12. (4) $\int_0^{2\pi}\frac{(\sin 3\theta)^2}{1-2a\cos\theta+a^2}\,\mathrm{d}\theta,\ |a|<1$。

解： 当 $a\neq 0$ 时，有

$$\int_0^{2\pi}\frac{(\sin 3\theta)^2}{1-2a\cos\theta+a^2}\,\mathrm{d}\theta \overset{z=\mathrm{e}^{\mathrm{i}\theta}}{=} \oint_{|z|=1}\frac{\left(\frac{z^3-z^{-3}}{2\mathrm{i}}\right)^2}{1-2a\frac{z+z^{-1}}{2}+a^2}\frac{\mathrm{d}z}{\mathrm{i}z}$$

$$=-\frac{1}{4\mathrm{i}}\oint_{|z|=1}\frac{(z^6-1)^2}{z^6(z-az^2-a+a^2z)}\,\mathrm{d}z$$

$$=-\frac{1}{4\mathrm{i}}\oint_{|z|=1}\frac{(z^6-1)^2}{z^6(1-az)(z-a)}\,\mathrm{d}z$$

$$=-\frac{1}{4\mathrm{i}}\cdot 2\pi\mathrm{i}\left\{\mathrm{Res}\left[\frac{(z^6-1)^2}{z^6(1-az)(z-a)},a\right]+\mathrm{Res}\left[\frac{(z^6-1)^2}{z^6(1-az)(z-a)},0\right]\right\}$$

$$=-\frac{\pi}{2}\left\{\mathrm{Res}\left[\frac{(z^6-1)^2}{z^6(1-az)(z-a)},a\right]+\mathrm{Res}\left[\frac{(z^6-1)^2}{z^6(1-az)(z-a)},0\right]\right\}。$$

现在

$$\mathrm{Res}\left[\frac{(z^6-1)^2}{z^6(1-az)(z-a)},a\right]=\frac{(a^6-1)^2}{a^6(1-a^2)}=\frac{1}{a^6}(1-a^2)(1+a^2+a^4)^2。$$

又

$$\frac{(z^6-1)^2}{z^6(1-az)(z-a)}=\frac{z^{12}-2z^6+1}{z^6(1-az)(z-a)}$$

$$=\frac{z^6}{(1-az)(z-a)}-\frac{2}{(1-az)(z-a)}+\frac{1}{z^6(1-az)(z-a)}。$$

所以

$$\operatorname{Res}\left[\frac{\left(z^6-1\right)^2}{z^6(1-az)(z-a)},0\right]=\operatorname{Res}\left[\frac{1}{z^6(1-az)(z-a)},0\right]。$$

注意到

$$\begin{aligned}\frac{1}{z^6(1-az)(z-a)}&=\frac{1}{z^6}\left(1+az+a^2z^2+a^3z^3+\cdots\right)\left(-\frac{1}{a}\right)\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\cdots\right)\\&=\cdots-\frac{1}{a}\frac{1}{z^6}\left(\frac{1}{a^5}+\frac{a}{a^4}+\frac{a^2}{a^3}+\frac{a^3}{a^2}+\frac{a^4}{a^1}+a^5\right)z^5+\cdots\\&=\cdots-\left(\frac{1}{a^6}+\frac{1}{a^4}+\frac{1}{a^2}+1+a^2+a^4\right)\frac{1}{z}+\cdots,\qquad 0<|z|<|a|<1。\end{aligned}$$

故

$$\begin{aligned}\operatorname{Res}\left[\frac{\left(z^6-1\right)^2}{z^6(1-az)(z-a)},0\right]&=\operatorname{Res}\left[\frac{1}{z^6(1-az)(z-a)},0\right]\\&=-\left(\frac{1}{a^6}+\frac{1}{a^4}+\frac{1}{a^2}+1+a^2+a^4\right)\\&=-\frac{1}{a^6}\left(1+a^6\right)\left(1+a^2+a^4\right)。\end{aligned}$$

因此

$$\begin{aligned}\int_0^{2\pi}\frac{(\sin 3\theta)^2}{1-2a\cos\theta+a^2}\,\mathrm{d}\theta&=\frac{\pi}{2a^6}\left[\left(1+a^6\right)\left(1+a^2+a^4\right)-\left(1-a^2\right)\left(1+a^2+a^4\right)^2\right]\\&=\left(1+a^2+a^4\right)\pi。\end{aligned}$$

当 $a=0$ 时，有

$$\int_0^{2\pi}\left(\sin 3\theta\right)^2\ \mathrm{d}\theta \overset{z=\mathrm{e}^{\mathrm{i}\theta}}{=} \oint_{|z|=1}\left(\frac{\mathrm{z}^3-\mathrm{z}^{-3}}{2\mathrm{i}}\right)^2\frac{\mathrm{dz}}{\mathrm{iz}}$$

$$=-\frac{1}{4\mathrm{i}}\oint_{|z|=1}\frac{\left(\mathrm{z}^6-1\right)^2}{z^7}\ \mathrm{dz}$$

$$=-\frac{1}{4\mathrm{i}}\oint_{|z|=1}\left(\mathrm{z}^5-2\frac{1}{z}+\frac{1}{z^7}\right)\mathrm{dz}$$

$$=-\frac{1}{4\mathrm{i}}\oint_{|z|=1}\left(\mathrm{z}^5-2\frac{1}{z}+\frac{1}{z^7}\right)\mathrm{dz}$$

$$=-\frac{1}{4\mathrm{i}}\left(0-2\cdot 2\pi\mathrm{i}+0\right)=\pi。$$