2.6 Rate of Change Problems

Earlier in this chapter, the connection between calculus and physics was examined in relation to velocity and acceleration. There are many other applications of calculus to physics, such as the analysis of the change in the density of materials and the rate of flow of an electrical current. But calculus is applied far beyond the realm of physics. In biology, derivatives are used to determine growth rates of populations or the rate of concentration of a drug in the bloodstream. In chemistry, derivatives are used to analyse the rate of reaction of chemicals. In the world of business and economics, rates of change pertaining to profit, revenue, cost, price, and demand are measured in terms of the number of items sold or produced. This section will focus on applying derivatives to solve problems involving rates of change in the social and physical sciences.

Rates of Change in Business and Economics

The primary goal of most businesses is to generate profits. To achieve this goal, many different aspects of the business need to be considered and measured. For instance, a business has to determine the price for its products that will maximize profits. If the price of a product or service is set too high, fewer consumers are willing to buy it. This often results in lower revenues and lower profits. If the price is set too low, the cost of producing large quantities of the item may also result in reduced profits. A delicate balance often exists between the cost, revenue, profit, and demand functions.

Functions Pertaining to Business

- The **demand function**, or **price function**, is $p(x)$, where p is the number of units of a product or service that can be sold at a particular price, x.
- The **revenue function**, is $R(x) = xp(x)$, where x is the number of units of a product or service sold at a price per unit of $p(x)$.
- The **cost function**, $C(x)$, is the total cost of producing x units of a product or service.
- The **profit function**, $P(x)$, is the profit from the sale of x units of a product or service. The profit function is the difference between the revenue function and the cost function: $P(x) = R(x) - C(x)$.

Derivatives of Business Functions

Economists use the word *marginal* to indicate the derivative of a business function.

- $C'(x)$ or $\frac{dC}{dx}$ is the **marginal cost function** and refers to the instantaneous rate of change of total cost with respect to the number of items produced.
- $R'(x)$ or $\frac{dR}{dx}$ is the **marginal revenue function** and refers to the instantaneous rate of change of total revenue with respect to the number of items sold.
- $P'(x)$ or $\frac{dP}{dx}$ is the **marginal profit function** and refers to the instantaneous rate of change of total profit with respect to the number of items sold.

Example 1 Apply Mathematical Modelling to Determine the Demand Function

A company sells 1500 movie DVDs per month at \$10 each. Market research has shown that sales will decrease by 125 DVDs per month for each \$0.25 increase in price.

a) Determine the demand, or price, function.

b) Determine the marginal revenue when sales are 1000 DVDs per month.

c) The cost of producing x DVDs is $C(x) = -0.004x^2 + 9.2x + 5000$. Determine the marginal cost when production is 1000 DVDs per month.

d) Determine the actual cost of producing the 1001st DVD.

e) Determine the profit and marginal profit from the monthly sales of 1000 DVDs.

Solution

a) Let p be the price of one movie DVD.
Let x be the number of DVDs sold per month.
Let n be the number of \$0.25 increases in price.

Two equations can be derived from this information:

① $x = 1500 - 125n$ and

② $p = 10 + 0.25n$

You want to express p in terms of x.

From ①, you have $n = \dfrac{1500 - x}{125}$.

Substitute this expression into ②.

$$\begin{aligned} p &= 10 + 0.25\left(\frac{1500 - x}{125}\right) \\ &= 10 + 0.002(1500 - x) \\ &= 10 + 3 - 0.002x \\ &= 13 - 0.002x \end{aligned}$$

The demand function is $p(x) = 13 - 0.002x$. This function gives the price for one DVD when x of them are being sold.

b) The revenue function is

$$\begin{aligned} R(x) &= xp(x) \\ &= x(13 - 0.002x) \\ &= 13x - 0.002x^2 \end{aligned}$$

Take the derivative to determine the marginal revenue function for this situation.

$$\begin{aligned} R'(x) &= 13 - 0.004x \\ R'(1000) &= 13 - 0.004(1000) \\ &= 9 \end{aligned}$$

When sales are at 1000 DVDs per month, revenue is increasing at the rate of \$9.00 per additional DVD.

c)

$$\begin{aligned} C(x) &= -0.004x^2 + 9.2x + 5000 \\ C'(x) &= -0.008x + 9.2 \\ C'(1000) &= -0.008(1000) + 9.2 \\ &= 1.20 \end{aligned}$$

When production is at 1000 DVDs per month, the marginal cost is \$1.20.

d) The cost of producing the 1001st DVD is

$$\begin{aligned} C(1001) - C(1000) &= [-0.004(1001)^2 + 9.2(1001) + 5000] \\ &\quad - [-0.004(1000)^2 + 9.2(1000) + 5000] \\ &= 10\,201.196 - 10\,200.00 \\ &= 1.196 \end{aligned}$$

The actual cost of producing the 1001st DVD is \$1.196. Notice the similarity between the marginal cost of the 1000th DVD and the actual cost of producing the 1001st DVD. For large values of x, the marginal cost when producing x items is approximately equal to the cost of producing one more item, the $(x + 1)$th item.

e) The profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 13x - 0.002x^2 - (-0.004x^2 + 9.2x + 5000) \\ &= 0.002x^2 + 3.8x - 5000 \\ P(1000) &= 0.002(1000)^2 + 3.8(1000) - 5000 \\ &= 800 \\ P'(x) &= 0.004x + 3.8 \\ P'(1000) &= 0.004(1000) + 3.8 \\ &= 7.80 \end{aligned}$$

When 1000 DVDs per month are sold, the total profit is \$800, and the marginal profit is \$7.80 per DVD.

Example 2 Apply Mathematical Modelling to Determine the Revenue Function

An ice cream shop sells 150 cookies 'n' cream ice cream cakes per month at a price of \$40 each. A customer survey indicates that for each \$1 decrease in price, sales will increase by 5 cakes.

a) Determine a revenue function based on the number of price decreases.

b) Determine the marginal revenue for the revenue function developed in part a).

c) When is this marginal revenue function equal to zero? What is the total revenue at this time? How can the owners use this information?

Solution

a) It is not always necessary to involve the demand function as in Example 1.
Revenue = price × sales.
Let n represent the number of \$1 decreases in the cake price.
The price is $p = 40 - n$.
For each decrease in price, cake sales increase by 5, so sales $= 150 + 5n$.
The revenue function is $R(n) = (40 - n)(150 + 5n)$.

b)
$$\begin{aligned} R'(n) &= (-1)(150 + 5n) + (40 - n)(5) \\ &= -150 - 5n + 200 - 5n \\ &= -10n + 50 \end{aligned}$$

c) Solve $R'(n) = 0$.

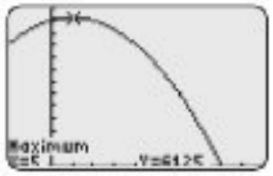
$$\begin{aligned} -10n + 50 &= 0 \\ -10n &= -50 \\ n &= 5 \end{aligned}$$

The marginal revenue is zero when there are five \$1 decreases in the price of the cakes. Cakes then sell for \$35.

$$\begin{aligned} R(5) &= (40 - 5)(150 + 5(5)) \\ &= (35)(175) \\ &= 6125 \end{aligned}$$

When the price is \$35, total revenue is \$6125.

As shown below, you can use a graphing calculator to verify that the maximum point on the graph of $R(n)$ occurs at $x = 5$.



Window variables:
$x \in [-10, 50]$, Xscl $= 5$,
$y \in [0, 6500]$, Yscl $= 500$

The owners of the ice cream shop should realize that decreasing the price further will lead to increased sales, but decreased total revenue.

Example 3 Apply Derivatives to Kinetic Energy

Kinetic energy, K, is the energy due to motion. When an object is moving, its kinetic energy is determined by the formula $K(v) = 0.5mv^2$, where K is in joules, m is the mass of the object, in kilograms, and v is the velocity of the object, in metres per second.

Suppose a ball with a mass of 350 g is thrown vertically upward with an initial velocity of 40 m/s. Its velocity function is $v(t) = 40 - 9.8t$, where t is time, in seconds.

a) Express the kinetic energy of the ball as a function of time.

b) Determine the rate of change of the kinetic energy of the ball at 3 s.

Solution

a) Substitute $m = 0.350$ kg and $v(t) = 40 - 9.8t$ into $K(v) = 0.5mv^2$.

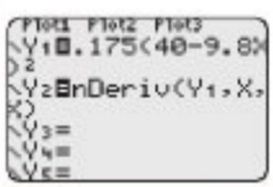
$$K[v(t)] = 0.5(0.350)(40 - 9.8t)^2$$
$$K(t) = 0.175(40 - 9.8t)^2$$

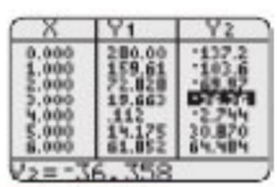
b) Differentiate.

$$\begin{aligned} K'(t) &= 0.175(2)(40 - 9.8t)(-9.8) \\ &= -3.43(40 - 9.8t) \\ K'(3) &= -3.43(40 - 9.8(3)) \\ &= -36.358 \end{aligned}$$

At 3 s, the rate of change of the kinetic energy of the ball is -36.358 J/s. The negative value indicates that the kinetic energy is decreasing.

Confirm this value using a graphing calculator as shown below.





Example 4 Apply Derivatives to Electrical Currents

In a certain electrical circuit, the resistance, R, in ohms, is represented by the function $R = \frac{150}{I}$, where I is the current, in amperes (A). Determine the rate of change of the resistance with respect to the current when the current is 10 A.

Solution

$$R = 150I^{-1} \quad \text{Express } R = \frac{150}{I} \text{ as a power with a negative exponent.}$$

$$\frac{dR}{dI} = (-1)150I^{-2}$$

$$\frac{dR}{dI} = -\frac{150}{I^2}$$

$$\left.\frac{dR}{dI}\right|_{I=10} = -\frac{150}{10^2}$$

$$= -1.5$$

When the current is 10 A, the rate of decrease of the resistance is 1.5 Ω/A.

CONNECTIONS

Ohm's law is $R = \frac{V}{I}$. It is named after its discoverer, Georg Ohm, who published it in 1827.

Derivatives and Linear Density

Derivatives can be used in the analysis of different types of densities. For instance, population density refers to the number of people per unit area. Colour density, used in the study of radiographs, refers to the depth of colour per unit area. Linear density refers to the mass of an object per unit length: linear density $= \frac{\text{mass}}{\text{length}}$.

Consider a linear object, such as a rod or wire. If it is made of the exact same material along its entire length, it is said to be made out of homogenous material (*homogeneous* means the same or similar). In cases like this, the linear density of the object is constant at every point. This would not be true of an object made of nonhomogenous materials, in which case the linear density would vary along the object's length.

Suppose the function $f(x)$ gives the mass, in kilograms, of the first x metres of the object. For the part of the object that lies between $x = x_1$ and $x = x_2$, the average linear density (or mass per unit length) is defined as $\frac{f(x_1) - f(x_2)}{x_1 - x_2}$. The corresponding derivative function $\rho(x) = f'(x)$ is the linear density, the rate of change of density at a particular length x.

CONNECTIONS

ρ is the Greek letter rho.

Example 5 Represent Linear Density as a Rate of Change

The mass, in kilograms, of the first x metres of a wire is represented by the function $f(x) = \sqrt{3x + 1}$.

a) Determine the average linear density of the part of the wire from $x = 5$ to $x = 8$.

b) Determine the linear density at $x = 5$ and at $x = 8$. Compare the densities at the two points. What do these values confirm about the wire?

Solution

a)

$$
\begin{aligned}
f(x) &= \sqrt{3x+1} \\
\text{average linear density} &= \frac{f(x_1) - f(x_2)}{x_1 - x_2} \\
&= \frac{f(8) - f(5)}{8 - 5} \\
&= \frac{\sqrt{3(8)+1} - \sqrt{3(5)+1}}{8-5} \\
&= 0.33
\end{aligned}
$$

The average linear density for this part of the wire is approximately 0.33 kg/m.

b)

$$
\begin{aligned}
f(x) &= \sqrt{3x+1} \\
&= (3x+1)^{\frac{1}{2}} \\
\rho(x) &= f'(x) \\
&= \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3) \\
&= \frac{3}{2\sqrt{3x+1}} \\
\rho(5) &= \frac{3}{2\sqrt{3(5)+1}} \\
&= \frac{3}{2(4)} \\
&= 0.375 \\
\rho(8) &= \frac{3}{2\sqrt{3(8)+1}} \\
&= 0.3
\end{aligned}
$$

The linear density at $x = 5$ is 0.375 kg/m, and at $x = 8$ it is 0.3 kg/m. Since the two density values are different, they confirm that the material of which the wire is composed is nonhomogenous.

KEY CONCEPTS

- The cost function, $C(x)$, is the total cost of producing x units of a product or service.
- The revenue function, $R(x)$, is the total revenue (income) from the sale of x units of a product or service. The revenue function is the product of the demand function, $p(x)$, and the number of items sold: $R(x) = xp(x)$.
- The profit function, $P(x)$, is the total profit from the sales of x units of a product or service. The profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.
- The demand, or price, function, $p(x)$, is the price at which x units of a product or service can be sold.
- $C'(x)$ is the marginal cost function.
- $R'(x)$ is the marginal revenue function.
- $P'(x)$ is the marginal profit function.

Communicate Your Understanding

C1 What does the word *marginal* refer to in economics and business problems?

C2 The demand function is also referred to as the price function. Explain why this is appropriate.

C3 What is the difference between negative marginal revenue and positive marginal revenue?

C4 Why is it true to say that for certain items the actual cost of producing the 1001st item is the same as the marginal cost of producing 1000 items? Explain your answer.

A Practice

1. The demand function for a DVD player is $p(x) = \frac{575}{\sqrt{x}} - 3$, where x is the number of DVD players sold and p is the price, in dollars. Determine the following:

a) the revenue function

b) the marginal revenue function

c) the marginal revenue when 200 DVD players are sold

2. Refer to question 1. If the cost, in dollars, of producing x DVD players is $C(x) = 2000 + 150x - 0.002x^2$, determine the following:

a) the profit function

b) the marginal profit function

c) the marginal profit for the sale of 500 DVD players

3. The cost, in dollars, of making x large combo pizzas at a local pizzeria is modelled by the function $C(x) = -0.001x^3 + 0.025x^2 + 4x$, and the price per large combo pizza is $17.50. Determine the following:

a) the revenue function

b) the marginal revenue function

c) the profit function

d) the marginal profit function

e) the marginal revenue and marginal profit for the sale of 300 large combo pizzas

B Connect and Apply

4. The mass, in grams, of the first x metres of a wire is represented by the function $f(x) = \sqrt{2x - 1}$.

a) Determine the average linear density of the part of the wire from $x = 1$ to $x = 8$.

b) Determine the linear density at $x = 5$ and at $x = 8$, and compare the densities at the two points. What do these values confirm about the wire?

5. A paint store sells 270 pails of paint per month at a price of $32 each. A customer survey indicates that for each $1.20 decrease in price, sales will increase by 6 pails of paint.

a) Determine the demand, or price, function.

b) Determine the revenue function.

c) Determine the marginal revenue function.

d) Solve $R'(x) = 0$. Interpret the meaning of this value for this situation.

e) What is the price that corresponds to the value found in part d)? How can the paint store use this information?

6. The mass, in kilograms, of the first x metres of a metal rod is represented by the function $f(x) = (x - 0.5)^3 + 5x$.

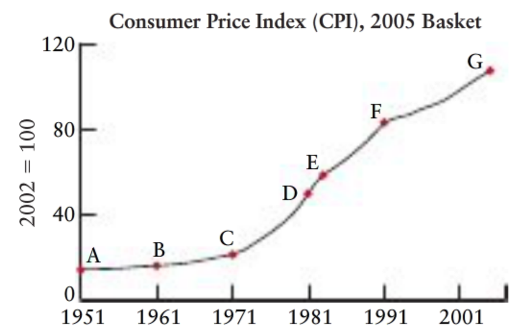
a) Determine the average linear density of the part of the rod from $x = 1$ to $x = 3$.

b) Determine the linear density at $x = 2$.

CONNECTIONS

Find out more about the consumer price index online. The Statistics Canada web site might be a good place to begin your search.

7. The following graph represents Canada's consumer price index (CPI) between 1951 and 2007. The CPI is an index number measuring the average price of consumer goods and services purchased by households. The percent change in the CPI is a measure of inflation.



Source: Statistics Canada. Table 326-0021 - Consumer price index (CPI), 2005 basket, annual (2002 = 100 unless otherwise noted), CANSIM (database).

a) Consider the interval from A to G.

i) Is the CPI increasing or decreasing over this interval? Justify your answer.

ii) Is the rate of growth positive or negative during this period? Explain.

b) Place each interval in order from the lowest rate of growth to the highest rate of growth. Explain your reasoning. State the years for each interval.

i) A to B ii) B to C iii) C to D

iv) D to E v) E to F vi) F to G

c) Compare the rate of inflation from 1951 to 1975 with the rate of inflation after 1975. What conclusions can be made? Explain.

d) Has the rate of inflation been increasing or decreasing since 1981? Justify your answer.

8. A yogurt company estimates that the revenue from selling x containers of yogurt is $4.5x$. Its cost for producing this number of containers of yogurt is $C(x) = 0.0001x^2 + 2x + 3200$.

a) Determine the marginal cost of producing 4000 containers of yogurt.

b) Determine the marginal profit from selling 4000 containers of yogurt.

c) What is the selling price of a container of yogurt? Explain.

9. The total cost, C, in dollars, of operating a factory that produces kitchen utensils is $C(x) = 0.5x^2 + 40x + 8000$, where x is the number of items produced, in thousands.

a) Determine the marginal cost of producing 5000 items and compare this with the actual cost of producing the 5001st item.

b) The average cost is found by dividing the total cost by the number of items produced. Determine the average cost of producing 5000 items. Compare this value to those found in part a). What do you notice?

c) Determine the rate of change of the average cost of producing 5000 items. Interpret the meaning of this value.

10. A demographer develops the function $P(x) = 12\ 500 + 320x - 0.25x^3$ to represent the population of the town of Calcville x years from today.

a) Determine the present population of the town.

b) Predict the rate of change of the population in 3 years and in 8 years.

c) When will the population reach 16 294?

d) When will the rate of growth of the population be 245 people per year?

e) Is the rate of change of the population increasing or decreasing? Explain.

11. The cost, in dollars, of producing x hot tubs is represented by the function $C(x) = 3450x - 1.02x^2$, $0 \leq x \leq 1500$.

a) Determine the marginal cost at a production level of 750 hot tubs. Explain what this means to the manufacturer.

b) Find the cost of producing the 751st hot tub.

c) Compare and comment on the values you found in parts a) and b).

d) Each hot tub is sold for \$9200. Write an expression that represents the total revenue from the sale of x hot tubs.

e) Determine the rate of change of profit for the sale of 750 hot tubs.

12. A certain electrical current, I, in amperes, can be modelled by the function $I = \dfrac{120}{R}$, where R is the resistance, in ohms. Determine the rate of change of the current with respect to the resistance when the resistance is 18 Ω.

13. An iron bar with an initial temperature of 10°C is heated such that its temperature increases at a rate of 4°C/min. The temperature, C, in degrees Celsius, at any time, t, in minutes, after heat is applied is given by the function $C = 10 + 4t$. The equation $F = 1.8C + 32$ is used to convert from C degrees Celsius to F degrees Fahrenheit. Determine the rate of change of the temperature of the bar with respect to time, in degrees Fahrenheit, after 4 min.

14. The size of the pupil of a certain animal's eye, in millimetres, is given by the function $f(x) = \dfrac{155x^{-0.5} + 85}{3x^{-0.5} + 18}$, where x is the intensity of light the pupil is exposed to. Is the rate of change of the size of the animal's pupil positive or negative? Interpret this result in terms of the response of the pupil to light.

15. A company's revenue for selling x items of a commodity, in thousands of dollars, is represented by the function $R(x) = \dfrac{15x - x^2}{x^2 + 15}$.

a) Determine the rate of change of revenue for the sale of 1000 items and of 5000 items.

b) Compare the values found in part a). Explain their meaning.

c) Determine the number of items that must be sold to obtain a \$0 rate of change in revenue.

d) Determine the revenue for the value found in part c). Explain the significance of this value.

16. When a person coughs, the airflow to the lungs is increased because the cough dislodges particles that may be blocking the windpipe, thereby increasing the radius of the windpipe. Suppose the radius of a windpipe, when there is no cough, is 2.5 cm. The velocity of air moving through the windpipe at radius r can be modelled by the function $V(r) = cr^2(2.5 - r)$ for some constant, c.

a) Determine the rate of change of the velocity of air through the windpipe with respect to r when $r = 2.75$ cm.

b) Determine the value of r that results in $V'(r) = 0$. Interpret the meaning of this value for this situation.

17. A coffee shop sells 500 mocha lattes a month at \$4.75 each. The results of a customer survey indicate that sales of mocha lattes would increase by 125 per month for each \$0.25 decrease in price.

a) Determine the demand, or price, function.

b) Determine the revenue and marginal revenue from the monthly sales of 350 mocha lattes.

c) The cost of producing x mocha lattes is $C(x) = -0.0005x^2 + 3.5x + 400$. Determine the marginal cost of producing the 350 mocha lattes.

d) Determine the actual cost of producing the 351st mocha latte.

e) Determine the profit and marginal profit from the monthly sales of 350 mocha lattes.

f) Determine the average revenue and average profit for the sale of 360 mocha lattes. Compare these values with the results of parts b) and e). Explain any similarities, or give reasons for differences.

18. The mass, in grams, of a compound being formed during a chemical reaction is modelled by the function $M(t) = \dfrac{6.3t}{t + 2.2}$, where t is the time after the start of the reaction, in seconds.

a) Determine the rate of change of the mass after 6 s.

b) Is the rate of change of the mass ever negative? Explain.

C Extend and Challenge

19. The wholesale demand function of a personal digital assistant (PDA) is $p(x) = \dfrac{650}{\sqrt{x}} - 4.5$, where x is the number of PDAs sold, and p is the wholesale price, in dollars.

a) Determine the revenue function.

b) Determine the marginal revenue for the sale of 500 PDAs.

c) If it costs \$125 to produce each PDA, determine the profit function.

d) Determine the marginal profit for the sale of 500 PDAs.

20. A patient's reaction to an antibacterial drug is represented by the function $r = \dfrac{m^2}{a}\left(\dfrac{1}{b} - \dfrac{m}{c}\right)$, where r is the time it takes for the body to react, in minutes, m is the amount of drug absorbed by the blood, in millilitres, and a, b, and c are positive constants. Determine $\dfrac{dr}{dm}$, the sensitivity of the patient to the drug, when 15 mL of the drug is administered.

21. Many sports involve hitting a ball with a striking object, such as a racquet, club, or bat. The velocity of the ball after being hit is represented by the function $u(W) = \dfrac{WV(1+c) + v(cW - w)}{W + w}$, where w is the weight of the ball (in grams), v is the velocity of the ball before it is hit (in metres per second), W is the weight of the striking object (in grams), $-V$ is the velocity of the striking object (in metres per second) before the collision (the negative value indicates that the striking object is moving in the opposite direction), and c is the coefficient of restitution, or bounciness, of the ball.

a) Show that $\dfrac{du}{dW} = \dfrac{V(1+c)w + cvw + vw}{(W+w)^2}$.

b) A baseball with mass 0.15 kg, coefficient of restitution of 0.575, and speed of 40 m/s is struck with a bat of mass m and speed 35 m/s (in the opposite direction to the ball's motion).

i) Determine the velocity of the ball after being hit, in terms of m.

ii) Determine the rate of change of the velocity of the ball when $m = 1.05$ kg.

22. **Math Contest** A water tank that holds V_0 litres of water drains in T minutes. The volume of water remaining in the tank after t minutes is given by the function $V = V_0\left(1 - \dfrac{t}{T}\right)^2$. The rate at which water is draining from the tank is the slowest when t equals

A 0 **B** $\dfrac{T}{2}$ **C** $\dfrac{(\sqrt{2}-1)T}{\sqrt{2}}$ **D** T **E** ∞

23. **Math Contest** In a certain chemical reaction, $X + Y \rightarrow Z$, the concentration of the product Z at time t is given by the function $z = \dfrac{c^2kt}{ckt + 1}$, where c and k are positive constants. The rate of reaction $\dfrac{dz}{dt}$ can be written as

A $k(c - z)$

B $\dfrac{k}{c - z}$

C $k(c - z)^2$

D $\dfrac{k}{(c - z)^2}$

E none of the above

CAREER CONNECTION

Prakesh took a 4-year Bachelor of Science degree at the University of Guelph, specializing in microbiology. Since graduating, Prakesh has worked as a public health microbiologist. He detects and identifies micro-organisms, such as bacteria, fungi, viruses, and parasites, that are associated with infectious and communicable diseases. Prakesh uses derivatives in his work to help determine the growth rate of a bacterial culture when variables, such as temperature or food source are changed. He can then help the culture to increase its rate of growth of beneficial bacteria, or decrease the rate of growth of harmful bacteria.