Exercises

Introduction to Calculus: Limits and Rates of Change

1. An ambulance is called to the scene of an accident 4 kilometres away from the ambulance station. The distance, in kilometres, that the ambulance has traveled to the scene, t seconds after leaving the station, is given in the following table:

Time (s)	0	30	60	90	120	150	180
Distance (km)	0	0.5	1.5	1.75	2.75	3.25	4.0

   a. Over which 30-second interval(s) did the ambulance travel the fastest?

   b. Over which 30-second interval(s) did the ambulance travel the slowest?

   c. Determine the average speed of the ambulance, in kilometres per hour, over its entire trip to the accident.

2. An egg is dropped from a third story window. The distance between the egg and the ground t seconds after the egg is dropped is observed at various times and recorded in given table.

Time (s)	0	0.64	0.90	1.11	1.28	1.43
Distance (m)	10	8	6	4	2	0

   a. The distance between the egg and the ground is decreasing with respect to time. Is the distance changing at a constant rate? Explain.

   b. What is the average rate of change of the distance between the egg and the ground over the entire egg drop?

   c. Over which of the five time intervals is the average rate of change of the distance between the egg and the ground the largest in magnitude?

   d. Estimate the velocity of the egg 1.11 seconds after the drop.

3. Determine the slope of the secant to the given curve between the specified values of x.

   a. $y = x^2 - 3,\ x = 1,\ x = 3$

   b. $y = 2^x - 4,\ x = 2,\ x = 3$

   c. $y = 2x^2 + 8x - x^3,\ x = 0,\ x = 2$

   d. $y = 2\sqrt{x+3},\ x = -2,\ x = 6$

   e. $y = \dfrac{1}{x-2},\ x = \dfrac{5}{2},\ x = 3$

   f. $y = \sin(x),\ x = \dfrac{\pi}{2},\ x = \pi$

4. Approximate the slope of the tangent to the given curve at the specified value of x, by finding the slopes of the secant lines between x and each of the points $x \pm 0.1, x \pm 0.01,$ and $x \pm 0.001$.

a. $y = 3 - x^2,\ x = 1$

b. $y = 3^x + 1,\ x = 2$

c. $y = \sqrt{2+x},\ x = 0$

d. $y = \cos(2x),\ x = \frac{\pi}{2}$

5. Find the exact value of the slope of the tangent to the given curve at the specified value of x. For part b) and part e), compare your answer to the approximation found in Exercise 4.

   a. $y = x - \dfrac{1}{2},\ x = -1$

   b. $y = 3 - x^2,\ x = 1$

   c. $y = 2x^2 + 1,\ x = 3$

   d. $y = -5x^2 + 30x + 2,\ x = 4$

   e. $y = \sqrt{2+x},\ x = 0$

6. The altitude of a rock climber t hours after she begins her ascent up a mountain is modelled by the equation $a(t) = -10t^2 + 60t$, where the altitude, $a(t)$, is measured in metres.

   a. Determine the altitude of the rock climber 2 hours after she begins her climb.

   b. Determine the altitude of the rock climber 3 hours after she begins her climb.

   c. Determine the average rate of change of the altitude of the rock climber between 2 and 3 hours after she begins her climb.

   d. Determine the instantaneous rate of change of the altitude of the rock climber 3 hours after she begins her climb.

   e. What is the significance of the instantaneous rate of change value found in part d)? Explain what this value tells us about the rock climber's travel at this point.

7. A particle is traveling back and forth along a straight path. The particle is released at $t = 0$, and the position, s(t), of the particle relative to its starting position, at time $t \geq 0$ (in seconds), is given by the formula $s(t) = t^3 - 5t^2 + 4t$ (in metres). If $s(t) > 0$, then the particle is $|s(t)|$ metres *ahead* of its starting position at time t; if $s(t) < 0$ then the particle is $|s(t)|$ metres *behind* its starting position at time t.

   a. Calculate $s(1)$. What is the physical interpretation of this value?

   b. Calculate $s(2)$. What is the physical interpretation of this value?

   c. Determine the average rate of change of the particle's position, relative to its starting position, between 1 and 2 seconds.

   d. What does a negative average rate of change tell us about how the particle is traveling?

   e. Estimate the instantaneous rate of change of the particle's position, relative to its starting position, 2 seconds after it is released.

8. A glass of cold water is removed from the fridge and left on the kitchen counter. The temperature of the water, t minutes after being placed on the counter, is measured at various times and the data is recorded in the following table.

Time (min)	**Temperature** ($^\circ\mathrm{C}$)
0	6
1	8
4	10
9	12
16	14
25	16
36	18

a. Graph this data.

b. Observe that the temperature of the water is increasing over time. Is the temperature increasing at a constant rate? Explain.

c. Determine the average rate of change of the temperature of the water between 4 and 9 minutes, and between 9 and 16 minutes.

d. Over which interval(s) is the average rate of change of the temperature of the water the highest? Justify your answer both numerically and graphically.

e. Estimate the instantaneous rate of change of the temperature of the water 9 minutes after it has been placed on the table, using the following methods:

   i. Average the rate of change data collected in part c).

   ii. Draw a smooth curve through the data points on the graph, and sketch the tangent line to the curve at $t = 9$. Estimate the slope of this tangent line.

f. The given data points lie on a curve of the form $T(t) = a\sqrt{t} + b$ for some integers a and b.

   i. Using the given table of data, find the values of a and b.

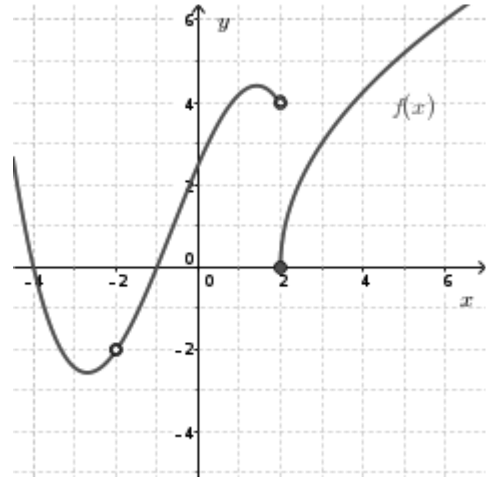
   ii. Find the instantaneous rate of change of the temperature of the water at $t = 9$, and compare this value with the answers from e).

9. The path of a robot along a track is modelled by the curve $y = 4x^2 + 6$. As the robot moves, it passes through the point $P\,(-1, 10)$. At this point, it attempts to shoot a ball at a target located at the point $\left(\frac{1}{4}, 0\right)$. If the ball travels along the tangent line to the curve at point P, will the ball hit the target? Support your answer with a sketch.

10. On an evening walk, a man passes under a streetlight and notices that the length of his shadow increases as he walks away from the base of the streetlight. If the man is $1.75\ \mathrm{m}$ tall and the streetlight is $6\ \mathrm{m}$ tall, determine how the length of the man's shadow is changing, in terms of the rate at which he is walking away from the streetlight. Support your answer with a diagram.

Evaluating Limits Graphically

1. Given the graph of $f(x)$, evaluate the following expressions involving $f(x)$.



   a. $\lim\limits_{x \to -1} f(x)$

   b. $\lim\limits_{x \to -2} f(x)$

   c. $\lim\limits_{x \to 2^+} f(x)$

   d. $\lim\limits_{x \to -2^-} f(x)$

   e. $f(-2)$

   f. $\lim\limits_{x \to 2} f(x)$

   g. $\lim\limits_{x \to -2^+} f(x)$

   h. $\lim\limits_{x \to 2^-} f(x)$

   i. $f(2)$

2. Graph the following piecewise function, then evaluate each limit below.

$$f(x) = \begin{cases} -x+4 & \text{if } x \leq 3 \\ x-5 & \text{if } x > 3 \end{cases}$$

   a. $\lim\limits_{x \to 3^-} f(x)$

   b. $\lim\limits_{x \to 3^+} f(x)$

   c. $\lim\limits_{x \to 3} f(x)$

3. Sketch the graph of a function that has the following characteristics:

   - $\lim\limits_{x \to -1^-} f(x) = 3$
   - $\lim\limits_{x \to -1^+} f(x) = 1$
   - $\lim\limits_{x \to 3} f(x) = 2$
   - $f(3)$ does not exist

4. Evaluate the following limits, given the graph.

   a. $\lim\limits_{x \to 1} f(x)$

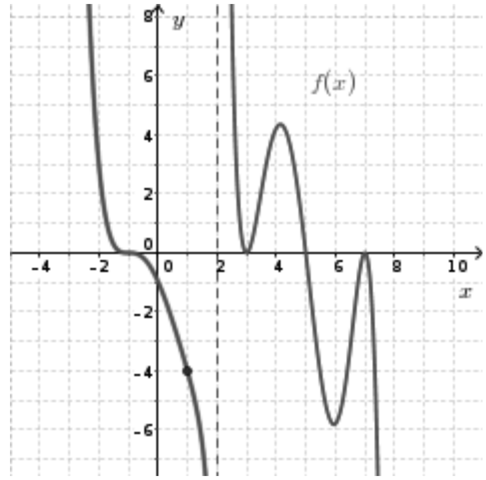
   b. $\lim\limits_{x \to 3^-} f(x)$

   c. $\lim\limits_{x \to 3^+} f(x)$

   d. $\lim\limits_{x \to 2^-} f(x)$

   e. $\lim\limits_{x \to 2^+} f(x)$

f. $\lim_{x \to 2} f(x)$



5. Sketch the graph of a function that has the following characteristics:
   - $\lim_{x \to 3^-} f(x) \to +\infty$
   - $\lim_{x \to -1^+} f(x) \to -\infty$
   - $\lim_{x \to 1} f(x) = 1$
   - $f(0) = 0$

6. Consider the piecewise function $f(x)$ defined below, where A is a constant.

$$f(x) = \begin{cases} A^2x - 4A & \text{if } x \geq 2 \\ -2 & \text{if } x < 2 \end{cases}$$

   Determine all values of A so that $\lim_{x \to 2} f(x)$ exists.

7. Consider the following piecewise function $f(x)$, where A and B are constants.

$$f(x) = \begin{cases} Ax + B & \text{if } x < -2 \\ x^2 + 2Ax - B & \text{if } -2 \leq x < 1 \\ 4 & \text{if } x > 1 \end{cases}$$

   Determine all values of the constants A and B so that $\lim_{x \to -2} f(x)$ and $\lim_{x \to 1} f(x)$ both exist.

8. The greatest integer function is defined as $f(x) = [x] = n$, where n is an integer such that $n \leq x < n + 1$.
   a. Sketch the graph of $f(x) = [x]$.

   b. For what values of p do the following one sided limits exist?
      i. $\lim_{x \to p^-} f(x)$
      ii. $\lim_{x \to p^+} f(x)$

   c. For what values of p do the right and left hand limits exist, but $\lim_{x \to p^-} f(x) \neq \lim_{x \to p^+} f(x)$?

   d. For what values of p does $\lim_{x \to p} f(x)$ exist?

Properties of Limits

1. Given that $\lim\limits_{x\to a} f(x) = 4$ and $\lim\limits_{x\to a} g(x) = -2$, find the following limits:
   a. $\lim\limits_{x\to a} (f(x) + g(x))$
   b. $\lim\limits_{x\to a} f(x)g(x)$
   c. $\lim\limits_{x\to a} 4f(x)$
   d. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$
   e. $\lim\limits_{x\to a} \dfrac{\sqrt{f(x)}}{g(x)}$
   f. $\lim\limits_{x\to a} \dfrac{f(x)+2}{2-2g(x)}$

2. Find the following limits.
   a. $\lim\limits_{x\to 5} (2x+3)$
   b. $\lim\limits_{x\to 2} (-x^2+3x-2)$
   c. $\lim\limits_{t\to -1} 5(t-2)(t-3)$
   d. $\lim\limits_{x\to 3} \dfrac{x+3}{x-2}$
   e. $\lim\limits_{x\to 0} 2(2x-1)^3$
   f. $\lim\limits_{x\to -3} (5-x)^{\frac{4}{3}}$

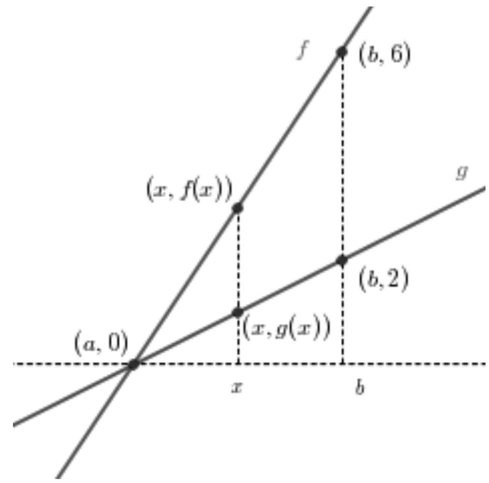
3. Suppose we have that $\lim\limits_{x\to 0} f(x) = 2$ and $\lim\limits_{x\to 0} g(x) = -10$. State the limit properties that are used to accomplish steps (a), (b), and (c) of the following calculation:

$$\begin{aligned}
\lim_{x\to 0} \frac{3f(x)-g(x)}{(g(x)+2)^{\frac{1}{3}}} &= \frac{\left(\lim\limits_{x\to 0}(3f(x)-g(x))\right)}{\lim\limits_{x\to 0}(g(x)+2)^{\frac{1}{3}}} && \text{(a)} \\
&= \frac{\lim\limits_{x\to 0} 3f(x) - \lim\limits_{x\to 0} g(x)}{\left(\lim\limits_{x\to 0}(g(x)+2)\right)^{\frac{1}{3}}} && \text{(b)} \\
&= \frac{3\lim\limits_{x\to 0} f(x) - \lim\limits_{x\to 0} g(x)}{\left(\lim\limits_{x\to 0} g(x) + \lim\limits_{x\to 0} 2\right)^{\frac{1}{3}}} && \text{(c)} \\
&= \frac{3(2)-(-10)}{(-10+2)^{\frac{1}{3}}} \\
&= \frac{16}{-2} \\
&= -8
\end{aligned}$$

4. If $\lim\limits_{x \to 1} f(x) = -2$ and $\lim\limits_{x \to 1} g(x) = 3$, then what is the value of $\lim\limits_{x \to 1} \dfrac{[f(x)]^3 + [g(x)]^2}{5 - 2g(x)}$?

5. If $\lim\limits_{x \to a} f(x) = 4$ and $\lim\limits_{x \to a} g(x) = -1$, then what is the value of $\lim\limits_{x \to a} \sqrt{\dfrac{2\sqrt{5 + g(x)}}{3f(x) + 2g(x) + 6}}$?

6. Let $a < b$ be real numbers. Consider two linear functions as shown in the graph. Evaluate $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$.

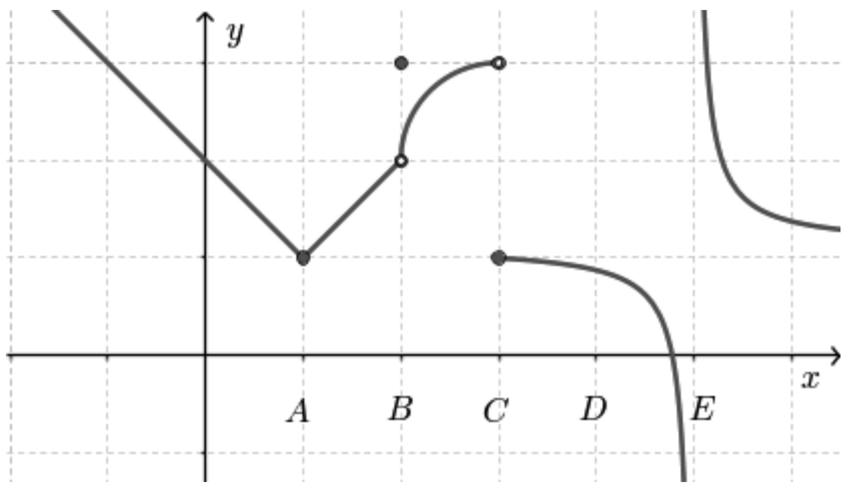


7. Give an example of functions $f(x)$ and $g(x)$ such that $\lim\limits_{x \to 0} (f(x) + g(x))$ exists but $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 0} g(x)$ do not exist.

8. Give an example of a function such that $\lim\limits_{x \to 0} [f(x)]^2$ exists but $\lim\limits_{x \to 0} f(x)$ does not exist.

Continuity

1. Sketch a possible graph of a function with the given properties.
   a. $f(3)$ is undefined and $\lim_{x\to 3} f(x) = -1$.
   b. $f(1) = 3$ and $\lim_{x\to 1} f(x)$ does not exist.
   c. $f(-1) = 3$ and $\lim_{x\to -1} f(x) = -3$
2. Sketch the graph of a function f such that $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$ both exist and are equal, but f is discontinuous at $x = -1$.
3. The graph of $y = f(x)$ is shown. Determine whether the function is continuous at the indicated points. State the type of discontinuity (removable, jump, infinite, or none of these).



   a. $x = A$
   b. $x = B$
   c. $x = C$
   d. $x = D$
   e. $x = E$
4. Determine the points at which the following functions are discontinuous. Describe the type(s) of discontinuity at the points from the following list: removable, jump, infinite, or none of these.
   a. $f(x) = \dfrac{x^2 + x - 2}{x^2 + 3x + 2}$
   b. $f(x) = \begin{cases} x + 2 & \text{if } x \leq 3 \\ x & \text{if } x > 3 \end{cases}$
   c. $f(x) = \dfrac{x + 1}{|x - 2|}$
5. Find all values of x for which the given function is continuous.
   a. $f(x) = 2^x$
   b. $f(x) = \dfrac{x^2 + 5}{x^2 - 5x}$
   c. $f(x) = \dfrac{16x}{x^2 + 16}$
   d. $f(x) = \sqrt{2x + 3}$

6. Sketch the following piecewise functions and determine whether each function is continuous for all real numbers x. Justify your answers.

   a. $f(x) = \begin{cases} x-2 & \text{if } x \geq 0 \\ -(x+3) & \text{if } x < 0 \end{cases}$

   b. $g(x) = \begin{cases} x+4 & \text{if } x \leq -1 \\ x^2-2x & \text{if } x > -1 \end{cases}$

7. Your bank account is continuously accruing compounded interest. If you deposit $\$500$ into your account at noon, and the amount of money in your account is plotted against time t, where t is the number of hours after noon, what type of discontinuity will appear at noon?

8. a. Find all values of a such that the function

$$f(x) = \begin{cases} x^2-4x & \text{if } x < a \\ -4 & \text{if } x \geq a \end{cases}$$

   is continuous for all values of x.

   b. Find all values of a and b such that the function

$$g(x) = \begin{cases} a^2x+2 & \text{if } x > 3 \\ 5 & \text{if } x = 3 \\ x^2-bx+a & \text{if } x < 3 \end{cases}$$

   is continuous.

9. For each of the following sets of properties, sketch a function that satisfies the properties given.

   a. $f(1) = 0$, $\lim\limits_{x \to 1^-} f(x) = -1$, and $\lim\limits_{x \to 1^+} f(x) = 1$

   b. $f(x) = -1$ for $-1 \leq x \leq 3$, $\lim\limits_{x \to -1^-} f(x) = 0$, and $\lim\limits_{x \to 3^+} f(x) = 1$

10. For each of the following, sketch the graph of a function $f(x)$ that satisfies the given description.

   a. f is continuous for all $x \neq 2$, and has a removable discontinuity at $x = 2$.

   b. The domain of f is $\{x \mid 0 \leq x \leq 1, x \in \mathbb{R}\}$, f is continuous from the right at $x = 0$, continuous on $0 < x < 1$, and has an infinite discontinuity at $x = 1$.

   c. f is continuous for all $x \neq -1, \frac{1}{2}$, has a removable discontinuity at $x = -1$ and an infinite discontinuity at $x = \frac{1}{2}$.

   d. $f(2) = -3$, f is continuous for all $x \neq 1, 3,$ has a jump discontinuity at $x = 3$ and an infinite discontinuity at $x = 1$.

11. For each of the descriptions in question 10, find an explicit equation of a function $f(x)$ that satisfies the given description. If possible, try and find a function that is not a piecewise function.