2.2 The Product Rule

The manager of a miniature golf course is planning to raise the ticket price per game. At the current price of \$6.50, an average of 81 rounds is played each day. The manager's research suggests that for every \$0.50 increase in price, an average of four fewer games will be played each day. Based on this information, the revenue from sales is represented by the function $R(n) = (81 - 4n)(6.50 + 0.50n)$, where n represents the number of \$0.50 increases in the price.

The manager can use derivatives to determine the price that will provide maximum revenue. To do so, he could use the differentiation rules explored in Section 2.1, but only after expanding and simplifying the expression. In this section, you will explore the product rule, which is a more efficient method for differentiating a function like the one described above. Not only is this method more efficient in these cases, but it can also be used to differentiate functions that cannot be expanded and simplified.

Investigate — Does the derivative of a product of two functions equal the product of their derivatives?

Complete each step for the functions $f(x) = x^3$ and $g(x) = x^4$.

1. Determine the following:

a) $f'(x)$ and $g'(x)$

b) $f'(x) \times g'(x)$, in simplified form

2. Determine the following:

a) $f(x) \times g(x)$, in simplified form

b) $\dfrac{d}{dx}[f(x) \times g(x)]$, in simplified form

3. **Reflect** Does the above result verify that $f'(x) \times g'(x)$ does not equal $\dfrac{d}{dx}[f(x)g(x)]$.

4. **Reflect** You know that the derivative of x^7 is $7x^6$. Can you combine $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ in an expression that will give the result $7x^6$? Do you think this will always work?

The Product Rule

If $P(x) = f(x)g(x)$, where $f(x)$ and $g(x)$ are differentiable functions, then $P'(x) = f'(x)g(x) + f(x)g'(x)$.

In Leibniz notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

CONNECTIONS

A simple way to remember the product rule is to express the product, P, in terms of f (first term) and s (second term): $P = fs$.
The product rule then becomes $P' = f's + fs'$.
In words: "the derivative of the first times the second, plus the first times the derivative of the second."

Proof:

Use the first principles definition for $P(x) = f(x)g(x)$.

$$\begin{aligned} P'(x) &= \lim_{h\to 0}\frac{P(x+h) - P(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \qquad P(x+h) = f(x+h)g(x+h). \end{aligned}$$

Reorganize this expression as two fractions, one involving $f'(x)$ and one involving $g'(x)$, by adding and subtracting $f(x+h)g(x)$ in the numerator.

$$\begin{aligned} &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0}\frac{[f(x+h)g(x+h) - f(x+h)g(x)]}{h} + \lim_{h\to 0}\frac{[f(x+h)g(x) - f(x)g(x)]}{h} && \text{Separate the limit into two fractions.} \\ &= \lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0}\frac{g(x)[f(x+h) - f(x)]}{h} && \text{Common factor each numerator.} \\ &= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} && \text{Apply limit rules.} \\ &= f(x)g'(x) + g(x)f'(x) && \text{Recognize the expressions for } g'(x) \text{ and} \\ &= f'(x)g(x) + f(x)g'(x) && f'(x) \text{ and rearrange.} \end{aligned}$$

Notice that $\lim_{h\to 0} g(x) = g(x)$, since $g(x)$ is a constant with respect to h, and that $\lim_{h\to 0} f(x+h) = f(x)$, since $f(x)$ is continuous.

Reflect on the steps in the preceding proof by considering these questions:

- What special form of zero is added to the numerator in the proof?
- Why is it necessary to add this form of zero?
- How is factoring used in the proof?
- Why is it important to separate the expression into individual limits?

Example 1 Apply the Product Rule

Use the product rule to differentiate each function.

a) $p(x) = (3x - 5)(x^2 + 1)$

Check your result algebraically.

b) $y = (2x + 3)(1 - x)$

Check your result graphically and numerically using technology.

Solution

a) $p(x) = (3x - 5)(x^2 + 1)$

Let $f(x) = 3x - 5$ and $g(x) = x^2 + 1$.

Then $f'(x) = 3$ and $g'(x) = 2x$.

Apply the product rule, $p'(x) = f'(x)g(x) + f(x)g'(x)$.

$$\begin{aligned} p'(x) &= 3(x^2 + 1) + (3x - 5)(2x) \\ &= 3x^2 + 3 + 6x^2 - 10x \\ &= 9x^2 - 10x + 3 \end{aligned}$$

Checking the solution algebraically,

$$\begin{aligned} p(x) &= (3x - 5)(x^2 + 1) \\ &= 3x^3 + 3x - 5x^2 - 5 \\ &= 3x^3 - 5x^2 + 3x - 5 \\ p'(x) &= 9x^2 - 10x + 3 \end{aligned}$$

b) $y = (2x + 3)(1 - x)$

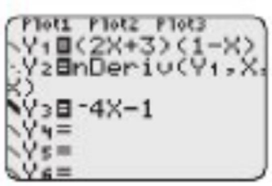
Let $f(x) = 2x + 3$ and $g(x) = 1 - x$.

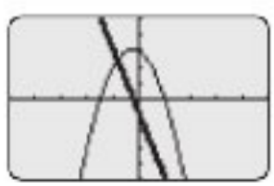
Then $f'(x) = 2$ and $g'(x) = -1$.

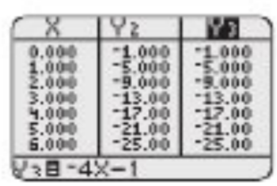
Apply the product rule, $y' = f'(x)g(x) + f(x)g'(x)$

$$\begin{aligned} y' &= 2(1 - x) + (2x + 3)(-1) \\ &= 2 - 2x - 2x - 3 \\ &= -4x - 1 \end{aligned}$$

To check the solution using a graphing calculator, enter the functions as shown and press GRAPH. The straight line represents both **Y2** and **Y3,** confirming that $-4x - 1$ is equal to the derivative. The table of values confirms this result numerically.







Example 2 Find Equations of Tangents Using the Product Rule

Determine the equation of the tangent to the curve $y = (x^2 - 1)(x^2 - 2x + 1)$ at $x = 2$. Use technology to confirm your solution.

Solution

Apply the product rule.

$$\frac{dy}{dx} = \left[\frac{d}{dx}(x^2 - 1)\right](x^2 - 2x + 1) + (x^2 - 1)\left[\frac{d}{dx}(x^2 - 2x + 1)\right]$$

$$\frac{dy}{dx} = (2x)(x^2 - 2x + 1) + (x^2 - 1)(2x - 2)$$

There is no need to simplify the expression to calculate the slope. It is quicker to simply substitute $x = 2$ and then simplify the expression.

$$\begin{aligned} m &= \left.\frac{dy}{dx}\right|_{x=2} \\ &= 2(2)[2^2 - 2(2) + 1] + (2^2 - 1)[2(2) - 2] \\ &= 10 \end{aligned}$$

Determine the y-coordinate of the tangent point by substituting the x-value into the original function.

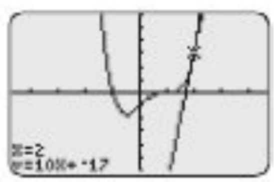
$$\begin{aligned} y &= f(2) \\ &= (2^2 - 1)(2^2 - 2(2) + 1) \\ &= 3 \end{aligned}$$

The tangent point is (2, 3).

The equation of the line through (2, 3) with slope 10 is

$$\begin{aligned} y - 3 &= 10(x - 2) \\ y &= 10x - 17 \end{aligned}$$

This answer can be confirmed using the Tangent function on a graphing calculator.



Example 3 Apply Mathematical Modelling to Develop a Revenue Function

The student council is organizing its annual trip to an out-of-town concert. For the past three years, the cost of the trip was \$140 per person. At this price, all 200 seats on the train were filled. This year, the student council plans to increase the price of the trip. Based on a student survey, the council estimates that for every \$10 increase in price, 5 fewer students will attend the concert.

a) Write an equation to represent revenue, R, as a function of the number of \$10 increases, n.

b) Determine an expression, in simplified form, for $\frac{dR}{dn}$ and interpret its meaning for this situation.

c) What is the rate of change in revenue when the price of the trip is \$200? How many students will attend the concert at this price?

Solution

a) Revenue, R, is the product of the price per student and the number of students attending. So, $R = \text{price} \times \text{number of students}$. The following table illustrates how each \$10 increase will affect the price, the number of students attending, and the revenue.

Number of \$10 Increases, *n*	Cost/Student (\$)	Number of Students	Revenue (\$)
0	140	200	140×200
1	$140 + 10(1)$	$200 - 5(1)$	$(140 + 10(1))(200 - 5(1))$
2	$140 + 10(2)$	$200 - 5(2)$	$(140 + 10(2))(200 - 5(2))$
3	$140 + 10(3)$	$200 - 5(3)$	$(140 + 10(3))(200 - 5(3))$
⋮	⋮	⋮	⋮
n	$140 + 10n$	$200 - 5n$	$(140 + 10n)(200 - 5n)$

The revenue function is $R(n) = (140 + 10n)(200 - 5n)$.

b) Apply the product rule.

$$\begin{aligned}\frac{dR}{dn} &= \left[\frac{d}{dn}(140 + 10n)\right](200 - 5n) + (140 + 10n)\left[\frac{d}{dn}(200 - 5n)\right]\\ &= (10)(200 - 5n) + (140 + 10n)(-5)\\ &= 2000 - 50n - 700 - 50n\\ &= 1300 - 100n\end{aligned}$$

$\frac{dR}{dn} = 1300 - 100n$ represents the rate of change in revenue for each \$10 increase.

c) The price of the trip is \$200 when there are 6 increases of \$10. Evaluate the derivative for $n = 6$.

$$\begin{aligned}R'(6) &= 1300 - 100(6)\\ &= 700\end{aligned}$$

When the price of the trip is \$200, the rate of increase in revenue is \$700 per price increase. The number of students attending when $n = 6$ is $200 - 5(6) = 170$. Thus, 170 students will attend the concert when the price is \$200.

KEY CONCEPTS

- The Product Rule

  If $h(x) = f(x)g(x)$, where $f(x)$ and $g(x)$ are differentiable functions, then $h'(x) = f'(x)g(x) + f(x)g'(x)$.

  In Leibniz notation,

  $$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

Communicate Your Understanding

C1 Why is the following statement false? "The product of the derivatives of two functions is equal to the derivative of the product of the two functions." Support your answer with an example.

C2 What response would you give to a classmate who makes the statement, "I don't need to learn the product rule because I can always expand each product and then differentiate"?

C3 Why is it unnecessary to simplify the derivative before substituting the value of x to calculate the slope of a tangent?

C4 How can the product rule be used to differentiate $(2x - 5)^3$?

A Practise

1. Differentiate each function using the two different methods presented below. Compare your answers in each case.

i) Expand and simplify each binomial product and then differentiate.

ii) Apply the product rule and then simplify.

a) $f(x) = (x + 4)(2x - 1)$

b) $h(x) = (5x - 3)(1 - 2x)$

c) $h(x) = (-x + 1)(3x + 8)$

d) $g(x) = (2x - 1)(4 - 3x)$

2. Use the product rule to differentiate each of the following functions.

a) $f(x) = (5x + 2)(8x - 6)$

b) $h(t) = (-t + 4)(2t + 1)$

c) $p(x) = (-2x + 3)(x - 9)$

d) $g(x) = (x^2 + 2)(4x - 5)$

e) $f(x) = (1 - x)(x^2 - 5)$

f) $h(t) = (t^2 + 3)(3t^2 - 7)$

3. Differentiate.

a) $M(u) = (1 - 4u^2)(u + 2)$

b) $g(x) = (-x + 3)(x - 10)$

c) $p(n) = (5n + 1)(-n^2 + 3)$

d) $A(r) = (1 + 2r)(2r^2 - 6)$

e) $b(k) = (-0.2k + 4)(2 - k)$

4. The derivative of the function $h(x) = f(x)g(x)$ is given in the form $h'(x) = f'(x)g(x) + f(x)g'(x)$. Determine $f(x)$ and $g(x)$ for each derivative.

a) $h'(x) = (10x)(21 - 3x) + (5x^2 + 7)(-3)$

b) $h'(x) = (-12x^2 + 8)(2x^2 - 4x) + (-4x^3 + 8x)(4x - 4)$

c) $h'(x) = (6x^2 - 1)(0.5x^2 + x) + (2x^3 - x)(x + 1)$

d) $h'(x) = (-3x^3 + 6)\left(7x - \frac{2}{3}x^2\right) + \left(-\frac{3}{4}x^4 + 6x\right)\left(7 - \frac{4}{3}x\right)$

B Connect and Apply

5. Determine $f'(-2)$ for each function.

a) $f(x) = (x^2 - 2x)(3x + 1)$

b) $f(x) = (1 - x^3)(-x^2 + 2)$

c) $f(x) = (3x - 1)(2x + 5)$

d) $f(x) = (-x^2 + x)(5x^2 - 1)$

e) $f(x) = (2x - x^2)(7x + 4)$

f) $f(x) = (-5x^3 + x)(-x + 2)$

6. Determine the equation of the tangent to each curve at the indicated value.

a) $f(x) = (x^2 - 3)(x^2 + 1)$, $x = -4$

b) $g(x) = (2x^2 - 1)(-x^2 + 3)$, $x = 2$

c) $h(x) = (x^4 + 4)(2x^2 - 6)$, $x = -1$

d) $p(x) = (-x^3 + 2)(4x^2 - 3)$, $x = 3$

7. Determine the point(s) on each curve that correspond to the given slope of the tangent.

a) $y = (-4x + 3)(x + 3)$, $m = 0$

b) $y = (5x + 7)(2x - 9)$, $m = \dfrac{2}{5}$

c) $y = (2x - 1)(-4 + x^2)$, $m = 3$

d) $y = (x^2 - 2)(2x + 1)$, $m = -2$

8. Differentiate.

a) $y = (5x^2 - x + 1)(x + 2)$

b) $y = (1 - 2x^3 + x^2)\left(\dfrac{1}{x^3} + 1\right)$

c) $y = -x^2(4x - 1)(x^3 + 2x + 3)$

d) $y = \left(2x^2 - \sqrt{x}\right)^2$

e) $y = (-3x^2 + x + 1)^2$

9. Recall the problem introduced at the start of this section: The manager of a miniature golf course is planning to raise the ticket price per game. At the current price of \$6.50, an average of 81 rounds is played each day. The manager's research suggests that for every \$0.50 increase in price, an average of 4 fewer games will be played each day. The revenue from sales is represented by the function $R(n) = (81 - 4n)(6.50 + 0.50n)$, where n represents the number of \$0.50 increases in the price. By finding the derivative, the manager can determine the price that will provide the maximum revenue.

a) Describe two methods that could be used to determine $R'(n)$. Apply your methods and then compare the answers. Are they the same?

b) Evaluate $R'(4)$. What information does this value give the manager?

c) Determine when $R'(n) = 0$. What information does this give the manager?

d) Sketch the graph of $R(n)$. Determine the maximum revenue from the graph. Compare this value to your answer in part c). What do you notice?

e) Describe how the derivative could be used to find the value in part d).

10. Some years ago, an orchard owner began planting ten saplings each year. The saplings have begun to mature, and the orchard is expanding at a rate of ten fruit-producing trees per year. There are currently 120 trees in the apple orchard, producing an average yield of 280 apples per tree. Also, because of improved soil conditions, the average annual yield has been increasing at a rate of 15 apples per tree.

a) Write an equation to represent the annual yield, Y, as a function of t years from now.

b) Determine $Y'(2)$ and interpret its meaning for this situation.

c) Evaluate $Y'(6)$. Explain what this value represents.

11. The owner of a local hair salon is planning to raise the price for a haircut and blow dry. The current rate is \$30 for this service, with the salon averaging 550 clients a month. A survey indicates that the salon will lose 5 clients for every incremental price increase of \$2.50.

    a) Write an equation that models the salon's monthly revenue, R, as a function of x, where x represents the number of \$2.50 increases in the price.

    b) Use the product rule to determine $R'(x)$.

    c) Evaluate $R'(3)$ and interpret its meaning for this situation.

    d) Solve $R'(x) = 0$.

    e) Explain how the owner can use the result of part d). Justify your answer graphically.

12. a) Determine the equation of the tangent to the graph of $f(x) = 2x^2(x^2 + 2x)(x - 1)$ at the point $(-1, 4)$.

    b) **Use Technology** Use a graphing calculator to sketch a graph of the function and the tangent.

13. a) Determine the points on the graph of $f(x) = (3x - 2x^2)^2$ where the tangent line is parallel to the x-axis.

    b) **Use Technology** Use a graphing calculator to sketch a graph of the function and the tangents.

14. The gas tank of a parked pickup truck develops a leak. The amount of gas, in litres, remaining in the tank after t hours is represented by the function
$V(t) = 90\left(1 - \dfrac{t}{18}\right)^2, 0 \le t \le 18.$

    a) How much gas was in the tank when the leak developed?

    b) How fast is the gas leaking from the tank after 12 h?

    c) How fast is the gas leaking from the tank when there are 40 L of gas in the tank?

15. The fish population in a lake can be modelled by the function $p(t) = 15(t^2 + 30)(t + 8)$, where t is time, in years, from now.

    a) What is the current fish population?

    b) Determine the rate of change of the fish population in 3 years.

    c) Determine the rate of change of the fish population when there are 5000 fish in the lake.

    d) When will the fish population double from its current level? What is the rate of change in the fish population at this time?

16. **Chapter Problem** The owners of Mooses, Gooses, and Juices are considering an increase in the price of their frozen fruit smoothies. At the current price of \$1.75 they sell on average 150 smoothies a day. Their research shows that every \$0.25 increase in the price of a smoothie will result in a decrease of 10 sales per day.

    a) Write an equation to represent the revenue, R, as a function of n, the number of \$0.25 price increases.

    b) Compare the rate of change of revenue when the price increases by \$0.25, \$0.75, \$1.00, \$1.25, and \$1.50.

    c) When is $R'(n) = 0$? Interpret the meaning of this value for this situation.

    d) If it costs \$0.75 to make one smoothie, what will be the rate of change in profit for each price increase indicated in part b)?

    e) What price will result in a maximum profit? Justify your answer. How can this be confirmed using derivatives?

    f) Compare your answers in part b) and part d). Give reasons for any similarities or differences.

C Extend and Challenge

17. a) Use the product rule to differentiate each function. Do not simplify your final answer.

 i) $y = (x^2 - 3x)^2$

 ii) $y = (2x^3 + x)^2$

 iii) $y = (-x^4 + 5x^2)^2$

 What do you notice about the two parts that make up the derivative?

 b) Make a conjecture about $\frac{d}{dx}[f(x)]^2$.

 c) Verify whether your conjecture is true by replacing $g(x)$ with $f(x)$ in the product rule and then comparing the result to your conjecture.

 d) Use your result in part c) to differentiate the functions in part a). Compare your derivatives to those found in part a). What do you notice?

18. a) Use the product rule to show that $(fgh)' = f'gh + fg'h + fgh'$.

 b) Apply the above result to differentiate $f(x) = (x^2 + 4)(3x^4 - 2)(5x + 1)$.

 c) Describe another method for finding the derivative in part b). Apply the method you have described.

 d) Verify that the derivatives in parts b) and c) are the same.

19. a) Make a conjecture about $\frac{d}{dx}[f(x)]^3$.

 b) Use the results of question 18 part a), replacing both $g(x)$ and $h(x)$ with $f(x)$, to verify whether your conjecture is true.

 c) Use the results of part b) to differentiate each function.

 i) $y = (4x^2 - x)^3$

 ii) $y = (x^3 + x)^3$

 iii) $y = (-2x^4 + x^2)^3$

20. Determine expressions for each derivative, given that $f(x)$ and $g(x)$ are differentiable functions.

 a) $h(x) = x^3 f(x)$

 b) $p(x) = g(x)(x^4 - 3x^2)$

 c) $q(x) = (-3x^4 - 8x^2 + 5x + 6)f(x)$

 d) $r(x) = f(x)(2x^3 + 5x^2)^2$

21. a) Use the results of questions 17 and 19 to make a conjecture about $\frac{d}{dx}[f(x)]^n$.

 b) Test your conjecture by differentiating $y = (2x^3 + x^2)^n$, for $n = 4$, 5, and 6.

22. **Math Contest** Let f be a function such that $\frac{d}{dx}[(x^2 + 1)f(x)] = 2xf(x) + 3x^4 + 3x^2$. Which of the following could $f(x)$ be?

 A $6x$

 B $3x^2$

 C x^3

 D $12x^3$

 E $3x^4$

23. **Math Contest** Let p be a polynomial function with $p(a) = 0 = p'(a)$ for some real a. Which of the following *must* be true?

 A $p(x)$ is divisible by $x + a$

 B $p(x)$ is divisible by $x^2 + a^2$

 C $p(x)$ is divisible by $x^2 - a^2$

 D $p(x)$ is divisible by $x^2 + 2ax + a^2$

 E $p(x)$ is divisible by $x^2 - 2ax + a^2$