Data Structure

Heap

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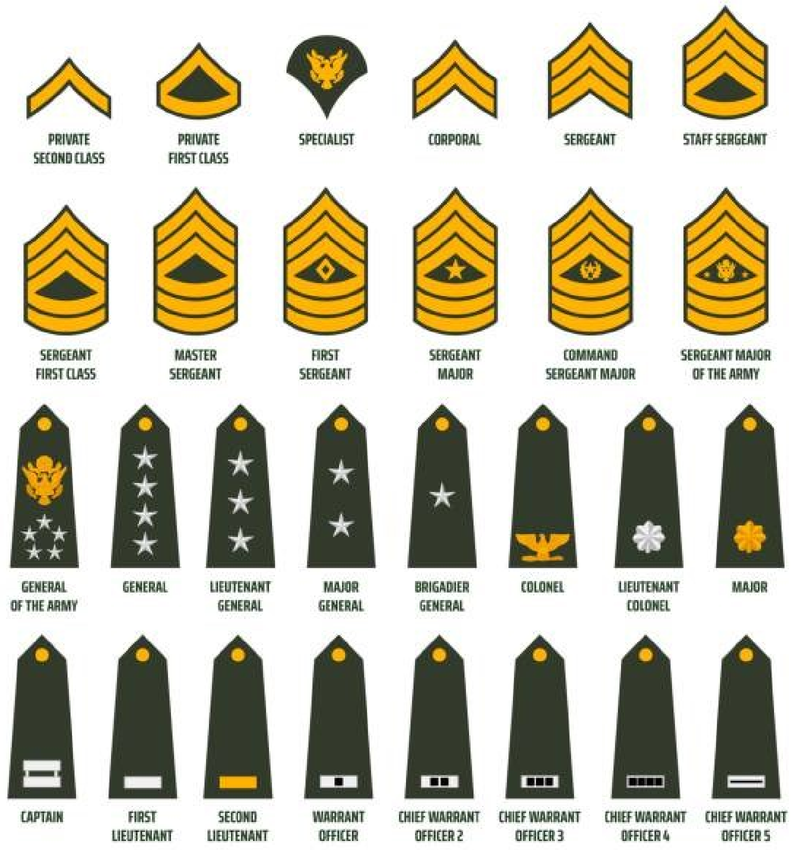


DS&A. Chapter 8.3 Heap

Priority Queue

- A priority queue is a collection of elements, that provides insertion and removal of elements in order of priority
  - operations
    - `insert (elem, key)`
    - `min()`
    - `removeMin ()`

- A priority queue manages elements according to their priorities, not their positions or the order of their arrivals
  - e.g., Suppose a certain flight is fully booked an hour prior to departure. Because of the possibility of cancellations, the airline maintains a priority queue of standby passengers hoping to get a seat. The priority of each passenger is determined by the fare paid and the frequent-flyer status.

Key and Comparator

- Each element is assigned with a key which defines the ranking (or ordering)
  - The ordering of keys must be totally defined without any contradiction (i.e., total ordering)
    - Reflexive property: $k \leq k$
    - Antisymmetric property: if $k_1 \leq k_2$ and $k_2 \leq k_1$, then $k_1 = k_2$
    - Transitive property: if $k_1 \leq k_2$ and $k_2 \leq k_3$, then $k_1 \leq k_3$

- A comparator is a function that receives two key objects and determines the ordering in them
  - e.g., a geometric algorithm may compare points p and q in 2D space, by their x-coordinate (that is, p ≤ q if p.x ≤ q.x), to sort them from left to right,
  - e.g., another algorithm may compare them by their y-coordinate (that is, p ≤ q if p.y ≤ q.y), to sort them from bottom to top.

Priority Queue with Lists

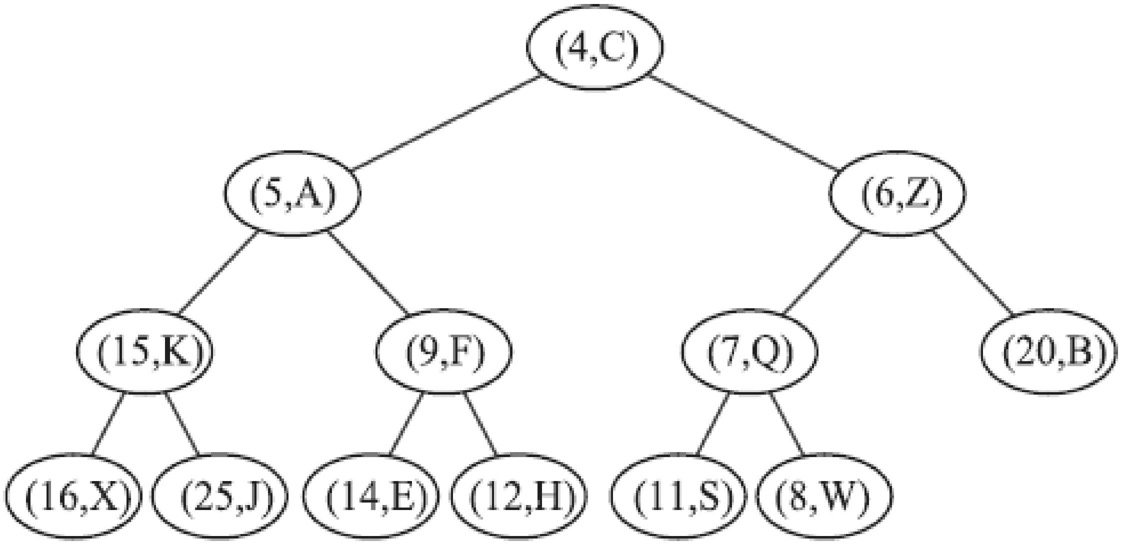
- Implementation with an unsorted list
  - insertion: O(1)
  - removeMin: O(n)

- Implementation with a sorted list
  - insertion: O(n)
  - removeMin: O(1)

Heap

- A heap is a complete binary tree such that the element at a node always precedes those of the children

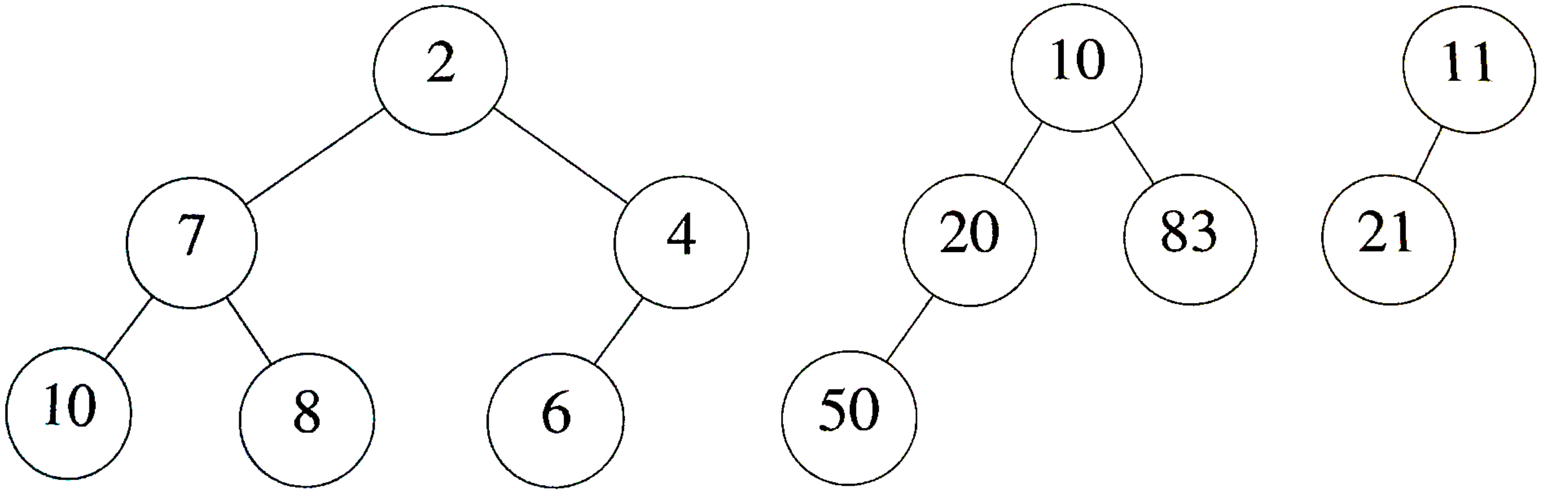
- A heap provides both insertion and removal in O(log *n*) which significantly improves list-based priority queue
  - the height of a complete binary tree is $\lfloor \log n \rfloor$



Heap-order Property

- In a min-heap T, for every node v other than the root, the key associated with v is greater than or equal to the key associate with v's parent.

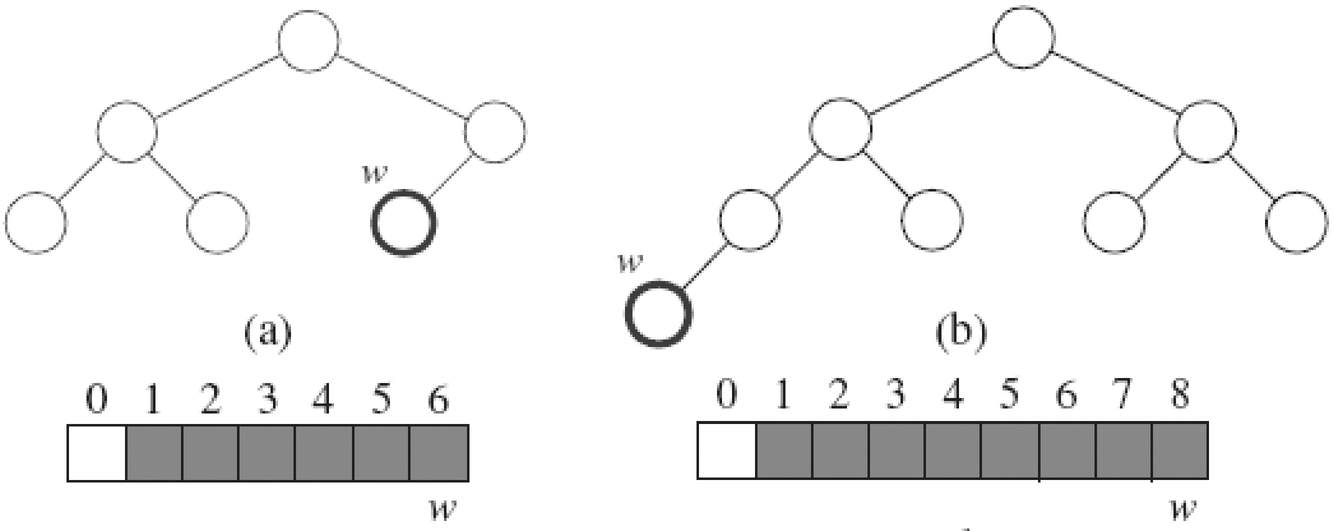
- By the heap-order property:
  - an element with the minimum key is always placed at the root,
  - the key encountered on a path from the root to a leaf node are in non-decreasing order



Complete Binary Tree

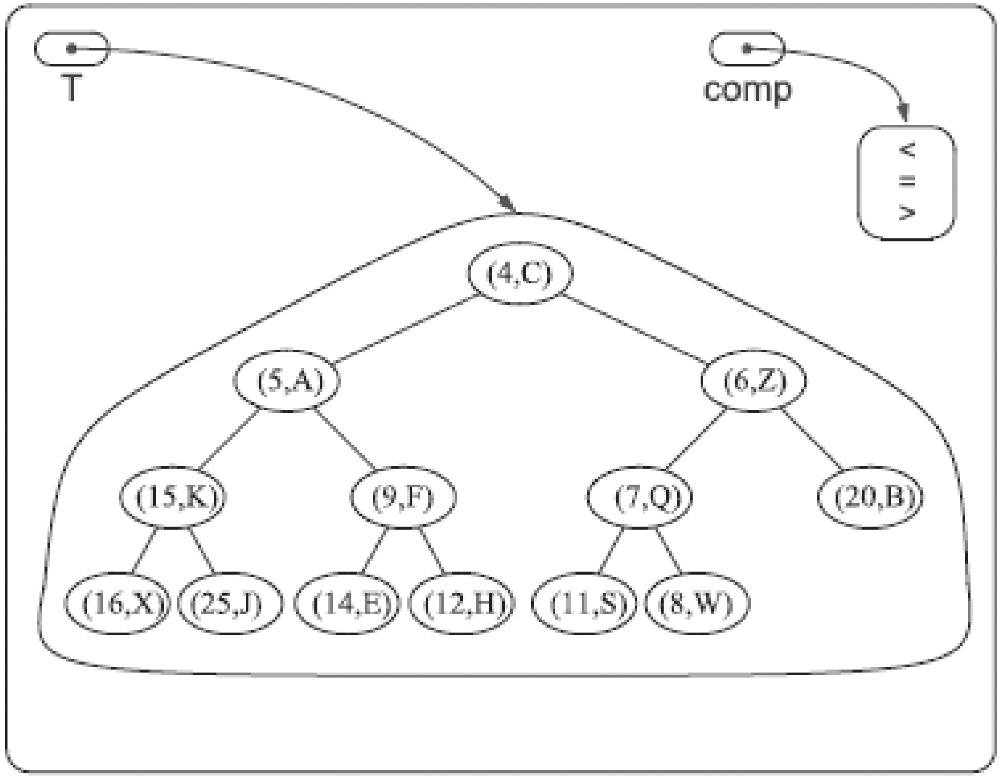
- an array (or vector) is especially suitable for representing a complete binary tree

- the level number of a node, $f(n)$ in a binary tree is defined as follows:
  - if n is the root, $f(n) = 1$
  - if n is the left child of node u, $f(n) = 2f(u)$
  - if n is the right child of node u, $f(n) = 2f(u) + 1$



Priority Queue with Heap

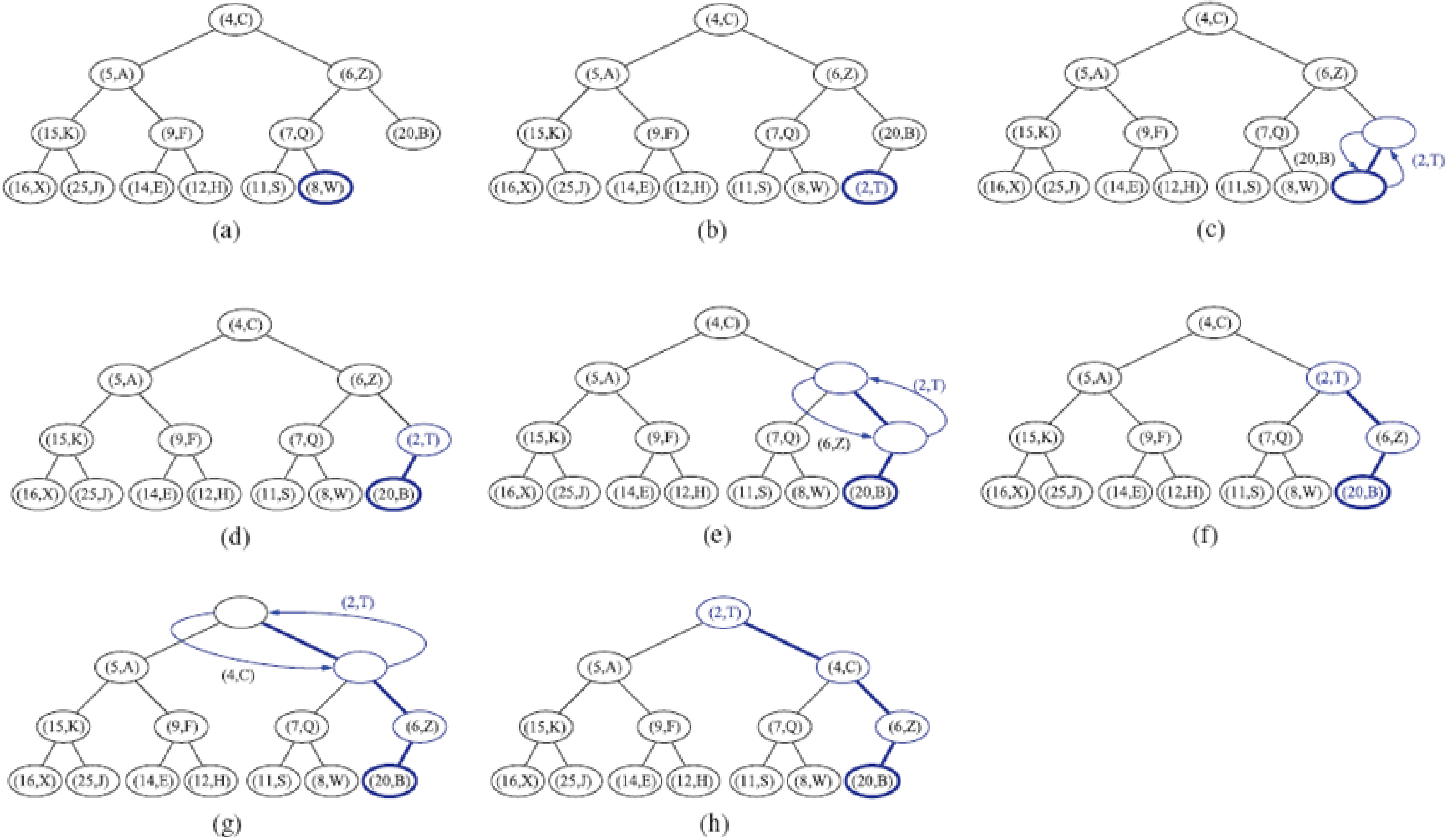
- a heap-based priority queue consists of:
  - heap: a complete binary tree whose nodes store the elements and whose keys satisfy the heap-order property
  - comp: a comparator that defines the total order relation among the keys



Insertion

1. add a new node n s.t. the new node becomes the last node
   - it keeps the tree complete
   - it may violate the heap-order property

2. if the key of n is less than that of its parent, swap n and the parent
   - repeat this step until the key of the parent node is less than that of n or n becomes the root
     - the heap-order property will be satisfied again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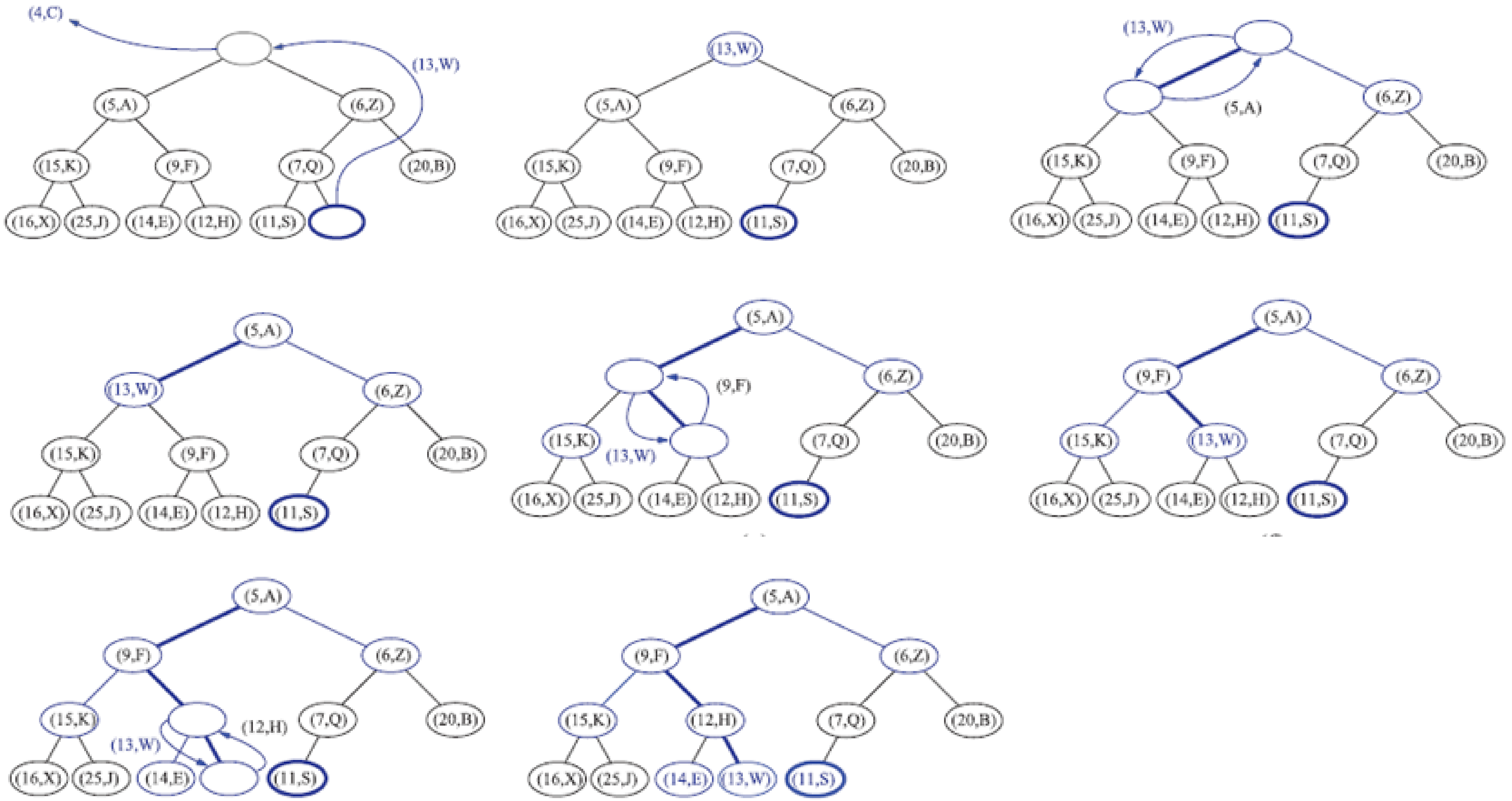
Removal

- An element with the minimal key is always found at the root

- After removal, the root must be replaced with another while keeping the tree complete and keeping the heap-order property
  1. move the last node n to the root
     - keep the tree complete
     - the heap-order property may be violated
  2. swap n and a child with the least key if the key of n is greater than that of one or two children
     - repeat this step until the tree restores the heap-order property
     - this process is called as heapify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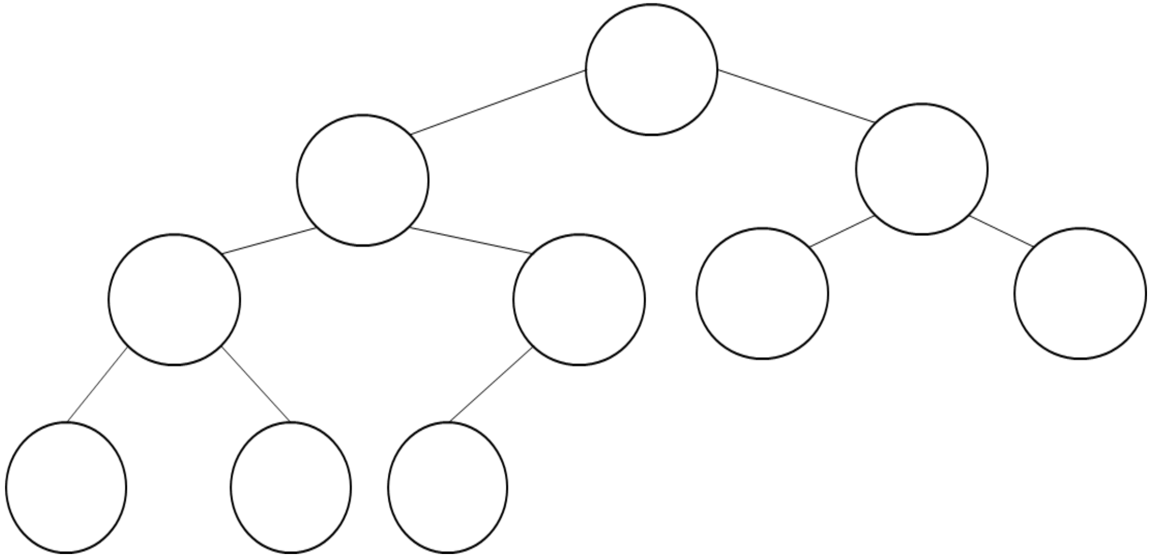
Heap Sort

```
heapsort(elem * a, int n) {
   for (i = 2 ; i <= n ; i++)
      insert(a, i) ;
   for (i = n - 1 ; i > 1 ; i--) {
      swap(&(a[1]),&(a[i+1])) ;
      heapify(a, n) ;
   }
}
```



ex. (26, 5, 77, 1, 61, 11, 59, 15, 48, 19)