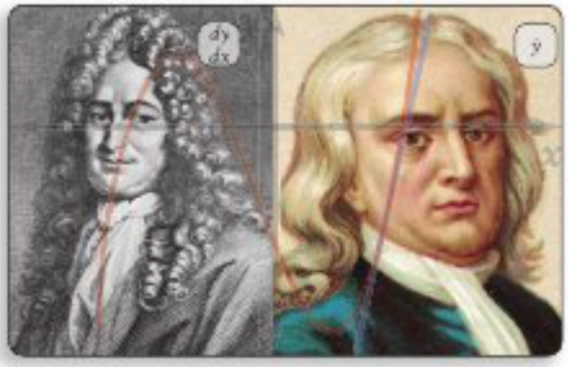
1.5 Introduction to Derivatives

Throughout this chapter, you have examined methods for calculating instantaneous rate of change. The concepts you have explored to this point have laid the foundation for you to develop a sophisticated operation called **differentiation** —one of the most fundamental and powerful operations of calculus. It is a concept that was developed over two hundred years ago by two men: Sir Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). The output of this operation is called the **derivative** . The derivative can be used to calculate the slope of the tangent to *any* point in the function's domain.



Investigate How can you create a derivative function on a graphing calculator?

Tools

- graphing calculator

CONNECTIONS

To see how to create a derivative function using a slider in *The Geometer's Sketchpad®*, go to *www.mcgrawhill.ca/links/calculus12*, and follow the links to Section 1.5.

1. Graph $y = x^2$ using a graphing calculator.
2. Use the Tangent operation to graph the tangent at each x-value in the table, recording the corresponding tangent slope from the equation that appears on the screen.

x	Slope (m) of the Tangent
−4	
−3	
−2	
−1	
0	
1	
2	
3	
4	

3. Press STAT to access the **EDIT** menu, and then select **1:Edit** to edit a list of values. Enter the values from the table into the lists **L1** and **L2**. Create a scatter plot. What do the y-values of this new graph represent with respect to the original graph?
4. **Reflect** What type of function does the scatter plot represent? What type of regression should you select from the STAT **CALC** menu for this data?
5. Perform the selected regression and record the equation.

6. **Reflect** The regression equation represents the derivative function. Compare the original equation and the derivative equation. What relationship do you notice?
7. Repeat steps 1 to 6 for the equations $y = x$ and $y = x^3$.
8. **Reflect** Based on your results in steps 1 to 7, what connection can be made between the graph of $y = f(x)$ and the derivative graph, $y = f'(x)$, when $f(x)$ is
   a) linear? b) quadratic? c) cubic?
9. Predict what the derivative of a constant function will be. Support your prediction with an example.
10. **Reflect** Refer to the tangent slopes recorded in the table for each of the original functions in steps 1 and 7. What is the connection between
    a) the sign of the slopes and the behaviour of the graph of the function for the corresponding x-values?
    b) the behaviour of the function for x-values where the slope is 0?

CONNECTIONS

$f'(x)$, read "f prime of x," is one of a few different notations for the derivative. This form was developed by the French mathematician Joseph Louis Lagrange (1736–1813). Another way of indicating the derivative is simply to write y'. You will see different notation for the derivative later in this section.

The derivative of a function can also be found using the **first principles definition of the derivative**. To understand this definition, recall the equation formula for the slope of a tangent at a specific point a:

$m_{\tan} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$. If you replace the variable a with the independent variable x, you arrive at the first principles definition of the derivative.

First Principles Definition of the Derivative

The derivative of a function $f(x)$ is a new function $f'(x)$ defined by

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, if the limit exists.

When this limit is simplified by letting $h \to 0$, the resulting expression is expressed in terms of x. You can use this expression to determine the derivative of the function at *any* x-value that is in the function's domain.

Example 1 Determine a Derivative Using the First Principles Definition

a) State the domain of the function $f(x) = x^2$.

b) Use the first principles definition to determine the derivative of $f(x) = x^2$. What is the derivative's domain?

c) What do you notice about the nature of the derivative? Describe the relationship between the function and its derivative.

Solution

a) The quadratic function $f(x) = x^2$ is defined for all real numbers x, so its domain is $x \in \mathbb{R}$.

b) To find the derivative, substitute $f(x+h) = (x+h)^2$ and $f(x) = x^2$ into the first principles definition, and then simplify.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}\\
&= \lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h}\\
&= \lim_{h\to 0}\frac{2xh+h^2}{h}\\
&= \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}} \qquad \text{Divide by } h \text{ since } h \neq 0.\\
&= \lim_{h\to 0}(2x+h)\\
&= 2x
\end{aligned}$$

The derivative of $f(x) = x^2$ is $f'(x) = 2x$. Its domain is $x \in \mathbb{R}$.

c) Notice that the derivative is also a function. The original function, $f(x) = x^2$, is quadratic. Its derivative, $f'(x) = 2x$, is linear. The derivative represents the slope of the tangent, or instantaneous rate of change on the curve. So, you can substitute any value, x, into the derivative to find the instantaneous rate of change at the corresponding point on the graph of the original function.

Example 2 Apply the First Principles Definition to Determine the Equation of a Tangent

a) Use first principles to differentiate $f(x) = x^3$. State the domain of the function and of its derivative.

b) Graph the original function and the derivative function.

c) Determine the following and interpret the results.

i) $f'(-2)$ **ii)** $f'(0)$ **iii)** $f'(1)$

d) Determine the equations of the tangent lines that correspond to the values you found in part c).

e) Use a graphing calculator to draw the function $f(x) = x^3$. Use the Tangent operation to confirm the equation line for one of the derivatives you calculated in part c).

Solution

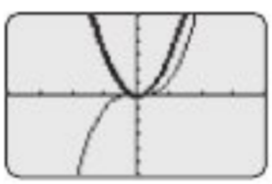
a) The cubic function $f(x) = x^3$ is defined for all real numbers x, so the domain is $x \in \mathbb{R}$.

Find the derivative by substituting $f(x + h) = (x + h)^3$ and $f(x) = x^3$ in the first principles definition.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{(x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3) - x^3}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} \\
&= \lim_{h \to 0} 3x^2 + 3xh + h^2 \\
&= 3x^2
\end{aligned}
$$

The derivative of $f(x) = x^3$ is $f'(x) = 3x^2$. The limit exists for any value of x, so the domain of $f'(x) = 3x^2$ is $x \in \mathbb{R}$.

b) On a graphing calculator, enter the equations **Y1** $= x^3$ and **Y2** $= 3x^2$. Select a thick line to graph the derivative. Press GRAPH.



Window variables:
$x \in [-4, 4]$, $y \in [-6, 6]$

c) Substitute into the derivative equation to calculate the slope values.

i) $f'(-2) = 3(-2)^2 = 12$

The slope of the tangent to $f(x) = x^3$ at $x = -2$ is equal to 12.

ii) $f'(0) = 3(0)^2 = 0$

The slope of the tangent to $f(x) = x^3$ at $x = 0$ is equal to 0.

iii) $f'(1) = 3(1)^2 = 3$

The slope of the tangent to $f(x) = x^3$ at $x = 1$ is equal to 3.

d) You know the slope for each specified x-value. Determine the tangent point by calculating the corresponding y-value.

i) When $x = -2$, $y = -8$. The tangent point is $(-2, -8)$.
Substitute into the intercept equation of a line, $y - y_1 = m(x - x_1)$.

Technology Tip

You can select different line styles for your graph by moving to the first column in the **Y=** screen and pressing ENTER. Pressing ENTER repeatedly cycles through the possible styles.

$$y - (-8) = 12(x - (-2))$$
$$y = 12x + 24 - 8$$
$$= 12x + 16$$

The equation of the tangent is $y = 12x + 16$.

ii) When $x = 0$, $y = 0$

Substitute $(x, y) = (0, 0)$ and $m = 0$.

The equation of the tangent is $y = 0$. This line is the x-axis.

iii) When $x = 1$, $y = 1$

Substitute $(x, y) = (1, 1)$ and $m = 3$.

$$y - 1 = 3(x - 1)$$
$$y = 3x - 2$$

The equation of the tangent is $y = x - 2$.

e) Verify the tangent equation for $x = -2$.

Graph the function **Y1** $= x^3$.

Access the Tangent operation and enter −2. Press ENTER.

Window variables:
$x \in [-5, 5]$, $y \in [-12, 10]$.

The equation of the tangent at $x = -2$ appears in the bottom left corner of the calculator screen.

Leibniz Notation expresses the derivative of the function $y = f(x)$ as $\frac{dy}{dx}$, read as "dee y by dee x." Leibniz's form can also be written $\frac{d}{dx}f(x)$.

The expression $\left.\frac{dy}{dx}\right|_{x=a}$ in Leibniz notation means "determine the value of the derivative when $x = $ a."

Both Leibniz notation and Lagrange notation can be used to express the derivative. At times, it is easier to denote the derivative using a simple form, such as $f'(x)$. But in many cases Leibniz notation is preferable because it clearly indicates the relationship that is being considered. To understand this, keep in mind that $\frac{dy}{dx}$ does not denote a fraction. It symbolizes the change in one variable, y, with respect to another variable, x. The variables used depend on the relationship being considered. For example, if you were considering a function that modelled the relationship between the volume of a gas, V, and temperature, t, your notation would be $\frac{dV}{dt}$, or volume with respect to temperature.

Example 3 Apply the Derivative to Solve a Rate of Change Problem

The height of a javelin tossed into the air is modelled by the function $H(t) = -4.9t^2 + 10t + 1$, where H is height, in metres, and t is time, in seconds.

a) Determine the rate of change of the height of the javelin at time t. Express the derivative using Leibniz notation.

b) Determine the rate of change of the height of the javelin after 3 s.

Solution

a) First find the derivative of the function using the first principles definition. To do this, substitute the original function into the first principles definition of the derivative and then simplify.

$$\begin{aligned}
\frac{dH}{dt} &= \lim_{h\to 0}\frac{H(t+h)-H(t)}{h}\\
&= \lim_{h\to 0}\frac{[-4.9(t+h)^2+10(t+h)+1]-(-4.9t^2+10t+1)}{h}\\
&= \lim_{h\to 0}\frac{[-4.9(t^2+2th+h^2)+10t+10h+1]-(-4.9t^2+10t+1)}{h}\\
&= \lim_{h\to 0}\frac{-4.9t^2-9.8th-4.9h^2+10t+10h+1+4.9t^2-10t-1}{h}\\
&= \lim_{h\to 0}\frac{-4.9h^2-9.8th+10h}{h}\\
&= \lim_{h\to 0}\frac{h(-4.9h-9.8t+10)}{h}\\
&= \lim_{h\to 0} -4.9h-9.8t+10\\
\frac{dH}{dt} &= -9.8t+10
\end{aligned}$$

b) Once you have determined the derivative function, you can substitute any value of t within the function's domain. For $t = 3$,

$$\begin{aligned}
\left.\frac{dH}{dt}\right|_{x=3} &= -9.8(3)+10\\
&= -19.4
\end{aligned}$$

The instantaneous rate of change of the height of the javelin at 3 s is -19.4 m/s.

CONNECTIONS

Part b) is an example of a case where it might have been simpler to use the notation $H'(3)$ to denote the derivative.

Example 4 Differentiate a Simple Rational Function

a) Differentiate the function $y = \frac{1}{x}$. Express the derivative using Leibniz notation.

b) Use a graphing calculator to graph the function and its derivative.

c) State the domain of the function and the domain of the derivative. How is the domain reflected in the graphs?

Solution

a) Using the first principles definition, substitute $f(x+h) = \frac{1}{x+h}$ and $f(x) = \frac{1}{x}$ into the first principles definition.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{x+h} - \frac{1}{x}\right) \times \frac{1}{h} \\
&= \lim_{h\to 0} \left[\left(\frac{1}{x+h} \times \frac{x}{x}\right) - \left(\frac{1}{x} \times \frac{x+h}{x+h}\right)\right] \times \frac{1}{h} \\
&= \lim_{h\to 0} \left[\frac{x}{(x+h)x} - \frac{(x+h)}{x(x+h)}\right] \times \frac{1}{h} \\
&= \lim_{h\to 0} \left[\frac{x - (x+h)}{(x+h)x}\right] \times \frac{1}{h} \\
&= \lim_{h\to 0} \frac{x - x - h}{(x+h)xh} \\
&= \lim_{h\to 0} \frac{-\cancel{h}}{(x+h)x\cancel{h}} \\
&= \lim_{h\to 0} \frac{-1}{(x+h)x} \\
&= -\frac{1}{x^2} \\
\frac{dy}{dx} &= -\frac{1}{x^2}
\end{aligned}$$

To divide, multiply by the reciprocal of the denominator.

Multiply by 1 to create a common denominator

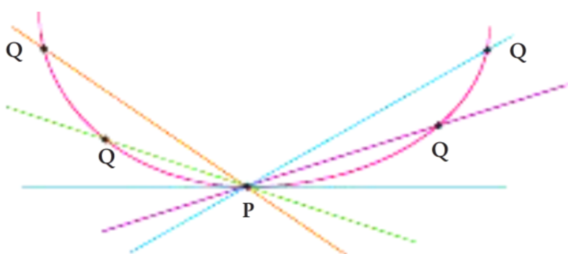
b) On a graphing calculator, enter the equations $Y1 = \frac{1}{x}$ and $Y2 = -\frac{1}{x^2}$.
Select a thick line to graph the derivative. Press GRAPH.

Window variables:
$x \in [-2, 2]$, $y \in [-5, 5]$

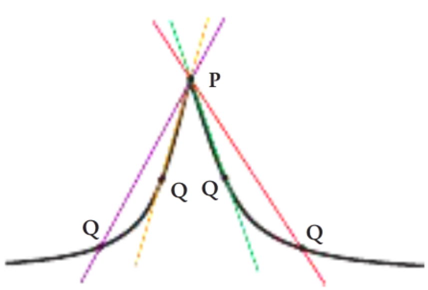
c) The function $y = \frac{1}{x}$ and the derivative function $\frac{dy}{dx} = -\frac{1}{x^2}$ are both undefined when the denominator is 0, so the domain of the function and its derivative is $\{x \in \mathbb{R} \mid x \neq 0\}$. Zero is not in the domain because both graphs have a vertical asymptote at $x = 0$.

A derivative may not exist at every point on a curve. For example, discontinuous functions are **non-differentiable** at the point(s) where they are discontinuous. The function in Example 4 is not differentiable at $x = 0$.

There are also continuous functions that may not be differentiable at some points. Consider the graph of the two continuous functions below. On Curve A, the slope of the secant approaches the slope of the tangent to P, as Q comes closer to P from both sides. This function is differentiable at P. However, this is not the case for Curve B. The limit of the slopes of the secants as Q approaches P from the left is different from the limit of the slopes of the secants as Q approaches P from the right. This function is non-differentiable at P even though the function is continuous.



Curve A



Curve B

CONNECTIONS

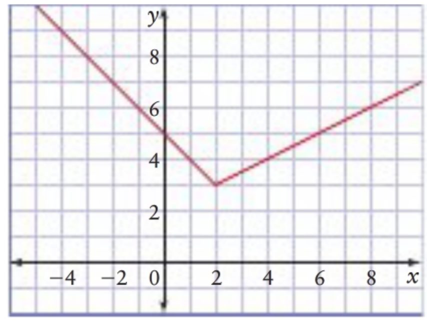
To see a second method for graphing this function and its derivative using *The Geometer's Sketchpad®*, go to *www.mcgrawhill.ca/links/calculus12* and follow the links to Section 1.5.

CONNECTIONS

To see an animated example of the derivative as the left-hand and right-hand limits approach the point at which the function is non-differentiable, go to *www.mcgrawhill.ca/links/calculus12* and follow the links to Section 1.5.

Example 5 Recognize and Verify Where a Function Is Non-Differentiable

A piecewise function f is defined by $y = -x + 5$ for $x \leq 2$ and $y = 0.5x + 2$ for $x > 2$. The graph of f consists of two line segments that form a vertex, or corner, at (2, 3).



a) From the graph, what is the slope as x approaches 2 from the left? What is the slope as x approaches 2 from the right? What does this tell you about the derivative at $x = 2$?

b) Use the first principles definition to prove that the derivative $f'(2)$ does not exist.

c) Graph the slope of the tangent for each x on the function. How does this graph support your results in parts a) and b)?

Solution

a) From the graph you can see that for $x < 2$, the slope of the graph is -1. The slope for $x > 2$ is 0.5. The slopes are not approaching the same value as you approach $x = 2$, so you can make the conjecture that the derivative does not exist at that point.

b) Using the first principles definition,

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}.$$

$f(2+h)$ has different expressions depending on whether $h < 0$ or $h > 0$, so you will need to compute the left-hand and right-hand limits.

For the left-hand limit, when $h < 0$,

$$f(2+h) = [-(2+h) + 5] = -h + 3.$$

$$\begin{aligned}\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} &= \lim_{h \to 0^-} \frac{-h + 3 - 3}{h} \\ &= \lim_{h \to 0^-} -1 \\ &= -1\end{aligned}$$

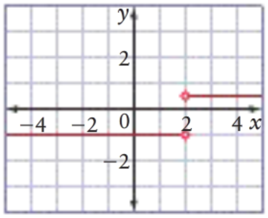
For the right-hand limit, when $h > 0$,

$$f(2+h) = [0.5(2+h) + 2] = 0.5h + 3.$$

$$\begin{aligned}\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \to 0^+} \frac{0.5h + 3 - 3}{h} \\ &= \lim_{h \to 0^+} 0.5 \\ &= 0.5\end{aligned}$$

Since the left-hand and right-hand limits are not equal, the derivative does not exist at $f(2)$.

c) Graphing the slope of the tangent at each point on f gives



When $x < 2$, the slope of the tangent to f is -1. When $x > 2$, the slope of the tangent to f is 0.5. So, the graph of the derivative consists of two horizontal lines. There is a break in the derivative graph at $x = 2$, where the slope of the original function f abruptly changes from -1 to 0.5. The function f is non-differentiable at this point. The open circles on the graph indicate this.

KEY CONCEPTS

- The derivative of $y = f(x)$ is a new function $y = f'(x)$, which represents the slope of the tangent, or instantaneous rate of change, at any point on the curve of $y = f(x)$.
- The derivative function is defined by the first principles definition for the derivative, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, if the limit exists.
- Different notations for the derivative of $y = f(x)$ are $f'(x)$, y', $\dfrac{dy}{dx}$, and $\dfrac{d}{dx} f(x)$.
- If the derivative does not exist at a point on the curve, the function is non-differentiable at that x-value. This can occur at points where the function is discontinuous or in cases where the function has an abrupt change, which is represented by a cusp or corner on a graph.

Communicate Your Understanding

C1 Discuss the differences and similarities between the formula $m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ and the first principles definition for the derivative.

C2 What does the derivative represent? What does it mean when we say that the derivative describes a new function? Support your answer with an example.

C3 What is the relationship between the domain of the original function and the domain of the corresponding derivative function? Provide an example to support your answer.

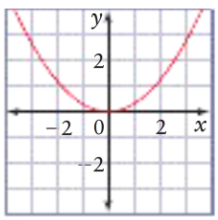
C4 Is the following statement true: "A function can be both differentiable and non-differentiable"? Justify your answer.

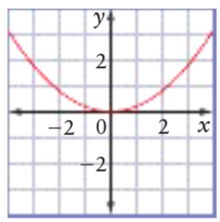
C5 Which of the following do *not* represent the derivative of y with respect to x for the function $y = f(x)$? Justify your answer.

a) $f'(x)$ **b)** y' **c)** $\dfrac{dx}{dy}$ **d)** $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ **e)** $\dfrac{dy}{dx}$

A Practise

1. Match graphs a, b, and c of $y = f(x)$ with their corresponding derivatives, graphs A, B, and C. Give reasons for your choice.

a)


A


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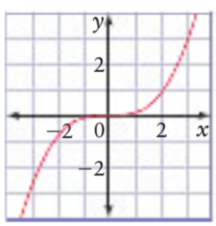
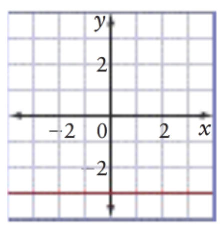
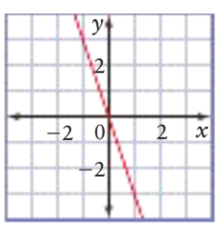
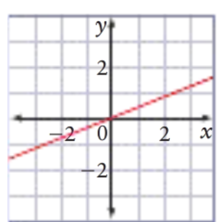


B


c)


C


2. **a)** State the derivative of $f(x) = x^3$.

b) Evaluate each derivative.

i) $f'(-6)$ **ii)** $f'(-0.5)$

iii) $f'\left(\dfrac{2}{3}\right)$ **iv)** $f'(2)$

c) Determine the equation of the tangent at each x-value indicated in part b).

3. Explain, using examples, what is meant by the statement "The derivative does not exist."

4. **a)** State the derivative of $f(x) = x$.

b) Evaluate each derivative.

i) $f'(-6)$ **ii)** $f'(-0.5)$

iii) $f'\left(\dfrac{2}{3}\right)$ **iv)** $f'(2)$

5. Each derivative represents the first principles definition for some function $f(x)$. State the function.

a) $f'(x) = \lim\limits_{h \to 0} \dfrac{3(x + h) - 3x}{h}$

b) $f'(x) = \lim\limits_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

c) $f'(x) = \lim\limits_{h \to 0} \dfrac{4(x + h)^3 - 4x^3}{h}$

d) $f'(x) = -6 \lim\limits_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

e) $f'(x) = \lim\limits_{h \to 0} \dfrac{\dfrac{5}{x + h} - \dfrac{5}{x}}{h}$

f) $f'(x) = \lim\limits_{h \to 0} \dfrac{\sqrt{x + h} - \sqrt{x}}{h}$

6. **a)** State the derivative of $f(x) = \dfrac{1}{x}$.

b) Evaluate each derivative.

i) $f'(-6)$ **ii)** $f'(-0.5)$

iii) $f'\left(\dfrac{2}{3}\right)$ **iv)** $f'(2)$

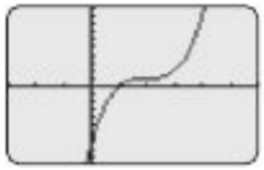
c) Determine the equation of the tangent at each x-value indicated in part b).

7. State the domain on which each function is differentiable. Explain your reasoning.

a)

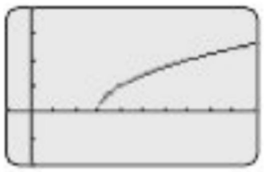
Window variables:
$x \in [-8, 6]$, $y \in [-2, 8]$

b)



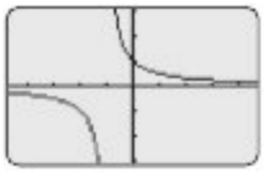
Window variables:
$x \in [-3, 6]$, $y \in [-10, 10]$

c)



Window variables:
$x \in [-1, 10]$, $y \in [-2, 4]$

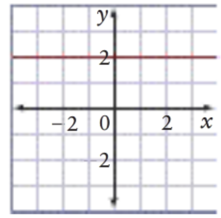
d)



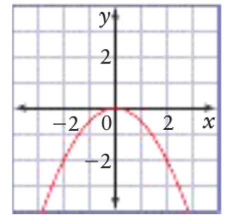
Window variables:
$x \in [-4.7, 4.7]$,
$y \in [-3.1, 3.1]$

8. Each graph represents the derivative of a function $y = f(x)$. State whether the original function is constant, linear, quadratic, or cubic. How do you know?

a)



b)



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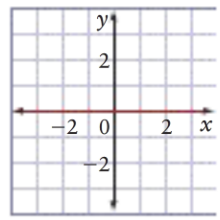

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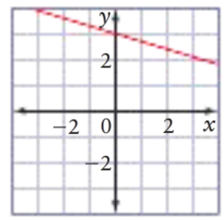


B Connect and Apply

9. a) Use the first principles definition to differentiate $y = x^2$.

b) State the domain of the original function and of the derivative function.

c) What is the relationship between the original function and its derivative?

10. a) Use the first principles definition to find $\dfrac{dy}{dx}$ for each function.

i) $y = -3x^2$ ii) $y = 4x^2$

b) Compare these derivatives with the derivative of $y = x^2$ in question 9. What pattern do you observe?

c) Use the pattern you observed in part b) to predict the derivative of each function.

i) $y = -2x^2$ ii) $y = 5x^2$

d) Verify your predictions using the first principles definition.

11. a) Use the first principles definition to determine the derivative of the constant function $y = -4$.

b) Will your result in part a) be true for any constant function? Explain.

c) Use the first principles definition to determine the derivative of any constant function $y = c$.

12. a) Expand $(x + h)^3$.

b) Use the first principles definition and your result from part a) to differentiate each function.

i) $y = 2x^3$ ii) $y = -x^3$

13. a) Compare the derivatives in question 12 part b) with the derivative of $y = x^3$ found in Example 2. What pattern do you observe?

b) Use the pattern to predict the derivative of each function.

i) $y = -4x^3$ ii) $y = \dfrac{1}{2}x^3$

c) Verify your predictions using the first principles definition.

14. Use the first principles definition to determine $\frac{dy}{dx}$ for each function.

a) $y = 8x$ **b)** $y = 3x^2 - 2x$

c) $y = 7 - x^2$ **d)** $y = x(4x + 5)$

e) $y = (2x - 1)^2$

15. a) Expand $(x + h)^4$.

b) Use the first principles definition and your result from part a) to differentiate each function.

i) $y = x^4$ **ii)** $y = 2x^4$ **iii)** $y = 3x^4$

c) What pattern do you observe in the derivatives?

d) Use the pattern you observed in part c) to predict the derivative of each function.

i) $y = -x^4$ **ii)** $y = \frac{1}{2}x^4$

e) Verify your predictions using the first principles definition.

16. The height of a soccer ball after it is kicked into the air is given by $H(t) = -4.9t^2 + 3.5t + 1$, where H is the height, in metres, and t is time, in seconds.

a) Determine the rate of change of the height of the soccer ball at time t.

b) Determine the rate of change of the height of the soccer ball at 0.5 s.

c) When does the ball momentarily stop? What is the height of the ball at this time?

17. a) Use the first principles definition to determine $\frac{dy}{dx}$ for $y = x^2 - 2x$.

b) Sketch the function in part a) and its derivative.

c) Determine the equation of the tangent to the function at $x = -3$.

d) Sketch the tangent on the graph of the function.

18. a) Use the first principles definition to differentiate each function.

i) $y = \frac{2}{x}$

ii) $y = -\frac{1}{x}$

iii) $y = \frac{3}{x}$

iv) $y = -\frac{4}{3x}$

b) What pattern do you observe in the derivatives?

c) State the domain of each of the original functions and of each of their derivative functions.

19. a) Use the pattern you observed in question 18 to predict the derivative of each function.

i) $y = \frac{5}{x}$ **ii)** $y = -\frac{3}{5x}$

b) Verify your predictions using the first principles definition.

20. A function is defined for $x \in \mathbb{R}$, but is not differentiable at $x = 2$.

a) Write a possible equation for this function, and draw a graph of it.

b) Sketch a graph of the derivative of the function to verify that it is not differentiable at $x = 2$.

c) Use the first principles definition to confirm your result in part b) algebraically.

21. Chapter Problem Alicia found some interesting information regarding trends in Canada's baby boom that resulted when returning soldiers started families after World War II. The following table displays the number of births per year from January 1950 to December 1967.

a) Enter the data into a graphing calculator to draw a scatter plot for the data. Let 1950 represent year 0.

Year	Number of Births
1950	372 009
1951	381 092
1952	403 559
1953	417 884
1954	436 198
1955	442 937
1956	450 739
1957	469 093
1958	470 118
1959	479 275
1960	478 551
1961	475 700
1962	469 693
1963	465 767
1964	452 915
1965	418 595
1966	387 710
1967	370 894

Source: Statistics Canada. "Table B1-14: Live births, crude birth rate, age-specific fertility rates, gross reproduction rate and percentage of births in hospital, Canada, 1921 to 1974." *Section B: Vital Statistics and Health* by R. D. Fraser, Queen's University. Statistics Canada Catalogue no. 11-516-XIE. Available at *www.statcan.ca/english/freepub/11-516-XIE/sectionb/sectionb.htm.*

b) Use the appropriate regression to determine the equation that best represents the data. Round the values to whole numbers.

c) Use the first principles definition to differentiate the equation.

d) Determine the instantaneous rate of change of births for each of the following years.

i) 1953 **ii)** 1957 **iii)** 1960

iv) 1963 **v)** 1966

e) Interpret the meaning of the values found in part d).

f) Use a graphing calculator to graph the original equation and the derivative equation you developed in parts c) and d).

g) **Reflect** Why would it be useful to know the equation of the tangent at any given year?

Achievement Check

22. a) Use the first principles definition to differentiate $y = 2x^3 - 3x^2$.

b) Sketch the original function and its derivative.

c) Determine the instantaneous rate of change of y when $x = -4, -1, 0$, and 3.

d) Interpret the meaning of the values you found in part c).

23. a) Predict $\dfrac{dy}{dx}$ for each polynomial.

i) $y = x^2 + 3x$

ii) $y = x - 2x^3$

iii) $y = 2x^4 - x + 5$

b) Verify your predictions using the first principles definition.

C Extend and Challenge

24. a) Use the first principles definition to differentiate each function.

i) $y = \dfrac{1}{x^2}$ **ii)** $y = \dfrac{1}{x^3}$ **iii)** $y = \dfrac{1}{x^4}$

b) State the domain of the original function and of the derivative function.

c) What pattern do you observe in the derivatives in part a)? Why does this pattern make sense?

25. Use Technology a) Use a graphing calculator to graph the function $y = 3|x - 2| + 1$. Where is it non-differentiable?

b) Use the first principles method to confirm your answer to part a).

Technology Tip

To enter an absolute value, press MATH and then ▶ to select **NUM.** Select **1:abs(** and press ENTER.

26. a) Use the results of the investigations and examples you have explored in this section to find the derivative of each function.

i) $y = 1$ **ii)** $y = x$ **iii)** $y = x^2$

iv) $y = x^3$ **v)** $y = x^4$

b) Describe the pattern for the derivatives in part a).

c) Predict the derivative of each function.

i) $y = x^5$ **ii)** $y = x^6$

d) Use the first principles definition to verify whether your predictions were correct.

e) Write a general rule to find the derivative of $y = x^n$, where n is a positive integer.

f) Apply the rule you created in part e) to some polynomial functions of your choice. Verify your results using the first principles definition.

27. a) A second form of the first principles definition for finding the derivative at $x = a$ is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. Use this form of the definition to determine the derivative of $y = x^2$. What do you need to do to the numerator to reduce the expression and determine the limit?

b) Use the definition in part a) to determine the derivative of each function.

i) $y = x^3$ **ii)** $y = x^4$ **iii)** $y = x^5$

c) What are the advantages of using this second form of the first principles definition? Explain.

28. Use the first principles definition to determine the derivative of each function.

a) $f(x) = \dfrac{x + 2}{x - 1}$ **b)** $f(x) = \dfrac{3x - 1}{x + 4}$

29. Differentiate each function. State the domain of the original function and of the derivative function.

a) $f(x) = \sqrt{x + 1}$ **b)** $f(x) = \sqrt{2x - 1}$

30. a) Use Technology Use a graphing calculator to graph the function $y = x^{\frac{2}{3}}$. Where is the function non-differentiable? Explain.

b) Use the first principles definition to confirm your answer to part a).

31. Math Contest If the terms 2^a, 3^b, 4^c form an arithmetic sequence, determine all possible ordered triples (a, b, c), where a, b, and c are positive integers.

32. Math Contest A triangle has side lengths of 1 cm, 2 cm, and $\sqrt{3}$ cm. A second triangle has the same area as the first and has side lengths x, x, and x. Determine the value of x.

33. Math Contest Determine the value of the expression $\dfrac{x^{12} + 3x^{11} + 2x^{10}}{x^{11} + 2x^{10}}$ when $x = 2008$.

CAREER CONNECTION

Tanica completed a 4-year bachelor of science in chemical engineering at Queen's University. She works for a company that designs and manufactures environmentally friendly cleaning products. In her job, Tanica and her team are involved in the development, safety testing, and environmental assessment of new cleaning products. During each of these phases, she monitors the rates of change of many types of chemical reactions. Tanica then analyses the results of these data in order to produce a final product.

Extension

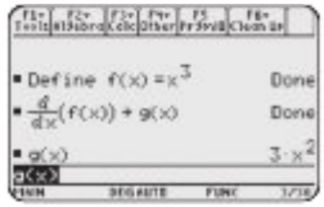
Use a Computer Algebra System to Determine Derivatives

Tools

- calculator with computer algebra system

1. A computer algebra system (CAS) can be used to determine derivatives. To see how this can be done, consider the function $y = x^3$.

- Turn on the CAS. If necessary, press the HOME key to display the home screen.
- Clear the CAS variables by pressing 2ND, then F1, to access the **F6 Clean Up** menu. Select **2:NewProb** and press ENTER. It is wise to follow this procedure every time you use the CAS.
- From the **F4** menu, select **1:Define**.
- Type $f(x) = x$^3, and press ENTER.
- From the **F3** menu, select **1:*d*(differentiate**.
- Type $f(x)$, x). Press STO. Type $g(x)$. Press ENTER.



The **CAS** will determine the derivative of $f(x)$ and store it in $g(x)$. You can see the result by typing $g(x)$ and pressing ENTER.

2. You can evaluate the function and its derivative at any x-value.

- Type $f(2)$, and press ENTER.
- Type $g(2)$, and press ENTER.

3. You can use the CAS to determine the equation of the tangent to $f(x)$ at $x = 2$.

- Use the values from step 2 to fill in $y = mx + b$.

This is a simple example, so the value of b can be determined by inspection. However, use the CAS to solve for b.

- From the **F2** menu, select **1:solve(**.
- Type $8 = 24 + b$, b). Press ENTER.

The equation of the tangent at $x = 2$ is $y = 12x - 16$.

Technology Tip

When you use the **SOLVE** function on a CAS, you must specify which variable you want the CAS to solve for. Since a CAS can manipulate algebraic symbols, it can also solve equations that consist of symbols. For example, to solve $y = mx + b$ for b using the CAS,

- From the **F2** menu, select **1: solve(**.
- Type $y = m \times x + b$, b). Press ENTER.

Note that the CAS has algebraically manipulated the equation to solve for b.

Problems

1. **a)** Use a CAS to determine the equation of the tangent to $y = x^4$ at $x = -1$.

b) Check your answer to part a) algebraically, using paper and pencil.

c) Graph the function and the tangent in part a) on the same graph.

2. **a)** Use a CAS to determine the equation of the tangent to $y = x^3 + x$ at $x = 1$.

b) Graph the function and the tangent in part a) on the same graph.