Chapter 2

Derivatives

In Chapter 1, you learned that instantaneous rate of change is represented by the slope of the tangent at a point on a curve. You also learned that you can determine this value by taking the derivative of the function using the first principles definition of the derivative. However, mathematicians have derived a set of rules for calculating derivatives that make this process more efficient. You will learn to use these rules to quickly determine instantaneous rate of change.

By the end of this chapter you will

- verify the power rule for functions of the form $f(x) = x^n$, where n is a natural number
- verify the constant, constant multiple, sum, and difference rules graphically and numerically, and read and interpret proofs involving $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ of the constant, constant, power, and product rules
- determine algebraically the derivatives of polynomial functions, and use these derivatives to determine the instantaneous rate of change at a point and to determine point(s) at which a given rate of change occurs
- verify that the power rule applies to functions of the form $f(x) = x^n$, where n is a rational number, and verify algebraically the chain rule using monomial functions and the product rule using polynomial functions
- solve problems, using the product and chain rules, involving the derivatives of polynomial functions, rational functions, radical functions, and other simple combinations of functions
- make connections between the concept of motion and the concept of the derivative in a variety of ways
- make connections between the graphical or algebraic representations of derivatives and real-world applications
- solve problems, using the derivative, that involve instantaneous rate of change, including problems arising from real-world applications, given the equation of a function

Prerequisite Skills

Identifying Types of Functions

1. Identify the type of function (polynomial, rational, logarithmic, etc.) represented by each of the following. Justify your response.

   a) $f(x) = 5x^3 + 2x - 4$

   b) $y = \sin x$

   c) $g(x) = -2x^2 + 7x + 1$

   d) $f(x) = \sqrt{x}$

   e) $h(x) = 5^x$

   f) $q(x) = \dfrac{x^2+1}{3x-2}$

   g) $y = \log_3 x$

   h) $y = (4x+5)(x^2-2)$

Determining Slopes of Perpendicular Lines

2. For each function, state the slope of a line that is perpendicular to it.

   a) $y = 2x + 9$ b) $y = -5x - 3$

   c) $\dfrac{2}{3}x - y + 3 = 9$ d) $y = 26$

   e) $y = x$ f) $x = -3$

Using the Exponent Laws

3. Express each radical as a power.

   a) $\sqrt{x}$ b) $\sqrt[3]{x}$ c) $\left(\sqrt[4]{x}\right)^3$ d) $\sqrt[5]{x^2}$

4. Express each term as a power with a negative exponent.

   a) $\dfrac{1}{x}$ b) $-\dfrac{2}{x^4}$ c) $\dfrac{1}{\sqrt{x}}$ d) $\dfrac{1}{\left(\sqrt[3]{x}\right)^2}$

5. Express each quotient as a product by using negative exponents.

   a) $\dfrac{x^3-1}{5x+2}$ b) $\dfrac{3x^4}{\sqrt{5x+6}}$

   c) $\dfrac{(9-x^2)^3}{(2x+1)^4}$ d) $\dfrac{(x+3)^2}{\sqrt[3]{1-7x^2}}$

Simplify Expressions with Negative Exponents

6. Simplify. Express answers using positive exponents.

   a) $(x^2)^{-3}$ b) $\dfrac{2x^3 - x^2 + 3x}{x^3}$

   c) $\dfrac{x^5}{x^8}$ d) $x^{-\frac{1}{2}}(x-1)$

   e) $\dfrac{c^6}{c^{-3}}$ f) $(x^2+3)^{-\frac{3}{2}}(4x-3)^2$

Analysing Polynomial Graphs

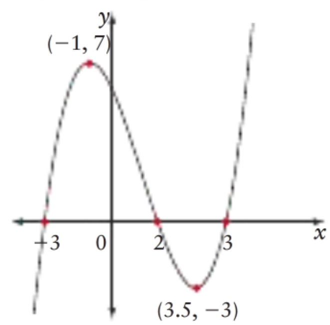
7. Maximum and minimum points and x-intercepts are indicated on each graph. Determine the intervals, or values of x, over which

   i) the function is increasing and decreasing

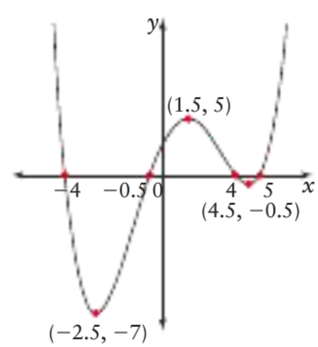
   ii) the function is positive and negative

   iii) the curve has zero slope, positive slope, and negative slope

   a)



   b)



Solving Equations

8. Solve.

a) $x^2 - 8x + 12 = 0$ **b)** $4x^2 - 16x - 84 = 0$

c) $5x^2 - 14x + 8 = 0$ **d)** $6x^2 - 5x - 6 = 0$

e) $x^2 + 5x - 4 = 0$ **f)** $2x^2 + 13x - 6 = 0$

g) $4x^2 = 9x - 3$ **h)** $-x^2 + 7x = 1$

Factoring Polynomials

9. Solve using the factor theorem.

a) $x^3 + 3x^2 - 6x - 8 = 0$

b) $2x^3 - x^2 - 5x - 2 = 0$

c) $3x^3 + 4x^2 - 35x - 12 = 0$

d) $5x^3 + 11x^2 - 13x - 3 = 0$

e) $3x^3 + 2x^2 - 7x + 2 = 0$

f) $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$

Simplify Expressions

10. Expand and simplify.

a) $(x^2 + 4)(5) + 2x(5x - 7)$

b) $(9 - 5x^3)(14x) + (-20x^3)(7x^2 + 2)$

c) $(3x^4 - 6x)(6x^2 + 5) + (12x^3 - 6)(2x^3 + 5x)$

11. Factor first and then simplify.

a) $8(x^3 - 1)^5(2x + 7)^3 + 15x^2(x^3 - 1)^4(2x + 7)^4$

b) $6(x^3 + 4)^{-1} - 3x^2(6x - 5)(x^3 + 4)^{-2}$

c) $2x^{\frac{7}{2}} - 2x^{\frac{1}{2}}$

d) $1 + 2x^{-1} + x^{-2}$

12. Determine the value of y when $x = 4$.

a) $y = 6u^2 - 1$, $u = \sqrt{x}$

b) $y = \dfrac{5}{u^3}$, $u = 9 - 2x$

c) $y = -u^2 + 3u + 1$, $u = 5x - 18$

Creating Composite Functions

13. Given $f(x) = x^3 + 1$, $g(x) = \dfrac{1}{x - 2}$, and $h(x) = \sqrt{1 - x^2}$, determine

a) $f \circ g(x)$ **b)** $g \circ h(x)$

c) $h[f(x)]$ **d)** $g[f(x)]$

14. Express each function $h(x)$ as a composition of two simpler functions $f(x)$ and $g(x)$.

a) $h(x) = (2x - 3)^2$ **b)** $h(x) = \sqrt{2 + 4x}$

c) $h(x) = \dfrac{1}{3x^2 - 7x}$ **d)** $h(x) = \dfrac{1}{(x^3 - 4)^2}$

CHAPTER PROBLEM

Five friends in Ottawa have decided to start a fresh juice company with a Canadian flavour. They call their new enterprise Mooses, Gooses, and Juices. The company specializes in making and selling a variety of fresh fruit drinks, smoothies, frozen fruit yogurt, and other fruit snacks. The increased demand for these healthy products has had a positive influence on sales, and business is expanding. How can the young entrepreneurs use derivatives to analyse their costs, revenues, profits, and employee productivity, thereby increasing their chance for success?