应用随机过程

（Chapter one Introduction）

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Chapter 1 Outline of Introduction

- Overview
- Preliminary probability knowledge
- Introduction of Stochasitc Processes
- Discrete random variables
- Continuous random variables
- Some mathematical background

Chapter Objective

- What is Stochastic Process
  - ✓ Overview
  - ✓ Grasp the variables transformation about Stochastic processes
- How to study Stochastic Process
  - ✓ Understand relation between each chapter
  - ✓ Understand the basic knowledge, such as, competing exponential random variables and compound random variables.

1.1 Preliminary probability knowledge

- **Sample space and events**

The set of all possible outcomes of an experiment is known as **sample space** of the experiment and is denoted by S.

Any subset E of the sample space S is known as an **event.**

1.1 Preliminary probability knowledge

- **Operations of event:**

(i) $E \cup F$: is referred to as the **union** of the event E and the event F. The event $E \cup F$ will occur if either E or F occurs.

(ii) $E \cap F$: is referred to as the **intersection** of E and F. The event EF will occur only if both E and F occur. If $EF = \Phi$, then E and F are said to be mutually exclusive

(iii) $\overline{E}$: is referred to as the **complement** of E The will occur only if E does not occur.

1.1 Preliminary probability knowledge

- **Probability defined on events**

Consider an experiment whose sample space is S.

For each event E of the sample space S, we assume that a number $P(E)$ is defined and satisfied that following three conditions:

(i) $0 \leqslant P(E) \leqslant 1$

(ii) $P(S) = 1$

(iii) For any sequence of events $E_1, E_2 \cdots$ that are mutually exclusive, i.e. events for which $E_n E_m = \Phi$ when $n \neq m$, then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

We refer to $P(E)$ as the probability of the event E.

$P(E \cup F) = P(E) + P(F) - P(EF)$.

1.1 Preliminary probability knowledge

- **Conditional probability**

Conditional probability is denoted by $P(E|F)$. It states that E occurs given that F has occurred.

$$P(E|F)=\frac{P(EF)}{P(F)} \quad \text{or} \quad P(EF)=P(F)P(E|F)$$

If E and F are independent, then

$$P(EF)=P(E)P(F) \qquad P(E|F)=P(E)$$

1.1 Preliminary probability knowledge

- **Random variable**

The real-valued functions defined on the sample space are known as random variables. Discrete random variable: take on either a finite or a countable number of possible values.

Continuous random variable: take on a continue of possible values.

1.1 Preliminary probability knowledge

- **Distribution function**

Distribution function $F(\cdot)$ of the random variable X is defined for any real number b by $F(b)=P\{X \leqslant b\}$

(i) For a discrete random variable X, the distribution function F can be expressed as

$$F(x_i)=\sum_{all x \leq x_i} p(x)$$

where $p(x_i)$ is the probability mass function of X, $p(x_i)=P\{X=x_i\}$.

1.1 Preliminary probability knowledge

(ii)For a continuous random variable X, the distribution function F can be expressed as

$$F(x_i)=\int_{-\infty}^{x_i} f(x)dx$$

where $f(x)$ is called probability density function of X.

1.1 Preliminary probability knowledge

- Expectation of a random variable

(i) If X is discrete random variable having a probability mass function $p(x)$, then the expected value of X is defined by

$$E[X]=\sum_{i} x_i p(x_i)$$

(ii) If X is continuous random variable having a probability density function $f(x)$, then the expected value of X is defined by

$$E[X]=\int_{-\infty}^{+\infty} x f(x) dx$$

1.1 Preliminary probability knowledge

(iii) Expectation of $g(X)$

(a) If X is a discrete random variable with probability mass function $p(x)$,then for any real-valued function $g(X)$,

$$E[g(X)] = \sum_i g(x_i)p(x_i)$$

(b) If X is continuous random variable with probability density function $f(x)$, then for any real-valued function $g(X)$,

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx$$

1.1 Preliminary probability knowledge

The expected value of a random variable X, $E[X]$, is also referred to as the mean or the first moment of X.

The quantity $E[X_n]$, $n \geqslant 1$, is called the nth moment of X.

$$E\left[X^n\right]=\begin{cases}\sum_i x_i^n p(x) \\ \int_{-\infty}^{+\infty} x^n f(x)dx\end{cases}$$

1.1 Preliminary probability knowledge

Variance of a random variable

$$Var(X) = E[(X - E[X])^2] = E([X^2]) - (E[X])^2$$

Conditional expectation of a random variable

Discrete case: $E[X|Y = y] = \sum_x x\, P\{X = x|Y = y\} = \sum_x x\, p_{X|Y}(x|y)$

Continuous case: $E[X|Y = y] = \int_{-\infty}^{+\infty} x f(x|y)\, dx$

Conditional variance of a random variable

$$Var(X|Y = y) = E\left[(X - E[X|Y = y])^2 | Y = y\right] = E\left[X^2|Y = y\right] - (E[X|Y = y])^2$$

1.1 Preliminary probability knowledge

- Compound random variable

Let $\{X_i\}$ be a sequence of i.i.d. (independently and identically distributed), nonnegative, and integer-valued random variables. Let N be a nonnegative and integer-valued random variable. The compound random variable S_N is defined as the sum of $X_1, \cdots X_N$, this random variable is often called the random sum

$$
\begin{aligned}
E[S_N] &= E[X_1]E[N] \\
Var[S_N] &= Var[X_1]E[N] + E^2[X_1]Var[N]
\end{aligned}
$$

1.1 Preliminary probability knowledge

- **Jointly distributed random variable**

For any two random variables X and Y, the joint cumulative probability distribution function of X and Y is defined by

$F(a,b)=P\{X\leqslant a,\ Y\leqslant b\}$, $-\infty<a$ and $b<\infty$

X and Y are both discrete random variables:

$$F(a,b)=\sum_{x<a}\sum_{y<b}p(x,y)$$

X and Y are both continuous random variables:

$$F\{X\in A,Y\in B\}=\int_B\int_A f(x,y)dxdy$$

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1.2 Introduction of stochastic processes

Let $X(t)$ denotes the state of a system at time t. The collection of the random variables $X=\{X(t), t\in T\}$ is called a Stochastic Process.

The set T is called the index set. If we assume that $X(t)$ takes values in a set S for every $t\in T$, then S is called the state space of the process X.

A realization of a stochastic process X is called a sample path of the process.

1.2 Introduction of stochastic processes

- **Classification:**

(i) Discrete-time process with a discrete state space.
The index set T is countable, the state space is countable

(ii) Continuous-time process with a discrete state space
The index set T is an interval of the real line, the state space is countable.

(iii) Discrete-time with a continuous state space
The index set T is countable, the state space S is an interval of a real line.

(iv) Continuous-time process with a continuous state space
The index set T is continuous, the state space S is continuous

Hits

- 了解随机过程的介绍
- 掌握概率论的基本知识

1.3 Generating function for discrete random variables

$\{a_n\}$ denote a sequence of numbers. The Z-transform of $\{a_n\}$ is defined by

$$a^g(z)=\sum_{n=0}^{\infty}a_n z^n$$

Definition: Let X denote a discrete random variable and $a_n = P\{X=n\}$, then we define the probability generating function for

Random variable X is: $P(z) = a^g(z) = E[z^X]$ $(|z| \leqslant 1)$

Define the kth derivative of by $P_X^{(k)}(z)=\frac{d^k}{dz^k}P_X(z)$

1.3 Generating function for discrete random variables

- **Compound random variables**

Let $\{X_i\}$ be a sequence of i.i.d, nonnegative, and integer-valued random variables with a compound probability generating function $P_X(z)$. Let N be a nonnegative and integer valued random variable with a probability generating function $\pi_N(z)$. Assume that N is independent of $\{X_i\}$. The compound random variable S_N is defined as the sum of $x_1,\cdots,x_n$. This random variable is often called the random sum. We let $H_s(z)$ denote the probability generating function of S_N.

1.3 Generating function for discrete random variables

Now we see that

$$H_S(z) = E[z^S] = E_N[E[z^S|N]] = E_N[E[z^{X_1+\dots X_N}|N]]$$

$$= E_N[E[z^{X_1+\dots X_N}]] \quad \text{(by independent if } N \text{ and } \{X_i\})$$

$$= E_N[E[z^{X_1}]\dots E[z^{X_N}]] \quad \text{(by independent of } X_1,\cdots,X_N)$$

$$= E_N[(P_X(z))^N] = \pi_N(P_X(z))$$

TABLE 1.1 A Table of Generating Functions

The Sequence $\{a_n\}$	Generating Function $a^g(z)=\sum_{n=0}^{\infty} a_n z^n$
1. $\{\alpha a_n\}$	$\alpha a^g(z)$
2. $\{\alpha a_n+\beta b_n\}$	$\alpha a^g(z)+\beta b^g(z)$, where $b^g(z)=\sum_{n=0}^{\infty} b_n z^n$
3. $\left\{\sum_{m=0}^{n} a_m b_{n-m}\right\}$ Convolution	$a^g(z)b^g(z)$
4. $\{a^n\}$	$\dfrac{1}{1-az}$
5. $\left\{\dfrac{1}{k!}(n+1)(n+2)\cdots(n+k)a^n\right\}$	$\dfrac{1}{(1-az)^{k+1}}$
6. $\{b_n\}$, where $b_n=0$ if $n<k$; $=a_{n-k}$ if $n\ge k$; and k is a positive integer	$z^k a^g(z)$
7. $\{b_n\}$, where $b_n=0$ if $n<0$; $=a_{n+k}$ if $n\ge 0$; and k is a positive integer	$\dfrac{1}{z^k}\left[a^g(z)-a_0-a_1z-\cdots-a_{k-1}z^{k-1}\right]$
8. $\left\{\sum_{m=0}^{n} a_m\right\}$	$\dfrac{1}{1-z}a^g(z)$
9. $\{b_n\}$, where $b_n=a_0$ if $n=0$; $=a_n-a_{n-1}$ if $n\ge 1$	$(1-z)a^g(z)$
10. $\{A^n\}$, where A is a square matrix	$\sum_{n=0}^{\infty}(zA)^n=[I-Az]^{-1}$, where I is an identity matrix

1.3 Generating function for discrete random variables

- Example

Let N be the number of times a person will visit a store in a year. Assume that N follows the geometric distribution $p\{N = n\} = (1-\theta)\theta^n$ n=0,1,···. During each visit with probability p the person buys something. Purchase will be made during a visit are probabilistically independent and whether a purchase will be made during a visit is independent of number of times the person visits the store in a year. We let have X_i=1 if the person buys something during the ith visit and 0 otherwise. Then we have $S = X_1 + \ldots + X_N$. The probability generating function of X_i is $p_X(z) = q + pz$

1.3 Generating function for discrete random variables

Solution:

$$H_S(z) = \pi_N(p_X(z)) = \frac{1-\theta}{1-\theta p_X(z)} = \frac{1-\theta}{1-\theta[q+pz]}$$

$$= \frac{1-\theta}{(1-q\theta)-p\theta z} = \frac{\dfrac{1-\theta}{1-q\theta}}{1-(\dfrac{p\theta}{1-q\theta})z} = \frac{1-Q}{1-Qz}$$

Where we let $Q = \dfrac{p\theta}{1-q\theta}$

1.4 Generating function for continuous random variables

Let f be any real-valued function defined on [0,∞).

The Laplace transform of *f* is defined as

$$f_X^e(s) = \int_0^{\infty} e^{-sX} f(X)dX = E\left[e^{-sX}\right]$$

Define the nth derivative of the Laplace transform $f_x^e(s)$,with respect to s by

$$f_x^e(s) = \frac{d_n}{ds^n} f_X^e(s) = (-1)^n E\left[X^n e^{-sX}\right]$$

From the equation, we conclude that

$$E\left[x^n\right] = (-1)^n f^{(n)}(0)$$

TABLE 1.2 A Table of Laplace Transforms

The Function $f(t)$	Laplace Transform $f^e(s)=\int_0^\infty e^{-st}f(t)dt$
1. $\alpha f(t)$	$\alpha f^e(s)$
2. $\alpha f(t)+\beta g(t)$	$\alpha f^e(s)+\beta g^e(s)$ where $g^e(s)=\int_0^\infty e^{-st}g(t)dt$
3. $\int_0^\infty f(\tau)g(t-\tau)d\tau$	$f^e(s)g^e(s)$
4. e^{-at}	$\dfrac{1}{s+a}$
5. $\dfrac{1}{k!}t^k e^{-at}$	$\dfrac{1}{(s+a)^{k+1}}$
6. $f(t-\tau)$ $(\tau>0)$	$e^{s\tau}fe(s)$
7. $f(t+\tau)$ $(\tau>0)$	$e^{s\tau}\left[f^e(s)-\int_0^\tau e^{-st}f(t)dt\right]$
8. $\int_0^t f(\tau)d\tau$	$\dfrac{1}{s}f^e(s)$
9. $\dfrac{d}{dt}f(t)$	$sf^e(s)-f(0)$
10. e^{At} where A is a square matrix	$\int_0^\infty e^{-st}e^{At}dt=[sI-A]^{-1}$, where I is an identity matrix

$X(s)$ $X(t)$

$p=1$ $a=1$

te^{-at}

$\dfrac{1}{(s+a)^2}$

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1.4 Laplace transforms for continuous random variables

- **Competing exponential random variables**

Let X_1 and X_2 denote the occurrence times of events 1 and 2, respectively, where $X_1 \sim exp(u_1)$ and $X_2 \sim exp(u_2)$. Assume that X_1 and X_2 are independent. Let X be the first occurrence time, that is , $X= \min\{X_1 , X_2\}$. Hence the two events are competing for the first occurrence.

For examples, assume that a piece of equipment contains two key components. Let X_1 and X_2 denote their respective lifetime and assume that the two lifetimes follow exponential distributions with respective parameters u_1 and u_2. if one component fails, then the equipment fails; the equipment lifetime is given by X.

1.4 Laplace transforms for continuous random variables

- The Erlang random variable

Let $X_1, \cdots X_n$ be i.i.d. random variables whose common density is exponential with parameter $\lambda > 0$. Let $S = X_1 + \cdots + X_n$. Then S is called an Erlang random variable with parameters (n, λ). Clearly S is the convolution of n i.i.d. random variable. Using the Laplace transform of S:

$$f_S^e(s) = \left(\frac{\lambda}{s + \lambda}\right)^n = \lambda^n \frac{1}{(s + \lambda)^{(n-1)+1}}$$

1.4 Laplace transforms for continuous random variables

The moment of *S* are found by noting

$$f_S^{(1)}(s) = \lambda^n(-n)\,(s+\lambda)^{-(n+1)} \text{ and } f_S^{(2)}(s) = \lambda^n(-n)\,(-(n+1)\,(s+\lambda)^{-(n+2)})$$

$$E[S] = -f_S^{(1)}(0) = \frac{n}{\lambda}$$

$$E[S^2] = f_S^{(2)}(0) = \frac{n(n+1)}{\lambda^2}$$

$$Var[S] = \frac{n}{\lambda^2}$$

1.4 Laplace transforms for continuous random variables

For random variable X, the moment generating function is:

$$M(t) = E\left[e^{tX}\right] = \begin{cases} \sum_{x=-\infty}^{\infty} e^{tx} p(x) \\ \int_{-\infty}^{+\infty} e^{tx} f(x) dx \end{cases}$$

Relation between moments of random variable X and a moment generating function $M(t)$: By successive differentiating $M(t)$ with respect to t and setting the resulting expressions equal to zero, we find a formula for finding moments of X, that is :

$$E\,[X^n] = M^{(n)}(0)$$

1.5 Some mathematical background

- **Right and left continuity and limits**

A function F(t) is defined as right-continuous if

$$\lim_{t \to \tau} F(t) = F(\tau)$$

For all τ （"$t \to \tau$" means that t approaches τ from the right)

1.5 Some mathematical background

- **Riemann-Stieltjes integrals**

Let g be a continuous function and F be a non-decreasing function. A subdivision of interval (a,b) is a set of numbers $\{x_0, x_1, \cdots x_n\}$. The subdivision divides the interval into n disjoint subintervals $(x_0, x_1), \cdots, (x_{n-1}, x_n)$. The Riemann-Stieltjies integral of g with respect to F from a to b is

$$\int_a^b g(x)dF(x) = \lim_{\Delta \to 0} \sum_{k=1}^{n} g(\zeta_k)[F(x_k) - F(x_{k-1})]$$

where $x_{k-1} < \zeta \le x_k, k = 1, \cdots, n$ and $\Delta = \max\{x_1 - x_0, \cdots, x_n - x_{n-1}\}$

1.5 Some mathematical background

- **Taylor series expansion**

Let $f(x)$ be a continuous function possessing n+1 derivatives for all x in the interval $[a,b]$. For any $0 \leq h \leq b - a$, we have, for some s between a and $a+h$,

$$f(a+h) = f(a) + \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!} h^{i} + \frac{f^{(n+1)}(s)}{(n+1)!} h^{n+1}$$

Where $f^{(i)}(a)$ denotes the ith derivative of $f(x)$ with respect to x evaluated at a.

1.5 Some mathematical background

- **Little-oh functions**

For any function f, we say that it is a little-oh function if f possesses the following property:

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

Hits

- 掌握离散变量的概率生成函数和连续变量的拉普拉斯变换
- 掌握重点几个案例，例如复合随机变量和竞争型指数分布等
- 掌握基本的数学计算方法