# 2.3 Velocity, Acceleration, and Second Derivatives

When Sir Isaac Newton was working on his "method of fluxions," he recognized how these concepts could be applied to the study of objects in motion. This section will explore the use of derivatives to analyse the motion of objects travelling in a straight line. Three related concepts will be considered in relation to this type of motion:

- **Displacement** – the distance and direction an object has moved from an origin over a period of time
- **Velocity** – the rate of change of displacement of an object with respect to time
- **Acceleration** – the rate of change of velocity with respect to time

## Investigate A What is the derivative of a derivative?

**Method 1:** ***Use Paper and Pencil***

**1. a)** Determine the derivative of $y = x^3$.

**b)** Determine the derivative of the derivative you found in part a).

**c)** **Reflect** How is the result in part b) related to the original function?

**d)** **Reflect** Why does it make sense to call your result in part b) a **second derivative**?

**2. a)** Sketch the graphs of $y$, $y'$, and $y''$.

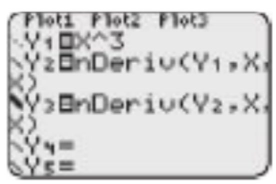
**b)** **Reflect** Describe how the graphs show the relationships among the three functions.

CONNECTIONS

There are several notations for the second derivative, including $y''$, $f''(x)$, $\dfrac{d^2y}{dx^2}$, $D^2f(x)$, and $D_x^2f(x)$.

**Method 2:** ***Use Technology***

**1. a)** Consider the function $y = x^3$. Use a graphing calculator to determine the derivative of the derivative of this function. Enter the information as shown, and then change the window variables to $x \in [-4, 4]$, $y \in [-20, 20]$, Yscl = 2. Before pressing GRAPH, draw a sketch to predict the shape of the graphs of Y1, Y2, and Y3.




**b)** Press GRAPH. Are the graphs you predicted accurate?

**c)** **Reflect** What is the relationship between the three graphs?

**2. a)** Determine the equations of the graphs of **Y2** and **Y3**.

**b)** **Reflect** Is it possible to differentiate a derivative? Why does it make sense to call **Y3** a second derivative?

**Tools**

- graphing calculator
- graphing software

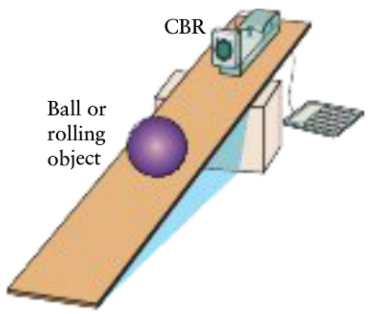
## Investigate B What is the relationship between displacement, velocity, and acceleration?

In this activity, you will use a motion sensor to gather displacement and time data for a ball rolling up and down a ramp. You will investigate the displacement-time, velocity-time, and acceleration-time graphs, and form connections with derivatives.

**Tools**

- graphing calculator
- Calculator-Based Ranger (CBR™)
- calculator-to-CBR™ cable
- ramp at least 3 m long
- large ball, such as a basketball

**1.** Set up the ramp as shown.




**2.** Prepare the CBR™ and calculator to collect data.

- Connect the CBR™ to the calculator using the calculator-to-CBR™ cable. Ensure that both ends of the cable are firmly in place.
- Press APPS. Select **2:CBL/CBR**. Press ENTER.
- To access the programs available, select **3:RANGER**.
- When the **RANGER** menu is displayed, press ENTER.
- From the **MAIN MENU** screen, select **1:SETUP/SAMPLE**.


- To change the **TIME (S)** setting, move the cursor down to **TIME (S)** and enter 5. Press ENTER.
- Move the cursor up to **START NOW** at the top of the screen, and press ENTER.

**3.** Collect the data.

- Align the CBR™ on the ramp, as shown in step 1.
- Press ENTER, and roll the ball.

**Technology Tip**

All settings, except **TIME (S)**, can be changed by using the cursor keys to position the ▶ beside the current option and pressing ENTER to cycle through the choices.

**Technology Tip**

The CBR™ is most accurate in an interval from about 1 m to 3 m. Practise rolling the ball such that it stays within this interval on the ramp.

4. The displacement-time graph will be displayed. If necessary, the graph can be redrawn to display the part that represents the motion in more detail.

   Press ENTER to display the **PLOT MENU** screen, and choose **4:PLOT TOOLS**. Select **1:SELECT DOMAIN**. Move the cursor to the point where the motion begins and press ENTER. Move the cursor to the point where the motion ends and press ENTER.

5. Use the **TRACE** feature to investigate displacement and time along the curve.

   **a)** Determine the time when the ball was closest to the CBR™. Describe what point this represents on the curve.

   **b)** For what time interval was the displacement increasing? For what time interval was it decreasing?

   **c)** **Reflect** Which direction was the ball rolling during the intervals in part b)?

6. **a)** Sketch a graph of your prediction for the corresponding velocity-time graph. Give reasons for your prediction.

   **b)** Return to the **PLOT MENU** screen. Select **2:VEL-TIME** to display the velocity-time graph. How does your sketch from part a) compare to the actual graph?

   **c)** **Reflect** Explain the significance of the time you found in step 5 part a), in regard to the velocity-time graph.

   **d)** **Reflect** How are the intervals you found in step 5 part b) reflected on the velocity-time graph? Explain.

   **e)** **Reflect** Think about rates of change and derivatives. What is the relationship between the distance-time graph and the velocity-time graph? Justify your answer.

7. **a)** Sketch a graph of your prediction for the corresponding acceleration-time graph. Give reasons for your prediction.

   **b)** Return to the **PLOT MENU** screen. Select **3:ACCEL-TIME** to display the acceleration-time graph. How does your sketch from part a) compare to the actual graph?

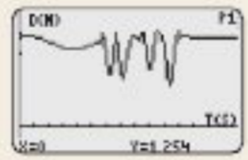
   **c)** For what time interval was the acceleration positive? For what interval was it negative? How do these intervals reflect the motion of the ball?

   **d)** **Reflect** Think about rates of change and derivatives. What is the relationship between the velocity-time graph and the acceleration-time graph? Justify your answer.

8. **Reflect** How can derivatives be used to determine the relationships between displacement-time, velocity-time, and acceleration-time graphs? Justify your reasoning.

**Technology Tip**

If you see any spikes, like those shown here, or other "artifacts" on the graph, it means that the CBR™ is intermittently losing the signal. Ensure that it is aimed properly, and that the ball you are using is big enough to reflect the sound waves. If necessary, repeat the data collection step until you have a graph free of artifacts.




**Technology Tip**

The data for $t$, $d$, $v$, and $a$ are stored in **L1**, **L2**, **L3**, and **L4** respectively.

## Example 1 Apply Derivative Rules to Determine the Value of a Second Derivative

Determine $f''(2)$ for the function $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$.

### Solution

$$f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$$

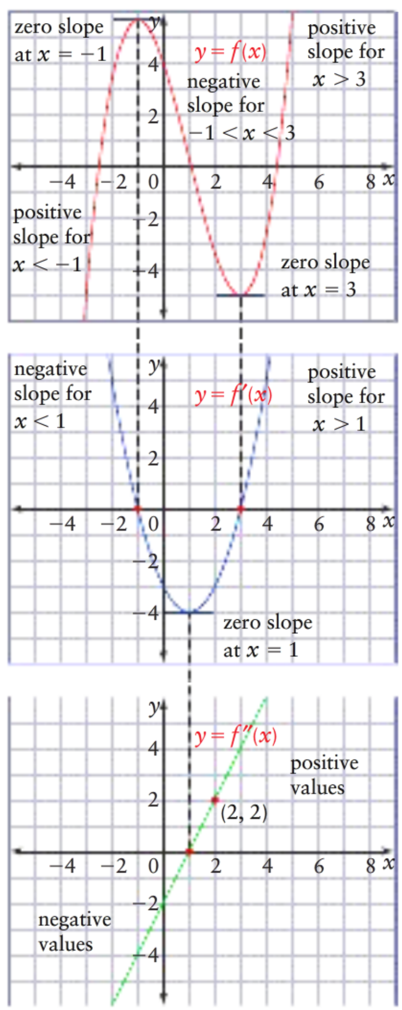
$$f'(x) = \frac{1}{3}(3x^2) - 2x - 3$$

$$= x^2 - 2x - 3$$

$$f''(x) = 2x - 2 \qquad \text{Differentiate } f'(x)$$

$$f''(2) = 2(2) - 2 = 2$$

The following graphs show the relationship between the original function, $f(x)$, the derivative, $f'(x)$, and the second derivative, $f''(x)$. The point (2, 2) on the graph of $f''(x)$ corresponds to $f''(2) = 2$.




Negative $y$-values for $-1 < x < 3$ on the derivative graph correspond to the negative slopes on the original graph. Positive $y$-values for $x < -1$ and $x > 3$ on the derivative graph correspond to positive slopes on the original graph. The $x$-intercepts, $x = -1$ and 3, of the derivative graph correspond to points on $f(x)$ that have zero slope.

Negative $y$-values for $x < 1$ on the second derivative graph correspond to the negative slopes on the derivative graph. Positive $y$-values for $x > 1$ on the second derivative graph correspond to positive slopes on the derivative graph. The $x$-intercept, $x = 1$, of the second derivative graph corresponds to the point on $f'(x)$ that has zero slope.

## Displacement, Velocity, and Acceleration

| | Displacement ($s$) | Velocity ($v$) | Acceleration ($a$) |
|---|---|---|---|
| Definition in Words | Distance an object has moved from the origin over a period of time ($t$) | The rate of change of displacement ($s$) with respect to time ($t$) | The rate of change of velocity ($v$) with respect to time ($t$) |
| Relationship | $s(t)$ | $s'(t) = v(t)$ | $v'(t) = a(t)$ and $s''(t) = a(t)$ |
| Typical Units | m | m/s | m/s$^2$ |

When describing motion, there are two terms that are sometimes misused in everyday speech: speed and velocity. These are sometimes used interchangeably, but they are, in fact, different.

**Speed** is a scalar quantity. It describes the magnitude of motion, but does not describe direction. **Velocity**, on the other hand, is a vector quantity. It has both magnitude and direction. The answer in a velocity problem will be either a negative or positive value. The sign indicates the direction the object is travelling. That is, the original position of the object is considered the origin. One direction from the origin is assigned positive values, and the opposite direction is assigned negative values, depending on what makes sense for the question.

### Example 2 Solve a Velocity and Acceleration Problem Involving a Falling Object

A construction worker accidently drops a hammer from a height of 90 m while working on the roof of a new apartment building. The height of the hammer, $s$, in metres, after $t$ seconds is modelled by the function $s(t) = 90 - 4.9t^2$, $t \geq 0$.

**a)** Determine the average velocity of the hammer between 1 s and 4 s.

**b)** Explain the significance of the sign of your result in part a).

**c)** Determine the velocity of the hammer at 1 s and at 4 s.

**d)** When will the hammer hit the ground?

**e)** Determine the impact velocity of the hammer.

**f)** Determine the acceleration function. What do you notice? Interpret its meaning for this situation.

## Solution

**a)** Average velocity $= \dfrac{\Delta s}{\Delta t}$

$$= \frac{s(4) - s(1)}{4 - 1}$$

$$= \frac{[90 - 4.9(4)^2] - [90 - 4.9(1)^2]}{4 - 1}$$

$$= -\frac{73.5}{3}$$

$$= -24.5$$

The average velocity of the hammer between 1 s and 4 s is $-24.5$ m/s.

**b)** In this type of problem, the movement in the upward direction is commonly assigned positive values. Therefore, the negative answer indicates that the motion of the hammer is downward. (Note that the speed of the hammer is 24.5 m/s.)

CONNECTIONS

Earlier in the chapter, air resistance was defined as a force that counters the effects of gravity as falling objects encounter friction with air molecules. When this force, also called atmospheric drag, becomes equal to the force of gravity, a falling object will accelerate no further. Its velocity remains constant after this point. This maximum velocity is called terminal velocity.

**c)** $v(t) = s'(t)$

$$= \frac{d}{dt}(90 - 4.9t^2)$$

$$= -9.8t$$

Substitute $t = 1$ and $t = 4$.

$$v(1) = -9.8(1) = -9.8$$

$$v(4) = -9.8(4) = -39.2$$

The velocity of the hammer at 1 s is $-9.8$ m/s, and at 4 s it is $-39.2$ m/s. Once again, the negative answers indicate downward movement.

**d)** The hammer hits the ground when the displacement is zero.
Solve $s(t) = 0$.

$$90 - 4.9t^2 = 0$$

$$t^2 = \frac{90}{4.9}$$

$$\doteq 18.37$$

$$t \doteq \pm 4.29$$

Since $t \geq 0$, $t = 4.29$.
The hammer takes approximately 4.3 s to hit the ground.

**e)** The impact velocity is the velocity of the hammer when it hits the ground.

$$v(4.3) = -9.8(4.3) = -42.14$$

The impact velocity of the hammer is about 42 m/s.

**f)** The acceleration function is the derivative of the velocity function.

$$\begin{aligned} a(t) &= v'(t) \\ &= \frac{d}{dt}(-9.8t) \\ &= -9.8 \end{aligned}$$

The hammer falls at a constant acceleration of $-9.8$ m/s$^2$. This value is the acceleration due to gravity for any falling object on Earth (when air resistance is ignored).

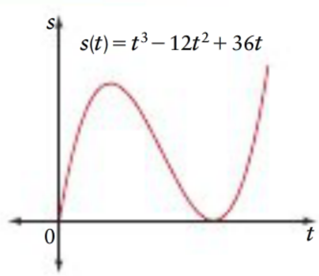
In general, if the acceleration and velocity of an object have the same sign at a particular time, then the object is being pushed in the direction of the motion, and the object is speeding up. If the acceleration and velocity have opposite signs at a particular time, the object is being pushed in the opposite direction to its motion, and it is slowing down.

An object is speeding up, at time $t$ if $v(t) \times a(t) > 0$.

An object is slowing down, at time $t$ if $v(t) \times a(t) < 0$.

## Example 3 Relate Velocity and Acceleration

The position of a particle moving along a straight line is represented by the function $s(t) = t^3 - 12t^2 + 36t$, where distance, $s$, is in metres, time, $t$, is in seconds, and $t \geq 0$.

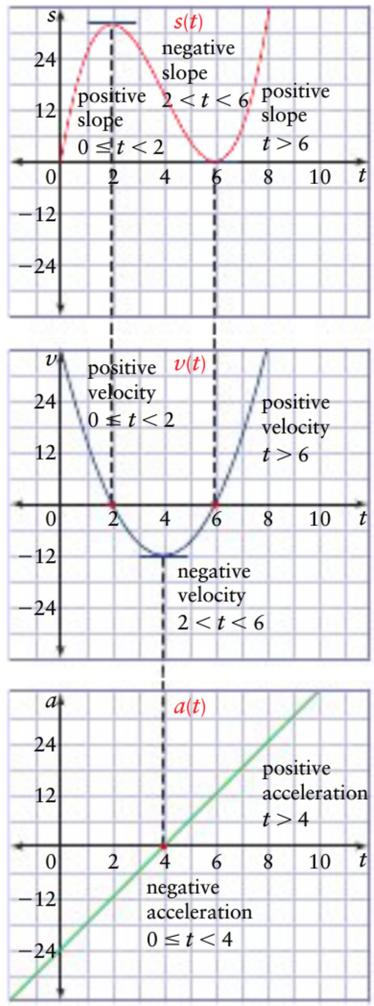



**a)** The graph of the position function is given. Sketch the graph of the velocity and acceleration functions.

**b)** Determine when the particle is speeding up and slowing down. How does this relate to the slope of the position function?

## Solution

**a)**




Begin with $t = 2$ and $t = 6$ on the graph of $s(t)$, where the slope of the tangent is zero. Mark these as the $t$-intercepts of the derivative graph, $v(t)$.

The graph of $s(t)$ has positive slope over the intervals $[0, 2)$ and $(6, 8)$, so the graph of $v(t)$ is positive (lies above the $t$-axis) over these intervals. The graph of $s(t)$ has negative slope over the interval $(2, 6)$, so the graph of $v(t)$ is negative (lies below the $t$-axis) over this interval.

Begin with $t = 4$ on the graph of $v(t)$, where the slope of the tangent is zero. Mark this as the $t$-intercept of the derivative graph, $a(t)$.

The graph of $v(t)$ has negative slope over the interval $[0, 4)$, so the graph of $a(t)$ is negative over this interval. The graph of $v(t)$ has positive slope over the interval $(4, 8)$, so the graph of $a(t)$ is positive over this interval.

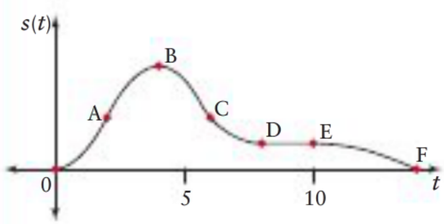
**b)** The graph of $v(t)$ changes sign at the intercepts $t = 2$ and $t = 6$. The graph of $a(t)$ changes sign at the intercept $t = 4$. The signs of $v(t)$ and $a(t)$ are easily observed by determining whether the respective graph lies below or above the $t$-axis. Consider the following four time intervals: $[0, 2)$, $(2, 4)$, $(4, 6)$, $(6,8)$. The following chart summarizes this information.

| Interval | $v(t)$ | $a(t)$ | $v(t) \times a(t)$ | Motion of Particle | Description of slope of $s(t)$ |
|---|---|---|---|---|---|
| [0, 2) | + | − | − | slowing down and moving forward | positive slope that is decreasing |
| (2, 4) | − | − | + | speeding up and moving in reverse | negative slope that is decreasing |
| (4, 6) | − | + | − | slowing down and moving in reverse | negative slope that is increasing |
| (6, 8) | + | + | + | speeding up and moving forward | positive slope that is increasing |

Therefore, the particle is slowing down between 0 s and 2 s and again between 4 s and 6 s. The particle is speeding up between 2 s and 4 s and after 6 s.

## Example 4 Analyse and Interpret a Position-Time Graph

The graph shows the position function of a motorcycle. Describe the slope of the graph, in terms of being positive, negative, increasing, or decreasing, over the interval between consecutive pairs of points, beginning at the origin. For each interval, determine the sign of the velocity and acceleration by considering the slope of the graph.




### Solution

The analysis of this situation is organized in the following chart.

| Interval | Slope of Graph | Velocity | Acceleration |
|---|---|---|---|
| 0 to A | positive slope that is increasing | + | + |
| A to B | positive slope that is decreasing | + | − |
| B to C | negative slope that is decreasing | − | − |
| C to D | negative slope that is increasing | − | + |
| D to E | slope = zero, horizontal segment | 0 | 0 |
| E to F | negative slope that is decreasing | − | − |

## KEY CONCEPTS

- The second derivative of a function is determined by differentiating the first derivative of the function.
- For a given position function $s(t)$, its velocity function is $v(t)$, or $s'(t)$, and its acceleration function is $a(t)$, $v'(t)$, or $s''(t)$.
- When $v(t) = 0$, the object is at rest, or stationary. There are many instances where an object will be momentarily at rest when changing directions. For example, a ball thrown straight upward will be momentarily at rest at its highest point, and will then begin to descend.
- When $v(t) > 0$, the object is moving in the positive direction.
- When $v(t) < 0$, the object is moving in the negative direction.
- When $a(t) > 0$, the velocity of an object is increasing (i.e., the object is accelerating).
- When $a(t) < 0$, the velocity of an object is decreasing (i.e., the object is decelerating).
- An object is speeding up if $v(t) \times a(t) > 0$ and slowing down if $v(t) \times a(t) < 0$.

## Communicate Your Understanding

**C1** Under what conditions is an object speeding up? Under what conditions is it slowing down? Support your answers with examples.

**C2** Give the graphical interpretation of positive velocity and negative velocity.

**C3** How are speed and velocity similar? How are they different?

**C4** What is the relationship between the degrees of $s(t)$, $v(t)$, and $a(t)$, if $s(t)$ is a polynomial function?

## A Practise

**1.** Determine the second derivative of each function.

**a)** $y = 2x^3 + 21$

**b)** $s(t) = -t^4 + 5t^3 - 2t^2 + t$

**c)** $h(x) = \frac{1}{6}x^6 - \frac{1}{5}x^5$

**d)** $f(x) = \frac{1}{4}x^3 - 2x^2 + 8$

**e)** $g(x) = x^5 + 3x^4 - 2x^3$

**f)** $h(t) = -4.9t^2 + 25t + 4$

**2.** Determine $f''(3)$ for each function.

**a)** $f(x) = 2x^4 - 3x^3 + 6x^2 + 5$

**b)** $f(x) = 4x^3 - 5x + 6$

**c)** $f(x) = -\frac{2}{5}x^5 - x^3 + 0.5$

**d)** $f(x) = (3x^2 + 2)(1 - x)$

**e)** $f(x) = (6x - 5)(x^2 + 4)$

**f)** $f(x) = 4x^5 - \frac{1}{2}x^4 - 3x^2$

3. Determine the velocity and acceleration functions for each position function $s(t)$. Where possible, simplify the functions before differentiating.

**a)** $s(t) = 5 + 7t - 8t^3$

**b)** $s(t) = (2t + 3)(4 - 5t)$

**c)** $s(t) = -(t + 2)(3t^2 - t + 5)$

**d)** $s(t) = \dfrac{-2t^4 - t^3 + 8t^2}{4t^2}$

4. Determine the velocity and acceleration at $t = 2$ for each position function $s(t)$, where $s$ is in metres and $t$ is in seconds.

**a)** $s(t) = t^3 - 3t^2 + t - 1$

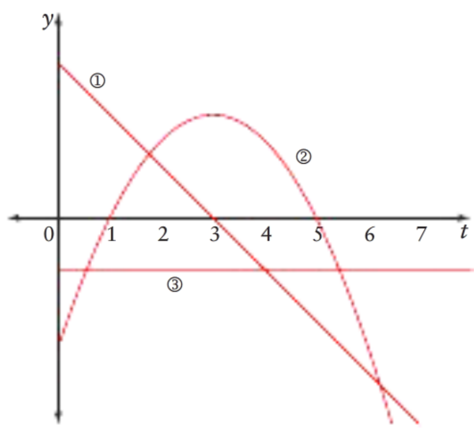
**b)** $s(t) = -4.9t^2 + 15t + 1$

**c)** $s(t) = t(3t + 5)(1 - 2t)$

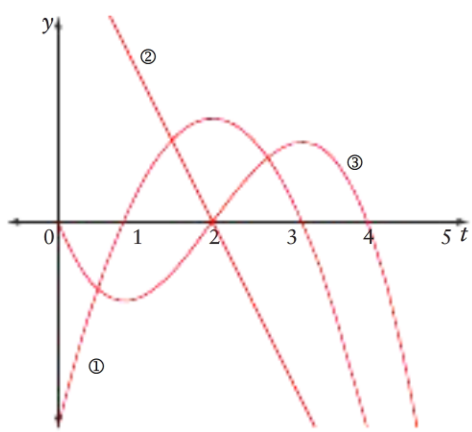
**d)** $s(t) = (t^2 - 2)(t^2 + 2)$

5. In each graph, identify which curve or line represents $y = s(t)$, $y = v(t)$, and $y = a(t)$. Justify your choices.

**a)**




**b)**




6. Copy and complete the chart for each graph in question 5.

**a)**

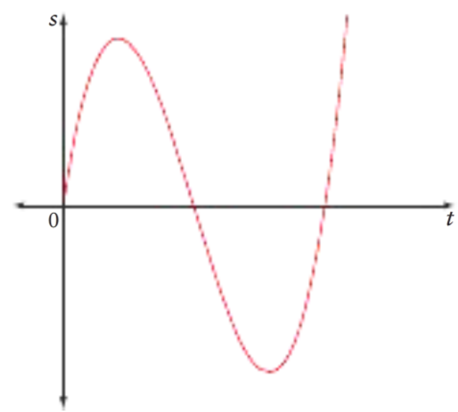
| Interval | $v(t)$ | $a(t)$ | $v(t) \times a(t)$ | Motion of Object | Description of slope of $s(t)$ |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |

**b)**

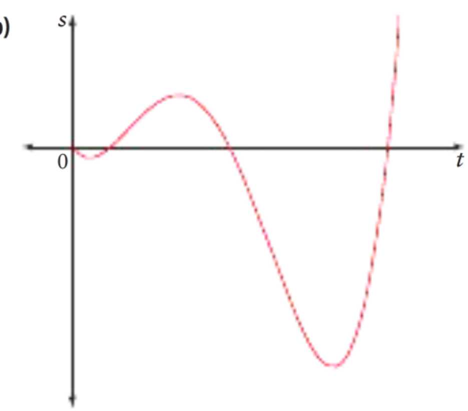
| Interval | $v(t)$ | $a(t)$ | $v(t) \times a(t)$ | Motion of Object | Description of slope of $s(t)$ |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |

7. For each position function $y = s(t)$ given, sketch the graphs of $y = v(t)$ and $y = a(t)$.

**a)**




**b)**

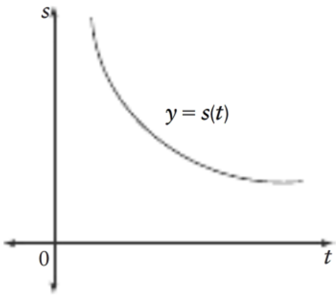


8. Answer the following for each graph of $y = s(t)$. Explain your reasoning.

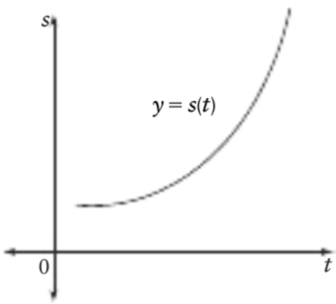
   i) Is velocity increasing, decreasing, or constant?

   ii) Is acceleration positive, negative, or zero?

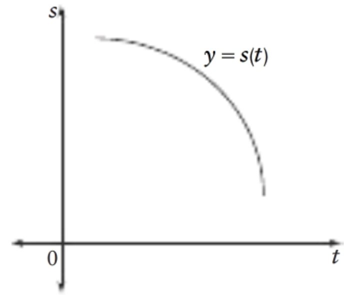
a)



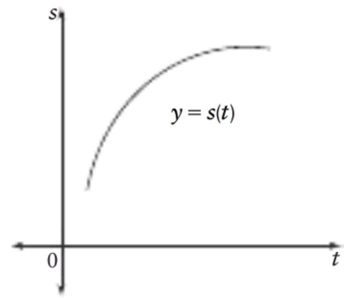

b)



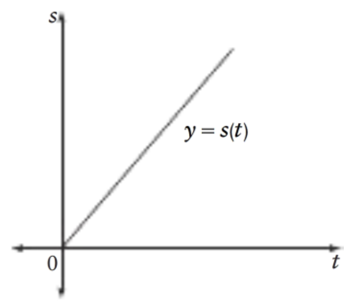

c)




d)

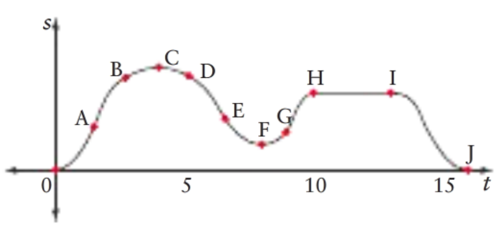



e)




## B Connect and Apply

9. The following graph shows the position function of a bus during a 15-min trip.




   a) What is the initial velocity of the bus?

   b) What is the bus's velocity at C and at F?

   c) Is the bus going faster at A or at B? Explain.

   d) What happens to the motion of the bus between H and I?

   e) Is the bus speeding up or slowing down at A, B, and D?

   f) What happens at J?

10. Refer to the graph in question 9. Is the acceleration positive, zero, or negative during the following intervals?

   a) 0 to A

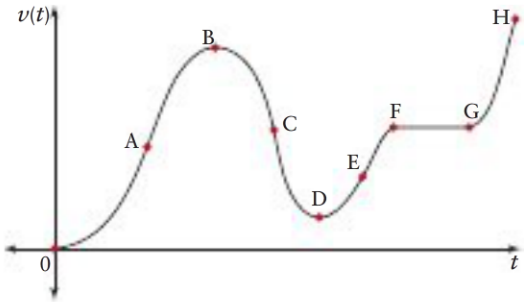
   b) C to D

   c) E to F

   d) G to H

   e) F to G

11. The following is the graph of a velocity function.




a) State whether the acceleration is positive, negative, or zero for the following intervals or points.

i) 0 to B

ii) B to D

iii) D to F

iv) F to G

v) G to H

vi) at A and at D

b) Describe similarities and differences in the acceleration for the given intervals. Justify your response.

i) 0 to A, D to E, and G to H

ii) A to B and E to F

iii) B to C and C to D

12. A water bottle rolls off a rooftop patio from a height of 80 m. The distance, $s$, in metres, the bottle is above the ground after $t$ seconds is modelled by the function $s(t) = 80 - 4.9t^2$, $t \geq 0$.

a) Determine the average velocity of the bottle between 1 s and 3 s.

b) Determine the velocity of the bottle at 3 s.

c) When will the bottle hit the ground?

d) Determine the impact velocity of the bottle.

13. During a fireworks display, a starburst rocket is shot upward with an initial velocity of 34.5 m/s from a platform 3.2 m high. The height, in metres, of the rocket after $t$ seconds is represented by the function $h(t) = -4.9t^2 + 34.5t + 3.2$.


a) Determine the velocity and acceleration of the rocket at 3 s.

b) When the rocket reaches its maximum height, it explodes to create a starburst display. How long does it take for the rocket to reach its maximum height?

c) At what height does the starburst display occur?

d) Sometimes the rockets malfunction and do not explode. How long would it take for an unexploded rocket to return to the ground?

e) At what velocity would it hit the ground?

14. Consider the motion of a truck that is braking while moving forward. Justify your answer to each of the following.

a) Is the velocity positive or negative?

b) Is the velocity increasing or decreasing?

c) Is the acceleration positive or negative?

## C Extend and Challenge

15. A bald eagle flying horizontally at 48 km/h drops its prey from a height of 50 m.

a) State the equation representing the horizontal displacement of the prey while held by the eagle.

b) Determine the velocity and acceleration functions for the prey's horizontal displacement.

c) State the equation representing the vertical displacement of the prey as it falls. (Assume the acceleration due to gravity is $-9.8$ m/s$^2$.)

d) Determine the velocity and acceleration functions for the prey's vertical displacement.

e) What is the prey's vertical velocity when it hits the ground?

f) When is the vertical speed greater than the horizontal speed?

g) Develop an equation to represent the total velocity. Determine the velocity.

h) Develop an equation to represent the total acceleration.

i) Determine the prey's acceleration 4 s after being dropped.

**16.** The position function of an object moving along a straight line is represented by the function $s(t) = 2t^3 - 15t^2 + 36t + 10$, where $t$ is in seconds and $s$ in metres.

a) What is the velocity of the object after 1 s and after 4 s? What is the object's acceleration at these times?

b) When is the object momentarily at rest? What is the object's position when stopped?

c) When is the object moving in a positive direction? When is it moving in a negative direction?

d) Determine the total distance travelled by the object during the first 7 s.

e) Sketch a diagram to illustrate the motion of the object.

**17.** The height of any object that is shot into the air is given by the position function $h(t) = 0.5gt^2 + v_0t + s_0$, where $h$ is in metres and $t$ is in seconds, $s_0$ is the initial height of the object, $v_0$ is the initial velocity of the object, and $g$ is the acceleration due to gravity ($g = -9.8\,\text{m/s}^2$).

a) Determine the velocity function and the acceleration function for $h(t)$.

b) An arrow is shot upward at 17.5 m/s from a position in a tree 4 m above the ground. State the position, velocity, and acceleration functions for this situation.

c) Suppose a flare is shot upward, and after 2 s its velocity is 10.4 m/s and its height is 42.4 m. Determine the position, velocity, and acceleration functions for this situation.

**18.** The position function $s(t) = 0.5gt^2 + v_0t + s_0$ is also used to represent the motion of an object moving along a straight line, where $s$ is in metres and $t$ is in seconds, and where $s_0$ is the initial position of the object, $v_0$ is the initial velocity of the object, and $g$ is the acceleration. The driver of a pickup truck travelling at 86.4 km/h suddenly notices a stop sign and applies the brakes, resulting in a constant deceleration of 12 m/s$^2$.

a) Determine the position, velocity, and acceleration function for this situation.

b) How long does it take for the truck to stop?

**19.** **Math Contest** If $p$ and $q$ are two polynomials such that $p''(x) = q''(x)$ for $x \in \mathbb{R}$, which of the following *must* be true?

**A** $p(x) = q(x)$ for $x \in \mathbb{R}$

**B** $p'(x) = q'(x)$ for $x \in \mathbb{R}$

**C** $p(0) - q(0) = 0$

**D** The graph of $y = p(x) - q(x)$ is a horizontal line.

**E** The graph of $y = p'(x) - q'(x)$ is a horizontal line.

**20.** **Math Contest**

$f^{(n)}(x) = \dfrac{d}{dx}\left(\dfrac{d}{dx}\left(\dfrac{d}{dx}\left(\ldots\dfrac{d}{dx}(f(x)\ldots)\right)\right)\right)$ denotes the $n$th derivative of the function $f(x)$. If $f'(x) = g(x)$ and $g'(x) = -f(x)$, then $f^{(n)}(x)$ is equal to

**A** $\dfrac{1}{2}[1 + (-1)^n]f(x) + \dfrac{1}{2}[1 + (-1)^{n-1}]g(x)$

**B** $\dfrac{(-1)^n}{2}[1 + (-1)^n]f(x) + \dfrac{(-1)^{n-1}}{2}[1 + (-1)^{n-1}]g(x)$

**C** $\dfrac{(-1)^{n-1}}{2}[1 + (-1)^n]f(x) + \dfrac{(-1)^n}{2}[1 + (-1)^{n-1}]g(x)$

**D** $\dfrac{i^n}{2}[1 + (-1)^n]f(x) + \dfrac{i^{n-1}}{2}[1 + (-1)^{n-1}]g(x)$, where $i = \sqrt{-1}$

**E** $\dfrac{i^{n-1}}{2}[1 + (-1)^{n-1}]f(x) + \dfrac{i^n}{2}[1 + (-1)^n]g(x)$, where $i = \sqrt{-1}$