2.4

The Chain Rule

Andrew and David are both training to run a marathon, a long-distance running event that covers a distance of 42.195 km (26.22 mi). They both go for a run on Sunday mornings at precisely 7 A.M. Andrew's house is 22 km south of David's house. One Sunday morning, Andrew leaves his house and runs north at 9 km/h. At the same time, David leaves his house and runs west at 7 km/h. The distance between the two runners can be modelled by the function $s(t) = \sqrt{130t^2 - 396t + 484}$, where s is in kilometres and t is in hours. You can use differentiation to determine the rate at which the distance between the two runners is changing. This rate of change is given by $s'(t)$.

To this point, you have not learned a differentiation rule that will help you find the derivative of this type of function. You could use the first principles limit definition, but that would be complicated. Notice, however, that $s(t)$ is the composition of a root function and a polynomial function: $s(t) = f \circ g(t) = f(g(t))$, where $f(t) = \sqrt{t}$, and $g(t) = 130t^2 - 396t + 484$. Both $f'(t)$ and $g'(t)$ are easily computed using the derivative rules you already know. This section will develop a general rule that can be used to find the derivative of composite functions. This rule is called the **chain rule**.

Investigate How are composite functions differentiated?

1. Consider the function $f(x) = (8x^3)^{\frac{1}{3}}$.

a) Simplify $f(x)$, using the laws of exponents.

b) Use your result from part a) to determine $f'(x)$.

2. a) Let $g(x) = x^{\frac{1}{3}}$. Determine $g'(x)$.

b) Let $h(x) = 8x^3$. Replace x with $8x^3$ in your expression for $g'(x)$ from part a). This will give an expression for $g'[h(x)]$. Do not simplify.

c) **Reflect** Why is it appropriate to refer to the expression $g'[h(x)]$ as a composite function?

d) Determine $h'(x)$.

3. a) Use the results of step 2 parts b) and d) to write an expression for the product $g'[h(x)] \times h'(x)$.

b) Simplify your answer from part a).

c) Compare the derivative result from step 1 part b) with the derivative result from step 3 part b). What do you notice?

4. a) **Reflect** Use the above results to write a rule for differentiating a composite function $f(x) = g \circ h(x)$. This rule is called the chain rule. What operation forms the "chain"? Explain.

b) Write the rule from part a) in terms of the "outer function" and the "inner function."

c) Use your rule to differentiate $f(x) = (2x^3 - 5)^2$.

d) Verify the accuracy of your rule by differentiating $f(x) = (2x^3 - 5)^2$ using the product rule.

The Chain Rule

Given two differentiable functions $g(x)$ and $h(x)$, the derivative of the composite function $f(x) = g[h(x)]$ is $f'(x) = g'[h(x)] \times h'(x)$.

A composite function $f(x) = (g \circ h)(x) = g[h(x)]$ consists of an outer function, $g(x)$, and an inner function, $h(x)$. The chain rule is an efficient way of differentiating a composite function by first differentiating the outer function with respect to the inner function, and then multiplying by the derivative of the inner function.

Example 1 Apply the Chain Rule

Differentiate each function using the chain rule.

a) $f(x) = (3x - 5)^4$

b) $f(x) = \sqrt{4 - x^2}$

Solution

a) $f(x) = (3x - 5)^4$ is a composite function, $f(x) = g[h(x)]$, with $g(x) = x^4$ and $h(x) = 3x - 5$. Then $g'(x) = 4x^3$, $g'[h(x)] = 4(3x - 5)^3$, and $h'(x) = 3$.

$$\begin{aligned} f'(x) &= g'[h(x)]h'(x) && \text{Apply the chain rule.} \\ &= 4(3x - 5)^3(3) \\ &= 12(3x - 5)^3 \end{aligned}$$

b) $f(x) = \sqrt{4 - x^2}$ is a composite function, $f(x) = g[h(x)]$, with $g(x) = x^{\frac{1}{2}}$ and $h(x) = 4 - x^2$. Then $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $g'[h(x)] = \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}$, and $h'(x) = -2x$.

$$\begin{aligned} f'(x) &= g'[h(x)]h'(x) \qquad \text{Apply the chain rule.} \\ &= \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x) \\ &= -x(4 - x^2)^{-\frac{1}{2}} \\ &= \frac{-x}{\sqrt{4 - x^2}} \end{aligned}$$

Alternate Form of the Chain Rule

Consider the function in Example 1 part b): $f(x) = \sqrt{4 - x^2}$.

Let $y = \sqrt{u}$. Let $u = 4 - x^2$.

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ $\qquad$ $\frac{du}{dx} = -2x$

The product of these two derivatives, $\frac{dy}{du} \times \frac{du}{dx}$, results in the derivative of y with respect to x, $\frac{dy}{dx}$.

Therefore, $\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(-2x)$. Replacing u with $4 - x^2$, the result is $\frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$.

Leibniz Form of the Chain Rule

If $y = f(u)$ and $u = g(x)$ are differentiable functions, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

The Leibniz form of the chain rule can be expressed in words as follows: If y is a function in u, and u is a function in x, then the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x.

This form of the chain rule is easily remembered if you think of each term as a fraction. You can then cancel du. Keep in mind, however, that this is only a memory device, and does not reflect the mathematical reality. These terms are not really fractions because du has not been defined.

Example 2 Represent the Chain Rule in Leibniz Notation

a) If $y = -\sqrt{u}$ and $u = 4x^3 - 3x^2 + 1$, determine $\frac{dy}{dx}$.

b) If $y = u^{-3}$ and $u = 2x - x^3$, determine $\left.\frac{dy}{dx}\right|_{x=2}$.

Solution

a)
$$y = -\sqrt{u} \text{ and } u = 4x^3 - 3x^2 + 1$$
$$y = -u^{\frac{1}{2}}$$
$$\frac{dy}{du} = -\frac{1}{2}u^{-\frac{1}{2}} \text{ and } \frac{du}{dx} = 12x^2 - 6x$$
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= -\frac{1}{2}u^{-\frac{1}{2}}(12x^2 - 6x)\\ &= -\frac{1}{2}(4x^3 - 3x^2 + 1)^{-\frac{1}{2}}(12x^2 - 6x)\\ &= \frac{-12x^2 + 6x}{2\sqrt{4x^3 - 3x^2 + 1}}\end{aligned}$$

Substitute $u = 4x^3 - 3x^2 + 1$ to express the answer in terms of x.

b) For $y = u^{-3}$, $\frac{dy}{du} = -3u^{-4}$.

For $u = 2x - x^3$, $\frac{du}{dx} = 2 - 3x^2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= -3u^{-4}(2 - 3x^2)\\ &= \frac{-3(2 - 3x^2)}{u^4}\end{aligned}$$

Notice that it was not necessary to write the expression entirely in terms of x, because you can determine the value of u when $x = 2$ and then substitute as shown below.

When $x = 2$, $u = 2(2) - 2^3 = -4$.

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=2} &= \frac{-3(2 - 3(2)^2)}{(-4)^4}\\ &= \frac{-3(2 - 12)}{256}\\ &= \frac{15}{128}\end{aligned}$$

Example 2 illustrates a special case of the chain rule that occurs when the outer function is a power function, such as $y = u^n$ or $[g(x)]^n$. The derivative consists of using the power rule first.

Power of a Function Rule

If $y = u^n$ and $u = g(x)$, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \text{ or } \frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x).$$

Example 3 Combine the Chain Rule and the Product Rule

Determine the equation of the tangent to $f(x) = 3x(1-x)^2$ at $x = 0.5$.

Solution

Differentiate the function.

$$f(x) = 3x(1-x)^2$$

$$f'(x) = (1-x^2)\frac{d}{dx}(3x) + (3x)\frac{d}{dx}(1-x)^2 \qquad \text{Apply the product rule first.}$$

$$= 3(1-x)^2 + 3x[2(1-x)(-1)] \qquad \text{Apply the power of a function rule.}$$

Determine the slope at $x = 0.5$ by substituting into the derivative function.

$$\begin{aligned} f'(0.5) &= 3(1-0.5)^2 + 3(0.5)[2(1-0.5)(-1)] \\ &= 0.75 - 1.5 \\ &= -0.75 \end{aligned}$$

The slope of the tangent at $x = 0.5$ is -0.75.

Calculate the tangent point by substituting the x-value into the original function.

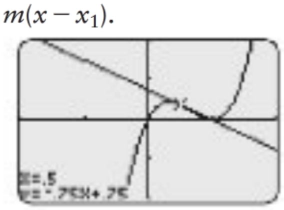
$$\begin{aligned} f(0.5) &= 3(0.5)(1-0.5)^2 \\ &= 1.5(0.25) \\ &= 0.375 \end{aligned}$$

The tangent point is $(0.5, 0.375)$.

Substitute $m = -0.75$ and $(0.5, 0.375)$ into $y - y_1 = m(x - x_1)$.

$$\begin{aligned} y - 0.375 &= -0.75(x - 0.5) \\ y &= -0.75x + 0.375 + 0.375 \\ y &= -0.75x + 0.75 \end{aligned}$$

You can check your answer using the Tangent function on a graphing calculator.

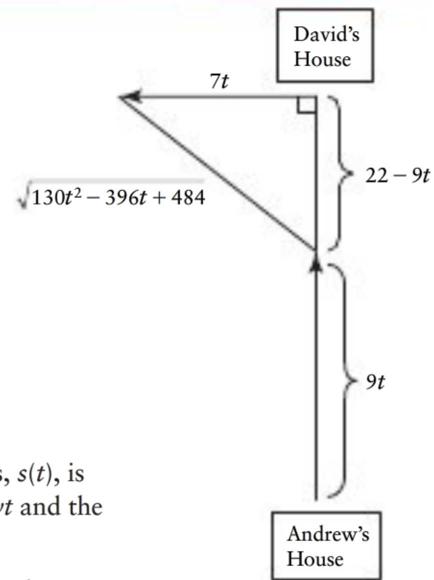


Technology Tip

To get the screen shown here, you will have to change the number of decimal places to 2.

Example 4 Apply the Chain Rule to Solve a Rate of Change Problem

The chain rule can be used to solve the problem presented at the beginning of this section. Andrew and David both leave their houses at 7 A.M. for their Sunday run. Andrew's house is 22 km south of David's house. Andrew runs north at 9 km/h, while David runs west at 7 km/h. Determine the rate of change of the distance between the two runners after 1 h.



Solution

The distance between the two runners, $s(t)$, is obtained by using two formulas: $d = vt$ and the Pythagorean theorem.

Andrew runs at 9 km/h, so the distance he runs is $9t$, with t measured in hours. Since his house is 22 km from David's, the distance he is from David's house as he runs is represented by $22 - 9t$. David runs at 7 km/h, so the distance he runs is represented by $7t$.

$$s(t) = \sqrt{(22 - 9t)^2 + (7t)^2} \qquad \text{Pythagorean theorem.}$$

$$= \sqrt{130t^2 - 396t + 484}$$

$$= (130t^2 - 396t + 484)^{\frac{1}{2}}$$

$$s'(t) = \frac{1}{2}(130t^2 - 396t + 484)^{-\frac{1}{2}}\frac{d}{dt}(130t^2 - 396t + 484) \qquad \text{Power of a function rule.}$$

$$= \frac{1}{2\sqrt{130t^2 - 396t + 484}}(260t - 396)$$

$$s'(1) = \frac{260(1) - 396}{2\sqrt{130(1)^2 - 396(1) + 484}}$$

$$= \frac{-136}{2\sqrt{218}}$$

$$\doteq -4.6$$

After 1 h, the distance between Andrew and David is decreasing at 4.6 km/h.

KEY CONCEPTS

- **The Chain Rule**

  The chain rule is used to differentiate composite functions, $f = g \circ h$. Given two differentiable functions $g(x)$ and $h(x)$, the derivative of the composite function $f(x) = g[h(x)]$ is $f'(x) = g'[h(x)] \times h'(x)$.

- **The Chain Rule in Leibniz Notation**

  If $y = f(u)$ and $u = g(x)$ are differentiable functions, then

  $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

- **The Power of a Function Rule**

  If $y = u^n$ and $u = g(x)$, then

  $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \text{ or } \frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x).$$

Communicate Your Understanding

C1 Use an example to explain the meaning of the statement, "The derivative of a composite function is equal to the derivative of the outer function multiplied by the derivative of the inner function."

C2 Can the product rule be used to verify the chain rule? Support your answer with an example.

C3 Would it be true to say, "The power of a function rule is separate and distinct from the chain rule"? Use an example to justify your response.

C4 What operation creates the "chain" in the chain rule?

A Practise

1. Determine the derivative of each function by using the following methods.

i) Use the chain rule, and then simplify.

ii) Simplify first, and then differentiate.

a) $f(x) = (2x)^3$

b) $g(x) = (-4x^2)^2$

c) $p(x) = \sqrt{9x^2}$

d) $f(x) = (-16x^2)^{\frac{3}{4}}$

e) $q(x) = (8x)^{\frac{2}{3}}$

2. Copy and complete the following table.

$f(x) = g[h(x)]$	$g(x)$	$h(x)$	$h'(x)$	$g'[h(x)]$	$f'(x)$
a) $(6x - 1)^2$					
b) $(x^2 + 3)^3$					
c) $(2 - x^3)^4$					
d) $(-3x + 4)^{-1}$					
e) $(7 + x^2)^{-2}$					
f) $\sqrt{x^4 - 3x^2}$					

3. Differentiate, expressing each answer using positive exponents.

a) $y=(4x+1)^2$
b) $y=(3x^2-2)^3$
c) $y=(x^3-x)^{-3}$
d) $y=(4x^2+3x)^{-2}$

4. Express each function as a power with a rational exponent, and then differentiate. Express each answer using positive exponents.

a) $y=\sqrt{2x-3x^5}$
b) $y=\sqrt{-x^3+9}$
c) $y=\sqrt[3]{x-x^4}$
d) $y=\sqrt[5]{2+3x^2-x^3}$

5. Express each of the following as a power with a negative exponent, and then differentiate. Express each answer using positive exponents.

a) $y=\dfrac{1}{(-x^3+1)^2}$
b) $y=\dfrac{1}{(3x^2-2)}$
c) $y=\dfrac{1}{\sqrt{x^2+4x}}$
d) $y=\dfrac{1}{\sqrt[3]{x-7x^2}}$

6. a) Use two different methods to differentiate $f(x)=\sqrt{25x^4}$.

b) **Reflect** Explain why you prefer one of the methods in part a) over the other.

c) **Reflect** Can both methods that you described in part a) be used to differentiate $f(x)=\sqrt{25x^4-3}$? Explain.

B Connect and Apply

7. Determine $f'(1)$.

a) $f(x)=(4x^2-x+1)^2$
b) $f(x)=(3-x+x^2)^{-2}$
c) $f(x)=\sqrt{4x^2+1}$
d) $f(x)=\dfrac{5}{\sqrt[3]{2x-x^2}}$

8. Using Leibniz notation, apply the chain rule to determine $\dfrac{dy}{dx}$ at the indicated value of x.

a) $y=u^2+3u,\ u=\sqrt{x},\ x=4$
b) $y=\sqrt{u},\ u=2x^2+3x+4,\ x=-3$
c) $y=\dfrac{1}{u^2},\ u=x^3-5x,\ x=-2$
d) $y=u(2-u^2),\ u=\dfrac{1}{x},\ x=2$

9. Determine the equation of the tangent to the curve $y=(x^3-4x^2)^3$ at $x=3$.

10. Determine the equation of the tangent to the curve $y=\dfrac{1}{\sqrt[5]{5x^3-2x^2}}$ at $x=2$.

11. The position function of a moving particle is given by $s(t)=\sqrt[3]{t^5-750t^2}$, where s is in metres and t is in seconds. Determine the velocity of the particle at 5 s.

12. Determine the point(s) on the curve $y=x^2(x^3-x)^2$ where the tangent line is horizontal.

13. **Chapter Problem** The owners of Mooses, Gooses, and Juices are interested in analysing the productivity of their staff. The function $N(t)=150-\dfrac{600}{\sqrt{16+3t^2}}$ models the total number of customers, N, served by the staff after t hours during an 8-h workday $(0\leq t\leq 8)$.

a) Determine $N'(t)$. What does the derivative represent for this situation?

b) Determine $N(4)$ and $N'(4)$. Interpret the meaning of each of these values for this situation.

c) Solve $N(t)=103$. Interpret the meaning of your answer for this situation.

d) Determine $N'(t)$ for the value you found in part c). Compare this value with $N'(4)$. What conclusion, if any, can be made from comparing these two values?

14. The population of a small town is modelled by the function $P(t)=\dfrac{1250}{1+0.01t}$, where P is the number of people, and t is time, in years, $t\geq 0$.

Determine the instantaneous rate of change of the population after 2 years, 4 years, and 7 years.

15. The formula for the volume of a cube in terms of its side length, s, is $V(s) = s^3$. If the side length is expressed in terms of a variable, x, measured in metres, such that $s = 3x^2 - 7x + 1$, determine $\left.\dfrac{dv}{dx}\right|_{x=3}$. Interpret the meaning of this value for this situation.

16. Express $y = \dfrac{4x - x^3}{(3x^2 + 2)^2}$ as a product and then differentiate. Simplify your answer using positive exponents.

Achievement Check

17. The red squirrel population in a neighbourhood park can be modelled by the function $p(t) = \sqrt{210t + 44t^2}$, where p is the number of red squirrels, and t is time, in years.
 - **a)** Determine the rate of growth of the squirrel population after 2 years.
 - **b)** When will the population reach 60 squirrels?
 - **c)** What is the rate of change of the population at the time in part b)?
 - **d)** When is the rate of change of the squirrel population approximately 7 squirrels per year?

C Extend and Challenge

18. Determine the equations of the tangents to the curve $y = x^3\sqrt{8x^2 + 1}$ at the points where $x = 1$ and $x = -1$. How are the tangent lines related? Explain why this relationship is true at all points with corresponding positive and negative values $x = a$ and $x = -a$.

19. Determine $f'(2)$ for $f(x) = g[h(x)]$, given $g(2) = 5$, $g'(2) = -3$, $g'(-6) = -3$, $h(2) = -6$, and $h'(2) = 4$.

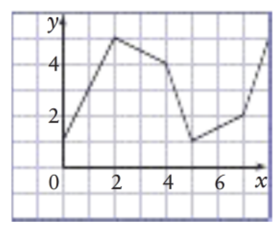
20. Determine $\dfrac{d^2y}{dx^2}$ for the function $y = \sqrt{2x + 1}$.

21. Consider the statement $\dfrac{d}{dx} f \circ g(x) = \dfrac{d}{dx} g \circ f(x)$. Determine examples of two differentiable functions $f(x)$ and $g(x)$ to show
 - **a)** when the statement *is not* true
 - **b)** when the statement *is* true

22. If $f(x) = x^2$, $g(x) = \dfrac{1}{x}$, and $h(x) = \sqrt{x^2 + 2x}$, determine the derivative of each composite function.
 - **a)** $y = f \circ g \circ h(x)$
 - **b)** $y = g \circ f \circ h(x)$
 - **c)** $y = g \circ h \circ f(x)$
 - **d)** $y = h \circ g \circ f(x)$

23. Determine a rule to differentiate a composite function of the form $y = f \circ g \circ h(x)$ given that f, g, and h are all differentiable functions.

24. **Math Contest** This figure shows the graph of $y = f(x)$.



If the function F is defined by $F(x) = f[f(x)]$, then $F(1)$ equals

A -1 **B** 2 **C** 4 **D** 4.5 **E** undefined

25. **Math Contest** Let f be a function such that $f'(x) = \dfrac{1}{x}$. What is the derivative $(f^{-1})'(x)$ of its inverse? Hint: If $y = f^{-1}(x)$, you can write $f(y) = x$.

A 0 **B** x **C** $-\dfrac{1}{x^2}$ **D** $f(x)$ **E** $f^{-1}(x)$