Chapter Five

（Chapter Five Continuous-Time Markov chain）

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Chapter 5 Continuous-Time Markov chain

- 5.1 Continuous-time Markov chain
- 5.2 Kolmogorov differential equations
- 5.3 Limiting probabilities
- 5.4 Absorbing continuous Markov chains
- 5.5 Phase-type distributions (optional)

5.1 Continuous-time Markov chain

- **Definition of Continuous-time Markov chain**

Consider a continuous-time stochastic process $X = \{X(t),\ t \geq 0\}$ with state space $S = \{0, 1, 2 \dots\}$ thus the process X is a continuous-time Markov chain if, for $i, j \in S$

$$P\{ X(t+s) = j \mid X(s) = i,\ X(u) = x(u), 0 \leq u < s\} = P\{X(t+s) = j \mid X(s) = i\}$$

for all s ⩾ 0, t ⩾ 0, and x(u), 0 ⩽ u < s.

5.1 Continuous-time Markov chain

- **Transition probability function**

Homogeneous continuous-time Markov chain

If the transition probability function $P\left\{X\left(t+s\right)=j \mid X\left(s\right)=i\right\}=P_{ij}\left(t\right)$ is independent of s.

Properties of transition probability function

$$P\left\{X\left(t+s\right)=j \mid X\left(s\right)=i\right\}=P_{ij}\left(t\right)$$

a) $P_{ij}\left(0\right)=\delta_{ij}$, $\delta_{ij}=\begin{cases} 1 & i=j \\ 0 & otherwise \end{cases}$ b) $P_{ij}\left(t\right)\geq 0$

c) $\sum_{j\in S} P_{ij}\left(t\right)=1$ d) $P_{ij}\left(t+s\right)=\sum_{k\in S} P_{ik}\left(t\right)P_{kj}\left(s\right)$

5.1 Continuous-time Markov chain

- **Transition probability function**

Over small interval h:

$$P_{ij}(h) = P_{ij}(0) + q_{ij}h + o(h) = \delta_{ij} + q_{ij}h + o(h)$$

at which $q_{ij} = \lim_{h\to 0} \frac{P_{ij}(\mathrm{h}) - \delta_{ij}}{h}$

q_{ij} is called the transition rate of moving from state i to state j.

Case 1: If $i = j$, $q_{ii} = \lim_{h\to 0} \frac{P_{ii}(\mathrm{h}) - 1}{h}$

Define $q_{ii} = -v_i$

v_i is the leaving rate associated with state i, then

$$v_i = \lim_{h\to 0} \frac{1 - P_{ii}(h)}{h}$$

5.1 Continuous-time Markov chain

- **Transition probability function**

Case 2: If $i \neq j$, $q_{ij} = \lim\limits_{h \to 0} \dfrac{P_{ii}(h)}{h}$ $\quad$ $q_{ij} = v_i p_{ij}$

5.1 Continuous-time Markov chain

- **Infinitesimal generator Q**

For all i, and form a matrix $Q = \{q_{ij}\}$

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ . \\ . \end{array} \begin{bmatrix} -v_0 & q_{01} & q_{02} & . & . \\ q_{10} & -v_1 & q_{12} & . & . \\ q_{20} & q_{21} & -v_2 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

Properties of Q matrix:

(1) $q_{ij} > 0, (i \neq j)$, $q_{ii} <= 0$;

(2) the row sums are zeros

The sojourn time of X in state i follows the exponential distribution with parameter v_i mean= $1/v_i$.

5.1 Continuous-time Markov chain

- **Birth and death processes**

A continuous-time Markov chain is called a birth and death process if state space $S = \{0,1\ldots\}$ and $q_{ij} = 0$ if $|i-j| > 1$

when the process is in state i, the transition rates are

$$q_{i,i+1} = \lambda_i \qquad i = 0,1,2\ldots$$

$$q_{i,i-1} = \mu_i \qquad i = 1,2,3\ldots$$

$$q_{ij} = 0 \quad \text{, otherwise}$$

$\{\lambda_i\}$ are called the birth rates and $\{\mu_i\}$ are called death rate.

5.1 Continuous-time Markov chain

- **Birth and death processes**

$$Q = \begin{bmatrix} -v_0 & \lambda_0 & 0 & 0 & . & . \\ \mu_1 & -v_1 & \lambda_1 & 0 & . & . \\ 0 & \mu_2 & -v_2 & \lambda_2 & 0 & . \\ 0 & 0 & \mu_3 & -v_3 & \lambda_3 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}$$

$$v_i = \lambda_i + \mu_i,\ p_{i,i+1} = {\lambda_i}/{v_i},\ p_{i,i-1} = {\mu_i}/{v_i},\ i > 0 \Rightarrow v_0 = \lambda_0,\ p_{01} = 1$$

5.1 Continuous-time Markov chain

- **Birth and death processes**

Example 5.1.1

Consider a barber shop with two barbers and two waiting chairs. Customers arrive at a rate of five per hour. Each barber serves customers at a rate of two per hour. Customers arriving to a fully occupied shop leave without being served. We assume that arrivals are Poisson and service times are exponential and the arrival process is independent of service times.
Write the infinitesimal generator.

5.1 Continuous-time Markov chain

- **Birth and death processes**

Solution:

$$Q = \begin{bmatrix} -5 & 5 & & & \\ 2 & -7 & 5 & & \\ & 4 & -9 & 5 & \\ & & 4 & -9 & 5 \\ & & & 4 & -4 \end{bmatrix}$$

5.2 Kolmogorov differential equations

- **Proof of the Kolmogrov equation**

Discrete-time Markov chain: Chapman-Kolmogorov equation
Continuous-time Markov chain: Kolmogorov differential equation

From the Chapman Kolmogorov equation:

$$P_{ij}(t+h)=\sum_{k\in S}P_{ik}(h)P_{kj}(t)=\sum_{k\in,k\neq i}P_{ik}(h)P_{kj}(t)+P_{ii}(h)P_{ij}(t)$$

Subtracting $P_{ij}(t)$ from both sides yields:

$$P_{ij}(t+h)-P_{ij}(t)=\sum_{k\in S,k\neq i}P_{ik}(h)P_{kj}(t)-\left[1-P_{ii(h)}\right]P_{ij(t)}$$

5.2 Kolmogorov differential equations

- Proof of the Kolmogrov equation

$$\lim_{h\to 0}\frac{P_{ij}(t+h)-P_{ij}(t)}{h}=\lim_{h\to 0}\frac{\sum\limits_{k\in S,k\neq i}P_{jk}(h)P_{kj}(t)}{h}-\lim_{h\to 0}(\frac{1-P_{ii}(h)}{h})P_{ij}(t)$$

$$=\sum_{k\in,\mathrm{k}\neq\mathrm{i}}\lim_{h\to 0}\frac{P_{ik(h)}}{h}P_{kj}(t)-\lim_{h\to 0}\left(\frac{1-P_{ii(h)}}{h}\right)P_{ij}(t)$$

Backward Kolmogorov equation

$$P_{ij}^{'}=\sum_{k\in S,k\neq i}q_{ij}P_{jk}(t)-v_iP_{ij}(t)\quad (t\geq 0,i,j\in S)$$

Initial condition: $P_{ij}(0)=\delta_{ij}\ \ i,j\in S$

In matrix form: $\dfrac{dP(t)}{dt}=QP(t)\qquad t\geq 0$

$$P(0)=I$$

5.2 Kolmogorov differential equations

- **Proof of the Kolmogrov equation**

Forward Kolmogorov equation

$$P_{ij}^{'} = \sum_{k \in S, k \neq j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j \quad (t \geq 0, i, j \in S)$$

Initial condition: $P_{ij}(0) = \delta_{ij} \ i, j \in S$

In matrix form:

$$\frac{dP(t)}{dt} = P(t)Q \qquad t \geq 0$$

$$P(0) = I$$

5.2 Kolmogorov differential equations

- **Approaches to solve the Kolmogorov differential equations:**

(1).Numerical methods

(2).Algebraic methods

(3).Uniformization method.

$P_{ij}(t) \to P_{ij}^e(s)$ in matrix form: $P^e(s) = \{P_{ij}^e(s)\}$

$$\frac{dP(t)}{dt} = QP(t) \to sP^e(s) - P(0) = QP^e(s)$$

$$[sI - Q]P^e(s) = I \to P^e(s) = [sI - Q]^{-1}$$

$$P(t) = e^{Qt} = \sum_{n=0}^{\infty} \frac{(Qt)^n}{n!}$$

5.2 Kolmogorov differential equations

- **Approaches to solve the Kolmogorov differential equations:**

Example (5.2.1)

Consider a two-state continuous-time Markov chain with state space $S=\{0,1,\dots\}$ and infinitesimal generator

$$Q=\begin{bmatrix}-\lambda & \lambda \\ \mu & -\mu\end{bmatrix}$$

Find the transition probability function.

5.3 Limiting probability

- **Proof of limiting probability**

$$P_j = \lim_{t\to\infty} P_{ij}(t)$$

Consider forward equation: $P'_{ij} = \sum_{k\in S, k\neq j} P_{ik}(t) q_{kj} - P(t) v_j$

$$\lim P'_{ij}(t) = \lim_{t\to\infty}\left[\sum_{\substack{k\neq j\\ t\to\infty}} P_{jk}(t) q_{kj} - P_{ij(t)} v_j\right] = \sum_{k\neq j} P_k q_{kj} - P_j v_j = 0$$

5.3 Limiting probability

- **Calculation of the limiting probability**

$$\begin{cases} P_j v_j = \sum_{k \neq j} P_k q_{kj} \\ \sum_{j=0}^{\infty} P_j = 1 \end{cases}$$

In matrix form $\begin{cases} PQ = 0 \\ Pe = 1 \end{cases}$

Limiting probabilities P_j

5.3 Limiting probability

- **Calculation of the limiting probability**

Example (review the example of barber shop)

$$\begin{cases} PQ = 0 \\ Pe = 1 \end{cases} \rightarrow \begin{aligned} P_0 &= 0.0649 \\ P_1 &= 0.1622 \\ P_2 &= 0.2027 \\ P_3 &= 0.2534 \\ P_4 &= 0.3168 \end{aligned}$$

5.4 Absorbing continuous Markov chains

- **Definition and calculation of the elementary matrix**

In a continuous-time Markov chain, an absorbing state i is one whose transition rate ν_i is zero.

For continuous-time Markov chain:
The process with at least one absorbing state is called an absorbing continuous-time Markov chain.
Infinitesimal generator Q in a canonical form:

$$Q = \begin{array}{cc} & \begin{array}{cc} T^c & T \end{array} \\ \begin{array}{c} T^c \\ T \end{array} & \begin{bmatrix} 0 & 0 \\ R & V \end{bmatrix} \end{array}$$

5.4 Absorbing continuous Markov chains

- **Definition and calculation of the elementary matrix**

The matrix of transition functions in a canonical form:

$$P(t) = \begin{array}{c} \\ T^c \\ T \end{array} \begin{array}{c} \begin{array}{cc} T^c & T \end{array} \\ \begin{bmatrix} 0 & 0 \\ S(t) & T(t) \end{bmatrix} \end{array}$$

Define the matrix of the expected durations:

$$\int_0^t P(t)\,dx = H(t) = \begin{bmatrix} tI & 0 \\ M(t) & N(t) \end{bmatrix}$$

5.4 Absorbing continuous Markov chains

- **Definition and calculation of the elementary matrix**

$N_{ij}(t)$ gives the expected time the process spent in transient state j by time t given the process starts from transient state i at time 0.

$M_{ij}(t)$ gives the expected time the process spent in absorbing state j by time t given the process starts from transient state i at time 0.

$$H(t)=\int_0^t P(t)\,dx=\int_0^t \sum_{k=0}^{\infty}\frac{x^k Q^k}{k!}dx=\sum_{k=0}^{\infty}\frac{Q^k}{k!}\int_0^t x^k dx=\sum_{k=0}^{\infty}\frac{x^{k+1}Q^k}{(k+1)!}$$

$$QH(t)=\sum_{k=0}^{\infty}\frac{(Qx)^{k+1}}{(k+1)!}=P(t)-I$$

$$QH(t)=\begin{bmatrix}0 & 0\\ tR+VM(t) & VN(t)\end{bmatrix}\quad P(t)-I=\begin{bmatrix}0 & 0\\ S(t) & T(t)-I\end{bmatrix}$$

$$VN(t)=T(t)-I\Rightarrow N(t)=V^{-1}\left[T(t)-I\right]$$

5.4 Absorbing continuous Markov chains

- **Definition and calculation of the elementary matrix**

From forward Kolmogorov differential equation:

$$\frac{dP(t)}{dt} = P(t)Q \Rightarrow P(t) = \int_0^t P(t)Qdx$$

$$S(t) = N(t)R$$

$$S(t) = V^{-1}\left[T(t) - I\right]R$$

Since $\begin{bmatrix} 0 & 0 \\ tR + VM(t) & VN(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ S(t) & T(t) - I \end{bmatrix}$

Thus $tR + VM(t) = S(t)$

Then $M(t) = V^{-1}\left[S(t) - tR\right]$

5.5 Phase-type distributions

- **Definition of Phase-type distributions**

Consider finite-state absorbing continuous-time Markov chain with a single absorbing state.

We have infinitesimal generator as

$$Q = \begin{bmatrix} 0 & O \\ T^0 & T \end{bmatrix} = \begin{array}{c|cccccc} & 0 & 1 & 2 & . & . & m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & T_{10} & T_{11} & T_{12} & . & . & T_{1m} \\ 2 & T_{20} & T_{21} & T_{22} & . & . & T_{2m} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ m & T_{m0} & T_{m1} & T_{m2} & . & . & T_{mm} \end{array}$$

We assume Q is irreducible. It can be show that the matrix T is nonsigular.

5.5 Phase-type distributions

- **Definition of Phase-type distributions**

$\{\alpha_i\}$: elements of the starting probability vector, denote the row vector.

τ : time to absorption given the starting-state probability vector

F : distribution function

Let $x_j(t) = P\{X(t) = j\}$ and row vector $x(t) = \{x_1(t), \ldots, x_m(t)\}$

Forward Kolmogorov equation is given in matrix notation as

$$x^{'}(t) = x(t)T \ and \ x(0) = \alpha$$

Define the Laplace transform $x^e(s) = \int_0^\infty e^{-st} x(t)\,dt$

Taking the Laplace transform $sx^e(s) - x(0) = x^e(s)T \ or \ x^e(s)[sI - T] = \alpha$

Inverse of $[Is - T]$ exist for $\mathrm{Re}(s) \geq 0$ hence $x^e(s) = \alpha[sI - T]^{-1}$.

Taking the inverse Laplace transform $x(t) = \alpha \exp(Tt), t \geq 0$.

5.5 Phase-type distributions

- **Definition of Phase-type distributions**

$$F(t)=P\{\tau\le t\}=1-x(t)e=1-\alpha\exp(Tt)e \qquad t\ge 0 \qquad (1)$$

Now we have phase-type distribution with representation (α,T) with order m, we use $PH(\alpha,T)$ to denote it.

$$t=0 \qquad F(0)=1-\alpha e=\alpha_0$$

$$t>0 \quad f(t)=\frac{d}{dt}F(t)=\frac{d}{dt}\left[1-\alpha\exp(Tt)e\right]$$

$$=-\alpha\frac{d}{dt}\exp(Tt)e=-\alpha\exp(Tt)Te \quad (2)$$

5.5 Phase-type distributions

- **Definition of Phase-type distributions**

Given that $Qe = 0 \; or \; T^0 + Te = 0$

$$f(t) = \alpha \exp(Tt) T^0 = x(t) T^0 \quad t > 0 \quad (3)$$

Define the Laplace transform $f^e(s) = \int_0^\infty e^{-st} f(t) dt$

From Equation 3, we obtain

$$f^e(s) = \alpha [sI - T]^{-1} T^0 \quad (4)$$

Considered $\frac{d}{dt} X^{-1} = -X^{-1} \frac{dX}{dt} X^{-1}$ we get

$$\frac{d}{ds} f^e(s) = \alpha \left(\frac{d}{ds} [sI - T]^{-1} \right) = \alpha \left(-[sI - T]^{-1} \frac{d}{ds} [sI - T]^{-1} [sI - T]^{-1} \right) T^0$$

$$= \alpha \left(-[sI - T]^{-1} [sI - T]^{-1} \right) T^0 \qquad (5)$$

5.5 Phase-type distributions

- **Definition of Phase-type distributions**

Using Equation 5, we obtain

$$E[\tau] = -\frac{d}{ds} f^{e}(s)|_{s=0} = \alpha\left[(-T)^{-1}(-T)^{-1}\right]T^{0}$$

Recall that $Te = -T^{0}$ and hence $e = -T^{-1}T^{0}$

$$E[\tau] = -\alpha T^{-1}e \qquad (6)$$

Iterating the given procedure,

$$E\left[\tau^{i}\right] = (-1)^{i}\, i!\left(\alpha T^{-i}e\right) \qquad for\ i \geq 1$$

5.5 Uniformization

- **Definition**

Virtual transitions: transitions that return to the same state in a single transition.

The epochs of state change are governed by an independent Poisson process $\{N(t), t \geq 0\}$ with rate v.

At a state-change epoch, if the current State is i the process selects its next destination j with probability p_{ij}, the process stays in state i until it reaches the next state change at which time it enters state j, and the scenario is repeated ad infinitum (forever). The stochastic process $\{X(t), t \geq 0\}$ so constructed is called a Markov chain subordinated to a Poisson process.

5.5 Uniformization

- **Definition**

Its transition function is given by

$$p_{ij}(t) = P\{X(t) = j, N(t) = n \mid X(0) = i\}$$

$$= \sum_{m=0}^{\infty} P\{X(t) = j, N(t) = n \mid X(0) = i\}$$

$$= \sum_{m=0}^{\infty} P\{X(t) = j \mid N(t) = n, X(0) = i\} P\{N(t) = n \mid X(0) = i\}$$

$$= \sum_{n=0}^{\infty} p_{ij}^{(n)} e^{-vt} \frac{(vt)^n}{n!} \qquad (1)$$

The process $\{X(t), t \geq 0\}$ is a Markov process because

$$\{X(s+t) = j \mid X(u), u \leq s\} = \sum_{n=0}^{\infty} p_{X(s),j}^{(n)} e^{-vt} \frac{(vt)^n}{n!}$$

5.5 Uniformization

- **Related process and the original Markov chain**

We now consider a continuous-time Markov chain in which $v_i \le v$ for all i (the case in which transition rates may differ among states but are bounded from above). When in State i, the process leaves the state at a rate v. Consider a related process. In this related process, transitions out of each state occur at a constant rate of v. However, when the process is in state i only a fraction v_i/v are transitions out of i to state $j \ne i$ and the rest are transitions back to state i again (the Virtual transitions).

5.5 Uniformization

- **Related process and the original Markov chain**

For this related process, the transition probabilities are then given by

$$\tilde{P}_{ij} = \begin{cases} \dfrac{v_i}{v} p_{ij} & if \; i \neq j \\ 1 - \dfrac{v_i}{v} & if \; i = j \end{cases} \qquad (2)$$

Where $\{p_{ij}\}$ are the transition probabilities of the embedded Markov chain underlying the continuous-time Markov chain defined at the onset of Section 5.3.

$$\tilde{P}_{ij}(t) = \sum_{n=0}^{\infty} \tilde{p}_{ij}^{(n)} e^{-vt} \frac{(vt)^n}{n!} \qquad (3)$$

5.5 Uniformization

- **Related process and the original Markov chain**

Next we need to find out whether this related process is equivalent to the original continuous-time Markov chain. In other words, is $P_{ij}(t) = \tilde{P}_{ij}(t)$ for all i,j, and $t \geq 0$?

Given $P(t) = \exp(Qt)$ we have

$$\tilde{P} = I + \frac{1}{v} Q \qquad (4) \qquad \text{or} \qquad Q = v\tilde{P} - vI$$

So we find

$$P(t) = e^{Qt} = e^{\left(v\tilde{P} - vI\right)t} = e^{-vtI} e^{vt\tilde{P}} = \sum_{n=0}^{\infty} \frac{(-vtI)^n}{n!} \sum_{n=0}^{\infty} \frac{\left(vt\tilde{P}\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-vt)^n}{n!} I \sum_{n=0}^{\infty} \frac{(vt)^n}{n!} \tilde{P}^n = \sum_{n=0}^{\infty} e^{-vt} \frac{(vt)^n}{n!} \tilde{P}^{(n)} = \tilde{P}(t)$$

Hence the two processes are probabilistically equivalent.