应用随机过程

（Chapter Two Poisson Process）

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Chapter 2 Outline of Poisson Process

- Poisson Process Definition
- Properties of Poisson Process
- Nonhomogeneous Poisson Process
- Compound Poisson Process
- Filtered Poisson Process

Course Objective

- What is Poisson Process
  - ✓ Acquire the four typical properties of Poisson Process
  - ✓ Decide the adequacy of using it to approximate actual arrival behaviors
- How to obtain the analytical equation of Poisson Process
  - ✓ Understand mathematical thought from intuitively literal describing to rigorously theoretical deduction
  - ✓ Grasp the probability generating function and Laplace transform for random variables

2.1 Poisson Process Definition

Consider a counting process $N=\{N(t), t>0\}$, where $N(t)$ denotes the number of arrivals in the interval $(0, t]$

- **Definition**

A counting process $N=\{N(t), t>0\}$ is a Poisson Process with rate $\lambda>0$, if it possesses the following properties:

(i) $N(0)=0$,

(ii) It satisfies the stationary and independent increment properties,

(iii) $P\{N(h)=1\}=\lambda h+o(h)$,

(iv) $P(N(h)\geqslant 2\}=o(h)$.

2.1 Poisson Process Definition

Now we aim to prove $P\left\{N(t)=n\right\}=e^{-\lambda t}\dfrac{(\lambda t)^n}{n!},n=0,1,2,\cdots$

Let $P_n(t)=P\left\{N(t)=n\right\}$

1、For n=0

$$P_0(t+h)=P\left\{N(t+h)=0\right\}=P\left\{N(t)=0,N(t+h)-N(t)=0\right\}$$

$$=P\left\{N(t)=0\right\}P\left\{N(t+h)-N(t)=0\right\}$$ By the independent-increment property

$$=P_0(t)P\left\{N(h)=0\right\}$$ By the stationary-increment property

$$=P_0(t)\left[1-\lambda h+o(h)\right]$$ By Property (iii)

We obtain $\dfrac{P_0(t+h)-P_0(t)}{h}=-\lambda P_0(t)+\dfrac{o(h)}{h}$

Taking the limit as $h\to 0$ yields $P_0'(t)=-\lambda P_0(t)$..........(1)

2.1 Poisson Process Definition

2、For $n \geq 1$, we condition on the number of arrivals by time t and write

$$P_n(t+h) = P\{N(t+h) = n\}$$

$$= \sum_{i=0}^{n} P_{n-i}(t) P_i(h)$$

By independent-increment and stationary-increment properties

$$= P_n(t) P_0(h) + P_{n-1}(t) P_1(h) + \sum_{i=2}^{n} P_{n-i}(t) P_i(h)$$

$$= P_n(t)\left[1 - \lambda h + o(h)\right] + P_{n-1}(t)\left[\lambda h + o(h)\right] + o(h)$$

Stage 1. Achieve the linear differential-difference equations

2.1 Poisson Process Definition

We obtain $$\frac{P_n(t+h)-P_n(t)}{h}=-\lambda P_n(t)+\lambda P_{n-1}(t)+\frac{o(h)}{h}$$

Taking the limit as $h \to 0$ yields

$$P_n'(t)=-\lambda P_n(t)+\lambda P_{n-1}(t),(n=1,2,\cdots)\text{.............}(2)$$

Define the probability generating function for random variable $N(t)$

$$P^g(z,t)=\sum_{n=0}^{\infty} z^n P_n(t),|z|<1$$

$$P^{(1)}(z,t)=\partial P^g(z,t)/\partial t$$

2.1 Poisson Process Definition

Differentiating the preceding equation with respect to t and using Equations (1) and (2), we obtain

$$P^{(1)}(z,t) = \sum_{n=0}^{\infty} z^n P_n'(t) = -\lambda \sum_{n=0}^{\infty} z^n P_n(t) + \lambda z \sum_{n=0}^{\infty} z^n P_n(t)$$

$$= -\lambda P^g(z,t) + \lambda z P^g(x,t) = \lambda(z-1) P^g(z,t)$$

The boundary condition of the equation is given by

$$P^g(z,0) = \sum_{n=0}^{\infty} z^n P\{N(0)=\mathrm{n}\} = z^0 P\{N(0)=0\} = 1$$

For notational convenience, we let $f(t) = P^g(z,t)$ and $a = \lambda(z-1)$,

$$f'(t) = af(t) \text{ and } f(0) = 1$$

2.1 Poisson Process Definition

$$sf^{e}(s) - f(0) = af^{e}(s)$$

We obtain $f^{e}(s) = 1/(s-a)$

Inverting the transform gives

$$f(t) = P^{g}(z,t) = e^{at} = e^{\lambda(z-1)t}, t \geq 0$$

Consequently, we conclude that

$$P^{g}(z,t) = e^{-\lambda t}e^{\lambda zt} = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda zt)^{n}}{n!} = \sum_{n=0}^{\infty}e^{-\lambda t}\frac{(\lambda t)^{n}}{n!}z^{n}$$

$$P^{g}(z,t) = \sum_{n=0}^{\infty}z^{n}P_{n}(t)$$

$$P_{n}(t) = e^{-\lambda t}\frac{(\lambda t)^{n}}{n!}$$

Stage 2. Apply Laplace transform to solve the earlier equations

Hits

- 完全理解泊松过程的定义

2.2 Properties of Poisson Process

- **Interarrival time distribution**

Let S_n denote the epoch of the nth arrival of N and define S_0=0. The interarrival time X_n is then given by $S_n - S_{n-1}$. Then we have

$$S_n = \sum_{k=1}^{n} X_k \quad \text{n=1,2,...}$$

$\{X_n\}$ are i.i.d random variables. What is the distribution of $\{X_n\}$?

Hints: a given interarrival time longer than t means that there is no event in the period of t

2.2 Properties of Poisson Process

A key identity enables to obtain the distribution of $N(t)$ is

$$\{X_1 > t\} \Leftrightarrow \{N(t) = 0\}$$

$$P\{X_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

For any $s>0$ and $t>0$, we see that

$$P\{X_2 > t | X_1 = s\} = P\{0 \text{ events in } (s, s+t] | X_1 = s\}$$

$$P\{X_2 > t | X_1 = S\} = P\{0 \text{ events in } (s, s+t]\}$$

$$P\{X_2 > t | X_1 = S\} = P\{0 \text{ events in } (0, t]\} = e^{-\lambda t}$$

Using the identity $\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$

$$P\{S_n \leq t\} = P\{N(t) \geq n\} = \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

2.2 Properties of Poisson Process

- **Generating arrival times of a Poisson process by computer simulation**

Assume that the Poisson process has a rate λ. To generate arrival times $\{S_n\}$, we can successively generate the exponential interarrival times $\{X_n\}$. The generation of an exponential variate X with parameter λ can be done by the inverse transform method:

(i) Generate $U \sim U(0,1)$

(ii) Let $X = F^{-1}(U)$, where $X \sim F$.

Since $F_X(x) = 1 - e^{-\lambda x}$, $x \geq 0$, it is easy to verify that $X = -\frac{1}{\lambda}\log(1-U)$

Furthermore, since $1-U \sim U(0,1)$, we also have $X = -\frac{1}{\lambda}\log(U)$

2.2 Properties of Poisson Process

- **Generation the Poisson arrival count by computer simulation**

For fixed t, we may at times want to simulate the random variable $N(t)$, the number of arrivals by time t. By definition, we have $N(t) = \max\{n \mid S_n \le t\}$,using the result given above example, we see that

$$N(t) = \max\{n \mid \sum_{k=1}^{n} -\frac{1}{\lambda}\log(U_k) \le t\} = \max\{n \mid \sum_{k=1}^{n}\log(U_k) \ge -\lambda t\}$$
$$= \max\{n \mid \log(U_1 \cdots U_n) \ge -\lambda t\} = \max\{n \mid U_1 \cdots U_n \ge e^{-\lambda t}\}$$

Where U_k denotes the kth standard uniform variate generated

Hints: in simulation, we generate successive $\{U_k\}$ until the last condition is violated for the first time. Let U_N be the last uniform variate so obtained; the simulated $N(t)$ is then given by N-1.

2.2 Properties of Poisson Process

- **Past arrival times given $N(t)$**

Let $Y_1, \cdots, Y_n$ be i.i.d. random variables with common density f, $Y_{(1)}, \cdots Y_{(n)}$ are the corresponding n order statistic. Then the joint density of $\{Y_{(i)}\}$ is given by

$$f_{Y_{(1)},\ldots Y_{(n)}}(y_1, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i)$$

Given that $N(t)=n$, we show that the n arrival times $S_1, \cdots S_n$ have the same distribution as the order statistics corresponding to the n i.i.d. samples from $U(0,t)$. That is

$$f_{S_1,\ldots S_n|N(t)}(t_1, \cdots, t_n \mid n) = \frac{n!}{t^n} \qquad 0 < t_1 < \cdots < t_n < t$$

2.2 Properties of Poisson Process

- Proof:

$$P\{t_i \le S_i \le t_i + h_i, i = 1, \cdots, n \mid N(t) = n\}$$

$$= \frac{p\{\, one\ event\ \ in\ (t_i, t_i + h_i], 1 \le i \le n, no\ events\ elsewhere\ \ in\ (0, t]\}}{p\{N(t) = n\}}$$

$$= \frac{\lambda h_1 e^{-\lambda h_1} \cdots \lambda h_n e^{-\lambda h_n} e^{-\lambda(t - h_1 - \cdots h_n)}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}} = \frac{n!}{t^n} h_1 \cdots h_n$$

Dividing the last equality by $h_1, \cdots h_n$ yields

$$\frac{P\{t_i \le S_i \le t_i + h_i, i = 1, \cdots, n \mid N(t) = n\}}{h_1 \cdots h_2} = \frac{n!}{t^n}$$

2.2 Properties of Poisson Process

- **Example 1**

A cable TV company collects \$1/unit time from each subscriber. Subscribers sign up in accordance with a Poisson process with rate λ. What is the expected total revenue received in $(0, t]$?

Hints: Depends on the total number of subscribers and their arriving time

2.2 Properties of Poisson Process

Solution:

Let $N(t)$ denote the number of subscribers, and S_i denote the arrival time of the ith customer. The revenue generated by this customer in $(0,t]$ is t-S_i. Adding the revenues generated by all arrivals in $(0,t]$, we obtain the expected total revenue

$$E\left[\sum_{i=1}^{N(t)}(t-S_i)\right]$$

We first find the previous expectation by conditioning on $N(t)$

$$E\left[\sum_{i=1}^{N(t)}(t-S_i)\,\middle|\,N(t)=n\right]=E\left[\sum_{i=1}^{n}(t-S_i)\,\middle|\,N(t)=n\right]$$

$$=nt-E\left[\sum_{i=1}^{n}S_i\,\middle|\,N(t)=n\right]$$

2.2 Properties of Poisson Process

$U_1, \cdots U_n$ be i.i.d. random variables which follow $U(0,t)$. so

$$E\left[\sum_{i=1}^{n} S_i \middle| N(t) = n\right] = E\left[\sum_{i=1}^{n} U_i\right] = \sum_{i=1}^{n} E[U_i] = n\left(\frac{t}{2}\right)$$

$$E\left[\sum_{i=1}^{N(t)} (t - S_i) \middle| N(t) = n\right] = N(t)\left(\frac{t}{2}\right)$$

Calculate the expectation by conditional expectation:

$$E\left[\sum_{i=1}^{N(t)} (t - S_i)\right] = \frac{E[N(t)t]}{2} = \frac{1}{2}\lambda t^2$$

2.2 Properties of Poisson Process

- **Decomposition of Poisson process**

A Poisson process $N=\{N(t), t \geqslant 0\}$ with rate λ. We consider the case in which if an arrival occurs at time S,

it is a type-1 arrival with probability $P(s)$ and a type-2 arrival with probability 1-$P(s)$. The type of arrival depends on the epoch of arrival.

2.2 Properties of Poisson Process

- **Proposition**

Let $N_i = \{N_i(t),\ t \geqslant 0\}$, $i = 1$ and 2. where $N_i(t)$ denotes the number of type-i arrivals in $(0, t]$. $N_1(t)$ and $N_2(t)$ are two independent Poisson random variables with means λpt and λqt,

$$p\{N_1(t) = n, N_2(t) = m\} = \left[e^{-\lambda pt}\frac{(\lambda pt)^n}{n!}\right]\left[e^{-\lambda qt}\frac{(\lambda qt)^m}{m!}\right]$$

where $p = \frac{1}{t}\int_0^t p(s)ds$ and $q = 1 - p$

2.2 Properties of Poisson Process

- **Example 2**

Cars arrive at Galveston Beach during spring break. Assume that the interarrival time of cars follows an exponential distribution with parameter λ and the sojourn time of a car on the beach follows a probability distribution G. Also, we assume that the sojourn times are independent of each other and the arrival process, the beach can hold an unlimited number of cars, and at time 0 there are no cars on the beach. Let $N_1(t)$ denote the number of cars that have left the beach at time t and $N_2(t)$ the number of cars still at the beach at time t. What can be said about the two random variables $N_1(t)$ and $N_2(t)$?

Find means of $N_1(t)$ and $N_2(t)$.

2.2 Properties of Poisson Process

Solution:

X denotes sojourn time

$N_1(t)$: numbers of car left the beach at time t
$N_2(t)$: numbers of car still at the beach at time t

$\{X < t\text{-}s\} \Leftrightarrow$ {Car at time t will be type1}

$P\{X < t\text{-}s\} = P$\{Car at time t will be type 1\}
$P\{X < t\text{-}s\} = G(t\text{-}s) = P(s)$

$$E[N_1] = \lambda p t = \lambda \int_0^t G(t-s)ds \qquad E[N_2] = \lambda t(1-p) = \lambda t - \lambda \int_0^t G(t-s)ds$$

where $p = \dfrac{1}{t}\int_0^t G(t-s)ds$

Hints

- 理解间隔时间分布和泊松过程之间的关系
- 掌握泊松过程的分解过程

2.3 Nonhomogeneous Poisson process

- **Definition:**

The counting process $N=\{N(t),\ t\geqslant 0\}$ is called a non-homogeneous Poisson process with intensity function $\{\lambda(t),\ t\geqslant 0\}$ if it possesses the following properties:

(i) $N(0)=0$

(ii) It satisfies the independent-increment property

(iii) $P\{N(t+h)-N(t)=1\}=\lambda(t)h+o(h)$

(iv) $P\{N(t+h)-N(t)\geqslant 2\}=o(h)$

2.3 Nonhomogeneous Poisson process

- **Comparison** with Poisson process:

(i) Non-homogeneous Poisson process does not satisfy the stationary-increment property

(ii) Constant arrival rate λ of a Poisson process is replaced by a time-varying intensity function $\lambda(t)$

2.3 Nonhomogeneous Poisson process

- **Proposition**

If $N = \{N(t),\ t \geqslant 0\}$ is a non-homogeneous Poisson process, then

$$P\{N(t+s) - N(t) = n\} = e^{-[m(t+s)-m(t)]}\frac{[m(t+s) - m(t)]^n}{n!}$$

where $m(t) = \int_0^t \lambda(\mu)d\mu$

The number of arrivals in interval $(t,\ t+s]$ follows a Poisson distribution with parameter $m(s+t)$-$m(t)$, and $m(s+t)$-$m(t)$ is the expected number of arrivals in the interval $(t,\ t+s]$.

2.3 Nonhomogeneous Poisson process

- **Derivation of the formula**

For n=0

$$p_0(s+h) = p\{N(t+s+h) - N(t) = 0\}$$

$$= p\{0\ events\ \ in\,(t, t+s],\, 0\ event\ in\,(t+s, t+s+h]\}$$

$$= p\{0\ event\ in\,(t, t+s]\} \times p\{0\ event\ in\,(t+s, t+s+h]\}$$

$$= p_0(s)\,[1 - \lambda(t+s)h] + o(h)$$

We see that the second equality of the previous formula holds because of the independent-increment property, and the third equality results from properties (iii) and (iv). This leads to

$$p_0^{'}(s) = -\lambda(t+s)p_0(s)$$

2.3 Nonhomogeneous Poisson process

- For $n \geq 1$, using the law of total probability, properties (i)-(iii), and the nature of little-oh functions to obtain

$$p_n(s + h) = p\{N(t + s + h) - N(t) = n\}$$

$$= p\{n - 1\, events\;\; in\,(t, t + s]\} \times p\{1\, event\;\; in\,(t + s, t + s + h]\} + p\{n\, events\;\; in\,(t, t + s]\} \times p\{0\, event\;\; in\,(t + s, t + s + h]\} + o(h)$$

$$= p_{n-1}(s)\,[\lambda(t + s)h + o(h)] + p_n(s)\,[1 - \lambda(t + s)h + o(h)] + o(h)$$

Similarly, this leads to

$$p_n'(s) = -\lambda(t+s)p_n(s) + \lambda(t+s)p_{n-1}(s) \qquad n \geq 1$$

2.3 Nonhomogeneous Poisson process

- The probability generating function

$$P^g(z,s) = \sum_{n=0}^{\infty} z^n P_n(s)$$

$$P^{(1)}(z,s) = -\lambda(t+s)P^g(a,s) + z\lambda(t+s)P^g(z,s)$$

$$= [-\lambda(t+s) + z\lambda(t+s)]P^g(z,s) \qquad (2.3.1)$$

where $P^{(1)}(z,s) \equiv \partial P^g(z,s)/\partial s$

We let $a(s) = [-\lambda(t+s) + z\lambda(t+s)]$ and $f(s) = P^g(z,s)$

Then equation 2.3.1 reduces to $f'(s) = a(s)f(s)$

2.3 Nonhomogeneous Poisson process

Integrating the preceding yields $\log f(s) = \int_0^s a(u)du$

Hence the solution of the differential equation in its original notations is given by

$$\log P^g(z,s) = \int_0^s [-\lambda(t+u) + z\lambda(t+u)]du$$

$$= -[m(t+s) - m(t)] + z[m(t+s) - m(t)]$$

Exponentiating the preceding expression gives

$$P^g(z,s) = e^{-[m(t+s)-m(t)]+z[m(t+s)-m(t)]} = e^{-[m(t+s)-m(t)]}e^{z[m(t+s)-m(t)]}$$

$$= e^{-[m(t+s)-m(t)]}\sum_{n=0}^{\infty}\frac{[m(t+s)-m(t)]^n}{n!}z^n$$

2.3 Nonhomogeneous Poisson process

- **Generating arrival times of a nonhomogeneous Poisson process by computer simulation-method 1**

Consider a nonhomogeneous Poisson process N with intensity function $\lambda(t)$. Assume that $\lambda(t) \leq \lambda$ for all $t \geq 0$. We use the scheme developed in example 2.2.1 to generate a Poisson arrival sequence $\{S_i\}$. The arrival at S_i will be counted as an arrival of N with probability $\lambda(S_i)/\lambda$. Such a process is called thinning in the sense that the newly created arrival stream has been thinned out from the original nonhomogeneous Poisson process N. This is true because the thinned sequence inherits all the properties of a Poisson process except the stationary-increment assumption.

2.3 Nonhomogeneous Poisson process

To check whether property (iv) of a nonhomogeneous Poisson process in the current situation is satisfied, we define A={one arrival of N in $(t, t+h]$} and B={a Poisson arrival in $(t, t+h]$}. Then we see that

$$P\{A \cap B\} = P\{A|B\}P\{B\} = \frac{\lambda(t)}{\lambda}[\lambda h + o(h)] = \lambda(t)h + o(h)$$

So the results obtains from this sampling procedure will indeed produce simulated arrival times from the nonhomogeneous Poisson process.

2.3 Nonhomogeneous Poisson process

- **Generating arrival times of a nonhomogeneous Poisson process by computer simulation-method 2**

A second method to generate arrival times from a nonhomogeneous Poisson process. Suppose we have already obtained n samples $S_1 = s_1, \cdots S_n = s_n$, we are about to generate the next arrival time S_{n+1} . Let $\tau = S_{n+1} - S_n$, which follows the following distribution

$$F_\tau(x|S_n = t) = 1 - e^{-[m(t+x)-m(t)]} \qquad x>0$$

So, to obtain the next sample $S_{n+1}=s_{n+1}$, we simply take a sample from the preceding distribution.

2.3 Nonhomogeneous Poisson process

- **Convert nonhomogeneous Poisson to homogeneous Poisson**

If we have a set of arrival times $\{S_i\}$ of a nonhomogeneous Poisson processes with intensity function $\lambda(t)$, we can convert them to a corresponding set of arrival times $\{Z_i\}$, where the latter are samples from a Poisson process with parameter $\lambda = 1$. This is done by setting $Z_i = m(S_i)$, where $m(t)$ is the integrated intensity function. To prove the validity of the transformation, we use the second characterization of a Poisson process. To see whether the second property holds for the transformed data $\{Z_i\}$, we define the respective $M = \{M(u), u > 0\}$, where M(u) denotes the number of arrivals $\{Z_i\}$ in the interval of (0, u].

2.3 Nonhomogeneous Poisson process

Counting process $M=\{M(u), u\geq 0\}$,where $M(u)$ denotes the number of arrivals $\{Z_i\}$ in interval $(0,u]$, and need to show that

$$E\left[M(u+s)-M(u)\middle|M(v), v\leq u\right]=s$$

for all $u, s\geq 0.$ we see that

$$\begin{aligned}
&E\left[M(u+s)-M(u)\middle|M(v), v\leq u\right] \\
&=E\left[N\left(m^{-1}(u+s)\right)-N\left(m^{-1}(u)\right)\middle|N\left(m^{-1}(v)\right), m^{-1}(v)\leq m^{-1}(u)\right] \\
&=E\left[N\left(m^{-1}(u+s)\right)-N\left(m^{-1}(u)\right)\right] \\
&=m\left(m^{-1}(u+s)-m\left(m^{-1}(u)\right)\right)=u+s-u=s.
\end{aligned}$$

2.3 Nonhomogeneous Poisson process

- **Example 4** (Example 2.3.5)

In the study of the use patterns of a Hewlett Packard computer designed for online analysis of electrocardiograms, arrival data have been analyzed for developing an input processes for subsequent uses in computer simulation and analytical model building.

Solution: in modeling the arrival process, a piece-wise polynomial is used to approximate the intensity function $\lambda(t)$

2.3 Nonhomogeneous Poisson process

- **Example 5** (Example 2.3.8)

Consider a service system with s identical servers. The arrivals to the system follow a nonhomogeneous Poisson process with intensity function $\lambda(t)$. The service time of each server follows an exponential distribution with parameter u, when $k(\leq s)$ servers are busy at the same time, we assume that the k service times are mutually independent and independent of the arrival process. Moreover, when k servers are busy simultaneously at any epoch, the time S for the first service completion to occur follows an exponential distribution with parameter ku-this is due to the memoryless property of the exponential distribution and the fact that S is the minimum of k exponential random variables each with parameter u.

2.3 Nonhomogeneous Poisson process

Solution: let $X(t)$ denote the number of customers in the system at time t and assume $X(0)=0$. Define $p_n(t)=p\{X(t)=n\}$. In the following, we derive a system of differential-difference equations characterizing $\{p_n(t)\}$. The general approach is similar to the derivations of the Poisson and nonhomogeneous Poisson processes.

2.3 Nonhomogeneous Poisson process

- **Example 6** (Example 2.3.9)

Consider the computer system for processing electrocardiograms presented in examples 2.3.5. we start with the case in which the arrival process follows the nonhomogeneous Poisson process with the intensity function $\lambda(t)$. An arriving person seeing all waiting spaces are occupied will leave have no influence on the future of the system. Since the arrival rate exceeds the service rate, the system will be full from time to time. We are interested in obtaining the probability that an arrival will be lost as a function of time of day.

Solution: the system of difference-differential equations for this queue is slightly different. The dimension of the $Q(t)$ matrix is finite and of size 5*5 and there is only one server.

2.3 Nonhomogeneous Poisson process

- **Departure Process from an M/G/∞ Queue**

In an $M/G/\infty$ queue, the arrival process is Poisson with rate λ, the service time distribution is given by G, and there is an infinite number of servers in the system. We assume that service times are mutually independent and independent of the arrival process. Let $M(t)$ denote the number of service completions in $(0,t]$.

Show that the departure process $M=\{M(t),\ t\geqslant 0\}$ from this queue is a nonhomogeneous Poisson process with intensity function $\lambda(t)=\lambda G(t)$.

2.3 Nonhomogeneous Poisson process

- **Solution:**

Let $D(s, s+r)$ denotes the number of service completions in the interval $(s, s+r]$ in $(0,t]$, y: arrival time, S: service time

Type-1 arrival: arrive at time y and its service completion occurs in $(s, s+r]$

To show that $E[D(s, s+r)] = \lambda \int_s^{s+r} G(y) dy$

Case 1: If $y \leqslant s$: $\{s-y < S < s+r-y\} \Leftrightarrow$ {arrival is type 1}
$P(\mathrm{y}) = P\{s-y < S < s+r-y\} = G(s+r-y) - G(s-y)$

Case 2: If $s < y \leqslant s+r$: $\{S < s+r-y\} \Leftrightarrow$ {arrival is type 1}
$P(\mathrm{y}) = P\{S < s+r-y\} = G(s+r-y)$

Case 3: If $s+r < y \leqslant t$: $P(y)=0$
$E[D(s, s+r)] = \lambda pt = \lambda \int_0^t P(y) dy = \lambda \int_0^{s+r} G(s+r-y) dy - \lambda \int_0^s G(s-y) dy$

2.3 Nonhomogeneous Poisson process

- **Example 7** (Example 2.3.10)

In an $M/G/\infty$ queue, the arrival process is Poisson with rate λ, the service time distribution is given by G, and there is an infinite number of servers in the system. We assume that service times are mutually independent and independent of the arrival process. Let $M(t)$ denote the number of service completion in $(0,t]$.

Solution: in this example we show that the departure process

$$M = \{M(t), t \geq 0\}$$

from this queue is a nonhomogeneous Poisson process with intensity function $\lambda(t) = \lambda G(t)$

Hints

- 理解非均匀泊松过程的定义，以及推导过程，并且通过与泊松过程的定义进行比较，充分理解间隔时间，到达时刻等具体知识点。
- 理解如何利用非均匀泊松过程转变成泊松过程，进而来验证原过程为非均匀的泊松过程
- 理解书中的具体案例

2.4 compound Poisson process

- **Definition:**

A stochastic Process $\{X(t),\ t \geqslant 0\}$ is said to be a compound Poisson process if it can be represented as $X(t)=\sum_{n=1}^{N(t)} Y_n$ where $\{N(t),\ t \geqslant 0\}$ is a Poisson process, and $\{Y_n,\ n \geqslant 1\}$ be i.i.d. random variables. The process N and the sequence Y_n are assumed to be independent.

If $Y_n \equiv 1$, then $X(t)=N(t)$ $\Rightarrow$ usual Poisson process.

2.4 compound Poisson process

Probability generating function of compound Poisson processes $X(t)$

$$H_t(z) = E[z^{X(t)}] = \pi_N(P_Y(z)) = e^{\lambda t[P_Y(z)-1]}$$

Mean, variance and q value of composite Poisson process

$$H_t^{(1)}(z) = e^{\lambda t[P_Y(z)-1]}\lambda t[P_Y^{(1)}(z)]$$

$$H_t^{(2)}(z) = e^{\lambda t[P_Y(z)-1]}\lambda t[P_Y^{(2)}(z)] + e^{\lambda t[P_Y(z)-1]}(\lambda t[P_Y^{(1)}(z)])^2$$

Hence

$$E[X(t)] = H^{(1)}(1) = \lambda t E[Y]$$

2.4 compound Poisson process

And $H_t^{(2)}(1) = \lambda t[P_Y^{(2)}(1)] + (\lambda t E[Y])^2$

We write

$$E[X(t)^2] = H_t^{(2)}(1) + H_t^{(1)}(1) = \lambda t[E[Y^2] - E[Y]] + (\lambda t E[Y])^2 + \lambda t E[Y]$$
$$= \lambda t E[Y^2] + (\lambda t E[Y])^2$$

This gives $Var[X(t)] = \lambda t E[Y^2]$

For a compound Poisson process, the variance to mean ratio, defined as q, is given by

$$q = \frac{Var[X(t)]}{E[X(t)]} = \frac{E[Y^2]}{E[Y]}$$

2.4 compound Poisson process

- **The stuttering Poisson process**

For a compound Poisson process $X(t)$, consider the situation in which $\{Y_n\}$ follow a geometric distribution with $P\{Y=y\} = (1-\rho)\rho^{y-1}$, $y=1,2\cdots\cdots$, and probability generating function

$$P_Y(z) = \frac{(1-\rho)z}{1-\rho z}$$

where $0 < \rho < 1$. Find $E[X(t)]$ and $Var[X(t)]$.

Solution:

$$\mathrm{E}[Y] = \frac{1}{1-\rho},\quad E[Y^2] = \frac{1+\rho}{(1-\rho)^2},\quad Var[Y] = \frac{\rho}{(1-\rho)^2}$$

$E[X(t)]=\lambda t/(1-\rho)$ and $Var[X(t)]=\lambda t(1+\rho)/(1-\rho)^2$

2.5 Filtered Poisson process

- **Definition**

A stochastic process $X=\{X(t),\ t \geqslant 0\}$ is called a filtered Poisson process if

$$X(t) = \sum_{n=1}^{N(t)} \omega(t, S_n, Y_n)$$

Where

t is current time

$\{S_n\}$ are the arrival times

$N(t)$ is a Poisson process with rate λ

$\{Y_n\}$ be i.i.d. continuous random variables, Y_i is associated with the ith arrival of N

ω is called the **response function**.

2.5 Filtered Poisson process

It is a function of t, S_n and Y_n

the following form of the response function is often used :

$\omega(t,\tau, y) = \omega_0(t-\tau, y)$ where τ is the arrival time before t.

let $s=t-\tau$

if $\omega_0(s, y) = \begin{cases} 1 & s > 0 \\ 0 & otherwise \end{cases}$ $\Longrightarrow$ Usual Poisson process

if $\omega_0(s, y) = \begin{cases} y & s > 0 \\ 0 & otherwise \end{cases}$ $\Longrightarrow$ Compound Poisson process

2.5 Filtered Poisson process

- **Example 8**

In the $M/G/\infty$ queue, let y be the length of the service time, Y_n represents the service time of the nth customer. Let $s = t\text{-}\tau$, where τ is an arrival time before t, S_n represents nth arrival time. $X(t)$ represents the number of customers in the system at time t, please write the response function of $X(t)$?

2.5 Filtered Poisson process

Solution:

we define the response function: $\omega_0(s,y) = \begin{cases} 1 & y > s > 0 \\ 0 & otherwise \end{cases}$

The filtered Poisson process $X(t)$ is defined: $X(t) = \sum_{n=1}^{N(t)} \omega_0(s,\ y)$

Example 9

A cable TV company collects \$1/unit time from each subscriber. Subscribers sign up in accordance with a Poisson process with rate λ. What is the expected total revenue received in $(0,t]$?

2.5 Filtered Poisson process

Solution:

Define $X(t)$,the total revenue received in (0,t], as a filtered Poisson process

$$X(t) = \sum_{n=1}^{N(t)} \omega(t, S_n, Y_n)$$

and define the response function

$$\varpi_0(s, y) = \begin{cases} s & if \ \ s \geq 0 \\ 0 & otherwise \end{cases}$$

$$\left[E[X(t)] = \lambda \int_0^t E[\omega_0(s, Y)] ds = \lambda \int_0^t s ds = \frac{\lambda t^2}{2} \right]$$

2.5 Filtered Poisson process

Expectation and Variance of filtered Poisson processes

$$E\left[X(t)\right] = \lambda \int_0^t E\left[\omega(t, \tau, Y)\right] d\tau$$

$$Var\left[X(t)\right] = \lambda \int_0^t E\left[\omega^2(t, \tau, Y)\right] d\tau$$

Hints

掌握复合泊松过程的概率生成函数，并利用概率生成函数计算均值和方差

理解过滤泊松过程的反映函数的定义，并能够结合实例自行列出反映函数

2.6 Two-Dimensional and Marked Poisson Process

- **Briefly show Two-Dimensional Poisson Process**

Consider a two-dimensional plane S. Let A be a subset of plane S. We envision points being scatted randomly over S and let $N(A)$ denote the number of points in A. The stochastic process $N(A)$ is called a **point process** in S. Let $|A|$ denote the size of the set A. In this case $|A|$ represents the area of A. Stochastic process $N = \{N(A), A \subset S\}$ is a **two-dimensional Poisson process** if

(i) $N(A)$ follows a Poisson distribution with mean $\lambda|A|$

(ii) the numbers of points occurring in disjoint subsets of S are mutually independent.

2.6 Two-Dimensional and Marked Poisson Process

The two-dimensional Poisson process can be generalized to a two-dimensional nonhomogeneous Poisson process. Let $\lambda(x, y)$ be the intensity function of the point process N. The process is a two-dimensional nonhomogeneous Poisson process if

(i) for each $A \subset S$, $N(A)$ follows a Poisson distribution with mean

$$\iint_A \lambda(x, y)dxdy$$

(ii) the numbers of points occurring in disjoint subsets of S are mutually independent.