RSF 能量泛函：

$$
\begin{aligned}
\mathcal{E}\left(\phi, f_1, f_2\right) = {} & \lambda_1 \int \int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) \left|I(\mathbf{y}) - f_1(\mathbf{x})\right|^2 M_1(\phi(\mathbf{y})) d\mathbf{y} d\mathbf{x} \\
& + \lambda_2 \int \int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) \left|I(\mathbf{y}) - f_2(\mathbf{x})\right|^2 M_2(\phi(\mathbf{y})) d\mathbf{y} d\mathbf{x} \\
& + \nu \int \left|\nabla H_\epsilon(\phi(\mathbf{x}))\right| d\mathbf{x} + \mu \mathcal{P}(\phi)
\end{aligned}
$$

固定 ϕ 对 $f_1(\mathbf{x})$ 和 $f_2(\mathbf{x})$ 分别求导并使得偏导数为零,

$$
\frac{\partial \mathcal{F}}{\partial f_1} = 0, \frac{\partial \mathcal{F}}{\partial f_2} = 0
$$

$$
\begin{aligned}
\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) \left(I(\mathbf{y}) - f_1(\mathbf{x})\right) M_1(\phi(\mathbf{y})) d\mathbf{y} = 0 \\
\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) \left(I(\mathbf{y}) - f_2(\mathbf{x})\right) M_2(\phi(\mathbf{y})) d\mathbf{y} = 0
\end{aligned}
$$

$$
\begin{gathered}
\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) I(\mathbf{y}) M_1(\phi(\mathbf{y})) d\mathbf{y} - \int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) f_1(\mathbf{x}) M_1(\phi(\mathbf{y})) d\mathbf{y} = 0 \\
\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) I(\mathbf{y}) M_1(\phi(\mathbf{y})) d\mathbf{y} = f_1(\mathbf{x}) \int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) M_1(\phi(\mathbf{y})) d\mathbf{y} \\
f_1(\mathbf{x}) = \frac{\displaystyle\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) I(\mathbf{y}) M_1(\phi(\mathbf{y})) d\mathbf{y}}{\displaystyle\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right) M_1(\phi(\mathbf{y})) d\mathbf{y}}
\end{gathered}
$$

变量为函数的卷积定义：

$$
\int\limits_{-\infty}^{\infty} g(p) h(t-p) dp = g(t) * h(t)
$$

$$
\begin{aligned}
f_1(\mathbf{x}) = \frac{K_\sigma\left(\mathbf{x}\right) * \left[I(\mathbf{x}) M_1(\phi(\mathbf{x}))\right]}{K_\sigma\left(\mathbf{x}\right) * M_1(\phi(\mathbf{x}))} \\
f_2(\mathbf{x}) = \frac{K_\sigma\left(\mathbf{x}\right) * \left[I(\mathbf{x}) M_2(\phi(\mathbf{x}))\right]}{K_\sigma\left(\mathbf{x}\right) * M_2(\phi(\mathbf{x}))}
\end{aligned}
$$

$Heaviside$ 函数 H 用如下光滑函数 H_ϵ 替代

$$H_\epsilon = \frac{1}{2}\left[1 + \frac{2}{\pi} arctan\left(\frac{x}{\epsilon}\right)\right]$$

H_ϵ 的导数为

$$\delta_\epsilon = H_\epsilon^{'}(x) = \frac{1}{\pi}\frac{1}{\epsilon^2 + x^2}$$

令:

$$\mathcal{P}(\phi) = \int \frac{1}{2}(|\nabla\phi(\mathbf{x}) - 1|)^2 d\mathbf{x}$$

RSF 能量泛函可写为

$$\begin{aligned}\mathcal{E}(\phi, f_1, f_2) = \lambda_1 \int\int K_\sigma(\mathbf{x} - \mathbf{y}) \left|I(\mathbf{y}) - f_1(\mathbf{x})\right|^2 H(\phi(\mathbf{y})) d\mathbf{y} d\mathbf{x} \\ + \lambda_2 \int\int K_\sigma(\mathbf{x} - \mathbf{y}) \left|I(\mathbf{y}) - f_2(\mathbf{x})\right|^2 (1 - H(\phi(\mathbf{y}))) d\mathbf{y} d\mathbf{x} \\ + \nu \int \left|\nabla H_\epsilon(\phi(\mathbf{x}))\right| d\mathbf{x} + \mu \int \frac{1}{2}(|\nabla\phi(\mathbf{x}) - 1|)^2 d\mathbf{x}\end{aligned}$$

欧拉-拉格朗日公式：
对于积分泛函

$$E(u) = \int_b^a F(x, u, u') dx$$

我们的目标则是寻找目标函数 $u(x)$ 使得能量函数 $E(u)$ 取极小值，假设函数 $u(x)$ 能够使得 $E(u)$ 取极小值，那么对函数 $u(x)$ 引入任意小的扰动后，或者说对于任意的 $u(x) + tv(x)$ 都有

$$E(u) \leq E(u + tv)$$

$$E(u + tv) = \int_b^a F(x, u + tv, (u + tv)') dx$$

其中 t 为常数，$v(a) = v(b) = 0$, 我们把 $E(u + tv)$ 看成一个关于变量 t 的函数 $\phi(t)$, 当 $t \to 0$ 时 $E(u) = E(u + tv)$，因此可以推断出泛函 E 的一阶变分 δE 也就是 $\phi(t)$ 在 $t = 0$ 处的一阶导

$$\delta E = \left.\frac{\partial\phi}{\partial t}\right|_{t=0} = 0$$

$$\begin{aligned}
\frac{\partial\phi}{\partial t}=\frac{\partial E(u+tv)}{\partial t} &= \int_b^a\left[\frac{\partial F(x,u+tv,(u+tv)')}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial F(x,u+tv,(u+tv)')}{\partial(u+tv)}\frac{\partial(u+tv)}{\partial t}\right.\\
&\quad\left.+\frac{\partial F(x,u+tv,(u+tv)')}{\partial((u+tv)')}\frac{\partial((u+tv)')}{\partial t}\right]dx\\
&=\int_b^a\left[\frac{\partial F(x,u+tv,(u+tv)')}{\partial(u+tv)}v+\frac{\partial F(x,u+tv,(u+tv)')}{\partial((u+tv)')}v'\right]dx\\
&=\int_b^a\left[\frac{\partial F}{\partial u}v+\frac{\partial F}{\partial u'}v'\right]dx\ (t\to 0)\\
&=\int_b^a\frac{\partial F}{\partial u}vdx+\int_b^a\frac{\partial F}{\partial u'}v'dx\\
&=\int_b^a\frac{\partial F}{\partial u}vdx+\int_b^a\frac{\partial F}{\partial u'}dv\\
&=\int_b^a\frac{\partial F}{\partial u}vdx+\frac{\partial F}{\partial u'}v\bigg|_a^b-\int_b^a vd\left(\frac{\partial F}{\partial u'}\right)\ (\text{分部积分法})\\
&=\int_b^a\frac{\partial F}{\partial u}vdx+\frac{\partial F}{\partial u'}v\bigg|_a^b-\int_b^a v\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)dx\\
&=\int_b^a\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]vdx+\frac{\partial F}{\partial u'}v\bigg|_a^b\\
&=\int_b^a\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]vdx\ \ \left(v(a)=v(b)=0,\frac{\partial F}{\partial u'}v\bigg|_a^b=0\right)
\end{aligned}$$

若原式

$$\frac{\partial\phi}{\partial t}=0$$

根据变分法基本预备定理，有

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=0$$

上式即积分泛函 $E(u)=\int_b^a F(x,u,u')dx$ 对应的欧拉-拉格朗日公式

对于包括多元函数的积分泛函，如 $E(u)=\int_b^a F(x,y,u,u_x,u_y)dx$, 可通过完全相同的方法推导得到其对应的欧拉-拉格朗日公式为

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)-\frac{d}{dy}\left(\frac{\partial F}{\partial u_y}\right)=0$$

通常情况下，$Euler-Lagrange$ 方程都是非线性的偏微分方程，数值计算比较困难, 可以引入一个时间辅助变量 t，将求解静态非线性偏微分方程问

题转换为一个动态偏微分方程问题, 当演化达到稳态，便可得到变分问题对应的 $Euler-Lagrange$ 方程的解，这就是梯度下降法。引入时间辅助变量 t，假设变分问题的解可以随着时间变化，即解可以表示为 $u(\cdot,t)$，则随时间变化的解 $u(\cdot,t)$ 的变化应该使得能量泛函 $E(u(\cdot,t))$ 逐渐减小, 即泛函 $E(u(\cdot,t))$ 的一阶变分 $\delta E<0$。假设扰动函数 $v(\cdot)$ 是 $u(\cdot,t)$ 从 t 到 $t+\Delta t$ 所产生的变化量，即

$$v=\frac{\partial u}{\partial t}\Delta t$$

那么

$$\delta E=\int_b^a\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]vdx=\Delta t\int_b^a\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]\frac{\partial u}{\partial t}dx<0$$

要使上式成立，只需

$$\frac{\partial u}{\partial t}=-\frac{\partial F}{\partial u}+\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)$$

此时 $\delta E<0$, 说明 $E(u(\cdot,t))$ 在逐渐减小，因此称上式是变分问题 $E(u)=\int_b^a F(x,u,u')dx$ 对应的梯度下降流。同理，可以得到多元函数的积分泛函 $E(u)=\int_b^a F(x,y,u,u_x,u_y)dx$ 对应的梯度下降流为

$$\frac{\partial u}{\partial t}=-\frac{\partial F}{\partial u}+\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)+\frac{d}{dy}\left(\frac{\partial F}{\partial u_y}\right)$$

在 RSF 能量泛函中

$$\begin{aligned}\mathcal{E}\left(\phi,f_1,f_2\right)=&\lambda_1\int\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right)\left|I(\mathbf{y})-f_1(\mathbf{x})\right|^2H_\epsilon(\phi(\mathbf{y}))d\mathbf{y}d\mathbf{x}\\&+\lambda_2\int\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right)\left|I(\mathbf{y})-f_2(\mathbf{x})\right|^2(1-H_\epsilon(\phi(\mathbf{y})))d\mathbf{y}d\mathbf{x}\\&+\nu\int\left|\nabla H_\epsilon(\phi(\mathbf{x}))\right|d\mathbf{x}+\mu\int\frac{1}{2}(\left|\nabla\phi(\mathbf{x})\right|-1)^2d\mathbf{x}\end{aligned}$$

$$\begin{cases}F=\lambda_1\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right)\left|I(\mathbf{y})-f_1(\mathbf{x})\right|^2H_\epsilon(\phi(\mathbf{y}))d\mathbf{y}\\\quad+\lambda_2\int K_\sigma\left(\mathbf{x}-\mathbf{y}\right)\left|I(\mathbf{y})-f_2(\mathbf{x})\right|^2(1-H_\epsilon(\phi(\mathbf{y})))d\mathbf{y}\\\quad+\nu\left|\nabla H_\epsilon(\phi(\mathbf{x}))\right|+\mu\frac{1}{2}(\left|\nabla\phi(\mathbf{x})\right|-1)^2\\\quad u=\phi\end{cases}$$

因此 RSF 能量泛函对应的梯度下降流为

$$\frac{\partial \phi}{\partial t} = -\frac{\partial F}{\partial \phi} + \frac{d}{dx}\left(\frac{\partial F}{\partial \phi_x}\right) + \frac{d}{dy}\left(\frac{\partial F}{\partial \phi_y}\right)$$

$$\begin{aligned}
\frac{\partial F}{\partial \phi} &= \lambda_1 \int K_\sigma(\mathbf{x}-\mathbf{y})\left|I(\mathbf{y}) - f_1(\mathbf{x})\right|^2 d\mathbf{y} \frac{\partial H_\epsilon(\phi)}{\partial \phi} \\
&+ \lambda_2 \int K_\sigma(\mathbf{x}-\mathbf{y})\left|I(\mathbf{y}) - f_2(\mathbf{x})\right|^2 d\mathbf{y} \frac{\partial(1 - H_\epsilon(\phi))}{\partial \phi} \\
&+ \nu \frac{\partial\left|\nabla H_\epsilon(\phi(\mathbf{x}))\right|}{\partial \phi} + \mu(|\nabla \phi(\mathbf{x})| - 1)\frac{\partial(|\nabla \phi(\mathbf{x})| - 1)}{\partial \phi} \\
&= \delta_\epsilon(\phi)\left(\lambda_1 \int K_\sigma(\mathbf{x}-\mathbf{y})\left|I(\mathbf{y}) - f_1(\mathbf{x})\right|^2 d\mathbf{y} - \lambda_2 \int K_\sigma(\mathbf{x}-\mathbf{y})\left|I(\mathbf{y}) - f_2(\mathbf{x})\right|^2 d\mathbf{y}\right) \\
&= \delta_\epsilon(\phi)(\lambda_1 e_1 - \lambda_2 e_2)
\end{aligned}$$

其中

$$e_1(x) = \int K_\sigma(\mathbf{x}-\mathbf{y})\left|I(\mathbf{y}) - f_1(\mathbf{x})\right|^2 d\mathbf{y},\ e_2(x) = \int K_\sigma(\mathbf{x}-\mathbf{y})\left|I(\mathbf{y}) - f_2(\mathbf{x})\right|^2 d\mathbf{y}$$

$$\begin{aligned}
\frac{\partial F}{\partial \phi_x} &= \nu \frac{\partial\left|\nabla H_\epsilon(\phi(\mathbf{x}))\right|}{\partial \phi_x} + \mu(|\nabla \phi(\mathbf{x})| - 1)\frac{\partial(|\nabla \phi(\mathbf{x})| - 1)}{\partial \phi_x} \\
&= \nu \frac{\partial \sqrt{\left(\frac{\partial H_\epsilon(\phi)}{\partial x}\right)^2 + \left(\frac{\partial H_\epsilon(\phi)}{\partial y}\right)^2}}{\partial \phi_x} + \mu(|\nabla \phi| - 1)\frac{\partial \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}}{\partial \phi_x} \\
&= \nu \frac{\partial \sqrt{\left(\frac{\partial H_\epsilon(\phi)}{\partial \phi}\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial H_\epsilon(\phi)}{\partial \phi}\frac{\partial \phi}{\partial y}\right)^2}}{\partial \phi_x} + \mu(|\nabla \phi| - 1)\frac{\partial \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}}{\partial \phi_x} \\
&= \nu \frac{\partial \sqrt{(\delta_\epsilon(\phi)\phi_x)^2 + (\delta_\epsilon(\phi)\phi_y)^2}}{\partial \phi_x} + \mu(|\nabla \phi| - 1)\frac{\partial \sqrt{(\phi_x)^2 + (\phi_y)^2}}{\partial \phi_x} \\
&= \nu \frac{1}{2}\frac{2(\delta_\epsilon(\phi)\phi_x)\delta_\epsilon(\phi)}{\sqrt{(\delta_\epsilon(\phi)\phi_x)^2 + (\delta_\epsilon(\phi)\phi_y)^2}} + \mu(|\nabla \phi| - 1)\frac{1}{2}\frac{2\phi_x}{\sqrt{\phi_x^2 + \phi_y^2}} \\
&= \nu \frac{\delta_\epsilon(\phi)\phi_x}{\sqrt{\phi_x^2 + \phi_y^2}} + \mu(|\nabla \phi| - 1)\frac{\phi_x}{\sqrt{\phi_x^2 + \phi_y^2}} \\
&= \nu \frac{\delta_\epsilon(\phi)\phi_x}{|\nabla \phi|} + \mu(|\nabla \phi| - 1)\frac{\phi_x}{|\nabla \phi|} \\
&= \nu \delta_\epsilon(\phi)\frac{\phi_x}{|\nabla \phi|} + \mu \phi_x - \mu \frac{\phi_x}{|\nabla \phi|}
\end{aligned}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial \phi_x}\right) = \nu\delta_\epsilon(\phi)\frac{d}{dx}\left(\frac{\phi_x}{|\nabla\phi|}\right) + \mu\phi_{xx} - \mu\frac{d}{dx}\left(\frac{\phi_x}{|\nabla\phi|}\right)$$

同理

$$\frac{d}{dy}\left(\frac{\partial F}{\partial \phi_y}\right) = \nu\delta_\epsilon(\phi)\frac{d}{dy}\left(\frac{\phi_y}{|\nabla\phi|}\right) + \mu\phi_{yy} - \mu\frac{d}{dy}\left(\frac{\phi_y}{|\nabla\phi|}\right)$$

那么

$$\begin{aligned}\frac{d}{dx}\left(\frac{\partial F}{\partial \phi_x}\right) + \frac{d}{dy}\left(\frac{\partial F}{\partial \phi_y}\right) &= \nu\delta_\epsilon(\phi)\left[\frac{d}{dx}\left(\frac{\phi_x}{|\nabla\phi|}\right) + \frac{d}{dy}\left(\frac{\phi_y}{|\nabla\phi|}\right)\right] + \mu(\phi_{xx} + \phi_{yy}) \\ &\quad -\mu\left[\frac{d}{dx}\left(\frac{\phi_x}{|\nabla\phi|}\right) + \frac{d}{dy}\left(\frac{\phi_y}{|\nabla\phi|}\right)\right] \\ &= \nu\delta_\epsilon(\phi)div\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + \mu\left(\nabla^2\phi - div\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right)\end{aligned}$$

将

$$\frac{\partial F}{\partial \phi} = \delta_\epsilon(\phi)(\lambda_1 e_1 - \lambda_2 e_2)$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial \phi_x}\right) + \frac{d}{dy}\left(\frac{\partial F}{\partial \phi_y}\right) = \nu\delta_\epsilon(\phi)div\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + \mu\left(\nabla^2\phi - div\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right)$$

代入

$$\frac{\partial \phi}{\partial t} = -\frac{\partial F}{\partial \phi} + \frac{d}{dx}\left(\frac{\partial F}{\partial \phi_x}\right) + \frac{d}{dy}\left(\frac{\partial F}{\partial \phi_y}\right)$$

可得，RSF 能量泛函对应的梯度下降流方程为

$$\frac{\partial \phi}{\partial t} = -\delta_\epsilon(\phi)(\lambda_1 e_1 - \lambda_2 e_2) + \nu\delta_\epsilon(\phi)div\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + \mu\left(\nabla^2\phi - div\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right)$$