Homework 4

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Math 370, Fall 2024

March 30, 2025

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Problem 3. Classify the integers $2, 3, ..., 21$ as abundant, deficient or perfect.

Solution.

	Deficient		**Perfect**		**Abundant**
2	prime				
3	prime				
4	$\sigma(4) = \sigma(2^2) = 7 < 8$				
5	prime				
		6	$\sigma(6) = \sigma(2) \cdot \sigma(3) = 12 = 12$		
7	prime				
8	$\sigma(8) = \sigma(2^3) = 15 < 16$				
9	$\sigma(9) = \sigma(3^2) = 13 < 18$				
10	$\sigma(10) = \sigma(2) \cdot \sigma(5) = 18 < 20$				
11	prime				
				12	$\sigma(12) = \sigma(3) \cdot \sigma(2^2) = 28 > 24$
13	prime				
14	$\sigma(14) = \sigma(2) \cdot \sigma(7) = 24 < 28$				
15	$\sigma(15) = \sigma(3) \cdot \sigma(5) = 24 < 30$				
16	$\sigma(16) = \sigma(2^4) = 31 < 32$				
17	prime				
				18	$\sigma(18) = \sigma(2) \cdot \sigma(3^2) = 39 > 36$
19	prime				
				20	$\sigma(20) = \sigma(2^2) \cdot \sigma(5) = 42 > 40$
21	$\sigma(21) = \sigma(3) \cdot \sigma(7) = 32 < 42$				

Problem 5. If $\sigma(n) = kn$, then n is called a *k-perfect number*. Verify that 672 is a 3-perfect and $2,178,540 = 2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$ is 4-perfect.

Solution.

672 is a 3-perfect because we can see that

$$\sigma(672) = \sigma(3) \cdot \sigma(2^5) \cdot \sigma(7) = 4 \cdot 63 \cdot 8 = 3 \cdot (3 \cdot 7 \cdot 5 \cdot 2^5) = 3(672) \checkmark$$

The second one we can see

$$\begin{aligned}
2,178,540 &= 2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \\
\sigma(2,178,540) &= \sigma(2^2) \cdot \sigma(3^2) \cdot \sigma(5) \cdot \sigma(7^2) \cdot \sigma(13) \cdot \sigma(19) \\
\sigma(2,178,540) &= 7 \cdot 13 \cdot 6 \cdot 57 \cdot 14 \cdot 20 \\
\sigma(2,178,540) &= 2^2(2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19) \\
\sigma(2,178,540) &= 4(2,178,540) \ \checkmark
\end{aligned}$$

Problem 6. Show that no number of the form 2^a3^b is 3-perfect.

Proof. Let $n = 2^a3^b$, for contradiction, let this be 3-perfect, so $\sigma(n) = 3n$, so we

$$\begin{aligned}
\sigma(n) &= 3n \\
\sigma(2^a) \cdot \sigma(3^b) &= 2^a3^{b+1} \\
(2^{a+1} - 1)\left(\frac{3^{b+1} - 1}{2}\right) &= 2^a3^{b+1} \\
(2^{a+1} - 1)(3^{b+1} - 1) &= 2^{a+1}3^{b+1} \\
2^{a+1}3^{b+1} - 2^{a+1} - 3^{b+1} + 1 &= 2^{a+1}3^{b+1} \\
-2^{a+1} - 3^{b+1} + 1 &= 0 \\
2^{a+1}3^{b+1} &= 1
\end{aligned}$$

But if this was true, then a and b would both need to be -1, which is not possible. So we can se that no number of the form 2^a3^b is 3-perfect. □

Problem 7. Let us say that n is *superperfect* if and only if $\sigma(\sigma(n)) = 2n$. Show that if $n = 2^k$ and $2^{k+1} - 1$ is prime, then n is superperfect.

Proof. Let $n = 2^k$, and let $2^{k+1} - 1$ be prime and look at $\sigma(\sigma(n))$,

$$\begin{aligned}
\sigma(\sigma(n)) &= \sigma(2^{k+1} - 1) \\
&= 2^{k+1} - 1 + 1 \\
&= 2^{k+1} \\
&= 2(2^k) \\
\sigma(\sigma(n)) &= 2n
\end{aligned}$$

This is the definition of superperfect, so we can see that n is superperfect. □

Problem 13. Show that all even perfect numbers end in 6 or 8.

Proof. Let n be any even perfect number. We know from class that every even perfect number is of the form $2^{k-1}(2^k - 1)$ where $2^k - 1$ is prime. In homework 2, we showed that for $2^k - 1$ to be prime, k must be prime. Lets look at two cases, where k is even, and when k is odd.

When k is even and prime, $k=2$, so then $n = 2^{2-1}(2^2-1) = 2(4-1) = 6$, so it ends in a 6. The other case, when k is odd, means that in 2^{k-1} $k-1$ is even. Then look mod 10.

a	$2^a \pmod{10}$
1	$2 \pmod{10}$
2	$4 \pmod{10}$
3	$8 \pmod{10}$
4	$6 \pmod{10}$
5	$2 \pmod{10}$

We can see this means that $2^{k-1} \equiv 4 \pmod{10}$ if $k \equiv 3 \pmod 4$ or $2^{k-1} \equiv 6 \pmod{10}$, when $k \equiv 1 \pmod 4$. Then, we can check each case alone,

$k \equiv 3 \pmod 4$

We know $2^{k-1} \equiv 4 \pmod{10}$, and we can see that $2^k - 1 \equiv 7 \pmod{10}$ so

$$\begin{aligned}(2^{k-1})(2^k-1) &\equiv 4 \cdot 7 &\pmod{10}\\ (2^{k-1})(2^k-1) &\equiv 8 &\pmod{10}\end{aligned}$$

$k \equiv 1 \pmod 4$

We know $2^{k-1} \equiv 6 \pmod{10}$, and we can see that $2^k - 1 \equiv 1 \pmod{10}$ so

$$\begin{aligned}(2^{k-1})(2^k-1) &\equiv 6 \cdot 1 &\pmod{10}\\ (2^{k-1})(2^k-1) &\equiv 6 &\pmod{10}\end{aligned}$$

□

Problem 14. If n is an even perfect number and $n > 6$, show that the sum of its digits is congruent to $1 \pmod 9$.

Proof. Let n be any even perfect number. We know from class that every even perfect number is of the form $2^{k-1}(2^k-1)$ where 2^k-1 is prime. We also know that the sum of the digits of a number is equal to what that number is congruent to mod 9.

We know that $n = 2^{k-1}(2^k-1)$ creates $n=6$ when $k=2$, so because $n>6$ we know $k>2$, and we know that k will be odd. Then, lets look at this mod 9. We can see that $2^k \pmod 9$ will repeat mod 6, as follows

a	$2^a \pmod 9$
1	$2 \pmod 9$
2	$4 \pmod 9$
3	$8 \pmod 9$
4	$7 \pmod 9$
5	$5 \pmod 9$
6	$1 \pmod 9$

So, then

$k \pmod 6$	$2^{k-1} \pmod 9$	$(2^k-1) \pmod 9$	$n \pmod 9$
3	4	7	1
5	7	4	1
1	1	1	1

So we can see that it is always congruent to 1 mod 9. □

For problems (2)-(4), the "daughter" function F is given, and you are asked to find the "mother," f. That is, given the function $F(n)$, find the function $f(n)$ satisfying

$$F(n) = \sum_{d|n} f(d)$$

You may find it easiest to define f piecewise but you may define it however you wish, just make sure that your definition is clear from the PPF of the input.

2

Problem. Find the "mother" of $F(n) = \mu(n)$

Solution.

Starting with cases, we can see that we have

$$\mu(n) = \sum_{d|n} f(n)$$

$$\begin{aligned}
&n = 1 \\
&\qquad \mu(1) &&= 1 &&= f(1) && f(1) &&= 1 \\
&n = p \\
&\qquad \mu(p) &&= -1 &&= f(1) + f(p) && f(p) &&= -2 \\
&n = p^2 \\
&\qquad \mu(p^2) &&= 0 &&= f(1) + f(p) + f(p^2) && f(p^2) &&= 1 \\
&n = p^3 \\
&\qquad \mu(p^3) &&= 0 &&= f(1) + f(p) + f(p^2) + f(p^3) && f(p^3) &&= 0
\end{aligned}$$

We can see the pattern of this means that

$$f(p^k) = \begin{cases} 1 & \text{if } k = 2 \text{ or if } p^k = 1 \\ -2 & \text{if } k = 1 \\ 0 & \text{if } k > 2 \end{cases}$$

Then, overall we can say that

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-2)^k & \text{if } n = p_1 \cdots p_k \cdot p_{k+1}^2 \cdots p_r^2 \\ 0 & \text{if } p^3 | n \end{cases}$$

3

Problem. Find the "mother" of $F(n) = (\mu(n))^2$

Solution.

$$\mu * (\mu(n))^2 = \sum_{d|n} (\mu(d))^2 \cdot \mu\left(\frac{n}{d}\right)$$

We can just look at p^k

$$\begin{aligned}
\mu * (\mu(p^k))^2 &= \sum_{i=0}^{k} (\mu(p^i))^2 \cdot \mu(p^{k-i}) \\
&= \mu(1)^2\mu(p^k) + \mu(p)^2\mu(p^{k-1}) + \mu(p^2)^2\mu(p^{k-2}) + \cdots + \mu(p^{k-1})^2\mu(p) + \mu(p^k)^2\mu(1) \\
&= \mu(1)^2\mu(p^k) + \mu(p)^2\mu(p^{k-1}) + \mu(p^2)^2\mu(p^{k-2}) + \cdots + \mu(p^{k-1})^2\mu(p) + \mu(p^k)^2\mu(1) \\
\mu * (\mu(p^k))^2 &= \begin{cases} -1 & \text{for k=2} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Then this extends to

$$\mu * (\mu(n))^2 = \begin{cases} 1 & \text{when } n = 1 \\ (-1)^k & \text{when } n = p_1^2 p_2^2 \cdots p_k^2 \\ 0 & \text{anything else} \end{cases}$$

4

Problem. Find the "mother" of $F(n) = 1$ if n is odd, and 0 if n is even.

Solution.

$$F * \mu(n) = \sum_{d|n} F(d) \cdot \mu\left(\frac{n}{d}\right)$$

We can just look at p^k first assuming p is odd

$$F * \mu(p^k) = \sum_{i=0}^{k} F(p^i) \cdot \mu(p^{k-i})$$

$$F * \mu(p^k) = \sum_{i=0}^{k} 1 \cdot \mu(p^{k-i})$$

$$F * \mu(p^k) = \begin{cases} -1 & \text{when } k = 1 \\ 0 & \text{for anything else} \end{cases}$$

If p is ever even, then it is 0, so generalized

$$F * \mu(n) = \begin{cases} 1 & \text{when } n = 1 \\ (-1)^k & \text{when } n = p_1 p_2 \cdots p_k \text{ where all } p_i \neq 2 \\ 0 & \text{if } 2|n \text{ or } p^2|n \text{ for any } p \end{cases}$$

For problems (5)-(9) let $*$ denote the Dirichlet convolution operation. That is

$$f * g(n) = \sum_{d|n} f(d) \cdot g\left(\frac{n}{d}\right)$$

Find the following functions (these should all be functions that you can easily write down or define):

5

Problem. Find $\phi * \sigma$

Solution.

$$\phi * \sigma(n) = \sum_{d|n} \phi(d) \cdot \sigma\left(\frac{n}{d}\right)$$

Look at just p

$$\begin{aligned}
\phi * \sigma(p) &= \phi(p) \cdot \sigma(1) + \phi(1) \cdot \sigma(p) \\
&= p - 1 + \frac{p^2 - 1}{p - 1} \\
&= p - 1 + \frac{(p-1)(p+1)}{p - 1} \\
&= p - 1 + p + 1 \\
&= 2p
\end{aligned}$$

Look at just p^2

$$\begin{aligned}
\phi * \sigma(p^2) &= \sum_{d|p^2} \phi(d) \cdot \sigma\left(\frac{n}{d}\right) \\
&= \phi(1) \cdot \sigma(p^2) + \phi(p) \cdot \sigma(p) + \phi(p^2) \cdot \sigma(1) \\
&= \frac{p^3 - 1}{p - 1} + (p - 1) \cdot \frac{p^2 - 1}{p - 1} + p(p - 1) \\
&= p^2 + p + 1 + p^2 - 1 + p^2 - p \\
&= 3p^2
\end{aligned}$$

Look at just p^3

$$\begin{aligned}
\phi * \sigma(p^3) &= \sum_{d|p^3} \phi(d) \cdot \sigma\left(\frac{n}{d}\right) \\
&= \phi(1) \cdot \sigma(p^3) + \phi(p) \cdot \sigma(p^2) + \phi(p^2) \cdot \sigma(p) + \phi(p^3) \cdot \sigma(1) \\
&= \frac{p^4 - 1}{p - 1} + (p - 1) \cdot \frac{p^3 - 1}{p - 1} + p(p - 1) \cdot \frac{p^2 - 1}{p - 1} + p^2(p - 1) \\
&= \frac{p^4 - 1}{p - 1} + (p^3 - 1) + p \cdot (p^2 - 1) + p^2(p - 1) \\
&= \frac{p^4 - 1}{p - 1} + p^3 - 1 + p^3 - p + p^3 - p^2 \\
&= p^3 + p^2 + p + 1 + 3p^3 - p^2 - p - 1 \\
&= 4p^3
\end{aligned}$$

We can see that $\phi * \sigma(p^k) = (k+1)p^k$. So we can write that when $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$

$$\phi * \sigma(n) = \prod_{i=1}^{k} (e_i + 1) p_i^{e_i}$$

Or we could also see it as $\tau(n) \cdot N(n)$.

6

Problem. Find $\tau * \phi$

Solution.

$$\tau * \phi(n) = \sum_{d|n} \tau(d) \cdot \phi\left(\frac{n}{d}\right)$$

Look at just p

$$\begin{aligned} \tau * \phi(p) &= \tau(1) \cdot \phi(p) + \tau(p) \cdot \phi(1) \\ &= (p-1) + (1+1) \\ &= p+1 \end{aligned}$$

Look at just p^2

$$\begin{aligned} \tau * \phi(p^2) &= \tau(1) \cdot \phi(p^2) + \tau(p) \cdot \phi(p) + \tau(p^2) \cdot \phi(1) \\ &= p(p-1) + (1+1)(p-1) + (2+1) \\ &= p^2 - p + 2p - 2 + 3 \\ &= p^2 + p + 1 \end{aligned}$$

Look at just p^3

$$\begin{aligned} \tau * \phi(p^2) &= \tau(1) \cdot \phi(p^3) + \tau(p) \cdot \phi(p^2) + \tau(p^2) \cdot \phi(p) + \tau(p^3) \cdot \phi(1) \\ &= p^2(p-1) + (2)(p(p-1)) + (3)(p-1) + (4) \\ &= p^3 - p^2 + 2p^2 - 2p + 3p - 3 + 4 \\ &= p^3 + p^2 + p + 1 \end{aligned}$$

So we can see that $\tau * \phi(p^k) = \sum_{i=0}^{k} p^i$. So we can write that when $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$

$$\tau * \phi(n) = \prod_{i=1}^{k} \sum_{a=0}^{e_i} p_i^a$$

However, this is the same as $\sigma(n)$.

7

Problem. Find $N * N$

Solution.

$$N * N(n) = \sum_{d|n} N(d) \cdot N\left(\frac{n}{d}\right)$$

Look at just p^k

$$\begin{aligned} N * N(p) &= \sum_{i=0}^{k} N(p^i) \cdot N(p^{k-i}) \\ &= \sum_{i=0}^{k} p^i \cdot p^{k-i} \\ &= \sum_{i=0}^{k} p^k \\ &= (k+1)p^k \end{aligned}$$

So we can write that when $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$

$$N * N(n) = \prod_{i=1}^{k} (e_i + 1)p_i^{e_i}$$

Or we could also see it as $\tau(n) \cdot N(n)$.

8

Problem. Find $\phi * 1$

Solution.

$$\phi * 1(n) = \sum_{d|n} \phi(d) \cdot 1\left(\frac{n}{d}\right)$$

Look at just p^k

$$\begin{aligned} \phi * 1(p^k) &= \sum_{d|p^k} \phi(d) \cdot 1 \\ &= \phi(1) + \phi(p) + \phi(p^2) + \cdots + \phi(p^k) \\ &= 1 + (p-1) + (p^2 - p) + \cdots + (p^k - p^{k-1}) \\ &= p^k \end{aligned}$$

So we can see that $\phi * 1(n) = n$ or simply the function N.

9

Problem. Find $\sigma * \mu$

Solution.

$$\sigma * \mu(n) = \sum_{d|n} \sigma(d) \cdot \mu\left(\frac{n}{d}\right)$$

Look at just p

$$\begin{aligned}\sigma * \mu(p) &= \sum_{d|p} \sigma(d) \cdot \mu\left(\frac{n}{d}\right)\\ &= \sigma(1) \cdot \mu(p) + \sigma(p) \cdot \mu(1)\\ &= -1 + \frac{p^2-1}{p-1}\\ &= -1 + p + 1\\ &= p\end{aligned}$$

Look at just p^2

$$\begin{aligned}\sigma * \mu(p^2) &= \sum_{d|p^2} \sigma(d) \cdot \mu\left(\frac{n}{d}\right)\\ &= \sigma(1) \cdot \mu(p^2) + \sigma(p) \cdot \mu(p) + \sigma(p^2) \cdot \mu(1)\\ &= 0 + -1 \cdot \frac{p^2-1}{p-1} + \frac{p^3-1}{p-1}\\ &= -(p+1) + (p^2+p+1)\\ &= p^2\end{aligned}$$

Look at just p^3

$$\begin{aligned}\sigma * \mu(p^3) &= \sum_{d|p^3} \sigma(d) \cdot \mu\left(\frac{n}{d}\right)\\ &= \sigma(1) \cdot \mu(p^3) + \sigma(p) \cdot \mu(p^2) + \sigma(p^2) \cdot \mu(p) + \sigma(p^3) \cdot \mu(1)\\ &= 0 + 0 + \frac{p^3-1}{p-1} \cdot (-1) + \frac{p^4-1}{p-1}\\ &= -(p^2+p+1) + (p^3+p^2+p+1)\\ &= p^3\end{aligned}$$

So we can see that $\sigma * \mu(p^k) = p^k$, so $\sigma * \mu(n) = n$ or the function N.

10

Problem. Prove that there are infinitely many integers n such that $\mu(n) + \mu(n+1) = 0$.

Proof. Assume to the contrary that we know all of the integers n such that $\mu(n) + \mu(n+1) = 0$ and that they are finite, so that a is the largest possible integer where this is true. Then let $c > a$, and let $c + 1$ be the square of any odd prime. meaning $c + 1 = p^2$. Then, looking at this mod 4, we know that $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$, and then $p^2 = 1 \pmod 4$ either way. Then,

$$\begin{aligned} c + 1 &= 1 && \pmod 4 \\ c &= 0 && \pmod 4 \end{aligned}$$

Meaning that c is divisible by 4, and therefore $\mu(c) = 0$, but we already knew that $\mu(c+1) = 0$, meaning that we know that a is not the largest, and therefore the set of integers is infinite. □

11

Problem. Prove that there are infinitely many integers n such that $\mu(n) + \mu(n+1) + \mu(n+2) = 0$.

Proof. We can look at when they are all 0 due to a unique repeated prime factor in each, so we can create

$$\begin{aligned} n &\equiv 0 \pmod{p_1^2} \\ n + 1 &\equiv 0 \pmod{p_2^2} \\ n + 2 &\equiv 0 \pmod{p_3^2} \end{aligned}$$

By the Chinese Remainder Theorem, there exists a solution mod $p_1^2 p_2^2 p_3^2$, so we know that there are infinitely many solutions of integers. □

12

Problem. Prove that there are infinitely many integers n such that $\mu(n) + \mu(n+1) = -1$.

Proof. **Not Finished** □

13

Problem. Prove that

$$\sum_{d|n} \frac{(\mu(d))^2}{\phi(d)} = \frac{n}{\phi(n)}$$

Proof. We want to start with

$$\sum_{d|n} \frac{(\mu(d))^2}{\phi(d)}$$

Any divisors with powers over 2 will be canceled by $\mu(d) = 0$, so we only need to look at the divisors of the form $p_1 \cdots p_k$. Then, since $\mu(d)$ is squared, all of the numerators we need to look at will be 1. So all we need to look at is

$$\sum_{d^\star|n} \frac{1}{\phi(d)} \quad d^\star \text{ is } d \text{ of the form } p_1 \cdots p_k$$

Not Finished □

14

Problem. Use Sage to find

(a) The first 10 abundant numbers

(b) The relative frequency of abundance for the first n positive integers, for n =100, 1000, 10000, 100000. That is, how many integers (out of n) are abundant.

Solution.

(a)

````python
def abund(n):
    sigma_n = sigma(n,1)
    if sigma_n > 2*n:
        return True
    else:
        return False

abundant_nums_10 = []

n=1
while len(abundant_nums_10)<10:
    test = abund(n)
    if test == True:
        abundant_nums_10.append(n)
        n+= 1
    else:
        n+= 1

print(f'The first 10 abundant numbers are {abundant_nums_10}.')
```

    The first 10 abundant numbers are [12, 18, 20, 24, 30, 36, 40, 42, 48, 54].
````

(b)

```python
def abund(n):
    sigma_n = sigma(n,1)
    if sigma_n > 2*n:
        return True
    else:
        return False

def frequency(upper):
    a = 1
    abundant_nums = []
    while a <= upper:
        test = abund(a)
        if test == True:
            abundant_nums.append(a)
            a+= 1
        else:
            a+= 1
    freq = len(abundant_nums)
    return freq

```
```

```


for n in [100,1000,10000,100000]:
    freq_n = frequency(n)
    print(f'The frequency of abundant numbers in the first {n} integers is {freq_n} integers out of {n}.')
```

    The frequency of abundant numbers in the first 100 integers is 22 integers out of 100.
    The frequency of abundant numbers in the first 1000 integers is 246 integers out of 1000.
    The frequency of abundant numbers in the first 10000 integers is 2488 integers out of 10000.
    The frequency of abundant numbers in the first 100000 integers is 24795 integers out of 100000.
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## 15 (5 pts Extra Credit)

**Problem.** A positive integer $n$ is "perfectly crazy" if $\phi(n)^{\sigma(n)^{\tau(n)}} = n^2$. Find all perfectly crazy numbers.

*Solution.*

## 16 (5 pts Extra Credit)

**Problem.** Let $P(n)$ be the product of the positive integers which are $\leq n$ and relative prime to $n$. Prove that

$$P(n) = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)}$$

*Proof.* $\square$
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