Homework 2

Requirement

- The part marked optional is not compulsory; do it for your interest.
- Compile everything (including your derivations, answers, and source codes) to a pdf file. In the main text, provide your name, student ID, and your email address (marked assignment may be sent to your email). Name your pdf file in the form of "`StudentID_name_2nd_homework.pdf`".
- Deadline: 23:59, 27 Oct 2023.
- To be submitted individually.
- Submission method will be announced later.

Exercises

Exercise 1:

Recall the Dirichlet distribution that we learnt from Part 2b,

$$\mathrm{Dir}(\boldsymbol{p} \mid \boldsymbol{\alpha}) = \frac{1}{\Delta(\boldsymbol{\alpha})} \prod_{k=1}^{K} p_k^{\alpha_k - 1},$$

$$\text{where } \ \Delta(\boldsymbol{\alpha}) := \int_{\sum_k p_k = 1, p_k \geq 0} \prod_k p_k^{\alpha_k - 1} \mathrm{d}\boldsymbol{p} = \frac{\prod_{k=1}^{K} \Gamma(\alpha_k)}{\Gamma\left(\sum_{k=1}^{K} \alpha_k\right)}.$$

Questions:

1. Verify that the expectation of the component p_k of a Dirichlet random vector $\boldsymbol{p} \sim \mathrm{Dir}(\boldsymbol{p} \mid \boldsymbol{\alpha})$ is

$$\mathbb{E}(p_k) = \frac{\alpha_k}{\sum_{j=1}^{K} \alpha_j}.$$

(Hint: make use of the recurrence relation of the Gamma function: $\Gamma(x+1) = x\Gamma(x)$.)

2. Let $\tilde{\boldsymbol{n}} \in \mathbb{N}^K$, $\boldsymbol{\alpha}, \boldsymbol{\theta} \in \mathbb{R}^K$, and $\boldsymbol{\theta} \sim \mathrm{Dir}(\boldsymbol{\theta} \mid \boldsymbol{\alpha})$. Verify the following identity that we have used in LDA:

$$p(\boldsymbol{z} \mid \boldsymbol{\alpha}) = \int_{\sum_k \theta_k = 1, \theta_k \geq 0} \mathrm{Dir}(\boldsymbol{\theta} \mid \boldsymbol{\alpha}) \left(\prod_{k=1}^{K} \theta_k^{\tilde{n}_k} \right) \mathrm{d}\boldsymbol{\theta} = \frac{\Delta(\tilde{\boldsymbol{n}} + \boldsymbol{\alpha})}{\Delta(\boldsymbol{\alpha})}.$$

Exercise 2:

(Based on Exercise 3.15 of BRML(Barber, 2020))

A belief network models the relation between the variables *oil*, *inf*, *eh*, *bp*, *rt* which stand for the price of oil, inflation rate, economy health, British Petroleum Stock price, retailer stock price. Each variable takes the states "low", "high", except for bp which has states "low", "high", "normal". The belief network model for these variables has tables

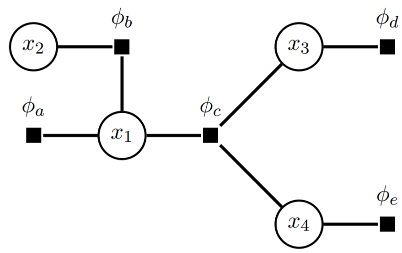
$p(eh = \text{low}) = 0.2$	
$p(bp = \text{low} \mid oil = \text{low}) = 0.9$	$p(bp = \text{normal} \mid oil = \text{low}) = 0.1$
$p(bp = \text{low} \mid oil = \text{high}) = 0.1$	$p(bp = \text{normal} \mid oil = \text{high}) = 0.4$
$p(oil = \text{low} \mid eh = \text{low}) = 0.9$	$p(oil = \text{low} \mid eh = \text{high}) = 0.05$
$p(rt = \text{low} \mid inf = \text{low}, eh = \text{low}) = 0.9$	$p(rt = \text{low} \mid inf = \text{low}, eh = \text{high}) = 0.1$
$p(rt = \text{low} \mid inf = \text{high}, eh = \text{low}) = 0.1$	$p(rt = \text{low} \mid inf = \text{high}, eh = \text{high}) = 0.01$
$p(inf = \text{low} \mid oil = \text{low}, eh = \text{low}) = 0.9$	$p(inf = \text{low} \mid oil = \text{low}, eh = \text{high}) = 0.1$
$p(inf = \text{low} \mid oil = \text{high}, eh = \text{low}) = 0.1$	$p(inf = \text{low} \mid oil = \text{high}, eh = \text{high}) = 0.01$

Questions:

1. Write down the joint distribution of all the 5 variables.
2. Draw a belief network for this distribution.
3. Are oil price and retailer stock price independent? How many pathways are there between them? Under what conditions (observations) will they become independent?
4. Without any observation, what is the probability that the oil price is "low", and the probability that inflation is "low" (aka the marginal probabilities $p(oil = \text{low}), p(inf = \text{low})$)?
5. Convert the belief network into a factor graph, and write down the potential function of each factor constituting the joint distribution.
6. (Optional) Given that the BP stock price is "normal" and the retailer stock price is "high", what is the probability that inflation is high? If you find the calculation by hand is too tedious, you can resort to an algorithm. Below are some options:
   - For Matlab users, you can use the `BRMLtoolbox` provided by Prof. David Barber for BRML (the textbook and the software can be found at `http://web4.cs.ucl.ac.uk/staff/D.Barber/pmwiki/pmwiki.php?n=Brml.HomePage`). Take a look at the demos `demoBurglar.m` and `demoChestClinic.m` (descriptions of the problem can be found in Chap. 3 of the textbook) for a similar type of problem.
   - For Python users, the `pgmpy` package (`https://pgmpy.org/index.html`) seems useful.
   - Run (loopy) belief propagation on the corresponding factor graph. When loops are present, you may need to iteratively passing the messages for many rounds until convergence.
   - Other tools that you find useful.

Exercise 3:

The following factor graph represents a probability distribution $p(\boldsymbol{x}) = \frac{1}{Z}\prod_{m\in\{a,b,c,d,e\}} \phi_m(\underline{\boldsymbol{x}}_m)$ over four binary variables $x_i \in \{0,1\}$:



The factors $\phi_a, \phi_b, \phi_d, \phi_e$ are defined as follows:

x_1	ϕ_a
0	2
1	1

x_1	x_2	ϕ_b
0	0	5
1	0	2
0	1	2
1	1	6

x_3	ϕ_d
0	1
1	2

x_4	ϕ_e
0	1
1	3

and $\phi_c(x_1, x_3, x_4) = 1$ if $x_1 = x_3 = x_4$, and is zero otherwise. Intuitively, factor ϕ_a favors variable x_1 to be in the state $x_1 = 0$ (as $\phi_a(x_1 = 0) > \phi_a(x_1 = 1)$), ϕ_d favors $x_3 = 1$, ϕ_c favors x_1, x_3, x_4 to have the same value, etc.

Though the problem is of small size and exact inference is not difficult, we aim to apply the belief propagation (sum-product message-passing here) algorithm for solving it to improve our understanding.

Questions:

1. Initialize the messages from leaf nodes. How should we set the values of the following messages?
   - $\mu_{x_2\to\phi_b}(x_2)$, for $x_2 = 0$ and $x_2 = 1$,
   - $\tilde{\mu}_{\phi_a\to x_1}(x_1)$, for $x_1 = 0$ and $x_1 = 1$,
   - $\tilde{\mu}_{\phi_d\to x_3}(x_3)$, for $x_3 = 0$ and $x_3 = 1$,
   - $\tilde{\mu}_{\phi_e\to x_4}(x_4)$, for $x_4 = 0$ and $x_4 = 1$.
2. Choose x_1 as the root node (this is arbitrary), pass the messages one step inward by computing the values of
   - $\tilde{\mu}_{\phi_b\to x_1}(x_1)$, for $x_1 = 0$ and $x_1 = 1$,
   - $\mu_{x_3\to\phi_c}(x_3)$, for $x_3 = 0$ and $x_3 = 1$,
   - $\mu_{x_4\to\phi_c}(x_4)$, for $x_4 = 0$ and $x_4 = 1$.
3. Computing the values of $\tilde{\mu}_{\phi_c\to x_1}(x_1)$, for $x_1 = 0$ and $x_1 = 1$.
4. Now you have all the (factor-to-variable) messages arriving at x_1, what is the marginal probability of $p(x_1 = 0)$? (Do not forget that $p(x_1)$ is normalized.)
5. To compute the marginal probabilities of x_2, x_3, x_4, we need to compute the messages outgoing from the root node x_1. For example, compute $\mu_{x_1\to\phi_c}(x_1)$ and then $\tilde{\mu}_{\phi_c\to x_3}(x_3)$. Based on the messages arriving at x_3, compute the marginal probability of $p(x_3 = 0)$.

6. (Optional) Can you use a similar message-passing procedure to compute the conditional probability $p(x_1|x_2 = 0)$?

   Hint: $p(x_1|x_2 = 0) = \frac{p(x_1,x_2=0)}{p(x_2=0)} \propto p(x_1, x_2 = 0) = \sum_{x_3,x_4} p(x_1, x_2 = 0, x_3, x_4)$. Apply belief propagation to the graphical model $\tilde{p}(x_1, x_3, x_4) := p(x_1, x_2 = 0, x_3, x_4)$, and normalize the obtained "marginal" $\tilde{p}(x_1)$ (this is the probability obtained by fixing $x_2 = 0$).

Exercise 4:
(By Prof. Jason Eisner)

Imagine that you are a climatologist studying the history of global warming. You cannot find any records of the weather in Baltimore for the summer of 2007, but you do find Jason Eisner's diary, which lists how many ice creams Jason ate every day that summer.

day #	1	2	3	4	5	6	7	8	9	10	11
ice cream	**2**	**3**	**3**	**2**	**3**	**2**	**3**	**2**	**2**	**3**	**1**
day #	12	13	14	15	16	17	18	19	20	21	22
ice cream	**3**	**3**	**1**	**1**	**1**	**2**	**1**	**1**	**1**	**3**	**1**
day #	23	24	25	26	27	28	29	30	31	32	33
ice cream	**2**	**1**	**1**	**1**	**2**	**3**	**3**	**2**	**3**	**2**	**2**

Your goal is to use these observations to estimate the temperature every day. You will simplify this weather task by assuming there are only two kinds of days: cold (C) and hot (H).

Given a sequence of observations $x_{1:N}$ (here $N = 33$), each observation $x_i \in \mathbb{N}$ corresponds to the number of ice creams eaten on a given day, the task is to infer the correct "hidden" weather states $z_i \in \{\mathrm{C}, \mathrm{H}\}$ which caused Jason to eat ice creams. Here assume that the weather condition is the only factor influencing Jason to eat ice creams, and that the weather conditions of consecutive days form a Markov chain. Therefore, you have a hidden Markov model (HMM)

$$p(x_{1:N}, z_{1:N}) = p(z_1) \prod_{n=2}^{N} p(z_n \mid z_{n-1}) \prod_{n=1}^{N} p(x_n \mid z_n).$$

Based on Jason's dairy and Baltimore's general summer weather conditions, you have an initial guess of the parameters:

$p(x_n \mid z_n)$	$z_n = \mathrm{C}$	$z_n = \mathrm{H}$
$x_n = 1$	0.7	0.1
$x_n = 2$	0.2	0.2
$x_n = 3$	0.1	0.7

$p(z_n \mid z_{n-1})$	$z_{n-1} = \mathrm{C}$	$z_{n-1} = \mathrm{H}$
$z_n = \mathrm{C}$	0.8	0.2
$z_n = \mathrm{H}$	0.2	0.8

Further assume that $p(z_1 = \mathrm{H}) = 0.5$.

Questions:

1. Write a computer program to solve the inference task of smoothing, i.e., to compute the univariate posterior $p(z_i \mid x_{1:N})$, $\forall i$.

   (Hint: use the forward-backward algorithm.)

2. Plot x_i and z_i versus time i, and interpret the results.

3. (Optional) Write a computer program to solve the inference task of decoding, i.e., to compute $\arg\max_{z_{1:N}} p(z_{1:N} \mid x_{1:N})$.

   (Hint: use the Viterbi algorithm.)