应用随机过程

（Chapter Three Renewal Process）

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Chapter 3 Outline of Renewal Process

- Introduction to Renewal Process
- Renewal-type Equations
- Excess Life, Current Life and Total Life
- Renewal Reward Process
- Stationary and Transient Renewal Process
- Regenerative processes

Renewal Process is the counting process **distinct fro**m Poisson Process

Course Objective

- What is Renewal Process
  - ✓ Acquire the definition of Renewal Process
  - ✓ Know the adequacy of using renewal function to approximate actual arrival behaviors
- How to model renewal process in practice
  - ✓ Understand the renewal-type equation and its solution to model the practical case and the concept of stopping time
  - ✓ Understand renewal reward processes
  - ✓ Grasp to distinguish limiting theorems, stationary and transient renewal processes

3.1 Definition of Renewal Process

Let $\{N(t), t \geqslant 0\}$ be a counting process and let X_n denotes time between the nth and $(n\text{-}1)$st event of this process, $n \geqslant 1$.
The sequence $\{X_1, X_2, \cdots\}$ is i.i.d random variable with common distribution F, mean μ, and variance σ^2 .

Let S_n denote the arrival time of the nth event , $S_n = \sum_{i=1}^{n} X_i$

Let $N(t) = \max\{n \mid S_n \leq t\}$

Hints: the stochastic process $\{N(t), t \geqslant 0\}$ is a renewal process

3.1 Definition of Renewal Process

- **Renewal Function**

A key identity enables to obtain the distribution of $N(t)$ is

$$\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$$

$$P\{N(t) \geq n\} = P\{S_n \leq t\} = F_n(t)$$

Remark : the distribution of S_n, F_n is the n-fold convolution of F.

$$\mathrm{P}\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$$

$$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$

$$= F_n(t) - F_{n+1}(t)$$

3.1 Definition of Renewal Process

- **Renewal function**

Renewal function $M(t)$: $M(t) = E\left[N(t)\right]$

The relation between $M(t)$ and F_n :

$$M(t) = E\left[N(t)\right] = \sum_{n=1}^{\infty} nP\left\{N(t) = n\right\} = \sum_{n=1}^{\infty} n\left(F_n(t) - F_{n+1}(t)\right) = \sum_{n=1}^{\infty} F_n(t)$$

Renewal density $m(t)$ $\quad m(t) = \dfrac{dM(t)}{dt}$

$$m(t) = \sum_{n=1}^{\infty} f_n(t)$$

Where f_n is the density of F_n

3.1 Definition of Renewal Process

- Laplace transform of $m(t)$ and $f(t)$

$$m^e(s) = \sum_{n=1}^{\infty} \left[f^e(s) \right]^n = \frac{f^e(s)}{1 - f^e(s)}$$

where

$$f^e(s) = \int_0^{\infty} e^{-st} f(t)\,dt,\; m^e(s) = \int_0^{\infty} e^{-st} m(t)\,dt$$

and

$$M^e(s) = \int_0^{\infty} e^{-st} M(t)\,dt$$

3.1 Definition of Renewal Process

- **Example 1** （EX3.1.1 on book)

  $N(t)$ is a Possion process with λ , find $E(N(t))$

  Solution :

  Distribution of interarrival time : $f(t) = \lambda e^{-\lambda t}$

$$f^e(s) = \frac{\lambda}{\lambda + s} \Rightarrow m^e(s) = \frac{\lambda}{s}$$

$$\Downarrow$$

$$E[N(t)] = M(t) = \lambda t \Leftarrow M^e(s) = \frac{\lambda}{\lambda s^2}$$

3.1 Definition of Renewal Process

- **Example 2(** EX 3.1.2 on book**)**

Consider a renewal process with interarrival time distribution

$$f(t) = te^{-t}, \qquad t \geq 0$$

the Laplace transform of f is given by $f^{e}(s) = \dfrac{1}{(s+1)^2}$
Find the renewal function.

Solution:

$$m^{e}(s) = \frac{1}{s(s+2)} \Rightarrow M^{e}(s) = \frac{1}{s^2(s+2)} = \left(-\frac{1}{4}\right)\left(\frac{1}{s}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{s^2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{s+2}\right)$$

$$M(t) = \left(-\frac{1}{4}\right) + \left(\frac{1}{2}\right)t + \left(\frac{1}{4}\right)e^{-2t}$$

3.2 Renewal type function

- **Renewal-type equation**

By conditioning on the first renewal epoch X_1, define a renewal-type equation

$$g(t) = h(t) + \int_0^t g(t-x) f(x) dx$$

where the functions $h(t)$ and $f(t)$ are known, and $g(t)$ is unknown. Solution of $g(t)$ is given by

$$g(t) = h(t) + \int_0^t h(t-x) m(x) dx$$

3.2 Renewal type function

- Proof:

Take Laplace transform on both sides: $g^e(s) = h^e(s) + g^e(s) f^e(s)$

$$g^e(s) = \frac{h^e(s)}{1 - f^e(s)}$$

$$= h^e(s)\left\{1 + f^e(s) + \left[f^e(s)\right]^2 + \cdots\right\}$$

$$= h^e(s) + h^e(s) m^e(s)$$

$$g(t) = h(t) + \int_0^t h(t-x) m(x)\, dx$$

3.2 Renewal type function

Example 3(EX 3.2.2 on book**)**

If $\{X_n\}$ are identically distributed random variables with a common mean μ, and N is independent of $\{X_n\}$, then we know that $E[S_n] = \mu E[N]$ where $S_n = X_1 + X_2 + \cdots + X_N$.

In a renewal process, we see that $S_{N(t)} = X_1 + X_2 + \cdots + X_{N(t)}$ denotes the time of the last renewal before t. Show that $E\left[S_{N(t)+1}\right] = \mu\left(M(t) + 1\right)$

Solution:

$M(t)$ is the mean number of renewals by time t

$E\left[S_{N(t)+1}\right]$ is the expected time of the first renewal after t

Let $g(t) = E\left[S_{N(t)+1}\right]$

3.2 Renewal type function

Conditioning on the time of the first renewal, this yields:

$$E\left[S_{N(t)+1} \mid X_1 = x\right] = \begin{cases} x & x > t \\ x + g(t-x) & x \le t \end{cases}$$

Computing expectations by conditioning:

$$g(t) = \int_0^\infty E\left[S_{N(t)+1} \mid X_1 = x\right] f(x)\,dx$$

$$= \int_0^\infty x f(x)\,dx + \int_0^t g(t-x) f(x)\,dx$$

$$= \mu + \int_0^t g(t-x) f(x)\,dx$$

Let $h(t) = \mu$

$$g(t) = h(t) + \int_0^t h(t-x) m(x)\,dx = \mu + \int_0^t \mu m(x)$$

$$= \mu\left[1 + M(t)\right] = E\left[S_{N(t)+1}\right]$$

3.2 Renewal type function

Stopping time

An integer-valued random variable N is a stopping time with respect to i.i.d. random variables $\{X_n\}$, if the occurrence or nonoccurrence of the event $\{N=n\}$, is independent of $X_{n+1}, X_{n+2}, \cdots\cdots$.

Proposition 1

Assume that $\{X_n\}$ are i.i.d. random variables, $E[X]<\infty$, when N is a stopping time with respect to $\{X_n\}$, $E[N]<\infty$

let $S_N = \sum_{n=1}^{N} X_n$, then $E[S]=E[X]E[N]$.

3.2 Renewal type function

- **Example** 5(EX 3.2.3 on book)

For example: $\{X_i\}$ be i.i.d. random variables with $P\{X_1=1\}= p$ $P\{X_1= -1\} = q$. Let $N = \min\{n: X_1+\cdots X_n=1\}$. Find $E[S_N]$.

Solution:

$E[X_1] = p\text{-}q <\infty$, $E[N] <\infty$, then $E[S_N]=(p\text{-}q)E[N]$

3.2 Renewal type function

- **Example 4(** EX 3.2.4 on book**)**

Consider a single round of offensive assault in a basketball game. At time 0, the offensive team attempts a shot and fails to score. With probability p the offensive team retains control of the ball. If so, the team waits a random time X_1 before score, then the process repeats itself. Assume that $\{X_i\}$ are i.i.d. random variables with a common density f.

Each shot other than the first one made at time 0 is called a reattempt. Let $R(t)$ denote the expected number of reattempts made by time t. we now derive a closed-form expression for $R(t)$.

3.2 Renewal type function

Conditioning on X_1, we write

$$R\left(t \middle| X_1 = x\right) = \begin{cases} \left[1 + qR\left(t-x\right)\right]p & \text{if } x \le t \\ 0 & \text{otherwise} \end{cases}$$

Applying the law of total probability, we obtain

$$R(t) = \int_0^t p\left[1 + qR(t-x)\right] f(x)dx = pF\left(t\right) + pq\int_0^t R\left(t-x\right) f\left(x\right)dx$$

Due to the appearance of the term *pq*, the preceding integral equation is not in the form of (3);we can use the transform approach to solve the problem. Define

$$r^e\left(s\right) = \int_0^\infty e^{-st} r\left(t\right)dt \quad \text{and} \quad R^e\left(s\right) = \int_0^\infty e^{-st} R\left(t\right)dt$$

where $r\left(t\right) = dR\left(t\right)/dt$

3.2 Renewal type function

Taking the derivative of $R(t)$ with respect to t, we obtain

$$r(t) = pf(t) + pq\int_0^t r(t-x)\, f(x)\, dx$$

The Laplace transform of the previous equation is

$$r^e(s) = pf^e(s) + pqr^e(s)\, f^e(s) \qquad \text{and} \qquad r^e(s) = \frac{pf^e(s)}{1 - pqf^e(s)}$$

We consider the case in which f is exponential with parameter λ.

3.2 Renewal type function

Since $f^e(s) = \lambda / (s+\lambda)$, the preceding expression reduces to

$$r^e(s) = \frac{p\lambda}{s+\lambda(1-pq)}$$

This gives $R^e(s) = \frac{1}{s}\frac{p\lambda}{s+\lambda(1+pq)} = \frac{p}{1-pq}\left[\frac{1}{s} - \frac{1}{s+\lambda(1-pq)}\right]$

The last equality is obtained from a partial fraction expansion. Inverting the preceding expression, we find

$$R(t) = \left[\frac{p}{1-pq}\right]\left[1-e^{-\lambda(1-pq)t}\right] \qquad t \geq 0$$

Hints

- 完全理解更新过程的具体细节，重点在更新函数和更新密度函数的求解。
- 灵活运用更新类型函数求解具体问题。

3.3 Excess life, Current life, and Total life

- **Definition**

Excess life : $Y(t) = S_{N(t)+1} - t$

Current life: $A(t) = t - S_{N(t)}$

Total life: $T(t) = Y(t) + A(t) = X_{N(t)+1}$

Excess-life Distribution

Let V_t denote the distribution function of the excess-life random variable $Y(t)$, this is , $V_t(x) = p\{Y(t) \le x\}$. we now define the complementary distribution $\overline{V}_t(x) = p\{Y(t) > x\}$. Conditioning on the epoch on the epoch of the first arrival X_1, we can write

$$p\{Y(t) > x | X_1 = z\} = \begin{cases} 1 & \text{if } z > t + x \\ 0 & \text{if } t < z \le t + x \\ \overline{V}_{t-z}(x) & \text{if } 0 < z \le t \end{cases}$$

Hints: applying renewal type function to solve problem of excess-life distribution

3.3 Excess life, Current life, and Total life

Current-life distribution

Recall that the current life at t is defined as $A(t) = t - S_{N(t)}$.

Let $U_t(x) = p\{A(t) \le x\}$ denote the distribution function of $A(t)$. one way to obtain the distribution function is to use what we know about $Y(t)$, the excess life at t. This is done by noting that $A(t) > x \Leftrightarrow Y(t-x) > x$ where $t > x$. In other words, the length of current life at t is the same as the length of excess life at t-x. Using this relation, we find the distribution function of the current life at time t

$$U_t(x) = \begin{cases} F(t) - \int_0^{t-x} \left[1 - F(t-y)\right] m(y)\,dy & \text{if } x < t \\ 1 & \text{if } x \ge t \end{cases}$$

3.3 Excess life, Current life, and Total life

- **Total-life distribution**

Let L_t be the distribution function of the total life $T(t)$ and

$\overline{L_t} = p\{T(t) > x\}$. Conditioning on X_1, we can write

$$p\{T(t) > x | X_1 = z\} = \begin{cases} 1 & if\ z > \max(x,t) \\ \overline{L}_{t-z}(x) & if\ z < t \\ 0 & otherwise \end{cases}$$

Summarizing the two result, the total-life distribution at time t is given by

$$L_t(x) = \begin{cases} \int_{t-x}^{t} \left[F(x) - F(t-y)\right] m(y)\,dy & x < t \\ F(x) + M(t)\left[F(x) - 1\right] & x \ge t \end{cases}$$

3.4 Renewal reward processes

- **Definition**

Consider a renewal process $\{N(t),\ t \geqslant 0\}$ having interarrival times X_n, $n \geqslant 1$, and suppose that each time a renewal occurs we receive a reward. Then $N(t)$ is known as a **renewal reward process.**

R_n denotes the reward earned at the time of the nth renewal.

Assume that R_n, $n \geqslant 1$, are independent and identically distributed, but R_n may (and usually will) depend on X_n, the length of the nth renewal interval. $E[R_n] < \infty$ for all n.

Let $R(t) = \sum_{n=1}^{N(t)} R_n$ = the total reward earned in $(0, t]$

3.4 Renewal reward processes

- **Proposition 2**

If $E[R] < \infty$, and $E[X] < \infty$, then $\lim_{t \to \infty} \frac{E[R(t)]}{t} = \frac{E[R_1]}{E[X_1]}$

If we call a renewal a cycle, then the proposition can be interpreted as:

$$\text{The long - run average reward per unit time}$$

$$= \frac{\text{Expected reward received per cycle}}{\text{Expected cycle length}}$$

3.4 Renewal reward processes

- **Example 6**

Suppose that customers arrive at a railway station in accordance with a renewal process having a mean interarrival time μ. Whenever there are N customers waiting in the station, a train leaves. If the station incurs a cost at the rate of nc dollars per unit time whenever there are n customers waiting, what is the average cost incurred by the station?

3.4 Renewal reward processes

Solution:

E[cycle length]$=N\mu$

Let X_n denote the time between the nth and $(n+1)$th arrival in a cycle. E[cost per cycle] $= E[cX_1+2cX_2+\ldots\ldots+(N-1)cX_{N-1}]$

$E[X]=\mu$ $\Rightarrow$ E[cost per cycle] $= c\mu\dfrac{N}{2}(N-1)$

$$\frac{\text{Expected reward received per cycle}}{\text{Expected cycle length}}$$

$$=\frac{c\mu N(N-1)}{2N\mu}=\frac{c(N-1)}{2}$$

3.4 Renewal reward processes

Example 7(EX 3.4.3 on book)

Consider a stage with a large number of high-intensity light bulbs. Suppose we replace each bulb when it fails. In addition, all bulbs are replaced every T time units to take advantage of the economies of scale. This type of replacement policy is called the block replacement policy. Let c_1 be the unit cost of replacement at the block replacement time and c_2 the unit cost of replacement at failure. Under this cost structure, we can simply look at the costs of each socket holding a bulb independently. If replacement were done only at failure, then they would have formed a renewal process with interarrival time distribution F, the time-to-failure distribution of the bulb.

3.4 Renewal reward processes

The renewal function $M(t)$ would give the expected number of failures by time t. However under the block replacement policy the aforementioned renewal process is terminated and restarted every T time units. Hence it is natural to introduce another renewal process with a constant interarrival time T.

The expected cost per renewal for this second renewal process is then $c_1 + c_2 M(t)$. Let $C(T)$ be the long-run expected average cost per unit time. For this renewal reward process, we obtain

$$C(T) = \frac{c_1 + c_2 M(T)}{T}$$

3.4 Renewal reward processes

Setting $dC(T)/dT = 0$, we get the necessary condition for T that minimizes $C(T)$:

$$Tm(T) - M(T) = \frac{c_1}{c_2}$$

We now consider the case in which F follows the gamma distribution

$$f(t) = te^{-t} \qquad t \geq 0$$

Then, after some algebra, Equation 3.4.3 reduces to

$$e^{-2T}(1+2T) = \left\{1 - \left(\frac{4c_1}{c_2}\right)\right\}$$

3.4 Renewal reward processes

Class discussion

If , then the right side of Equation 3.4.4 will always be nonpositive. However, the left side of Equation 3.4.4 will always be nonnegative for . Thus no finite T will satisfy Equation 3.4.4. This implies that the optimal T will be infinite and replacements at failure will be the minimum cost solution. On the other hand if , then the left side of Equation 3.4.4 is strictly decreasing in T from an initial value of 1. Since the right side is a constant, a unique value of T exists.

3.5 Limiting theorems, Stationary and Transient Renewal Process

- Three renewal theorem

(i)Elementary renewal theorem $\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mu}$

where $M(t)$ is the renewal function, $M(t)=E[N(t)]$; $\mu=E[X_n]$, $\{X_n\}$ interarrival time of $N(t)$

(ii) Blackwell's renewal theorem $\lim_{t \to \infty} M(t) - M(t-a) = \frac{a}{\mu}$

(iii) Key renewal theorem $\lim_{t \to \infty} g(t) = \frac{\int_0^{\infty} h(t)dt}{\mu}$

Where μ is the mean interarrival time.

3.5 Limiting theorems, Stationary and Transient Renewal Process

Stationary Renewal Process

A renewal process is a stationary if $m(t) = \frac{1}{\mu}$ for all t.

Transient Renewal Process

A renewal process is a transient if the interarrival time distribution is defective in the sense that $F(\infty) < 1$

3.5 Limiting theorems, Stationary and Transient Renewal Process

- **Example 8** (EX 3.5.1 on book)

Suppose that a pedestrian standing on a corner wants to cross the street. Let $\{S_n\}$ be the successive epoch at which cars pass by the pedestrian. Assume that the interarrival times $\{X_n\}$ associated with $\{S_n\}$ are i.i.d random variables with a common distribution G. To cross the street, the pedestrian needs s amount of time. Hence the pedestrian starts crossing the street at $L= S_n$ such that $X_1 \le s, X_2 \le s, \ldots, X_n \le s$,and $X_{n+1} > s$. Therefore L is the lifetime of a transient renewal process.

3.5 Limiting theorems, Stationary and Transient Renewal Process

Solution:

For this transient renewal process, the interarrival time distribution F is given by

$$F(t) = \begin{cases} G(t) & \text{if } t \le s \\ G(s) & \text{if } t > s \end{cases}$$

Therefore F is the distribution function associated with a defective random variable. We obtain

$$p\{L \le t\} = [1 - G(s)][1 + M(t)] \qquad \text{and} \qquad E[L] = \frac{1}{1 - G(s)} \int_0^s [G(s) - G(z)]\,dz$$

3.5 Limiting theorems, Stationary and Transient Renewal Process

Consider the case in which car arrivals follow a Poisson process with mean interarrival time of five seconds and the street crossing time is ten seconds. Then we have

$$E[L] = \frac{1}{e^{-2}} \int_0^{10} \left\{ \left[1 - e^{-2}\right] - \left[1 - e^{-2z}\right] \right\} dz$$

$$= e^2 \left[\int_0^{10} e^{-2z} dz - 10e^{-2} \right] = 5\left(e^2 - 1\right) - 10$$

The mean waiting time needed to cross the street is about twenty-two seconds.

3.6 Regenerative Process

- **Definition**

Consider a stochastic process $Z=\{Z(t),\ t\geqslant 0\}$ with state space $S=\{0,1,\cdots\}$ having the property that the process starts afresh at $S_1,S_2,$..By "the process starts afresh at S_n" we mean the process Z that originates at S_{n-1}.Such a process Z is called a regenerative process,$\{S_n\}$ the regeneration epochs, and $\{X_n\}$ the regeneration cycles, where $X_n = S_n - S_{n-1}$ and $S_0=0$. For such a regenerative process, we can envision that $\{S_n\}$ are the arrived epochs of a renewal process with interarrival times $\{X_n\}$.

3.6 Regenerative Process

Again we assume that $\{X_n\}$ follow distribution F with a finite mean u. Applying the law of total probability, we obtain the renewal-type equation

$$g(t) = P\{Z(t) = j \mid X_1 > t\}P\{X_1 > t\} + \int_0^t g(t - x)f(x)dx$$

$$= P\{X_1 > t, Z(t) = j\} + \int_0^t g(t - x)f(x)dx$$

3.6 Regenerative Process

- **Example 9** (EX 3.6.1 on book)

Assume that visitors arrive at the San Diego Zoo in accordance with a Poisson process with rate . They will wait for jitneys to take them for guided tours. Assume that the interarrival times $\{X_n\}$ of jitneys at the zoo entrance are i.i.d. random variables with a common density f and each jitney can accommodate all waiting visitors. Let $Z(t)$ denote the number of visitors waiting at the entrance at time t. We see that $\{Z(t), t \geqslant 0\}$ is a regenerative process with regeneration points defined at each jitney arrival time. This is because a Poission process is memory-less. At each jitney arrival time, the system (waiting stand) is emptied an the whole process starts afresh. A typical sample path during a regeneration cycle is shown in figure 3.21.

3.6 Regenerative Process

Solution:

To obtain the limiting distribution for $Z(t)$, we define T_i as the length of the interval in a cycle in which there are i visitors waiting. Applying the law of total probability, the conditional expectation for the expected length of T_i is given by

$$E\left[T_i \middle| X_1 = x\right] = \sum_{j=i}^{\infty} E\left[T_i \middle| X_1 = x, N\left(x\right) = j\right] P\left\{N\left(x\right) = j\right\},$$

Where $N(x)$ is the number of visitors arrival in $(0,x]$.

For a Poisson process, we know that conditioning on $N(x)$=j the arrival times are j ordered statistics from a uniform distribution over $(0,x]$.

3.6 Regenerative Process

Hence the expression simplifies to

$$E\left[T_i \middle| X_1 = x\right] = \sum_{j=i}^{\infty} \frac{x}{j+1} e^{-\lambda x} \frac{(\lambda x)^j}{j!}$$

Unconditioning on X_1, we find the mean length of the interval in a cycle with *i* waiting visitors

$$E\left[T_i\right] = \int_0^{\infty} f(x) \sum_{j=i}^{\infty} \frac{x}{j+1} e^{-\lambda x} \frac{(\lambda x)^j}{j!}\, dx$$

According the property of regenerative process ,we conclude that

$$\lim_{t \to \infty} P\{Z(t) = \mathrm{i}\} = \frac{E\left[T_i\right]}{\mu}$$

3.7 Discrete Renewal Process

- **Definition**

We consider a renewal process in which interarrival times $\{X_n\}$ are i.i.d. nonnegative integer-value random variables. We assume that interarrival time probability are given by $f_k=P\{X_1=k\},k=0,1,\cdots$ with distribution function $F(k)=f_0+f_1+\cdots+f_k$ and $f_0=1$. As before, we also assume that $E[X_1]<\infty$

Let $N(n)$ denote the number of renewals by time n(excluding the initial at time 0). The counting process $N=\{N(n),n=0,1,2\cdots\}$ is defined on the set of nonnegative integers. It is called the discrete-time renewal process or discrete renewal process.

Hints

- 掌握更新报酬过程，学会利用公式求解。
- 理解更新定理。
- 区分和辨别稳态更新过程和瞬态更新过程。