Assignment 3

Exercice 1

1) The overall system is LTI, thus if we delay the input, we also delay the output.
Since $h(t) = \delta(t-\tau)$, $r(t)$ is delayed by τ, hence after MF, if we sample at $t = kT + \tau$, the samples Y_k will be ISI-free.

2) a) $R_p(t) = \int p(\alpha+t) p(\alpha)^* d\alpha = \langle p(\alpha+t), p(\alpha) \rangle = \langle p(\alpha), p(\alpha+t) \rangle^*$.

$R_p^*(t) = \langle p(\alpha), p(\alpha+t) \rangle \Rightarrow R_p^*(-t) = \int p(\alpha) p(\alpha-t)^* d\alpha$

Let $\alpha = \alpha' + t \Rightarrow R_p^*(-t) = \int p(\alpha'+t) p(\alpha')^* d\alpha' = R_p(t)$ //

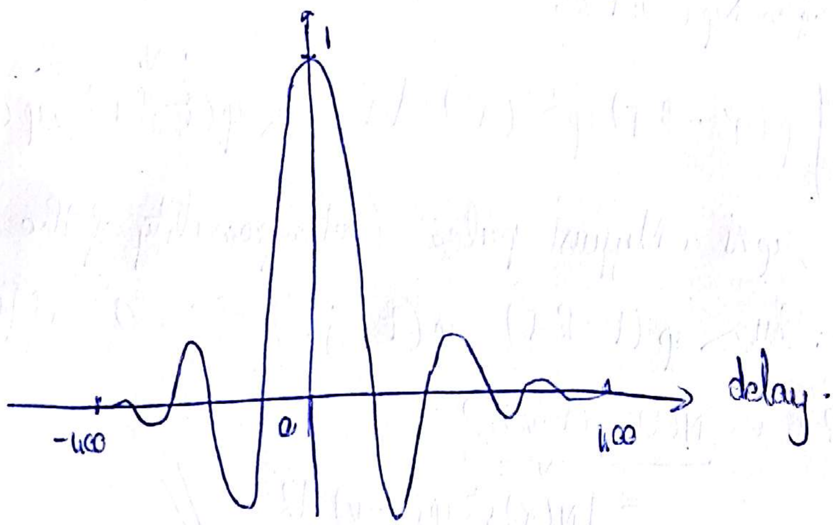
b) First note that $R_p(0) = \int p(\alpha) p^*(\alpha) d\alpha = \langle p(\alpha), p(\alpha) \rangle = \|p\|^2$.

By Cauchy-Schwarz ineq: $|R_p(t)| = |\langle p(\alpha+t), p(\alpha) \rangle| \leq \underbrace{\|p(\alpha+t)\|}_{\|p\|} \cdot \underbrace{\|p(\alpha)\|}_{\|p\|}$

$\Leftrightarrow |R_p(t)| \leq \|p\|^2$ //

self similarity funct° of RRC pulse.

c)



Exercise 2:

② From eq. [7] : $p(v) = \langle r(t), p(t-v) \rangle = R_p(v-\hat{\tau}) + \langle s_{data}(t-\hat{\tau}), p(t-v) \rangle + \langle N(t), p(t-v) \rangle$

a) we choose $p(t) = s_0 \varphi(t)$, thus:

$$R_p(v-\hat{\tau}) = \int s_0 \varphi(\alpha + v - \hat{\tau}) \varphi^*(\alpha) . s_0^* d\alpha = |s_0|^2 \int \varphi(\alpha + v - \hat{\tau}) \varphi^*(\alpha) = |s_0|^2 R_\varphi(v-\hat{\tau}$$

$$\langle s_{data}(t-\hat{\tau}), p(t-v) \rangle = \int s_{data}(\alpha - \hat{\tau}) p^*(\alpha - v) d\alpha = \int \sum_{k \geq l} s_k \varphi(\alpha - \hat{\tau} - kT) p^*(\alpha - v) d\alpha .$$

$$\langle s_{data}(t-\hat{\tau}), p(t-v) \rangle = \sum_{k \geq l} s_k s_0^* \int \varphi(\alpha - \hat{\tau} - kT) \varphi^*(\alpha - v) d\alpha ,$$

let $\alpha' = \alpha - v \iff \alpha = \alpha' + v \iff d\alpha = d\alpha'$:

$$\langle s_{data}(t-\hat{\tau}), p(t-v) \rangle = \sum_{k \geq l} s_k s_0^* \int \varphi(\alpha' + v - \hat{\tau} - kT) \varphi^*(\alpha') d\alpha' = \sum_{k \geq l} s_k s_0^* R_\varphi(v - \hat{\tau} - kT)$$

since $l = 1 \Rightarrow k > 0 \quad (k \geq 1)$. //

b) let's evaluate $\langle s_{data}(t-\hat{\tau}), p(t-v) \rangle$ when $v = \hat{\tau}$:

$$\langle s_{data}(t-\hat{\tau}), p(t-\hat{\tau}) \rangle = \sum_{k>0} s_k s_0^* R_\varphi(-kT) .$$

$k > 0$!

Note that: $R_\varphi(-kT) = \int \varphi(\alpha - kT) \varphi^*(\alpha) d\alpha = \langle \varphi(t-kT), \varphi(t) \rangle = 0$

$R_\varphi(-kT) = 0$ because, φ is a Nyquist pulse (orthogonality of the shifted pulses): $\langle \varphi(t-kT), \varphi(t-jT) \rangle = 0$ if $\begin{cases} k, j \in \mathbb{Z} \\ k \neq j \end{cases}$.

Thus: $p(v) = |s_0|^2 R_\varphi(v-\hat{\tau}) + \underbrace{\langle N(t), p(t-v) \rangle}_{= \int N(\alpha) s_0^* \varphi(\alpha - v) d\alpha}$ //

c) It doesn't work, some sent symbols are lost. Indeed the estimated channel delay is way larger than the actual channel delay. This comes from the fact that $v = \hat{\tau}$ is only a local maximum of $p(v)$, contribution of other data symbol for $v >> \hat{\tau}$ is non-negligible.

③ $p(t) = \sum_{i=0}^{l-1} B_i \varphi(t-kT)$.

1) $R_p(t) = \int p(\alpha+t)\, p(\alpha)^* d\alpha = \int \sum_{i=0}^{l-1} B_i \varphi(t-kT+\alpha) \cdot \sum_{j=0}^{l-1} B_j^* \varphi^*(\alpha - kT)\, d\alpha$.

$R_p(t) = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} B_i B_j^* \int \varphi(t-kT+\alpha)\, \varphi(\alpha-kT)\, d\alpha$, let $\alpha = \alpha' + kT \Rightarrow d\alpha = d\alpha'$.

$R_p(t) = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} B_i B_j^* \times R_\varphi(t)$.

$\Rightarrow E[B_i B_j] = \begin{cases} 1 & i=j \ (\text{since } E[X_{i,i}] = \frac{1}{2}\times 1 + \frac{1}{2}\times(-1)^2 = 1) \\ 0 & i \neq j \ (\text{since } i\neq j : E[X_{i,j}] = 1\times 1/4 + (-1)\times(-1)\times 1/4 + (-1)\times 1\times 1/4 + 1\times(-1)\times 1/4 = 0). \end{cases}$

Thus:

$E[R_p(t)] = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} E[B_i B_j] R_\varphi(t) = l \times R_\varphi(t)$.

φ has a sharp self-similarity function means that $R_\varphi(t)$ is large around 0 and negligible when we deviate from 0. Thus from [*], $R_p(t)$ is on average large around 0 and negligible for value of t not around 0.

2) $\langle q(t), p(t-v) \rangle = \int q(\alpha)\, p^*(\alpha - v)\, d\alpha = \int q(\alpha) \cdot \sum_{i=0}^{l-1} B_i^* \varphi^*(\alpha - v - kT)\, d\alpha$.

$$\langle q(t), p(t-v) \rangle = \underbrace{\sum_{i=0}^{l-1} B_i^*}_{X} \cdot \underbrace{\int q(\alpha)\, \varphi^*(\alpha - v - kT)\, d\alpha}_{Y}.$$

since random processes $p(t)$ and $q(t)$ are independant, so are X and Y

Hence $E[\langle q(t), p(t-v)\rangle] = E[X \cdot Y] \underset{X \text{ and } Y \text{ are indep.}}{=} E[X] \cdot E[Y]$

$E[X] = E\left[\sum_{i=0}^{l-1} B_i^*\right] \underset{\substack{\text{linearity of expectat}^{\circ} \\ \text{and } B_i \text{'s are real valued}}}{=} \sum_{i=0}^{l-1} E[B_i] = 0$

Thus $E[\langle q(t), p(t-v) \rangle] = 0$.

Exercise 3: (we assume b_i's are real valued)

① $y(t) = r(t) * \varphi^*(-t) = \int r(\alpha)\varphi^*(\alpha-t)\,d\alpha = \int p(\alpha-\tau)\varphi^*(\alpha-t)\,d\alpha + \int q(\alpha)\varphi^*(\alpha-t)\,d\alpha$.

$\int q(\alpha)\varphi^*(\alpha-t)\,d\alpha = \int q(\alpha'+t)\varphi^*(\alpha')\,d\alpha' = R_{q,\varphi}(t)$ (take $\alpha = \alpha'+t \Rightarrow d\alpha = d\alpha'$).

$\int p(\alpha-\tau)\varphi^*(\alpha-t)\,d\alpha = \int \sum_{i=0}^{l-1} b_i\,\varphi(\alpha-\tau-iT)\varphi^*(\alpha-t)\,d\alpha. = \sum_{i=0}^{l-1} b_i \int \varphi(\alpha'+t-\tau-iT)\varphi^*(\alpha')\,d\alpha'$

let $\alpha = \alpha'+t$

$\int p(\alpha-\tau)\varphi^*(\alpha-t)\,d\alpha = \sum_{i=0}^{l-1} b_i R_\varphi(t-\tau-iT)$.

$\Rightarrow y(t) = \sum_{i=0}^{l-1} b_i R_\varphi(t-\tau-iT) + R_{q,\varphi}(t)$.

② $\rho(v) = \langle r(t), p(t-v)\rangle = \langle p(t-\tau), p(t-v)\rangle + \langle q(t), p(t-v)\rangle$.

$\langle q(t), p(t-v)\rangle = \int q(\alpha)p^*(\alpha-v)\,d\alpha = \int q(\alpha)\sum_{i=0}^{l-1} b_i\,\varphi^*(\alpha-v-iT)\,d\alpha$

$= \sum_{i=0}^{l-1} b_i \int q(\alpha)\varphi^*(\alpha-v-iT)\,d\alpha = \sum_{i=0}^{l-1} b_i \int q(\alpha'+v+iT)\varphi^*(\alpha')\,d\alpha'$.

$\langle q(t), p(t-v)\rangle = \sum_{i=0}^{l-1} b_i R_{q,\varphi}(v+iT)$.

$\langle p(t-\tau), p(t-v)\rangle = \int p(\alpha-\tau)p^*(\alpha-v)\,d\alpha = \int p(\alpha'+v-\tau)\,p^*(\alpha')\,d\alpha'$

$= R_p(v-\tau) = \sum_{i=0}^{l-1}\sum_{j=0}^{l-1} b_i b_j \times R_\varphi(v-\tau)$.

$\Rightarrow \rho(v) = \sum_{i=0}^{l-1}\sum_{j=0}^{l-1} b_i b_j R_\varphi(v-\tau) + \sum_{i=0}^{l-1} b_i R_{q,\varphi}(v+iT)$.

$\rho(mT_s) = \sum_{i=0}^{l-1}\sum_{j=0}^{l-1} b_i b_j R_\varphi(mT_s-\tau) + \sum_{i=0}^{l-1} b_i R_{q,\varphi}(mT_s+iT)$.

Let's calculate $\langle y[m+k], \hat{b}_k\rangle = \sum_k y[m+k]\cdot\hat{b}_k = \sum_k \sqrt{T_s}\,y((m+k)T_s)\cdot\hat{b}_k$.

$\langle y[m+k], \hat{b}_k\rangle = \sum_k \sum_{i=0}^{l-1} b_i \hat{b}_k \sqrt{T_s}\cdot R_\varphi((m+k)T_s-\tau-iT) + \sum_k \hat{b}_k\sqrt{T_s}\cdot R_{q,\varphi}((m+k)T_s)$.

$= \sum_k \sum_{i=0}^{l-1} b_i \hat{b}_k \sqrt{T_s}\cdot R_\varphi((m+k)\frac{T}{U}-\tau-iT) + \sum_k \hat{b}_k\sqrt{T_s}\cdot R_{q,\varphi}(mT_s+k\frac{T}{U})$.

$= \sum_{i=0}^{l-1}\sum_{j=0}^{l-1} b_i b_j^2 \sqrt{T_s}\, R_\varphi(mT_s-\tau+iT-iT) + \sum_{i=0}^{l-1} b_i \sqrt{T_s}\, R_{q,\varphi}(mT_s+iT)$.

$$\langle y[m+k], \hat{\ell}_k \rangle = \left(\sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \ell_i \ell_j R_{\varphi}(mT_s - i^2) + \sum_{i=0}^{L-1} \ell_i R_{q,\varphi}(mT_s + iT) \right) \times \sqrt{T_s} .$$

$$\Rightarrow \langle y[m+k], \hat{\ell}_k \rangle = \sqrt{T_s} \times \rho(mT_s) .$$