Homework 1

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1 Class Questions

Problem 1. How many zeroes are at the end of 1000!

Solution.
The number of multiples of 5's less than 1000 is the same a the number of 0s at the end of 1000! as every multiple of 10 will add a 0 automatically, and every multiple of 5 but not 10, just needs an even number to make a multple of 10, and there will be more even numbers than 5's. Then 1000/5=200. So the number of zeros at the end of 1000! is **200**.

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Problem 2. Calculate (3141) and (10001,100083).

Solution.
Using the Euclidean algorithm we can see

$$\begin{aligned} 100083 &= (10001)(10) + 73 \\ 10001 &= (73)(137) + 0 \end{aligned}$$

So $(10001, 100083) = 73$.

Problem 4. Find x and y such that $4144x + 7696y = 592$.

Solution.
Diving by the GCD we get

$$\begin{aligned} 4144x + 7696y &= 592 \\ 7x + 13y &= 1 \end{aligned}$$

Then, use UA

$$\begin{aligned} 13 &= 7(1) + 6 \\ 7 &= 6(1) + 1 \\ 6 &= 1(6) + 0 \end{aligned}$$

Then, doing it back we get

$$
\begin{aligned}
1 &= 7 - 6(1) \\
1 &= 7 - (13 - 7(1)) \\
1 &= 7 - 13 + 7 \\
1 &= 7(2) + 13(-1)
\end{aligned}
$$

Multiply by 592 and we get,

$$592 = 4144(2) + 7696(-1)$$

So an integer solution would be $x = 2$ and $y = -1$.

Problem 5. If $N = abc + 1$, prove that $(N, a) = (N, b) = (N, c) = 1$.

Proof.
Let $N = abc + 1$. WLOG, $(N, a) = d$ and assume to the contrary that $d \neq 1$. This would mean that $d|N$ and $d|a$, then $\exists k$ where $N = dk$ and $\exists m$ where $a = dm$ meaning that

$$
\begin{aligned}
N &= abc + 1 \\
dk &= abc + 1 \\
dk &= d(mbc) + 1
\end{aligned}
$$

But we can see that $d|1$ must be true, meaning that $d = 1$ which is a contradiction. This means that $(N, a) = (N, b) = (N, c) = 1$. $\square$

Problem 6. Find two different solutions of $299x + 247y = 13$.

Solution.
Dividing by the GCD, we can get the following

$$
\begin{aligned}
299x + 247y &= 13 \\
23x + 19y &= 1
\end{aligned}
$$

Then using $y = t$ and $x = \frac{1-23t}{19}$, we can see $t = 5$ would have an integer solutoins. So a solution for this would be $x = -6$ and $y = 5$.

Problem 7. Prove that if $a|b$ and $b|a$ then $a = b$ or $a = -b$.

Proof. Let $a|b$ and $b|a$, this means that $\exists m, nin\mathbb{N}$ s.t. $bm = a$ and $an = b$, then

$$
\begin{aligned}
bm &= a \\
anm &= a \\
nm &= 1
\end{aligned}
$$

But this means that $m, n = \pm 1$. So $a = b$ or $a = -b$. $\square$

Problem 9. Prove that $((a,b),b) = (a,b)$.

Solution.

Let $((a,b),b) = d$. This means that $d|(a,b)$ and $d|b$. Let $(a,b) = c$, meaning that $c|a$ and $c|b$. We can see that $d|c$ and $d|b$, since $d|c$ and $c|a$ we know that $d|a$ and $d|b$. But since $(a,b) = c$ this means that $d \leq c$. However, since $d|c$ this means that $d = c$, so $((a,b),b) = (a,b)$.

Problem 12. Prove: If $a|b$ and $c|d$, then $ac|bd$.

Proof. Let $a|b$ and $c|d$, this then means that $\exists i,j$ s.t. $ai = b$ and $cj = d$. Then do the following,

$$\begin{aligned} cj &= d \\ (cj)b &= bd \\ (cj)ai &= bd \\ ac(ij) &= bd \end{aligned}$$

Which is the definition of $ac|bd$. $\square$

Problem 13. Prove: If $d|a$ and $d|b$ then $d^2|ab$.

Proof. Let $d|a$ and $d|b$, this then means that $\exists i,j \in \mathbb{N}$ s.t. $di = a$ and $dj = b$. Then do the following,

$$\begin{aligned} di &= a \\ b(di) &= ab \\ (dj)(di) &= ab \\ d^2(ij) &= ab \end{aligned}$$

Which is the definition of $d^2|ab$. $\square$

Problem 14. Prove: If $c|ab$ and $(c,a) = d$, then $c|db$.

Proof. Let $c|ab$ and $(c,a) = d$. $c|ab$ means that $\exists n \in \mathbb{N}$ s.t. $cn = ab$. Then $c, a = d$ means that $\exists i,j \in \mathbb{N}$ s.t. $d(i) = c$ and $d(j) = a$ and any other divisor of c and a are $\leq d$.

It follows that $d|ab$. Since $c|ab$

Then assume $j \nmid c$, that would mean that

$\square$

Problem 15.

(a) If $x^2 + ax + b = 0$ has an integer root, show that it divides b.

(b) If $x^2 + ax + b = 0$ has a rational root, show that it is in fact an integer.

Solution.

(a)

(b)

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Problem 2. Find the prime-power decompositions of 2345, 45670, and 999999999999. (Note that $101|1000001$).

Solution.
2345 - $5 \cdot 7 \cdot 67$

Problem 8. If $d|ab$, does it follow that $d|a$ or $d|b$?

Solution.
No, a counter example would be $a = 6$, $b = 2$, and $d = 4$. $4|12$, but $4 \nmid 6$ and $4 \nmid 2$.

Problem 10. Prove that $n(n + 1)$ is never a square for $n > 0$.

Proof. Let $n > 0$. Assume to the contrary that $k^2 = n(n + 1)$ for some k. This means that $n|k^2$ and $(n + 1)|k^2$. □

4 Sage Work

Problem A. How many primes are there less than 10^6?

Solution.

```
upper = 10^6
primes = list(filter(is_prime, [1..upper]))
count = len(primes)

print(f'There are {count} primes less than {upper}.')
```

```
    There are 78498 primes less than 1000000.
```

Problem B. Find x, y such that $2015x + 93y = 31$

Solution.

```
GCD, x, y = xgcd(2015,93)

print(f'One solution for 2015x+93y-31 is x={x} and y={y}.')
```

```
    One solution for 2015x+93y-31 is x=-1 and y=22
```

Problem C (Extra Credit). Let $r(n)$ be the number formed by repeating n 1s. For example $r(5) = 11111$. Find $\gcd(r(2025), r(103))$.

Solution.

```python
def r(n):
    answer = 0
    while n>0:
        answer = answer + 10**(n-1)
        n = n-1
    return answer

outcome = gcd(r(2025),r(103))

print(f'The gcd of r(2025) and r(103) is {outcome}.')
```

```
    The gcd of r(2025) and r(103) is 1.
```