
Fast Two-Time-Scale Noisy EM Algorithms

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Abstract

Using the Expectation-Maximization (EM) algorithm is the most popular choice for current latent data model learning tasks. For today's modern and complex models, variants of the EM have been initially introduced by [17], using incremental updates to scale to large datasets, and by [21, 6], using Monte-Carlo (MC) approximations to bypass the impossible conditional expectation of the latent data for most nonconvex models. In this paper, we propose a general class of methods called Two-Time-Scale EM Methods based on double stages of stochastic updates to tackle an essential large and nonconvex optimization task for latent data models. We motivate the choice of a double dynamics by invoking the variance reduction virtue of each stage of the method on both sources of noise: the incremental update and the MC approximation. We establish finite-time and global convergence bounds for nonconvex objective functions. Numerical applications are also presented in this article to illustrate our findings.

1 Introduction

Learning latent data models is critical for modern machine learning problems, see [15] for references. We formulate the training of such model as an empirical risk minimization problem:

$$\min_{\theta \in \Theta} \bar{L}(\theta) := r(\theta) + L(\theta) \quad \text{with} \quad L(\theta) = \frac{1}{n} \sum_{i=1}^n L_i(\theta) := \frac{1}{n} \sum_{i=1}^n \{ -\log g(y_i; \theta) \}, \quad (1)$$

We denote the observations by $\{y_i\}_{i=1}^n$, $\Theta \subset \mathbb{R}^d$ is the convex parameters space. We consider a smooth convex regularization noted $r : \Theta \rightarrow \mathbb{R}$ and $g(y; \theta)$ is the (incomplete) likelihood of each observation. The objective function $\bar{L}(\theta)$ is possibly *nonconvex* and is assumed to be lower bounded.

In the latent variable model, $g(y_i; \theta)$, is the marginal of the complete data likelihood defined as $f(z_i, y_i; \theta)$, i.e. $g(y_i; \theta) = \int_{\mathbb{Z}} f(z_i, y_i; \theta) \mu(dz_i)$, where $\{z_i\}_{i=1}^n$ are the latent variables. In this paper, we make the assumption of a complete model belonging to the curved exponential family:

$$f(z_i, y_i; \theta) = h(z_i, y_i) \exp(\langle S(z_i, y_i) | \phi(\theta) \rangle - \psi(\theta)), \quad (2)$$

where $\psi(\theta)$, $h(z_i, y_i)$ are scalar functions, $\phi(\theta) \in \mathbb{R}^k$ is a vector function, and $S(z_i, y_i) \in \mathbb{R}^k$ is the complete data sufficient statistics. Full batch EM [7] is the method of reference for that kind of task and is a two steps procedure. The **E-step** amounts to computing the conditional expectation of the complete data sufficient statistics,

$$\bar{s}(\theta) = \frac{1}{n} \sum_{i=1}^n \bar{s}_i(\theta) \quad \text{where} \quad \bar{s}_i(\theta) = \int_{\mathbb{Z}} S(z_i, y_i) p(z_i | y_i; \theta) \mu(dz_i). \quad (3)$$

The **M-step** is given by

$$\text{M-step: } \hat{\theta} = \bar{\theta}(\bar{s}(\theta)) := \arg \min_{\vartheta \in \Theta} \{ r(\vartheta) + \psi(\vartheta) - \langle \bar{s}(\theta) | \phi(\vartheta) \rangle \}, \quad (4)$$

Two caveats of this method are the following: (a) with the explosion of data, the first step of the EM is computationally inefficient as it requires a full pass over the dataset at each iteration and (b) the complexity of modern models makes the expectation in (3) intractable. So far, both challenges have been addressed separately, to the best of our knowledge, as detailed the sequel.

Prior Work Inspired by stochastic optimization procedures, [17] and [4] developed respectively an incremental and an online variant of the E-step in models where the expectation is computable then extensively used and studied in [18, 13, 3]. Some improvements of that methods have been provided and analyzed, globally and in finite-time, in [10] where variance reduction techniques taken from the optimization literature have been efficiently applied to scale the EM algorithm to large datasets.

Regarding the computation of the expectation under the posterior distribution, the first method was the Monte-Carlo EM (MCEM) introduced in the seminal paper [21] where a MC approximation for this expectation is computed. A variant of that method is the Stochastic Approximation of the EM (SAEM) in [6] leveraging the power of Robbins-Monro type of update [20] to ensure pointwise convergence of the vector of estimated parameters rather using a decreasing stepsize than increasing the number of MC samples. The MCEM and the SAEM have been successfully applied in mixed effects models [14, 8, 2] or to do inference for joint modeling of time to event data coming from clinical trials in [5], among other applications. Recently, an incremental variant of the SAEM was proposed in [12] showing positive empirical results but its analysis is limited to asymptotic consideration. Gradient-based methods have been developed and analyzed in [22] but they remain out of the scope of this paper as they tackle the high-dimensionality issue.

Contributions This paper *introduces* and *analyzes* a new class of methods which purpose is update two proxies for target expected quantities in a two-time-scale manner. Those approximated quantities are then used to optimize (1) for current modern examples and settings using EM-fashion Maximization step. The main contributions of the paper are:

- We propose a two-time-scale method based on Stochastic Approximation (SA), to alleviate the problem of MC computation, and on Incremental updates, to scale to large datasets. We describe in details the edges of each level of our method based on variance reduction arguments. The derivation of such class of algorithms has two advantages. First, it naturally leverages variance reduction and Robbins-Monro type of updates to tackle large-scale and highly nonlinear learning tasks. Then, it gives a simple formulation as a *scaled-gradient method* which makes the global analysis and the implementation accessible.
- We also establish global (independent of the initialization) and finite-time (true at each iteration) upper bounds on a classical suboptimality condition in the nonconvex literature, *i.e.*, the second order moment of the gradient of the objective function.

In Section 2 we formalize both incremental and Monte-Carlo variants of the EM. Then, we introduce our two-time-scale class of EM algorithms for which we derive several global statistical guarantees in Section 3 for possibly *nonconvex* functions. Section 4 is devoted to numerical illustrations.

2 Two-Time-Scale Stochastic EM Algorithms

We recall and formalize in this section the different methods found in the literature that aim to solving the large-scale problem and the intractable expectation. We then provide the general framework of our method that efficiently tackles the optimization problem (1).

2.1 Monte Carlo Integration and Stochastic Approximation

As mentioned in the introduction, for complex and possibly nonlinear models, the expectation under the posterior distribution defined in (3) is not tractable. In that case, the first solution involves computing a Monte Carlo integration of that latter term. For all $i \in \llbracket 1, n \rrbracket$, draw for $m \in \llbracket 1, M \rrbracket$, samples $z_{i,m} \sim p(z_i | y_i; \theta)$ and compute the MC integration \tilde{s} of the deterministic quantity $\bar{s}(\theta)$:

$$\text{MC-step : } \tilde{s} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}, y_i) \quad (5)$$

and then update the parameter $\hat{\theta} = \bar{\theta}(\tilde{s})$. This algorithm bypasses the intractable expectation issue but is rather computationally expensive in order to reach point wise convergence (M needs to be

large). An alternative to that stochastic algorithm is to use a Robbins-Monro (RM) type of update. We denote, at iteration k , the following quantity

$$\tilde{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}^{(k)}, y_i) \quad \text{where} \quad z_{i,m}^{(k)} \sim p(z_i | y_i; \theta^{(k)}) \quad (6)$$

Then, the RM updated of the sufficient statistics $\hat{s}^{(k+1)}$ reads:

$$\text{SA-step : } \hat{s}^{(k+1)} = \hat{s}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{s}^{(k)}) \quad (7)$$

where $\{\gamma_k\}_{k \geq 1} \in (0, 1)$ is a sequence of decreasing step sizes to ensure asymptotic convergence. This is called the Stochastic Approximation of the EM (SAEM) and has been shown to converge to a maximum likelihood of the observations under very general conditions [6]. In the simulation step (6), since the loss function between the observed data y_i and the latent variable z_i can be nonconvex, sampling from the posterior distribution $p(z_i | y_i; \theta)$, under the current model θ , requires using an inference algorithm. [11] proved almost sure convergence of the sequence of parameters obtained by this algorithm coupled with an MCMC procedure during the simulation step. In simple scenarios, the samples $\{z_{i,m}\}_{m=0}^{M-1}$ are conditionally independent and identically distributed with distribution $p(z_i, \theta)$. Nevertheless, in most cases, sampling exactly from this distribution is not an option and the Monte Carlo batch is sampled by Monte Carlo Markov Chains (MCMC) algorithm. In the SA-step, the sequence of decreasing positive integers $\{\gamma_k\}_{k \geq 1}$ controls the convergence of the algorithm. In practice, γ_k is set equal to 1 during the first few iterations to let the algorithm explore the parameter space without memory and converge quickly to a neighborhood of the target estimate. The Stochastic Approximation is performed during the remaining iterations where $\gamma_k = 1/k^\alpha$, where $\alpha \in (0, 1)$, ensuring the almost sure convergence of the estimate. It is inappropriate to start with small values for step size γ_k and large values for the number of simulations M_k . Rather, it is recommended that one decrease γ_k and keep a constant and small number of MC samples M_k which shows a great advantage over the MC-step (5), which requires large M_k to converge.

This Robbins-Monro type of update represents the *first level* of our algorithm, needed to temper the variance and noise implied by MC integration. In the next section, we derive variants of this algorithm to adapt to the sheer size of data of today's applications and formalize the *second level* of our class of Two-Time-Scale EM methods.

2.2 Incremental and Bi-Level Noisy EM Methods

Strategies to scale to large datasets include classical incremental and variance reduced variants. We will explicit a general update that will cover those variants and that represents the *second level* of our algorithm, namely the incremental update of the noisy statistics $\tilde{S}^{(k)}$ inside the RM type of update.

$$\text{Incremental-step : } \tilde{S}^{(k+1)} = \tilde{S}^{(k)} + \rho_{k+1}(\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}), \quad (8)$$

Note $\{\rho_k\}_{k \geq 1} \in (0, 1)$ is a sequence of step sizes, $\mathcal{S}^{(k)}$ is a proxy for $\tilde{S}^{(k)}$, If the stepsize is equal to one and the proxy $\mathcal{S}^{(k)} = \hat{S}^{(k)}$, i.e., computed in a full batch manner as in (6), then we recover the SAEM algorithm. Also if $\rho_k = 1$, $\gamma_k = 1$ and $\mathcal{S}^{(k)} = \tilde{S}^{(k)}$, then we recover the MCEM [21].

We now introduce three variants of the SAEM update depending on different definitions of the proxy $\mathcal{S}^{(k)}$ and the choice of the stepsize ρ_k . Let $i_k \in \llbracket 1, n \rrbracket$ be a random index drawn at iteration k and $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$ be the iteration index where $i \in \llbracket 1, n \rrbracket$ is last drawn prior to iteration k . For iteration $k \geq 0$, the fiTTSEM method draws *two* indices *independently* and uniformly as $i_k, j_k \in \llbracket 1, n \rrbracket$. In addition to τ_i^k which was defined w.r.t. i_k , we define $t_j^k = \{k' : j_{k'} = j, k' < k\}$ to be the iteration index where the sample $j \in \llbracket 1, n \rrbracket$ is last drawn as j_k prior to iteration k . With the initialization $\bar{\mathcal{S}}^{(0)} = \bar{s}^{(0)}$, we use a slightly different update rule from SAGA inspired by [19]. Then, we obtain:

$$(i\text{SAEM [12]}) \quad \mathcal{S}^{(k+1)} = \mathcal{S}^{(k)} + \frac{1}{n}(\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)}) \quad (9)$$

$$(vr\text{TTSEM}) \quad \mathcal{S}^{(k+1)} = \tilde{S}^{(\ell(k))} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\ell(k))}) \quad (10)$$

$$(fi\text{TTSEM}) \quad \mathcal{S}^{(k+1)} = \bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}), \quad \bar{\mathcal{S}}^{(k+1)} = \bar{\mathcal{S}}^{(k)} + n^{-1}(\tilde{S}_{j_k}^{(k)} - \tilde{S}_{j_k}^{(t_{j_k}^k)}) \quad (11)$$

116 where $\tilde{S}_{i_k}^{(k)}$ is the MC approximation of the expectation $\bar{S}_{i_k}(\boldsymbol{\theta}^{(k)})$:

$$\tilde{S}_{i_k}^{(k)} = \frac{1}{M_k} \sum_{m=1}^{M_k} S(z_{i_k, m}^{(k)}, y_{i_k}) \quad \text{with} \quad z_{i_k, m}^{(k)} \sim p(z_{i_k} | y_{i_k}; \boldsymbol{\theta}^{(k)}) \quad (12)$$

117 The stepsize is set to $\rho_{k+1} = 1$ for the iSAEM method and we initialize with $\boldsymbol{S}^{(0)} = \tilde{S}^{(0)}$; $\rho_{k+1} = \gamma$
 118 is constant for the vrTTSEM and fTTSEM methods. Moreover, for vrTTSEM we set an epoch size
 119 of m and define $\ell(k) := m \lfloor k/m \rfloor$ as the first iteration number in the epoch that iteration k is in.

120 **Two-Time-Scale Noisy EM methods:** We now introduce the general method derived using the two
 121 variance reduction techniques described above. Algorithm 1 leverages both levels (7) and (8) in
 122 order to output a vector of fitted parameters $\hat{\boldsymbol{\theta}}^{(K)}$ where K is a randomly chosen termination point.

Algorithm 1 Two-Time-Scale Noisy EM methods.

- 1: **Input:** initializations $\hat{\boldsymbol{\theta}}^{(0)} \leftarrow 0$, $\hat{\mathbf{s}}^{(0)} \leftarrow \hat{S}^{(0)}$, $K_{\max} \leftarrow \text{max. iteration number}$.
- 2: Set the terminating iteration number, $K \in \{0, \dots, K_{\max} - 1\}$, as a discrete r.v. with:

$$P(K = k) = \frac{\gamma_k}{\sum_{\ell=0}^{K_{\max}-1} \gamma_{\ell}} = \frac{\gamma_k}{P_{\max}}. \quad (13)$$

- 3: **for** $k = 0, 1, 2, \dots, K$ **do**
- 4: Draw index $i_k \in \llbracket 1, n \rrbracket$ uniformly (and $j_k \in \llbracket 1, n \rrbracket$ for fTTSEM).
- 5: Compute $\hat{S}_{i_k}^{(k)}$ using the MC-step (5), for the drawn indices.
- 6: Compute the surrogate sufficient statistics $\boldsymbol{S}^{(k+1)}$ using (9) or (10) or (11).
- 7: Compute $\hat{S}^{(k+1)}$ and $\hat{\mathbf{s}}^{(k+1)}$ using respectively (8) and (7):

$$\begin{aligned} \tilde{S}^{(k+1)} &= \tilde{S}^{(k)} + \rho_{k+1} (\boldsymbol{S}^{(k+1)} - \tilde{S}^{(k)}) \\ \hat{\mathbf{s}}^{(k+1)} &= \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}) \end{aligned} \quad (14)$$

- 8: Compute $\hat{\boldsymbol{\theta}}^{(k+1)} = \bar{\boldsymbol{\theta}}(\hat{\mathbf{s}}^{(k+1)})$ via the M-step (4).
 - 9: **end for**
 - 10: **Return:** $\hat{\boldsymbol{\theta}}^{(K)}$.
-

123 The update in (14) is said to have two-time-scales as the step sizes satisfy $\lim_{k \rightarrow \infty} \gamma_k / \rho_k < 1$ such that
 124 $\tilde{S}^{(k+1)}$ is updated at a faster time-scale, determined by ρ_k , than $\hat{\mathbf{s}}^{(k+1)}$, determined by γ_k . The next
 125 section presents the main results of this paper and establishes global and finite-time bounds for the
 126 three different updates of our two-time-scale scheme.

127 3 Finite Time Analysis of the Two-Time-Scale Scheme

128 Following [4], it can be shown that stationary points of the objective function (1) corresponds to the
 129 stationary points of the following *nonconvex* Lyapunov function:

$$\min_{\mathbf{s} \in \mathcal{S}} V(\mathbf{s}) := \bar{\mathcal{L}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) = \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\bar{\boldsymbol{\theta}}(\mathbf{s})) \quad (15)$$

130 that we propose to study in this article. Several critical assumptions required to derive convergence
 131 guarantees read as follows:

132 **H1.** The sets \mathcal{Z}, \mathcal{S} are compact. There exists constants $C_{\mathcal{S}}, C_{\mathcal{Z}}$ such that:

$$C_{\mathcal{S}} := \max_{\mathbf{s}, \mathbf{s}' \in \mathcal{S}} \|\mathbf{s} - \mathbf{s}'\| < \infty, \quad C_{\mathcal{Z}} := \max_{i \in \llbracket 1, n \rrbracket} \int_{\mathcal{Z}} |S(z, y_i)| \mu(dz) < \infty. \quad (16)$$

133 **H2.** The conditional distribution is smooth on $\text{int}(\Theta)$. For any $i \in \llbracket 1, n \rrbracket$, $z \in \mathcal{Z}$, $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \text{int}(\Theta)^2$,
 134 we have $|p(z|y_i; \boldsymbol{\theta}) - p(z|y_i; \boldsymbol{\theta}')| \leq L_p \|\boldsymbol{\theta} - \boldsymbol{\theta}'\|$.

135 We also recall from the introduction that we consider curved exponential family models. besides:

136 **H3.** For any $\mathbf{s} \in \mathcal{S}$, the function $\boldsymbol{\theta} \mapsto L(\mathbf{s}, \boldsymbol{\theta}) := \mathbf{r}(\boldsymbol{\theta}) + \psi(\boldsymbol{\theta}) - \langle \mathbf{s} | \phi(\boldsymbol{\theta}) \rangle$ admits a unique global
 137 minimum $\bar{\boldsymbol{\theta}}(\mathbf{s}) \in \text{int}(\Theta)$. In addition, $\mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))$ is full rank, L_{ϕ} -Lipschitz and $\bar{\boldsymbol{\theta}}(\mathbf{s})$ is L_{θ} -Lipschitz.

We denote by $H_L^\theta(\mathbf{s}, \boldsymbol{\theta})$ the Hessian (w.r.t to $\boldsymbol{\theta}$ for a given value of \mathbf{s}) of the function $\boldsymbol{\theta} \mapsto L(\mathbf{s}, \boldsymbol{\theta}) = \mathbf{r}(\boldsymbol{\theta}) + \psi(\boldsymbol{\theta}) - \langle \mathbf{s} | \phi(\boldsymbol{\theta}) \rangle$, and define

$$\mathbf{B}(\mathbf{s}) := \mathbf{J}_\phi^\theta(\bar{\boldsymbol{\theta}}(\mathbf{s})) \left(H_L^\theta(\mathbf{s}, \bar{\boldsymbol{\theta}}(\mathbf{s})) \right)^{-1} \mathbf{J}_\phi^\theta(\bar{\boldsymbol{\theta}}(\mathbf{s}))^\top. \quad (17)$$

H4. It holds that $v_{\max} := \sup_{\mathbf{s} \in \mathcal{S}} \|\mathbf{B}(\mathbf{s})\| < \infty$ and $0 < v_{\min} := \inf_{\mathbf{s} \in \mathcal{S}} \lambda_{\min}(\mathbf{B}(\mathbf{s}))$. There exists a constant L_B such that for all $\mathbf{s}, \mathbf{s}' \in \mathcal{S}^2$, we have $\|\mathbf{B}(\mathbf{s}) - \mathbf{B}(\mathbf{s}')\| \leq L_B \|\mathbf{s} - \mathbf{s}'\|$.

The class of algorithms we develop in this paper are two-time-scale where the first stage corresponds to the variance reduction trick used in [10] in order to accelerate incremental methods and reduce the variance induced by the index sampling. The second stage is the Robbins-Monro type of update that aims to reduce the variance induced by the MC approximations As the expectations (3) are never available, we introduce the errors when approximating the quantity $\bar{\mathbf{s}}_i(\hat{\boldsymbol{\theta}}(\hat{\mathbf{s}}^{(k-1)}))$ at iteration $k+1$:

$$\eta_i^{(r)} := \tilde{\mathbf{s}}_i^{(r)} - \bar{\mathbf{s}}_i(\vartheta^{(r)}) \quad \text{for all } i \in \llbracket 1, n \rrbracket, r > 0 \quad \text{and } \vartheta \in \Theta \quad (18)$$

For instance, we consider that the MC approximation is unbiased if for all $i \in \llbracket 1, n \rrbracket$ and $m \in \llbracket 1, M \rrbracket$, the samples $z_{i,m} \sim p(z_i | y_i; \boldsymbol{\theta})$ are i.i.d. under the posterior distribution, i.e., $\mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r] = 0$ where \mathcal{F}_r is the filtration up to iteration r . The following results are derived under the assumption of control of the fluctuations implied by the approximation stated as follows:

H5. There exist a positive sequence of MC batch size $\{M_r\}_{r>0}$ and constants (C, C_η) such that for all $k > 0, i \in \llbracket 1, n \rrbracket$ and $\vartheta \in \Theta$:

$$\mathbb{E} \left[\left\| \eta_i^{(r)} \right\|^2 \right] \leq \frac{C_\eta}{M_r} \quad \text{and} \quad \mathbb{E} \left[\left\| \mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r] \right\|^2 \right] \leq \frac{C}{M_r} \quad (19)$$

We can prove two important results on the Lyapunov function. The first one suggests smoothness:

Lemma 1. [10] Assume H1-H4. For all $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$ and $i \in \llbracket 1, n \rrbracket$, we have

$$\|\bar{\mathbf{s}}_i(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \bar{\mathbf{s}}_i(\bar{\boldsymbol{\theta}}(\mathbf{s}'))\| \leq L_s \|\mathbf{s} - \mathbf{s}'\|, \quad \|\nabla V(\mathbf{s}) - \nabla V(\mathbf{s}')\| \leq L_V \|\mathbf{s} - \mathbf{s}'\|, \quad (20)$$

where $L_s := C_Z L_p L_\theta$ and $L_V := v_{\max}(1 + L_s) + L_B C_S$.

and the second one suggests a growth condition on the gradient of V depending on the mean field of the algorithm:

Lemma 2. Assume H3, H4. For all $\mathbf{s} \in \mathcal{S}$,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (21)$$

Proof of this Lemma can be found in Appendix A.

3.1 Global Convergence of Incremental Noisy EM Algorithms

We present in this section a finite-time analysis of the incremental variant of the Stochastic Approximation of the EM algorithm. We want to draw the attention of the readers that the word "global" here does not mean for a global optimum of the nonconvex function, but of the independence of our analysis on the initialization and the iteration k (finite time).

The first intermediate result, see proof in Appendix B, is the computation of the quantity $\hat{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$, which corresponds to the drift term of (7) and reads as follows:

Lemma 3. The update (9) is equivalent to the following update on the resulting statistics

$$\hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}) \quad \text{where} \quad \tilde{\mathbf{S}}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{S}}_i^{(\tau_i^{k+1})} \quad (22)$$

Also:

$$\mathbb{E} [\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] = \mathbb{E} [\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{S}}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] + \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}] \quad (23)$$

where $\bar{\mathbf{s}}^{(k)}$ is defined by (3) and $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$.

171 The following main result for the iSAEM algorithm, which proof can be found in Appendix C, is
 172 derived under a control of the Monte Carlo fluctuations as described by assumption H 5. Typically,
 173 the controls exhibited above are of interest when the number of MC samples M_k increase with k .

174 **Theorem 1.** Assume H1-H5. Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of
 175 positive step sizes and consider the iSAEM sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = 1$ for any
 176 $k > 0$. We also set $c_1 = v_{\min}^{-1}$, $\alpha = \max\{8, 1 + 6v_{\min}\}$, $\bar{L} = \max\{L_s, L_V\}$, $\gamma_{k+1} = \frac{1}{k^\alpha \alpha c_1 \bar{L}}$ where
 177 $a \in (0, 1)$, $\beta = \frac{c_1 \bar{L}}{n}$. Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$.

$$v_{\max}^{-2} \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[\left\| \nabla V(\hat{\mathbf{s}}^{(k)}) \right\|^2 \right] \leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right] \quad (24)$$

178 3.2 Global Convergence of Two-Time-Scale Noisy EM Algorithms

179 We now proceed by giving our main result regarding the global convergence of the fiTTSEM algo-
 180 rithm. Two important auxiliary Lemmas, which proofs are given in Appendix D.1, are need in order
 181 to derive our finite-time bound. The first one derives an identity for the quantity $\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}\|^2]$
 182 using the vrTTSEM update:

183 **Lemma 4.** For any $k \geq 0$ and consider the vrTTSEM update in (10) with $\rho_k = \rho$, it holds for all
 184 $k > 0$

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &\quad + 2(1 - \rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{\mathbf{S}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (25)$$

185 where we recall that $\ell(k)$ is the first iteration number in the epoch that iteration k is in.

186 The second one derives an identity for the quantity $\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}\|^2]$ using the fiTTSEM update:

187 **Lemma 5.** For any $k \geq 0$ and consider the fiTTSEM update in (11) with $\rho_k = \rho$, it holds for all
 188 $k > 0$

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + 2(1 - \rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{\mathbf{S}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (26)$$

189 Recalling that K is an independent discrete r.v. drawn from $\{1, \dots, K_{\max}\}$ with distribution
 190 $\{\gamma_k / P_{\max}, 0 \leq k \leq K_{\max} - 1\}$, as in (13), we have

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] = \frac{1}{P_{\max}} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \quad (27)$$

191

192 We now state the main result regarding the vrTTSEM method, see proof in Appendix E:

193 **Theorem 2.** Assume H1-H5. Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of
 194 positive step sizes and consider the vrTTSEM sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = \rho$ for
 195 any $k > 0$. Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$. Setting $\bar{L} = \max\{L_s, L_V\}$, $\rho = \frac{\mu}{c_1 L n^{2/3}}$,
 196 $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$, a constant $\mu \in (0, 1)$, $\gamma_{k+1} = \frac{1}{k^a \bar{L}}$ where $a \in (0, 1)$, we have the following bound:

$$\begin{aligned} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] &\leq \frac{2n^{2/3} \bar{L}}{\mu P_{\max} v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{2n^{2/3} \bar{L}}{\mu P_{\max} v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[\tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right] \right] \end{aligned} \quad (28)$$

197 We now state the main result regarding the fiTTSEM method.

Theorem 3. Assume *H1-H5*. Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of positive step sizes and consider the *fTTSEM* sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = \rho$ for any $k > 0$. Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$. By setting $\alpha = \max\{2, 1 + 2v_{\min}\}$, $\bar{L} = \max\{L_s, L_V\}$, $\beta = \frac{1}{\alpha n}$, $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$, $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$, $\alpha \geq 2$ and $\gamma_{k+1} = \frac{1}{k^a \alpha c_1 \bar{L}}$ where $a \in (0, 1)$, we have the following bound:

$$\begin{aligned} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] &\leq \frac{4\alpha \bar{L} n^{2/3}}{\mathbf{P}_{\max} v_{\min}^2 v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{4\alpha \bar{L} n^{2/3}}{\mathbf{P}_{\max} v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[\Xi^{(k+1)} + \Gamma_{k+1} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right] \right] \end{aligned} \quad (29)$$

Proof of this Theorem can be found in Appendix F. Note that in those two bounds, the quantities $\tilde{\eta}^{(k+1)}$ and $\Xi^{(k+1)}$ depends only on the MC fluctuations $\mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right]$ and some constants.

Remarks: The following remarks are worth noting on the quantity $\mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right]$:

- This term is the price we pay for the two-time-scale dynamics and corresponds to the gap between the two asynchronous updates (one is on $\hat{\mathbf{s}}^{(k)}$ and the other on $\tilde{\mathbf{S}}^{(k)}$).
- It is trivial to see that if $\rho = 1$, i.e., there is no variance reduction, then for any $k > 0$

$$\mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right] = \mathbb{E} \left[\left\| \mathcal{S}^{(k+1)} - \tilde{\mathbf{S}}^{(k+1)} \right\|^2 \right] = 0 \quad \text{with} \quad \hat{\mathbf{s}}^{(0)} = \tilde{\mathbf{S}}^{(0)} = 0$$

which strengthen the fact that this quantity characterizes the impact of the variance reduction technique introduced in our two stages class of methods.

The following lemma, which proof can be found in Appendix D.2, characterizes this gap:

Lemma 6. Consider a decreasing stepsize $\gamma_k \in (0, 1)$ and a constant $\rho \in (0, 1)$, then the following inequality holds:

$$\mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right] \leq \frac{\rho}{1 - \rho} \sum_{\ell=0}^k (1 - \gamma_{\ell})^2 (\mathcal{S}^{(\ell)} - \tilde{\mathbf{S}}^{(\ell)}) \quad (30)$$

where $\mathcal{S}^{(k)}$ is defined either by (10) (*vrTTSEM*) or (11) (*fTTSEM*).

In the next section, we illustrate the benefits of our two-time-scale class of algorithms on several numerical applications.

4 Numerical Examples

For the sake of space, we provide details on the experiments in Appendix G.

4.1 Gaussian Mixture Models

We begin by a simple and illustrative example. The authors acknowledge that the following model can be trained using deterministic EM-type of algorithms but propose to apply stochastic methods, including theirs, and to compare their performances. Given n observations $\{y_i\}_{i=1}^n$, we want to fit a Gaussian Mixture Model (GMM) whose distribution is modeled as a Gaussian mixture of M components, each with a unit variance. Let $z_i \in \llbracket M \rrbracket$ be the latent labels of each component, the complete log-likelihood is defined as:

$$\log f(z_i, y_i; \boldsymbol{\theta}) = \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) [\log(\omega_m) - \mu_m^2/2] + \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) \mu_m y_i + \text{constant} . \quad (31)$$

where $\boldsymbol{\theta} := (\boldsymbol{\omega}, \boldsymbol{\mu})$ with $\boldsymbol{\omega} = \{\omega_m\}_{m=1}^{M-1}$ are the mixing weights with the convention $\omega_M = 1 - \sum_{m=1}^{M-1} \omega_m$ and $\boldsymbol{\mu} = \{\mu_m\}_{m=1}^M$ are the means. We use the penalization $r(\boldsymbol{\theta}) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \log \text{Dir}(\boldsymbol{\omega}; M, \epsilon)$ where $\delta > 0$ and $\text{Dir}(\cdot; M, \epsilon)$ is the M dimensional symmetric Dirichlet distribution with concentration parameter $\epsilon > 0$. The constraint set on $\boldsymbol{\theta}$ is given by

$$\Theta = \{\omega_m, m = 1, \dots, M-1 : \omega_m \geq 0, \sum_{m=1}^{M-1} \omega_m \leq 1\} \times \{\mu_m \in \mathbb{R}, m = 1, \dots, M\}. \quad (32)$$

229 In the following experiments on synthetic data,
 230 we generate 30 synthetic datasets of size $n =$
 231 10^5 from a GMM model with $M = 2$ com-
 232 ponents with two mixtures with means $\mu_1 =$
 233 $-\mu_2 = 0.5$. We run the bEM method until con-
 234 vergence (to double precision) to obtain the ML
 235 estimate μ^* averaged on 50 datasets. We com-
 236 pare the bEM, iEM (incremental EM), SAEM,
 237 iSAEM, vrTTSEM and fiTTSEM methods in
 238 terms of their precision measured by $|\mu - \mu^*|^2$.
 239 We set the stepsize of the SA-step of all
 240 method as $\gamma_k = 1/k^\alpha$ with $\alpha = 0.5$, and the
 241 stepsizes of the Incremental-step for vrTTSEM and the fiTTSEM to a constant stepsize equal to
 242 $1/n^{2/3}$. The number of MC samples is fixed to $M = 10$ chains. Figure 1 shows the convergence
 243 of the precision $|\mu - \mu^*|^2$ for the different methods against the epoch(s) elapsed (one epoch equals
 244 n iterations). Besides, vrTTSEM and fiTTSEM methods outperform the other stochastic methods,
 245 supporting the benefits of our scheme.

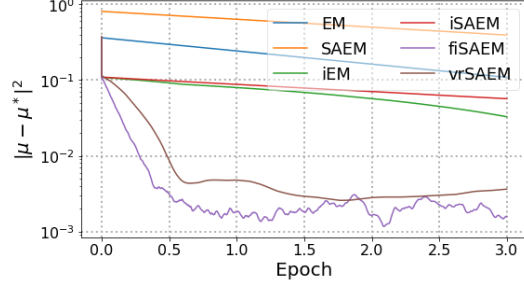


Figure 1: Precision $|\mu^{(k)} - \mu^*|^2$ per epoch

246 4.2 Deformable Template Model for Image Analysis

247 Let $(y_i, i \in \llbracket 1, n \rrbracket)$ be observed gray level images defined on a grid of pixels. Let $u \in \mathcal{U} \subset \mathbb{R}^2$
 248 denotes the pixel index on the image and $x_u \in \mathcal{D} \subset \mathbb{R}^2$ its location. The model used in this
 249 experiment suggests that each image y_i is a deformation of a template, noted $I : \mathcal{D} \rightarrow \mathbb{R}$, common
 250 to all images of the dataset:

$$y_i(u) = I(x_u - \Phi_i(x_u, z_i)) + \varepsilon_i(u) \quad (33)$$

251 where $\phi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a deformation function, z_i some latent variable parameterizing this defor-
 252 mation and $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$ is an observation error.

253 The template model, given $\{p_k\}_{k=1}^{k_p}$ landmarks on the template, a fixed known kernel \mathbf{K}_p and a
 254 vector of parameters $\beta \in \mathbb{R}^{k_p}$ is defined as follows:

$$I_\xi = \mathbf{K}_p \beta, \quad \text{where} \quad (\mathbf{K}_p \beta)(x) = \sum_{k=1}^{k_p} \mathbf{K}_p(x, p_k) \beta_k \quad (34)$$

255 Besides, we parameterize the deformation model given some landmarks $\{g_k\}_{k=1}^{k_g}$ and a fixed kernel
 256 \mathbf{K}_g as:

$$\Phi_i = \mathbf{K}_g z_i \quad \text{where} \quad (\mathbf{K}_g z_i)(x) = \sum_{k=1}^{k_g} \mathbf{K}_g(x, g_k) \left(z_i^{(1)}(k), z_i^{(2)}(k) \right) \quad (35)$$

257 where we put a Gaussian prior on the latent variables, $z_i \sim \mathcal{N}(0, \Gamma)$ and $z_i \in (\mathbb{R}^{k_g})^2$. The vector
 258 of parameters we ought to estimate is thus $\theta = (\beta, \Gamma, \sigma)$.

259 **Numerical Experiment:** We apply model (33) and our algorithms to a collection of handwritten
 260 digits, called the US postal database [9], featuring $n = 1\,000$ (16×16)-pixel images for each hand-
 261 written digit from 0 to 9. The main difficulty with these data comes from the geometric dispersion
 262 within each class of digit as shown Figure 2 for digit 5. We thus ought to use our deformable tem-
 263 plate model in order to account for both sources of variability: the intrinsic template to each class of
 264 digit and the small and local deformation in each observed image.



Figure 2: Training set of the USPS database (20 images for figt 5)

265 Figure ?? shows the resulting synthetic images for digit 5 through several epochs and for the bEM,
 266 SAEM and the TTS methods. We choose Gaussian kernels for both, \mathbf{K}_p and \mathbf{K}_g , defined on \mathbb{R}^2 and
 267 centered on the landmark points $\{p_k\}_{k=1}^{k_p}$ and $\{g_k\}_{k=1}^{k_g}$ with standard deviations 0.4. $k_p = k_g = 6$

landmarks are chosen randomly on the image for the training. Average of the images in the digit class is plotted on the left hand side in order to show how fast each method achieves a sharper template *w.r.t.* the average.

4.3 PK Model with Absorption Lag Time

This numerical example was conducted in order to characterize the pharmacokinetics (PK) of orally administered drug to simulated patients, using a population pharmacokinetic approach. $M = 50$ synthetic datasets were generated for $n = 500$ patients with 10 observations (concentration measures) per patient. The goal is to model the evolution of the concentration of the absorbed drug using a nonlinear and latent data model.

The model: We consider a one-compartment PK model for oral administration with an absorption lag-time (T^{lag}), assuming first-order absorption and linear elimination processes. The final model includes the following variables: ka the absorption rate constant, V the volume of distribution, k the elimination rate constant and T^{lag} the absorption lag-time. We also add several covariates to our model such as D the dose of drug administered, t the time at which measures are taken and the weight of the patient influencing the volume V . More precisely, the log-volume $\log(V)$ is a linear function of the log-weight $\log(wt/70) = \log(wt/70)$. Let $z_i = (T_i^{\text{lag}}, ka_i, V_i, k_i)$ be the vector of individual PK parameters, different for each individual i . The final model reads:

$$y_{ij} = f(t_{ij}, z_i) + \varepsilon_{ij} \quad \text{where} \quad f(t_{ij}, z_i) = \frac{D ka_i}{V(ka_i - k_i)} (e^{-ka_i(t_{ij} - T_i^{\text{lag}})} - e^{-k_i(t_{ij} - T_i^{\text{lag}})}) \quad (36)$$

where y_{ij} is the j -th concentration measurement of the drug of dosage D injected at time t_{ij} for patient i . We assume in this example that the residual errors ε_{ij} are independent and normally distributed with mean 0 and variance σ^2 . Lognormal distributions are used for the four PK parameters.

Monte Carlo study: We conduct a Monte Carlo study to showcase the benefits of our scheme.

$M = 50$ datasets have been simulated using the following PK parameters values: $T_{\text{pop}}^{\text{lag}} = 1$, $ka_{\text{pop}} = 1$, $V_{\text{pop}} = 8$, $k_{\text{pop}} = 0.1$, $\omega_{T^{\text{lag}}} = 0.4$, $\omega_{ka} = 0.5$, $\omega_V = 0.2$, $\omega_k = 0.3$ and $\sigma^2 = 0.5$. We define the mean square distance over the M replicates $E_k(\ell) = \frac{1}{M} \sum_{m=1}^M \left(\theta_k^{(m)}(\ell) - \theta^* \right)^2$ and plot it against the epochs (passes over the data) Figure 3. Note that the MC-step (5) is performed using a Metropolis Hastings procedure since the posterior distribution under the model θ noted $p(z_i|y_i, \theta)$ is intractable due to the nonlinearity of the model (36). Figure 3 shows clear advantage of variance reduced methods (vrTTSEM and fiTTSEM) avoiding the twists and turns displayed by the incremental and the batch methods.

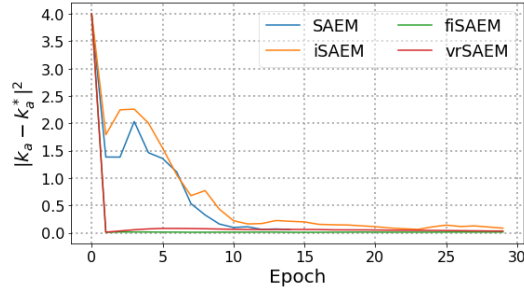


Figure 3: TO COMPLETE

5 Conclusion

References

- [1] S. Allasonnière, Y. Amit, and A. Trouvé. Towards a coherent statistical framework for dense deformable template estimation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 69(1):3–29, 2007.
- [2] C. Baey, S. Trevezas, and P.-H. Cournède. A non linear mixed effects model of plant growth and estimation via stochastic variants of the em algorithm. *Communications in Statistics-Theory and Methods*, 45(6):1643–1669, 2016.
- [3] O. Cappé. Online em algorithm for hidden markov models. *Journal of Computational and Graphical Statistics*, 20(3):728–749, 2011.
- [4] O. Cappé and E. Moulines. On-line expectation–maximization algorithm for latent data models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 71(3):593–613, 2009.
- [5] A. Chakraborty and K. Das. Inferences for joint modelling of repeated ordinal scores and time to event data. *Computational and mathematical methods in medicine*, 11(3):281–295, 2010.
- [6] B. Delyon, M. Lavielle, and E. Moulines. Convergence of a stochastic approximation version of the em algorithm. *Ann. Statist.*, 27(1):94–128, 03 1999. doi: 10.1214/aos/1018031103. URL <https://doi.org/10.1214/aos/1018031103>.
- [7] A. P. Dempster, N. M. Laird, and D. B. Rubin. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the royal statistical society. Series B (methodological)*, pages 1–38, 1977.
- [8] J. P. Hughes. Mixed effects models with censored data with application to hiv rna levels. *Biometrics*, 55(2):625–629, 1999.
- [9] J. J. Hull. A database for handwritten text recognition research. *IEEE Transactions on pattern analysis and machine intelligence*, 16(5):550–554, 1994.
- [10] B. Karimi, H.-T. Wai, É. Moulines, and M. Lavielle. On the global convergence of (fast) incremental expectation maximization methods. In *Advances in Neural Information Processing Systems*, pages 2833–2843, 2019.
- [11] E. Kuhn and M. Lavielle. Coupling a stochastic approximation version of em with an mcmc procedure. *ESAIM: Probability and Statistics*, 8:115–131, 2004.
- [12] E. Kuhn, C. Matias, and T. Rebafka. Properties of the stochastic approximation em algorithm with mini-batch sampling. *arXiv preprint arXiv:1907.09164*, 2019.
- [13] P. Liang and D. Klein. Online em for unsupervised models. In *Proceedings of human language technologies: The 2009 annual conference of the North American chapter of the association for computational linguistics*, pages 611–619, 2009.
- [14] C. E. McCulloch. Maximum likelihood algorithms for generalized linear mixed models. *Journal of the American statistical Association*, 92(437):162–170, 1997.
- [15] G. McLachlan and T. Krishnan. *The EM algorithm and extensions*, volume 382. John Wiley & Sons, 2007.
- [16] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Springer Science & Business Media, 2012.
- [17] R. M. Neal and G. E. Hinton. A view of the EM algorithm that justifies incremental, sparse, and other variants. In *Learning in graphical models*, pages 355–368. Springer, 1998.

- 346 [18] H. D. Nguyen, F. Forbes, and G. J. McLachlan. Mini-batch learning of exponential family
347 finite mixture models. *Statistics and Computing*, pages 1–18, 2020.
- 348 [19] S. J. Reddi, S. Sra, B. Póczos, and A. Smola. Fast incremental method for nonconvex opti-
349 mization. *arXiv preprint arXiv:1603.06159*, 2016.
- 350 [20] H. Robbins and S. Monro. A stochastic approximation method. *The annals of mathematical*
351 *statistics*, pages 400–407, 1951.
- 352 [21] G. C. Wei and M. A. Tanner. A monte carlo implementation of the em algorithm and the poor
353 man’s data augmentation algorithms. *Journal of the American statistical Association*, 85(411):
354 699–704, 1990.
- 355 [22] R. Zhu, L. Wang, C. Zhai, and Q. Gu. High-dimensional variance-reduced stochastic gradient
356 expectation-maximization algorithm. In *Proceedings of the 34th International Conference on*
357 *Machine Learning-Volume 70*, pages 4180–4188. JMLR. org, 2017.

A Proof of Lemma 2

Lemma. Assume H3, H4. For all $\mathbf{s} \in \mathcal{S}$,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (37)$$

Proof Using H3 and the fact that we can exchange integration with differentiation and the Fisher's identity, we obtain

$$\begin{aligned} \nabla_{\mathbf{s}} V(\mathbf{s}) &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^\top \left(\nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{L}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^\top \left(\nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^\top \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^\top \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^\top (\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))), \end{aligned} \quad (38)$$

Consider the following vector map:

$$\mathbf{s} \rightarrow \nabla_{\boldsymbol{\theta}} L(\mathbf{s}, \boldsymbol{\theta})|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})} = \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^\top \mathbf{s}. \quad (39)$$

Taking the gradient of the above map w.r.t. \mathbf{s} and using assumption H3, we show that:

$$\mathbf{0} = -\mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \underbrace{\left(\nabla_{\boldsymbol{\theta}}^2 (\psi(\boldsymbol{\theta}) + \mathbf{r}(\boldsymbol{\theta}) - \langle \phi(\boldsymbol{\theta}) | \mathbf{s} \rangle) \right)|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})}}_{=\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \boldsymbol{\theta})} \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s}). \quad (40)$$

The above yields

$$\nabla_{\mathbf{s}} V(\mathbf{s}) = \mathbf{B}(\mathbf{s})(\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))) \quad (41)$$

where we recall $\mathbf{B}(\mathbf{s}) = \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \left(\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \bar{\boldsymbol{\theta}}(\mathbf{s})) \right)^{-1} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^\top$. The proof of (37) follows directly from the assumption H4. \square

B Proof of Lemma 3

Lemma. Assume H??. The update (9) is equivalent to the following update on the resulting statistics

$$\hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}) \quad (42)$$

Also:

$$\mathbb{E} [\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] = \mathbb{E} [\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] + \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}] \quad (43)$$

where $\bar{\mathbf{s}}^{(k)}$ is defined by (3) and $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$.

Proof From update (9), we have:

$$\begin{aligned} \tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= \tilde{\mathbf{S}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \frac{1}{n} \left(\tilde{S}_{i_k}^{(k+1)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)} \right) \\ &= \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{\mathbf{S}}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left(\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \tilde{S}_{i_k}^{(k+1)} \right) \end{aligned} \quad (44)$$

Since $\tilde{S}_{i_k}^{(k+1)} = \bar{s}_{i_k}(\boldsymbol{\theta}^{(k)}) + \eta_{i_k}^{(k+1)}$ we have

$$\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{\mathbf{S}}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left(\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{s}_{i_k}(\boldsymbol{\theta}^{(k)}) \right) + \frac{1}{n} \eta_{i_k}^{(k+1)} \quad (45)$$

Taking the full expectation of both side of the equation leads to:

$$\begin{aligned} \mathbb{E} [\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] &= \mathbb{E} [\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] \\ &\quad - \frac{1}{n} \mathbb{E} \left[\mathbb{E} [\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{s}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k] \right] + \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}] \end{aligned} \quad (46)$$

The following equalities:

$$\mathbb{E} [\tilde{S}_i^{(\tau_i^k)} | \mathcal{F}_k] = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} \quad \text{and} \quad \mathbb{E} [\bar{s}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k] = \bar{\mathbf{s}}^{(k)} \quad (47)$$

concludes the proof of the Lemma. \square

377 C Proof of Theorem 1

378 **Theorem.** Assume *H1-H5*. Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of
 379 positive step sizes and consider the iSAEM sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = 1$ for any
 380 $k > 0$. We also set $c_1 = v_{\min}^{-1}$, $\alpha = \max\{8, 1 + 6v_{\min}\}$, $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$, $\gamma_{k+1} = \frac{1}{k^\alpha \alpha c_1 \bar{L}}$ where
 381 $a \in (0, 1)$, $\beta = \frac{c_1 \bar{L}}{n}$. Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$.

$$v_{\max}^{-2} \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[\left\| \nabla V(\hat{\mathbf{s}}^{(k)}) \right\|^2 \right] \leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right] \quad (48)$$

382 **Proof** We begin our proof by giving this auxiliary Lemma setting an upper bound for the quantity
 383 $\mathbb{E} \left[\left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right]$

384 **Lemma 7.** For any $k \geq 0$ and consider the iSAEM update in (9), it holds that

$$\begin{aligned} \mathbb{E} \left[\left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] &\leq 4\mathbb{E} \left[\left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] + \frac{2L_{\mathbf{s}}^2}{n^3} \sum_{i=1}^n \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \right\|^2 \right] \\ &\quad + 2\frac{C_\eta}{M_k} + 4\mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \end{aligned} \quad (49)$$

385 **Proof** Applying the iSAEM update yields:

$$\begin{aligned} \mathbb{E} \left[\left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] &= \mathbb{E} \left[\left\| \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} - \frac{1}{n} (\tilde{S}_{i_k}^{(\tau_i^k)} - \tilde{S}_{i_k}^{(k)}) \right\|^2 \right] \\ &\leq 4\mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] + 4\mathbb{E} \left[\left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &\quad + \frac{2}{n^2} \mathbb{E} \left[\left\| \bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)} \right\|^2 \right] + 2\frac{C_\eta}{M_k} \end{aligned} \quad (50)$$

386 The last expectation can be further bounded by

$$\frac{2}{n^2} \mathbb{E} \left[\left\| \bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)} \right\|^2 \right] = \frac{2}{n^3} \sum_{i=1}^n \mathbb{E} \left[\left\| \bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)} \right\|^2 \right] \stackrel{(a)}{\leq} \frac{2L_{\mathbf{s}}^2}{n^3} \sum_{i=1}^n \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \right\|^2 \right], \quad (51)$$

387 where (a) is due to Lemma 1 and which concludes the proof of the Lemma.

388 □

389 Under the smoothness of the Lyapunov function V (cf. Lemma 1), we can write:

$$V(\hat{\mathbf{s}}^{(k+1)}) \leq V(\hat{\mathbf{s}}^{(k)}) + \gamma_{k+1} \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \quad (52)$$

390 Taking the expectation on both sides yields:

$$\mathbb{E} \left[V(\hat{\mathbf{s}}^{(k+1)}) \right] \leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(k)}) \right] + \gamma_{k+1} \mathbb{E} \left[\langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} \left[\left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \quad (53)$$

391 Using Lemma 3, we obtain:

$$\begin{aligned}
& \mathbb{E} \left[\langle \tilde{S}^{(k+1)} - \hat{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] = \\
& \mathbb{E} \left[\langle \bar{s}^{(k)} - \hat{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] + \left(1 - \frac{1}{n} \right) \mathbb{E} \left[\left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \right\rangle \right] + \frac{1}{n} \mathbb{E} \left[\langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] \\
& \stackrel{(a)}{\leq} -v_{\min} \mathbb{E} \left[\left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] + \left(1 - \frac{1}{n} \right) \mathbb{E} \left[\left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \right\rangle \right] + \frac{1}{n} \mathbb{E} \left[\langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] \\
& \stackrel{(b)}{\leq} -v_{\min} \mathbb{E} \left[\left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] \\
& + \frac{\beta(n-1)+1}{2n} \mathbb{E} \left[\left\| \nabla V(\hat{s}^{(k)}) \right\|^2 \right] + \frac{1}{2n} \mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right] \\
& \stackrel{(a)}{\leq} \left(v_{\max}^2 \frac{\beta(n-1)+1}{2n} - v_{\min} \right) \mathbb{E} \left[\left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] + \frac{1}{2n} \mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right]
\end{aligned} \tag{54}$$

392 where (a) is due to the growth condition (2) and (b) is due to Young's inequality (with $\beta \rightarrow 1$). Note

393 $a_k = \gamma_{k+1} \left(v_{\min} - v_{\max}^2 \frac{\beta(n-1)+1}{2n} \right)$ and

$$\begin{aligned}
a_k \mathbb{E} \left[\left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] & \leq \mathbb{E} \left[V(\hat{s}^{(k)}) - V(\hat{s}^{(k+1)}) \right] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} \left[\left\| \tilde{S}^{(k+1)} - \hat{s}^{(k)} \right\|^2 \right] \\
& + \frac{\gamma_{k+1}(1 - \frac{1}{n})}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] + \frac{\gamma_{k+1}}{2n} \mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right]
\end{aligned} \tag{55}$$

394 We now give an upper bound of $\mathbb{E} \left[\left\| \tilde{S}^{(k+1)} - \hat{s}^{(k)} \right\|^2 \right]$ using Lemma 7 and plug it into (55):

$$\begin{aligned}
(a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E} \left[\left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] & \leq \mathbb{E} \left[V(\hat{s}^{(k)}) - V(\hat{s}^{(k+1)}) \right] \\
& + \gamma_{k+1} \left(\frac{1}{2\beta} \left(1 - \frac{1}{n} \right) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] \\
& + \gamma_{k+1} \left(\gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right] \\
& + \frac{\gamma_{k+1}^2 L_V L_s^2}{n^3} \sum_{i=1}^n \mathbb{E} \left[\left\| \hat{s}^{(k)} - \hat{s}^{(\tau_i^k)} \right\|^2 \right]
\end{aligned} \tag{56}$$

395 Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\left\| \hat{s}^{(k+1)} - \hat{s}^{(\tau_i^{k+1})} \right\|^2 \right] = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{n} \mathbb{E} \left[\left\| \hat{s}^{(k+1)} - \hat{s}^{(k)} \right\|^2 \right] + \frac{n-1}{n} \mathbb{E} \left[\left\| \hat{s}^{(k+1)} - \hat{s}^{(\tau_i^k)} \right\|^2 \right] \right) \tag{57}$$

396 where the equality holds as i_k and j_k are drawn independently. For any $\beta > 0$, it holds

$$\begin{aligned}
& \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\
&= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\
&= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 - 2\gamma_{k+1}\langle \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\
&\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}\|^2 + \gamma_{k+1}\beta\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right]
\end{aligned} \tag{58}$$

397 where the last inequality is due to the Young's inequality. Subsequently, we have

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\
&\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}\left[(1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}\|^2\right]
\end{aligned} \tag{59}$$

398 Observe that $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)})$. Applying Lemma 7 yields

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\
&\leq (\gamma_{k+1}^2 + \frac{n-1}{n} \frac{\gamma_{k+1}}{\beta}) \mathbb{E}[\|\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right] \\
&\leq 4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}\left[\|\eta_{i_k}^{(k)}\|^2\right] \\
&\quad + 4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{S}}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] \\
&\quad + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2\right]
\end{aligned} \tag{60}$$

399 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2] \tag{61}$$

400 From the above, we get

$$\begin{aligned}
\Delta^{(k+1)} &\leq (1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}))\Delta^{(k)} + 4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
&\quad + 2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}\left[\|\eta_{i_k}^{(k)}\|^2\right] + 4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{S}}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right]
\end{aligned} \tag{62}$$

401 Setting $c_1 = v_{\min}^{-1}$, $\alpha = \max\{8, 1 + 6v_{\min}\}$, $\bar{L} = \max\{L_s, L_V\}$, $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$, $\beta = \frac{c_1 \bar{L}}{n}$,

402 $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 6$, $\alpha \geq 8$, we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \leq 1 - \frac{c_1(k\alpha - 1) - 4}{k\alpha n c_1} \leq 1 - \frac{2}{k\alpha n c_1} \tag{63}$$

403 which shows that $1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \in (0, 1)$ for any $k > 0$. Denote $\Lambda_{(k+1)} =$
 404 $\frac{1}{n} - \gamma_{k+1}\beta - \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})$ and note that $\Delta^{(0)} = 0$, thus the telescoping sum yields:

$$\begin{aligned} \Delta^{(k+1)} \leq & 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] + 2 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E} \left[\left\| \eta_{i_\ell}^{(\ell)} \right\|^2 \right] \\ & + 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^\ell)} - \bar{\mathbf{s}}^{(\ell)} \right\|^2 \right] \end{aligned} \quad (64)$$

405 Note $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$ Summing on both sides over $k = 0$ to $k = K_{\max} - 1$ yields:

$$\begin{aligned} & \sum_{k=0}^{K_{\max}-1} \Delta^{(k+1)} \\ &= 4 \sum_{k=0}^{K_{\max}-1} (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2 \sum_{k=0}^{K_{\max}-1} (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E} \left[\left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\ &+ \sum_{k=0}^{K_{\max}-1} 4 (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &\leq \sum_{k=0}^{K_{\max}-1} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{k=0}^{K_{\max}-1} \frac{2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E} \left[\left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \end{aligned} \quad (65)$$

406 We recall (56) where we have summed on both sides from $k = 0$ to $k = K_{\max} - 1$:

$$\begin{aligned} & \sum_{k=0}^{K_{\max}-1} (a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E} \left[\left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \left(\frac{1}{2\beta} (1 - \frac{1}{n}) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \left(\gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \Delta^{(k)} \end{aligned} \quad (66)$$

407 Plugging (65) into (66) results in:

$$\begin{aligned} & \sum_{k=0}^{K_{\max}-1} \tilde{\alpha}_k \mathbb{E} \left[\left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\beta}_k \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E} \left[\left\| \eta_{i_k}^{(k)} \right\|^2 \right] \end{aligned} \quad (67)$$

408 where:

$$\begin{aligned}\tilde{\alpha}_k &= a_k - 2\gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^2} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \\ \tilde{\beta}_k &= \gamma_{k+1} \left(\frac{1}{2\beta} (1 - \frac{1}{n}) + 2\gamma_{k+1} L_V \right) - \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^2} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \\ \tilde{\Gamma}_k &= \gamma_{k+1} \left(\gamma_{k+1} L_V + \frac{1}{2n} \right) + \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^2} \frac{2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}}\end{aligned}$$

409 and

$$\begin{aligned}a_k &= \gamma_{k+1} \left(v_{\min} - v_{\max}^2 \frac{\beta(n-1) + 1}{2n} \right) \\ \Lambda_{(k+1)} &= \frac{1}{n} - \gamma_{k+1}\beta - \frac{2\gamma_{k+1} L_{\mathbf{s}}^2}{n^2} (\gamma_{k+1} + \frac{1}{\beta}) \\ c_1 &= v_{\min}^{-1}, \alpha = \max\{8, 1 + 6v_{\min}\}, \bar{L} = \max\{L_{\mathbf{s}}, L_V\}, \gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}, \beta = \frac{c_1 \bar{L}}{n}\end{aligned}$$

410 When, for any $k > 0$, $\tilde{\alpha}_k \geq 0$, we have by Lemma 2 that:

$$\sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[\left\| \nabla V(\hat{\mathbf{s}}^{(k)}) \right\|^2 \right] \leq v_{\max}^2 \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[\left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \quad (68)$$

411 which yields an upper bound of the gradient of the Lyapunov function V along the path of the
412 iSAEM update and concludes the proof of the Theorem. \square

413 D Proofs of Auxiliary Lemmas

414 D.1 Proof of Lemma 4 and Lemma 5

415 **Lemma.** For any $k \geq 0$ and consider the vrTTSEM update in (10) with $\rho_k = \rho$, it holds for all
416 $k > 0$

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &\quad + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (69)$$

417 where we recall that $\ell(k)$ is the first iteration number in the epoch that iteration k is in.

418 **Proof** Beforehand, we provide a rewriting of the quantity $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$ that will be useful through-
419 out this proof:

$$\begin{aligned} \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathbf{S}^{(k+1)}) \\ &= -\gamma_{k+1} \left((1-\rho) \left[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[\hat{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)} \right] \right) \end{aligned} \quad (70)$$

420 We observe, using the identity (70), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] \quad (71)$$

421 For the latter term, we obtain its upper bound as

$$\begin{aligned} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] &= \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \tilde{S}_i^{\ell(k)}) - (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{\ell(k)}) \right\|^2 \right] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{\ell(k)}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \stackrel{(b)}{\leq} L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{\ell(k)}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (72)$$

422 where (a) uses the variance inequality and (b) uses Lemma 1. Substituting into (71) proves the
423 lemma. \square

424 **Lemma.** For any $k \geq 0$ and consider the fiTTSEM update in (11) with $\rho_k = \rho$, it holds for all $k > 0$
425

$$\begin{aligned} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_k)}\|^2] \\ &\quad + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (73)$$

426 **Proof** Beforehand, we provide a rewriting of the quantity $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$ that will be useful through-
427 out this proof:

$$\begin{aligned} \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) \\ &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathbf{S}^{(k+1)}) \\ &= -\gamma_{k+1} \left((1-\rho) \left[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[\hat{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)} \right] \right) \\ &= -\gamma_{k+1} \left((1-\rho) \left[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} - (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_k)}) \right] \right) \end{aligned} \quad (74)$$

428 We observe, using the identity (74), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] \quad (75)$$

429 For the latter term, we obtain its upper bound as

$$\begin{aligned}\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] &= \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{S}}_i^{(k)}) - (\tilde{\mathbf{S}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)})\right\|^2\right] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(\ell(k))}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]\end{aligned}\quad (76)$$

430 where (a) uses the variance inequality. We can further bound the last expectation using Lemma 1:

$$\mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)}\|^2] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2] \stackrel{(a)}{\leq} \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \quad (77)$$

431 Substituting into (75) proves the lemma. \square

432 D.2 Proof of Lemma 6

433 **Lemma.** Consider a decreasing stepsize $\gamma_k \in (0, 1)$ and a constant ρ , then the following inequality
434 holds:

$$\mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \leq \frac{\rho}{1-\rho} \sum_{\ell=0}^k (1-\gamma_\ell)^2 (\mathbf{S}^{(\ell)} - \tilde{\mathbf{S}}^{(\ell)}) \quad (78)$$

435 where $\mathbf{S}^{(k)}$ is defined either by (11) (fTTSEM) or (10) (vrTTSEM)

436 **Proof** We begin by writing the two-time-scale update:

$$\begin{aligned}\tilde{\mathbf{S}}^{(k+1)} &= \tilde{\mathbf{S}}^{(k)} + \rho(\mathbf{S}^{(k+1)} - \tilde{\mathbf{S}}^{(k)}) \\ \hat{\mathbf{s}}^{(k+1)} &= \hat{\mathbf{s}}^{(k)} + \gamma_{k+1}(\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)})\end{aligned}\quad (79)$$

437 where $\mathbf{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{S}}_i^{(t_i^k)} + (\tilde{\mathbf{S}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)})$ according to (11). Denote $\delta^{(k+1)} = \hat{\mathbf{s}}^{(k+1)} -$
438 $\tilde{\mathbf{S}}^{(k+1)}$. Then from (79), doing the subtraction of both equations yields:

$$\delta^{(k+1)} = (1 - \gamma_{k+1})\delta^{(k)} + \frac{\rho}{1-\rho} (1 - \gamma_{k+1})(\mathbf{S}^{(k+1)} - \tilde{\mathbf{S}}^{(k+1)}) \quad (80)$$

439 Using the telescoping sum and noting that $\delta^{(0)} = 0$, we have

$$\delta^{(k+1)} \leq \frac{\rho}{1-\rho} \sum_{\ell=0}^k (1 - \gamma_{\ell+1})^2 (\mathbf{S}^{(\ell+1)} - \tilde{\mathbf{S}}^{(\ell+1)}) \quad (81)$$

440 \square

441 D.3 Additional Intermediary Result

442 **Lemma 8.** At iteration $k + 1$, the drift term of update (11), with $\rho_{k+1} = \rho$, is equivalent to the
443 following :

$$\begin{aligned}\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)} &= \rho(\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}) + \rho\eta_{i_k}^{(k+1)} + \rho \left[(\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)}] \right] \\ &\quad + (1 - \rho) (\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)})\end{aligned}\quad (82)$$

444 where we recall that $\eta_{i_k}^{(k+1)}$, defined in (19), which is the gap between the MC approximation and
445 the expected statistics.

446 **Proof** Using the fiTTSEM update $\tilde{S}^{(k+1)} = (1-\rho)\tilde{S}^{(k)} + \rho\mathcal{S}^{(k+1)}$ where $\mathcal{S}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})$ leads to the following decomposition:

$$\begin{aligned}
& \tilde{S}^{(k+1)} - \hat{s}^{(k)} \\
&= (1-\rho)\tilde{S}^{(k)} + \rho\left(\overline{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})\right) - \hat{s}^{(k)} + \rho\overline{\mathcal{S}}^{(k)} - \rho\overline{\mathcal{S}}^{(k)} \\
&= \rho(\overline{\mathcal{S}}^{(k)} - \hat{s}^{(k)}) + \rho(\tilde{S}_{i_k}^{(k)} - \overline{\mathcal{S}}_{i_k}^{(k)}) + (1-\rho)\left(\tilde{S}^{(k)} - \hat{s}^{(k)}\right) + \rho\left(\overline{\mathcal{S}}^{(k)} - \overline{\mathcal{S}}^{(k)} + (\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})\right) \\
&= \rho(\overline{\mathcal{S}}^{(k)} - \hat{s}^{(k)}) + \rho\eta_{i_k}^{(k+1)} - \rho\left[(\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}]\right] \\
&+ (1-\rho)\left(\tilde{S}^{(k)} - \hat{s}^{(k)}\right)
\end{aligned}$$

448 where we observe that $\mathbb{E}[\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] = \overline{\mathcal{S}}^{(k)} - \overline{\mathcal{S}}^{(k)}$ and which concludes the proof.

449 *Important Note:* Note that $\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}$ is not equal to $\eta_{i_k}^{(k+1)}$, defined in (19), which is the gap
450 between the MC approximation and the expected statistics. Indeed $\tilde{S}_{i_k}^{(t_{i_k}^k)}$ is not computed under the
451 same model as $\overline{\mathcal{S}}_{i_k}^{(k)}$. □

452 E Proof of Theorem 2

453 **Theorem.** Assume H1-H5. Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of
 454 positive step sizes and consider the vrTTSEM sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = \rho$ for
 455 any $k > 0$.

456 Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$. By setting $\bar{L} = \max\{L_S, L_V\}$, $\rho = \frac{\mu}{c_1 \bar{L} n^{2/3}}$, $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$
 457 and a constant $\mu \in (0, 1)$ and $\gamma_{k+1} = \frac{1}{k^a \bar{L}}$ where $a \in (0, 1)$, we have the following bound:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[\tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right] \right] \end{aligned} \quad (83)$$

458 **Proof** Using the smoothness of V and update (10), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) + \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}\|^2 \end{aligned} \quad (84)$$

459 Denote $\mathbf{H}_{k+1} := \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}$ the drift term of the fiTTSEM update in (7) and $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$.
 460 Taking expectations on both sides show that

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] \\ &\stackrel{(a)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}(1 - \rho) \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1} \rho \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\stackrel{(b)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} \rho \mathbb{E}[\langle \mathbf{h}_k | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1}(1 - \rho) \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad - \gamma_{k+1} \rho \mathbb{E}[\langle \eta_{i_k}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\stackrel{(c)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - (\gamma_{k+1} \rho v_{\min} + \gamma_{k+1} v_{\max}^2) \mathbb{E}[\|\mathbf{h}_k\|^2] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\quad - \gamma_{k+1} \rho \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \gamma_{k+1}(1 - \rho) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] \end{aligned} \quad (85)$$

461 where we have used (70) in (a) and $\mathbb{E}[\mathbf{S}^{(k+1)}] = \bar{\mathbf{s}}^{(k)} + \mathbb{E}[\eta_{i_k}^{(k+1)}]$ in (b), the growth condition in
 462 Lemma 2 and the Young's inequality with the constant equal to 1 in (c).

463 Furthermore, for $k+1 \leq \ell(k) + m$ (i.e., $k+1$ is in the same epoch as k), we have

$$\begin{aligned} &\mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] = \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} + \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \rangle] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 \\ &\quad - 2\gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \rho(\mathbf{h}_k - \eta_{i_k}^{(k+1)}) + (1 - \rho)(\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}) \rangle] \\ &\leq \mathbb{E}[(1 + \gamma_{k+1}\beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2 \\ &\quad + \frac{\gamma_{k+1}\rho}{\beta} \|\eta_{i_k}^{(k+1)}\|^2 + \frac{\gamma_{k+1}(1 - \rho)}{\beta} \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2], \end{aligned} \quad (86)$$

464 where we first used (70) and the last inequality is due to the Young's inequality.

465 Consider the following sequence

$$R_k := \mathbb{E}[V(\hat{\mathbf{s}}^{(k)}) + b_k \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \quad (87)$$

466 where $b_k := \bar{b}_{k \bmod m}$ is a periodic sequence where:

$$\bar{b}_i = \bar{b}_{i+1}(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2) + \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2, \quad i = 0, 1, \dots, m-1 \quad \text{with } \bar{b}_m = 0. \quad (88)$$

467 Note that \bar{b}_i is decreasing with i and this implies

$$\bar{b}_i \leq \bar{b}_0 = \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2 \frac{(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2)^m - 1}{\gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2}, \quad i = 1, 2, \dots, m. \quad (89)$$

468 For $k+1 \leq \ell(k) + m$, we have the following inequality

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2) \|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[\rho \left\|\eta_{i_k}^{(k+1)}\right\|^2 - (1-\rho) \left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\frac{\gamma_{k+1}\rho}{\beta} \left\|\eta_{i_k}^{(k+1)}\right\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2\right] \end{aligned} \quad (90)$$

469 And using Lemma 4 we obtain:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2 - \gamma_{k+1}^2\rho^2 L_V) \|\mathbf{h}_k\|^2 + \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2 \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2\right) \|\mathbf{h}_k\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[(\rho + \rho^2\gamma_{k+1} L_V) \left\|\eta_{i_k}^{(k+1)}\right\|^2 - (1-\rho - (1-\rho)^2\gamma_{k+1} L_V) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2\right) \left\|\eta_{i_k}^{(k+1)}\right\|^2 + \left(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2(1-\rho)^2\right) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2\right] \end{aligned} \quad (91)$$

470 Rearranging the terms yields:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2)) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\quad + \underbrace{\left(b_{k+1}(1 + \gamma\beta + 2\gamma^2\rho^2 L_{\mathbf{s}}^2) + \gamma^2\rho^2 L_V L_{\mathbf{s}}^2\right)}_{=b_k \text{ since } k+1 \leq \ell(k) + m} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] + \tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)} \end{aligned} \quad (92)$$

471 where

$$\begin{aligned} \tilde{\eta}^{(k+1)} &= \left(\gamma_{k+1}(\rho + \rho^2\gamma_{k+1} L_V) + b_{k+1}(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2)\right) \mathbb{E}\left[\left\|\eta_{i_k}^{(k+1)}\right\|^2\right] \\ \chi^{(k+1)} &= \left(b_{k+1}(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2(1-\rho)^2) - \gamma_{k+1}(1-\rho - (1-\rho)^2\gamma_{k+1} L_V)\right) \\ \tilde{\chi}^{(k+1)} &= \chi^{(k+1)} \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2\right] \end{aligned} \quad (93)$$

472 This leads, using Lemma 2, that for any γ_{k+1} , ρ and β such that $\rho v_{\min} + v_{\max}^2 -$
473 $\gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2) > 0$,

$$\begin{aligned} v_{\max}^2 \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] \leq \frac{R_k - R_{k+1}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))} \\ &\quad + \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))} \end{aligned} \quad (94)$$

474 We first remark that

$$\begin{aligned} & \gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2)) \\ & \geq \frac{\gamma_{k+1}\rho}{c_1}(1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1)) \end{aligned} \quad (95)$$

475 where $c_1 = v_{\min}^{-1}$. By setting $\bar{L} = \max\{L_s, L_V\}$, $\beta = \frac{c_1\bar{L}}{n^{1/3}}$, $\rho = \frac{\mu}{c_1\bar{L}n^{2/3}}$, $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$ and
 476 $\{\gamma_{k+1}\}$ any sequence of decreasing stepsizes in $(0, 1)$, it can be shown that there exists $\mu \in (0, 1)$,
 477 such that the following lower bound holds

$$\begin{aligned} & 1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1) \geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \bar{b}_0(\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ & \geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{L_V\mu^2}{c_1^2n^{\frac{4}{3}}}\frac{(1 + \gamma\beta + 2\gamma^2L_s^2)^m - 1}{\gamma\beta + 2\gamma^2L_s^2}(\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ & \stackrel{(a)}{\geq} 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{\mu}{c_1^2}(e - 1)(1 + \frac{2\mu}{n}) \geq 1 - \mu - \mu(1 + 2\mu)\frac{e - 1}{c_1^2} \stackrel{(b)}{\geq} \frac{1}{2} \end{aligned} \quad (96)$$

478 where the simplification in (a) is due to

$$\frac{\mu}{n} \leq \gamma\beta + 2\gamma^2L_s^2 \leq \frac{\mu}{n} + \frac{2\mu^2}{c_1^2n^{\frac{4}{3}}} \leq \frac{\mu c_1^2 + 2\mu^2}{c_1^2} \frac{1}{n} \text{ and } (1 + \gamma\beta + 2\gamma^2L_s^2)^m \leq e - 1. \quad (97)$$

479 and the required μ in (b) can be found by solving the quadratic equation.

480 Finally, these results yield:

$$v_{\max}^2 \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \leq \frac{2(R_0 - R_{K_{\max}})}{v_{\min}\rho} + 2 \sum_{k=0}^{K_{\max}-1} \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{v_{\min}\rho} \quad (98)$$

481 Note that $R_0 = \mathbb{E}[V(\hat{s}^{(0)})]$ and if K_{\max} is a multiple of m , then $R_{K_{\max}} = \mathbb{E}[V(\hat{s}^{(K_{\max})})]$. Under the
 482 latter condition, we have

$$\sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{s}^{(0)}) - V(\hat{s}^{(K_{\max})})] + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} [\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}] \quad (99)$$

483 This concludes our proof.

484 □

485 F Proof of Theorem 3

486 **Theorem.** Assume H1-H5. Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of
 487 positive step sizes and consider the fTTSEM sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = \rho$ for
 488 any $k > 0$.

489 Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$. By setting $\alpha = \max\{2, 1 + 2v_{\min}\}$, $\bar{L} = \max\{L_s, L_V\}$,
 490 $\beta = \frac{c_1 \bar{L}}{n}$, $\rho = \frac{1}{n^{2/3}}$, $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$, $\alpha \geq 2$ and $\gamma_{k+1} = \frac{1}{k^a \alpha c_1 \bar{L}}$ where $a \in (0, 1)$, we
 491 have the following bound:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[\Xi^{(k+1)} + \Gamma_{k+1} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right] \end{aligned} \quad (100)$$

492 **Proof** Using the smoothness of V and update (11), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) + \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \end{aligned} \quad (101)$$

493 Denote $\mathbf{H}_{k+1} := \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}$ the drift term of the fTTSEM update in (7) and $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$.
 494 Using Lemma 8 and the additional following identity:

$$\mathbb{E} \left[(\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] = 0 \quad (102)$$

495 we have:

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] \\ &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} \rho \mathbb{E}[\langle \mathbf{h}_k | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1} \mathbb{E} \left[\langle \rho \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] + (1 - \rho) \mathbb{E}[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}] | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2 \\ &\stackrel{(a)}{\leq} -v_{\min} \gamma_{k+1} \rho \mathbb{E}[\|\mathbf{h}_k\|^2] - \gamma_{k+1} \mathbb{E} \left[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2 \right] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2 \\ &\stackrel{(b)}{\leq} -(v_{\min} \gamma_{k+1} \rho + \gamma_{k+1} v_{\max}^2) \mathbb{E}[\|\mathbf{h}_k\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2 \end{aligned} \quad (103)$$

496 where $\xi^{(k+1)} = \mathbb{E} \left[\left\| \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] \right\|^2 \right]$. **Bounding** $\mathbb{E}[\|\mathbf{H}_{k+1}\|^2]$ Using Lemma 5, we obtain:

$$\begin{aligned} &\gamma_{k+1} (v_{\min} \rho + v_{\max}^2 - \gamma_{k+1} \rho^2 L_V) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\leq \mathbb{E} [V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)})] + \tilde{\xi}^{(k+1)} + \left((1 - \rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V \rho^2 L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \end{aligned} \quad (104)$$

497 where $\tilde{\xi}^{(k+1)} = \gamma_{k+1}^2 \rho^2 \mathbb{L}_V \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \frac{\gamma_{k+1}\rho^2}{2} \xi^{(k+1)}$. Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \right) \quad (105)$$

498 where the equality holds as i_k and j_k are drawn independently. Next,

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \end{aligned} \quad (106)$$

499 Note that $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) = -\gamma_{k+1}\mathbf{H}_{k+1}$ and that in expectation we recall
500 that $\mathbb{E}[\mathbf{H}_{k+1}|\mathcal{F}_k] = \rho\mathbf{h}_k + \rho\mathbb{E}[\eta_{i_k}^{(k+1)}|\mathcal{F}_k] + (1-\rho)\mathbb{E}[\tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)}]$ where $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$. Thus,
501 for any $\beta > 0$, it holds

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\quad + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned} \quad (107)$$

502 where the last inequality is due to the Young's inequality. Plugging this into (105) yields:

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\quad + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned} \quad (108)$$

503 Subsequently, we have

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}[(1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2] \\ &\quad + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned} \quad (109)$$

504 We now use Lemma 5 on $\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 = \gamma_{k+1}^2 \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2$ and obtain:

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\
& \leq \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \left(\frac{\gamma_{k+1}^2 \rho^2 L_s^2}{n} + \frac{(n-1)(1+\gamma_{k+1}\beta)}{n^2}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\
& \quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left(2\gamma_{k+1}^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\
& \leq \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \left(\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 L_s^2}{n}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\
& \quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left(2\gamma_{k+1}^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]
\end{aligned} \tag{110}$$

505 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \tag{111}$$

506 From the above, we get

$$\begin{aligned}
\Delta^{(k+1)} & \leq \left(1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 L_s^2\right) \Delta^{(k)} + \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
& \quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \gamma_{k+1} \left(2\gamma_{k+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]
\end{aligned} \tag{112}$$

507 Setting $c_1 = v_{\min}^{-1}$, $\alpha = \max\{2, 1+2v_{\min}\}$, $\bar{L} = \max\{L_s, L_V\}$, $\gamma_{k+1} = \frac{1}{k}$, $\beta = \frac{1}{\alpha n}$, $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$,
508 $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$, $\alpha \geq 2$, we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 L_s^2 \leq 1 - \frac{1}{n} + \frac{1}{\alpha k n} + \frac{1}{\alpha^2 c_1^2 k^2 n^{4/3}} \leq 1 - \frac{c_1(k\alpha - 1) - 1}{k \alpha n c_1} \leq 1 - \frac{1}{k \alpha n c_1} \tag{113}$$

509 which shows that $1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 L_s^2 \in (0, 1)$ for any $k > 0$. Denote $\Lambda_{(k+1)} = \frac{1}{n} -$
510 $\gamma_{k+1}\beta - \gamma_{k+1}^2 \rho^2 L_s^2$ and note that $\Delta^{(0)} = 0$, thus the telescoping sum yields:

$$\begin{aligned}
\Delta^{(k+1)} & \leq \sum_{\ell=0}^k \omega_{k,\ell} \left(2\gamma_{\ell+1}^2 \rho^2 + \frac{\gamma_{\ell+1}^2 \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] \\
& \quad + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1} (1-\rho)^2 \left(2\gamma_{\ell+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\tilde{S}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1} \tilde{\epsilon}^{(\ell+1)}
\end{aligned} \tag{114}$$

511 where $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$ and $\tilde{\epsilon}^{(\ell+1)} = \left(2\gamma_{\ell+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(\ell+1)}\|^2]$.

512 Summing on both sides over $k = 0$ to $k = K_{\max} - 1$ yields:

$$\begin{aligned}
\sum_{k=0}^{K_{\max}-1} \Delta^{(k+1)} & \leq \sum_{k=0}^{K_{\max}-1} \frac{2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta}}{\Lambda_{(k+1)}} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
& \quad + \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right)}{\Lambda_{(k+1)}} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}}{\Lambda_{(k+1)}} \tilde{\epsilon}^{(k+1)}
\end{aligned} \tag{115}$$

513 We recall (104) where we have summed on both sides from $k = 0$ to $k = K_{\max} - 1$:

$$\begin{aligned}
& \mathbb{E}[V(\hat{\mathbf{s}}^{(K_{\max})}) - V(\hat{\mathbf{s}}^{(0)})] \\
& \leq \sum_{k=0}^{K_{\max}-1} \left\{ \gamma_{k+1}(-v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}\rho^2 L_V \mathbb{E}[\|\mathbf{h}_k\|^2] + \gamma^2 L_V \rho^2 L_{\mathbf{s}}^2 \Delta^{(k)} \right\} \\
& + \sum_{k=0}^{K_{\max}-1} \left\{ \tilde{\xi}^{(k+1)} + \left((1-\rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \right\} \\
& \leq \sum_{k=0}^{K_{\max}-1} \left\{ -\gamma_{k+1}(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 L_V + \frac{\rho^2 \gamma_{k+1}^2 L_V L_{\mathbf{s}}^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right\} \mathbb{E}[\|\mathbf{h}_k\|^2] \\
& + \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma_{k+1} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]
\end{aligned} \tag{116}$$

where

$$\Xi^{(k+1)} = \tilde{\xi}^{(k+1)} + \frac{\gamma_{k+1}^3 L_V \rho^2 L_{\mathbf{s}}^2}{\Lambda_{(k+1)}} \tilde{c}^{(k+1)}$$

and

$$\Gamma_{k+1} = \left((1-\rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) + \frac{\gamma_{k+1}^3 L_V \rho^2 L_{\mathbf{s}}^2 (1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta} \right)}{\Lambda_{(k+1)}}$$

514 We now analyse the following quantity

$$\begin{aligned}
& -\gamma_{k+1}(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 L_V + \frac{\rho^2 \gamma_{k+1}^2 L_V L_{\mathbf{s}}^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \\
& = \gamma_{k+1} \left[-(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_{\mathbf{s}}^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right]
\end{aligned} \tag{117}$$

515 Furthermore, we recall that $c_1 = v_{\min}^{-1}$, $\alpha = \max\{2, 1 + 2v_{\min}\}$, $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$, $\gamma_{k+1} = \frac{1}{k}$,

516 $\beta = \frac{1}{\alpha n}$, $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$, $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$, $\alpha \geq 2$. Then,

$$\begin{aligned}
& \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_{\mathbf{s}}^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\frac{1}{n} - \gamma_{k+1}\beta - \gamma_{k+1}^2 \rho^2 L_{\mathbf{s}}^2} \\
& \leq \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L}(k\alpha^2 c_1^2 n^{4/3})^{-1} \left(\frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k\alpha c_1^2 \bar{L} n^{1/3}} \right)}{\frac{1}{n} - \frac{1}{k\alpha n} - \frac{1}{k^2 \alpha^2 c_1^2 n^{4/3}}} \\
& = \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L} \left(\frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k\alpha c_1^2 \bar{L} n^{1/3}} \right)}{(k\alpha c_1 n^{1/3})(k\alpha - 1)c_1 - 1} \\
& \stackrel{(a)}{\leq} \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\frac{1}{k\alpha c_1^2 \bar{L} n^{1/3}} \left(\frac{2}{k\alpha n} + 1 \right)}{2(\alpha c_1 n^{1/3}) - 1} \\
& \leq \frac{1}{k^2 \alpha c_1^2 \bar{L} n^{4/3}} + \frac{1}{4k\alpha^2 c_1^3 \bar{L} n^{2/3}} \\
& \leq \frac{3/4}{\alpha c_1^2 \bar{L} n^{2/3}}
\end{aligned} \tag{118}$$

where (a) is due to $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ and $k\alpha c_1 n^{1/3} \geq 1$. Note also that

$$-(v_{\min}\rho + v_{\max}^2) \leq -\rho v_{\min} = -\frac{1}{\alpha c_1^2 \bar{L} n^{2/3}}$$

which yields that

$$\left[-(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}\rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right] \leq -\frac{1/4}{\alpha c_1^2 \bar{L} n^{2/3}}$$

517 Using the Lemma 2, we know that $v_{\max}^2 \|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2 \leq \|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2$ and using (118) on (116)
 518 yields:

$$\begin{aligned} v_{\max}^2 \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2} \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma_{k+1} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \end{aligned} \quad (119)$$

519 proving the final bound on the gradient of the Lyapunov function:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2 v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma_{k+1} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \end{aligned} \quad (120)$$

520

□

G Practical Implementations of Two-Time-Scale EM Methods

G.1 Application on GMM

G.1.1 Explicit Updates

We first recognize that the constraint set for θ is given by

$$\Theta = \Delta^M \times \mathbb{R}^M. \quad (121)$$

Using the partition of the sufficient statistics as $S(y_i, z_i) = (S^{(1)}(y_i, z_i)^\top, S^{(2)}(y_i, z_i)^\top, S^{(3)}(y_i, z_i)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$, the partition $\phi(\theta) = (\phi^{(1)}(\theta)^\top, \phi^{(2)}(\theta)^\top, \phi^{(3)}(\theta)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$ and the fact that $\mathbb{1}_{\{M\}}(z_i) = 1 - \sum_{m=1}^{M-1} \mathbb{1}_{\{m\}}(z_i)$, the complete data log-likelihood can be expressed as in (2) with

$$\begin{aligned} s_{i,m}^{(1)} &= \mathbb{1}_{\{m\}}(z_i), \quad \phi_m^{(1)}(\theta) = \left\{ \log(\omega_m) - \frac{\mu_m^2}{2} \right\} - \left\{ \log(1 - \sum_{j=1}^{M-1} \omega_j) - \frac{\mu_M^2}{2} \right\}, \\ s_{i,m}^{(2)} &= \mathbb{1}_{\{m\}}(z_i) y_i, \quad \phi_m^{(2)}(\theta) = \mu_m, \quad s_i^{(3)} = y_i, \quad \phi^{(3)}(\theta) = \mu_M, \end{aligned} \quad (122)$$

and $\psi(\theta) = -\left\{ \log(1 - \sum_{m=1}^{M-1} \omega_m) - \frac{\mu_M^2}{2\sigma^2} \right\}$. We also define for each $m \in \llbracket 1, M \rrbracket$, $j \in \llbracket 1, 3 \rrbracket$, $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$. Consider the following latent sample used to compute an approximation of the conditional expected value $\mathbb{E}_\theta[\mathbb{1}_{\{z_i=m\}} | y = y_i]$:

$$z_{i,m} \sim \mathbb{P}(z_i = m | y_i; \theta) \quad (123)$$

where $m \in \llbracket 1, M \rrbracket$, $i \in \llbracket 1, n \rrbracket$ and $\theta = (\mathbf{w}, \boldsymbol{\mu}) \in \Theta$.

In particular, given iteration $k + 1$, the computation of the approximated quantity $\tilde{S}_{i_k}^{(k)}$ during Incremental-step updates, see (8) can be written as

$$\tilde{S}_{i_k}^{(k)} = \left(\underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1}), \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})}_{:=\tilde{s}_{i_k}^{(1)}}, \underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1})y_{i_k}, \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})y_{i_k}}_{:=\tilde{s}_{i_k}^{(2)}}, \underbrace{y_{i_k}}_{:=\tilde{s}_{i_k}^{(3)}(\theta^{(k)})} \right)^\top. \quad (124)$$

Recall that we have used the following regularizer:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m), \quad (125)$$

It can be shown that the regularized M-step in (4) evaluates to

$$\bar{\theta}(\mathbf{s}) = \begin{pmatrix} (1 + \epsilon M)^{-1} (s_1^{(1)} + \epsilon, \dots, s_{M-1}^{(1)} + \epsilon)^\top \\ ((s_1^{(1)} + \delta)^{-1} s_1^{(2)}, \dots, (s_{M-1}^{(1)} + \delta)^{-1} s_{M-1}^{(2)})^\top \\ (1 - \sum_{m=1}^{M-1} s_m^{(1)} + \delta)^{-1} (s^{(3)} - \sum_{m=1}^{M-1} s_m^{(2)}) \end{pmatrix} = \begin{pmatrix} \bar{\omega}(\mathbf{s}) \\ \bar{\boldsymbol{\mu}}(\mathbf{s}) \\ \bar{\mu}_M(\mathbf{s}) \end{pmatrix}. \quad (126)$$

where we have defined for all $m \in \llbracket 1, M \rrbracket$ and $j \in \llbracket 1, 3 \rrbracket$, $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$.

G.1.2 Model Assumptions (GMM example)

We use the GMM example to illustrate the required assumptions.

Many practical models can satisfy the compactness of the sets as in Assumption H1. For instance, the GMM example satisfies (16) as the sufficient statistics are composed of indicator functions and observations as defined Section G.1 Equation (122).

Assumptions H2 and H3 are standard for the curved exponential family models. For GMM, the following (strongly convex) regularization $\mathbf{r}(\theta)$ ensures H3:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m)$$

544 since it ensures $\theta^{(k)}$ is unique and lies in $\text{int}(\Delta^M) \times \mathbb{R}^M$. We remark that for H2, it is possible to
 545 define the Lipschitz constant L_p independently for each data y_i to yield a refined characterization.

546 Again, H4 is satisfied by practical models. For GMM, it can be verified by deriving the closed form
 547 expression for $B(s)$ and using H1.

548 Under H1 and H3, we have $\|\hat{s}^{(k)}\| < \infty$ since S is compact and $\hat{\theta}^{(k)} \in \text{int}(\Theta)$ for any $k \geq 0$ which
 549 thus ensure that the EM methods operate in a closed set throughout the optimization process.

550 G.1.3 Algorithms updates

551 In the sequel, recall that, for all $i \in \llbracket n \rrbracket$ and iteration k , the computed statistic $\tilde{S}_{i_k}^{(k)}$ is defined by
 552 (124). At iteration k , the several E-steps defined by (9) or (10) and (11) leads to the definition of the
 553 quantity $\hat{s}^{(k+1)}$. For the GMM example, after the initialization of the quantity $\hat{s}^{(0)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(0)}$,
 554 those E-steps break down as follows:

555 **Batch EM (EM):** for all $i \in \llbracket 1, n \rrbracket$, compute $\bar{s}_i^{(k)}$ and set

$$\hat{s}^{(k+1)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(k)}. \quad (127)$$

556 where $\bar{s}_i^{(k)}$ are computed using the exact conditional expected value $\mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i]$:

$$\tilde{\omega}_m(y_i; \theta) := \mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i] = \frac{\omega_m \exp(-\frac{1}{2}(y_i - \mu_i)^2)}{\sum_{j=1}^M \omega_j \exp(-\frac{1}{2}(y_i - \mu_j)^2)}, \quad (128)$$

557 **Incremental EM (iEM):** draw an index i_k uniformly at random on $\llbracket n \rrbracket$, compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} + \frac{1}{n} (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\tau_i^k)}) = n^{-1} \sum_{i=1}^n \bar{s}_i^{(\tau_i^k)}. \quad (129)$$

558 **batch SAEM (SAEM):** draw an index i_k uniformly at random on $\llbracket n \rrbracket$, compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} \tilde{S}^{(k)}. \quad (130)$$

559 where $= \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k)}$ with $\tilde{S}_i^{(k)}$ defined in (124).

560 **Incremental SAEM (iSAEM):** draw an index i_k uniformly at random on $\llbracket n \rrbracket$, compute $\bar{s}_{i_k}^{(k)}$ and set
 561

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} + \frac{1}{n} (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_i^k)})). \quad (131)$$

562 **Variance Reduced Two-Time-Scale EM (vrTTSEM):** draw an index i_k uniformly at random on
 563 $\llbracket n \rrbracket$, compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} (1 - \rho) + \rho (\tilde{S}^{(\ell(k))} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\ell(k))}))). \quad (132)$$

564 **Fast Incremental Two-Time-Scale EM (fitTTSEM):** draw an index i_k uniformly at random on $\llbracket n \rrbracket$,
 565 compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} (1 - \rho) + \rho (\bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}))). \quad (133)$$

566 Finally, the k -th update reads $\hat{\theta}^{(k+1)} = \bar{\theta}(\hat{s}^{(k+1)})$ where the function $s \rightarrow \bar{\theta}(s)$ is defined by (126).

567 G.2 Deformable Template Model for Image Analysis

568 G.2.1 Model and Updates

569 The complete model belongs to the curved exponential family, see [1], which vector of sufficient
570 statistics $S = (S_1(z), S_2(z), S_3(z))$ read:

$$\begin{aligned} S_1(z) &= \frac{1}{n} \sum_{i=1}^n S_1(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top y_i \\ S_2(z) &= \frac{1}{n} \sum_{i=1}^n S_2(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top (\mathbf{K}_p^{z_i}) \\ S_3(z) &= \frac{1}{n} \sum_{i=1}^n S_3(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n z_i^t z_i \end{aligned} \quad (134)$$

571 where for any pixel $u \in \mathbb{R}^2$ and $j \in \llbracket 1, k_g \rrbracket$ we noted:

$$\mathbf{K}_p^{z_i}(x_u, j) = \mathbf{K}_p^{z_i}(x_u - \phi_i(x_u, z_i), p_j) \quad (135)$$

572 Finally, the Two-Time-Scale M-step yields the following parameter updates:

$$\bar{\theta}(\hat{s}) = \begin{pmatrix} \beta(\hat{s}) = \hat{s}_2^{-1}(z) \hat{s}_1(z) \\ \Gamma(\hat{s}) = \frac{1}{n} \hat{s}_3(z) \\ \sigma(\hat{s}) = \beta(\hat{s})^\top \hat{s}_2(z) \beta(\hat{s}) - 2\beta(\hat{s}) \hat{s}_1(z) \end{pmatrix} \quad (136)$$

573 where $\hat{s} = (\hat{s}_1(z), \hat{s}_2(z), \hat{s}_3(z))$ is the vector of statistics obtained via the SA-step (7) and using the
574 MC approximation of the sufficient statistics $(S_1(z), S_2(z), S_3(z))$ defined in (140).

575 G.3 Application on PK Model

576 G.3.1 Model and Explicit Updates

577 Lognormal distributions are used for the four PK parameters:

$$\log(T_i^{\text{lag}}) \sim \mathcal{N}(\log(T_{\text{pop}}^{\text{lag}}), \omega_{T^{\text{lag}}}^2), \log(ka_i) \sim \mathcal{N}(\log(ka_{\text{pop}}), \omega_{ka}^2), \quad (137)$$

$$\log(V_i) \sim \mathcal{N}(\log(V_{\text{pop}}), \omega_V^2), \log(k_i) \sim \mathcal{N}(\log(k_{\text{pop}}), \omega_k^2). \quad (138)$$

578 The complete model belongs to the curved exponential family, which vector of sufficient statistics
579 $S = (S_1(z), S_2(z), S_3(z))$ read:

$$S_1(z) = \frac{1}{n} \sum_{i=1}^n z_i, \quad S_2(z) = \frac{1}{n} \sum_{i=1}^n z_i^\top z_i, \quad S_3(z) = \frac{1}{n} \sum_{i=1}^n (y_i - f(t_i, z_i))^2 \quad (139)$$

580 where we have noted y_i and t_i the vector of observations and time for each patient i .

581 We recall that the complete model (y, z) defined by (36) and (??) belongs to the curved exponential
582 family, which vector of sufficient statistics $S = (S_1(z), S_2(z), S_3(z))$ read:

$$S_1(z) = \frac{1}{n} \sum_{i=1}^n z_i, \quad S_2(z) = \frac{1}{n} \sum_{i=1}^n z_i^\top z_i, \quad S_3(z) = \frac{1}{n} \sum_{i=1}^n (y_i - f(t_i, z_i))^2 \quad (140)$$

583 where we have noted y_i and t_i the vector of observations and time for each patient i . At iter-
584 ation k , and setting the number of MC samples to 1 for the sake of clarity, the MC sampling
585 $z_i^{(k)} \sim p(z_i | y_i, \theta^{(k)})$ is performed using a Metropolis-Hastings procedure detailed in algorithm 2.
586 The quantities $\tilde{S}^{(k+1)}$ and $\hat{s}^{(k+1)}$ are then updated according to the different methods. Finally the
587 maximization step yields:

$$\bar{\theta}(s) = \begin{pmatrix} \hat{s}_1^{(k+1)} \\ \hat{s}_2^{(k+1)} - \hat{s}_1^{(k+1)} (\hat{s}_1^{(k+1)})^\top \\ \hat{s}_3^{(k+1)} \end{pmatrix} = \begin{pmatrix} \overline{z_{\text{pop}}}(\hat{s}^{(k+1)}) \\ \overline{\omega_z}(\hat{s}^{(k+1)}) \\ \overline{\sigma}(\hat{s}^{(k+1)}) \end{pmatrix}. \quad (141)$$

588 G.3.2 Metropolis Hastings algorithm

589 During the simulation step of the MISSO method, the sampling from the target distribution
 590 $\pi(z_i, \theta) := p(z_i | y_i, \theta)$ is performed using a Metropolis Hastings (MH) algorithm [16] with pro-
 591 posal distribution $q(z_i, \delta)$ where $\theta = (z_{\text{pop}}, \omega_z)$ and δ is the vector of parameters of the proposal
 592 distribution. Commonly they parameterize a Gaussian proposal. The MH algorithm is summarized
 593 in 2.

Algorithm 2 MH algorithm

```

1: Input: initialization  $z_{i,0} \sim q(z_i; \delta)$ 
2: for  $m = 1, \dots, M$  do
3:   Sample  $z_{i,m} \sim q(z_i; \delta)$ 
4:   Sample  $u \sim \mathcal{U}([0, 1])$ 
5:   Calculate the ratio  $r = \frac{\pi(z_{i,m}; \theta) / q(z_{i,m}; \delta)}{\pi(z_{i,m-1}; \theta) / q(z_{i,m-1}; \delta)}$ 
6:   if  $u < r$  then
7:     Accept  $z_{i,m}$ 
8:   else
9:      $z_{i,m} \leftarrow z_{i,m-1}$ 
10:  end if
11: end for
12: Output:  $z_{i,M}$ 

```
