基本练习 2 答案

1. Let $M=\left[a_{ij}\right]$ be an $n\times n$ matrix, and e_j be the j-th column of the $n\times n$ identity matrix. Then which of the following multiplication produces the j-th row of the matrix M __A_ ___?

(A) $e_j^T M$;　　（B）Me_j;　　（C）$e_j M$;　　（D）Me_j^T.

2. Assume $\{\alpha_1,\ \alpha_2,\ \alpha_3\}$ is linearly independent, then the one of following set of vectors is linearly dependent [C]

（A）$\alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_3+\alpha_1$　　（B）$\alpha_1, \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3$

（C）$\alpha_1-\alpha_2,\ \alpha_2-\alpha_3,\ \alpha_3-\alpha_1$　　（D）$\alpha_1+\alpha_2,\ 2\alpha_2+\alpha_3, 3\alpha_3+\alpha_1$

3. Let $\alpha_1,\alpha_2,\alpha_3$ be an orthonormal base of R^3, and $\xi=\alpha_1-\alpha_2+\alpha_3$, $\eta=a\alpha_1+b\alpha_2-c\alpha_3$. Then [D]

(A) $\xi\perp\eta\Leftrightarrow a+b+c=0$, (B) $\xi\perp\eta\Leftrightarrow a-b+c=0$.

(C) $\xi\perp\eta\Leftrightarrow a+b-c=0$, (D) $\xi\perp\eta\Leftrightarrow b+c-a=0$

4. For $n\times n$ matrices A and B, which of the following statement is **not** always true ___A____?

(A) If $AB=O$, then $B=O$ or $A=O$, where O denotes a zero matrix of order n;

(B). $\det(AB)=\det(A)\det(B)$;

(C). If $\det(AB)\neq 0$, then A and B are both nonsingular;

(D). If B is nonsingular and $AB=O$, then $A=O$

5. If $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 2$,then $\begin{vmatrix} 2a_{21} & 2a_{22} & 0 \\ \frac{1}{2}a_{11} & \frac{1}{2}a_{12} & 0 \\ 1 & 7 & -2 \end{vmatrix} =$ __D__.

A. -2, B. 2 , C -4, D 4.

6 Assume that $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -1 \\ -3 & 2 & 4 \\ 0 & 6 & 1 \end{bmatrix}$, then $X =$ ____A_

A. $\begin{pmatrix} -3 & 4 & 2 \\ 5 & -1 & 1 \\ 0 & 1 & 6 \end{pmatrix}$, B. $\begin{pmatrix} -3 & 2 & 4 \\ 5 & 1 & -1 \\ 0 & 6 & 1 \end{pmatrix}$, C. $\begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & -1 \\ 6 & 0 & 1 \end{pmatrix}$, D. $\begin{pmatrix} 0 & 6 & 1 \\ 5 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$.

7.Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} - a_{21} & a_{32} - a_{22} & a_{33} - a_{23} \end{bmatrix}$ and

$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. For which of the following elementary matrices

P_2 such that $P_2 P_1 A = B$ ____C____?

A. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$; D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

8. Find the distance from the point P (–9, –2, 4) to the plane $-9x+2y+6z+9=0$. 【C】

(A) $\frac{10}{11}$ (B) $\frac{4}{11}$ (C) 10 (D) 4

9. Find the symmetric equations of the line of intersection of the planes $x-y+z=0$ and $x+2y+3z-6=0$. 【 B】

(A) $\frac{x+2}{-5} = \frac{y+2}{2} = \frac{z}{3}$ (B) $\frac{x-2}{-5} = \frac{y-2}{-2} = \frac{z}{3}$

(C) $\frac{x+2}{5} = \frac{y+2}{2} = \frac{z}{-3}$ (D) $\frac{x-2}{-5} = \frac{y-2}{2} = \frac{z}{-3}$

10. Find the vector projection of $\mathbf{u}=\mathbf{i}+\mathbf{j}-2\mathbf{k}$ on $\mathbf{v}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.【 B 】

(A) $\mathrm{pr}_{\mathbf{v}}\mathbf{u}=-2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ (B) $\mathrm{pr}_{\mathbf{v}}\mathbf{u}=-\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$

(C) $\mathrm{pr}_{\mathbf{v}}\mathbf{u}=-2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ (D) $\mathrm{pr}_{\mathbf{v}}\mathbf{u}=-\frac{2}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$

11. If $\mathbf{v}=2\mathbf{i}-5\mathbf{j}+\mathbf{k}$, find a vector perpendicular to both $\mathbf{v}$ and $\mathbf{i}+\mathbf{j}$.

【 A 】

(A) $-\mathbf{i}+\mathbf{j}+7\mathbf{k}$ (B) $-6\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ (C) $\mathbf{i}-\mathbf{j}+8\mathbf{k}$ (D) $-\mathbf{i}-\mathbf{j}+7\mathbf{k}$

12. $\|\mathbf{u}+\mathbf{v}\|=\|\mathbf{u}\|+\|\mathbf{v}\|$ if and only if 【 D 】

(A) either $\mathbf{u}$ or $\mathbf{v}$ is a zero vector.

(B) both $\mathbf{u}$ and $\mathbf{v}$ are zero vectors.

(C) $\mathbf{u}$ and $\mathbf{v}$ are parallel.

(D) $\mathbf{u}$ and $\mathbf{v}$ point in the same direction.

13. Find the equation of the plane through the point P (–2, 1, 3) and perpendicular to the line: $x=10+8t, y=-5+8t, z=2-t$.

【C 】

(A) $8x+8y-z-19=0$ (B) $8x+8y+z-11=0$

(C) $8x+8y-z+11=0$ (D) $8x+8y-z-11=0$

14. **Let** $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2$ **be** 4×1 **matrices such that the determinant** $|\alpha_1,\alpha_2,\alpha_3,\beta_1|=m$ **and** $|\alpha_1,\alpha_2,\beta_2,\alpha_3|=n$ **. Then the determinant** $|\alpha_1,\alpha_2,\beta_1+\beta_2,\alpha_3|=$ (C).

(A) $m+n$ (B) $-(m+n)$; (C) $n-m$; (D) $m-n$.

15 Let A be an $n\times n$ matrix with $|A|=5$, then $|(2A^{-1})^T|=$___A_____.

(A) $2^n\cdot\frac{1}{5}$; (B) $\frac{5}{2^n}$; (C) $\frac{2}{5^n}$ (D) $2\cdot 5^n$

16.Let A be a square matrix of order 3 such that $|A|=3$. Then $\left||A|A^T\right|=$

__**81**________

17.Let A be an n by n($n\geq 2$) nonsingular matrix, A^* the adjoint matrix of A.Then (C)

(A) $(A^*)^*=|A|^{n-1}A$, (B) $(A^*)^*=|A|^{n+1}A$, (C) $(A^*)^*=|A|^{n-2}A$, (D) $(A^*)^*=|A|^{n+2}A$.

18. Let $A=\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$. If $R(A^*)=1$, then (B).

(A) $a=b$ or $a+2b=0$; (B) $a\neq b$ and $a+2b=0$;
(C) $a\neq b$ and $a+2b\neq 0$; (D) $a=b$ or $a+2b\neq 0$.

19. Let A be an $m\times n$ matrix, β be an $m\times 1$ matrix, and $X=\left(x_1,x_2,\cdots,x_n\right)^{\mathrm{T}}$.

Then (A) is correct.

(A) If $R(A)=m$, then $Ax=\beta$ has solution ;

(B) If $R(A)<m$, then $Ax=\beta$ has no solution;

(C) If $R(A)=n$, then $Ax=\beta$ has solution;

(D) If $R(A)<n$, then $Ax=\beta$ has no solution..

20. Suppose the vector set (1): $\alpha_1,\alpha_2,\cdots,\alpha_r$ can be linearly representated by the vector set (2): $\beta_1,\beta_2,\cdots,\beta_s$. Then (D) is correct.

(A) When $r<s$, the set (2) has to be linearly dependent.

(B) When $r>s$, the set (2) has to be linearly dependent.

(C) When $r<s$, the set (1) has to be linearly dependent.

(D) When $r>s$, the set (1) has to be linearly dependent.

21. Let A be an $m\times n$ matrix, B an $n\times m$ matrix.

Then (A) is correct.

(A) If $n<m$, then the determinant $|AB|=0$,

(B) If $m<n$, then. $|AB|=0$;

(C) If $n<m$, then $|AB|\neq 0$;

(D) If $m<n$, then $|AB|\neq 0$.

22.Suppose $AB=\begin{bmatrix}2 & 2 & 4\\ 0 & 8 & 4\\ 0 & 12 & 6\end{bmatrix}$ **with** $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 6\end{bmatrix}$. **Then** $A^{-1}=$___;$|B|=$_0_.

23. Suppose A is a 3×3 matrix and $|A^{-1}|=\frac{1}{2}$, then the determinant of its adjugate (adjoint) matrix A^* is given by $|A^*|=$ ___**4**_____.

24. Let A be an 3×3 matrix with $|A|=2$. Then $|A^*+A^{-1}|=$__$\mathbf{\frac{27}{2}}$___.

25. .If $\begin{vmatrix}a & b & c\\ d & e & f\\ h & i & j\end{vmatrix}=2$, then $\begin{vmatrix}3a+c & c & b\\ 3d+f & f & e\\ 3h+j & j & i\end{vmatrix}=$__**−6**____.

26. Let A,B be square matrices of order 3 and satisfy

$AB+I=A^2+B$, where $A=\begin{pmatrix}2 & 0 & -1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{pmatrix}$.

Then B $=$_$A+I=\begin{pmatrix}3 & 0 & -1\\ 0 & 3 & 0\\ 1 & 0 & 2\end{pmatrix}$_____.

27. 已知 $X\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}=\begin{pmatrix}2a_1 & 2b_1 & 2c_1\\ a_2 & b_2 & c_2\\ a_2-a_3 & b_2-b_3 & c_2-c_3\end{pmatrix}$，则 $X=$__$\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & -1\end{pmatrix}$

28. $f(x_1,x_2,x_3)=(x_1,x_2,x_3)\begin{pmatrix}1&0&2\\0&2&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ is a quadratic form，

then the matrix representation of $f(x_1,x_2,x_3)$ is___

$\frac{A+A^T}{2}=\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$___

29. Let $A=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$， $B=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$， and matrix X satisfies:

$AXA+BXB=AXB+BXA+I$. Then $X=\begin{pmatrix}1&2&5\\0&1&2\\0&0&1\end{pmatrix}$,

$A(XA-XB)+B(XB-XA)=I, (A-B)X(A-B)=I$ ___________

30. If the sum of coefficients on every row of matrix A is zero, and $r(A)=n-1$， then the general solution of $Ax=0$ is_ $t(1,1,\ldots,1)^T$

31. Let A,B be two 3×3 matrices such that $A^2B-A-B=I_3$.

If $A=\begin{pmatrix}1&0&1\\0&2&0\\-2&0&1\end{pmatrix}$， then $|B|=_\frac{1}{2}$______

32. The necessary and sufficient condition for the set of vectors

$\alpha_1=(1,\ 1,\ 0,\ 0)^T, \alpha_2=(0,\ k,\ 1,\ 1)^T, \alpha_3=(0,\ 0,\ 1,\ k)^T$,

$\alpha_4=(k,\ 0,\ 0,\ 1)^T$ linearly independent is that $k\neq 0,2$_.

33. $f(x_1,x_2,x_3)=a(x_1^2+x_2^2+x_3^2)+4x_1x_2+4x_1x_3+4x_2x_3$ can be linearly transformed to $f=6y_1^2$, then $a=$__2__

34. Let $A=\begin{bmatrix}1&1&0\\1&2&-2\\1&0&1\end{bmatrix}$, $B=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$ and $C=\begin{bmatrix}-2&1&2\\3&-1&-2\\2&-1&-1\end{bmatrix}$. Compute

(1) CA; (2) ABC; (3) $(ABC)^{100}$.

解：(1) $CA=I$; (2) $ABC=\begin{pmatrix}3&-1&-2\\2&0&-2\\2&-1&-1\end{pmatrix}$; (3) $(\boldsymbol{ABC})^{100}=\boldsymbol{ABC}$

35. Let A,B,C be three square matrices of order 3.

（a）Simplifies the formula $(BC^T-I)^T(AB^{-1})^T+[(BA^{-1})^T]^{-1}$;

（b）When $A=\begin{pmatrix}1&0&1\\0&1&0\\1&0&2\end{pmatrix}$, $C=\begin{pmatrix}-1&0&0\\0&1&2\\0&2&1\end{pmatrix}$, find the result of (a).

解:

$$(BC^T-I)^T(AB^{-1})^T+[(BA^{-1})^T]^{-1}=$$

$$[(BC^T-I)^T+I](AB^{-1})^T=CB^T(B^{-1})^TA^T=CA^T$$

$$=\begin{pmatrix}-1&0&-1\\2&1&4\\1&2&2\end{pmatrix}$$

36. For a given vector set $\{a_1,a_2,a_3,a_4\}$, where

$$a_1=\begin{bmatrix}1\\0\\0\\2\end{bmatrix},a_2=\begin{bmatrix}1\\2\\6\\4\end{bmatrix},a_3=\begin{bmatrix}1\\-2\\t\\0\end{bmatrix},a_4=\begin{bmatrix}2\\4\\12\\t-1\end{bmatrix}.$$

(1). For what values of t, the vector set $\{a_1,a_2,a_3,a_4\}$ will be linearly dependent ?

(2). In each of the above cases, please find the corresponding maximum linearly independent vector subset.

Solution: From

$$[a_1,a_2,a_3,a_4]=\begin{bmatrix}1&1&1&2\\0&2&-2&4\\0&6&t&12\\2&4&0&t-1\end{bmatrix}\underset{r_{13}(-2)}{\overset{\sim}{\rightarrow}}\begin{bmatrix}1&1&1&2\\0&2&-2&4\\0&6&t&12\\0&2&-2&t-5\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&2\\0&1&-1&2\\0&6&t&12\\0&2&-2&t-5\end{bmatrix}$$

$$\underset{r_{23}(-6),r_{24}(-2)}{\overset{\sim}{\rightarrow}}\begin{bmatrix}1&1&1&2\\0&1&-1&2\\0&0&t+6&0\\0&0&0&t-9\end{bmatrix}$$

Therefore, in the case of $t=-6$ or $t=9$, the vector set is linearly dependent.

In the case of $t=-6$, a_1,a_2,a_4 forms a basis for the vector set.

In the case of $t=9$, a_1,a_2,a_4 forms a basis for the vector set.

37. Find the determinant of

$$A=\begin{pmatrix}1-a&a&0&0&0\\-1&1-a&a&0&0\\0&-1&1-a&a&0\\0&0&-1&1-a&a\\0&0&0&-1&1-a\end{pmatrix}$$

and find a such that $r(A)=4$_.

Solution: $a=1$

38. Compute the determinant $D_4=\begin{vmatrix}1&2&2&\dots&2\\2&2&2&\dots&2\\2&2&3&\dots&2\\\vdots&\vdots&\vdots&\ddots&\vdots\\2&2&2&2&n\end{vmatrix}$

39. Let A be an $n\times n$ matrix that satisfy $A^2+2A-3I=O, A\neq I$

Prove that both A and $A+4I$ are invertible, then express

A^{-1} by using matrices A and I.

Proof. From $A^2+2A-3I=O$, we have $A(A+2I)=3I$. So $A\dfrac{(A+2I)}{3}=I$.

It follows that $A^{-1}=\dfrac{(A+2I)}{3}$. Secondly,

$$(A+4I)(A-2I)=(A+4I)(A-2I)=A^2+2A-8I=A^2+2A-3I-5I=O-5I$$

we have

$$(A+4I)\frac{(A-2I)}{-5}=I\text{. So }(A+4I)^{-1}=\frac{(A-2I)}{-5}.$$

40. Given three points $A\,(1, 0, 0)$, $B\,(2, 0, -1)$ and $C\,(1, 4, 3)$.
(a) Find an equation of the plane through A, B and C.
(b) Find the area of the triangle ABC.
Sol: (a) Let $\mathbf{u}=\overrightarrow{AB}=\langle 1,0,-1\rangle$, $\mathbf{v}=\overrightarrow{AC}=\langle 0,4,3\rangle$.

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Then a normal vector of the plane is $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -1\\ 0 & 4 & 3\end{vmatrix}=4\mathbf{i}-3\mathbf{j}+4\mathbf{k}$.

Thus the equation of the plane is
$4(x-1)-3y+4z=0$ or $4x-3y+4z-4=0$.

(b) The area of the triangle $=\dfrac{1}{2}\|\mathbf{u}\times\mathbf{v}\|=\dfrac{\sqrt{16+9+16}}{2}=\dfrac{\sqrt{41}}{2}$.

41. The planes $x+2z+1=0$ and $2x+2y-z-3=0$ intersect in a line.
(a) Show that the planes are orthogonal.
(b) Find the symmetric equations for the line of intersection.
Sol: (a) The normal vectors for the two planes are $\mathbf{n}_1=\langle 1,0,2\rangle$ and $\mathbf{n}_2=\langle 2,2,-1\rangle$ respectively. It is easy to check that $\mathbf{n}_1\cdot\mathbf{n}_2=(1)(2)+(0)(2)+(2)(-1)=0$, so the planes are orthogonal.
(b) Method I: The direction vector for the line of intersection is

$$\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 2\\ 2 & 2 & -1\end{vmatrix}=-4\mathbf{i}+5\mathbf{j}+2\mathbf{k}.$$

Let $y=0$, then solving the system of equations $\begin{cases} x+2z+1=0 \\ 2x-z-3=0 \end{cases}$ gives a point (1, 0, −1) in the line of intersection.

The symmetric equations for the line are

$$\frac{x-1}{-4}=\frac{y}{5}=\frac{z+1}{2}.$$

Method II: Let $y=0$, then solving the system of equations $\begin{cases} x+2z+1=0 \\ 2x-z-3=0 \end{cases}$ gives a point P (1, 0, −1) in the line of intersection.

Let $z=0$, then solving the system of equations $\begin{cases} x=-1 \\ 2x+2y-3=0 \end{cases}$ gives another point Q (−1, 5/2, 0) in the line of intersection.

Then $\mathbf{v}=2\overrightarrow{PQ}=2\left\langle -2,\frac{5}{2},1\right\rangle=\langle -4,5,2\rangle$ is a direction vector of the line and the symmetric equations for the line are

$$\frac{x-1}{-4}=\frac{y}{5}=\frac{z+1}{2}.$$

42. Given two points $A(1,0,0)$ and $B(0,2,1)$. Find a point C on the $z-axis$ such that the area of $\blacktriangle ABC$ is minimal .

Sol: Since point C is on $z-axis$, its coordinate can be denoted by $<0,0,z>$.

Then the area of $\blacktriangle ABC$ equals $\frac{1}{2}\|AC\times BC\|$,

and $\|AC\times BC\|=\sqrt{5[(z-\frac{1}{5})^2+\frac{24}{25}]}$. The minimal value of the area is $\sqrt{\frac{6}{5}}$ and $z=\frac{1}{5}$.

43. Find the equation of a plane through the vertical line from point $(1,-1,1)$ to the line $\begin{cases} y-z+1=0 \\ x=0 \end{cases}$ and perpendicular to the plane $z=0$.

Sol: The vertical line is in fact the line through the point $(1,-1,1)$ and perpendicular to the line $\begin{cases} y-z+1=0 \\ x=0 \end{cases}$. Finding the intersection of the plane through the point $(1,-1,1)$ and perpendicular to the line $\begin{cases} y-z+1=0 \\ x=0 \end{cases}$, we may determine the above required line. The directional vector of the line $\begin{cases} y-z+1=0 \\ x=0 \end{cases}$ is $<0,1,1>$, we have the equation of the plane through the point $(1,-1,1)$ and perpendicular to the line $\begin{cases} y-z+1=0 \\ x=0 \end{cases}$, that is $y+z=0$, and the intersection is $(0,-\frac{1}{2},\frac{1}{2})$. Now, The plane satisfying the assumption can be considered as the plane through two points $(1,-1,1)$ and $(0,-\frac{1}{2},\frac{1}{2})$ and perpendicular to the plane $z=0$. Assume its equation is $Ax+By+D=0$ (perpendicular to the plane $z=0$). Since two points $(1,-1,1)$ and $(0,-\frac{1}{2},\frac{1}{2})$ lay on this plane, we get $A=3D, B=2D$. Then the equation is $x+2y+1=0$.

44. Find the distance from the point $P(3,-1,2)$ to the line $L: x=1,\ y=3t+2,\ z=3t+4$.

Sol: The directional vector v of the line L is $<0,3,3>$ and the point $Q(1,2,4)$ is on L. Connecting the points Q and P, we obtain a vector $QP = <2,-3,-2>$. Now the distance from the point $P(3,-1,2)$ to the line $L: x=1,\ y=3t+2,\ z=3t+4$. equals

$$d=\left|\frac{QP\times v}{\|v\|}\right|=\frac{3}{\sqrt{2}}$$

45. 已知矩阵 $A=\begin{pmatrix}1&2&a\\1&3&0\\2&7&-a\end{pmatrix}$ 可经列初等变换变为 $B=\begin{pmatrix}1&a&2\\0&1&1\\-1&1&1\end{pmatrix}$。

（1）求 a 的值，

（2）求矩阵 P 使得 $AP=B$。

Sol: 先计算 A 的秩，为2， B 的秩亦为2，得 $a=2$. 考虑 $AX=B$

$$(A\mid B)=\left(\begin{array}{ccc:ccc}1&0&6&3&4&4\\0&1&-2&-1&-1&-1\\0&0&0&0&0&0\end{array}\right).$$ 记 $B=(\beta_1,\beta_2,\beta_3)$，

由 $AX=\beta_i, i=1,2,3,$ 得 $A\begin{pmatrix}-6\\2\\1\end{pmatrix}=0$， $A\begin{pmatrix}3\\-1\\0\end{pmatrix}=\beta_1$， $A\begin{pmatrix}4\\-1\\0\end{pmatrix}=\beta_2,\beta_3$.

则 $X=\begin{pmatrix}3-6k_1&4-6k_2&4-6k_3\\-1+2k_1&-1+2k_2&-1+2k_3\\k_1&k_2&k_3\end{pmatrix}$ 为所求 P 的通解

由 $|X|\neq 0$，应有 $k_2\neq k_3$.

46. Find all the values of t such that the following quadratic form

$$q=tx_1^2+2x_1x_2+tx_2^2-2x_2x_3+x_3^2$$

is positive definite.

Sol: (1) $t>0$, (2) $t>1, t<-1$, (3) $t>\frac{1+\sqrt{5}}{2}$

47. Let $A=(a_{ij})_{3\times 3}$ be a real 3×3 matrix satisfying

(1) $a_{ij}=-A_{ij}$, where A_{ij} is the algebraic cofactor of a_{ij};

(2) $A_{11}\neq 0$.

Then find $\det A$, the determinant of the matrix A.

Sol: From $a_{ij}=-A_{ij}$, we know that $A^*=-A^T$.

From $A_{11}\neq 0$, we know that $a_{11}\neq 0$ and therefore,

$$|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}=-(a_{11}{}^2+a_{12}{}^2+a_{13}{}^2)<0$$

From $AA^*=|A|I$, we have $-AA^T=|A|I$.

Compute the determinant on both sides, we have

$-|A|^2=|A|^3$. Therefore, $|A|=-1$.

48. It is known the systems of linear equations

$\begin{cases} x_1+2x_2+3x_3=0 \\ 2x_1+3x_2+5x_3=0 \\ x_1+x_2+ax_3=0 \end{cases}$ and $\begin{cases} x_1+bx_2+cx_3=0 \\ 2x_1+b^2x_2+(c+1)x_3=0 \end{cases}$ have same solutions. Find a,b,c.

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 1 & a \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -1 & a-3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & a-2 \end{bmatrix}.$$

It is easily to see that the system (II) has only one or two nonzero rows after row operation and therefore has infinitely solution. Furtherly the system (I) should have infinitely solutions, therefore $a=2$.

As $a=2$

$$\begin{bmatrix}1&2&3\\2&3&5\\1&1&a\end{bmatrix}\sim\begin{bmatrix}1&2&3\\0&-1&-1\\0&-1&a-3\end{bmatrix}\sim\begin{bmatrix}1&2&3\\0&1&1\\0&0&a-2\end{bmatrix}\sim\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}$$

The solution of (I) can be denoted as $\begin{bmatrix}-1\\-1\\1\end{bmatrix}t, t\in R$

Replacing this solution to (II)

$$\begin{cases}-t-bt+ct=0\\-2t-b^2t+(c+1)t=0\end{cases} \text{ or } \begin{cases}-1-b+c=0\\-2-b^2+c+1=0\end{cases}$$

Therefore $\begin{cases}b=0\\c=1\end{cases} or \begin{cases}b=1\\c=2\end{cases}$

However as $\begin{cases}b=0\\c=1\end{cases}$, the coefficient matrix of system (II) has rank 1 and therefore it is conflicted with the solution of (I).

Thus the final solution is $\begin{cases}b=1\\c=2\end{cases}$. We can also testify the solution of (I) and (II) are the same as $\begin{cases}b=1\\a=2\\c=2\end{cases}$.

49. Let $\alpha_1=(a,2,10)^T,\alpha_2=(-2,1,5)^T,\alpha_3=(-1,2,4)^T$, and $\beta=(1,b,c)^T$. What conditions the parameters a,b,c satisfying make

(1) β can be linearly expressed by $\alpha_1,\alpha_2,\alpha_3$ uniquely?

(2) β can not be linearly expressed by $\alpha_1,\alpha_2,\alpha_3$?

(3) β can be linearly expressed by $\alpha_1,\alpha_2,\alpha_3$, but not uniquely, in this

case find the general expression.

Solition:

$$\begin{pmatrix} a & -2 & -1 & 1 \\ 2 & 1 & 2 & b \\ 10 & 5 & 4 & c \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 2 & b \\ a & -2 & -1 & 1 \\ 10 & 5 & 4 & c \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 2 & b \\ 0 & -2-\frac{1}{2}a & -1-a & 1-\frac{1}{2}ab \\ 0 & 0 & -6 & c-5b \end{pmatrix}$$

(1) $a \neq -4$, β can be linearly expressed by $\alpha_1,\alpha_2,\alpha_3$ uniquely;

(2) $a=-4, c-b+2\neq 0$, β can not be linearly expressed by $\alpha_1,\alpha_2,\alpha_3$;

(3) $a=-4, c-b+2=0$, β can be linearly expressed by $\alpha_1,\alpha_2,\alpha_3$, but not uniquely, i.e.,

$$\beta=\left[(-\frac{1}{3}-\frac{1}{6}b)-t\right]\alpha_1+2t\alpha_2+(\frac{1}{3}+\frac{2}{3}b)\alpha_3 \quad (t \text{ for any real number}).$$

general linear expression of β by $\alpha_1,\alpha_2,\alpha_3$.

50. Suppose the systems of linear equations

$$\begin{cases} x_1+x_2+x_3=0 \\ x_1+2x_2+ax_3=0 \\ x_1+4x_2+a^2x_3=0 \end{cases}$$ and $x_1+2x_2+x_3=a-1$ have common solutions. Then find the values of a and all the common solutions.

Sol: By the assumption**,** the system of linear equations

$$\begin{cases} x_1+x_2+x_3=0 \\ x_1+2x_2+ax_3=0 \\ x_1+4x_2+a^2x_3=0 \\ x_1+2x_2+x_3=a-1 \end{cases}$$ has solution. Consider

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & a & 0 \\ 1 & 4 & a^2 & 0 \\ 1 & 2 & 1 & a-1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & a-1 & 0 \\ 0 & 3 & a^2-1 & 0 \\ 0 & 1 & 0 & a-1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & a-1 & 0 \\ 0 & 0 & a^2-1 & -3(a-1) \\ 0 & 1 & 0 & a-1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & a-1 & 0 \\ 0 & 0 & a^2-1 & -3(a-1) \\ 0 & 0 & -(a-1) & (a-1) \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & a-1 & 0 \\ 0 & 0 & 0 & (a-2)(a-1) \\ 0 & 0 & a-1 & -(a-1) \end{pmatrix}$$

It is easy to see that $a=1$, the two system have the common solutions $t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$; $a=2$, the two system have only one common solutions $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

51. 设 A 是一个 $m\times n$ 矩阵，证明：$AX=b$ 有解当且仅当对任意 m 维列向量 Z，只要 $A^TZ=0$ 就有 $b^TZ=0$

Sol: "$\Rightarrow$" 若有 ξ 使得 $AX=b$，即 $A\xi=b, Z^TA=0$ 必有 $Z^Tb=0$。

"$\Leftarrow$" 若 $A^TZ=0(Z^TA=0)$，Z 必为零向量，A 必为行满秩矩阵，此时对任意 b，$AX=b$ 有解．若 Z 可为非零向量，取 $A^TX=0$ 的一个基

础解系作为 Z（此时 Z 可以是一个列数大于 1 的矩阵，设A的秩为r,Z 的秩为$m-r$）并考虑$Z^T X=0$. 此时A是$Z^T X=0$

一个基础解系，但b也是它的一个解. b可由的A列向量线性表出，$AX=b$有解.

52. 线性方程组

$$\begin{cases} a_{11}x_1+a_{11}x_2+\ldots+a_{11}x_n=b_1 \\ \vdots \\ a_{s1}x_1+a_{s2}x_2+\ldots+a_{sn}x_n=b_s \end{cases}$$

对任意常数$b_1,\ldots,b_s$ 都有解的充分必要条件是它的系数矩阵的秩为 $\boldsymbol{S}$.

53. 证明：$\boldsymbol{r(ABC)\geq r(AB)+r(BC)-r(B)}$.

54. 设$\boldsymbol{n}$阶方阵$\boldsymbol{A,B}$满足$\boldsymbol{(A+B)^2=A+B}$,且$\boldsymbol{r(A+B)=r(A)+r(B)}$,证明：$\boldsymbol{A^2=A,B^2=B,AB=BA=0}$.

55. 设$\boldsymbol{A,B}$都是$\boldsymbol{n}$阶方阵，且$\boldsymbol{A+B=AB}$，证明：$\boldsymbol{I-A}$可逆，且$\boldsymbol{AB=BA}$.