Chapter Four

（Chapter Four   Discrete-time Markov chain）

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Chapter 4 Discrete-time Markov chain

- Introduction of Discrete-time Markov chain
- Chapman-Kolmogorov equation
- Classification of states
- Periodic and Ergodic Markov chains
- Absorbing Markov chains

Course Objective

- What is discrete-time Markov chain
  - ✓ Markov property
  - ✓ Understanding the classification of state
- How to calculate the limiting probability
  - ✓ What is ergodic Markov chain
  - ✓ How to calculate the limiting probability

4.1 Introduction of Discrete-time Markov chain

- **Definition**

$X = \{X_n,\ \ n = 0,1,2 \dots\}$ where X_n denotes the state of the system at time n. X_n takes on values in the set S — state space. $S = \{0,1,2 \dots\}$.

The stochastic process X is a discrete-time Markov chain with state space S if

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0\} = P\{X_{n+1} = j | X_n = i\}$$

Eq.4-1-1

holds for all $i, j, i_0, i_1, \dots, i_{n-1}$ in S and all $n = 0,1, \dots$

4.1 Introduction of Discrete-time Markov chain

- **Transition probability:**
  In a Markov chain, the n-step transition probability is defined by

  $$p_{ij}^{(n)} = P\{X_{n+m} = j \mid X_m = i\} \qquad \text{Eq.4-1-2}$$

  This is the probability that the process goes from state i to state j in n transitions.

- **Transition probability matrix:**
  The matrix containing all transition probabilities

  $$\boldsymbol{P} = \{p_{ij}\}$$

4.1 Introduction of Discrete-time Markov chain

- **Example 4.1.1**

A Mouse in a Maze A mouse starts from cell I in the maze shown in Figure 4.1. A cat is hiding patiently in cell 7, and there is a piece of cheese in cell 9. In the absence of learning, when the mouse is in a given cell, it will choose the next cell to visit with probability . Where k is the number of adjoining cells. Assume that once the mouse finds either the piece of cheese or the cat, it will understandably stay there forever.

4.1 Introduction of Discrete-time Markov chain

- **Example 4.1.1:**

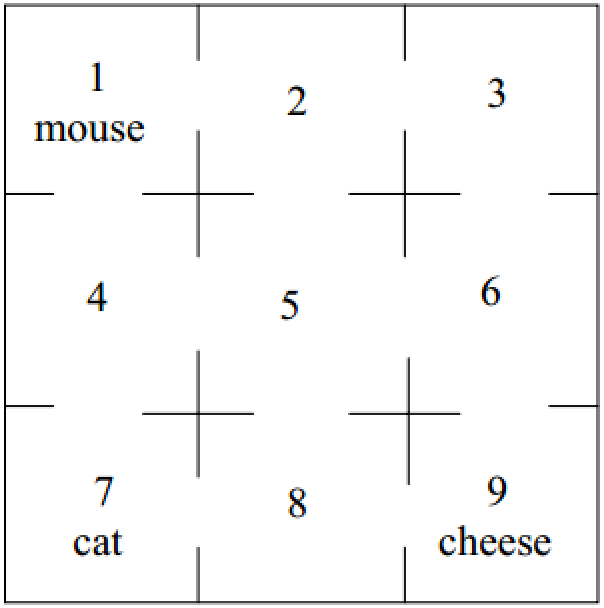

Figure 4.1

4.1 Introduction of Discrete-time Markov chain

- **Time homogeneous Markov chain**

  For all n, and all i and j in S, if

$$P\{X_{n+1} = j \mid X_n = i\} = p_{ij}^{(1)}$$

then the Markov chain is said to be time homogeneous

In matrix form

$$\begin{cases} P_{ij}^{(1)} = P_{ij} \\ P_{ij}^{(0)} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{cases} \xrightarrow{P^{(n)} = \{P_{ij}^{(n)}\}} \begin{cases} P^{(1)} = P \\ P^{(0)} = I \end{cases}$$

4.2 Chapman-Kolmogorov Equation

- **Chapman-Kolmogorov equations**

  Provides a method for computing the n-step transition probabilities.

$$p_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}$$

Proof:

$$p_{ij}^{(n+m)} = P\{X_{n+m} = j \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j, X_n = k \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j \mid X_n = k, X_0 = i\} P\{X_n = k \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}$$

In matrix form: $P^{(n+m)} = P^{(n)} P^{(m)} \Longrightarrow P^{(n)} = P^{n}$

4.3 Classification of state

- **Accessible i→ j**

  If for some $n \geq 0$, $p_{ij}^{(n)} > 0$, state j is said to be accessible from I

- **Communicate i↔ j**

  If i→ j and j→ i, we say that i and j communicate

- Properties :

  Reflexive: $i \leftrightarrow i$, all $i \geqslant 0$

  Symmetric: $i \leftrightarrow j$, $j \leftrightarrow i$

  Transitive: $i \leftrightarrow j$, $j \leftrightarrow k \Rightarrow i \leftrightarrow k$

4.3 Classification of state

- **Closed class**
- If i $\leftrightarrow$ j communicate, we say i and j in the same **class**.
- If $\forall i \in C,\ j \notin C,\ \mathrm{p}_{ij}^{(\mathrm{n})} = 0, n \geq 1$, C is called **closed class**.
- If only one state i in a closed class, then i is **absorbing state**
- If the only closed class of a Markov chain is the set of states in its state space S, the Markov chain is **irreducible**.

4.3 Classification of state

- Example 4.3.1

Consider a Markov chain with transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ . \\ . \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & . & . & N-1 & N \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$p + q = 1$$

4.3 Classification of state

- Example 4.3.1

Analyze the states in the Markov chain.

Where $p+q=1$ and $p>0$ and $q>0$. The chain is sometimes called a random walk with two absorbing barriers. We see that the chain has three classes $\{0\}$,$\{N\}$ and $\{1,\ldots,N-1\}$,and $\{0\}$ and $\{N\}$ are two closed classes.

4.3 Classification of state

- Example 4.2.2

Consider the Markov chain consisting of the three states 0,1,2 and having transition probability matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

Analyze the states in the Markov chain.

Solution: All the states communicate, an irreducible Markov chain.

4.3 Classification of state

- **First passage time and first-time-passage probability**

T_{ij} :first passage time from state i to j.

$f_{ij}^{(n)}$: first-time-passage probability into state j from i in n steps

$$f_{ij}^{(\mathrm{n})} = P\left\{T_{ij} = n \mid X_0 = i\right\} = P\left\{X_n = j, X_{n-1} \neq j, \ldots, \mathrm{X}_1 + j \mid X_0 = i\right\} \quad n = 1, 2, \ldots$$

Conditioning on $X_0 = i$, $f_{ij}(n)$ is calculated recursively:

$$f_{ij}^{(1)} = p_{ij}, \quad f_{ij}^{(\mathrm{n})} = \sum_{\substack{k=0 \\ k \neq j}}^{\infty} p_{ik} f_{kl}^{(\mathrm{n}-1)}$$

4.3 Classification of state

- **Recurrent and transient state**

T_{jj} : the recurrence time of state j.

$\mu_j = E\left[T_{jj}\right]$: the mean recurrence time of state j

f_{ij} : the reaching probability from state i to state j

$$f_{ij} = P\left\{T_{ij} < \infty\right\} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

State j is recurrent : if $f_{jj} = 1$, i.e. $P\left\{T_{jj} < \infty\right\} = 1$

State j is positive recurrent : if state j is recurrent and $\mu_j < \infty$.

state j is null recurrent : if state j is recurrent and $\mu_j \to \infty$.

State j is called transient : if $f_{jj} < 1$, i.e. $P\left\{T_{jj} < \infty\right\} < 1$

4.3 Classification of state

Corollary:

a) If state i is recurrent, and $i \leftrightarrow j$, then state j is recurrent.

b) If state i is recurrent, and $i \rightarrow j$, then $j \rightarrow i$

4.3 Classification of state

- **Example4.2.3:**
  Let the Markov chain consisting of the states 0,1,2,3 have the transition probability matrix, determine the recurrent states

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution:

Finite chain, irreducible

All states communicate

All states must be recurrent

4.3 Classification of state

- **Decomposition of state space**

  Define:

  T(i):TO-LIST, the set of all states that are accessible from i

  F(i): FROM-LIST, the set of states from which state i is accessible

  C(i): the class containing state i

  $C(i) = T(i) \cap F(i)$, if $C(i) = T(i)$, then C(i) is closed class

  E: closed class T: unclosed class

4.3 Classification of state

- **Procedure of decomposition:**

(i) For each state i, list T(i) and F(i), find C(i)

(ii) List closed class E and unclosed class T

(iii) Write the transition probabilities in the order E1, E2···. and set T

(iv) Combine states in each closed class into one state, that is make each closed class an absorbing state

(v) Write the canonical form of transition probability matrix

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$$

4.3 Classification of state

- **Periodic and aperiodic Markov chains**

  For state i, let d(i) denote the greatest common divisor of all integers $n \geqslant 1$ for which Then the integer **d(i)** is called the ***period of state i***.

  When a Markov chain is **irreducible, positive recurrent** and **periodic** with period d, we call it a **periodic Markov chain**.

- A state with a period of 1 is called **aperiodic.**

  An irreducible Markov chain whose states have a period of 1 is called an **aperiodic Markov chain**.

4.3 Classification of state

- **Example 4.3.1**

Consider a Markov chain with transition probability matrix

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

Find the period of states.

Solution:

Irreducible positive recurrent and periodic, d(i)=2

4.4 Ergodic Markov chains

- **Ergodic and ergodic Markov chains**
  Positive recurrent, aperiodic ---- **ergodic**.
  When a Markov chain is irreducible, positive recurrent and aperiodic, the chain is called an **ergodic Markov chain**.

4.4 Ergodic Markov chains

- **Limiting probability**

  For irreducible ergodic Markov chain, $\lim_{n\to\infty} p_{ij}^{(n)}$ exist and is independent of i. The limiting probability is denoted by

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)} \qquad j > 0$$

π_j :limiting distribution or stationary distribution

The limiting probability $\{\pi_j\}$ is the unique nonnegative solution of:

$$\text{(i)}\pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij}, \quad \text{j} \geq 0$$

$$\text{(ii)}\sum_{j=0}^{\infty} \pi_j = 1$$

Matrix form

$$\text{(i)}\pi = \pi P$$

$$\text{(ii)}\pi e = 1$$

4.4 Ergodic Markov chains

- Mean recurrence time to state j : $E[Tij] = \mu_{jj}$

$$\mu_{jj} = \frac{1}{\pi_j}$$

$$\begin{cases} if \ \ \mu_{jj} < \infty, \ state \ j \ is \ positive \ recurrent \\ if \ \ \mu_{jj} \to \infty, \ state \ j \ is \ null \ recurrent \end{cases}$$

$$\begin{cases} if \ \ \pi_j > 0, \ state \ j \ is \ positive \ recurrent \\ if \ \ \pi_j = 0, \ state \ j \ is \ null \ recurrent \end{cases}$$

4.4 Ergodic Markov chains

- **Example 4.4.2**

Consider a Markov chain with state space {0,1,2} having the transition probability matrix

$$P = \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.5 & 0.49 \end{bmatrix}$$

Find the limiting probabilities.

4.4 Ergodic Markov chains

- Solution:

$$\pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij}$$

$$\begin{cases} \pi_0 = 0.45\pi_0 + 0.05\pi_1 + 0.01\pi_2 \\ \pi_1 = 0.48\pi_0 + 0.70\pi_1 + 0.50\pi_2 \\ \pi_2 = 0.07\pi_0 + 0.25\pi_1 + 0.49\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

And finally we got $\begin{cases} \pi_0 = 0.07 \\ \pi_1 = 0.62 \\ \pi_2 = 0.31 \end{cases}$

4.4 Ergodic Markov chains

- **Example 4.3.3:**
  Consider a Markov chain with state space $S = \{0,1,\cdots\}$ and transition probability matrix P given by

$$
. \begin{bmatrix} q & p & 0 & . & . & . \\ q & 0 & p & 0 & . & . \\ 0 & q & 0 & p & 0 & . \\ . & 0 & q & 0 & p & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}
$$

  Where p>0, q>0, q>p, and p+q =1. Find the limiting probabilities

4.5 Absorbing Markov chains

- **Definition of absorbing Markov Chains**

  A Markov chain whose transition probability matrix can be written in the canonical form is called an absorbing Markov chain.

$$P = \begin{array}{c c} & \begin{array}{cc} T^c & T \end{array} \\ \begin{array}{c} T^c \\ T \end{array} & \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \end{array}$$

n-step transition matrix in canonical form:

$$P^{(n)} = P^n = \begin{array}{c c} & \begin{array}{cc} T^c & T \end{array} \\ \begin{array}{c} T^c \\ T \end{array} & \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \end{array}$$

Where $R_n = \left(I + Q + Q^2 + \ldots + Q^{n-1}\right)R$

4.5 Absorbing Markov chains

- **Definition of absorbing Markov Chains**

  When n approaches infinity, Q^n converges to a zero matrix.

  Mean number of visits to transient state j from transient state i before absorption.

4.5 Absorbing Markov chains

- **Calculation of the fundamental matrix**

Define the fundamental matrix of an absorbing Markov chain by

$W = I + Q + Q^2 + \ldots\ldots$

When the state space S is finite, we have $W = (I - Q)^{-1}$.

Proposition:

Let $i \in T$, $j \in T$, the elements in W are expressed as w_{ij}, then $w_{ij} = E\left[\tau_{ij}\right]$ where τ_{ij} represents the number of visits to state j before absorption given $X_0 = i$.

4.5 Absorbing Markov chains

- **Calculation of the fundamental matrix**

Proof:

Define an indicator function D, $D_{ij}^{(\mathrm{n})} = \begin{cases} 1 & \text{if } i \xrightarrow{n} j \\ 0 & \text{otherwise} \end{cases}$

$$E\left[\tau_{ij}\right] = E\left[\sum_{n=0}^{\infty} D_{ij}^{(\mathrm{n})}\right] = \sum_{n=0}^{\infty} E\left[D_{ij}^{(n)}\right]$$

$$= \sum_{n=0}^{\infty}\left[1 \cdot p_{ij}^{(n)} + 0 \cdot \left(1 - p_{ij}^{(n)}\right)\right]$$

$$= \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

$$= q_{ij}^{(0)} + \sum_{n=1}^{\infty} q_{ij}^{(n)}$$

$$= w_{ij}$$

4.5 Absorbing Markov chains

- **Calculation of the fundamental matrix**

  The fundamental matrix W states the mean numbers of visiting transient state j from transient state i before enter absorbing state. Mean number of visits to all transient states from transient state i before absorption

  τ_i : the number of visits to all transient states given $X_0 = i, \ i \in T$

  $E[\tau_i]$: the mean number of visits to all transient states from state i.

$$E[\tau_i] = \sum_{j=1}^{\infty} w_{ij}$$

4.5 Absorbing Markov chains

- **Calculation of the fundamental matrix**

The summation over ith row of the fundamental matrix W is the mean number of visits to all transient states from transient state i before enter absorbing state.

Probabilities from transient state i to absorbing state j.

$f_{ij}^{(n)}$: probability that the first passage from i to j occurs at the nth transition

In matrix form: $F^{(n)} = \left\{ f_{ij}^{(n)} \right\}$

4.5 Absorbing Markov chains

- **Calculation of the fundamental matrix**

  If $i \in T$, and $j \in T_c$, $F^{(n)} = Q^{n-1}R \qquad n \geq 1$

  $$F = \sum_{n=1}^{\infty} F^{(n)} = \sum_{n=1}^{\infty} Q^{n-1}R$$

  Limiting probabilities of absorbing Markov chain in canonical form:

  $$\lim_{n \to \infty} P^{(n)} = \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix}$$

4.5 Absorbing Markov chains

- **The variance of** τ_{ij}

  For i, j in T, we recall that τ_{ij} denotes the number of visits to state j before absorption given that $X_0 = i$. We now consider the computation of the variance of τ_{ij} for an absorbing Markov chain with a finite State space S.

$$Var[\tau_{ij}] = \mathrm{E}[\tau_{ij}^2] - \mathrm{E}^2[\tau_{ij}] = \mathrm{E}[\tau_{ij}^2] - W_{ij}^2$$

τ_{ij}

So next, we first derive an expression for the second moment of

.

4.5 Absorbing Markov chains

- **The variance of** τ_{ij}

We observe that, for i and j in T,

$$\tau_i^2 = \begin{cases} 1 & \text{with probability } p_{ik}, \text{ where } k \in T^c \\ [1+\tau_k]^2 & \text{with probability } p_{ik}, \text{ where } k \in T \end{cases}$$

It follows that

$$E[\tau_i^2] = \sum_{k \in T^c} p_{ik} + \sum_{k \in T} p_{ik} E[1+\tau_{kj}]^2 = \sum_{k \in S} p_{ik} + 2\sum_{k \in T} p_{ij}\delta_{ij} E[\tau_{kj}] + \sum_{k \in T} p_{ik} E[\tau_{kj}^2]$$

$$= 1 + 2\delta_{ij} \sum_{k \in T} p_{ij} E[\tau_{kj}] + \sum_{k \in T} p_{ik} E[\tau_{kj}^2]$$

4.5 Absorbing Markov chains

- **The variance of τ_{ij}**

Let column vector $\{\mathrm{h} = \mathrm{E}[\tau_i^2]\}$, then $h = e + 2Qv + Qh \; or \; h = W(2\mathrm{Qv} + \mathrm{e})$

Using the identity we obtain

$$h = 2(\mathrm{W} - \mathrm{I})\,\mathrm{v} + \mathrm{v} = (2\,\mathrm{W} - \mathrm{I})\,\mathrm{v}$$

Let column vector $v_\tau = \{\mathrm{Var}[\tau_i]\}$. Then the column vector containing the variances is given by $v_\tau = h - v\square.v$

4.5 Absorbing Markov chains

- **Discrete phase-time distribution**

Consider an absorbing Markov chain with state space $S = \{1, \ldots, \mathrm{m}, \mathrm{m}+1\}$ where state m+1 is absorbing and all other states are transient. Let the starting state probability vector s(0) be denoted by (α, α_{m+1}), where $\alpha = (\alpha_1, \ldots, \alpha_m)$

The transition probability matrix is stated in a canonical form:

$$P = \begin{bmatrix} Q & r \\ 0 & 1 \end{bmatrix}$$

Where r is a column vector of size m and Q is an $m \times m$ substochastic matrix.

4.5 Absorbing Markov chains

- **Discrete phase-time distribution**

Let τ be the time until absorption into state m + 1.

We find the probability mass function $p_k = P\{\tau = \mathrm{k}\}$ as follows:

$$p_0 = \alpha_{m+1}$$

$$p_k = \alpha Q^{k-1} r \qquad k = 1, 2, \ldots$$

The given probability distribution $\{\mathrm{p}_k\}$ is called the discrete phase-type distribution with representation (α, Q) and order m.

4.5 Absorbing Markov chains

- **Discrete phase-time distribution**

The probability generating function of the discrete phase-type is given by

$$P_\tau(\mathrm{z}) = \sum_{k=0}^{\infty} z^k p_k = \alpha_{m+1} + \sum_{k=1}^{\infty} z^k \alpha Q^{k-1} = \alpha_{m+1} + z\alpha \sum_{k=1}^{\infty} z^{k-1} Q^{k-1} r$$

$$= \alpha_{m+1} + z\alpha \sum_{k=0}^{\infty} (\mathrm{zQ})^k r = \alpha_{m+1} + z\alpha[\mathrm{I} - z\,\mathrm{Q}]^{-1} r$$

Differentiating the probability generating function with to z and using the identity

$$\frac{d}{dz} X^{-1} = -X^{-1}\left(\frac{d}{dz}\mathrm{X}\right)\mathrm{X}^{-1}$$

4.5 Absorbing Markov chains

- **Discrete phase-time distribution**

(we use X to denote X(z) for notational convenience), we find

$$\frac{d}{dz}P_\tau(\mathrm{z}) = \alpha\left[I - zQ\right]^{-1} r + z\alpha\left[-(\mathrm{I}-\mathrm{zQ})^{-1}\frac{d}{dz}[\mathrm{I}-\mathrm{zQ}]\left(I - zQ\right)^{-1}\right]r$$

$$= \alpha\left[I - zQ\right]^{-1} r + z\alpha\left[(\mathrm{I}-\mathrm{zQ})^{-1}\mathrm{Q}\left(I - zQ\right)^{-1}\right]r$$

Hence the first (factorial) moment is given by

$$E\left[\tau\right] = \frac{d}{dz}P_\tau\left(z\right)|_{z=1} = \alpha\left[I - Q\right]^{-1} + \alpha(\mathrm{I}-\mathrm{zQ})^{-1}\mathrm{Q}\left(I - zQ\right)^{-1} r$$

$$= \alpha\left[I - Q\right]^{-1}\left[I + Q\left(I - Q\right)^{-1}\right]r = \alpha\left[I - Q\right]^{-1}\left[I + Q + Q^2 + \cdots\right]r$$

$$= \alpha\left[I - Q\right]^{-1}\left[I - Q\right]^{-1} r = \alpha\left[I - Q\right]^{-1} e$$

Higher-order factorial moments can be found similarly. They are

$$P_\tau^{(\mathrm{k})} = k!\alpha Q^{k-1}\left[I - Q\right]^{-k} e \qquad k = 1, 2, \ldots$$

4.5 Absorbing Markov chains

- **Discrete phase-time distribution**

Finding moments of a discrete random variable typically involves summations of infinite numbers of terms. When the random variable is of type, we see that the operations reduce to simple matrix manipulations.

4.5 Absorbing Markov chains

- **Discrete phase-time distribution**

Exercise : The Coupon Collection Problem

Consider a football team that has fifty-two players on its roster. The team offers the following promotional campaign. Every box of cereal sold in the city contains a coupon the picture of one of its players. A person who collects a complete set of coupons (which contains at least one of each type) receives an award enabling him or her to attend all of the team's home games for free in the forthcoming season. Assume that each time one opens a box of cereal it is equally likely to one of the fifty-two types. Let T denote the number of coupons needed until one obtains a complete set. We are interested in E[T], Var[T], and the probability distribution of T.

4.5 Absorbing Markov chains

- **Discrete phase-time distribution**

Let X denote the number of additional coupons needed to obtain i_{th} distinct type given that the player has already amassed i-1 distinct types. For generality, we let m denote the number of distinct types of coupons (here we have m= 52). For $i = 1, 2, \ldots, m$, we see that $P\{X_i = n\} = (q_i)^{n-1} p_i, n = 1, 2, \ldots$ where

$$p_i = \frac{m - (i-1)}{m}$$

4.5 Absorbing Markov chains

- **Discrete phase-time distribution**

It is now obvious that T follows a generalized negative binomial distribution and has the phase-type representation described in Example 4.4.5. Using Equation 4.4.15, we the first two factorial moments of T. They in turn give E[T]=235.98 and a standard deviation of 64.5. With Equation 4.4.12, we compute the probability mass function for T in the Appendix. The function is shown in figure 4.13.

4.5 Markov Reward Processes

- **Definition of Gain rate**

Markov reward process: irreducible Markov chain in which each occupancy of state j generates a reward r_j , $0 \le r < \infty$.

Gain rate: the time average of the expected reward received per transition is a measure of rate associated with the reward generation process.

$$g = \sum_{j=0}^{\infty} r_j \pi_j$$

π_j :The long-run fraction of time the process is in state j.

4.5 Markov Reward Processes

- **Definition of Gain rate**

Suppose now that rewards received in future periods are discounted by a discount factor $0 < \alpha < 1$.

$$g = r + \alpha \Pr + (\alpha P)^2 r + \cdots = \sum_{k=0}^{\infty} (\alpha P)^k r = \sum_{k=0}^{\infty} Q^k r$$

Where $Q = \alpha P$, since $0 < \alpha < 1$, Q is a substochastic matrix.

When the state space is finite, we recall that $W = I + Q + Q^2 + \cdots$ and hence $g = (I - Q)^{-1} r$.

4.5 Markov Reward Processes

- **Definition of Reversible Discrete-Time Markov Chains**

A stochastic process $X = \{X(t), t \geq 0\}$ is reversible if $\{X(t_1), \ldots, X(t_n)\}$ follows the same distribution as $\{X(s-t_1), \ldots, X(s-t_n)\}$ for all $t_1, \ldots, t_n$ and s.

Let $X = \{X_n, n \geq 0\}$ be an irreducible stationary Markov chain With state space $S = \{0, 1, \ldots\}$ and transition probability matrix $P = \{p_{ij}\}$. If we construct a related process X^* by looking at the sample path of X in reversed order of the time axis, we see that

4.5 Markov Reward Processes

- **Definition of Reversible Discrete-Time Markov Chains**

$$P\{X_m = j \mid X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k\}$$

$$= \frac{P\{X_m = j, X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k\}}{P\{X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k\}}$$

$$= \frac{P\{X_{m+2} = i_2, \ldots, X_{m+k} = i_k \mid X_m = j, X_{m+1} = i\} P\{X_{m+1} = i \mid X_m = j\} P\{X_m = j\}}{P\{X_{m+2} = i_2, \ldots, X_{m+k} = i_k \mid X_{m+1} = i\} P\{X_{m+1} = i\}}$$

$$= \frac{P\{X_{m+2} = i_2, \ldots, X_{m+k} = i_k \mid X_{m+1} = i\} P\{X_{m+1} = i \mid X_m = j\} P\{X_m = j\}}{P\{X_{m+2} = i_2, \ldots, X_{m+k} = i_k \mid X_{m+1} = i\} P\{X_{m+1} = i\}}$$

$$= \frac{P\{X_{m+1} = i \mid X_m = j\} P\{X_m = j\}}{P\{X_{m+1} = i\}} = \frac{p_{ij}\pi_j}{\pi_i}$$

4.5 Markov Reward Processes

- **Definition of Reversible Discrete-Time Markov Chains**

Let $H=\{X_{m+1},\ldots,X_{m+k}\}$ then

$P\{X_m=j\mid X_{m+1}=i,H\}=P\{X_m=j\mid X_{m+1}=i\}=\dfrac{p_{ij}\pi_j}{\pi_i}$

That is the transition probability $\{p_{ij}^*\}$ of the reversed stochastic process X^* is

$$p_{ij}^*=\frac{p_{ij}\pi_j}{\pi_i} \qquad (1)$$

4.5 Markov Reward Processes

- **Reversible condition**

If the probability matrix $P^* = \{p_{ij}^*\} = P$ (or equivalently, $p_{ij}^* = p_{ij}$ for all i and j), we now argue that the Markov chain is reversible. Consider a sample path involving the observation of a sequence of states at epochs $\{j_0, j_1, \ldots, j_k\}$, $\{m, m+1, \ldots, m+k\}$.

The probability for observing this sample path is given by

$$P\{X_m = j_0, X_{m+1} = j_1, \ldots, X_{m+k} = j_k\} = \pi_{j0} p_{j_0, j_1} \cdots p_{j_{k-1}, j_k} \qquad (2)$$

Consider another sample of X involving the observation of the sequence $\{j_k, \mathrm{j}_{k-1}, \ldots, j_0\}$. Then corresponding probability is given by

$$P\{X_{m'} = j_k, X_{m'+1} = j_{k-1}, \ldots, X_{m'+k} = j_0\} = \pi_{jk} p_{j_k, j_{k-1}} \cdots p_{j_1, j_0} \qquad (3)$$

4.5 Markov Reward Processes

- **Reversible condition**

Using Equation (1) with for all i and j. By successive substitutions it can be shown that the right sides of Equations (2) and (3) are the same. Define $\tau = m + m^{'} + k$. We rewrite Equation (3) as

$$P\left\{X_{\tau-m-k} = j_k, X_{\tau-m-(k-1)} = j_{k-1}, \ldots, X_{\tau-m} = j_0\right\}$$

$$= P\left\{X_{\tau-m} = j_0, X_{\tau-m-1} = j_1, \ldots, X_{\tau-m-k} = j_k\right\}$$

Hence we have shown that $\left\{X_m, \ldots, X_{m+k}\right\}$ and $\left\{X_{\tau-m}, \ldots, X_{\tau-(m+k)}\right\}$ follow the same distribution and consequently the chain X is reversible.

4.5 Markov Reward Processes

- **Reversible condition**

Stationary Markov chain reversible condition

$$\pi_i p_{ij} = \pi_j p_{ij} \qquad \text{for all } i, j \in S$$

The preceding equations are called the derailed balance equations (see the balance equations $\pi = \pi P$).

Computationally the Markov chain is reversible and the $\{x_i\}$ obtained are stationary distributions of the chain, if we can solve the following system of linear equations:

$$x_i p_{ij} = x_j p_j \qquad \text{for all } i, j \quad \text{and} \quad \sum_{i \in S} x_i = 1 \qquad (5)$$