Homework for Chapter 1

Deadline: 2022/12/12 23:59

Question 1

(a) Let X_1 and X_2 be two independent geometric random variables having the same parameter p. Please compute the value of $P\{X_1 = i \mid X_1 + X_2 = n\}$.

(b) Continuously flip a coin, with p the probability of the head on top, until there are 2 heads on top within 3 flips. Let N equal to the number of flips (Note that if the first 2 flips are heads, $N = 2$). Find $E[N]$.

Question 2

Consider an experiment involving a fair coin and two biased dice, one with a probability of $1/3$ of coming up with an ace and other with a probability of $1/5$ of coming up with an ace. We start with a coin flip to determine which dice should be used to perform subsequent trials involving repeated tosses of the dice chosen. Let T denote the number of tosses of the die needed to produce an ace for the first time.

(a) Find the probability generating function of T, $P_T^g(z)$.

(b) Find $E[T]$.

(c) Find $\mathrm{Var}[T]$.

Question 3

Let $X_1, \cdots, X_N$ be i.i.d. continuous and positive-valued random variables whose common density is exponential with parameter $\lambda > 0$. Define $S = X_1 + X_2 + \cdots + X_N$, where N is a positive integer-valued random variable and follows the geometric distribution. Let $f_S^e(s)$ denote the Laplace transform of random variable S.

(a) What is the probability density function(pdf) of variable S?

(b) Define $Z = \min\{X_1, Y_1\}$ .X_1 and Y_1 are independent. The distribution and pdf of Y_1 are F, f, respectively. Use F, f and λ to represent the pdf of Z.

Question 4

Consider a first-come-first-served (FCFS) service system serving one customer at a time. In the service of a customer, the probability that there are k new arrivals is p_k. Let $P_X(z)$ represent the probability generating function associated with $\{p_k\}$, where X is the number of arrivals in a service interval. Assume that the arrival process and service times are independent, and the service times themselves are mutually independent. We call the first arrival to an empty system "the zeroth generation customer." All arrivals during the service of the zeroth generation customer are called "the first generation customers." Similarly, all arrivals during the services of the first generation customers are called "the second generation customers." For modeling purposes, we may view that all arrivals in the service of the i th customer of the nth generation as the $(n+1)$ st generation "offspring" produced by the customer. Let S_n denote the total size of the nth generation customers and let $S_{n.i}$ denote the size of the nth generation customers who are actually the offspring of the i th "parent" of the ($n-1$)st generation customer. In other words, we have $S_n = S_{n,1} + \cdots + S_{n,S_{n-1}}$. Moreover $\left\{S_{n,1} \ldots\ldots, S_{n,S_{n-1}}\right\}$ are i.i.d. random variables with a common distribution $\{p_k\}$, and S_{n-1} is independent of $\{$ $S_{n,1} \ldots\ldots, S_{n,S_{n-1}}$ $\}$. Let $P_{S_n}(z)$ denote the probability generating function of S_n.

(a) Find $P_{S_n}(z)$ as a function of $P_x(z)$ for $n \geq 1$.

(b) For the case with $p_0 = 0.5, p_1 = 0.3$ and $p_2 = 0.2$, find the probability distribution of the random variable S_2.

Solutions for Homework 1

Question 1

(a) Intuitively it would see that the first head would be equally likely to occur on either of trails $1, \ldots, \mathrm{n}-1$. That is, it is intuitive that

$$\mathrm{P}\left\{X_1=\mathrm{i} \mid X_1+X_2=\mathrm{n}\right\}=\frac{1}{n-1}, i=1, \ldots, n-1.$$

Formally,

$$\begin{aligned}
\mathrm{P}\left\{X_1=\mathrm{i} \mid X_1+X_2=\mathrm{n}\right\} &= \frac{P\left\{X_1=i, X_1+X_2=n\right\}}{P\left\{X_1+X_2=n\right\}} \\
&= \frac{P\left\{X_1=i, X_2=n-i\right\}}{P\left\{X_1+X_2=n\right\}} \\
&= \frac{p(1-p)^{i-1} p(1-p)^{n-i-1}}{\begin{pmatrix} n-1 \\ 1 \end{pmatrix} p(1-p)^{n-2} p} \\
&= \frac{1}{n-1}
\end{aligned}$$

(b) Let X be the first time that the head appears. Conditioning on the two flips after X, and deriving the equation of E[N | X] (h-head, t-tail):

$$\mathrm{E}[N \mid X]=\mathrm{E}[N \mid X, h, h] p^2+\mathrm{E}[N \mid X, h, t] p q+\mathrm{E}[N \mid X, t, h] p q+\mathrm{E}[N \mid X, t, t] q^2$$

Note that $q=1-p$.Now:

$$\mathrm{E}[N \mid X, h, h]=X+1, \quad \mathrm{E}[N \mid X, h, t]=X+1$$

$$\mathrm{E}[N \mid X, t, h]=X+2, \quad \mathrm{E}[N \mid X, t, t]=X+2+\mathrm{E}[N]$$

It can be solved that:

$$\mathrm{E}[N \mid X]=(X+1)\left(p^2+p q\right)+(X+2) p q+(X+2+\mathrm{E}[N]) q^2$$

Since X is a random variable in Geomatic Distribution:

$$\mathrm{E}[N]=1+p+q+2 p q+q^2 / p+2 q^2+q^2 \mathrm{E}[N]$$

$$E[N]=\frac{2+2 q+q^2 / p}{1-q^2}$$

Question 2

(a) We could get the probability of T :

$$P\{T=n\}=\frac{1}{2}\left(\frac{2}{3}\right)^{n-1}\frac{1}{3}+\frac{1}{2}\left(\frac{4}{5}\right)^{n-1}\frac{1}{5}, n\geq 1$$

The generating function of T is:

$$P_T^g(z)=\sum_{n=1}^{\infty}\left[\frac{1}{2}\left(\frac{2}{3}\right)^{n-1}\frac{1}{3}+\frac{1}{2}\left(\frac{4}{5}\right)^{n-1}\frac{1}{5}\right]z^n=\frac{z}{2}\left(\frac{1}{3-2z}+\frac{1}{5-4z}\right)$$

(b)

$$E[T]=P_T^{(1)}(1)=4$$

(c)

$$E\left[T^2\right]=P_T^{(2)}(1)+P_T^{(1)}(1)=26+4=30$$
$$\operatorname{Var}[T]=E\left[T^2\right]-(E[T])^2=30-16=14$$

Question 3

(a) p is the failure probability.

$f_S^e(s)=\pi_N\left(f_X^e(s)\right)$ in which π_N is the Z-transform of N

$$f_X^e(s)=\frac{\lambda}{s+\lambda}$$
$$\text{So, } f_S^e(s)=\frac{\lambda(1-p)}{s+\lambda-p\lambda}$$

Then, $f_s(t)=L^{-1}\left\{f_S^e(s)\right\}=\lambda(1-p)e^{-(\lambda-p\lambda)t}$

If your p means the probability of success, the answer will be: $f_s(t)=\lambda pe^{-\lambda pt}$

(b)

$$\begin{aligned}G&=P(z\leqslant x)\\&=P\left\{\min\left(X_1,Y_1\right)\leqslant x\right\}\\&=1-P\left\{\min\left(X_1,Y_1\right)>x\right)\\&=1-P\left\{X_1>x\right)P\left(Y_1>x\right)\\&=1-\left[1-P\left(X_1\leqslant x\right)\right]\left[1-P\left(Y_1\leqslant x\right)\right]\\&=1-\left[1-\left(1-e^{-\lambda(x)}\right)\right]\left[1-F(x)\right]\\&=1-e^{-\lambda x}(1-F(x))\end{aligned}$$

$$\bar{f}(x)=G'(x)=\lambda e^{-\lambda x}(1-F(x))+e^{-\lambda x}f(x)$$

Question 4

(a)

$$S_n = \sum_{i=1}^{S_{n-1}} S_{n,i}$$

Since the PGF of $S_{n,i}$ is $P_x(z)$:

$$P_{S_n}(z) = P_{S_{n-1}}(P_x(z)) = \underbrace{P_x(P_x(\cdots P_x(z)))}_{n}$$

(b)

$$P_x(z) = 0.3z + 0.2z^2 + 0.5$$

$$P_{S_2}(z) = 0.008z^4 + 0.024z^3 + 0.118z^2 + 0.15z + 0.7$$

So, we have:

$$\begin{cases} P(S_2 = 0) = 0.7 \\ P(S_2 = 1) = 0.15 \\ P(S_2 = 2) = 0.118 \\ P(S_2 = 3) = 0.024 \\ P(S_2 = 4) = 0.008 \end{cases}$$

Homework for Chapter 2

Deadline: 2022/12/12 23:59

Question 1

A gas station is providing the state auto inspection service for the general public. Arrivals of cars for inspection follow a Poisson process with rate λ. Each inspection takes a constant of c minutes (about thirty minutes in Texas). The gas station opens for business at 7 A.M.

(a) Find the probability that the second arriving car will not have to wait and also find the mean waiting time.

(b) Do the problem also for the case in which the inspection time follows an exponential distribution with parameter μ.

Question 2

Suppose that the process of ambulances carrying patients to the hospital follows the Poisson process with a rate of λ, and the probability that each ambulance carries i patients is $a_i,\ i = 1, 2, 3$. The time each patient spends in the hospital obeys a certain distribution function G. The patient's treatment process is independent of the patient's arrival process, and the hospital has a sufficient number of medical staff and medical facilities. $X(t)$ is defined as the number of patients still being treated in hospital at t. Find $E[X(t)]$.

Question 3

Consider a small supermarket system. The supermarket is self-service, with the goods stored in the shelves where people can pick their choices without having to wait in line. The arrivals to the supermarket can be modeled by a Poisson process with parameter λ. Each customer's total sojourn time follows a distribution function G. Once a customer finishes the selection, the customer joins a single waiting line leading to three cashiers. Any cashier who becomes free will serve whoever is at the head of the line. The service time at each cashier follows an exponential distribution with parameter μ. Service times at the cashiers are mutually independent and are independent of the arrival process leading to the cashiers. We consider the arrival rate to the supermarket is three customers per minute, the mean sojourn time is ten minutes per customer, the mean check-out time is one minute. Let S denote the sojourn time and G its distribution. We assume that S follows a two-stage Erlang distribution.

(a) Find $E[S]$ and $Var[S]$.

(b) Find the intensity function $\lambda(t)$ of the nonhomogeneous Poisson process.

Question 4

Ships arrive at the port according to a Poisson process with rate 5 per day. The ship's dwell time X (day) is subject to uniform distribution $U \sim [0, 10]$. Let $Y(t)$ denote the number of remaining ships at the port by time t.

(a) Please describe $Y(t)$ by the response function $W_0(s, t)$.

(b) Compute $E[Y(t)]$.

Solutions for Homework 2

Question 1

Let X_i denote the interarrival time between ith arriving car and (i-1)th . $\{X_i\}$ be i.i.d random variables.

(a) if $X_2 \geq c$, the second arriving will not have to wait,

$$\mathrm{P}\left\{X_2 \geq c\right\} = e^{-\lambda c}$$

if $X_2 \leq c$, the second arriving will have to wait $(\mathrm{c} - X_2)$ minutes, set $X_2 = t$, the mean waiting time is :

$$E(c-t) = \int_0^c (c-t)\lambda e^{-\lambda t}dt + \int_c^{+\infty} 0 * \lambda e^{-\lambda t}dt$$
$$= c + \frac{e^{-\lambda c}}{\lambda} - \frac{1}{\lambda}$$

(b) Let Y denote the inspection time for the first arriving car. Conditioning on $Y = y$. if $X_2 \geq y$,the second will not need to wait, $X_2 \leq y$, it will wait $(y-t)$ minutes. So the probability that it will not wait is

$$P\left\{X_2 \geq Y\right\} = \int_0^{+\infty} P\left\{x_2 \geq Y \mid Y = y\right\} f(y)dy$$
$$= \int_0^{+\infty} e^{-\lambda y}\mu e^{-\mu y}dy = \frac{\mu}{\lambda+\mu}$$

the mean waiting time is :

$$E(y-t) = \int_0^{+\infty}\int_0^{y}(y-t)\lambda e^{-\lambda t}dt\mu e^{-\mu y}dy$$
$$= \int_0^{+\infty}\left(y + \frac{1}{\lambda}e^{-\lambda y} - \frac{1}{\lambda}\right)\mu e^{-\mu y}dy$$
$$= \frac{\lambda}{\mu(\lambda+\mu)}$$

Question 2

令 $N_i(t)$ 定义为以概率为 $\alpha_i, i = 1, 2, \ldots$ 承载 i 个病人的救护车的数量。对于每一个 $N_i(t), i = 1, 2, \ldots$, 定义 $N_{i1}(t)$ 在 t 时刻已经离开医院的救护车的数量, 定义 $N_{i2}(t)$ 在 t 时刻仍然在医院的救护车的数量。那么: $N_{i1}(t)$ 服从参数为 λtp 的泊松分布, 其中,

$$p = \frac{1}{t}\int_0^t G(t-s)ds = \frac{1}{t}\int_0^t G(x)dx$$

$N_{i2}(t)$ 服从参数为 $\lambda t(1-p)$ 的泊松分布。令 $X(t)$ 定义为在 $(0, t]$ 时间内到达并且在时刻 t 仍然在医院的病人数, 则,

$$X(t) = \sum_{i=1} i\alpha_i N_{i2}(t)$$

$$E[X(t)] = E\left[\sum_{i=1} i\alpha_i N_{i2}(t)\right] = \sum_{i=1} i\alpha_i E\left[N_{i2}(t)\right]$$

$$= \sum_i i\alpha_i \lambda t(1-p) = \lambda \sum_{i=1} i\alpha_i \int_0^t (1-G(x))dx$$

Question 3

The arrival rate to cafeteria is 3 customers per minute $(\lambda = 3)$

The mean sojourn time (food selection time) is 10 minutes per customer

The mean check-out time is l minute, $(\mu = 1)$

Let S denote the sojourn time and G its distribution.

S follows a two-stage Erlang distribution with parameters $(n = 2, v)$

Erlang random variable with parameters (n, v)

$$\mathrm{E}[S] = \frac{n}{v}, \mathrm{var}[S] = \frac{n}{v^2}$$

$$\mathrm{E}[S] = \frac{2}{v} = 10 \Rightarrow v = \frac{1}{5} \Rightarrow \mathrm{var}[S] = 50$$

$$G(t) = P\{S \leq t\} = \sum_{k=2}^{\infty} e^{-0.2t}\frac{(0.2t)^k}{k!} = 1 - e^{-0.2t}[1 + 0.2t]$$

The arrival process is a nonhomogeneous Poisson process with intensity function:

$$\lambda(t) = \lambda G(t) = 3\left[1 - e^{-0.2t}(1 + 0.2t)\right]$$

Question 4

(a) 设 $X(t)$ 为 $(0,t]$ 内到达港口的船只数量, 则 $\{X(t), t \geq 0\}$ 为参数是 $\lambda = 5$ 的泊松过程, 设 S_n 为第 n 只船到达的时刻, X_n 为第 n 只船到达停留时间 $X \sim U[0,10]$, 于是有:

$$y(t) = \sum_{n=1}^{N(t)} W_n\left(t - S_n, X_n\right), t \geq 0$$

$$\text{其中} W_0(s,t) = \begin{cases} 1, 0 < s < t \\ 0, \text{其它} \end{cases}$$

$\{y(t), t \geq 0\}$ 是过滤型泊松过程。

(b)

$$\begin{aligned} \mathrm{E}[y(t)] &= \lambda \int_0^t E\left[W_0\left(t-\tau, X_t\right)\right] d\tau = \lambda \int_0^t P\left(X_t > t - \tau\right) d\tau \\ &= \lambda \int_0^t \left[1 - F_X(t-\tau)\right] d\tau \\ &= \begin{cases} 5t - \frac{t^2}{4}, 0 < t \leq 10 \\ 25, t > 10 \end{cases} \end{aligned}$$

Homework for Chapter 3

Deadline: 2022/12/22 23:59

Question 1

Assume that the telephone switchboard holds a lot of telephone lines, and the number of its calls follows a Poisson process with rate λ. The distribution function of the call duration is G. There is no phone line on the phone at $t = 0$, Let $M(t)$ represent the mathematical expectation of the calling numbers $N(t)$ on the phone at time t. Please write the expression for $M(t)$ according to the renewal-type equation.

Question 2

In a special game, one is equally likely to either win or lose 1 score. Let X be your cumulative scores if you use the strategy that quits playing if you win the first game, and plays two more games and then quits if you lose the first game.Use both Wald's equation and probability mass function to determine E[X].

Question 3

Potential customers come to a bank following a Poisson process with rate λ, and the bank has only one server. But the potential customers only enter the bank when the server is free, otherwise they leave. Suppose the distribution of service hours per customer is G.

(a) What is the rate of customers entering the bank?

(b) What is the percentage of potential customers that actually enter?

Question 4

J's car buying policy is to always buy a new car, repair all breakdowns that occur during the first T time units of ownership, and then junk the car and buy a new one at the first breakdown that occurs after the car has reached age T. Suppose that the time until the first breakdown of a new car is exponential with rate λ, and that each time a car is repaired the time until the next breakdown is exponential with rate μ.

(a) At what rate does J buy new cars?

(b) Supposing that a new car costs C and that a cost r is incurred for each repair, what is J's long run average cost per unit time?

Solutions for Homework 3

Question 1

[考察更新类型函数/更新方程]

令最初电话打来的时刻是 T_1, 所以有

$$M(t) = \int_0^\infty E\left[N(t) \mid T_1 = s\right] dF(s)$$

T_1 是均值为 $\frac{1}{\lambda}$ 的负指数分布, 即: $dF(t) = \lambda e^{-\lambda t} dt$。由 $N(t)$ 的定义, 有

$$M(t) = \int_0^t E\left[N(t) \mid T_1 = s\right] \lambda e^{-\lambda s} ds$$

此外因为次电话通话至时刻 t 的概率为 $1 - G(t-s)$。所以

$$E\left[N(t) \mid T_1 = s\right] = 1 \times [1 - G(t-s)] + m(t-s)$$

故 $^{M(t)}$ 满足的更新方程为:

$$M(t) = \int_0^t [1 - G(t-s)] \lambda e^{-\lambda s} ds + \int_0^t M(t-s) \lambda e^{-\lambda s} ds$$

Question 2

[考察Wald定理和停时]

With W_i equal to your winnings in game $i, i \geq 1$, and N the number of games played, it's easy to find that N is a stopping time. According to Wald's equation: $E[X] = E[N]E[W] = 0$.

With $p_i = P(X = i), p_1 = 1/2 + 1/8 = 5/8, p_{-1} = 1/4, p_{-3} = 1/8$, also verifying that $E[X] = 0$.

Question 3

[考察更新过程基本定理]

不妨设第一位顾客在 0 时刻来到银行, 即过程从第一位顾客来到时刻开始, 设平均服务时间为 μ_G 。

(a) 由于每位到达的顾客看见服务台空闲时才会进入银行, 因此下一位顾客进入银行的时间, 必须是从第一位顾客结束服务时算起的、第一个到达时间间隔 (因 Poisson过程时间间隔的无记忆性, 与第一位顾客结束服务前刚离去的顾客何时离去无关), 所以前后两个进入银行的顾客的到达时间的平均时间间隔为:

$$\mu = \mu_G + \frac{1}{\lambda}$$

从而由更新定理知, 经长时间后进入银行顾客流的到达率 $\frac{N(t)}{t}$ 是:

$$\frac{1}{\mu} = \frac{\lambda}{1 + \lambda\mu_G}$$

(b) 由于顾客的到达率为 λ (到达平均时间间隔 $\frac{1}{\lambda}$ 的倒数), 即单位时间内到达的顾客数为 λ; 而进入银行的顾客的速率为 $\frac{\lambda}{1+\lambda\mu_G}$, 即单位时间内进入银行的顾客数为 $\frac{\lambda}{1+\lambda\mu_G}$, 因此实际进入银行的顾客占潜在顾客的比例为:

$$\frac{\frac{\lambda}{1+\lambda\mu_G}}{\lambda} = \frac{1}{1 + \lambda\mu_G}$$

Question 4

[考察更新报酬过程]

(a) Let X denote the length of time that J keeps a car. Let I equal 1 if there is a breakdown by time T and equal 0 otherwise. Then

$$\begin{aligned}
E[X] &= E[X \mid I = 1]\left(1 - e^{-\lambda T}\right) + E[X \mid I = 0]e^{-\lambda T} \\
&= \left(T + \frac{1}{\mu}\right)\left(1 - e^{-\lambda T}\right) + \left(T + \frac{1}{\lambda}\right)e^{-\lambda T} \\
&= T + \frac{1 - e^{-\lambda T}}{\mu} + \frac{e^{-\lambda T}}{\lambda}
\end{aligned}$$

$1/E[X]$ is the rate that J buys a new car.

(b) Let W equal to the total cost involved with purchasing a car. Then, with Y equal to the time of the first breakdown

$$\begin{aligned}
E[W] &= \int_0^\infty E[W \mid Y = y]\lambda e^{-\lambda y} dy \\
&= C + \int_0^T r(1 + \mu(T - y))\lambda e^{-\lambda y} dy \\
&= C + r\left(1 - e^{-\lambda T}\right) + r\int_0^T \mu(T - y)\lambda e^{-\lambda y} dy
\end{aligned}$$

J's long run average cost is $E[W]/E[X]$.

Homework for Chapter 4

Deadline: 2022/12/22 23:59

Question 1

An urn always contains 2 balls. Ball colors are red and blue. At each stage a ball is randomly chosen and then replaced by a new ball, which with probability 0.8 is the same color, and with probability 0.2 is the opposite color, as the ball it replaces. If initially both balls are red, find the probability that the fifth ball selected is red.

Question 2

Given state space $S = \{1, 2, 3, 4\}$, and its transition probability matrix:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

(a) Give the classification of all the states with corresponding reason;

(b) Find all the *ergodic* states in the example.

Question 3

Consider there are a balls in the bag, and the colors of the balls are either black or white. Take a ball from the bag each time, and meanwhile put back a ball of another color so that the number of balls in the bag is invariant. Define X_n as the number of white balls remained in the bag after taking n balls.

(a) Write the state space of this Markov chain for the random variable X_n;
(b) Draw the Markov chain;
(c) Is there a stationary distribution? If yes, give your result.

Question 4

Consider a cycle of phosphorus in one grassland ecosystem. There are four states: phosphorus in the soil, phosphorus in the grass, phosphorus in the sheep, and phosphorus lost outside of the system. It is known that for every year, the probability of phosphorus in the soil being absorbed to the grass is 0.4, and the probability of phosphorus in the soil losing outside of the system is 0.2. For the phosphorus in the grass, the probability of eating by sheep is 0.6, and the probability of returning to the soil by rotting is 0.1, and the probability of still in the grass is 0.3. For phosphorus in the sheep, the probability of returning to the soil by excreting is 0.7, and the probability of still in the sheep is 0.2, and the probability of losing outside of the system is 0.1.

(a) Assume that the time unit is year, try to build the Markov model, and show the average time of the phosphorus in the soil before losing outside of the system;
(b) If at the beginning the proportion of phosphorus in the soil, in the grass, and in the sheep are $40\%, 40\%$ and 20%, respectively, show the time of phosphorus in the system.

Solutions for Homework 4

Question 1

定义 X_n 为罐中红球的数量，状态空间为 $\{0,1,2\}$，有转移概率矩阵：

$$\mathbf{P}=\begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}$$

由此我们有：

$$\begin{aligned} P(\text{fifth selection is red}) &= \sum_{i=0}^{2} P\left(\text{ fifth selection is red } \mid X_4=i\right) P\left(X_4=i \mid X_0=2\right) \\ &= (0)P_{2,0}^4+(0.5)P_{2,1}^4+(1)P_{2,2}^4 \\ &= 0.5P_{2,1}^4+P_{2,2}^4 \end{aligned}$$

计算 $\mathbf{P}^4$ 后易得：

$$P(\text{ fifth selection is red })=0.7048.$$

Question 2

(a) 因为对一切 $n\geq 1, f_{44}^{(n)}=0$, 所以状态 4 是非常返态。又 $f_{33}^{(1)}=\frac{2}{3}$, 所以状态 3 也是非常返态。而:

$$f_{11}=f_{11}^{(1)}+f_{11}^{(2)}=\frac{1}{2}+\frac{1}{2}=1$$

$$f_{22}=f_{22}^{(1)}+f_{22}^{(2)}+\cdots=0+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots=1,$$

所以状态 1 和状态 2 是常返态。

(b)

$$\mu_1 = \sum_{n=1}^{\infty} n f_{11}^{(n)} = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2} < \infty$$

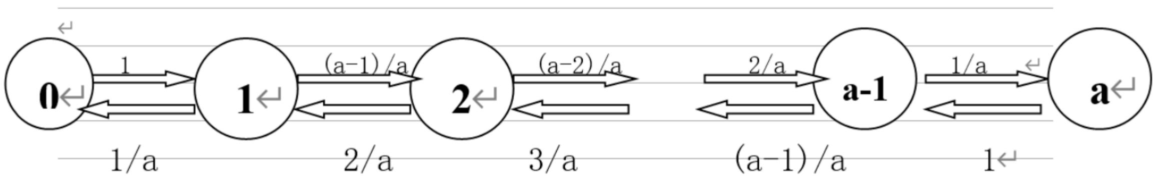
$$\mu_2 = \sum_{n=1}^{\infty} n f_{22}^{(n)} = 1 \cdot 0 + 2 \cdot \frac{1}{2} + 3 \cdot \left(\frac{1}{2}\right)^2 + \cdots = 3 < \infty$$

而其周期均为 1，故状态 1 与状态 2 是正常返态，且为遍历态。

Question 3

(a) $\{X_n, n = 0, 1, 2, \ldots\}$ 的状态空间是 $S = \{0, 1, 2, \ldots a\}$

(b) 如下图：



(c) 是平稳分布;

由状态转移图可见, $0, 1, 2, \cdots a$ 都是正常返状态, S 是唯一的一个正常返状态的不可约集, 因此平稳分布存在唯一, 设平稳分布为 $\pi = \{\pi_0, \pi_1, \pi_2, \ldots, \pi_a\}$, 求解方程组

$$\begin{cases} \pi = \pi P \\ \pi_0 + \pi_1 + \pi_2 + \ldots + \pi_a = 1 \end{cases}$$

得：

$$\pi_k = C_a^k \frac{1}{2^a}, k = 0, 1, 2, \ldots, a$$

所以平稳分布为

$$\pi = \left\{ \frac{1}{2^a}, C_a^1 \frac{1}{2^a}, C_a^2 \frac{1}{2^a}, \ldots, \frac{1}{2^a} \right\}$$

Question 4

定义状态空间:

1. phosphorus lost outside of the system.
2. phosphorus in the soil.

3. phosphorus in the grass.
4. phosphorus in the sheep;

建立 Markov model：

$$
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0.2 & 0.4 & 0.4 & 0 \\
0 & 0.1 & 0.3 & 0.6 \\
0.1 & 0.7 & 0 & 0.2
\end{bmatrix}
$$

根据矩阵性质，导出：

$$
Q = \begin{bmatrix}
0.4 & 0.4 & 0 \\
0.1 & 0.3 & 0.6 \\
0.7 & 0 & 0.2
\end{bmatrix}, \quad R = \begin{bmatrix}
0.2 \\ 0 \\ 0.1
\end{bmatrix}
$$

得 fundamental matrix：

$$
\begin{aligned}
w = (I-Q)^{-1} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= \begin{bmatrix}
0.6 & -0.4 & 0 \\
-0.1 & 0.7 & -0.6 \\
-0.7 & 0 & 0.8
\end{bmatrix}^{-1} \\
&= \begin{bmatrix}
4.1176 & 2.3539 & 1.7647 \\
3.6765 & 3.5294 & 2.6471 \\
3.6029 & 2.0588 & 2.7941
\end{bmatrix} = \begin{bmatrix}
t_{11} & t_{12} & t_{13} \\
t_{21} & t_{22} & t_{23} \\
t_{31} & t_{32} & t_{33}
\end{bmatrix}
\end{aligned}
$$

(a)

$$
w_1 = t_{11} + t_{12} + t_{13} = 4.1176 + 2.3539 + 1.7647 = 8.2352
$$

(b) 磷在系统里的时间：

$$
\begin{aligned}
t &= 0.4w_1 + 0.4w_2 + 0.2w_3 \\
&= 0.4\left(t_{11} + t_{12} + t_{13}\right) + 0.4\left(t_{21} + t_{22} + t_{23}\right) + 0.2\left(t_{31} + t_{32} + t_{33}\right) \\
&= 8.9264
\end{aligned}
$$

第五章练习题

1、 Consider in a taxi station the number of arrival taxis follows a Poisson process with rate $\lambda_1 = 1$ per minute; the number of arrival customers follows another Poisson process with rate $\lambda_2 = 2$ per minute. Taxis will stop if no customer is waiting there, no matter there is a taxi or not; customers will not stop if no taxi is waiting there ; customers will take a taxi if the taxi is already waiting there.

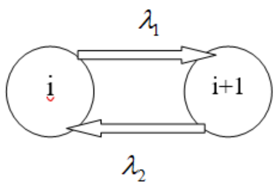
Question:

(1)Give the flow chart and the Q matrix regarding to the number of waiting taxis (8 points);

(2) Find the mean number of waiting taxis (8 points);

(3)Compute the number of customers take taxis during the customers waiting at the station (9 points). （2012）

Solution:



系统的 Q 矩阵为

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & -3 & 1 & 0 & 0 & \cdots \\ 0 & 2 & -3 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

(2)其平稳分布为

$$p_n = \left(1 - \frac{1}{2}\right)\frac{1}{2^n} = \frac{1}{2^{n+1}}, n = 0,1,2,\ldots$$

在车站上等候出租车的平均数为

$$E = \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{\infty} n\frac{1}{2^{n+1}} = \frac{1}{2}\bullet\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 1$$

(3)在到站的潜在顾客中雇得出租车的平均数是

$$\sum_{n=0}^{\infty} np_n = \sum_{n=0}^{\infty} n \frac{1}{2^{n+1}} = \frac{1}{2} \bullet \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 1$$

2、 Consider the customers who enter a service system follow a Poisson process with arrival rate λ, there is only one waiter in the system, and the service time follows the exponential distribution with parameter μ. If the system is idle, the arrived customer will be served immediately. Otherwise he has to queue up. If there are already two customers waiting in the system, the new customer will leave and never come back. Let X(t) be the number of customers in this system.

(1) Write the state space of this Markov chain (5 points)

(2) Draw the Markov chain and give the infinitesimal generator Q (10 points)

(3) Describe the stable probability distribution of each state (10 points). （2011）

Solution:

(1) $S=\{0,1,2,3\}$

(2) draw the Markov chain

$$Q=\begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 \\ 0 & \mu & -(\mu+\lambda) & \lambda \\ 0 & 0 & \mu & -\mu \end{pmatrix}$$

(3) $\pi_0=\left[\rho^0+\rho^1+\rho^2+\rho^3\right]^{-1}=\left[\frac{1-\rho^4}{1-\rho}\right]^{-1}$

得出$\pi_k = \rho^k \pi_0 = \rho^k \left[\frac{1-\rho^4}{1-\rho}\right]^{-1}, k = 0,1,2,3$

3、There are two types of customers. Type 1 and 2 customers arrive to the system in accordance with independent Poisson processes with respective rate λ_1 and λ_2. There are two servers in this system. A type 1 arrival will enter service with server 1 if that server is free. If server 1 is busy and server 2 is free, then type 1 arrival will enter service with server 2. If both servers are busy, then the type 1 arrival will go away. A type 2 customer can only be served by server 2. If server 2 is free when a type 2 customer arrives, then the customer enters service. If server 2 is busy when a type 2 arrives, then that customer goes away. Once a customer finishes his service by either server, he/she departs the system. Service times at server 1 and 2 are exponential with rate μ_1 and μ_2 respectively. Suppose we want to find the average number of customers in the system.

（a）Define a Markov chain with 4 states and draw its transition diagram. (10 points)
(**Hints**: Define the state of Markov chain in terms of both states of two servers.)

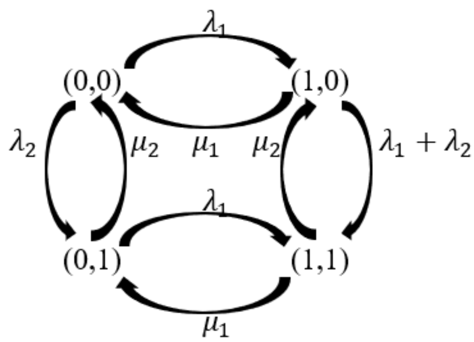
（b）Give the solution formula of the limiting probabilities. Do not attempt to solve them. (5 points) (**Hints**: Define $\pi(i,j) = \lim_{t\to\infty} P\{(X_1(t), X_2(t)) = (i,j)\}$.)

（c）In terms of the given limiting probabilities, what is the average number of customers in the system? (5 points)

（d）In terms of the previous limiting probabilities, what is the fraction of server 2's customers that are type 1? (5 points)
(**Hints**: Find the rate at which a type 1 or 2 customer enters service with server 2)

Solution:

（a）状态空间表示为$S = \{(0,0), (0,1), (1,0), (1,1)\}$，(0,0)表示系统是空的，(1,0)表示 1 个顾客在服务线 1，服务线 2 则是空的，其余状态类似。转移图：



（b）极限概率求解方程：

$$(\lambda_1 + \lambda_2)P_{(0,0)} = \mu_1 P_{(1,0)} + \mu_2 P_{(0,1)}$$

$$(\lambda_1 + \lambda_2 + \mu_1)P_{(1,0)} = \lambda_1 P_{(0,0)} + \mu_2 P_{(1,1)}$$

$$(\lambda_1+\mu_2)P_{(0,1)}=\lambda_2 P_{(0,0)}+\mu_1 P_{(1,1)}$$

$$(\mu_1+\mu_2)P_{(1,1)}=\lambda_1 P_{(0,1)}+(\lambda_1+\lambda_2)P_{(1,0)}$$

$$P_{(0,0)}+P_{(1,0)}+P_{(0,1)}+P_{(1,1)}=1$$

（c）系统中顾客的平均数量：

$$\mathrm{L}=P_{(1,0)}+P_{(0,1)}+2P_{(1,1)}$$

（d）类型Ⅰ顾客进入服务线 2 的速率: $\lambda_1 P_{(1,0)}$

类型Ⅱ顾客进入服务线 2 的速率: $\lambda_2(P_{(0,0)}+P_{(1,0)})$

在服务线 2 的类型Ⅰ顾客的比例是: $\lambda_1 P_{(1,0)}/[\lambda_1 P_{(1,0)}+\lambda_2(P_{(0,0)}+P_{(1,0)})]$