Week 3 H/W

Problem 1. Simplify the expression

$$\log_2\left(\frac{8\sqrt{2}}{16}\right) + \log_2(32) - 2\log_2(4) = \frac{8\sqrt{2}}{16} = \frac{8 \cdot 2^{\frac{1}{2}}}{16} =$$

$$= \frac{8}{16} \cdot 2^{\frac{1}{2}} = \frac{1}{2} \cdot 2^{\frac{1}{2}} = 2^{-1} \cdot 2^{\frac{1}{2}} = 2^{-\frac{1}{2}} = \log_2(2^{-\frac{1}{2}}) =$$

$$= -\frac{1}{2}; \quad \log_2(32) = 5; \quad 2\log_2(4) = 4$$

$$-\frac{1}{2} + 5 - 4 = 0.5$$

Problem 2. Solve for x: $\log_3(x-1) + \log_3(x+1) = 2$

Logarithmic property:
$\log_b(A) + \log_b(B) = \log_b(AB)$

$$\log_3((x-1)(x+1)) = 3^2$$

$$\log_3((x-1)(x+1)) = 2$$

$$(x-1)(x+1) = x^2 - 1$$

$$\log_3(x^2-1) = 2$$

$$x^2 - 1 = 3^2 = 9$$

$$x^2 = 10$$

$$x = \pm\sqrt{10}$$

Problem 3. Compound Interest Exercise

Initial investment = 10000 \$

annual interest rate = 6%; $\frac{1}{4}$

Compound Interest Formula

A - final amount n

P - initial balance t.

r - interest rate

$$\boxed{A = P\left(1 + \frac{r}{n}\right)^{nt}}$$

$$20000 = 10000\left(1+\frac{0.06}{4}\right)^{4t}$$

$$\frac{\cancel{20000}}{\cancel{10000}} = \frac{\cancel{10000}(1.015)^{4t}}{\cancel{10000}}$$

$$2 = (1.015)^{4t}$$

$$\ln(2) = \ln((1.015^{4t}))$$

$$\ln(2) = 4t\cdot\ln(1.015)$$

$$t = \frac{\ln(2)}{4\ln(1.015)} \approx 11.64 \text{ years}$$

Problem 4. Radioactive Decay Exercise

A RadioActive substance decays according to the formula:

$$N(t) = N_0 e^{-kt}$$

N_0 - initial amount

k - decay constant

t - time in years

half time $t_{\frac{1}{2}}$; $N(t_{\frac{1}{2}}) = \frac{N_0}{2}$

$$\frac{N_0}{2} = N_0 e^{-kt_{\frac{1}{2}}}$$

$$\frac{1}{2} = e^{-k5}$$

$$\ln\left(\frac{1}{2}\right) = -k5$$

$$-0.6931 = -5k$$

$$k = \frac{0.6931}{5} \approx 0.1386$$

The decay constant k is ≈ 0.1386 per year

Problem 5. Radioactive Decay Exercise

$N_0 = 100$ grams

k = 70 grams in 3 hours

$t = ?$ in 20 grams

$$N(t) = N_0 e^{-kt}$$

$$N(3) = 100 e^{-k\cdot 3}$$

$$\frac{70}{100} = \frac{\cancel{100}e^{-3k}}{\cancel{100}}$$

$$0.7 = e^{-3k}$$

$\ln(0.7) = -3k$

$-0.3567 = -3k$

$k = \frac{0.3567}{3} \approx 0.1189$

now $N(t) = 20$

$20 = 100 e^{-0.1189 t}$

$0.2 = e^{-0.1189 t}$

$\ln(0.2) = -0.1189 t$

$-1.6094 = -0.1189 t$

$t = \frac{1.6094}{-0.1189} \approx 13.54$ hours

Problem 6. Geometric

Find the unit vector in the direction from $A(1,2,3)$ to $B(4,6,9)$

$\overrightarrow{AB} = \langle 4-1, 6-2, 9-3 \rangle = \langle 3, 4, 6 \rangle$

Magnitude of $\overrightarrow{AB}$: $|AB| = \sqrt{3^2+4^2+6^2} = \sqrt{9+16+36} = \sqrt{61} \approx 7.81$

$$\vec{U} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \underline{\left\langle \frac{3}{\sqrt{61}}; \frac{4}{\sqrt{61}}; \frac{6}{\sqrt{61}} \right\rangle}$$

Problem 7. Matrix form

Express the vector $\vec{u} = 7\hat{i} - 2\hat{j} + 4\hat{k}$ in matrix form and find its magnitude

$\vec{u} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix}$; Magnitude of $\vec{u}$:

$\vec{u} = \sqrt{7^2 - 2^2 + 4^2} = \sqrt{49+4+16} = \sqrt{69} \approx$

≈ 8.31

Problem 8. Adding and Scaling Vectors

$\vec{a} = \langle 2, -1, 3 \rangle$ and $\vec{b} = \langle -1, 4, 2 \rangle$, compute $3\vec{a} - 2\vec{b}$

$\vec{3a} = \langle 6, -3, 9 \rangle$

$2\vec{b} = \langle -2, 8, 4 \rangle$

$\vec{3a} - \vec{2b} = \langle 6+2, -3-8, 9-4 \rangle = \langle 8, -11, 5 \rangle$

Problem 9. Dot Product

Find angle between vectors $\vec{p} = \langle 1, 2, 3 \rangle$ and $\vec{q} = \langle 4, -5, 6 \rangle$

$\vec{p} \cdot \vec{q} = \langle 1 \cdot 4 + 2 \cdot (-5) + 3 \cdot 6 \rangle = 4 - 10 + 18 = 12$

Magnitude: $\vec{p} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

$\vec{q} = \sqrt{4^2 + (-5)^2 + 6^2} = \sqrt{16+25+36} = \sqrt{77}$

Angle θ: $\cos\theta = \dfrac{\vec{p} \cdot \vec{q}}{|\vec{p}||\vec{q}|} = \dfrac{12}{\sqrt{14}\sqrt{77}} = \dfrac{12}{\sqrt{1078}} = 0.3647$

θ across$(0.3647) \approx 68.58^\circ$

Problem 10. Dot Product Application

Determine if the vectors $\vec{u} = \langle 2, -1, 4 \rangle$ and $\vec{v} = \langle -8, 4, -16 \rangle$ are orthnogonal.

Dot product: $\vec{u} \cdot \vec{u} = \langle 2 \cdot (-8) + (-1)(4) + 4 \cdot (-16) \rangle = \langle -16 + -4 - 64 \rangle = -84$

Since $\vec{u} \cdot \vec{u} \neq 0$, they are not orthnogonal

Problem 11. Adding and Substracting Matrices

$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix}$

Compute $2A - 3B$

$2A = 2 \cdot \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix}$;

$3B = 3 \cdot \begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 15 \\ -6 & 3 \end{bmatrix}$;

$\begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 12 & 15 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 4-12 & -2-15 \\ 0-(-6) & 6-3 \end{bmatrix} = \begin{bmatrix} -8 & -17 \\ 6 & 3 \end{bmatrix}$

Problem 12. Multiplying Matrices

$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $D = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$; $E = CD$

$$E = \begin{bmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Problem 13. Row Operations

Use Gaussian elimination to solve system

$$\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 14 \\ -3x + 2y - 2z = -10 \end{cases} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 14 \\ -3 & 2 & -2 & -10 \end{array}\right] \xrightarrow[R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & 2 \\ -3 & 2 & -2 & -10 \end{array}\right] \Rightarrow$$

$$\xrightarrow[R_3 + 3R_1 = R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & 2 \\ 0 & 5 & 1 & 8 \end{array}\right]$$

Step 2: Use R_2 to eliminate y from R_3:

- Multiply R_2 by $\frac{5}{-3}$ and add to R_3:

$$R_3 = R_3 + \left(\frac{5}{-3} R_2\right)$$

Compute:

- new $R_3[2]$: $5 + \left(\frac{5}{-3} \cdot -3\right) = 5 + 5 = 10$
- new $R_3[3]$: $1 + \left(\frac{5}{-3} \cdot 1\right) = 1 - \frac{5}{3} = -\frac{2}{3}$
- new $R_4[4]$: $8 + \left(\frac{5}{-3} \cdot 2\right) = 8 - \frac{10}{3} = \frac{24}{3} - \frac{10}{3} = \frac{14}{3}$

$$R_3 = \left[0, 0, -\frac{2}{3}, 1, \frac{14}{3}\right]$$

Step 3: Solve for z:

$$-\frac{2}{3} z = \frac{14}{3}$$

$$z = \frac{14}{3} : \left(-\frac{2}{3}\right) = \frac{14}{3} \cdot \left(-\frac{3}{2}\right) = -7$$

$x + (-3) + (-7) = 6$

$x = 16$

$-3y + 1(-7) = 2$

$-3y = 9$

$y = -3$

Problem 14. Reduced Row Enchelon form

Find Row Enchelon form of matrix:

$$B = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

- $R_1 = R_1 + R_3 \cdot 1$

$$-1 + 1(1) = -1 + 1 = 0$$

$$0 + 1(-1) = 0 - 1 = -1$$

- $R_1[4] = R_1[4] + 1 \cdot R_3[4]$:

$$0 + 1(-1) = -1$$

$$R_1 = [1, 2, 0, -1]$$

$$R_2 = [0, 1, 0, 8]$$

- $R_1 = R_1 - 2R_2$: $2 - 2(1) = 2 - 2 = 0$

$$-1 - 2(8) = -1 - 16 = -17$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Problem 15. Matrix Inverse and RRef Relationship

$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, find A^{-1} using row operations

$$[A|I]: \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

- Step 1: Make $a_{11} = 1$
- $R_1 = R_1 : 2$

$$R_1 = [1, \tfrac{1}{2} \,|\, \tfrac{1}{2}, 0]$$

- $R_2 = R_2 \cdot 2$: $R_2 = [0, 1 | -5, 2]$
- $R_1 = R_1 - (\frac{1}{2} R_2)$:

$$1 - \frac{1}{2}(0) = 1$$

$$\frac{1}{2} - \frac{1}{2}(1) = \frac{1}{2} - \frac{1}{2} = 0$$

$$\frac{1}{2} - \frac{1}{2}(-5) = \frac{1}{2} + \frac{5}{2} = 3$$

$$0 - \frac{1}{2}(2) = 0 - 1 = -1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right] \; ; \; A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$