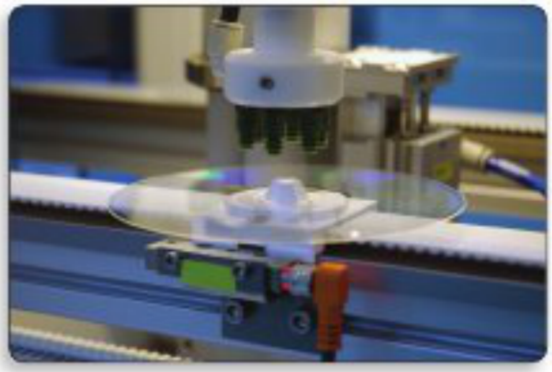
# 2.1 Derivative of a Polynomial Function

There are countless real-world situations that can be modelled by polynomial functions. Consider the following:



- A recording studio determines that the cost, in dollars, of producing $x$ music CDs is modelled by the function $C(x) = 85\ 000 + 25x + 0.015x^2$.
- The amount of fuel required to travel a distance of $x$ kilometres by a vehicle that consumes 8.5 L/100 km is represented by the function $f(x) = 0.085x$.
- The price of a stock, $p$, in dollars, $t$ years after it began trading on the stock exchange is modelled by the function $p(t) = 0.5t^3 - 5.7t^2 + 12t$.

Whether you are trying to determine the costs and price points that will maximize profits, calculating optimum fuel efficiency, or deciding the best time to buy or sell stocks, calculating instantaneous rates of change heighten the chances of making good decisions.

Instantaneous rate of change in each of the above situations can be found by differentiating the polynomial function. In this section, you will examine five rules for finding the derivative of functions: the constant rule, the power rule, the constant multiple rule, and the sum and difference rules.

## Investigate What derivative rules apply to polynomial functions?

**Tools**

- graphing calculator

As you work through this Investigate, you will explore five rules for finding derivatives. Create a table similar to the one below, and record the findings from your work.

| | Original Function | Derivative Function |
|---|---|---|
| **Constant Rule** | $y = c$ | |
| **Power Rule** | $y = x^n$ | |
| **Sum Rule** | $y = f(x) + g(x)$ | |
| **Difference Rule** | $y = f(x) - g(x)$ | |
| **Constant Multiple Rule** | $y = cf(x)$ | |

## A: The Constant Rule

**1. a)** Graph the following functions on a graphing calculator. What is the slope of each function at any point on its graph?

**i)** $y = 2$ **ii)** $y = -3$ **iii)** $y = 0.5$

**b) Reflect** Would the slope of the function $y = c$ be different for any value of $c \in \mathbb{R}$? Explain.

**c) Reflect** Write a rule for the derivative of a constant function $y = c$ for any $c \in \mathbb{R}$.

## B: The Power Rule

**1. a)** Use the nDeriv function to graph the derivative of $y = x$ as follows:

### Technology Tip

The **nDeriv** (numerical derivative) on a graphing calculator can be used to graph the derivative of a given function.

- Enter **Y1** $= x$.

  Move the cursor to **Y2**.

- Press MATH.

  Select **8:nDeriv(**.

- Press VARS to display the **Y-VARS** menu.

  Select **1: Function**, and then select **1:Y1**.

- Press , X, T, θ, n , X,T,θ,n ).

- Select a thick line for **Y2.** Press GRAPH.


**b)** Write the equation that corresponds to the graph of the derivative.

**c)** Press 2ND GRAPH on your calculator to display a table of values for the function and its derivative. How does the table of values confirm the equation in part b)?

**2.** Repeat step 1 for each function.

**a)** $\mathbf{Y1} = x^2$ **b)** $\mathbf{Y1} = x^3$ **c)** $\mathbf{Y1} = x^4$

**3.** Complete the following chart based on your findings in steps 1 and 2.

| $f(x)$ | $f'(x)$ |
|---|---|
| $x$ | |
| $x^2$ | |
| $x^3$ | |
| $x^4$ | |

**4. a) Reflect** Is there a pattern in the graphs and the derivatives in step 3 that could be used to formulate a rule for determining the derivative of a power, $f(x) = x^n$?

**b)** Apply your rule to predict the derivative of $f(x) = x^5$. Verify the accuracy of your rule by repeating step 1 for $\mathbf{Y1} = x^5$.

## C: The Constant Multiple Rule

**1.** **a)** Graph the following functions on the same set of axes.

$\mathbf{Y1} = x \qquad \mathbf{Y2} = 2x \qquad \mathbf{Y3} = 3x \qquad \mathbf{Y4} = 4x$

**b)** How are the graphs of **Y2, Y3,** and **Y4** related to the graph of **Y1**?

**c)** State the slope and the equation of the derivative of each function in part a). How are the derivatives related?

**d)** **Reflect** Predict a rule for the derivative of $y = cx$, for any constant $c \in \mathbb{R}$. Verify your prediction for other values of $c$.

**2.** **a)** Graph the following functions on the same set of axes.

$\mathbf{Y1} = x^2 \qquad \mathbf{Y2} = 2x^2 \qquad \mathbf{Y3} = 3x^2 \qquad \mathbf{Y4} = 4x^2$

**b)** How are the graphs of **Y2, Y3,** and **Y4** related to the graph of **Y1**?

**c)** Use what you learned about the power rule to predict the derivative of each function in part a). Confirm your prediction using the **nDeriv** function.

**d)** **Reflect** Predict a rule for the derivative of $y = cx^2$ for any constant $c \in \mathbb{R}$. Verify your prediction for other values of $c$.

**3.** **Reflect** Predict a general rule for the derivative of $f(x) = cx^n$, where $c$ is any real number. Verify your prediction using a graphing calculator.

## D: The Sum and Difference Rules

**1.** **a)** Given $f(x) = 4x$ and $g(x) = 7x$, determine

**i)** $f'(x)$, $g'(x)$, and $f'(x) + g'(x)$

**ii)** $h(x) = f(x) + g(x)$ and $h'(x)$

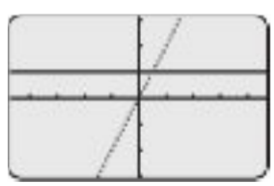
**b)** **Reflect** Compare your results in part a), i) and ii). Use these results to predict the derivative of $h(x) = x + x^2$.

**c)** Verify the accuracy of your prediction using a graphing calculator.

- Enter the two functions as shown in Screen 1 below.
- Press ZOOM and select **4:ZDecimal** to view Screen 2.
- Press 2ND GRAPH to see the table of values in Screen 3.

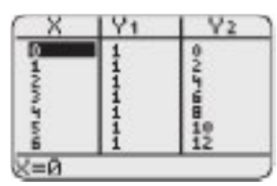
Screen 1


Screen 2



Window variables:
$x \in [-4.7, 4.7]$, $y \in [-3.1, 3.1]$

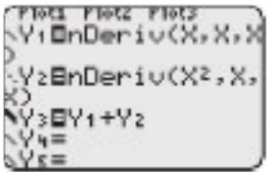
Screen 3



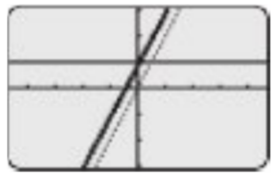
- Enter **Y1** + **Y2** as shown in Screen 4. Select a thick line to graph **Y3**.

View the graph and the table of values, shown in Screens 5 and 6. You will have to scroll to the right to see the **Y3** column.

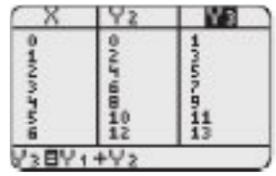
Screen 4




Screen 5



Screen 6




Window variables:
$x \in [-4.7, 4.7]$, $y \in [-3.1, 3.1]$

**d)** **Reflect** What do the table values in Screen 6 represent?

**e)** What is the relationship between **Y1**, **Y2**, and **Y3**?

**2.** Determine the derivative of the sum of the two functions by entering the function shown in Screen 7. View the graph and the table of values shown in Screens 8 and 9.

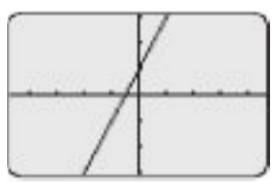
**a)** **Reflect** Compare Screens 6 and 9. What do the table values suggest to you?

**b)** Compare Screens 5 and 8. Describe how the graphs are related.

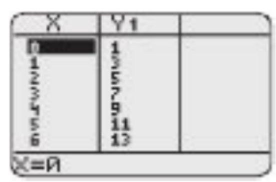
**c)** How do your results in parts a) and b) compare to the prediction you made in step 1 part b)?

Screen 7


Screen 8



Screen 9




Window variables:
$x \in [-4.7, 4.7]$, $y \in [-3.1, 3.1]$

**3.** **Reflect** Predict a general rule about the derivative of the sum of two functions.

**4.** Do you think there is a similar rule for the derivative of the difference of two functions? Describe how you could confirm your prediction.

## Derivative Rules

| Rule | Lagrange Notation | Leibniz Notation |
|---|---|---|
| **Constant Rule** If $f(x) = c$, where $c$ is a constant, then | $f'(x) = 0$ | $\frac{d}{dx}(c) = 0$ |
| **Power Rule** If $f(x) = x^n$, where $n$ is a positive integer, then | $f'(x) = nx^{n-1}$ | $\frac{d}{dx}x^n = nx^{n-1}$ |
| **Constant Multiple Rule** If $f(x) = cg(x)$, for any constant $c$, then | $f'(x) = cg'(x)$ | $\frac{d}{dx}cg(x) = c\frac{d}{dx}g(x)$ |
| **Sum Rule** If functions $f(x)$ and $g(x)$ are differentiable, and $h(x) = f(x) + g(x)$, then | $h'(x) = f'(x) + g'(x)$ | $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ |
| **Difference Rule** If functions $f(x)$ and $g(x)$ are differentiable, and $h(x) = f(x) - g(x)$, then | $h'(x) = f'(x) - g'(x)$ | $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$ |

The five differentiation rules you have explored can be proved using the first principles definition for the derivative. These proofs confirm that the patterns that you observed will apply to all functions. The following proofs of the constant rule and the power rule show how these proofs work. The constant multiple rule, sum, and difference rules can be similarly proved.

### The Constant Rule

If $f(x) = c$, where $c$ is a constant, then $f'(x) = 0$.

**Proof:**

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{c - c}{h} && f(x) \text{ is constant, so } f(x+h) \text{ is also equal to } c. \\
&= \lim_{h \to 0} \frac{0}{h} \\
&= \lim_{h \to 0} 0 \\
&= 0
\end{aligned}
$$

To reflect on how the preceding proof works, consider the following:

- Why are $f(x+h)$ and $f(x)$ equal? To support your answer, substitute a particular value for $c$ and work through the steps of the proof again.
- What would happen if you let $h \to 0$ before simplifying the expression?
- Describe how this proof can be verified graphically.

**The Power Rule**

If $f(x) = x^n$, where $n$ is a natural number, then $f'(x) = nx^{n-1}$.

**Proof:**

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \end{aligned}$$

Factor the numerator. Use $a^n - b^n$ with $a = (x+h)$ and $b = x$.

$$\begin{aligned} &= \lim_{h\to 0} \frac{[(x+h) - x][(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}]}{h} \\ &= \lim_{h\to 0} \frac{[h][(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}]}{h} \\ &= \lim_{h\to 0}[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}] \\ &= x^{n-1} + x^{n-2}x + \ldots + xx^{n-2} + x^{n-1} \\ &= x^{n-1} + x^{n-1} + \ldots + x^{n-1} + x^{n-1} && \text{Simplify using laws of exponents.} \\ &= nx^{n-1} && \text{There are } n \text{ terms.} \end{aligned}$$

This proves the power rule for any exponent $n \in \mathbb{N}$. A generalized version of the power rule is also valid. If $n$ is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$. This proof is explored in later calculus courses.

Reflect on the steps in the preceding proof by considering these questions:

- How does factoring help to prove the power rule?
- How are the laws of exponents applied in the proof?
- Why is it important to first simplify and reduce the expression before letting $h \to 0$?
- Why is it important to state that there are $n$ terms?

## Example 1 Justify the Power Rule for Rational Exponents Graphically and Numerically

**Tools**
- graphing calculator

**a)** Use the power rule with $n = \frac{1}{2}$ to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

**b)** Verify this derivative graphically and numerically.

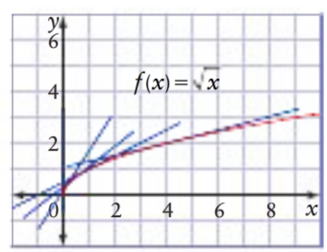
### Solution

**a)**

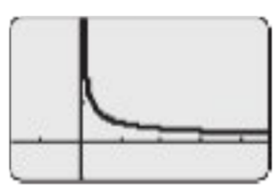
$$\begin{aligned} f(x) &= \sqrt{x} \\ &= x^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

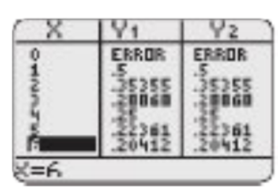
Apply the power rule.

**b)** Sketch the graph of the function $f(x) = \sqrt{x}$. The graph's domain is restricted because $x \geq 0$. At $x = 0$, the tangent is undefined. For values close to zero, the slope of the tangent is very large. As the $x$-values increase, the slope of the tangent becomes smaller, approaching zero.




You can verify these results using a graphing calculator. Enter the equations as shown and graph the functions. The graph of the derivative confirms that its value is very large when $x$ is close to zero, and gets smaller, approaching zero, as the $x$-value increases. Also, the fact that the two functions have identical graphs proves that the functions are equal. This is also confirmed by the table of values.




Window variables:
$x \in [-1, 4.7]$, $y \in [-1, 3.1]$

## Example 2 Rational Exponents and the Power Rule

Determine $\frac{dy}{dx}$ for each function. Express your answers using positive exponents.

**a)** $y = \sqrt[3]{x}$ **b)** $y = \frac{1}{x}$ **c)** $y = -\frac{1}{x^5}$

### Solution

First express the function in the form $y = x^n$, and then differentiate.

**a)**
$$\begin{aligned} y &= \sqrt[3]{x} \\ &= x^{\frac{1}{3}} \\ \frac{dy}{dx} &= \frac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{3x^{\frac{2}{3}}} \end{aligned}$$

**b)**
$$\begin{aligned} y &= \frac{1}{x} \\ &= x^{-1} \\ \frac{dy}{dx} &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

**c)**
$$\begin{aligned} y &= -\frac{1}{x^5} \\ &= (-1)x^{-5} \\ \frac{dy}{dx} &= 5x^{-6} \\ &= \frac{5}{x^6} \end{aligned}$$

## Example 3 Apply Strategies to Differentiate Polynomial Functions

Differentiate each function, naming the derivative rule(s) that are being used.

**a)** $y = 5x^6 - 4x^3 + 6$ **b)** $f(x) = -3x^5 + 8\sqrt{x} - 9.3$

**c)** $g(x) = (2x - 3)(x + 1)$ **d)** $h(x) = \frac{-8x^6 + 8x^2}{4x^5}$

### Solution

**a)**
$$\begin{aligned} y &= 5x^6 - 4x^3 \\ y' &= 5(6x^5) - 4(3x^2) \\ &= 30x^5 - 12x^2 \end{aligned}$$

Use the difference, constant multiple, and power rules.

**b)**
$$\begin{aligned} f(x) &= -3x^5 + 8\sqrt{x} - 9.3 \\ &= -3x^5 + 8x^{\frac{1}{2}} - 9.3 \end{aligned}$$

Express the root as a rational exponent.

$$f'(x) = -3(5x^4) + 8\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

Use the sum, constant multiple, power, and constant rules.

$$= -3(5x^4) + 8\left(\frac{1}{2\sqrt{x}}\right)$$

Express as a positive exponent.

$$= -15x^4 + \frac{4}{\sqrt{x}}$$

**c)** $g(x) = (2x - 3)(x + 1)$

$= 2x^2 + 2x - 3x - 3$

$= 2x^2 - x - 3$

$g'(x) = 4x - 1$ Use the difference, constant multiple, power, and constant rules.

**d)** $h(x) = \dfrac{-8x^6 + 8x^2}{4x^5}$

$= \dfrac{-8x^6}{4x^5} + \dfrac{8x^2}{4x^5}$

$= -2x + \dfrac{2}{x^3}$

$= -2x + 2x^{-3}$

$h'(x) = -2 - 6x^{-4}$ Use the difference, constant multiple, and power rules.

$= -2 - \dfrac{6}{x^4}$

## Example 4 Apply Derivative Rules to Determine the Equation of a Tangent

Determine the equation of the tangent to the curve $f(x) = 4x^3 + 3x^2 - 5$ at $x = -1$.

### Solution

**Method 1:** ***Use Paper and Pencil***

The derivative represents the slope of the tangent at any value $x$.

$f(x) = 4x^3 + 3x^2 - 5$

$f'(x) = 12x^2 + 6x - 0$ Use the sum and difference, constant multiple, power, and constant rules.

$= 12x^2 + 6x$

Substitute $x = -1$ into the derivative function.

$f'(-1) = 12(-1)^2 + 6(-1)$

$= 6$

When $x = -1$, the derivative, or slope of the tangent, is 6.

To find the point on the curve corresponding to $x = -1$, substitute $x = -1$ into the original function.

$f(-1) = 4(-1)^3 + 3(-1)^2 - 5$

$= -4 + 3 - 5$

$= -6$

The tangent point is $(-1, -6)$.

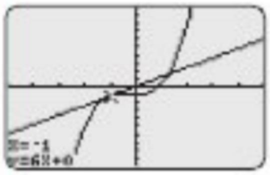
To find the equation of the tangent line, use the point-slope form for the equation of a line, $y - y_1 = m(x - x_1)$, substituting $m = 6$, $x_1 = -1$, and $y_1 = -6$.

$$y - (-6) = 6(x - (-1))$$
$$y = 6x + 6 - 6$$
$$y = 6x$$

The equation of the tangent line is $y = 6x$.

**Method 2:** ***Use Technology***

Use the Tangent operation on a graphing calculator to graph the tangent to the function **Y1** $= 4x^3 + 3x^2 - 5$ at $x = -1$.



Window variables:
$x \in [-5, 5]$, $y \in [-50, 50]$, Yscl $= 5$

**Tools**

- graphing calculator

**Technology Tip**

To get the result shown in this example, set the number of decimal places to 1, using the **MODE** menu on your graphing calculator.

## Example 5 Apply Derivative Rules to Solve an Instantaneous Rate of Change Problem

A skydiver jumps out of a plane from a height of 2200 m. The skydiver's height above the ground, in metres, after $t$ seconds is represented by the function $h(t) = 2200 - 4.9t^2$ (assuming air resistance is not a factor). How fast is the skydiver falling after 4 s?

### Solution

The instantaneous rate of change of the height of the skydiver at any point in time is represented by the derivative of the height function.

$$h(t) = 2200 - 4.9t^2$$
$$h'(t) = 0 - 4.9(2t)$$
$$= -9.8t$$

Substitute $t = 4$ into the derivative function to find the instantaneous rate of change at 4 s.

$$h'(4) = -9.8(4)$$
$$= -39.2$$

After 4 s, the skydiver is falling at a rate of 39.2 m/s.

**CONNECTIONS**

Earth's atmosphere is not empty space. It is filled with a mixture of gases, especially oxygen, that is commonly referred to simply as air. Falling objects are slowed by friction with air molecules. This resisting force is "air resistance." The height function used in this text ignores the effects of air resistance. Taking its effects into account would make these types of problems considerably more difficult.

## Example 6 Apply a Strategy to Determine Tangent Points for a Given Slope

Determine the point(s) on the graph of $y = x^2(x + 3)$ where the slope of the tangent is 24.

### Solution

Expand the function to put it in the form of a polynomial, and then differentiate.

$$\begin{aligned} y &= x^2(x + 3) \\ &= x^3 + 3x^2 \\ y' &= 3x^2 + 6x \end{aligned}$$

Since the derivative is the slope of the tangent, substitute $y' = 24$ and solve.

$$\begin{aligned} 24 &= 3x^2 + 6x \\ 0 &= 3x^2 + 6x - 24 \\ 0 &= 3(x^2 + 2x - 8) \\ 0 &= 3(x - 2)(x + 4) \end{aligned}$$

The equation is true when $x = 2$ or $x = -4$.

Determine $y$ by substituting the $x$-values into the original function.

For $x = 2$

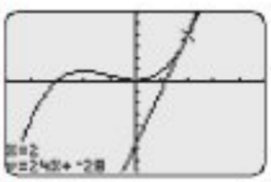
$$\begin{aligned} y &= 2^3 + 3(2)^2 \\ &= 8 + 12 \\ &= 20 \end{aligned}$$

For $x = -4$,

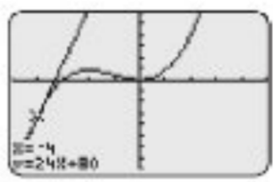
$$\begin{aligned} y &= (-4)^3 + 3(-4)^2 \\ &= -64 + 48 \\ &= -16 \end{aligned}$$

The two tangent points at which the slope is 24 are (2, 20) and (−4, −16).

Use the Tangent function on a graphing calculator to confirm these points. Enter the two $x$-values, noting the slope in the equation of the tangent at the bottom of the screens.




Window variables:
$x \in [-5, 5]$, $y \in [-30, 20]$, Yscl = 4




Window variables:
$x \in [-5, 5]$, $y \in [-40, 30]$, Yscl = 5

## KEY CONCEPTS

- Derivative rules simplify the process of differentiating polynomial functions.
- To differentiate a radical, first express it as a power with a rational exponent (e.g., $\sqrt[3]{x} = x^{\frac{1}{3}}$).
- To differentiate a power of $x$ that is in the denominator, first express it as a power with a negative exponent (e.g., $\frac{1}{x^2} = x^{-2}$).

## Communicate Your Understanding

**C1** How can you use slopes to prove that the derivative of a constant is zero?

**C2** How can the sum and difference rules help differentiate polynomial functions?

**C3** Why can you extend the sum and difference rules to three or more functions that are added and subtracted? Describe an example to support your answer.

**C4** What is the difference between proving a derivative rule and showing that it works for certain functions?

## A Practise

**1.** Which of the following functions have a derivative of zero?

**a)** $y = 9.8$ **b)** $y = 11$

**c)** $y = -4 + x$ **d)** $y = \frac{5}{9}x$

**e)** $y = \sqrt{7}$ **f)** $y = x$

**g)** $y = \frac{3}{4}$ **h)** $y = -2.8\pi$

**2.** For each function, determine $\frac{dy}{dx}$.

**a)** $y = x$ **b)** $y = \frac{1}{4}x^2$

**c)** $y = x^5$ **d)** $y = -3x^4$

**e)** $y = 1.5x^3$ **f)** $y = \sqrt[5]{x^3}$

**g)** $y = \frac{5}{x}$ **h)** $y = \frac{4}{\sqrt{x}}$

**3.** Determine the slope of the tangent to the graph of each function at the indicated value.

**a)** $y = 6, x = 12$

**b)** $f(x) = 2x^5, x = \sqrt{3}$

**c)** $g(x) = -\frac{3}{\sqrt{x}}, x = 4$

**d)** $h(t) = -4.9t^2, t = 3.5$

**e)** $A(r) = \pi r^2, r = \frac{3}{4}$

**f)** $y = \frac{1}{3x}, x = -2$

**4.** Determine the derivative of each function. State the derivative rules used.

**a)** $f(x) = 2x^2 + x^3$ **b)** $y = \frac{4}{5}x^5 - 3x$

**c)** $h(t) = -1.1t^4 + 78$ **d)** $V(r) = \frac{4}{3}\pi r^3$

**e)** $p(a) = \frac{a^5}{15} - 2\sqrt{a}$ **f)** $k(s) = -\frac{1}{s^2} + 7s^4$

## B Connect and Apply

**5. a)** Determine the point at which the slope of the tangent to each parabola is zero.

**i)** $y = 6x^2 - 3x + 4$

**ii)** $y = -x^2 + 5x - 1$

**iii)** $y = \frac{3}{4}x^2 + 2x + 3x^2 + 2x + 3$

**b) Use Technology** Graph each parabola in part a). What does the point found in part a) correspond to on each of these graphs?

**6.** Simplify, and then differentiate.

**a)** $f(x) = \dfrac{10x^4 - 6x^3}{2x^2}$

**b)** $g(x) = (3x + 4)(2x - 1)$

**c)** $p(x) = \dfrac{x^8 - 4x^6 + 2x^3}{4x^3}$

**d)** $f(x) = (5x + 2)^2$

**7. a)** Describe the steps you would follow to determine the equation of a tangent to a curve at a given $x$-value.

**b)** How is the derivative used to determine the tangent point when the slope of the tangent is known?

**8.** Consider the function $f(x) = (2x - 1)^2(x + 3)$.

**a)** Explain why the derivative rules from this section cannot be used to differentiate the function in the form in which it appears here.

**b)** Describe what needs to be done to $f(x)$ before it can be differentiated.

**c)** Apply the method you described in part b), and then differentiate $f(x)$.

**9. Use Technology** Verify algebraically, numerically, and graphically that the derivative of $y = \frac{1}{x}$ is $\frac{dy}{dx} = -\frac{1}{x^2}$.

**10.** The amount of water flowing into two barrels is represented by the functions $f(t)$ and $g(t)$. Explain what $f'(t)$, $g'(t)$, $f'(t) + g'(t)$, and $(f + g)'(t)$ represent. Explain how you can use this context to verify the sum rule.

**11.** A skydiver jumps out of a plane that is flying 2500 m above the ground. The skydiver's height above the ground, in metres, after $t$ seconds is $h(t) = 2500 - 4.9t^2$.

**a)** Determine the rate of change of the height of the skydiver after 5 s.

**b)** The skydiver's parachute opens at 1000 m above the ground. After how many seconds does this happen?

**c)** What is the rate of change of the height of the skydiver at the time found in part b)?

**12.** The following chart lists the acceleration due to gravity on several planets.

| Planet | Acceleration Due to Gravity (m/s$^2$) |
|---|---|
| Earth | 9.8 |
| Venus | 8.9 |
| Mars | 3.7 |
| Saturn | 10.5 |
| Neptune | 11.2 |

The height of a free-falling object on any planet is represented by the function $h(t) = -0.5gt^2 + k$, where $h$ is the height, in metres, $t$ is time, in seconds, $t \geq 0$, $g$ is the planet's acceleration due to gravity, in metres per second squared, and $k$ is the height, in metres, from which the object is dropped. Suppose a rock is dropped from a height of 250 m on each planet listed in the table. Use derivatives to determine the instantaneous rate of change of the height of the rock on each planet after 4 s.

**13. a)** Determine the slope of the tangent to the graph of $y = -6x^4 + 2x^3 + 5$ at the point $(-1, -3)$.

**b)** Determine the equation of the tangent at this point.

**c) Use Technology** Confirm your equation using a graphing calculator.

**14.** **a)** Determine the slope of the tangent to the graph of $y = -1.5x^3 + 3x - 2$ at the point $(2, -8)$.

**b)** Determine the equation of the tangent at this point.

**c)** **Use Technology** Confirm your equation using a graphing calculator.

**15.** A flaming arrow is shot into the air to mark the beginning of a festival. Its height, in metres, after $t$ seconds is modelled by the function $h(t) = -4.9t^2 + 24.5t + 2$.


**a)** Determine the height of the arrow at 2 s.

**b)** Determine the rate of change of the height of the arrow at 1, 2, 4, and 5 s.

**c)** What happens at 5 s?

**d)** How long does it take the arrow to return to the ground?

**e)** How fast is the arrow travelling when it hits the ground? Explain how you arrived at your answer.

**f)** Graph the function. Use the graph to confirm your answers in parts a) to e).

**16.** **a)** Determine the coordinates of the point(s) on the graph of $f(x) = x^3 - 7x$ where the slope of the tangent is 5.

**b)** Determine the equation(s) of the tangent(s) to the graph of $f(x)$ at the point(s) found in part a).

**c)** **Use Technology** Confirm your equation(s) using a graphing calculator.

**17.** **a)** Find the values of $x$ at which the tangents to the graphs of $f(x) = 2x^2$ and $g(x) = x^3$ have the same slope.

**b)** Determine the equations of the tangent lines to each curve at the points found in part a).

**c)** **Use Technology** Confirm your equations using a graphing calculator.

**18.** Use the first principles definition of the derivative and the properties of limits to prove the sum rule: $h'(x) = f'(x) + g'(x)$.

**19.** **Chapter Problem** The cost, in dollars, of producing $x$ frozen fruit yogurt bars can be modelled by the function $C(x) = 3450 + 1.5x - 0.0001x^2$, $0 \leq x \leq 5000$. The revenue from selling $x$ yogurt bars is $R(x) = 3.25x$.

**a)** Determine the cost of producing 1000 frozen fruit yogurt bars. What is the revenue generated from selling this many bars?

**b)** Compare the values for $C'(1000)$ and $C'(3000)$. What information do these values provide?

**c)** When is $C'(x) = 0$? Explain why this is impossible.

**d)** Determine $R'(x)$. What does this value represent?

**e)** The profit function, $P(x)$, is the difference of the revenue and cost functions. Determine the equation for $P(x)$.

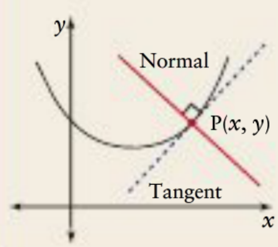
**f)** When is the profit function positive? When is it negative? What important information does this provide the owners?

**20.** **a)** Determine the slope of the tangent to the curve $f(x) = -2x^3 + 5x^2 - x + 3$ at $x = 2$.

**b)** Determine the equation of the normal to $f(x)$ at $x = 2$.

### CONNECTIONS

The normal to a curve at a point $(x, y)$ is the line perpendicular to, and intersecting, the curve's tangent at that point.




**21.** **a)** Determine the slope of the tangent to the curve $f(x) = -4x^3 + \frac{3}{x} + \sqrt{x} - 2$ at $x = 1$.

**b)** Determine the equation of the normal to $f(x)$ at $x = 1$.

**22. a)** Determine the equation of the tangent to the graph of $f(x) = \left(2 - \sqrt{x}\right)^2$ at $x = 9$.

**b) Use Technology** Confirm your equation using a graphing calculator.

**23. a)** Determine the equation of the tangent to the graph of $g(x) = \left(\frac{4}{x^3} + 1\right)(x - 3)$ at $x = -1$.

**b) Use Technology** Confirm your equation using a graphing calculator.

**24. a)** Determine the equations of the tangents to the points on the curve $y = -x^4 + 8x^2$ such that the tangent lines are perpendicular to the line $x = 1$.

**b) Use Technology** Verify your solution using a graphing calculator.

**25. a)** Show that there are no tangents to the curve $y = 6x^3 + 2x^2$ that have a slope of $-5$.

**b) Use Technology** Verify your solution using a graphing calculator.

## Achievement Check

**26.** The population of a bacteria colony is modelled by the function $p(t) = 200 + 20t - t^2$, where $t$ is time, in hours, $t \geq 0$, and $p$ is the number of bacteria, in thousands.

**a)** Determine the growth rate of the bacteria population at each of the following times.

**i)** 3 h **ii)** 8 h **iii)** 13 h **iv)** 18 h

**b)** What are the implications of the growth rates in part a)?

**c)** Determine the equation of the tangent to $p(t)$ at the point corresponding to $t = 8$.

**d)** When does the bacteria population stop growing? What is the population at this time?

**e)** Graph the growth function and its derivative. Describe how each graph reflects the rate of change of the bacteria population.

**f)** Determine the time interval over which the bacteria population

**i)** increases **ii)** decreases

## C Extend and Challenge

**27.** Determine the values of $a$ and $b$ for the function $f(x) = ax^3 + bx^2 + 3x - 2$ given that $f(2) = 10$ and $f'(-1) = 14$.

**28.** Determine the equations of two lines that pass through the point $(1, -5)$ and are tangent to the graph of $y = x^2 - 2$.

**29. a)** Determine the equations of the tangents to the cubic function $y = 2x^3 - 3x^2 - 11x + 8$ at the points where $y = 2$.

**b) Use Technology** Verify your solution using a graphing calculator.

**30.** Show that there is no polynomial function that has a derivative of $x^{-1}$.

**31. Math Contest** Consider the polynomials $p(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$ and $q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$, where $a_m \neq 0 \neq b_n$ and $m, n \geq 1$. If the equation $p(x) - q(x) = 0$ has $m + n$ real roots, which of the following must be true?

**i)** $p(x)$ and $q(x)$ have no common factors.

**ii)** The equation $p'(x) - q'(x) = 0$ has exactly $m + n - 1$ real roots.

**iii)** The equation $p(x) - q(x) = 0$ has infinitely many real roots.

**A** i) only **B** ii) only

**C** i) and ii) only **D** iii) only

**E** i) and iii) only

**32. Math Contest** If $p$ and $q$ are two polynomials such that $p'(x) = q'(x)$ for all real $x$ with $p(0) = 1$ and $q(0) = 2$, then which of the following best describes the intersection of the graphs of $y = p(x)$ and $y = q(x)$?

**A** They intersect at exactly one point.

**B** They intersect at at least one point.

**C** They intersect at at most one point.

**D** They intersect at more than one point.

**E** They do not intersect.

# Extension Problem Solving with a Computer Algebra System

You can use a Computer Algebra System (CAS) to solve exercise 16 from Section 2.1.

**16. a)** Determine the coordinates of the point(s) on the graph of $f(x) = x^3 - 7x$ where the slope of the tangent is 5.

**b)** Determine the equation(s) of the tangent(s) to the graph of $f(x)$ at the point(s) found in part a).

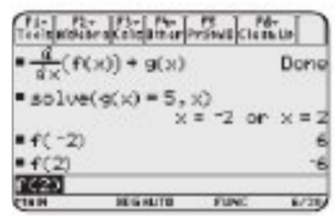
## Solution:

**a)** Turn on the CAS. If necessary press HOME to display the HOME screen. Ensure that the following parameters are correctly set.

- Clear the CAS variables by pressing 2ND F1 to access the F6 menu. Select **2:NewProb**. Press ENTER.
- Press MODE. Scroll down to **Exact/Approx**, and ensure that **AUTO** is selected. Enter the function and store it as $f(x)$.
- From the F4 menu, select **1:Define**. Enter the function $f(x) = x^3 - 7x$ and press ENTER.


Determine the derivative of the function, and the value(s) of $x$ when the derivative is 5.

- From the F3 menu, select **1:*d*( differentiate**. Enter $f(x)$, $x$).
- Press STO, and enter $g(x)$. Press ENTER.
- From the F2 menu, select **1:solve(**. Enter $g(x) = 5$, $x$), and press ENTER.



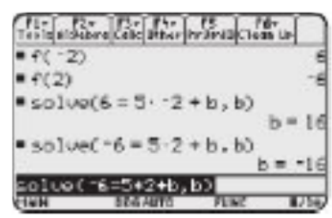

The slope of the tangent is equal to 5 when $x = -2$ and $x = 2$.

Determine $f(-2)$ and $f(2)$.

- Enter $f(-2)$, and press ENTER. Similarly, enter $f(2)$, and press ENTER.

The slope of the tangent is equal to 5 at the tangent points $(-2, 6)$ and $(2, -6)$.

**b)** The equation of a straight line is $y = mx + b$. Substitute $m = 5$, $x = -2$, and $y = 6$, and solve for $b$. When this is entered, $b = 16$. The equation of the tangent to the curve at $(-2, 6)$ is $y = 5x + 16$.




In a similar manner, substitute $m = 5$, $x = 2$, and $y = -6$, and solve for $b$. When this is entered, $b = -16$. The equation of the tangent to the curve at $(2, -6)$ is $y = 5x - 16$.

To view the form of the derivative, enter $g(x)$, and press ENTER.