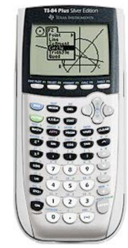
CLASS: MCV4U



LESSON: Introduction to Rates of Change and Limits

Chapter 1 – Prerequisite Skills

1. Slope

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Equation of a line: $y = mx + b$ or $ax + by + c = 0$

Don't forget slope-point form: $y - y_1 = m(x - x_1)$

$y = m(x - x_1) + y_1$

(x_1, y_1) is given

2. Function Notation

$y = mx + b$ can be written in function notation as $f(x) = mx + b$

Example: $f(x) = \begin{cases} x + 2 & \text{if } x \le 3 \\ 2x + 4 & \text{if } x > 3 \end{cases}$ Find f(-3) and f(5).

$f(-3) = -3 + 2 = -1$

$f(5) = 2(5) + 4 = 14$

3. Rationalizing the Denominator

$\sqrt{a} \cdot \sqrt{a} = a$

$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$

$$\frac{\sqrt{3}}{\sqrt{3}} \cdot \frac{6 + \sqrt{2}}{\sqrt{3}} = \frac{6\sqrt{3} + \sqrt{6}}{3}$$

$$\frac{6\sqrt{3} + \sqrt{6}}{3} =$$

$$\frac{\sqrt{3} + 2}{\sqrt{3} + 2} \cdot \frac{2\sqrt{3}}{\sqrt{3} - 2} = \frac{2(3) + 4\sqrt{3}}{3 - 4} = \frac{6 + 4\sqrt{3}}{-1} = -\frac{(6 + 4\sqrt{3})}{1} = -6 - 4\sqrt{3}$$

Conjugate

$\sqrt{3} + 2$

$(a + b)(a - b) = a^2 - b^2$

4. Rationalizing the Numerator

$\frac{\sqrt{3}}{\sqrt{3}}$ $\frac{\sqrt{3}}{6+\sqrt{2}} = \frac{3}{6\sqrt{3}+\sqrt{6}}$

5. Domain

Think of the following:

Restrictions – values of x that would make the denominator equal to zero
Square root signs – the number underneath the square root sign must be greater than or equal to zero.

a) $y = x^2$

No Restriction

$\therefore D: \{x \in R\}$

b) $y = \frac{x^2 + y}{(x-2)(x+3)}$

V.A @ $x=2$
$x=-3$

Restrictions
$x \neq 2, -3$

$\therefore D: \{x \in R \mid x \neq -3, 2\}$

c) $y = \sqrt{x-2}$

$x - 2 \geq 0$

$x \geq 2$

Restriction
$x \geq 2$

$\therefore D: \{x \in R \mid x \geq 2\}$

6. Completing the Square

Examples:

$f(x) = x^2 + 8x + 3$

$\frac{b}{2} = \frac{8}{2} = 4$

$\left(\frac{b}{2}\right)^2 = 4^2 = 16$

$= x^2 + 8x + 16 - 16 + 3$

$= (x+4)^2 - 16 + 3$

$\boxed{f(x) = (x+4)^2 - 13}$

$f(x) = 3x^2 + 12x - 2$

$= 3(x^2 + 4x) - 2$

$= 3[(x^2 + 4x + 4) - 4] - 2$

$= 3[(x+2)^2 - 4] - 2$

$= 3(x+2)^2 - 12 - 2$

$\boxed{f(x) = 3(x+2)^2 - 14}$

$\frac{b}{2}$

$\frac{4}{2} = 2$

$\left(\frac{b}{2}\right)^2$

$\left(\frac{4}{2}\right)^2 = 4$

Calculus is the study of change and therefore of motion. Sir Issac Newton is accredited for the invention of calculus in the seventeenth century, yet many mathematicians have since contributed to this branch of mathematics resulting in a very powerful present day mathematical tool.

Linear Functions:

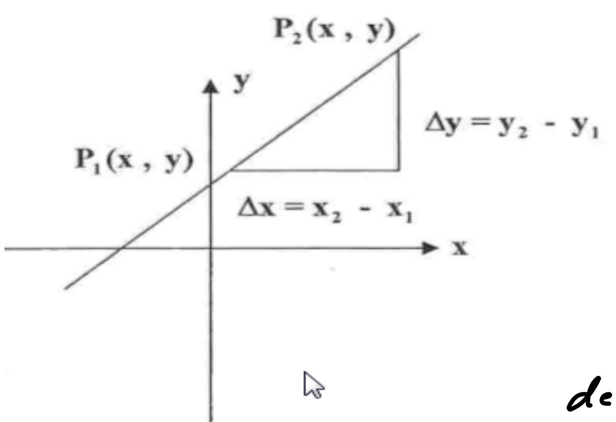
f(x)=mx + b (slope y-intercept form) or Ax + By + C = 0 (standard form)

$m = slope$

$b = y - \text{intercept}$

$m = \frac{-A}{B}$

$b = \frac{-C}{B}$



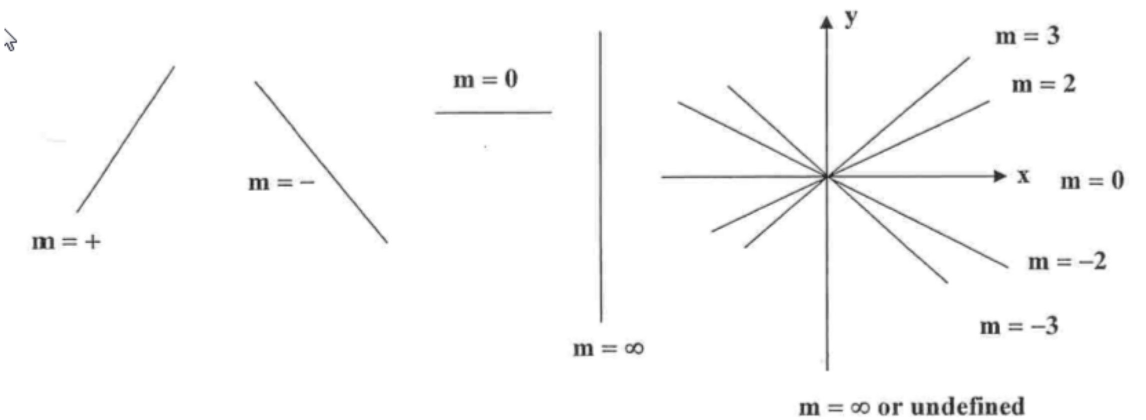
$$slope = m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

dependent variable

Therefore slope is the rate of change of y with respect to x.

Independent variable

Review of slope:



Recall parallel lines have equal slopes while perpendicular lines have slopes that are the negative reciprocals of each other.

The Slope Point Formula is used to find the equation of a line. We need to know 1) the lines slope and 2) a point on the line in order to find its equation.

The slope point formula is $y_2 - y_1 = m(x_2 - x_1)$

$y_2 = m(x_2 - x_1) + y_1$

Example:

a) Find the slope of the segment joining A(3,5) to B(-4,-7)

x_1 y_1 x_2 y_2

$m = \frac{-7-5}{-4-3}$

b) Find the equation

$m_{AB} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} =$

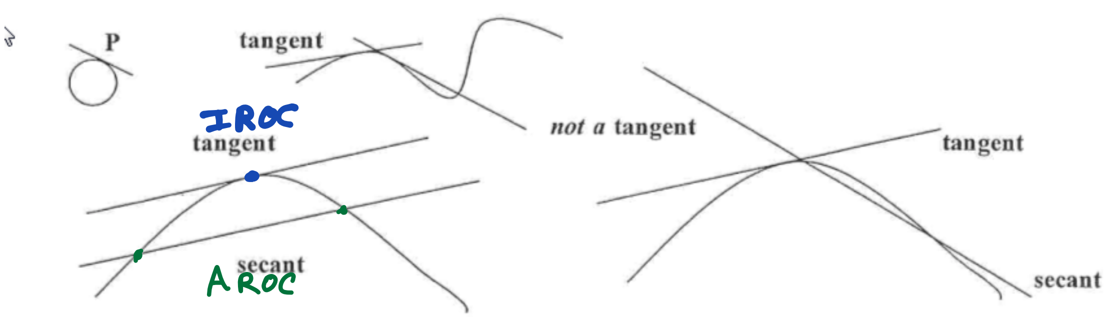
$y = \frac{12}{7}(x-3)+5$

$y = \frac{12}{7}x - \frac{36}{7} + \frac{35}{7}$

$y = \frac{12}{7}x - \frac{1}{7}$

a) $m = \frac{12}{7}$

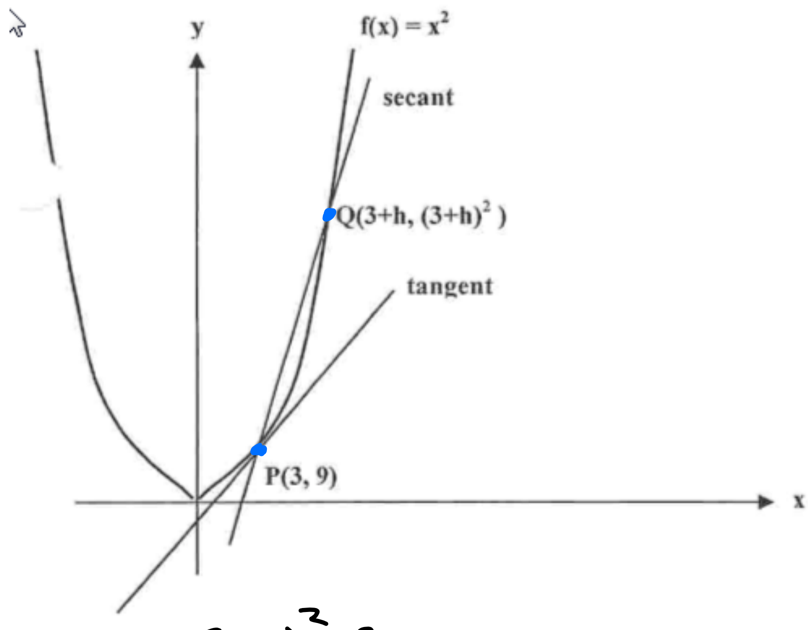
A tangent is a line that intersects a curve once and only once. It touches the curve but does not cross it. It may become a secant if it is extended such that it crosses the curve a second time.



Example1: Find the slope of the secant line PQ through the points P(3,9) and Q(3+h,$(3+h)^2$)

x_1 y_1

x_2 y_2



$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{(3+h)^2 - 9}{3+h-3}$$

$$= \frac{9+6h+h^2-9}{h}$$

$$= h(h+6)$$

$$= \frac{\cancel{h}(h+6)}{\cancel{h}}$$

$$= h+6$$

Example 2: Simplify:

$(d = 7(7+h)$

$$\frac{\frac{7}{7}\cdot\frac{5}{7+h} - \frac{5}{7}\cdot\frac{(7+h)}{(7+h)}}{h} = \frac{\frac{5(7)-5(7+h)}{7(7+h)}}{h}$$

Don't Expand the denominator

$$= \frac{\frac{35-35-5h}{7(7+h)}}{h}$$

$$= \frac{\frac{-5h}{7(7+h)}}{h}$$

$$= \frac{-5\cancel{h}}{7(7+h)} \times \frac{1}{\cancel{h}}$$

$$= \frac{-5}{7(7+h)}$$

$$\frac{\frac{1}{2}}{3} \qquad \frac{1}{2} \div \frac{3}{1} = \frac{1}{2} \times \frac{1}{3} =$$

Example 3: Rationalize the numerator:

$$\frac{\sqrt{25+h}-5}{h} = \frac{\cancel{25}+h-\cancel{25}}{h\left(\sqrt{25+h}+5\right)} = \frac{\cancel{h}}{\cancel{h}\left(\sqrt{25+h}+5\right)}$$

$$= \frac{1}{\left(\sqrt{25+h}+5\right)}$$

$$\left(\sqrt{25+h}+5\right)\left(\sqrt{25+h}-5\right)$$

$$= 25+h - 5\sqrt{25+h} + 5 + 5\sqrt{25+h} - 5 - 25$$

$$= h$$

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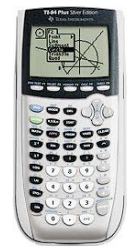

LESSON: The Tangent Investigation

What is calculus?

Two simple geometric problems originally led to the development of what is now called calculus. Both problems can be stated in terms of the graph of a function $y = f(x)$.

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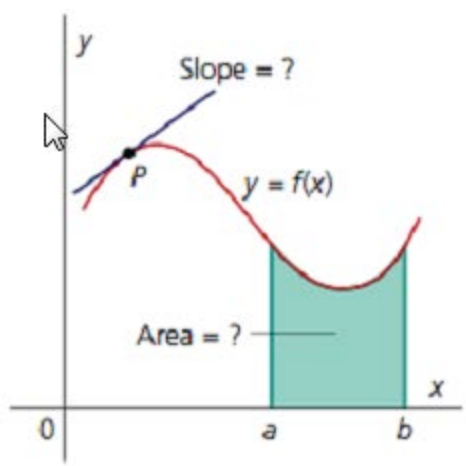
- **The problem of tangents:** What is the value of the slope of the tangent to the graph of a function at a given point ***P?***
- **The problem of areas:** What is the area under a graph of a function $y = f(x)$ between $x = a$ and $x = b$?

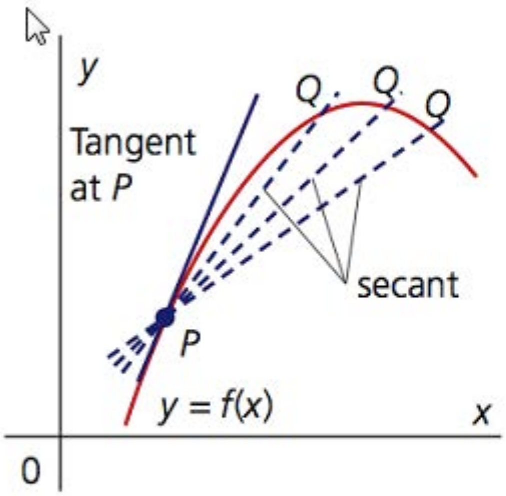


Calculus relies heavily upon two techniques: **differentiation and integration.** Out first step on the calculus journey is to investigate the role of the slope of the tangent at a given point on a curve.

Recall the equation of a line from the prerequisite skills lesson: $y = mx + b$. To find the slope of a tangent line, we could use the definition of slope as given below:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Unfortunately, we only have one point – the one on the line. Instead, we use a **secant line**. Imagine another point on the curve, labeled as Q. The line that joins P and Q is a secant line. As point Q slides back along the curve towards P, the secant line becomes a better approximation of the slope of the tangent line. In mathematical terms, as Q approaches P, the slope of the tangent is said to be the **limit** of the slopes of the secant.



To better understand secant lines and limits, complete the investigation below.

1. Find the y-coordinates of the following points that lie on the graph of the parabola $y = x^2$.

a. Q_1(3.5,) b. Q_2(3.1,) c. Q_3(3.01,) d. Q_4(3.001,)

2. Find the slopes of the secants through P(3,9) and each of the points Q_1, Q_2, Q_3, and Q_4.

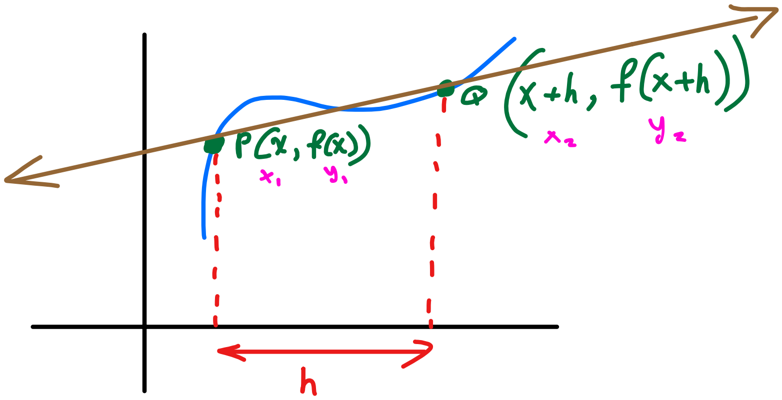
3. Find the y-coordinates of each point on the parabola and then repeat step 2 using the points.

a. Q_5(2.5,) b. Q_6(2.9,) c. Q_7(2.99,) d. Q_8(2.999,)

4. Use the results from steps 2 and 3 to estimate the slope of the tangent at point P(3,9).

5. Graph $y = x^2$ and the tangent to the graph at point P(3,9).

The Slope at a tangent at an arbitrary point.



IROC

$$m_{tangent} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

First Principles Definition of the Derivative

IROC

AROC

$$m_{secant} = \frac{f(x+h) - f(x)}{x+h - x}$$

$$m_{secant} = \frac{f(x+h)-f(x)}{h}$$

Example 1: Find the slope of the tangent to the graph of the parabola $f(x)=x^2$ at P(3,9).

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{9}+6h+h^2-\cancel{9}}{h}$$

$$= \lim_{h \to 0} \frac{6h+h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (6+h)$$

$h \to 0$

$= 6+0$

$m = 6$

Side

$f(x+h) = f(3+h)$

$= (3+h)^2$

$= 9+6h+h^2$

$f(x) = f(3)$

$= 9$

$\therefore$ The slope of the tangent of $f(x)$ at $x=3$ is 6.

Example 2: Determine the slope of the tangent to the rational function $f(x)=\frac{3x+6}{x}$ at point (2,6).

$$m=\lim_{h\to 0}\frac{\frac{12+3h}{2+h}-6}{h}$$

$$=\lim_{h\to 0}\frac{\frac{12+3h-12-6h}{2+h}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{-3h}{2+h}}{h}$$

$$=\lim_{h\to 0}\frac{-3\cancel{h}}{\cancel{h}(2+h)}$$

$$\lim_{h\to 0}-\frac{3}{2+h}$$

$h\to 0$

$m=-\frac{3}{2}$

Side

$$f(2+h)=\frac{3(2+h)+6}{2+h}$$

$$=\frac{6+3h+6}{2+h}$$

$$=\frac{12+3h}{2+h}$$

$f(2)=6$

$\therefore$ the slope of the tangent at $x=2$ is $-\frac{3}{2}$

Example 3: Find the slope of the tangent to the curve at the given point.

$f(x)=\frac{1}{\sqrt{x-1}}$ at (2,1)

$$m=\lim_{h\to 0}\frac{\frac{1}{\sqrt{1+h}}-1}{h}$$

conjugate

$$=\lim_{h\to 0}\frac{\frac{1-\sqrt{1+h}}{\sqrt{1+h}}}{h}\quad\frac{1+\sqrt{1+h}}{1+\sqrt{1+h}}$$

Be careful !

$$=\lim_{h\to 0}\frac{\frac{1-(1+h)}{\sqrt{1+h}\,(1+\sqrt{1+h})}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1-1-h}{\sqrt{1+h}\,(1+\sqrt{1+h})}}{h}$$

$$=\lim_{h\to 0}\frac{-\cancel{h}}{\sqrt{1+h}\,(1+\sqrt{1+h})}\times\frac{1}{\cancel{h}}$$

Side

$$f(2+h)=\frac{1}{\sqrt{(2+h)-1}}$$

$$=\frac{1}{\sqrt{1+h}}$$

$$f(2)=\frac{1}{\sqrt{2-1}}$$

$$=1$$

$h\to 0$

$$m=-\frac{1}{\sqrt{1}\,(1+\sqrt{1})}$$

$$m=-\frac{1}{2}$$

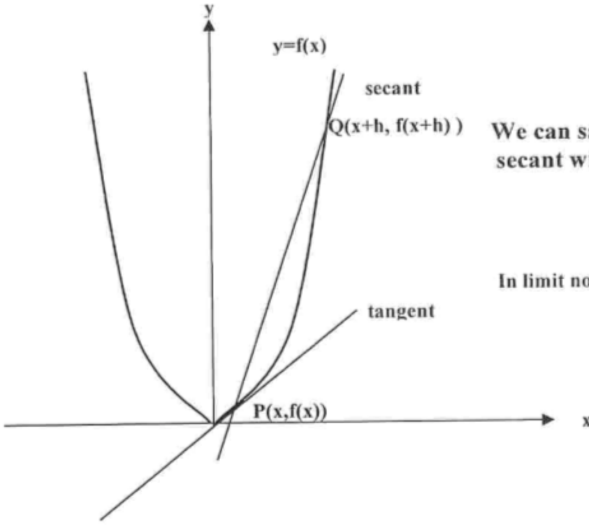
$\therefore$ The slope of the tangent at $x=2$ is $-\frac{1}{2}$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{1+h}\,(1+\sqrt{1+h})}$$

Calculating the Slope and Equation of the Tangent Line:
Recall from yesterday we can generalize;

$$m_{secant} = \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{x+h-x}$$

$$= \frac{f(x+h)-f(x)}{h}$$



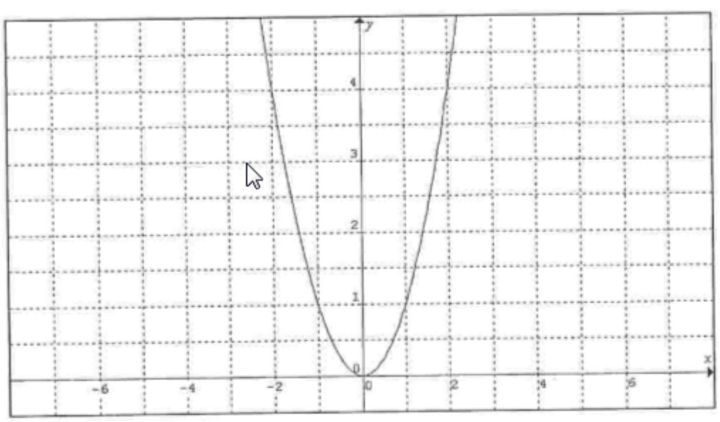
We can say that as Q approaches P the slope of the secant will approach the slope of the tangent line.

In limit notation $m_{tangent} = \lim_{Q\to P} m_{PQ}$

or

$$m_{tangent} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Example 1: Find the equation of a tangent line to the parabola $y = x^2$ at the point P(1,1).



Need m = ?
To find m → choose another point (Q)
So that $Q \neq P,\ x \neq 1$
QP is called the secant line

$$m_{PQ} = \frac{y-1}{x-1} = \frac{x^2-1}{x-1}$$

Let's say I chose: Q(3,9)

$$\therefore m_{PQ} = \frac{9-1}{3-1} = \frac{8}{2} = 4$$

Choosing Q(2 , 4),

$$\therefore m_{PQ} = \frac{4-1}{2-1} = \frac{3}{1} = 3$$

Choosing Q(1.5 , 2.25)

$$\therefore m_{PQ} = \frac{9-1}{3-1} = \frac{8}{2} = 4$$

approaching 1 from the right

$x > 1$	$m_{PQ} = \frac{x^2-1}{x-1}$
3	4
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

m is getting closer and closer to 2

approaching 1 from the left

$x < 1$	$m_{PQ} = \frac{x^2-1}{x-1}$
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

m is getting closer and closer to 2

This suggests the slope of the tangent line at point P(1,1) is 2.

Therefore the equation of tangent is:

$$m = \frac{\Delta y}{\Delta x}$$
$$2 = \frac{y-1}{x-1}$$
$$2x - 2 = y - 1$$
$$2x - y - 1 = 0$$

Thus we say the slope of the tangent line is the limit of the slopes of the secant lines. In mathematical symbols this is written.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$ **For our case**

$$\lim_{Q \to P} m_{PQ} = m_{\text{tangent}}$$ **For the general case or**

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = m_{\text{tangent}}$$ **Is called the first principles definition of the slope of the tangent.**

Examples: Find the slopes of the tangent to:

a) $y = -2x^2 + 7$ at point (3,-11)

Side

$$f(3+h) = -2(3+h)^2 + 7$$
$$-2(h^2 + 6h + 9) + 7$$
$$= -2h^2 - 12h - 18 + 7$$
$$= -2h^2 - 12h - 11$$

$$f(3) = -11$$

$$m = \lim_{h \to 0} \frac{-2h^2 - 12h - 11 + 11}{h}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}(h+6)}{\cancel{h}}$$

$$= \lim_{h \to 0} -2(h+6)$$

$h \to 0$

$$= -2(6)$$
$$= -12$$

∴ The slope of the tangent x = 3 is −12

Side

$$= \sqrt{2(h+1)-1}$$

$$= \sqrt{2h+2-1}$$

$$= \sqrt{2h+1}$$

$$y(1) = 1$$

b) $y = \sqrt{2x-1}$ at point (1,1)

$$m = \lim_{h \to 0} \frac{\sqrt{2h+1} - 1}{h}$$

$$= \lim_{h \to 0} \frac{2h+1-1}{h(\sqrt{2h+1}+1)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2h+1}+1}$$

$$= \lim_{h \to 0} \frac{2}{2}$$

$$= \lim_{h \to 0} 1$$

∴ the slope of the tangent at $x = 1$ is 1

c) $y = \dfrac{6}{x}$ at point (2,3) and state the equation of the tangent line.

$$f(2+h) = \frac{6}{2+h}$$

$$f(2) = 3$$

$$m = \lim_{h \to 0} \frac{\frac{6}{2+h} - 3}{h}$$

$$= \lim_{h \to 0} \frac{\frac{6-6-3h}{2+h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-3h}{2+h}}{h}$$

$$= \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h}(2+h)}$$

$h \to 0$

The slope of the tangent $x=2$ is $-\frac{3}{2}$

$$y = -\frac{3}{2}(x-2)+3$$

$$y = -\frac{3}{2}x+6$$

$= \lim_{h\to 0} \frac{\ }{2+h}$ ∴ The equation ... is $y = -\frac{3}{2}x + 6$.

$= \lim_{h\to 0} -\frac{3}{2}$

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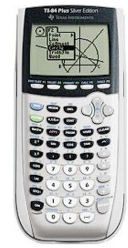


LESSON: Rates of Change

In calculus, we are often interested in how rapidly the dependent variable, y, changes when there is a change in the independent variable, x. This concept is called rate of change.

Examining the concept: Average Rate of Change

The table below shows the results of recording a student's temperature every 5s.

Time (s)	Temperature (˚F)	Average Rate of change
0	86.75	X
5	89.33	0.516 ˚F/s
10	91.54	0.442 ˚F/s
15	92.67	0.226 ˚F/s
20	93.54	0.174 ˚F/s
25	94.01	0.094˚ ˚F/s
30	94.49	$\frac{\Delta T}{\Delta t} = \frac{94.49-94.01}{30-25} = 0.096$ ˚F/s

What does the table tell us about the average rates of change?

The AROC of temperature decreases as time increases.

Average Rate of Change

The average rate of change of y = f(x) with respect to x over the interval from x_1 to x_2 is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

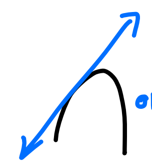
Average velocity $= \dfrac{\text{change in position}}{\text{change in time}} = \dfrac{\Delta s}{\Delta t}$

Prime

IROC

Derivative

$v(t) = s'(t)$



Example: The height of a model rocket in flight can be modeled by $h(t) = -4.9t^2 + 25t + 2$, where h is the height in metres at t seconds. Determine the average rate of change of the model rocket's height during (a) the 1st second (b) the 2nd second.

a) AROC $0 \le t \le 1$

$$= \frac{h(1) - h(0)}{1 - 0}$$

$$= \frac{-4.9(1)^2 + 25(1) + 2 - (-4.9(0)^2 + 25(0) + 2)}{1}$$

$$= \frac{22.1 - 2}{1}$$

$$= 20.1 \, m/s$$

b) AROC $1 \le t \le 2$

$$= \frac{h(2) - h(1)}{2 - 1}$$

$\vdots$

$= 10.3 \, m/s$

Examining the concept: Instantaneous Rate of Change

Using the example we just completed, we will now examine what happens to the average rate of change during the second interval, from 1s to 2s.

$h(t) = -4.9t^2 + 25t + 2$

Interval	Δh	Δt	Average Rate of change
$1 \le t \le 2$			
$1 \le t \le 1.5$			
$1 \le t \le 1.1$			
$1 \le t \le 1.01$			
$1 \le t \le 1.001$			

What happens to the average rate of change as the time interval decreases from $1 \le t \le 2$ to $1 \le t \le 1.001$?

Instantaneous Rate of Change

The **instantaneous rate of change** for any function y = f(x) is the limiting value of the sequence of the average rates of change as the interval between the x-coordinates of points (x_1, y_1) and (x_2, y_2) continuously decreases to 0. An instantaneous rate of change occurs at a single point.

$$\text{Instantaneous rate of change} = \lim_{x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{x_2 \to x_1} \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 2: The height, in metres, of a toy rocket launched at an initial upward velocity of 30m/s, from a height of 1m, is approximately given by $s(t) = -4.9t^2 + 30t + 1$, where t is measured in seconds. Find the instantaneous velocity of the rocket after 4s.

Solution: Use the formula for instantaneous velocity with t=4.

$$v(t) = \lim_{h \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \leftarrow \text{first principle formula}$$

$$s(4+h) = -4.9(4+h)^2 + 30(4+h) + 1$$
$$= -4.9h^2 - 9.2h + 42.6$$

$$s(4) = -4.9(4)^2 + 30(4) + 1$$
$$= 42.6$$

$$v(4) = \lim_{h \to 0} \frac{-4.9h^2 - 9.2h + \cancel{42.6} - \cancel{42.6}}{h}$$

$$= \lim_{h \to 0} \frac{-4.9h^2 - 9.2h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-4.9h - 9.2)}{\cancel{h}}$$

$h \longrightarrow 0$

$v(4) = -9.2$

$\therefore$ The velocity at 4s is 9.2m/s [$\downarrow$]

For any $y = f(x)$ the slope of the tangent at a point on f(x) gives the rate of change of y with respect to x at that point.

$$y' = f'(x) = m_{\text{tangent}} = \lim_{Q \to P} m_{PQ}$$

$$= m_{\text{tangent}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \textbf{the rate of change of y with respect to x}$$

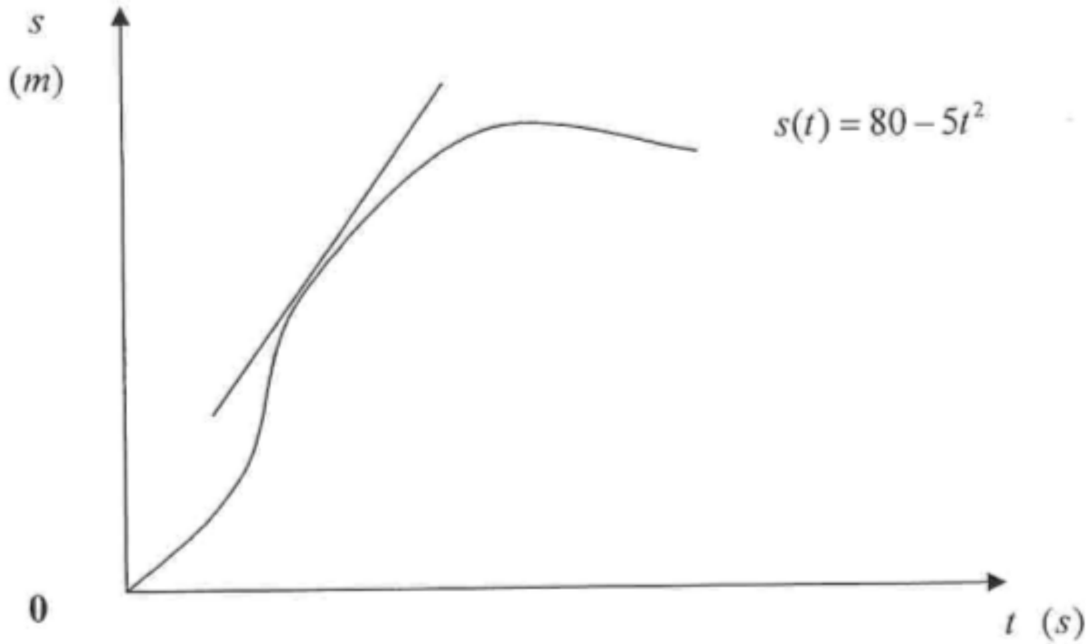
As the graph below describes, the function $s = f(t)$ says that the position s of a particle is a function of time t. The average velocity of a particle between 2 positions at 2 different times is: Average Velocity $= \dfrac{\Delta s}{\Delta t}$

Find the average velocity for s(t) between t=1 and t=3 seconds.

Prime

As with y' above $s' = f'(t)$
= the slope of the tangent at a point on s(t)
= the rate of change of position s with respect to time t at that point.
= velocity at that point in time

Note: Velocity of an object at one point in time is the slope of the s vs. t graph at that point in time. This is called the instantaneous velocity. (ie velocity at that instant)



$\mathbf{v = s' = s'(t)} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$ = **the rate of change of position with respect to time.**

Velocity is a vector quantity meaning it has a direction. This is obvious as the rate of change of position over time.

Example: A ball is tossed upwards so its position, s in metres, m and time, t in seconds is $s(t) = -5t^2 + 30t + 2$.

a) At what height was the ball tossed from?

$s(0) = -5(0)^2 + 30(0) + 2$

$= 2$ ∴ 2 meters

b) What is the velocity of the ball at t=4s?

$\mathbf{v = s'(t)} = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$

$v(4+h) = -5(4+h)^2 + 30(4+h) + 2$

$= -5h^2 - 10h + 42$

$v(4) = \lim_{t \to 0} \frac{-5h^2 - 10h + \cancel{42} - \cancel{42}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(-5h - 10)}{\cancel{h}}$

$= \lim_{h \to 0} \frac{-5h - 10}{1}$

$h \to 0$

∴ the velocity

$v(4) = -5(4)^2 + 30(4) + 2$

$= 42$

$= -10$ at $t=4$, is $10\,m/s\,[\downarrow]$

Example: The total cost in dollars of manufacturing x units of a product is given by: $C(x) = 10\sqrt{x} + 1000$

a) What is the cost of manufacturing 64 items?

$C(64) = 1080$

b) What is the rate of change of the total cost with respect to the number of units, x, being produced when x=64?

$(64, 1080)$

$C'(64) = \frac{5}{8}$

$C(h+64) = 10\sqrt{h+64} + 1000$

$C(64) = 1080$

$$v(64) = \lim_{h\to 0} \frac{10\sqrt{h+64} + 1000 - 1080}{h(10\sqrt{h+64} + 1000}$$

$$= \lim_{h\to 0} \frac{10\sqrt{h+64} - 80}{h} , \frac{10\sqrt{h+64} + 80}{}$$

$$= \lim_{h\to 0} \frac{10h + 640 - 640}{h(10\sqrt{h+64} + 80)}$$

$$= \lim_{h\to 0} \frac{10\cancel{h}}{\cancel{h}(10\sqrt{h+64} + 80)}$$

$$= \lim_{h\to 0} \frac{10}{160} = \frac{1}{16}$$

Recall that:

The **average** rate of change is given by $\frac{\Delta y}{\Delta x}$ *or* $\frac{f(b)-f(a)}{b-a}$ and represents the slope of the **secant** to the graph between 2 points.

The **instantaneous** rate of change is given by $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ and represents the slope of the tangent to the graph at a specific point.

Economics Notation:

$C(x)$ - is the **Cost** function where ***x*** represents the number of items produced.

$R(x)$ - is the **Revenue** function where **Revenue = (number of items sold)** $\times$ **(price per item)**

$P(x)$ - is the **Profit** function where **Profit = Revenue - Cost**

As a result:

average rate of change of cost $= \frac{\Delta C}{\Delta x}$

instantaneous rate of change of cost $= \lim_{h \to 0} \frac{C(x+h)-C(x)}{h} = C'(x)$

[Marginal Cost]

Furthermore, the instantaneous rate of change of cost, $C'(x)$, is more commonly referred to as **marginal cost**. Marginal cost denotes the **increase in total cost incurred by the production of one added unit**.

Important note: Using the derivative, $C'(x)$, provides really only an estimate of the marginal cost of the *x*th item. To get the true value, you could calculate the exact value using $C(x+1)-C(x)$, but this is often more tedious and complicated then just using the derivative. In any case, when you're talking about thousands or millions of items, using the slope of the tangent will give you a very good estimate of the marginal value.

e

1. An editor wants to publish a large quantity of art books . She estimates that the total cost of production in dollars of producing ***x*** copies is given by: $C(x) = 2400 + 30x - 0.1x^2$

   a) If $C(x)$ is graphed, would the points be joined? Explain. Can't graph because it will pr[illegible]
   b) Explain why $C(0) \neq 0$. y-intercept
   c) Calculate the cost of producing 100 books.

d) Calculate the average cost per book from the 50th to the 60th book.
e) Calculate the cost of producing the 101st book.
f) Find the marginal cost of producing 100 books.

Solutions:

1. $C(x) = 2400 + 30x - 0.1x^2$

a) If $C(x)$ is graphed, would the points be joined? Explain.
No since x must be whole numbers.

b) Explain why $C(0) \neq 0$.
Fixed costs not related to number of units of production (i.e. rent, utilities, etc.)

c) Calculate the cost of producing 100 books.
$C(100) = 2400 + 30(100) - 0.1(100)^2 = \$4400.00$

d) Calculate the average cost per book from the 50th to the 60th book.

$$\frac{\Delta C}{\Delta x} = \frac{C(60) - C(50)}{60 - 50} = \frac{\$3840 - 3650}{10} = \$19/\text{book}$$

e) Calculate the cost of producing the 101st book.

C(101) – C(100) = \$4409.90 - \$4400.00 = \$9.90. Thus it costs \$9.90 to produce the 101st book.

g) Find the marginal cost of producing 100 books.

$$C'(100) = \lim_{h \to 0}\left[\frac{C(100+h) - C(100)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{2400 + 30(100+h) - 0.1(100+h)^2 - 4400}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{2400 + 3000 + 30h - 0.1(10000 + 200h + h^2) - 4400}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{5400 + 30h - 1000 - 20h - 0.1h^2 - 4400}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{10h - 0.1h^2}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{h(10 - 0.1h}{h}\right]$$

$$= \lim_{h \to 0}\left[10 - 0.1h\right]$$

$$= 10$$

Thus the marginal cost of producing 100 books is \$10/book.

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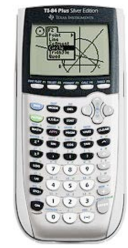


LESSON: The Limit of a Function

We begin investigating limits by examining the behaviour of a rational function.

$$f(x) = \frac{x^3 + x^2 - 4x - 4}{x - 2} \qquad x \neq 2$$

We know f(x) is undefined when x=2, but what happens to the function when x approaches (gets close to) 2.

x	f(x)
1.9	11.31
1.99	11.9301
1.999	11.993
2	Undefined
2.001	12.007
2.01	12.07
2.1	

$$\begin{array}{c|cccc} 2 & 1 & 1 & -4 & -4 \\ & \downarrow & 2 & 6 & 4 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

OK

$$\therefore f(x) = \frac{(x-2)(x^2+3x+2)}{x-2}$$

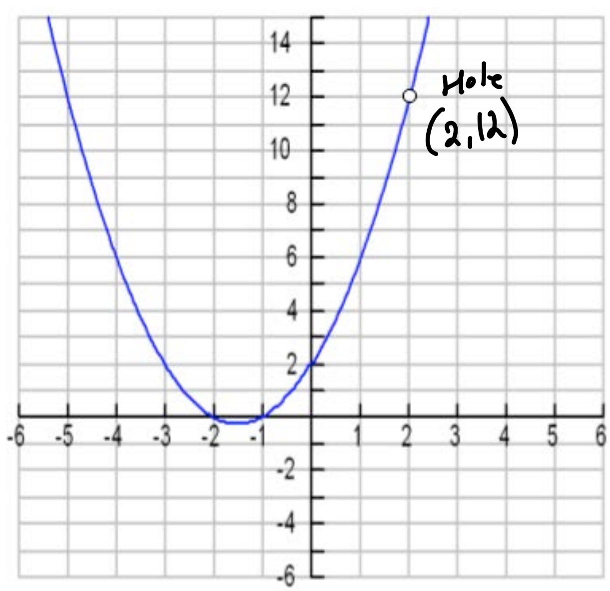
$$f(x) = \frac{\cancel{(x-2)}(x+1)(x+2)}{\cancel{x-2}}$$

Hole @ $x = 2$

$(2, 12)$

$$f(x) = (x+1)(x+2), x \neq 2$$

$$f(x) = \frac{x^3 + x^2 - 4x - 4}{x - 2} \qquad x \neq 2$$



$$\therefore \lim_{x \to 2} f(x) = 12$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

from left

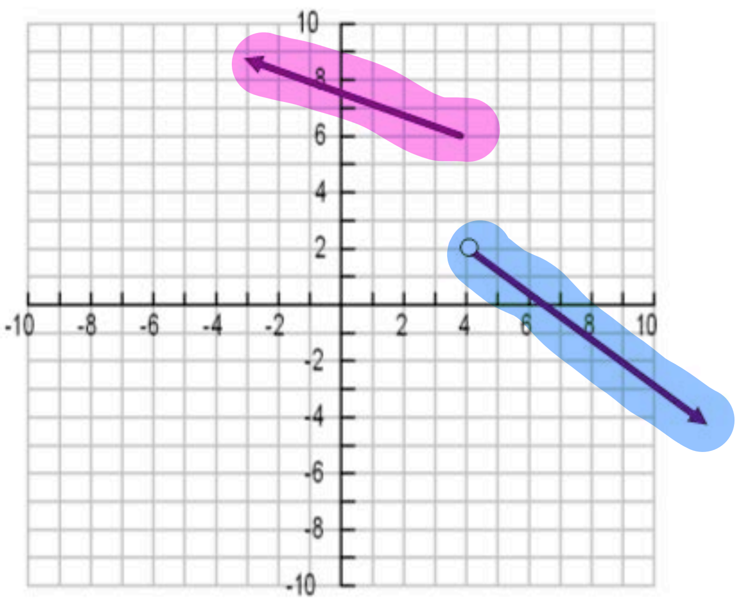
From Right

It will be at the assignment

One-Sided Limits

Left-hand Limit: $\lim_{x \to a^-} f(x)$ denotes the limit as x approaches a from the left side.

Right-hand Limit: $\lim_{x \to a^+} f(x)$ denotes the limit as x approaches a from the right side.



$\lim_{x \to 4^+} f(x) = 2$

$\lim_{x \to 4^-} f(x) = 6$

$\lim_{x \to 4} f(x) =$ doesn't exist "dne"

Two-Sided Limits

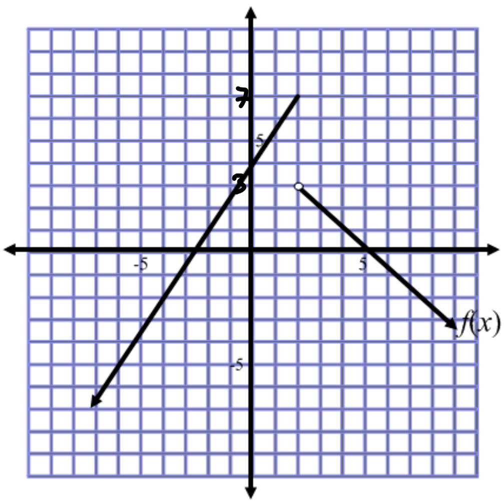
- If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$, then $\lim_{x \to a} f(x)$ exists and is equal to L.

- If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x)$ does not exists.

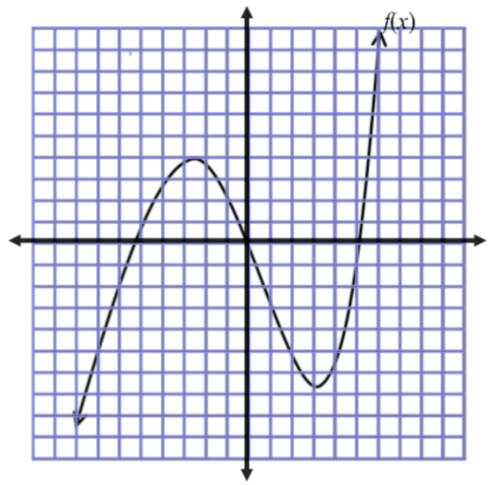
Limits can be found graphically and algebraically.

Ex 1. For the given functions, state the following limits.

a) $\lim\limits_{x \to 2} f(x)$ = dne



b) $\lim\limits_{x \to 0} f(x)$ = 0



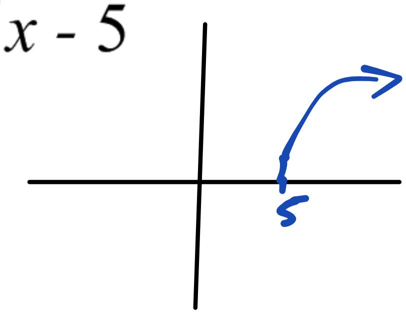
c) $\lim\limits_{x \to 10} (3x - 2)$

$x \to 10$

$= 3(10) - 2$

$= 28$

d) $\lim\limits_{x \to 5} \sqrt{x - 5}$



= dne

$\lim\limits_{x \to 5^+} f(x) = 0$

$\lim\limits_{x \to 5^-} f(x) = \text{dne}$

e) $f(x) = \begin{cases} x + 4, & \text{if } x < 1 \\ -x, & \text{if } x \geq 1 \end{cases}$; $\lim\limits_{x \to 1} f(x)$

$\left.\begin{array}{l} \lim\limits_{x \to 1^-} f(x) = 1 + 4 = 5 \\ \lim\limits_{x \to 1^+} f(x) = -1 \end{array}\right\}$ $\therefore \lim\limits_{x \to 1} f(x) = \text{dne}$

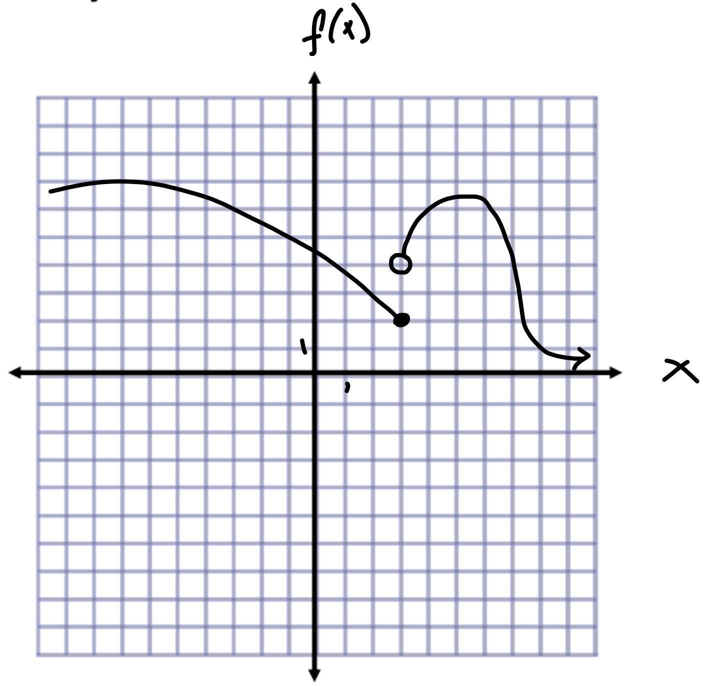
Ex 2. Sketch the graph of any function that satisfies the given conditions.

i) $f(3) = 2$

ii) $\lim_{x \to 3^-} f(x) = 2$

iii) $\lim_{x \to 3^+} f(x) = 4$

iv) $\lim_{x \to \infty} f(x) = 0$



Ex 3. Let $g(x) = Ax + B$, where A and B are constants. If $\lim_{x \to 1} g(x) = -2$ and $\lim_{x \to -1} g(x) = 4$, find the values of A and B.

$g(1) = -2$

$A(1) + B = -2$

$A + B = -2$ ①

$g(-1) = 4$

$A(-1) + B = 4$

$-A + B = 4$ ②

Elimination

①+②

$$\begin{aligned} A + B &= -2 \\ -A + B &= 4 \\ \hline 2B &= 2 \end{aligned}$$

$B = 1$

Sub. into ①

$A + 1 = -2$

$A = -3$

$\therefore g(x) = -3x + 1$

Direct Evaluation of Limits

You can only directly evaluate limits for the cases as listed below. We will soon see how you can transform limit questions into one of the cases below.

Type	Description	Example
"Plug In"	$\lim\limits_{x\to a} f(x)$, where a is in Domain of $f(x)$	$\lim\limits_{x\to 3}(3x+1)=3(3)+1=9+1=10$ $\lim\limits_{x\to -2}\frac{3x+1}{x-2}=\frac{3(-2)+1}{-4}=\frac{5}{4}$
$\frac{0}{\#}$	$\lim\limits_{x\to a} f(x)$, where a is in Domain of $f(x)$	$\lim\limits_{x\to -2}\frac{x+2}{x-2}=\frac{-2+2}{-2-2}=\frac{0}{-4}=0$
$\frac{\#}{0^+}$	The numerator approaches a number. The denominator approaches a very small positive number. Remember: the reciprocal of a very small positive number is a very large positive number (i.e. $+\infty$) For example, $\frac{3}{.00001}=300000$	$\lim\limits_{x\to 2^+}\frac{x+2}{x-2}=\frac{4}{0^+}=+\infty$
$\frac{\#}{0^-}$	The numerator approaches a number. The denominator approaches a very small negative number. Remember: the reciprocal of a very small negative number is a very small negative number (i.e. $-\infty$) For example, $\frac{3}{-0.00001}=-300000$	$\lim\limits_{x\to 2^-}\frac{x+2}{x-2}=\frac{4}{0^-}=-\infty$
$\frac{\#}{\pm\infty}$	The numerator approaches a number. The denominator increases without bound (i.e. it approaches a very large positive or negative number: $\pm\infty$ Remember: the reciprocal of a very large positive or negative number is 0. $\frac{3}{-300000}=-0.00001\cong 0$ $\frac{3}{300000}=+0.00001\cong 0$	$\lim\limits_{x\to +\infty}\frac{2}{x-2}=\frac{4}{+\infty}=0^+=0$ $\lim\limits_{x\to -\infty}\frac{2}{x-2}=\frac{4}{-\infty}=0^-=0$

At this point, we cannot evaluate directly limits of the form:

$$\frac{0}{0},\frac{\infty}{\infty},\frac{0}{\infty},\frac{\infty}{0},\infty\pm 0,\infty\pm\infty$$

All of the above are in ***Indeterminate form***. Other methods must first be used to transform them into a form where we can evaluate the limit directly.

Intuitive Definition of Limit

Let **y = f(x)** be a function. Suppose that **a** and **L** are numbers such that whenever **x** is close to **a** but not equal to **a**, **f(x)** is close to **L**; as **x** gets closer and closer to **a** but not equal to **a**, **f(x)** gets closer and closer to **L**; and

Suppose that **f(x)** can be made as close as we want to **L** by making **x** close to **a** but not equal to **a**.

Then we say that **the limit of f(x) as x approaches a is L** and we write

$$\lim_{x \to a} f(x) = L$$

You can think of the limit, L, as the “intended height”, y, of the function that it may, or may not, reach.

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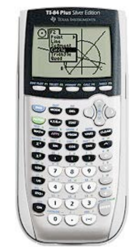

LESSON: Properties of Limits

Steps to determining the limit algebraically

Direct Substitution

- Substitute the value of x into the function

If you get:

i) a number, then that is the answer

Assignment

Ex1. evaluate the following.

a) $\lim_{x \to 0} \frac{x-1}{x+1}$

$x \to 0$

$= \frac{0-1}{0+1}$

$= -1$

b) $\lim_{x \to 2} \frac{x^4 - 3x + 1}{x^2(x-1)^3}$

$x \to 2$

$= \frac{16-6+1}{4(1)^3}$

$= \frac{11}{4}$

ii) $\frac{number}{0}$ then check the left and right limits by substituting in values very close to x=a.

This will come at test and more likely at the assignment

Ex.2 evaluate the following.

a) $\lim_{x \to 4} \frac{x+5}{x-4}$.

$x = 3.99$ $\lim_{x \to 4^-} \frac{x+5}{x-4} = -\infty$

$x = 4.001$ $\lim_{x \to 4^+} \frac{x+5}{x-4} = \infty$

$\therefore \lim_{x \to 4} \frac{x+5}{x-4} = dne$

b) $\lim_{x \to -2} \frac{1}{(x+2)(x-3)}$

$x = -2.001$ $\lim_{x \to -2^-} \frac{1}{(x+2)(x-3)} = \infty$

$x = -1.99$ $\lim_{x \to -2^+} \frac{1}{(x+2)(x-3)} = -\infty$

$\therefore \lim_{x \to -2} \frac{1}{(x+2)(x-3)} = dne$

iii) $\frac{0}{0}$ then you apply one of the following strategies:

- Factoring
- Rationalizing (multiply by the conjugate)
- Find a common denominator
- Introduce new variable

Ex.3 evaluate the following.

a) $\lim_{x \to -2} \frac{x+2}{x^2-4}$ * Factoring

$= \lim_{x \to -2} \frac{\cancel{x+2}^{\,1}}{\cancel{(x+2)}(x-2)}$

$= \lim_{x \to -2} \frac{1}{(x-2)}$

$x \to -2$

$= -\frac{1}{4}$

conjugate

b) $\lim_{t\to 0}\frac{\sqrt{2+t}-\sqrt{2}}{t} \cdot \frac{\sqrt{2+t}+\sqrt{2}}{\sqrt{2+t}+\sqrt{2}}$ Rationalize

$= \lim_{t\to 0}\frac{2+t-2}{t(\sqrt{2+t}+\sqrt{2})}$

$= \lim_{t\to 0}\frac{t}{t(\sqrt{2+t}+\sqrt{2})}$

$= \lim_{t\to 0}\frac{1}{(\sqrt{2+t}+\sqrt{2})}$

$t\to 0$

$= \frac{1}{\sqrt{2}+\sqrt{2}}$

$= \frac{1}{2\sqrt{2}}$

c) $\lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h}$ Find a common denominator

$= \lim_{h\to 0}\frac{\frac{1-1(1+h)}{1+h}}{h}$

$= \lim_{h\to 0}\frac{\frac{1-1-h}{1+h}}{h}$

$= \frac{-\frac{h}{1+h}}{h}$

$= \lim_{h\to 0} -\frac{h}{1+h}\times\frac{1}{h}$

$= \lim_{h\to 0} -\frac{\cancel{h}}{\cancel{h}(1+h)}$

$= \lim_{h\to 0} -\frac{1}{1+h}$

$h\to 0$

$= -\frac{1}{1}$

$= -1$

coming in the test

d) $\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ Introduce New Variable

let $u = \sqrt[3]{x}$

$x \to 1, u \to 1$

$u^3 = \left(\sqrt[3]{x}\right)^3$

$(u^3)^{\frac{1}{2}} = x^{\frac{1}{2}}$

$u^{\frac{3}{2}} = x^{\frac{1}{2}}$

conjugate

$= \lim_{u\to 1} \frac{\sqrt{u^3}-1}{u-1} \cdot \frac{\sqrt{u^3}+1}{\sqrt{u^3}+1}$

$= \lim_{u\to 1} \frac{u^3-1}{(u-1)(\sqrt{u^3}+1)}$

$= \lim_{u\to 1} \frac{\cancel{(u-1)}(u^2+u+1)}{\cancel{(u-1)}(\sqrt{u^3}+1)}$

$= \lim_{u\to 1} \frac{u^2+u+1}{\sqrt{u^3}+1}$

$u \to 1$

$= \frac{1^2+1+1}{1+1}$

$= \frac{3}{2}$

$x^3 + y^3 = (x+y)(x^2-xy+y^2)$

$x^3 - y^3 = (x-y)(x^2+xy-y^2)$

$\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{x-8}$

let $u = \sqrt[3]{x}$

$x \to 8, u \to 2$

$(u)^3 = (\sqrt[3]{x})^3$

$u^3 = x$

$= \lim_{x\to 2} \frac{u-2}{u^3-8}$

$= \lim_{x\to 2} \frac{(u-2)}{(u-2)(u^2+2u+4)}$

$= \lim_{x\to 2} \frac{1}{u^2+2u+4}$

$x \to 2$

$= \frac{1}{(2)^2+2(2)+4}$

$= \frac{1}{12}$

e)

$$\lim_{x \to 5} \sqrt{\frac{x^2}{x-1}}:$$

$x \to 5$

$$\lim_{x \to 5} \sqrt{\frac{x^2}{x-1}} = \sqrt{\lim_{x \to 5} \frac{x^2}{x-1}}$$

$$= \sqrt{\frac{\lim_{x \to 5} x^2}{\lim_{x \to 5}(x-1)}}$$

$$= \sqrt{\frac{25}{4}}$$

$$= \frac{5}{2}$$

$$= \sqrt{\frac{25}{4}}$$

$$= \frac{\sqrt{25}}{2}$$

$$= \frac{5}{2}$$

Properties of Limits
For any real number a, suppose f and g both have limits at $x = a$.
1. $\lim_{x \to a} k = k$ for any constant k
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a}[cf(x)] = c(\lim_{x \to a} f(x))$ for any constant c
5. $\lim_{x \to a}[f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$
6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$, for n a rational number

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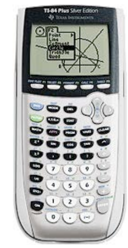
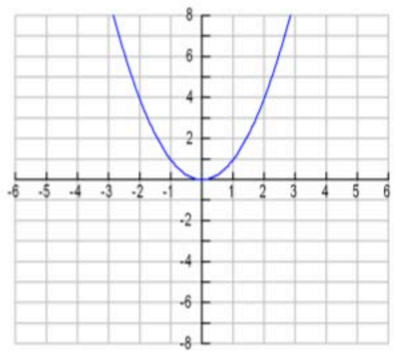


LESSON: Continuity

Continuous:

Informal Definition – you can trace the graph with your finger without ever lifting it off the paper.

a)



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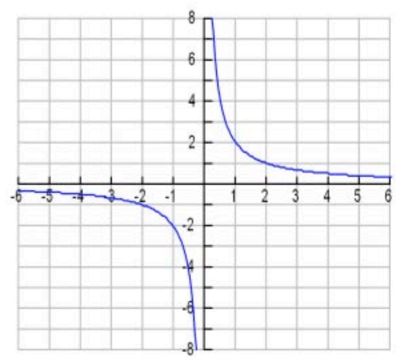
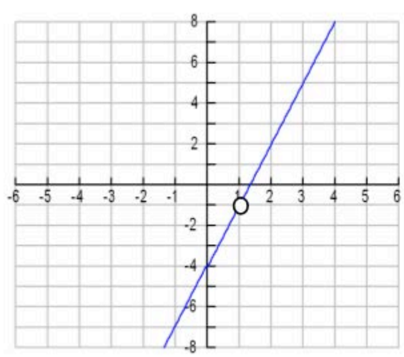
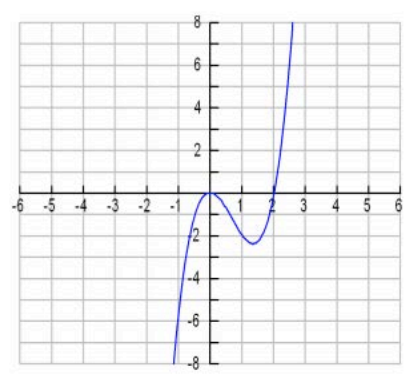


c)



d)



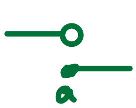
What types of functions are ALWAYS continuous – think last semester

- Polynomials
- Log functions
- Sin(x), cos(x)

It will be at the test

*If $f(x)$ is continuous at "a":

① $f(a)$ is defined in the domain of $f(x)$



② $\lim_{x \to a} f(x)$ exists

$\circled{3}$ $\lim_{x \to a} f(x) = f(a)$

a

Continuous:

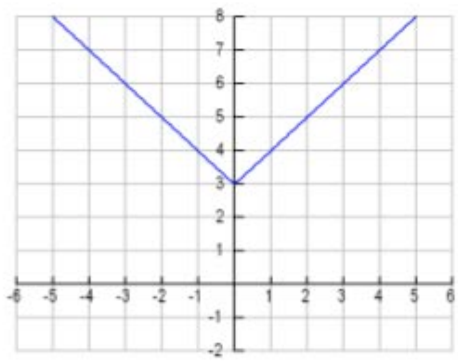
1 Formal Definition – the limit of the function at a point $x = a$ is equal to the value of the function at a.

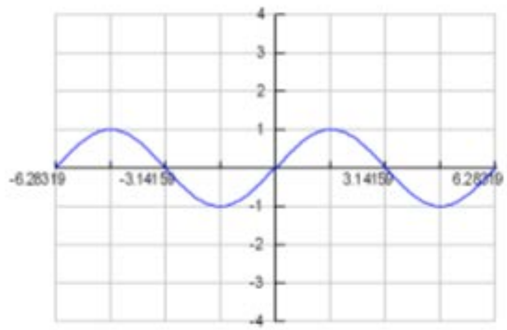
2 $\lim_{x \to a} f(x) = f(a)$

3 For the function to be continuous everywhere – this MUST be true for the entire domain.

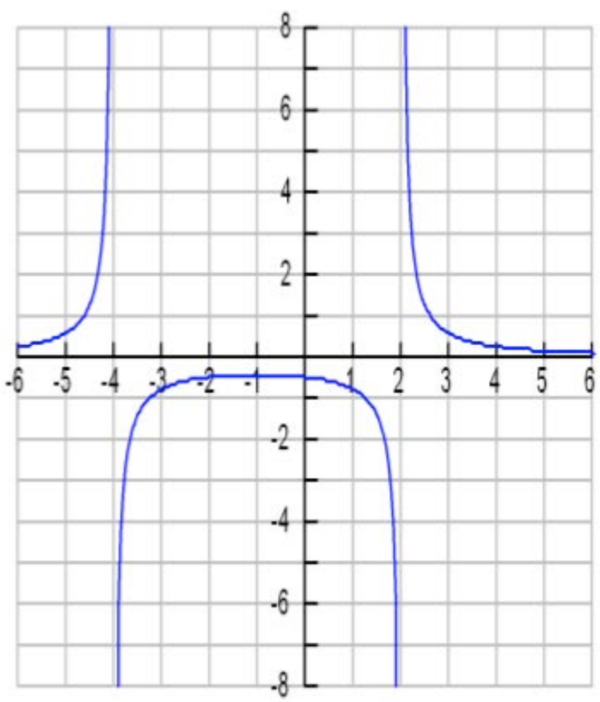
Smooth Curves:

Have no jagged points, no corners.





Recall: that rational functions have asymptotes.



a) determine the equation of the vertical asymptotes

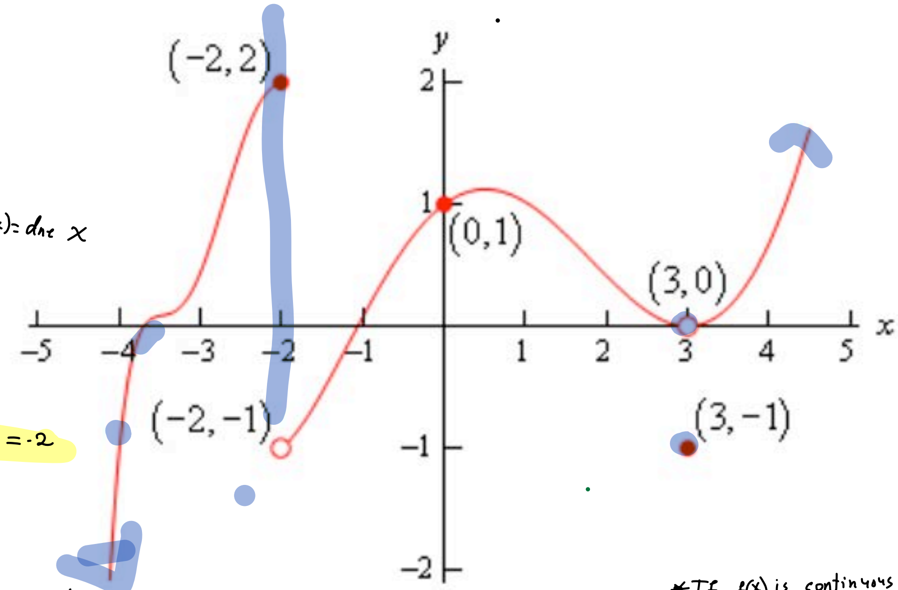
b) determine the left and right hand limits as x approaches the asymptotes

c) IS the function continuous? Use the formal definition of continuity to provide a GOOD explanation.

Similar in the test

Example: Given the graph of $f(x)$, shown below, determine if $f(x)$ is continuous at $x=-2$, $x=0$ and $x=3$

$\{x, x3$



at $x=-2$

① $f(-2)=2$ ✓

② $\lim_{x\to -2^-} f(x)=2$

$\lim_{x\to -2^+} f(x)=-1$

$\therefore \lim_{x\to -2} f(x) =$ dne ✗

③ $\lim_{x\to -2} f(x) \neq f(-2)$ ✗

$\therefore f(x)$ is not continuous at $x=-2$

at $x=0$

① $f(0)=1$ ✓

② $\lim_{x\to 0^-} f(x)=1$

$\lim_{x\to 0^+} f(x)=1$

$\therefore \lim_{x\to 0} f(x)=1$ ✓

③ $\lim_{x\to 0} f(x)=f(0)$ ✓

$\therefore f(x)$ is continuous at $x=0$

*If $f(x)$ is continuous at "a":

① $f(a)$ is defined in the domain of $f(x)$

② $\lim_{x\to a} f(x)$ exists

③ $\lim_{x\to a} f(x)=f(a)$

a

at $x=3$

① $f(3)=-1$ ✓

② $\lim_{x\to 3^-} f(x)=0$

$\lim_{x\to 3^-} f(x)=0$

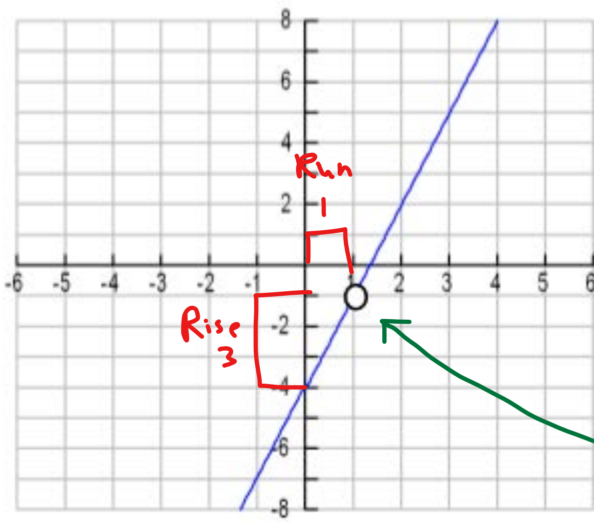
$\therefore \lim_{x\to 3} f(x)=0$ ✓

③ $\lim_{x\to 3} f(x) \neq f(3)$ ✗

$\therefore f(x)$ is discontinuous

at $x=3$

Example: The following function is discontinuous at $x=1$. How could we make this function continuous?



fill the hole

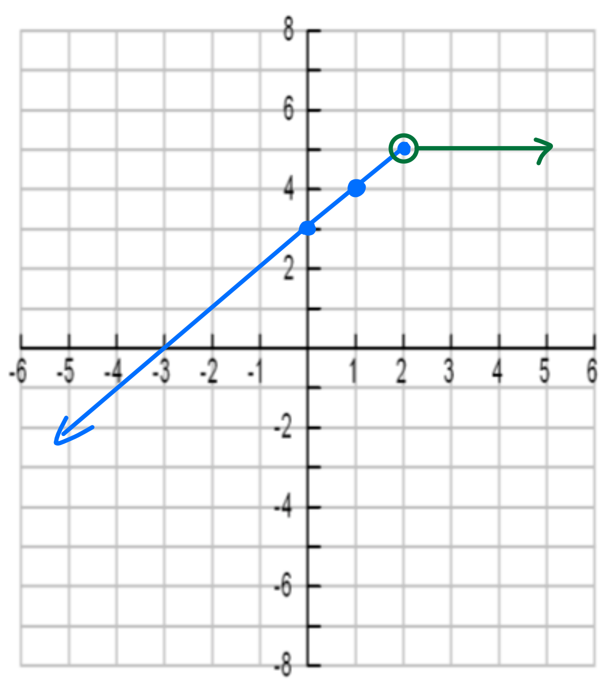
$$f(x)=\begin{cases} 3x-4, & x\neq 1 \\ -1, & x=1 \end{cases}$$

It is coming on the test

Example: Graph the function

$m=\frac{1}{1}$

$$f(x)=\begin{cases} x+3, & \text{if } x\le 2 \\ 5, & \text{if } x>2 \end{cases}$$



Is it continuous on its domain?

Yes

Example: Are the following functions continuous at $x=3$?

a) $f(x)=x^3-x$ Polynomial function

$\therefore$ yes

b) $f(x)=\dfrac{3+x}{3-x}$

V.A. at $x=3$ $\therefore$ No

c) $f(x)=\dfrac{x^2-2x-3}{x-3}$ and $\overbrace{f(3)=4}^{\text{Fills the}}$ hole

$\therefore$ yes

$$f(x)=\frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}}$$

Hole at $x=4$

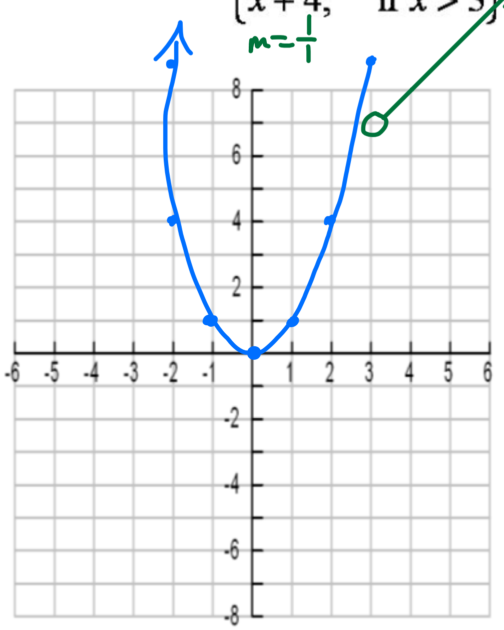
$(3,4)$

d)

$$f(x)=\begin{cases} x^2, & \text{if } x\le 3 \\ x+4, & \text{if } x>3 \end{cases}$$

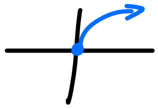
$m=\frac{1}{1}$

$\therefore$ No



Root Functions:

$\sqrt{x}$



$\lim_{x \to 0} \sqrt{x}$

$\lim_{x \to 0^-} f(x) = dne$ Because the graph doesn't have a left part.

$\lim_{x \to 0} \sqrt{x} = dne$

Example:

$f(x) = \sqrt{x-4}$

$\lim_{x \to 4} \sqrt{x-4} = dne$