Stat 412
lec 8

Modle $(M/M/C):(GD/N/\infty)$

$$\lambda_n = \lambda \ , \quad 0 \le n < N \qquad \mu_n = n\mu \ , \quad 0 \le n < c$$
$$\quad = 0 \ , \quad n \ge N \qquad \mu_n = c\mu \ , \quad c \le n \le N$$

From general solution of Marcovin model

$$P_n = \left(\prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}\right) P_0$$

$0 \le n < c$

$$P_n = \left(\prod_{i=0}^{n-1} \frac{\lambda}{(i+1)\mu}\right) P_0 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 = \frac{\rho^n}{n!} P_0$$

$c \le n \le N$

$$P_n = \left(\prod_{i=0}^{c-1} \frac{\lambda_i}{\mu_{i+1}} \prod_{i=c}^{n-1} \frac{\lambda_i}{\mu_{i+1}}\right) P_0 = \left(\prod_{i=0}^{c-1} \frac{\lambda}{(i+1)\mu} \prod_{i=c}^{n-1} \frac{\lambda}{c\mu}\right) P_0$$

$$= \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left(\frac{\lambda}{c\mu}\right)^{n-c} P_0 = \frac{\rho^c}{c!}\left(\frac{\rho}{c}\right)^{n-c} P_0 = \frac{\rho^n}{c!\, c^{n-c}} P_0$$

$$P_n = \frac{\rho^n}{n!} P_0 \ , \quad 0 \le n < c$$

$$\quad = \frac{\rho^n}{c!\, c^{n-c}} P_0 \ , \quad c \le n < N$$

$$\because \sum_{n=0}^{\infty} P_n = 1 \Rightarrow P_0 + \sum_{n=1}^{c-1} P_n + \sum_{n=c}^{N} P_n = 1$$

$$P_0 + \sum \frac{\rho^n}{n!} P_0 + \sum \frac{\rho^n}{c!\, c^{n-c}} P_0 = 1$$

$$P_0 \left[1 + \sum_{n=1}^{c-1} \frac{\rho^n}{n!} + \sum_{n=c}^{N} \frac{\rho^n}{c!\, c^{n-c}}\right] = 1$$

$$\therefore P_0 = \left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \sum_{n=c}^{N} \left(\frac{\rho}{c}\right)^{n-c}\right]^{-1}$$

$$P_0 = \left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \cdot \frac{1-(\rho/c)^{N-c+1}}{1-\rho/c} \right]^{-1} , \quad \rho/c \neq 1$$

$$P_0 = \left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \cdot (N-c+1) \right]^{-1} , \quad \rho/c = 1$$ (اشتق بسط ومقام بالنسبة ρ/c وعوض عنها بواحد)

$$L_q = \sum_{n=c}^{N} (n-c) P_n = \sum_{n=c}^{N} (n-c) \frac{\rho^n}{c!\, c^{n-c}} P_0$$

$$= \frac{\rho^c P_0}{c!} \sum_{n=c}^{N} \left(\frac{\rho}{c}\right)^{n-c} (n-c)$$

$$= \frac{\rho^c}{c!} \frac{\rho}{c} P_0 \sum_{n=c}^{N} (n-c) \left(\frac{\rho}{c}\right)^{n-c-1}$$

$$= \frac{\rho^c}{c!} \frac{\rho}{c} P_0 \sum_{n=c}^{N} \frac{d}{d(\rho/c)} \left(\frac{\rho}{c}\right)^{n-c}$$

$$= \frac{\rho^c}{c!} \frac{\rho}{c} P_0 \frac{d}{d\rho/c} \left[\sum_{n=c}^{N} \left(\frac{\rho}{c}\right)^{n-c} \right] = \frac{\rho^c}{c!} \frac{\rho}{c} P_0 \frac{d}{d\rho/c} \left(\frac{1-(\rho/c)^{N-c+1}}{1-\rho/c} \right)$$

$$= \frac{\rho^c}{c!} \frac{\rho}{c} P_0 \frac{(1-\rho/c)\left[-(N-c+1)(\rho/c)^{N-c}\right] - \left[1-(\rho/c)^{N-c+1}\right](-1)}{(1-\rho/c)^2}$$

$$= \frac{\rho^c}{c!} \frac{\rho}{c} P_0 \left[\frac{1-(\rho/c)^{N-c} - (N-c)(\rho/c)^{N-c} + (N-c)(\rho/c)^{N-c+1}}{(1-\rho/c)^2} \right] , \quad \rho/c \neq 1$$

تفاضل بسط ومقام

لو $\frac{\rho}{c} = 1$ بنستخدم لوبيتال بس بنفاضل مرتين وانعوض عن $\rho/c = 1$

أو

$$L_q = \sum_{n=c}^{N} (n-c) P_n = \sum_{n=c}^{N} \frac{\rho^n}{c!\, c^{n-c}} P_0$$

$$= \frac{\rho^c}{c!} P_0 \sum_{n=c}^{N} (n-c) \left(\frac{\rho}{c}\right)^{n-c} \xrightarrow[\rho/c=1]{} \frac{\rho^c}{c!} P_0 \underbrace{\sum_{n=c}^{N} (n-c)}_{\frac{(N-c)(N-c+1)}{2}}$$

From L_q we can get L_s, W_s, W_q; But we use $\underline{\lambda_{eff}}$

$$\lambda_{eff} = \lambda(1 - P_N)$$

$(1 - P_N) \Rightarrow$ احتمال الاتاحه

$$\lambda_{eff} = E(\lambda_n)$$

$$= \sum_{n=0}^{N} \lambda_n P_n = \lambda_0 P_0 + \lambda P_1 + \dots + \lambda P_{N-1} + 0$$

$$= \lambda(P_0 + P_1 + \dots + P_{N-1}) = \lambda(1 - P_N)$$

or

$$\lambda_{eff} = \mu(L_s - L_q) \Rightarrow L_s = L_q + \frac{\lambda_{eff}}{\mu} \quad \text{— (1)}$$

or

$$L_q = \sum_{n=C+1}^{N} (n - c) P_n = \sum_{n=C+1}^{N} n P_n - \sum_{n=C+1}^{N} c P_n$$

$$= \sum_{n=0}^{N} n P_n - \sum_{n=0}^{C} n P_n - c\left[\sum_{n=0}^{N} P_n - \sum_{n=0}^{C} P_n\right]$$

$$= L_s - \sum_{n=0}^{C} n P_n - C + \sum_{n=0}^{C} c P_n$$

$$= L_s - C + \sum_{n=0}^{C} (C - n) P_n \quad \left(\text{let } \bar{C} = \sum_{n=0}^{C} (c - n) P_n\right)$$

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$\bar{C}$ = expect no. of ideal Server

$$L_q = L_s - (C - \bar{C}) \quad \text{— (2)}$$

From (1) & (2)

$$\frac{\lambda_{eff}}{\mu} = C - \bar{C} \Rightarrow \lambda_{eff} = \mu(C - \bar{C})$$

Prove $\mu(C - \bar{C}) = \lambda(1 - P_N)$

if the system don't have place to wait this mean $N = C$ and the Model become $(M/M/C):(GD/C/\infty)$

$$\lambda_n = \lambda \quad 0 \le n < c \qquad , \qquad \mu_n = n\mu \quad , \quad 0 \le n \le c$$
$$= 0 \quad n \ge c \qquad \qquad = 0 \quad n > c$$

$$P_n = \left(\prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \right) P_0 = \left(\prod_{i=0}^{n-1} \frac{\lambda}{(i+1)\mu} \right) P_0$$

$$= \frac{\rho^n}{n!} P_0 \quad , \quad 1 \le n \le c$$

$$\sum_{n=0}^{c} P_n = 1 \Rightarrow \sum_{n=0}^{c} \frac{\rho^n}{n!} P_0 \Rightarrow P_0 = \left[\sum_{n=0}^{c} \frac{\rho^n}{n!} \right]^{-1}$$

$$P_c = \frac{\rho^c}{c!} P_0 = \frac{\rho^c / c!}{\sum_{n=0} \rho^n / n!}$$ this Called Erlang loss Formula

$$L_s = \sum_{n=0}^{c} n P_n = \sum_{n=0}^{c} n \frac{\rho^n}{n!} P_0 = \rho \sum_{n=1}^{c} \frac{\rho^{n-1}}{(n-1)!} P_0$$
$$= \rho [P_0 + P_1 + \dots + P_{c-1}]$$
$$= \rho (1 - P_c) = \rho \left(1 - \frac{\rho^c}{c!} P_0 \right)$$

$$W_s = \frac{L_s}{\lambda_{eff}} = \frac{\rho(1 - P_c)}{\lambda(1 - P_c)} = \frac{1}{\mu}$$

$$W_q = W_s - \frac{1}{\mu} = 0 \Rightarrow L_q = \lambda_{eff} W_q = 0$$