},z)L-c=0.$$

By (4) and (1), $\Pr(\theta=1|y(0),z)H-\Pr(\theta=0|y(0),z)L=0$. It then follows from

[33] Technically, the strategy "disinvesting at $\mathrm{d}t$" for $\mathrm{d}t>0$ small is not a well-defined best response. One can consider z as a type that would have been indifferent were he able to achieve the supremum of his expected payoff by disinvesting at $\mathrm{d}t$. This technical issue does not compromise the analysis as it arises only on a zero-measure set. The threshold z is well-defined as it is shown below that all types above z strictly prefer investing and all types below strictly prefer not investing.

MLRP that $y(0) < \underline{y}$. By Lemma 6,

$$\begin{aligned}
&\Pr(\theta = 1, s_{-i} \in [y(0), z] | z, s_{-i} \leq z) H - \Pr(\theta = 0, s_{-i} \in [y(0), z] | z, s_{-i} \leq z) L - c \\
>& \Pr(\theta = 1, s_{-i} \in [\underline{y}, z] | z, s_{-i} \leq z) H - \Pr(\theta = 0, s_{-i} \in [\underline{y}, z] | z, s_{-i} \leq z) L - c \\
=& \overline{U}(z).
\end{aligned}$$

This means z's expected payoff from investing in the initial stage is strictly higher than not investing. This is a contradiction as z is not indifferent. $\square$

Optimality

I verify that all types above z find it optimal to invest. The argument for all types below z is analogous and thus omitted.

Suppose the other player plays according to the strategy specified above. Fix a type $x > z$. Suppose x does not invest. x's continuation play is the same as z's. So x's expected payoff from not investing is

$$\Pr(s_{-i} > z | x) \left[\Pr(\theta = 1 | x, s_{-i} > z) H - \Pr(\theta = 0 | x, s_{-i} > z) L - c \right]. \tag{30}$$

Suppose x invests. If the other player invests, payoff realizes; if the other player does not invest, the game moves to the leader-follower continuation game in which x is the leader and optimally stops at $\sigma_L(x)$. So x's expected payoff from investing in the initial stage is

$$\begin{aligned}
&\Pr(s_{-i} > z | x) \left[\Pr(\theta = 1 | x, s_{-i} > z) H - \Pr(\theta = 0 | x, s_{-i} > z) L - c \right] \\
+& \Pr(s_{-i} \leq z | x) \Bigg(\Pr(s_{-i} \in [y(0), z] | x, s_{-i} \leq z) \\
&\qquad \cdot \left[\Pr(\theta = 1 | x, s_{-i} \in [y(0), z]) H - \Pr(\theta = 0 | x, s_{-i} \in [y(0), z]) L \right] \\
&\qquad + \Pr(s_{-i} < y(0) | x, s_{-i} \leq z) \mathcal{L}(x, \sigma_L(x)) - c \Bigg).
\end{aligned} \tag{31}$$

The difference in payoff between investing and not investing is (31)−(30), which is

$$\begin{aligned}\Pr(s_{-i} \le z | x)\Big(&\Pr(s_{-i} \in [y(0), z] | x, s_{-i} \le z) \\ &\cdot [\Pr(\theta = 1 | x, s_{-i} \in [y(0), z]) H - \Pr(\theta = 0 | x, s_{-i} \in [y(0), z]) L] \\ &+ \Pr(s_{-i} < y(0) | x, s_{-i} \le z) \mathcal{L}(x, \sigma_L(x)) - c\Big).\end{aligned}$$

By optimality and strict MLRP, for all $x > z$,

$$\mathcal{L}(x, \sigma_L(x)) \ge \lim_{\mathrm{d}t \to 0} \mathcal{L}(x, \mathrm{d}t) > \lim_{\mathrm{d}t \to 0} \mathcal{L}(z, \mathrm{d}t).$$

So to show x strictly prefers investing, it suffices to show

$$\begin{aligned}&\Pr(s_{-i} \in [y(0), z] | x, s_{-i} \le z) \\ &\quad \cdot [\Pr(\theta = 1 | x, s_{-i} \in [y(0), z]) H - \Pr(\theta = 0 | x, s_{-i} \in [y(0), z]) L] > c,\end{aligned}$$

which follows from z's indifference condition (2) and MLRP.

A.1.8 Proof of Theorem 1

Uniqueness

Lemma 10. *There exists a unique set of initial values $(z, x(0), y(0))$ that solves equations (1), (2), and (3).*

Proof. From (3), one can solve $y(0)$ as a function of z, denote it by $y_0(z)$, where $y_0(z) = l^{-1}\left((\rho_0/(1-\rho_0))(H/L)/l(z)\right)$ and is decreasing in z by MLRP.

The initial condition (2) can be written as $\mathcal{V}(z) = c$ where $\mathcal{V}(z)$ is defined as

$$\mathcal{V}(z) := q(z)\left(1 - \frac{F^1(y_0(z))}{F^1(z)}\right) H - (1 - q(z))\left(1 - \frac{F^0(y_0(z))}{F^0(z)}\right) L, \tag{32}$$

with

$$q(z) = \frac{\rho_0 f^1(z) F^1(z)}{\rho_0 f^1(z) F^1(z) + (1 - \rho_0) f^0(z) F^0(z)}.$$

As $z \to y(0)$, $\mathcal{V}(z) \to 0 < c$. As $z \to 1$, because the likelihood ratio is unbounded, $y_0(z) \to 0$ and $q(z) \to 1$. So $\lim_{z \to 1} \mathcal{V}(z) = H > c$. It follows from MLRP and IFRP that the derivative of $\mathcal{V}(z)$ with respect to z is positive. Because $\mathcal{V}(z)$ is continuous in z, by the intermediate value theorem, there exists a unique z^* such that $\mathcal{V}(z^*) = c$. $\square$

Therefore, the pair of initial values $(x(0), y(0))$ is unique. It follows from the Picard–Lindelöf theorem that there exists a unique solution $(x(t), y(t))$ to the differential system (4) and (5).

Existence

A solution $(x(t), y(t))$ to the differential system (4) and (5) is an equilibrium if and only if $x(t)$ is strictly increasing and $y(t)$ is strictly decreasing. Equation (4) and MLRP imply it suffices to establish conditions under which $x(t)$ is strictly increasing.

Claim 6. Suppose the initial value $x(0)$ satisfies

$$h(x(0)) < \frac{(H-c)L}{(L+c)H}. \tag{33}$$

Solution $x(t)$ to the differential equation (5) with initial value $x(0)$ is strictly increasing for $t \geq 0$ if and only if (33) holds.[34]

Proof of Claim 6. Because $x'(t) = \phi(x(t))$ with initial value $x(0)$ is a first-order autonomous differential equation where $\phi(\cdot)$ is continuous, the solution $x(t)$ is either constant or monotone. $x(t)$ is strictly increasing if and only if $\phi(x(0)) > 0$ (Teschl, 2012, Lemma 1.1), which reduces to (33). $\square$

Claim 7. There exists a unique $\bar{c} \in (0, H)$ such that

$$h(\mathcal{V}^{-1}(\bar{c})) = \frac{(H-\bar{c})L}{(L+\bar{c})H}, \tag{34}$$

where $\mathcal{V}$ is defined in (32).

Proof of Claim 7. The right-hand side of (34) as a function of c is continuous and strictly decreasing. For all $c \in (0, H)$, $(H-c)L/((L+c)H)$ is in $(0,1)$ and converges to 1 as $c \to 0$ and converges to 0 as $c \to H$. For the left-hand side of (34), from the proof of Lemma 10, for all $z > \underline{z} := l^{-1}(((\rho_0/(1-\rho_0))(H/L))^{1/2})$, $\mathcal{V}(z)$ is increasing

[34] It is worth noting that condition (33) coincides with the condition that $y(0)$, the lowest type who invests at $t=0$, gets positive payoff from investing at $t=0$. This implies all types above $y(0)$ gets positive payoff from investing at $t=0$.

in z. This implies $\mathcal{V}^{-1}(c)$ is increasing in c with $\mathcal{V}^{-1}(c) \to \underline{z}$ as $c \to 0$ and $\mathcal{V}^{-1}(c) \to 1$ as $c \to H$. By IHRP, $h(\mathcal{V}^{-1}(c))$ is strictly increasing in c, and by MLRP, $h(\mathcal{V}^{-1}(c))$ converges to some constant $\underline{h} \geq 0$ as $c \to 0$ and converges to $\bar{h} \leq 1$ as $c \to H$. □

The proof of Claim 7 also implies $c < \bar{c}$ if and only if

$$h(\mathcal{V}^{-1}(c)) < \frac{(H-c)L}{(L+c)H}$$

where by definition $x(0) = \mathcal{V}^{-1}(c)$. The result follows.

A.1.9 Proof of Lemma 5

The public belief is given by $\rho(t) := \Pr(\theta = 1 | s_L \geq x(t), s_F < y(t))$, which can be written as

$$\rho(t) = 1 \Big/ \left(1 + \frac{1-\rho_0}{\rho_0} \frac{F^0(y(t))(1-F^0(x(t)))}{F^1(y(t))(1-F^1(x(t)))}\right).$$

For all $t \geq 0$, $x(t), y(t) \in (0,1)$ which implies $\rho(t) \in (0,1)$.

Recall the equilibrium condition (4): for all $t \geq 0$,

$$\frac{1-\rho_0}{\rho_0} \frac{f^0(x(t))f^0(y(t))}{f^1(x(t))f^1(y(t)} = \frac{H}{L}. \tag{35}$$

(35) implies as $t \to \infty$, either $x(t)$ and $y(t)$ both converge to interior values, or they both converge to the extreme points 1 and 0.[35] As discussed in the proof of Lemma 4 (also see Teschl, 2012, Lemma 1.1), (i) if there exists $\bar{x} < 1$ such that $h(\bar{x}) = L(H-c)/(H(L+c))$, then $x(t) \to \bar{x} < 1$ and $y(t) \to \underline{y} > 0$ where $\bar{x}$ and $\underline{y}$ satisfy equation (35). (ii) Otherwise, $x(t) \to 1$ and $y(t) \to 0$.

Note that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1-\rho_0}{\rho_0} \frac{F^0(y(t))(1-F^0(x(t)))}{F^1(y(t))(1-F^1(x(t)))} \\
&= \lim_{t\to\infty} \frac{1-\rho_0}{\rho_0} \frac{f^0(x(t))f^0(y(t))}{f^1(x(t))f^1(y(t))} \cdot \underbrace{\frac{F^0(y(t))}{F^1(y(t))} \frac{f^1(y(t))}{f^0(y(t))}}_{=k(y(t))} \cdot \underbrace{\frac{1-F^0(x(t))}{1-F^1(x(t))} \frac{f^1(x(t))}{f^0(x(t))}}_{=h(x(t))},
\end{aligned} \tag{36}$$

where $k(\cdot)$ is the failure ratio and $h(\cdot)$ is the hazard ratio. By MLRP, $k(\cdot) > 1$ and

[35] To see this, by the consistency condition of posterior belief distribution (6), (35) can be written as $y(t) = (1-x(t))/(1-x(t)+Kx(t))$,where $K = (H/L)((1-\rho_0)/\rho_0)$,

$h(\cdot) < 1$; by assumption, $k(\cdot)$ and $h(\cdot)$ are both strictly increasing. So in case (i) where $x(t) \to \bar{x} < 1$ and $y(t) \to \underline{y} > 0$, $\lim_{t\to\infty} k(y(t))h(x(t))$ is positive and finite. In case (ii), as $y(t) \to 0$, $k(y(t))$ converges a finite number (larger than 1); as $x(t) \to 1$, $h(x(t))$ converges to a positive number (smaller than 1). Thus, $\lim_{t\to\infty} k(y(t))h(x(t))$ is positive and finite. So (36) is equal to $(H/L)\lim_{t\to\infty} k(y(t))h(x(t))$, where H/L, by (35), is the limit of the first term in (36). Therefore,

$$\lim_{t\to\infty} \rho(t) = \rho^* := 1/\Big(1 + (H/L)\lim_{t\to\infty} k(y(t))h(x(t))\Big) \in (0,1).$$

A.2 Proofs for Section 4

A.2.1 Preliminaries

Properties of precision and the ULR order

I derive some useful results regarding the precision and the ULR order. The proofs of these results are in the Online Appendix.

Lemma 11. *Suppose* $\hat{\mathbf{F}}$ *is more precise than* $\mathbf{F}$*. For all* $\mu \in (0,1)$*, the hazard ratio of* $\mathbf{F}$ *is higher than the hazard ratio of* $\hat{\mathbf{F}}$*:* $h_F(\mu) > h_{\hat{F}}(\mu)$*. In the symmetric setting, this implies the failure ratio of* $\mathbf{F}$ *is lower than the failure ratio of* $\hat{\mathbf{F}}$*:* $k_F(\mu) < k_{\hat{F}}(\mu)$*.*

Corollary 3. *Suppose* $\hat{\mathbf{F}}$ *is more precise than* $\mathbf{F}$*. For all* $\mu \in (0,1)$*,*

$$\frac{1-\hat{F}^0(\mu)}{1-\hat{F}^1(\mu)} < \frac{1-F^0(\mu)}{1-F^1(\mu)} \text{ and } \frac{\hat{F}^0(\mu)}{\hat{F}^1(\mu)} > \frac{F^0(\mu)}{F^1(\mu)}.$$

Lemma 12. *For any two distributions* F *and* $\hat{F}$*, if* $F \succ_{ULR} \hat{F}$*, then for any* $\lambda \in (0,1)$*,* $F \succ_{ULR} (1-\lambda)F + \lambda\hat{F} \succ_{ULR} \hat{F}$*.*

Uninformative and perfectly informative distributions

Denote the uninformative distribution by $\mathbf{F}_0 = (F_0^0, F_0^1)$ and the perfectly informative distribution by $\mathbf{F}_\infty = (F_\infty^0, F_\infty^1)$. By definition,

$$F_0^0(\mu) = F_0^1(\mu) = \begin{cases} 0 & \mu < 1/2 \\ 1 & \mu \geq 1/2 \end{cases};\ F_\infty^0(\mu) = 1 \forall \mu \geq 0 \text{ and } F_\infty^1(\mu) = \mathbf{1}_{\{1\}}(\mu).$$

A sequence of random variables $\{X_n\}$ is said to converge to X if the sequence of cumulative functions of X_n converges pointwise to the cumulative function of X. In the case where the limiting function is continuous, pointwise convergence can be strengthened to uniform convergence. In what follows, convergence is taken to be pointwise unless otherwise specified.

A.2.2 Proof of Proposition 3

First, I show that the initial value $x(0)$ is increasing in precision. Then, I show that the solution $x(t)$ to the differential equation (5) is increasing pointwise in precision. Equation (4) then implies $y(t)$ for $t \geq 0$ is decreasing in precision.

In the symmetric environment, (6) implies $f^1(\mu)/f^0(\mu) = \mu/(1-\mu)$. (5) reduces to $x(t) = 1 - y(t)$ for all $t \geq 0$ for any precision. Because $x(0) \geq y(0)$, $x(0) \geq 1/2$.

Initial value. For distribution $\mathbf{F}_\gamma$, denote the initial value under this distribution by $x_\gamma(0)$. Rewrite the initial condition (2) as $\mathcal{V}_\gamma(x_\gamma(0)) = c$, where

$$\mathcal{V}_\gamma(\mu) = \mathcal{U}^1_\gamma(\mu) - \mathcal{U}^0_\gamma(\mu),$$

with

$$\mathcal{U}^\theta_\gamma(\mu) = \frac{f^\theta_\gamma(\mu)F^\theta_\gamma(\mu)}{f^1_\gamma(\mu)F^1_\gamma(\mu) + f^0_\gamma(\mu)F^0_\gamma(\mu)}\left(1 - \frac{F^\theta_\gamma(1-\mu)}{F^\theta_\gamma(\mu)}\right).$$

As is shown in the proof of Lemma 10, $\mathcal{V}_\gamma(\mu)$ is increasing in μ for $\mu \geq 1/2$. Therefore, proving $x_\gamma(0)$ increasing in γ is equivalent to proving $\mathcal{V}_\gamma(\mu)$ decreasing in γ for a fixed μ. Figure 7 illustrates this result.

By symmetry, $\mathcal{U}^1_\gamma(\mu)/\mathcal{U}^0_\gamma(\mu) = \mu/(1-\mu)$. So $\mathcal{V}_\gamma(\mu) = \mathcal{U}^1_\gamma(\mu)(1-(1-\mu)/\mu)$. $\mathcal{U}^1_\gamma(\mu)$ can be written as $\mathcal{U}^1_\gamma(\mu) = q_\gamma(\mu)(1 - R_\gamma(\mu))$ where

$$q_\gamma(\mu) := \frac{f^1_\gamma(\mu)F^1_\gamma(\mu)}{f^1_\gamma(\mu)F^1_\gamma(\mu) + f^0_\gamma(\mu)F^0_\gamma(\mu)}, R_\gamma(\mu) := \frac{F^1_\gamma(1-\mu)}{F^1_\gamma(\mu)} = \frac{1-F^0_\gamma(\mu)}{F^1_\gamma(\mu)}. \tag{37}$$

Corollary 3 implies $q_\gamma(\mu)$ is decreasing in γ. The result then follows from $R_\gamma(\mu)$ increasing in γ, which is shown below.

Fix two distributions $\mathbf{F}_{\gamma_1}$ and $\mathbf{F}_{\gamma_2}$ with $\gamma_1 < \gamma_2$. Define $Q^0(\mu) := (1-F^0_{\gamma_1}(\mu))/(1-F^0_{\gamma_2}(\mu))$ and $P^1(\mu) := F^1_{\gamma_1}(\mu)/F^1_{\gamma_2}(\mu)$. Showing $R_{\gamma_2}(\mu) > R_{\gamma_1}(\mu)$ for $\mu \geq 1/2$ is

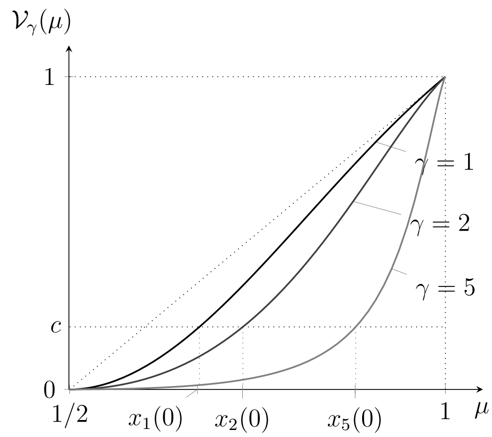

Figure 7: Illustration for $\mathcal{V}_\gamma(\mu)$ decreasing in γ for a fixed μ with posterior beliefs induced by $Beta(1+\gamma\theta, 1+\gamma(1-\theta))$.

equivalent to showing $Q^0(\mu) < P^1(\mu)$ for $\mu \geq 1/2$.

By Proposition 2 in Hopkins and Kornienko (2007), $P^1(\mu)$ is unimodal with a maximum at $\hat{\mu}_P \geq 1/2$, and $Q^0(\mu)$ is unimodal with a maximum at $\hat{\mu}_Q \leq 1/2$. In the symmetric environment, $Q^0(\mu)$ and $P^1(\mu)$ are symmetric around $1/2$, in particular, $Q^0(1/2) = P^1(1/2)$. So for all $\mu \in (1/2, \hat{\mu}_P)$, because $Q^0(\mu)$ is decreasing and $P^1(\mu)$ is increasing, $Q^0(\mu) < P^1(\mu)$. For all $\mu \geq \hat{\mu}_P$, by Corollary 1 in Hopkins and Kornienko (2007), $Q^0(\mu) < f^0_{\gamma_1}(\mu)/f^0_{\gamma_2}(\mu) = f^1_{\gamma_1}(\mu)/f^1_{\gamma_2}(\mu) < P^1(\mu)$.

Differential equation. The differential equation (5) for distribution with precision γ in the symmetric environment can be written as

$$x'_\gamma(t) = \phi_\gamma(x_\gamma(t)) = K^1/h^1_\gamma(x_\gamma(t)) - K^0/h^0_\gamma(x_\gamma(t))$$

with $K^1 = r(1-c)/2c$, $K^0 = r(1+c)/2c$.

Note that the posterior distribution conditional on state $\theta = 0$ satisfies the standard definition of the ULR order. Therefore, $F^0_{\gamma_1}(\mu) \succ_{\text{ULR}} F^0_{\gamma_2}(\mu)$. Then for $\mu \geq 1/2$, $h^0_{\gamma_1}(\mu) > h^0_{\gamma_2}(\mu)$ (Hopkins and Kornienko, 2007, Corollary 1). Together with Lemma 11,

$$\frac{h^1_{\gamma_1}(\mu)}{h^1_{\gamma_2}(\mu)} > \frac{h^0_{\gamma_1}(\mu)}{h^0_{\gamma_2}(\mu)} > 1.$$

$\phi_\gamma(\mu) > 0$ implies $K^1/h^1_\gamma(\mu) > K^0/h^0_\gamma(\mu)$ for $\gamma = \gamma_1, \gamma_2$. Then

$$\phi_{\gamma_2}(\mu) - \phi_{\gamma_1}(\mu) = \frac{K^1}{h^1_{\gamma_1}(\mu)}\left(\frac{h^1_{\gamma_1}(\mu)}{h^1_{\gamma_2}(\mu)} - 1\right) - \frac{K^0}{h^0_{\gamma_1}(\mu)}\left(\frac{h^0_{\gamma_1}(\mu)}{h^0_{\gamma_2}(\mu)} - 1\right) > 0.$$

The result follows from a standard comparison argument (Teschl, 2012, Theorem 1.3).

A.2.3 Proof of Proposition 4

Existence. To establish the existence of a dynamic equilibrium, first, there must exist a set of initial values $x(0)$ and $y(0)$ satisfying the initial conditions. Second, the solution $x(t)$ and $y(t)$ to the differential system must be monotone. Same as before, it suffices to analyze $x(t)$, as $y(t) = 1 - x(t)$ in equilibrium.

Step 1. There exists a (unique) initial value $x_\gamma(0)$ for all $\gamma > 0$.

For distribution $\mathbf{F}_\gamma$, write the initial condition as $\mathcal{V}_\gamma(\mu) = c$, where

$$\mathcal{V}_\gamma(\mu) = \underbrace{\frac{f^1_\gamma(\mu)F^1_\gamma(\mu)}{f^1_\gamma(\mu)F^1_\gamma(\mu) + f^0_\gamma(\mu)F^0_\gamma(\mu)}}_{=q_\gamma(\mu)}\left(1 - \frac{1 - F^0_\gamma(\mu)}{F^1_\gamma(\mu)}\right)\left(1 - \frac{1-\mu}{\mu}\right).$$

As is shown in the proof of Lemma 10, $\mathcal{V}_\gamma(\mu)$ is increasing in μ. In the symmetric environment, $\mathcal{V}_\gamma(\mu) \to 0$ as $\mu \to 1/2$ and $\mathcal{V}_\gamma(\mu) \to 1$ as $\mu \to 1$ for any $\gamma > 0$.

Take $\gamma \to 0$. For any $\mu \geq 1/2$, $F^1_\gamma(\mu)/F^0_\gamma(\mu) \to 1$, so $q_\gamma(\mu) \to \mu$. For any $\mu > 1/2$, $(1 - F^0_\gamma(\mu))/F^1_\gamma(\mu) \to 0$ so $\mathcal{V}_\gamma(\mu) \to 2\mu - 1$. At $\mu = 1/2$, $\mathcal{V}_\gamma(\mu) = 0$. Therefore, for all $\mu \geq 1/2$, $\mathcal{V}_\gamma(\mu) \to 2\mu - 1$.

Take $\gamma \to \infty$. For any $\mu < 1$, $F^1_\gamma(\mu)/F^0_\gamma(\mu) \to 0$ so $q_\gamma(\mu) \to 0$ and $\mathcal{V}_\gamma(\mu) \to 0$. At $\mu = 1$, $\mathcal{V}_\gamma(\mu) = 1$. So for all $\mu \leq 1$, $\mathcal{V}_\gamma(\mu) \to \mathbf{1}_{\{1\}}(\mu)$ pointwise.

For all $\gamma > 0$, for any $1/2 < \mu < 1$, $0 < \mathcal{V}_\gamma(\mu) < 2\mu - 1$. $\mathcal{V}_\gamma(\mu)$ is continuous in μ. There exists a unique $x_\gamma(0)$ such that $\mathcal{V}_\gamma(x_\gamma(0)) = c$. Moreover,

$$\lim_{\gamma\to 0} x_\gamma(0) = (c+1)/2 \text{ and } \lim_{\gamma\to\infty} x_\gamma(0) \to 1. \tag{38}$$

Step 2. A dynamic equilibrium exists when $\gamma \to \infty$.

Recall that if a dynamic equilibrium exists, then $x_\gamma(t)$, the solution to $x'_\gamma(t) = \phi_\gamma(x_\gamma(t))$, must be strictly increasing in t. The solution to $x'_\gamma(t) = \phi_\gamma(x_\gamma(t))$ is either monotone or constant in t (Teschl, 2012, Lemma 1.1). I show $x_\gamma(t)$ cannot be decreasing or constant when $\gamma \to \infty$.

Take $\gamma \to \infty$. Recall that $\phi_\gamma(x) = 0$ has at least one at most two solutions and one of them must be 1. If there exists $\bar{x}$ such that $h_\gamma(\bar{x}) = (1-c)/(1+c)$, then $\bar{x}$ is the other solution. Because $h_\gamma(x) \to 0$ for all $x < 1$, so if there exists $\bar{x}$ such that $h_\gamma(\bar{x}) = (1-c)/(1+c) > 0$, $\bar{x}$ can only be 1. This means if $x_\gamma(t)$ is decreasing in t, $x_\gamma(t)$ must be decreasing to 0 as t increases (Teschl, 2012, Lemma 1.1). However, as $x \to 0$, $h_\gamma(x) \to 0$ which means $\phi_\gamma(x) > 0$, a contradiction. If $x_\gamma(t)$ is constant, it must be $x_\gamma(t) = 1$ for all $t \geq 0$, which is a contradiction because $x_\gamma(0) < 1$.

Step 3. For $\gamma_1 < \gamma_2$, if a dynamic equilibrium exists under $\mathbf{F}_{\gamma_1}$, then a dynamic equilibrium exists under $\mathbf{F}_{\gamma_2}$.

As is shown in the proof of Theorem 1, the solution to $x'_\gamma(t) = \phi_\gamma(x_\gamma(t))$ is increasing if $h_\gamma(x_\gamma(0)) < (1-c)/(1+c)$. I show $h_{\gamma_2}(x_{\gamma_2}(0)) < h_{\gamma_1}(x_{\gamma_1}(0))$.

Take the mixture of the ex ante distributions F_{γ_1} and F_{γ_2} and denote it by $F_\lambda := (1-\lambda)F_{\gamma_1} + \lambda F_{\gamma_2}$ with $\lambda \in (0,1)$. By Lemma 12, $F_{\gamma_1} \succ_{\text{ULR}} F_\lambda \succ_{\text{ULR}} F_{\gamma_2}$. Denote the hazard ratio of F_λ by h_λ and the initial value by $x_\lambda(0)$.

Take the derivative of $h_\lambda(x_\lambda(0))$ with respect to λ,

$$\frac{\mathrm{d}h_\lambda(x_\lambda(0))}{\mathrm{d}\lambda} = \frac{\partial h}{\partial \mu}\frac{\mathrm{d}x_\lambda(0)}{\mathrm{d}\lambda} + \frac{\partial h}{\partial \lambda} = \frac{\partial h}{\partial \mu}\left(\frac{\mathrm{d}x_\lambda(0)}{\mathrm{d}\lambda} + \frac{\partial h/\partial\lambda}{\partial h/\partial\mu}\right) = \frac{\partial h}{\partial \mu}\left(-\frac{\partial \mathcal{V}/\partial\lambda}{\partial \mathcal{V}/\partial\mu} + \frac{\partial h/\partial\lambda}{\partial h/\partial\mu}\right).$$

Claim 8. For all $\mu \geq 1/2$,

$$-\frac{\partial \mathcal{V}/\partial\lambda}{\partial \mathcal{V}/\partial\mu} + \frac{\partial h/\partial\lambda}{\partial h/\partial\mu} < 0.$$

Claim 8 follows from properties of the ULR order, MLRP, IHRP, and symmetry. The proof is mostly algebraic and is left to the Online Appendix.

By IHRP, $\partial h/\partial\mu > 0$. So $\mathrm{d}h_\lambda(x_\lambda(0))/\mathrm{d}\lambda < 0$. The result follows.

To conclude the proof, let $\underline{\gamma}$ be the highest γ such that a dynamic equilibrium does not exist. That is, $\underline{\gamma} := \sup_\gamma\{\gamma : h_\gamma(x_\gamma(0)) \geq (1-c)/(1+c)\}$. A dynamic equilibrium exists if and only if $\gamma > \underline{\gamma}$. It remains to show $\underline{\gamma} \in (0,\infty)$. By step 2, $\underline{\gamma} < \infty$. As $\gamma \to 0$, by (38), $x_\gamma(0) \to (c+1)/2$ so $h_\gamma(x_\gamma(0)) \to 1$. So $\underline{\gamma} = 0$ if and only if $c = 0$. Therefore, for $c > 0$, $\underline{\gamma} \in (0,\infty)$.

Convergence. The probability of initial agreement is $\Pi_\gamma(x_\gamma(0))$ where $\Pi_\gamma(\mu) = 1 - 2(1 - F^1_\gamma(\mu))(1 - F^0_\gamma(\mu))$. The following claim establishes the result.

Claim 9. As $\gamma \to \infty$, $F^0_\gamma(x_\gamma(0)) \to 1$ and $F^1_\gamma(x_\gamma(0)) \to 0$.

Proof of Claim 9. Take $\gamma \to \infty$. Then $F^0_\gamma(\mu) \to 1$ uniformly for all $\mu > 0$. By (38), $x_\gamma(0) \to 1$. So $F^0_\gamma(x_\gamma(0)) \to 1$.

Rearrange the initial condition $\mathcal{V}_\gamma(x_\gamma(0)) = c$,

$$F^1_\gamma(x_\gamma(0))\left(1 - \frac{x_\gamma(0)}{2x_\gamma(0)-1}c\right) = F^0_\gamma(x_\gamma(0))\left(\frac{1-x_\gamma(0)}{2x_\gamma(0)-1}c - 1\right) + 1.$$

Take the limit of both sides as $\gamma \to \infty$. Because $F^0_\gamma(x_\gamma(0)) \to 1$ and $x_\gamma(0) \to 1$,

$$\lim_{\gamma\to\infty} F^1_\gamma(x_\gamma(0))\,(1-c) = 0.$$

Because $1 - c > 0$, it must be that $F^1_\gamma(x_\gamma(0)) \to 0$. $\square$

A.2.4 Proof of Proposition 5

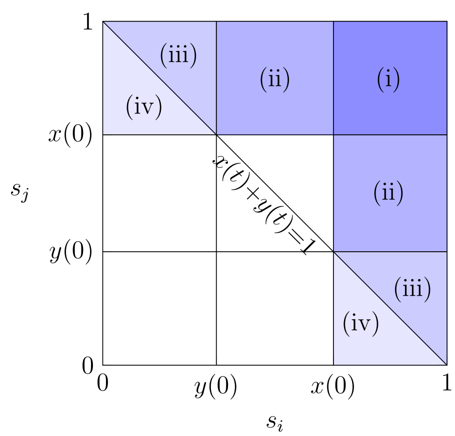

Figure 8: Partition of the type space in a symmetric environment with $c = 1/5, r = 1/5$ and posterior beliefs induced by $Beta(1+\theta, 1+(1-\theta))$.

Figure 8 partitions the type space $(0,1)^2$ using the equilibrium (inverse) strategies $x(t)$ and $y(t)$. If the pair of the players' types (s_i, s_j) is in area (i), both players invest initially and get $R^\theta - c$. Area (ii) is where one players invests initially, the other player follows suit. Both players get $R^\theta - c$. Area (iii) is the leader-follower continuation game where the follower invests before the leader disinvests. The leader gets $e^{-rt}R^\theta - c$ and the follower gets $e^{-rt}(R^\theta - c)$. Area (iv) is the leader-follower continuation game where the leader disinvests before the follower invests. The leader gets $-c$, the follower gets 0. Players do not invest and get payoff 0 in the white area.

Let $\mathcal{E}_\gamma(x_\gamma(0))$ denote the ex ante efficiency in a dynamic equilibrium with distribution $\mathbf{F}_\gamma$, where $\mathcal{E}_\gamma(\mu)$ is given by

$$\begin{aligned}
\mathcal{E}_\gamma(\mu) =&(1-F^1_\gamma(\mu))^2(1-c)-(1-F^0_\gamma(\mu))^2(1+c) && (\mathcal{E}_{\text{i}})\\
&+2(1-F^1_\gamma(\mu))(F^1_\gamma(\mu)-F^1_\gamma(1-\mu))(1-c) && (\mathcal{E}_{\text{ii-1}})\\
&-2(1-F^0_\gamma(\mu))(F^0_\gamma(\mu)-F^0_\gamma(1-\mu))(1+c) && (-\mathcal{E}_{\text{ii-0}})\\
&+\left((1-F^1_\gamma(\mu))F^1_\gamma(1-\mu)+(1-F^0_\gamma(\mu))F^0_\gamma(1-\mu)\right)(-c) && (\mathcal{E}_{\text{LF-c}})\\
&+\int_0^\infty\left(\int_0^\tau -e^{-rt}(2-c)f^1_\gamma(y_\gamma(t))y'_\gamma(t)\mathrm{d}t\right)f^1_\gamma(x_\gamma(\tau))x'_\gamma(\tau)\mathrm{d}\tau && (\mathcal{E}_{\text{LF-1}})\\
&-\int_0^\infty\left(\int_0^\tau -e^{-rt}(2+c)f^0_\gamma(y_\gamma(t))y'_\gamma(t)\mathrm{d}t\right)f^0_\gamma(x_\gamma(\tau))x'_\gamma(\tau)\mathrm{d}\tau. && (-\mathcal{E}_{\text{LF-0}})
\end{aligned}$$

Take $\gamma\to\infty$. By Claim 9,

$$\begin{aligned}
\mathcal{E}_{\text{i}}+\mathcal{E}_{\text{ii-1}}-\mathcal{E}_{\text{ii-0}}=&\left(1-F^1_\gamma(x_\gamma(0))\right)\left(F^1_\gamma(x_\gamma(0))-1+2F^0_\gamma(x_\gamma(0))\right)(1-c)\\
&-\left(1-F^0_\gamma(x_\gamma(0))\right)\left(F^0_\gamma(x_\gamma(0))-1+2F^1_\gamma(x_\gamma(0))\right)(1+c)\to 0
\end{aligned}$$

and

$$\mathcal{E}_{\text{LF-c}}=2\left(1-F^1_\gamma(x_\gamma(0))\right)\left(1-F^0_\gamma(x_\gamma(0))\right)(-c)\to 0.$$

Because $\mathcal{E}_{\text{LF-1}}$ and $\mathcal{E}_{\text{LF-0}}$ are both positive and decreasing in r (keeping $x_\gamma(t)$ and $y_\gamma(t)$ fixed), so $\mathcal{E}_{\text{LF-1}}-\mathcal{E}_{\text{LF-0}}$ is bounded above by $\mathcal{E}_{\text{LF-1}}$ evaluated at $r=0$:

$$\mathcal{E}_{\text{LF-1}}-\mathcal{E}_{\text{LF-0}}\le(2-c)\int_{x_\gamma(0)}^1\left(F^0_\gamma(x)-F^0_\gamma(x_\gamma(0))\right)f^1_\gamma(x)\mathrm{d}x,$$

and bounded below by $-\mathcal{E}_{\text{LF-0}}$ evaluated at $r=0$:

$$\mathcal{E}_{\text{LF-1}}-\mathcal{E}_{\text{LF-0}}\ge-(2+c)\int_{x_\gamma(0)}^1\left(F^1_\gamma(x)-F^1_\gamma(x_\gamma(0))\right)f^0_\gamma(x)\mathrm{d}x.$$

Because the integrands are finite and $x_\gamma(0)\to 1$, both the upper and the lower bounds converge to zero. So $\mathcal{E}_{\text{LF-1}}-\mathcal{E}_{\text{LF-0}}\to 0$. Figure 9 plots $\mathcal{E}_\gamma(x_\gamma(0))$ as a function of γ and illustrates that $\mathcal{E}_\gamma(x_\gamma(0))\to\mathcal{E}^*$ as $\gamma\to\infty$.

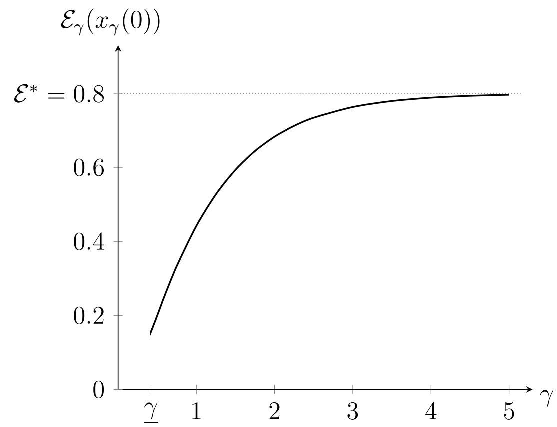

Figure 9: Ex ante efficiency in a symmetric environment with $c = 1/5, r = 1/5$ and posterior beliefs induced by signals distributed according to $Beta(1+\gamma\theta, 1+\gamma(1-\theta))$.

A.3 Proofs for Section 5

A.3.1 Proof of Proposition 6

No investment region. Let (s_i, s_j) be $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L < 0$. Because $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L < 0 < c$, by (7), eventually neither player invests in the constrained outcome.

I now show eventually neither player invests in equilibrium. Fix $s_i = \hat{s}_i$.

First, suppose $\hat{s}_i > x(0)$. $\hat{s}_i$ invests initially and becomes the leader. Recall the analysis of the leader's first-order condition (4) in Section 3.2.2. For a leader with type $\hat{s}_i$, he stays invested as long as s_j is such that $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L > 0$. So players eventually do not invest if s_j is such that $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L < 0$, which is given by the assumption.

Next, suppose $\hat{s}_i \in [y(0), x(0)]$. Players eventually do not invest if $s_j \leq x(0)$. By MLRP, $\Pr(\theta = 1|s_i, s_j)H - \Pr(\theta = 0|s_i, s_j)L$ is increasing in s_i and in s_j. The leader's first-order condition says $\Pr(\theta = 1|x(0), y(0))H - \Pr(\theta = 0|x(0), y(0))L = 0$. So for all $\hat{s}_i \geq y(0)$, $\Pr(\theta = 1|\hat{s}_i, x(0))H - \Pr(\theta = 0|\hat{s}_i, x(0))L \geq 0$. Therefore, if $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L < 0$, it must be $s_j < x(0)$.

Lastly, suppose $\hat{s}_i < y(0)$. Players eventually do not invest if s_j is such that $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L < 0$, which is given by the assumption.

Investment region. Define

$$\kappa := \Pr(\theta = 1|x(0), x(0))H - \Pr(\theta = 0|x(0), x(0))L.$$

To show eventually both players invest in the constrained outcome, it suffices to show $\kappa > c$. Recall the initial condition (2) is given by

$$\begin{aligned} c = &\Pr(s_j \in [y(0), x(0))|s_i = x(0), s_j \leq x(0)) \\ &\cdot \left[\Pr(\theta = 1|s_i = x(0), s_j \in [y(0), x(0)])H - \Pr(\theta = 0|s_i = x(0), s_j \in [y(0), x(0)])L\right], \end{aligned}$$

which implies the second line is larger than c. By Lemma 6, $\Pr(\theta = 1|s_i = x(0), s_j = x(0)) > \Pr(\theta = 1|s_i = x(0), s_j \in [y(0), x(0)])$. The result follows.

I now show both players eventually invest in equilibrium. Fix $s_i = \hat{s}_i$.

First, suppose $\hat{s}_i > x(0)$. $\hat{s}_i$ invests initially and becomes the leader. Both players eventually invest if s_j is such that $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L > 0$. By assumption, $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L \geq \kappa$. $\kappa > c > 0$ as shown above.

Next, suppose $\hat{s}_i \in [y(0), x(0)]$. Both players eventually invest if $s_j > x(0)$. By MLRP, For $\hat{s}_i \leq x(0)$, $\Pr(\theta = 1|\hat{s}_i, x(0)) < \Pr(\theta = 1|x(0), x(0))$, so if $\Pr(\theta = 1|\hat{s}_i, s_j) > \Pr(\theta = 1|x(0), x(0))$, it must be $s_j > x(0)$.

Lastly, suppose $\hat{s}_i < y(0)$. Both players eventually invest if s_j is such that $\Pr(\theta = 1|\hat{s}_i, s_j)H - \Pr(\theta = 0|\hat{s}_i, s_j)L > 0$, which is given by the assumption and $\kappa > 0$.

A.3.2 Proof of Proposition 7

Characterizing x_{ir} and y_{ir}. A similar argument to Lemma 9 in the Appendix shows in the equilibrium of the irreversible investment case, players do not invest after no initial investment. I maintain the assumption that players use symmetric monotonic strategies. By MLRP, there exists a threshold x_{ir} in the first stage of time 0 such that a player invests in the first stage if and only if his type is above x_{ir}, and a threshold y_{ir} in the second stage of time 0 such that a player invests if and only if his type is above y_{ir}. x_{ir}'s indifference condition is the same as the initial threshold z's indifference condition in the dynamic equilibrium analyzed in Section 3, which is given by $W_0(x_{\mathrm{ir}}, y_{\mathrm{ir}}) = c$, where

$$W_0(x, y) := \frac{\rho_0 f^1(x)(F^1(x) - F^1(y))H}{\rho_0 f^1(x)F^1(x) + (1-\rho_0)f^0(x)F^0(x)} - \frac{(1-\rho_0)f^0(x)(F^0(x) - F^0(y))L}{\rho_0 f^1(x)F^1(x) + (1-\rho_0)f^0(x)F^0(x)}.$$

Given the other player (leader)'s type is above x_{ir}, in the second stage, type y_{ir} (follower) is indifferent between investing right away and never investing. His indifference condition is given by $W_1(x_{\text{ir}}, y_{\text{ir}}) = c$, where $W_1(x,y) := \Pr(\theta = 1|s_F = y, s_L \geq x)H - \Pr(\theta = 0|s_F = y, s_L \geq x)L$, which is

$$W_1(x,y) := \frac{\rho_0 f^1(y)(1-F^1(x))H}{\rho_0 f^1(y)(1-F^1(x)) + (1-\rho_0)f^0(y)(1-F^0(x))} - \frac{(1-\rho_0)f^0(y)(1-F^0(x))L}{\rho_0 f^1(y)(1-F^1(x)) + (1-\rho_0)f^0(y)(1-F^0(x))}.$$

Comparing to $x(0)$ and $y(0)$. To simplify notation, define $W^*(x,y) := \Pr(\theta = 1|s_i = y, s_{-i} = x)H - \Pr(\theta = 0|s_i = y, s_{-i} = x)L$, which is

$$W^*(x,y) := \frac{\rho_0 f^1(y)f^1(x)H}{\rho_0 f^1(y)f^1(x) + (1-\rho_0)f^0(y)f^0(x)} - \frac{(1-\rho_0)f^0(y)f^0(x)L}{\rho_0 f^1(y)f^1(x) + (1-\rho_0)f^0(y)f^0(x)}.$$

Figure 10 plots curves $W_0(x,y) = c$, $W_1(x,y) = c$, and $W^*(x,y) = 0$ in the (x,y)-space. By definition, the intersection of $W_0(x,y) = c$ and $W_1(x,y) = c$ is the irreversible investment thresholds $(x_{\text{ir}}, y_{\text{ir}})$; the intersection of $W_0(x,y) = c$ and $W^*(x,y) = 0$ is the dynamic equilibrium initial investment thresholds $(x(0), y(0))$.

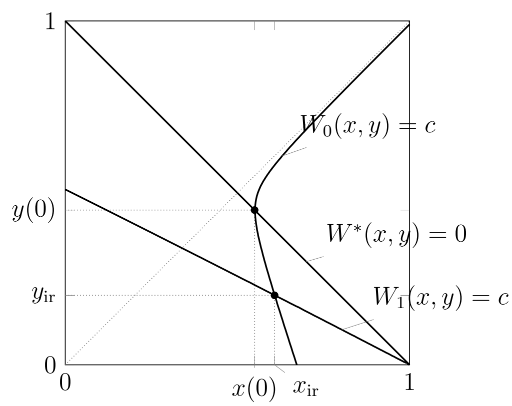

Figure 10: $(x(0), y(0))$ and $(x_{\text{ir}}, y_{\text{ir}})$ for $\rho_0 = 1/2, H = L = 1, c = 1/50$ and posterior beliefs distributed according to $Beta(1+\theta, 1+(1-\theta))$.

The proof of Proposition 7 relies on the following claim, the proof of which is mostly algebraic and is relegated to the Online Appendix.

Claim 10. For all $(x,y) \in (0,1)^2$, the following holds:

(i) $\partial W^*(x,y)/\partial x > 0$ and $\partial W^*(x,y)/\partial y > 0$;

(ii) $\partial W_1(x,y)/\partial x > 0$ and $\partial W_1(x,y)/\partial y > 0$;
(iii) $\partial W_0(x,y)/\partial y \geq 0$ if and only if $W^*(x,y) \leq 0$;
(iv) For $x > y$ and $W_0(x,y) > 0$, $\partial W_0(x,y)/\partial x > 0$.

The two additional conditions on (x,y) in (iv) are necessary conditions for both the dynamic equilibrium and the equilibrium with irreversible investment. I will only consider (x,y) such that these two conditions hold.

Because the initial condition $(x(0),y(0))$ is such that $W^*(x(0),y(0)) = 0$ and $W_0(x(0),y(0)) = c$, by Claim 10 (iii) and (iv), for $(x_{\text{ir}}, y_{\text{ir}})$ such that $W_0(x_{\text{ir}}, y_{\text{ir}}) = c$, either $x_{\text{ir}} > x(0), y_{\text{ir}} < y(0)$ or $x_{\text{ir}} > x(0), y_{\text{ir}} > y(0)$. Recall the analysis of the follower's first-order condition in the dynamic equilibrium in Section 3.2.2. The follower $y(0)$'s marginal cost from waiting, $rW_1(x(0),y(0))$, is strictly positive for otherwise $y(0)$ would not find it optimal to invest. This means $W_1(x(0),y(0)) > c$. By Claim 10 (ii), if $x_{\text{ir}} > x(0), y_{\text{ir}} > y(0)$, then $W_1(x_{\text{ir}}, y_{\text{ir}}) > W_1(x(0),y(0)) > c$, which contradicts the equilibrium condition $W_1(x_{\text{ir}}, y_{\text{ir}}) = c$. Therefore, it must be $x_{\text{ir}} > x(0), y_{\text{ir}} < y(0)$.

Ex ante efficiency. Let $\mathcal{E}^{\text{ir}}_\gamma(x^{\text{ir}}_\gamma, y^{\text{ir}}_\gamma)$ denote the ex ante efficiency in an equilibrium with irreversible investment with distribution $\mathbf{F}_\gamma$. $(x^{\text{ir}}_\gamma, y^{\text{ir}}_\gamma)$ is the equilibrium investment thresholds and $\mathcal{E}^{\text{ir}}_\gamma(x,y)$ is given by

$$\begin{aligned}\mathcal{E}^{\text{ir}}_\gamma(x,y) =& (1-F^1_\gamma(x))^2(1-c) - (1-F^0_\gamma(x))^2(1+c) \\ &+ 2\left((1-F^1_\gamma(x))(F^1_\gamma(x)-F^1_\gamma(y))(1-c) - (1-F^0_\gamma(x))(F^0_\gamma(x)-F^0_\gamma(y))(1+c)\right).\end{aligned}$$

References

Banerjee, Abhijit V. (1992) "A Simple Model of Herd Behavior," *The Quarterly Journal of Economics*, 107, 797–817.

Bar-Ilan, Avner and William C. Strange (1996) "Investment Lags," *The American Economic Review*, 86, 610–622.

Bikhchandani, Sushil, David Hirshleifer, and Ivo Welch (1992) "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades," *Journal of Political Economy*, 100, 992–1026.

Bobtcheff, Catherine, Jérôme Bolte, and Thomas Mariotti (2017) "Researcher's Dilemma," *The Review of Economic Studies*, 84, 969–1014.

Brindisi, Francesco, Boğaçhan Çelen, and Kyle Hyndman (2014) "The Effect of Endogenous Timing on Coordination under Asymmetric Information: An Experimental Study," *Games and Economic Behavior*, 86, 264–281.

Chamley, Christophe and Douglas Gale (1994) "Information Revelation and Strategic Delay in a Model of Investment," *Econometrica*, 1065–1085.

Dasgupta, Amil (2000) "Social Learning with Payoff Complementarities."

Dixit, Avinash K. and Robert S. Pindyck (1994) *Investment under uncertainty*, Princeton, N.J: Princeton University Press.

Gale, Douglas (1995) "Dynamic Coordination Games," *Economic theory*, 5, 1–18.

Giorgi, G. and S. Komlósi (1992) "Dini Derivatives in Optimization - Part I," *Rivista di Matematica per le Scienze Economiche e Sociali*, 15, 3–30.

Herrera, Helios and Johannes Hörner (2012) "A Necessary and Sufficient Condition for Information Cascades."

Hopkins, Ed and Tatiana Kornienko (2007) "Cross and Double Cross: Comparative Statics in First Price and All Pay Auctions," *The B.E. Journal of Theoretical Economics*, 7.

Kalashnikov, V. V. and S. T. Rachev (1985) "Characterization Problems in Queueing and Their Stability," *Advances in Applied Probability*, 17, 868–886.

Klein, Nicolas and Peter Wagner (2019) "Strategic Investment and Learning with Private Information."

Kováč, Eugen and Jakub Steiner (2013) "Reversibility in Dynamic Coordination Problems," *Games and Economic Behavior*, 77, 298–320.

McDonald, Robert L. and Daniel R. Siegel (1985) "Investment and the Valuation of Firms When There is an Option to Shut Down," *International Economic Review*, 26, 331.

Murto, Pauli and Juuso Välimäki (2013) "Delay and Information Aggregation in Stopping Games with Private Information," *Journal of Economic Theory*, 148, 2404–2435.

Ramos, Hector M., Jorge Ollero, and Miguel A. Sordo (2000) "A Sufficient Condition for Generalized Lorenz Order," *Journal of Economic Theory*, 90, 286–292.

Teschl, Gerald (2012) *Ordinary Differential Equations and Dynamical Systems*, Providence, R.I: American Mathematical Society.