2.5 Derivatives of Quotients

Suppose the function $V(t) = \dfrac{50\,000 + 6t}{1 + 0.4t}$ represents the value, in dollars, of a new car t years after it is purchased. You want to calculate the rate of change in the value of the car at 2 years, 5 years, and 7 years to determine at what rate the car is depreciating.

This problem is similar to the one considered in Section 1.1, though at that time you were only considering data and graphs. In this section, you will explore strategies for differentiating a function that has the form $q(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$.

Example 1 Differentiate a Quotient When the Denominator's Exponent is 1

Differentiate $q(x) = \dfrac{6x - 5}{x^3 + 4}$. State the domain of $q(x)$ and $q'(x)$.

Solution

$q(x) = (6x - 5)(x^3 + 4)^{-1}$ — Express as a product.

$$q'(x) = \left[\frac{d}{dx}(6x - 5)\right](x^3 + 4)^{-1} + (6x - 5)\left[\frac{d}{dx}(x^3 + 4)^{-1}\right]$$

Apply the product rule.

$$= (6)(x^3 + 4)^{-1} + (6x - 5)(-1)(x^3 + 4)^{-2}(3x^2)$$

Apply the chain rule.

$$= 3(x^3 + 4)^{-2}[2(x^3 + 4)] + (6x - 5)(-1)x^2]$$

Common factor $3(x^3 + 4)^{-2}$.

$$= 3(x^3 + 4)^{-2}(2x^3 + 8 - 6x^3 + 5x^2)$$

$$= \frac{3(-4x^3 + 5x^2 + 8)}{(x^3 + 4)^2}$$

The denominator cannot be zero, so

$$x^3 + 4 \neq 0$$
$$x^3 \neq -4$$
$$x \neq \sqrt[3]{-4}$$

The domain of $q(x)$ and $q'(x)$ is $\{x \in \mathbb{R} \mid x \neq \sqrt[3]{-4}\}$.

Differentiating a Simple Quotient Function

If $q(x) = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are differentiable functions, and $g'(x) \neq 0$, then $q(x)$ can be expressed as $q(x) = f(x)[g(x)]^{-1}$.

Then $q'(x) = f(x)(-1)[g(x)]^{-2}\, g'(x) + f'(x)[g(x)]^{-1}$.

Example 2 Differentiate a Quotient When the Denominator's Exponent is *n*

Differentiate $q(x) = \dfrac{x+3}{\sqrt{x^2-1}}$. State the domain of $q(x)$ and $q'(x)$.

Solution

$$q(x) = (x+3)(x^2-1)^{-\frac{1}{2}}$$ Express as a product.

$$q'(x) = \left[\frac{d}{dx}(x+3)\right](x^2-1)^{-\frac{1}{2}} + (x+3)\left[\frac{d}{dx}(x^2-1)^{-\frac{1}{2}}\right]$$ Apply the product rule.

$$= 1(x^2-1)^{-\frac{1}{2}} + (x+3)\left(-\frac{1}{2}\right)(x^2-1)^{-\frac{3}{2}}(2x)$$ Apply the chain rule.

$$= (x^2-1)^{-\frac{3}{2}}[(x^2-1) - x(x+3)]$$ Common factor $(x^2-1)^{-\frac{3}{2}}$

$$= (x^2-1)^{-\frac{3}{2}}(x^2-1-x^2-3x)$$

$$= \frac{-3x-1}{\left(\sqrt{x^2-1}\right)^3}$$

The denominator cannot be zero. Also, x^2-1 must be positive. So,

$x^2 - 1 > 0$
$x^2 > 1$
$x < -1$ or $x > 1$

The domain of $q(x)$ and $q'(x)$ is $\{x \in \mathbb{R} \mid x > 1 \text{ or } x < -1\}$.

Example 3 Find the Equation of the Tangent

Determine the equation of the tangent to the curve $y = \dfrac{x^2 - 3}{5 - x}$ at $x = 2$.

Solution

$$y = (x^2 - 3)(5 - x)^{-1}$$

$$\begin{aligned}
y' &= \left[\frac{d}{dx}(x^2 - 3)\right](5 - x)^{-1} + (x^2 - 3)\left[\frac{d}{dx}(5 - x)^{-1}\right] \\
&= (2x)(5 - x)^{-1} + (x^2 - 3)(-1)(5 - x)^{-2}(-1)
\end{aligned}$$

Use the unsimplified form to determine the slope of the tangent.

$$\begin{aligned}
f'(2) &= 2(2)(5 - 2)^{-1} + (2^2 - 3)(-1)(5 - 2)^{-2}(-1) \\
&= 4(3)^{-1} + (1)(3)^{-2} \\
&= \frac{4}{3} + \frac{1}{9} \\
&= \frac{13}{9}
\end{aligned}$$

Determine the y-coordinate for the function at $x = 2$ by substituting into the original function.

$$\begin{aligned}
y &= \frac{2^2 - 3}{5 - 2} \\
&= \frac{1}{3}
\end{aligned}$$

The tangent point is $\left(2, \frac{1}{3}\right)$.

Substitute $m = \frac{13}{9}$ and $(x_1, y_1) = \left(2, \frac{1}{3}\right)$ into $y - y_1 = m(x - x_1)$.

$$\begin{aligned}
y - \frac{1}{3} &= \frac{13}{9}(x - 2) \\
y &= \frac{13}{9}(x - 2) + \frac{1}{3} \\
&= \frac{13}{9}x - \frac{26}{9} + \frac{3}{9} \\
&= \frac{13}{9}x - \frac{23}{9}
\end{aligned}$$

The equation of the tangent is $y = \frac{13}{9}x - \frac{23}{9}$.

Example 4 Solve a Rate of Change Problem Involving a Quotient

Recall the problem introduced at the beginning of this section:

Suppose the function $V(t) = \dfrac{50\ 000 + 6t}{1 + 0.4t}$ represents the value, in dollars, of a new car t years after it is purchased.

a) What is the rate of change of the value of the car at 2 years, 5 years, and 7 years?

b) What was the initial value of the car?

c) Explain how the values in part a) can be used to support an argument in favour of purchasing a used car, rather than a new one.

Solution

a) $V(t) = (50\ 000 + 6t)(1 + 0.4t)^{-1}$

$$\begin{aligned}
V'(t) &= (50\ 000 + 6t)\frac{d}{dt}\left[(1 + 0.4t)^{-1}\right] + (1 + 0.4t)^{-1}\frac{d}{dt}\left[50\ 000 + 6t\right] \\
&= (50\ 000 + 6t)[-(1 + 0.4t)^{-2}(0.4)] + (6)(1 + 0.4t)^{-1} \\
&= (1 + 0.4t)^{-2}[-0.4(50\ 000 + 6t) + 6(1 + 0.4t)] \quad \text{Common factor } (1 + 0.4t)^{-2}. \\
&= (1 + 0.4t)^{-2}(-20\ 000 - 2.4t + 6 + 2.4t) \\
&= (1 + 0.4t)^{-2}(-19\ 994) \\
&= \frac{-19\ 994}{(1 + 0.4t)^2}
\end{aligned}$$

When $t = 2$, $V'(2) = \dfrac{-19\ 994}{[1 + 0.4(2)]^2}$
$= -6170.99$

After 2 years, the value of the car is decreasing at a rate of \$6170.99/year.

When $t = 5$, $V'(5) = \dfrac{-19\ 994}{[1 + 0.4(5)]^2}$
$= -2221.56$

After 5 years, the value of the car is decreasing at a rate of \$2221.56/year.

When $t = 7$, $V'(7) = \dfrac{-19\ 994}{[1 + 0.4(7)]^2}$
$= -1384.63$

After 7 years, the value of the car is decreasing at a rate of \$1384.63/year.

b) The initial value of the car occurs when $t = 0$.

$V(0) = 50\ 000$

The initial value of the car was \$50 000.

c) The value of the car decreases at a slower rate as the vehicle ages. This would suggest that it makes less sense to purchase a new vehicle because it depreciates rapidly in the first few years.

KEY CONCEPTS

- To find the derivative of a quotient $q(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$,
  - **i)** Express $q(x)$ as a product.
  - **ii)** Differentiate the resulting expression using the product and chain rules.
  - **iii)** Simplify the result to involve only positive exponents, if appropriate.

Communicate Your Understanding

C1 Describe the similarities and differences between the derivatives of $y = \frac{1}{x^2+1}$ and $y = \frac{x}{x^2+1}$.

C2 Predict a rule that could be used to determine the derivative of quotients of the form $y = \frac{1}{g(x)}$, $g(x) \neq 0$.

C3 "The derivative of $q(x) = \frac{f(x)}{g(x)}$ is $q'(x) = \frac{f'(x)}{g'(x)}$." Is this statement true or false? Use an example to explain your answer.

C4 Why is it important to state that $f(x)$ and $g(x)$ are two differentiable functions, and that $g(x) \neq 0$ when differentiating $q(x) = \frac{f(x)}{g(x)}$?

A Practise

1. Express each quotient as a product, and state the domain of x.

a) $q(x) = \frac{1}{3x+5}$

b) $f(x) = \frac{-2}{x-4}$

c) $g(x) = \frac{6}{7x^2+1}$

d) $r(x) = \frac{-2}{x^3-27}$

2. Differentiate each function in question 1. Do not simplify your answers.

3. Express each quotient as a product, and state the domain of x.

a) $q(x) = \frac{3x}{x+1}$

b) $f(x) = \frac{-x}{2x+3}$

c) $g(x) = \frac{x^2}{5x-4}$

d) $r(x) = \frac{8x^2}{x^2-9}$

4. Differentiate each function in question 3. Do not simplify.

B Connect and Apply

5. Differentiate.

a) $y = \dfrac{-x+3}{2x^2+5}$ **b)** $y = \dfrac{4x+1}{x^3-2}$

c) $y = \dfrac{9x^2-1}{1+3x}$ **d)** $y = \dfrac{x^4}{x^2-x+1}$

6. Determine the slope of the tangent to each curve at the indicated value of x.

a) $y = \dfrac{x^2}{6x+2}$, $x = -2$

b) $y = \dfrac{\sqrt{x}}{3x^2-1}$, $x = 1$

c) $y = \dfrac{4x+1}{x^2-1}$, $x = -3$

d) $y = \dfrac{2x}{x^2-x+1}$, $x = -1$

e) $y = \dfrac{x^3-3}{x^2+x-1}$, $x = 2$

7. a) Describe two different methods that can be used to differentiate $q(x) = \dfrac{-4x^3+5x^2-2x+6}{x^3}$. Differentiate using both methods and explain why you prefer one method over the other.

b) Can both methods that you described in part a) be used to differentiate $q(x) = \dfrac{-4x^3+5x^2-2x+6}{x^3+1}$? Explain.

8. Determine the points on the curve $y = \dfrac{x^2}{x+2}$ where the slope of the tangent line is -3.

9. Determine the equation of the tangent to the curve $y = \left(\dfrac{x^3-1}{x+2}\right)^2$ at the point where $x = -1$.

10. Alison has let her hamster run loose in her living room. The position function of the hamster is $s(t) = \dfrac{5t}{t^2+4}$, $t \geq 0$, where s is in metres, and t is in seconds.

a) How fast is the hamster moving after 1 s?

b) When does the hamster change direction?

11. The number of clients investing in a new mutual fund w weeks after it is introduced into the market is modelled by the function $C(w) = \dfrac{800w^2}{200+w^3}$, where C is the number of clients, and $w \geq 0$.

a) Determine $C'(1)$, $C'(3)$, $C'(5)$, and $C'(8)$. Interpret the meaning of these values for this situation.

b) **Use Technology** Use a graphing calculator to sketch the graph of $C(w)$. Explain how this graph can be used to determine when $C'(w)$ is positive, zero, and negative.

c) **Use Technology** Use a graphing calculator to sketch the graph of $C'(w)$. Use this graph to confirm your answers to part b).

d) Confirm your solution to part b) algebraically.

e) Interpret your results from part b).

12. **Chapter Problem** Mooses, Gooses, and Juices has launched a television and Internet advertising campaign to attract new customers. The predicted number of new customers, N, can be modelled by the function $N(x) = \dfrac{500x^2}{\sqrt{280+x^2}} + 10x$, where x is the number of weeks after the launch of the advertising campaign.

a) Determine the predicted number of new customers 8 weeks after the campaign launch.

b) Determine the predicted average number of new customers per week between 1 and 6 weeks.

c) Determine the rate of change of the predicted number of new customers at week 1 and at week 6.

d) Is the rate of change of new customers ever negative? Explain what this implies.

13. The value of an original painting t years after it is purchased is modelled by the function

$$V(t) = \frac{(2500 + 0.2t)(1 + t)}{\sqrt{0.5t + 2}},$$

where V is in dollars, and $t \geq 0$.

a) What was the purchase price of the painting?

b) Determine the rate of change of the value of the painting after t years.

c) Is the value of the painting increasing or decreasing? Justify your answer.

d) Compare $V'(2)$ and $V'(22)$. Interpret the meaning of these values.

C Extend and Challenge

14. a) Determine a pattern for the nth derivative of $f(x) = \frac{1}{ax + b}$. State the restriction on the denominator.

b) Use your pattern from part a) to determine the fourth derivative of $f(x) = \frac{1}{2x - 3}$.

15. Consider the function $p(x) = \frac{x^2 - 4}{x^2 + 4}$.

a) Determine the points on the graph of $p(x)$ that correspond to $p'(x) = 0$.

b) Determine the points on the graph of $p(x)$ that correspond to $p''(x) = 0$.

c) **Use Technology** Use graphing technology to sketch $p(x)$. What do the points found in parts a) and b) represent on the graph of $p(x)$? Explain.

d) **Use Technology** Use graphing technology to sketch $p''(x)$. What do the points found in part b) represent on the graph of $P'(x)$? Explain.

16. Given $f(x) = \frac{x}{\sqrt{x - 1}}$ and $g(x) = \frac{1}{x} + x$, determine the derivative of each composite function and state its domain.

a) $y = f \circ g(x)$ **b)** $y = g \circ f(x)$

17. **Math Contest** Consider the function $f(x) = \frac{ax + b}{cx + d}$. Depending on the choice of the real constants a, b, c, and d, which of the following are possible for the graph of $y = f(x)$?

i) The graph has no points with horizontal tangents.

ii) The graph has exactly one point with a horizontal tangent.

iii) The graph has infinitely many points with horizontal tangents.

A i) only

B ii) only

C iii) only

D i) and iii) only

E i), ii), and iii)

18. **Math Contest** Let p be a quadratic polynomial such that $p(0) = 0$. Consider the function $F(x) = \frac{p(x)}{x^2 + 1}$. If $F'(0) = 0$ and $F'(1) = 1$, then $p(x)$ equals

A $\frac{2x^2}{3}$

B $\frac{4x^2}{3}$

C $\frac{4x^2}{5}$

D $2x^2$

E $-2x^2$

Extension

The Quotient Rule

In the previous section, you differentiated rational functions by expressing the denominator in terms of a negative exponent and then used the product and chain rules to determine the derivative. Another method that can be used to differentiate functions of the form $q(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$, is the quotient rule.

Investigate What is the quotient rule for derivatives?

A: Develop the Quotient Rule From the Product Rule

Consider the quotient $Q(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are two differentiable functions, and $g(x) \neq 0$.

1. Multiply each side of the above equation by $g(x)$.
2. Differentiate both sides of the equation, using the product rule where needed.
3. Isolate $Q'(x)g(x)$ in the equation in step 2.
4. Substitute $Q(x) = \frac{f(x)}{g(x)}$ in the result of step 3.
5. Express the right side of the result of step 4 in terms of a common denominator.
6. Isolate $Q'(x)$ in the result of step 5.
7. Now look at another way to develop a derivative for $Q(x)$.

   $Q(x) = \frac{f(x)}{g(x)}$ can also be written as $Q(x) = f(x)g^{-1}(x)$.

   Differentiating this expression using the product and chain rules gives $Q'(x) = f(x)(-1)[g(x)]^{-2}\,g'(x) + f'(x)[g(x)]^{-1}$.

   Common factor $g(x)^{-2}$ from this expression. Simplify the resulting expression using only positive exponents.
8. **Reflect** Compare the results of steps 6 and 7.

B: Verify the Quotient Rule

Consider the function $Q(x) = \frac{x^3 - 4x^2}{3x^2 + x}$.

1. Differentiate $Q(x)$ by changing it to a product and differentiating the result, as was done in Section 2.5. Simplify your final answer using positive exponents.
2. Differentiate $Q(x)$ using the result from step 6 of Part A of this Investigate. Simplify your answer.
3. a) **Reflect** Compare your answers in steps 1 and 2. Describe the results.

   b) Repeat steps 1 and 2 for a quotient of your choice. Are the answers the same? Which method do you prefer? Why?

Quotient Rule

If $q(x) = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are differentiable functions and $g'(x) \neq 0$,

then $q'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

Example 1 Apply the Quotient Rule

Differentiate $q(x) = \dfrac{4x^3 - 7}{2x^2 + 3}$.

Solution

Apply the quotient rule.

$$q'(x) = \frac{(2x^2+3)\frac{d}{dx}(4x^3-7) - (4x^3-7)\frac{d}{dx}(2x^2+3)}{(2x^2+3)^2}$$

$$= \frac{(2x^2+3)(12x^2) - (4x^3-7)(4x)}{(2x^2+3)^2}$$

$$= \frac{(24x^4+36x^2) - (16x^4-28x)}{(2x^2+3)^2}$$

$$= \frac{8x^4+36x^2+28x}{(2x^2+3)^2}$$

Example 2 Combine the Chain Rule and the Quotient Rule

Differentiate $g(x) = \dfrac{4x+1}{\sqrt{1-x}}$.

Solution

$$g'(x) = \frac{(1-x)^{\frac{1}{2}}\frac{d}{dx}(4x+1) - (4x+1)\frac{d}{dx}(1-x)^{\frac{1}{2}}}{(1-x)}$$

Apply the quotient rule.

$$= \frac{(1-x)^{\frac{1}{2}}(4) - (4x+1)\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)}{(1-x)}$$

Apply the chain rule.

$$= \frac{\frac{1}{2}(1-x)^{-\frac{1}{2}}[(1-x)(8) + (4x+1)(-1)]}{(1-x)}$$

Common factor.

$$= \frac{8-8x-4x-1}{2(1-x)^{\frac{3}{2}}}$$

$$= \frac{7-12x}{2(1-x)^{\frac{3}{2}}}$$

A Practise

In the first three questions, you will revisit questions 4, 5, and 6 of section 2.5, using the quotient rule to differentiate.

1. Differentiate using the quotient rule.

a) $q(x) = \dfrac{3x}{x+1}$ **b)** $f(x) = \dfrac{-x}{2x+3}$

c) $g(x) = \dfrac{x^2}{5x-4}$ **d)** $r(x) = \dfrac{8x^2}{x^2-9}$

2. Differentiate using the quotient rule.

a) $y = \dfrac{-x+3}{2x^2+5}$ **b)** $y = \dfrac{4x+1}{x^3-2}$

c) $y = \dfrac{9x^2-1}{1+3x}$ **d)** $y = \dfrac{x^4}{x^2-x+1}$

3. Use the quotient rule to determine the slope of the tangent to each curve at the indicated value of x.

a) $y = \dfrac{x^2}{6x+2}$, $x = -2$

b) $y = \dfrac{\sqrt{x}}{3x^2-1}$, $x = 1$

c) $y = \dfrac{4x+1}{x^2-1}$, $x = -3$

d) $y = \dfrac{2x}{x^2-x+1}$, $x = -1$

e) $y = \dfrac{x^3-3}{x^2+x-1}$, $x = 2$

4. a) Given $y = \dfrac{1}{(x^2+3x)^5}$, determine the derivative by using the following two methods.

i) Use the quotient rule.

ii) First express the quotient using a negative exponent and then use the power of a function rule.

b) Which method in part a) is more efficient? Justify your answer.

c) Which method in part a) do you prefer? Explain.

5. Use the chain rule to determine

$\left.\dfrac{dy}{dx}\right|_{x=2}$ for $y = \dfrac{u^3}{u^2+1}$, $u = 3x - x^2$.

6. Determine the points on the curve $y = \dfrac{x^2}{2x+5}$ where the tangent line is horizontal.

7. Given $f(x) = 15x^5 - 9x^3$ and $g(x) = 3x^2$, is the following true? Justify your answer.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \left[\frac{d}{dx}f(x)\right] \div \left[\frac{d}{dx}g(x)\right]$$

8. The number, n, of new FAST cars sold by RACE dealership w weeks after going on the market is represented by the function $n(w) = \dfrac{300w^2}{1+w^2}$, where $0 \le w \le 10$.

a) At what rate is the number of sales changing after 1 week? At what rate are the sales changing after 5 weeks?

b) Does the number of sales per week decrease at any time during this 10-week period? Justify your answer.

9. The concentration of an antibiotic in the blood t hours after it is taken is represented by the function $c(t) = \dfrac{4t}{3t^2+4}$. Determine $c'(3)$ and interpret its meaning for this situation.

10. A chemical cleaner from a factory is accidently spilled into a nearby lake. The concentration of cleaner in the water t days after it is spilled is represented by the function $c(t) = \dfrac{6t}{2t^2+9}$, where c is in grams per litre.

a) At what rate is the concentration of the chemical cleaner changing after 1 day, 4 days, and 1 week?

b) When is the rate of change of concentration zero? When is it positive? When is it negative?

c) **Use Technology** Confirm your answers to part b) using the graphs of $c(t)$ and $c'(t)$.

d) Interpret the meaning of your answers to part b) for this situation.

e) Determine $c''(4)$ and interpret its meaning for this situation.