Homework 1

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1 Class Questions

Problem 1. How many zeroes are at the end of 1000!

Solution.
The number of multiples of 5's less than 1000 is the same a the number of 0s at the end of 1000! as every multiple of 10 will add a 0 automatically, and every multiple of 5 but not 10, just needs an even number to make a multpe of 10, and there will be more even numbers than 5's. Then 1000/5=200. So the number of zeros at the end of 1000! is **200**.

2 Page 9 Problems

Problem 2. Calculate (3141) and (10001,100083).

Solution.
Using the Euclidean algorithm we can see

$$100083 = (10001)(10) + 73$$
$$10001 = (73)(137) + 0$$

So $(10001, 100083) = 73$.

Problem 4. Find x and y such that $4144x + 7696y = 592$.

Solution.
Diving by the GCD we get

$$4144x + 7696y = 592$$
$$7x + 13y = 1$$

Then, use UA

$$13 = 7(1) + 6$$
$$7 = 6(1) + 1$$
$$6 = 1(6) + 0$$

Then, doing it back we get

$$\begin{aligned}
1 &= 7 - 6(1) \\
1 &= 7 - (13 - 7(1)) \\
1 &= 7 - 13 + 7 \\
1 &= 7(2) + 13(-1)
\end{aligned}$$

Multiply by 592 and we get,

$$592 = 4144(2) + 7696(-1)$$

So an integer solution would be $x = 2$ and $y = -1$.

Problem 5. If $N = abc + 1$, prove that $(N, a) = (N, b) = (N, c) = 1$.

Proof.
Let $N = abc + 1$. WLOG, $(N, a) = d$ and assume to the contrary that $d \neq 1$. This would mean that $d|N$ and $d|a$, then $\exists k$ where $N = dk$ and $\exists m$ where $a = dm$ meaning that

$$\begin{aligned}
N &= abc + 1 \\
dk &= abc + 1 \\
dk &= d(mbc) + 1
\end{aligned}$$

But we can see that $d|1$ must be true, meaning that $d = 1$ which is a contradiction. This means that $(N, a) = (N, b) = (N, c) = 1$. $\square$

Problem 6. Find two different solutions of $299x + 247y = 13$.

Solution.
Dividing by the GCD, we can get the following

$$\begin{aligned}
299x + 247y &= 13 \\
23x + 19y &= 1
\end{aligned}$$

Then, use UA

$$\begin{aligned}
23 &= 19(1) + 4 \\
19 &= 4(4) + 3 \\
4 &= 3(1) + 1 \\
3 &= 1(3) + 0
\end{aligned}$$

Then, doing it back we get

$$\begin{aligned}
1 &= 4 - 3(1) \\
1 &= 23 - 19 - (19 - 4(4)) \\
1 &= 23 - 19 - 19 + 4(23 - 19(1)) \\
1 &= 23 - 19 - 19 + 23(4) - 19(4) \\
1 &= 23(5) + 19(-6)
\end{aligned}$$

Multiply by 13 and we get,

$$13 = 299(5) + 247(-6)$$

So an integer solution would be $x = 5$ and $y = -6$.

Then, for the second solution, we could have

$$13 = 299(-242) + 247(293)$$

So a second integer solution would be $x = -242$ and $y = 293$.

Problem 7. Prove that if $a|b$ and $b|a$ then $a = b$ or $a = -b$.

Proof. Let $a|b$ and $b|a$, this means that $\exists m, nin\mathbb{N}$ s.t. $bm = a$ and $an = b$, then

$$\begin{aligned}
bm &= a \\
anm &= a \\
nm &= 1
\end{aligned}$$

But this means that $m, n = \pm 1$. So $a = b$ or $a = -b$. □

Problem 9. Prove that $((a, b), b) = (a, b)$.

Solution.

Let $((a, b), b) = d$. This means that $d|(a, b)$ and $d|b$. Let $(a, b) = c$, meaning that $c|a$ and $c|b$. We can see that $d|c$ and $d|b$, since $d|c$ and $c|a$ we know that $d|a$ and $d|b$. But since $(a, b) = c$ this means that $d \leq c$. However, since $d|c$ this means that $d = c$, so $((a, b), b) = (a, b)$.

Problem 12. Prove: If $a|b$ and $c|d$, then $ac|bd$.

Proof. Let $a|b$ and $c|d$, this then means that $\exists i, j$ s.t. $ai = b$ and $cj = d$. Then do the following,

$$\begin{aligned}
cj &= d \\
(cj)b &= bd \\
(cj)ai &= bd \\
ac(ij) &= bd
\end{aligned}$$

Which is the definition of $ac|bd$. □

Problem 13. Prove: If $d|a$ and $d|b$ then $d^2|ab$.

Proof. Let $d|a$ and $d|b$, this then means that $\exists i, j \in \mathbb{N}$ s.t. $di = a$ and $dj = b$. Then do the following,

$$\begin{aligned} di &= a \\ b(di) &= ab \\ (dj)(di) &= ab \\ d^2(ij) &= ab \end{aligned}$$

Which is the definition of $d^2|ab$. $\square$

Problem 14. Prove: If $c|ab$ and $(c, a) = d$, then $c|db$.

Proof. Since $(c, a) = d$, we know this means that $\exists x, y$ s.t. $cx + ay = d$, then

$$\begin{aligned} cx + ay &= d \\ bcx + aby &= db \end{aligned}$$

Given that $c|ab$, we know that $\exists n$ s.t. $cn = ab$,

$$\begin{aligned} bcx + cny &= db \\ c(bx + ny) &= db \end{aligned}$$

This is the definition of $c|db$. $\square$

Problem 15.

(a) If $x^2 + ax + b = 0$ has an integer root, show that it divides b.

(b) If $x^2 + ax + b = 0$ has a rational root, show that it is in fact an integer.

Solution.

(a) *Proof.* Let $x^2 + ax + b = 0$ have an integer root. This means that $x^2 + ax + b \equiv 0 \pmod{m}$ for all $m > 0$. So

$$\begin{aligned} x^2 + ax + b &\equiv 0 \pmod{x} \\ 0 + 0 + b &\equiv 0 \pmod{x} \\ b &\equiv 0 \pmod{x} \end{aligned}$$

This is the same as saying $x|b$.

(b) *Proof.* Let $x^2 + ax + b = 0$ have a rational root $x = {}^{p}/_{q}$; where $p, q \in \mathbb{N}$ (assume x is in lowest terms) and $q \neq 0$. Assume to the contrary that $x \notin \mathbb{Z}$, meaning $q \nmid p$. Then,

$$\begin{aligned}
\frac{p^2}{q^2} + \frac{ap}{q} + b &= 0 \\
p^2 + apq + bq^2 &= 0 \\
p^2 + apq + bq^2 &\equiv 0 \pmod{p} \\
0 + 0 + bq^2 &\equiv 0 \pmod{p} \\
bq^2 &\equiv 0 \pmod{p}
\end{aligned}$$

Since $q \nmid p$ it follows that $q^2 \nmid p$, but the above is a way to define $q^2|p$, so this is a contradiction, meaning that $q|p$, and therefore $x \in \mathbb{Z}$.

3 Page 19 Problems

Problem 2. Find the prime-power decompositions of 2345, 45670, and 999999999999. (Note that 101|1000001).

Solution.

2345:

$$\begin{aligned}
2345 &\equiv 5 && \pmod{5} \\
2345 &\equiv 0 && \pmod{5}
\end{aligned}$$

$$2345 = 5 \cdot 469$$

$$\begin{aligned}
469 &= 4 \cdot 100 + 6 \cdot 10 + 9 \\
469 &\equiv 4 \cdot 2 + 6 \cdot 3 + 2 && \pmod{7} \\
469 &\equiv 8 + 18 + 2 && \pmod{7} \\
469 &\equiv 1 + 4 + 2 && \pmod{7} \\
469 &\equiv 0 && \pmod{7}
\end{aligned}$$

$$2345 = 5 \cdot 7 \cdot 67$$

Since $8 < \sqrt{67} < 9$, we only need to look for primes up to 9, and we know 67 is not divisible by 1 to 9, so the prime power decomposition is $2345 = 5 \cdot 7 \cdot 67$.

45670:

$$\begin{aligned}
45670 &= 49000 - 3330 \\
&= (7^2 \cdot 1000) - (333 \cdot 10) \\
&= 10(70^2 - 3 \cdot 111) \\
&= 10(70^2 - 3^2 \cdot 37)
\end{aligned}$$

Since nothing can be furhter factored, we can see that that number will be prime

$$= 10(4900 - 333)$$

So $45670 = 5 \cdot 2 \cdot 4567$.

999999999999:

$$\begin{aligned}
999999999999 &= 10^12 - 1 \\
&= (10^6 - 1)(10^6 + 1) \\
&= (10^3 - 1)(10^3 + 1)(10^6 + 1) \\
&= (999)(1001)(1000001) \\
&= 3^2 \cdot 111 \cdot 1001 \cdot 101 \cdot 9901 \\
&= 3^2 \cdot (3 \cdot 37) \cdot (11 \cdot 91) \cdot 101 \cdot 9901 \\
&= 3^2 \cdot (3 \cdot 37) \cdot (11 \cdot 7 \cdot 13) \cdot 101 \cdot 9901
\end{aligned}$$

So $999999999999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 101 \cdot 9901$

Problem 8. If $d|ab$, does it follow that $d|a$ or $d|b$?

Solution.
No, a counter example would be $a = 6$, $b = 2$, and $d = 4$. $4|12$, but $4 \nmid 6$ and $4 \nmid 2$.

Problem 10. Prove that $n(n+1)$ is never a square for $n > 0$.

Proof. Assume to the contrary that $n(n+1) = k^2$ for some $k \in \mathbb{Z}$. Let the prime factorization of $k = p_1^{e_1} \cdot p_2^{e_2} \cdots p_m^{e_m}$ where $p_1 < p_2 < \cdots < p_m$ and $e_i > 0$. We know then that $k^2 = p_1^{2e_1} \cdot p_2^{2e_2} \cdots p_m^{2e_m}$. We can look at 2 cases for n and $n+1$.

First, if n is a square itself, then for $n(n+1)$ to be a square, $n+1$ must also be a square. But then we have $n^2 < n(n+1) < (n+1)^2$, and there is no perfect square in between n^2 and $(n+1)^2$.

Second, if n is not a square itself, then we can look at $(n, (n+1))$.

Let $(n, (n+1)) = d$, this then means that $d|n$ and $d|(n+1)$, so $\exists i, j \in \mathbb{Z}$ s.t. $di = n$ and $dj = n+1$. Then

$$\begin{aligned}
di &= n \\
di + 1 &= n + 1 \\
di + 1 &= dj \\
1 &= dj - di \\
1 &= d(j - i)
\end{aligned}$$

But this means that $d|1$, so $d = 1$. Since they are coprime, they have no overlapping primes in their prime decompositions, but then both n and $n+1$ would have to have prime decompositions with all exponents $2e_i$, meaning that they would have to be square, so this is a contradiction. □

4 Sage Work

Problem A. How many primes are there less than 10^6?

Solution.

```
upper = 10^6
primes = list(filter(is_prime, [1..upper]))
count = len(primes)

print(f'There are {count} primes less than {upper}.')
```

```
    There are 78498 primes less than 1000000.
```

Problem B. Find x, y such that $2015x + 93y = 31$

Solution.

```
GCD, x, y = xgcd(2015,93)

print(f'One solution for 2015x+93y-31 is x={x} and y={y}.')
```

```
    One solution for 2015x+93y-31 is x=-1 and y=22
```

Problem C (Extra Credit). Let $r(n)$ be the number formed by repeating n 1s. For example $r(5) = 11111$. Find $\gcd(r(2025), r(103))$.

Solution.

```
def r(n):
    answer = 0
    while n>0:
        answer = answer + 10**(n-1)
        n = n-1
    return answer

outcome = gcd(r(2025),r(103))

print(f'The gcd of r(2025) and r(103) is {outcome}.')
```

```
    The gcd of r(2025) and r(103) is 1.
```