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# Fast Two-Time-Scale Noisy EM Algorithms

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## Abstract

Training latent data models using the Expectation-Maximization (EM) algorithm is the most popular choice for current learning tasks. For today's modern and complex tasks, variants of the EM have been initially introduced by [15], using incremental updates to scale to large dataset, and by [19, 6], using Monte-Carlo (MC) approximations to bypass the impossible conditional expectation of the latent data for most nonconvex models. In this paper, we propose a general class of methods called Two-Time-Scale EM Methods based on double levels of stochastic updates to tackle a growingly common large and nonconvex optimization task for latent data models. We motivate the choice of a double dynamics by invoking the variance reduction virtue of each stage of the method on both sources of noise: the incremental update and the MC approximation. We establish finite-time and independent of the initialization convergence bounds for nonconvex objective function. Numerical applications are also presented in this article to illustrate our findings.

## 1 Introduction

Learning latent data models is critical for modern machine learning problems, see [14] for references. We formulate the training of such model as the following empirical risk minimization problem:

$$\min_{\theta \in \Theta} \bar{L}(\theta) := r(\theta) + L(\theta) \quad \text{with} \quad L(\theta) = \frac{1}{n} \sum_{i=1}^n L_i(\theta) := \frac{1}{n} \sum_{i=1}^n \{ -\log g(y_i; \theta) \}, \quad (1)$$

We denote the observations by  $\{y_i\}_{i=1}^n$ ,  $\Theta \subset \mathbb{R}^d$  is the convex parameters space. We consider a regularized model where  $r : \Theta \rightarrow \mathbb{R}$  is a smooth convex regularization function and for  $\theta \in \Theta$ ,  $g(y; \theta)$  is the (incomplete) likelihood of each individual observation. The objective function  $\bar{L}(\theta)$  is possibly *nonconvex* and is assumed to be lower bounded  $\bar{L}(\theta) > -\infty$  for all  $\theta \in \Theta$ .

In the latent variable model,  $g(y_i; \theta)$ , is the marginal of the complete data likelihood defined as  $f(z_i, y_i; \theta)$ , i.e.  $g(y_i; \theta) = \int_{\mathcal{Z}} f(z_i, y_i; \theta) \mu(dz_i)$ , where  $\{z_i\}_{i=1}^n$  are the (unobserved) latent variables. In this paper, we make the assumption of a complete model belonging to the curved exponential family, i.e.,

$$f(z_i, y_i; \theta) = h(z_i, y_i) \exp \left( \langle S(z_i, y_i) | \phi(\theta) \rangle - \psi(\theta) \right), \quad (2)$$

where  $\psi(\theta)$ ,  $h(z_i, y_i)$  are scalar functions,  $\phi(\theta) \in \mathbb{R}^k$  is a vector function, and  $S(z_i, y_i) \in \mathbb{R}^k$  is the complete data sufficient statistics.

Full batch EM [7] is the method of reference for that kind of task and is a two steps procedure. The **E-step** amounts to computing the conditional expectation of the complete data sufficient statistics,

$$\bar{s}(\theta) = \frac{1}{n} \sum_{i=1}^n \bar{s}_i(\theta) \quad \text{where} \quad \bar{s}_i(\theta) = \int_{\mathcal{Z}} S(z_i, y_i) p(z_i | y_i; \theta) \mu(dz_i). \quad (3)$$

31 The M-step is given by

$$\text{M-step: } \hat{\boldsymbol{\theta}} = \bar{\boldsymbol{\theta}}(\bar{\mathbf{s}}(\boldsymbol{\theta})) := \arg \min_{\boldsymbol{\vartheta} \in \Theta} \{ r(\boldsymbol{\vartheta}) + \psi(\boldsymbol{\vartheta}) - \langle \bar{\mathbf{s}}(\boldsymbol{\theta}) | \phi(\boldsymbol{\vartheta}) \rangle \}, \quad (4)$$

32 Two caveats of this method are the following: (a) with the explosion of data, the first step of the EM  
 33 is computationally inefficient as it requires a full pass over the dataset at each iteration and (b) the  
 34 complexity of modern models makes the expectation intractable. So far, both challenges have been  
 35 addressed separately, to the best of our knowledge, and we give an overview of current solutions in  
 36 the sequel.

37 **Prior Work** Inspired by stochastic optimization procedures, [15] and [4] developed respectively an  
 38 incremental and an online variant of the **E-step** in models where the expectation is computable then  
 39 extensively used and studied in [16, 12, 3]. Some improvements of that methods have been provided  
 40 and analyzed, globally and in finite-time, in [9] where variance reduction techniques taken from the  
 41 optimization literature have been efficiently applied to scale the EM algorithm to large datasets.

42 Regarding the computation of the expectation under the posterior distribution, the first method was  
 43 the Monte-Carlo EM (MCEM) introduced in the seminal paper [19] where a MC approximation  
 44 for this expectation is computed. A variant of that method is the Stochastic Approximation of the  
 45 EM (SAEM) in [6] leveraging the power of Robbins-Monro type of update [18] to ensure pointwise  
 46 convergence of the vector of estimated parameters rather using a decreasing stepsize than increasing  
 47 the number of MC samples. The MCEM and the SAEM have been successfully applied in mixed  
 48 effects models [13, 8, 2] or to do inference for joint modelling of time to event data coming from  
 49 clinical trials in [5], among other applications.

50 Recently, an incremental variant of the SAEM was proposed in [11] showing positive empirical  
 51 results but its analysis is limited to asymptotic consideration. Gradient-based methods have been  
 52 developed and analyzed in [20] but they remain out of the scope of this paper as they tackle the  
 53 high-dimensionality issue.

54 **Contributions** This paper *introduces* and *analyzes* a new class of methods which purpose is up-  
 55 date two proxies for target expected quantities in a two-time-scale manner. Those approximated  
 56 quantities are then used to optimize (1) for current modern examples and settings using EM-fashion  
 57 Maximization step.

58 The main contributions of the paper are:

- 59 • We propose a two-time-scale method based on Stochastic Approximation (SA), to alleviate  
 60 the problem of MC computation, and on Incremental updates, to scale to large datasets.  
 61 We describe in details the edges of each level of our method based on variance reduction  
 62 arguments. The derivation of such class of algorithms has two advantages. First, it naturally  
 63 leverages variance reduction and Robbins-Monro type of updates to tackle large-scale and  
 64 highly nonlinear learning tasks. Then, it gives a simple formulation as a *scaled-gradient*  
 65 *method* which makes the global analysis and the implementation accessible.
- 66 • We also establish global (independent of the initialization) and finite-time (true at each  
 67 iteration) upper bounds on a classical suboptimality condition in the nonconvex literature,  
 68 *i.e.*, the second order moment of the gradient of the objective function.

69 In Section 2 we give rigorous mathematical definitions of the various updates used for both incre-  
 70 mental and Monte-Carlo EMs and we introduce the main class of new algorithms, based on two  
 71 different dynamics, we are proposing to analyze and compare to baselines algorithms. Section 3  
 72 presents the main theoretical guarantees of this newly introduced two-time-scale class of algorithms.  
 73 Results are given both in finite-time and in the nonconvex setting. Finally, we illustrate the advan-  
 74 tages of our method in Section 4 on two numerical experiments.

## 75 2 Two-Time-Scale Stochastic EM Algorithms

76 We recall and formalize in this section the different methods found in the literature that aim to solv-  
 77 ing the large-scale problem and the intractable expectation. We then provide the general framework  
 78 of our method that efficiently tackles the optimization problem (1).

## 79 2.1 Monte Carlo Integration and Stochastic Approximation

80 As mentioned in the introduction, for complex and possibly nonlinear models, the expectation under  
 81 the posterior distribution defined in (3) is not tractable. In that case, the first solution involves  
 82 computing a Monte Carlo integration of that latter term. For all  $i \in \llbracket 1, n \rrbracket$ , draw for  $m \in \llbracket 1, M \rrbracket$ ,  
 83 samples  $z_{i,m} \sim p(z_i | y_i; \theta)$  and compute the MC integration  $\tilde{s}$  of the deterministic quantity  $\bar{s}(\theta)$ :

$$\text{MC-step : } \tilde{s} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}, y_i) \quad (5)$$

84 and then update the parameter  $\hat{\theta} = \bar{\theta}(\tilde{s})$ . This algorithm bypasses the intractable expectation issue  
 85 but is rather computationally expensive in order to reach point wise convergence ( $M$  needs to be  
 86 large). An alternative to that stochastic algorithm is to use a Robbins-Monro (RM) type of update.  
 87 We denote, at iteration  $k$ , the following quantity

$$\tilde{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}^{(k)}, y_i) \quad \text{where } z_{i,m}^{(k)} \sim p(z_i | y_i; \theta^{(k)}) \quad (6)$$

88 Then, the RM updated of the sufficient statistics  $\hat{s}^{(k+1)}$  reads:

$$\text{SA-step : } \hat{s}^{(k+1)} = \hat{s}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{s}^{(k)}) \quad (7)$$

89 where  $\{\gamma_k\}_{k \geq 1} \in (0, 1)$  is a sequence of decreasing step sizes to ensure asymptotic convergence.  
 90 This is called the Stochastic Approximation of the EM (SAEM) and has been shown theoretically  
 91 to converge to a maximum of the likelihood of the observations under very general conditions [6].  
 92 In the simulation step (6), since the relation between the observed data  $y_i$  and the latent variable  $z_i$   
 93 can be non linear, sampling from the posterior distribution  $p(z_i | y_i; \theta)$ , under the current model  $\theta$ ,  
 94 could require using an inference algorithm. [10] proved almost sure convergence of the sequence  
 95 of parameters obtained by this algorithm coupled with an MCMC procedure during the simulation  
 96 step. In simple scenarios, the samples  $\{z_{i,m}\}_{m=0}^{M-1}$  are conditionally independent and identically  
 97 distributed with distribution  $p(z_i, \theta)$ . Nevertheless, in most cases, sampling exactly from this dis-  
 98 tribution is not an option and the Monte Carlo batch is sampled by Monte Carlo Markov Chains  
 99 (MCMC) algorithm. In the SA-step, the sequence of decreasing positive integers  $\{\gamma_k\}_{k \geq 1}$  controls  
 100 the convergence of the algorithm. In practice,  $\gamma_k$  is set equal to 1 during the first few iterations  
 101 to let the algorithm explore the parameter space without memory and converge quickly to a neigh-  
 102 bourhood of the target estimate. The Stochastic Approximation is performed during the remaining  
 103 iterations where  $\gamma_k = 1/k^\alpha$ , where  $\alpha \in (0, 1)$ , ensuring the almost sure convergence of the esti-  
 104 mate. It is inappropriate to start with small values for step size  $\gamma_k$  and large values for the number  
 105 of simulations  $M_k$ . Rather, it is recommended that one decrease  $\gamma_k$  and keep a constant and small  
 106 number of MC samples  $M_k$  which shows a great advantage over the MC-step (5), which requires  
 107 large  $M_k$  to converge.

108 This Robbins-Monro type of update represents the *first level* of our algorithm, needed to temper  
 109 the variance and noise implied by MC integration. In the next section, we derive variants of this  
 110 algorithm to adapt to the sheer size of data of today's applications and formalize the *second level* of  
 111 our class of Two-Time-Scale EM methods.

## 112 2.2 Incremental and Bi-Level Inexact EM Methods

113 Strategies to scale to large datasets include classical incremental and variance reduced variants. We  
 114 will explicit a general update that will cover those variants and that represents the *second level* of our  
 115 algorithm, namely the incremental update of the noisy statistics  $\hat{S}^{(k)}$  inside the RM type of update.

$$\text{Incremental-step : } \tilde{S}^{(k+1)} = \tilde{S}^{(k)} + \rho_{k+1}(\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}), \quad (8)$$

116 Note  $\{\rho_k\}_{k \geq 1} \in (0, 1)$  is a sequence of step sizes,  $\mathcal{S}^{(k)}$  is a proxy for  $\tilde{S}^{(k)}$ , If the stepsize is equal  
 117 to one and the proxy  $\mathcal{S}^{(k)} = \hat{S}^{(k)}$ , i.e., computed in a full batch manner as in (6), then we recover  
 118 the SAEM algorithm. Also if  $\rho_k = 1$ ,  $\gamma_k = 1$  and  $\mathcal{S}^{(k)} = \tilde{S}^{(k)}$ , then we recover the Monte Carlo  
 119 EM algorithm.

120 We now introduce three variants of the SAEM update depending on different definitions of the  
 121 proxy  $\mathcal{S}^{(k)}$  and the choice of the stepsize  $\rho_k$ . Let  $i_k \in \llbracket 1, n \rrbracket$  be a random index drawn at iteration  
 122  $k$  and  $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$  be the iteration index where  $i \in \llbracket 1, n \rrbracket$  is last drawn  
 123 prior to iteration  $k$ . For iteration  $k \geq 0$ , the fiTTSEM method draws *two* indices *independently* and  
 124 uniformly as  $i_k, j_k \in \llbracket 1, n \rrbracket$ . In addition to  $\tau_i^k$  which was defined w.r.t.  $i_k$ , we define  $t_j^k = \{k' : j_{k'} = j, k' < k\}$  to be the iteration index where the sample  $j \in \llbracket 1, n \rrbracket$  is last drawn as  $j_k$  prior to  
 125 iteration  $k$ . With the initialization  $\overline{\mathcal{S}}^{(0)} = \hat{\mathbf{s}}^{(0)}$ , we use a slightly different update rule from SAGA  
 126 inspired by [17]. Then, we obtain:

$$(iSAEM [11]) \quad \mathcal{S}^{(k+1)} = \mathcal{S}^{(k)} + \frac{1}{n} (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)}) \quad (9)$$

$$(vrTTSEM) \quad \mathcal{S}^{(k+1)} = \tilde{S}^{(\ell(k))} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\ell(k))}) \quad (10)$$

$$(fiTTSEM) \quad \mathcal{S}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \quad (11)$$

$$\overline{\mathcal{S}}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + n^{-1} (\tilde{S}_{j_k}^{(k)} - \tilde{S}_{j_k}^{(t_{j_k}^k)}). \quad (12)$$

128 The stepsize is set to  $\rho_{k+1} = 1$  for the iSAEM method;  $\rho_{k+1} = \gamma$  is constant for the vrTTSEM and  
 129 fiTTSEM methods. Moreover, for iSAEM we initialize with  $\mathcal{S}^{(0)} = \tilde{S}^{(0)}$ ; for vrTTSEM we set an  
 130 epoch size of  $m$  and define  $\ell(k) := m \lfloor k/m \rfloor$  as the first iteration number in the epoch that iteration  
 131  $k$  is in.

### 132 2.3 Two-Time-Scale Noisy EM methods

133 We now introduce the general method derived using the two variance reduction techniques described  
 134 above. Algorithm 1 leverages both levels (7) and (8) in order to output a vector of fitted parameters  
 135  $\hat{\theta}^{(K)}$  where  $K$  is some randomly chosen termination point.

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#### Algorithm 1 Two-Time-Scale Noisy EM methods.

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- 1: **Input:** initializations  $\hat{\theta}^{(0)} \leftarrow 0, \hat{\mathbf{s}}^{(0)} \leftarrow \hat{S}^{(0)}, K_{\max} \leftarrow \text{max. iteration number}$ .
- 2: Set the terminating iteration number,  $K \in \{0, \dots, K_{\max} - 1\}$ , as a discrete r.v. with:

$$P(K = k) = \frac{\gamma_k}{\sum_{\ell=0}^{K_{\max}-1} \gamma_\ell}. \quad (13)$$

- 3: **for**  $k = 0, 1, 2, \dots, K$  **do**
- 4: Draw index  $i_k \in \llbracket 1, n \rrbracket$  uniformly (and  $j_k \in \llbracket 1, n \rrbracket$  for fiTTSEM).
- 5: Compute  $\hat{S}_{i_k}^{(k)}$  using the MC-step (5), for the drawn indices.
- 6: Compute the surrogate sufficient statistics  $\mathcal{S}^{(k+1)}$  using (9) or (10) or (11).
- 7: Compute  $\hat{S}^{(k+1)}$  and  $\hat{\mathbf{s}}^{(k+1)}$  using respectively (8) and (7):

$$\begin{aligned} \tilde{S}^{(k+1)} &= \tilde{S}^{(k)} + \rho_{k+1} (\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}) \\ \hat{\mathbf{s}}^{(k+1)} &= \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}) \end{aligned} \quad (14)$$

- 8: Compute  $\hat{\theta}^{(k+1)}$  via the M-step (4).
  - 9: **end for**
  - 10: **Return:**  $\hat{\theta}^{(K)}$ .
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136 The update in (14) is said to have two-time-scales as the step sizes satisfy  $\lim_{k \rightarrow \infty} \gamma_k / \rho_k < 1$  such that  
 137  $\tilde{S}^{(k+1)}$  is updated at a faster time-scale, determined by  $\rho_k$ , than  $\hat{\mathbf{s}}^{(k+1)}$ , determined by  $\gamma_k$ . The next  
 138 section presents the main results of this paper and establishes global and finite-time bounds for the  
 139 three different updates of our two-time-scale scheme..

### 3 Finite Time Analysis of the Two-Time-Scale Scheme

Following [4], it can be shown that stationary points of the objective function (1) corresponds to the stationary points of the following *nonconvex* Lyapunov function:

$$\min_{\mathbf{s} \in \mathcal{S}} V(\mathbf{s}) := \bar{L}(\bar{\boldsymbol{\theta}}(\mathbf{s})) = r(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\bar{\boldsymbol{\theta}}(\mathbf{s})) \quad (15)$$

We thus propose to study the latter minimization problem in the sequel.

An important assumption in order to derive convergence guarantees reads as follows:

**H1.** *The sets  $\mathcal{Z}, \mathcal{S}$  are compact. There exists constants  $C_S, C_Z$  such that:*

$$C_S := \max_{\mathbf{s}, \mathbf{s}' \in \mathcal{S}} \|\mathbf{s} - \mathbf{s}'\| < \infty, \quad C_Z := \max_{i \in \llbracket 1, n \rrbracket} \int_{\mathcal{Z}} |S(z, y_i)| \mu(dz) < \infty. \quad (16)$$

**H2.** *The conditional distribution is smooth on  $\text{int}(\Theta)$ . For any  $i \in \llbracket 1, n \rrbracket$ ,  $z \in \mathcal{Z}$ ,  $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \text{int}(\Theta)^2$ , we have  $|p(z|y_i; \boldsymbol{\theta}) - p(z|y_i; \boldsymbol{\theta}')| \leq L_p \|\boldsymbol{\theta} - \boldsymbol{\theta}'\|$ .*

We also recall from the introduction that we consider curved exponential family models. besides:

**H3.** *For any  $\mathbf{s} \in \mathcal{S}$ , the function  $\boldsymbol{\theta} \mapsto L(\mathbf{s}, \boldsymbol{\theta}) := r(\boldsymbol{\theta}) + \psi(\boldsymbol{\theta}) - \langle \mathbf{s} | \phi(\boldsymbol{\theta}) \rangle$  admits a unique global minimum  $\bar{\boldsymbol{\theta}}(\mathbf{s}) \in \text{int}(\Theta)$ . In addition,  $J_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))$  is full rank,  $L_{\phi}$ -Lipschitz and  $\bar{\boldsymbol{\theta}}(\mathbf{s})$  is  $L_{\theta}$ -Lipschitz.*

We denote by  $H_L^{\boldsymbol{\theta}}(\mathbf{s}, \boldsymbol{\theta})$  the Hessian (w.r.t to  $\boldsymbol{\theta}$  for a given value of  $\mathbf{s}$ ) of the function  $\boldsymbol{\theta} \mapsto L(\mathbf{s}, \boldsymbol{\theta}) = r(\boldsymbol{\theta}) + \psi(\boldsymbol{\theta}) - \langle \mathbf{s} | \phi(\boldsymbol{\theta}) \rangle$ , and define

$$\mathbf{B}(\mathbf{s}) := J_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \left( H_L^{\boldsymbol{\theta}}(\mathbf{s}, \bar{\boldsymbol{\theta}}(\mathbf{s})) \right)^{-1} J_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top}. \quad (17)$$

**H4.** *It holds that  $v_{\max} := \sup_{\mathbf{s} \in \mathcal{S}} \|\mathbf{B}(\mathbf{s})\| < \infty$  and  $0 < v_{\min} := \inf_{\mathbf{s} \in \mathcal{S}} \lambda_{\min}(\mathbf{B}(\mathbf{s}))$ . There exists a constant  $L_B$  such that for all  $\mathbf{s}, \mathbf{s}' \in \mathcal{S}^2$ , we have  $\|\mathbf{B}(\mathbf{s}) - \mathbf{B}(\mathbf{s}')\| \leq L_B \|\mathbf{s} - \mathbf{s}'\|$ .*

The class of algorithms we develop in this paper are two-time-scale where the first stage corresponds to the variance reduction trick used in [9] in order to accelerate incremental methods and reduce the variance induced by the index sampling. The second stage is the Robbins-Monro type of update that aims to reduce the variance induced by the MC approximations

Indeed the expectations (3) are never available and requires Monte Carlo approximation. Thus, at iteration  $k + 1$ , we introduce the errors when approximating the quantity  $\bar{s}_i(\hat{\boldsymbol{\theta}}(\hat{\mathbf{s}}^{(k-1)}))$ . For all  $i \in \llbracket 1, n \rrbracket$ ,  $r > 0$  and  $\vartheta \in \Theta$ , define:

$$\eta_i^{(r)} := \tilde{S}_i^{(r)} - \bar{s}_i(\vartheta^{(r)}) \quad (18)$$

For instance, we consider that the MC approximation is unbiased if for all  $i \in \llbracket 1, n \rrbracket$  and  $m \in \llbracket 1, M \rrbracket$ , the samples  $z_{i,m} \sim p(z_i|y_i; \boldsymbol{\theta})$  are i.i.d. under the posterior distribution, i.e.,  $\mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r] = 0$  where  $\mathcal{F}_r$  is the filtration up to iteration  $r$ . The following results are derived under the assumption of control of the fluctuations implied by the approximation stated as follows:

**H5.** *There exist a positive sequence of MC batch size  $\{M_r\}_{r>0}$  and constants  $(C, C_{\eta})$  such that for all  $k > 0$ ,  $i \in \llbracket 1, n \rrbracket$  and  $\vartheta \in \Theta$ :*

$$\mathbb{E} \left[ \left\| \eta_i^{(r)} \right\|^2 \right] \leq \frac{C_{\eta}}{M_r} \quad \text{and} \quad \mathbb{E} \left[ \left\| \mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r] \right\|^2 \right] \leq \frac{C}{M_r} \quad (19)$$

In that setting, we can prove two important results on the Lyapunov function. The first one suggests smoothness:

**Lemma 1.** [9] *Assume H1-H4. For all  $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$  and  $i \in \llbracket 1, n \rrbracket$ , we have*

$$\|\bar{s}_i(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \bar{s}_i(\bar{\boldsymbol{\theta}}(\mathbf{s}'))\| \leq L_s \|\mathbf{s} - \mathbf{s}'\|, \quad \|\nabla V(\mathbf{s}) - \nabla V(\mathbf{s}')\| \leq L_V \|\mathbf{s} - \mathbf{s}'\|, \quad (20)$$

where  $L_s := C_Z L_p L_{\theta}$  and  $L_V := v_{\max}(1 + L_s) + L_B C_S$ .

and the second one suggests a growth condition on the gradient of  $V$  depending on the mean field of the algorithm:

**Lemma 2.** *Assume H3, H4. For all  $\mathbf{s} \in \mathcal{S}$ ,*

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (21)$$

Proof of this Lemma can be found in Appendix A.

### 3.1 Global Convergence of Incremental Noisy EM Algorithms

We present in this section a finite-time analysis of the incremental variant of the Stochastic Approximation of the EM algorithm. We want to draw the attention of the readers that the word "global" here does not mean for a global optimum of the nonconvex function, but of the independence of our analysis on the initialization and the iteration  $k$  (finite time).

The first intermediate result is the computation of the quantity  $\hat{S}^{(k+1)} - \hat{s}^{(k)}$ , which corresponds to the drift term of (7) and reads as follows:

**Lemma 3.** *The update (9) is equivalent to the following update on the resulting statistics*

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{s}^{(k)}) \quad \text{where} \quad \tilde{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^{k+1})} \quad (22)$$

Also:

$$\mathbb{E}[\tilde{S}^{(k+1)} - \hat{s}^{(k)}] = \mathbb{E}[\bar{s}^{(k)} - \hat{s}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)}\right] + \frac{1}{n} \mathbb{E}[\eta_{i_k}^{(k+1)}] \quad (23)$$

where  $\bar{s}^{(k)}$  is defined by (3) and  $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$ .

Proof of this Lemma can be found in Appendix B.

The following main result for the iSAEM algorithm is derived under a control of the Monte Carlo fluctuations as described by assumption H 5. Typically, the controls exhibited below are of interest when the number of MC samples  $M_k$  increase with the iteration index  $f$ .

**Theorem 1.** *Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of positive step sizes and consider the iSAEM sequence  $\{\hat{s}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = 1$  for any  $k > 0$ . We also set  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ . Assume that  $\hat{s}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ .*

$$v_{\max}^{-2} \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \leq \mathbb{E}[V(\hat{s}^{(0)}) - V(\hat{s}^{(K)})] + \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \quad (24)$$

Proof of this Theorem can be found in Appendix C.

### 3.2 Global Convergence of Two-Time-Scale Noisy EM Algorithms

We now proceed by giving our main result regarding the global convergence of the fiTTSEM algorithm. Two important auxiliary Lemmas, which proofs are given in Appendix D.1, are need in order to derive our finite-time bound. The first one derives an identity for the quantity  $\mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k+1)}\|^2]$  using the vrTTSEM update:

**Lemma 4.** *For any  $k \geq 0$  and consider the vrTTSEM update in (10) with  $\rho_k = \rho$ , it holds for all  $k > 0$*

$$\begin{aligned} \mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 2\rho^2 \mathbb{E}[\|\hat{s}^{(k)} - \bar{s}^{(k)}\|^2] + 2\rho^2 L_s^2 \mathbb{E}[\|\hat{s}^{(k)} - \hat{s}^{(\ell(k))}\|^2] \\ &\quad + 2(1 - \rho)^2 \mathbb{E}[\|\hat{s}^{((k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (25)$$

where we recall that  $\ell(k)$  is the first iteration number in the epoch that iteration  $k$  is in.

The second one derives an identity for the quantity  $\mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k+1)}\|^2]$  using the fiTTSEM update:

**Lemma 5.** *For any  $k \geq 0$  and consider the fiTTSEM update in (11) with  $\rho_k = \rho$ , it holds for all  $k > 0$*

$$\begin{aligned} \mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 2\rho^2 \mathbb{E}[\|\hat{s}^{(k)} - \bar{s}^{(k)}\|^2] + 2\rho^2 \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{s}^{(k)} - \hat{s}^{(\tau_i^k)}\|^2] \\ &\quad + 2(1 - \rho)^2 \mathbb{E}[\|\hat{s}^{((k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (26)$$

206 We now state the main result regarding the vrTTSEM method.

207 **Theorem 2.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
 208 positive step sizes and consider the vrTTSEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = \rho$  for  
 209 any  $k > 0$ . Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ . By setting  $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$ ,  $\rho = \frac{\mu}{c_1 \bar{L} n^{2/3}}$ ,  
 210  $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$  and a constant  $\mu \in (0, 1)$ , we have the following bound:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[ \tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right] \end{aligned} \quad (27)$$

211 Proof of this Theorem can be found in Appendix E. We now state the main result regarding the  
 212 fiTTSEM method.

213 **Theorem 3.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
 214 positive step sizes and consider the fiTTSEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = \rho$  for  
 215 any  $k > 0$ . Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ . By setting  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  
 216  $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ ,  $\rho = \frac{1}{n^{2/3}}$ ,  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ , we have the following  
 217 bound:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[ \Xi^{(k+1)} + \Gamma_{k+1} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right] \end{aligned} \quad (28)$$

218 Proof of this Theorem can be found in Appendix F. Note that in those two bounds, the quantities  
 219  $\tilde{\eta}^{(k+1)}$  and  $\Xi^{(k+1)}$  are abstraction that depends only on the MC fluctuations  $\mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right]$  and some  
 220 constants.

221 **Remarks:** The following remarks are worth noting on the quantity  $\mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right]$ :

- 222 • This term is the price we pay for the two-time-scale dynamics and corresponds to the gap  
 223 between the two asynchronous updates (one is on  $\hat{\mathbf{s}}^{(k)}$  and the other on  $\tilde{S}^{(k)}$ ).
- It is trivial to see that if  $\rho = 1$ , i.e., there is no variance reduction, then for any  $k > 0$

$$\mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] = \mathbb{E} \left[ \left\| \mathcal{S}^{(k+1)} - \tilde{S}^{(k+1)} \right\|^2 \right] = 0 \quad \text{with} \quad \hat{\mathbf{s}}^{(0)} = \tilde{S}^{(0)} = 0$$

224 which strengthen the fact that this quantity characterizes the impact of the variance reduc-  
 225 tion technique introduced in our two stages class of methods.

226 The following lemma, which proof can be found in Appendix D.2, can be derived to characterize  
 227 this gap:

228 **Lemma 6.** Consider a decreasing stepsize  $\gamma_k \in (0, 1)$  and a constant  $\rho \in (0, 1)$ , then the following  
 229 inequality holds:

$$\mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \leq \frac{\rho}{1 - \rho} \sum_{\ell=0}^k (1 - \gamma_{\ell})^2 (\mathcal{S}^{(\ell)} - \tilde{S}^{(\ell)}) \quad (29)$$

230 where  $\mathcal{S}^{(k)}$  is defined either by (10) (vrTTSEM) or (11) (fiTTSEM).

231 In the next section, we illustrate the benefits of our two-time-scale class of methods on several  
 232 numerical applications.



## 4 Numerical Examples

### 4.1 Gaussian Mixture Models

We begin by a simple and illustrative example. The authors acknowledge that the following model can be trained using deterministic EM-type of algorithms but propose to apply stochastic methods, including theirs, and to compare their performances. Given  $n$  observations  $\{y_i\}_{i=1}^n$ , we want to fit a Gaussian Mixture Model (GMM) whose distribution is modeled as a Gaussian mixture of  $M$  components, each with a unit variance. Let  $z_i \in \llbracket M \rrbracket$  be the latent labels of each component, the complete log-likelihood is defined as:

$$\log f(z_i, y_i; \theta) = \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) [\log(\omega_m) - \mu_m^2/2] + \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) \mu_m y_i + \text{constant} . \quad (30)$$

where  $\theta := (\omega, \mu)$  with  $\omega = \{\omega_m\}_{m=1}^{M-1}$  are the mixing weights with the convention  $\omega_M = 1 - \sum_{m=1}^{M-1} \omega_m$  and  $\mu = \{\mu_m\}_{m=1}^M$  are the means. We use the penalization  $r(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \log \text{Dir}(\omega; M, \epsilon)$  where  $\delta > 0$  and  $\text{Dir}(\cdot; M, \epsilon)$  is the  $M$  dimensional symmetric Dirichlet distribution with concentration parameter  $\epsilon > 0$ . The constraint set on  $\theta$  is given by

$$\Theta = \{\omega_m, m = 1, \dots, M-1 : \omega_m \geq 0, \sum_{m=1}^{M-1} \omega_m \leq 1\} \times \{\mu_m \in \mathbb{R}, m = 1, \dots, M\}. \quad (31)$$

Exact two-time-scale updates are given in Appendix G.1.

In the following experiments on synthetic data, we generate samples from a GMM model with  $M = 2$  components with two mixtures with means  $\mu_1 = -\mu_2 = 0.5$ . We use  $n = 10^5$  synthetic samples and run the bEM method until convergence (to double precision) to obtain the ML estimate  $\mu^*$  averaged on 50 datasets. We compare the bEM, iEM (incremental EM), SAEM, iSAEM, vrTTSEM and fiTTSEM methods in terms of their precision measured by  $|\mu - \mu^*|^2$ . We set the stepsize of the SA-step of all method as  $\gamma_k = 1/k^\alpha$  with  $\alpha = 0.5$ , and the stepsizes of the Incremental-step for vrTTSEM and the fiTTSEM to a constant stepsize equal to  $1/n^{2/3}$ .

The number of MC samples is fixed to  $M = 10$  chains. Figure 1 shows the convergence of the precision  $|\mu - \mu^*|^2$  for the different methods against the epoch(s) elapsed (one epoch equals  $n$  iterations). We observe that the vrTTSEM and fiTTSEM methods outperform the other stochastic methods, supporting the benefits of our newly introduced scheme.

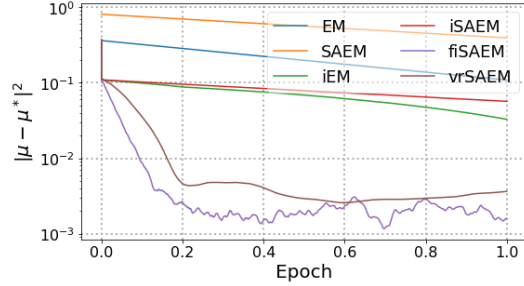


Figure 1: TO COMPLETE

### 4.2 Deformable Template Model for Image Analysis

Let  $(y_i, i \in \llbracket 1, n \rrbracket)$  be observed gray level images defined on a grid of pixels. Let  $u \in \mathcal{U} \subset \mathbb{R}^2$  denotes the pixel index on the image and  $x_u \in \mathcal{D} \subset \mathbb{R}^2$  its location. The model used in this experiment suggests that each image  $y_i$  is a deformation of a template, noted  $I : \mathcal{D} \rightarrow \mathbb{R}$ , common to all images of the dataset:

$$y_i(u) = I(x_u - \Phi_i(x_u, z_i)) + \varepsilon_i(u) \quad (32)$$

where  $\phi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a deformation function,  $z_i$  some latent variable parametrizing this deformation and  $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$  is an observation error.

The template model, given  $(p_k, k \in \llbracket 1, k_p \rrbracket)$  landmarks on the template, a fixed known kernel  $\mathbf{K}_p$  and a vector of parameters  $\beta \in \mathbb{R}^{k_p}$  is defined as follows:

$$I_\xi = \mathbf{K}_p \beta, \quad \text{where} \quad (\mathbf{K}_p \beta)(x) = \sum_{k=1}^{k_p} \mathbf{K}_p(x, p_k) \beta_k \quad (33)$$

Besides, we parameterize the deformation model given some landmarks  $(g_k, k \in \llbracket 1, k_g \rrbracket)$  and a fixed kernel  $\mathbf{K}_g$  as:

$$\Phi_i = \mathbf{K}_g z_i \quad \text{where} \quad (\mathbf{K}_g z_i)(x) = \sum_{k=1}^{k_g} \mathbf{K}_g(x, g_k) \left( z_i^{(1)}(k), z_i^{(2)}(k) \right) \quad (34)$$



274 where we put a Gaussian prior on the latent variables,  $z_i \sim \mathcal{N}(0, \Gamma)$  and  $z_i \in (\mathbb{R}^{k_g})^2$ . The vector of  
 275 parameters we ought to estimate is thus  $\theta = (\beta, \Gamma, \sigma)$ . The complete model belongs to the curved  
 276 exponential family, see [1], which vector of sufficient statistics  $S = (S_1(z), S_2(z), S_3(z))$  read:

$$\begin{aligned} S_1(z) &= \frac{1}{n} \sum_{i=1}^n S_1(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top y_i \\ S_2(z) &= \frac{1}{n} \sum_{i=1}^n S_2(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top (\mathbf{K}_p^{z_i}) \\ S_3(z) &= \frac{1}{n} \sum_{i=1}^n S_3(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n z_i^t z_i \end{aligned} \quad (35)$$

277 where for any pixel  $u \in \mathbb{R}^2$  and  $j \in \llbracket 1, k_g \rrbracket$  we noted:

$$\mathbf{K}_p^{z_i}(x_u, j) = \mathbf{K}_p^{z_i}(x_u - \phi_i(x_u, z_i), p_j) \quad (36)$$

278 Finally, the Two-Time-Scale M-step yields the following parameter updates:

$$\bar{\theta}(\hat{s}) = \begin{pmatrix} \beta(\hat{s}) = \hat{s}_2^{-1}(z) \hat{s}_1(z) \\ \Gamma(\hat{s}) = \frac{1}{n} \hat{s}_3(z) \\ \sigma(\hat{s}) = \beta(\hat{s})^\top \hat{s}_2(z) \beta(\hat{s}) - 2\beta(\hat{s}) \hat{s}_1(z) \end{pmatrix} \quad (37)$$

279 where  $\hat{s} = (\hat{s}_1(z), \hat{s}_2(z), \hat{s}_3(z))$  is the vector of statistics obtained via the **SA-step** (7) and using the  
 280 MC approximation of the sufficient statistics  $(S_1(z), S_2(z), S_3(z))$  defined in (42).

281 **Comparison using epochs credit**

282 **Comparison using number of training samples credit**

### 283 4.3 PK Model with Absorption Lag Time

284 This numerical example was conducted in order to characterize the pharmacokinetics of orally ad-  
 285 ministered drug to simulated patients, using a population pharmacokinetic approach.  $M = 50$  syn-  
 286 thetic datasets were generated for  $n = 500$  patients with 10 observations (concentration measures)  
 287 per patient.

288 **The model:** We consider a one-compartment pharmacokinetics (PK) model for oral administration  
 289 with an absorption lag-time ( $T^{\text{lag}}$ ), assuming first-order absorption and linear elimination processes.  
 290 The final model includes  $ka$  is the absorption rate constant,  $V$  the volume of distribution,  $k$  the  
 291 elimination rate constant. We also add several covariates to our model such as  $D$  the dose of drug  
 292 administered,  $t$  the time at which measures are taken and the weight such as  $V$  is function of it.  
 293 More precisely, the log-volume  $\log(V)$  is a linear function of the log-weight  $lw70 = \log(wt/70)$ .  
 294 The final reads:

$$f(t, ka, V, k) = \frac{D ka}{V(ka - k)} (e^{-ka(t - T^{\text{lag}})} - e^{-k(t - T^{\text{lag}})}) , \quad (38)$$

295 Here,  $T^{\text{lag}}$ ,  $ka$ ,  $V$  and  $k$  are PK parameters that can change from one individual to another.

296 Let  $z_i = (T_i^{\text{lag}}, ka_i, V_i, k_i)$  be the vector of individual PK parameters for individual  $i$ . The model  
 297 for the  $j$ -th measured concentration, noted  $y_{ij}$ , for individual  $i$  writes:

$$y_{ij} = f(t_{ij}, z_i) + \varepsilon_{ij} . \quad (39)$$

298 We assume in this example that the residual errors are independent and normally distributed with  
 299 mean 0 and variance  $\sigma^2$ . Lognormal distributions are used for the three PK parameters:

$$\log(T_i^{\text{lag}}) \sim \mathcal{N}(\log(T_{\text{pop}}^{\text{lag}}), \omega_{T^{\text{lag}}}^2) \sim \mathcal{N}(\log(ka_{\text{pop}}), \omega_{ka}^2) , \quad (40)$$

$$\log(V_i) \sim \mathcal{N}(\log(V_{\text{pop}}), \omega_V^2) , \log(k_i) \sim \mathcal{N}(\log(k_{\text{pop}}), \omega_k^2) . \quad (41)$$

300 The complete model belongs to the curved exponential family, which vector of sufficient statistics  
 301  $S = (S_1(z), S_2(z), S_3(z))$  read:

$$S_1(z) = \frac{1}{n} \sum_{i=1}^n z_i, \quad S_2(z) = \frac{1}{n} \sum_{i=1}^n z_i^\top z_i, \quad S_3(z) = \frac{1}{n} \sum_{i=1}^n (y_i - f(t_i, z_i))^2 \quad (42)$$

302 where we have noted  $y_i$  and  $t_i$  the vector of observations and time for each patient  $i$ .

303 **Monte Carlo study:** We conduct a Monte  
 304 Carlo study to showcase the benefits of our  
 305 scheme.

306  $M = 50$  datasets have been simulated using  
 307 the following PK parameters values:  $T_{\text{pop}}^{\text{lag}} =$   
 308  $1$ ,  $ka_{\text{pop}} = 1$ ,  $V_{\text{pop}} = 8$ ,  $k_{\text{pop}} = 0.1$ ,  
 309  $\omega_{T^{\text{lag}}} = 0.4$ ,  $\omega_{ka} = 0.5$ ,  $\omega_V = 0.2$ ,  $\omega_k =$   
 310  $0.3$  and  $\sigma^2 = 0.5$ . We define the mean  
 311 square distance over the  $M$  replicates  $E_k(\ell) =$

312  $\frac{1}{M} \sum_{m=1}^M \left( \theta_k^{(m)}(\ell) - \theta^* \right)^2$  and plot it against

313 the epochs (passes over the data) Figure 2. Note that the MC-step (5) is performed using a Metropo-  
 314 lis Hastings procedure since the posterior distribution under the model  $\theta$  noted  $p(z_i|y_i, \theta)$  is in-  
 315 tractable due to the nonlinearity of the model (38). Figure 2 shows

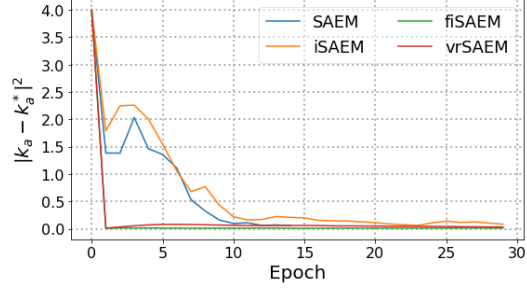


Figure 2: TO COMPLETE

## 316 5 Conclusion

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## 367 A Proof of Lemma 2

368 **Lemma.** Assume H3, H4. For all  $\mathbf{s} \in \mathcal{S}$ ,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (43)$$

369 **Proof** Using H3 and the fact that we can exchange integration with differentiation and the Fisher's  
370 identity, we obtain

$$\begin{aligned} \nabla_{\mathbf{s}} V(\mathbf{s}) &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left( \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{L}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left( \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} (\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))), \end{aligned} \quad (44)$$

371 Consider the following vector map:

$$\mathbf{s} \rightarrow \nabla_{\boldsymbol{\theta}} L(\mathbf{s}, \boldsymbol{\theta})|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})} = \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \mathbf{s}. \quad (45)$$

372 Taking the gradient of the above map w.r.t.  $\mathbf{s}$  and using assumption H3, we show that:

$$\mathbf{0} = -\mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \underbrace{\left( \nabla_{\boldsymbol{\theta}}^2 (\psi(\boldsymbol{\theta}) + \mathbf{r}(\boldsymbol{\theta}) - \langle \phi(\boldsymbol{\theta}) | \mathbf{s} \rangle) \right)|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})}}_{=\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \boldsymbol{\theta})} \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s}). \quad (46)$$

373 The above yields

$$\nabla_{\mathbf{s}} V(\mathbf{s}) = \mathbf{B}(\mathbf{s})(\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))) \quad (47)$$

374 where we recall  $\mathbf{B}(\mathbf{s}) = \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \left( \mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \bar{\boldsymbol{\theta}}(\mathbf{s})) \right)^{-1} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top}$ . The proof of (43) follows directly  
375 from the assumption H4.  $\square$

## 376 B Proof of Lemma 3

377 **Lemma.** Assume H??. The update (9) is equivalent to the following update on the resulting statistics

$$\hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}) \quad (48)$$

379 Also:

$$\mathbb{E} [\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] = \mathbb{E} [\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] + \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}] \quad (49)$$

380 where  $\bar{\mathbf{s}}^{(k)}$  is defined by (3) and  $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$ .

381 **Proof** From update (9), we have:

$$\begin{aligned} \tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= \tilde{\mathbf{S}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \frac{1}{n} \left( \tilde{S}_{i_k}^{(k+1)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)} \right) \\ &= \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{\mathbf{S}}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left( \tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \tilde{S}_{i_k}^{(k+1)} \right) \end{aligned} \quad (50)$$

382 Since  $\tilde{S}_{i_k}^{(k+1)} = \bar{s}_{i_k}(\boldsymbol{\theta}^{(k)}) + \eta_{i_k}^{(k+1)}$  we have

$$\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{\mathbf{S}}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left( \tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{s}_{i_k}(\boldsymbol{\theta}^{(k)}) \right) + \frac{1}{n} \eta_{i_k}^{(k+1)} \quad (51)$$

383 Taking the full expectation of both side of the equation leads to:

$$\begin{aligned} \mathbb{E} [\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] &= \mathbb{E} [\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] \\ &\quad - \frac{1}{n} \mathbb{E} \left[ \mathbb{E} [\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{s}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k] \right] + \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}] \end{aligned} \quad (52)$$

384 The following equalities:

$$\mathbb{E} [\tilde{S}_i^{(\tau_i^k)} | \mathcal{F}_k] = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} \quad \text{and} \quad \mathbb{E} [\bar{s}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k] = \bar{\mathbf{s}}^{(k)} \quad (53)$$

385 concludes the proof of the Lemma.  $\square$

## C Proof of Theorem 1

**Theorem.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of positive step sizes and consider the iSAEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = 1$  for any  $k > 0$ . We also set  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ . Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ .

$$v_{\max}^{-2} \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \nabla V(\hat{\mathbf{s}}^{(k)}) \right\|^2 \right] \leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right] \quad (54)$$

**Proof** We begin our proof by giving this auxiliary Lemma setting an upper bound for the quantity  $\mathbb{E} \left[ \left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right]$

**Lemma 7.** For any  $k \geq 0$  and consider the iSAEM update in (9), it holds that

$$\begin{aligned} \mathbb{E} \left[ \left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] &\leq 4\mathbb{E} \left[ \left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] + \frac{2L_s^2}{n^3} \sum_{i=1}^n \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \right\|^2 \right] \\ &\quad + 2\frac{C_\eta}{M_k} + 4\mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \end{aligned} \quad (55)$$

**Proof** Applying the iSAEM update yields:

$$\begin{aligned} \mathbb{E} \left[ \left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] &= \mathbb{E} \left[ \left\| \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} - \frac{1}{n} (\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \tilde{S}_{i_k}^{(k)}) \right\|^2 \right] \\ &\leq 4\mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] + 4\mathbb{E} \left[ \left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &\quad + \frac{2}{n^2} \mathbb{E} \left[ \left\| \bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)} \right\|^2 \right] + 2\frac{C_\eta}{M_k} \end{aligned} \quad (56)$$

The last expectation can be further bounded by

$$\frac{2}{n^2} \mathbb{E} \left[ \left\| \bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)} \right\|^2 \right] = \frac{2}{n^3} \sum_{i=1}^n \mathbb{E} \left[ \left\| \bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)} \right\|^2 \right] \stackrel{(a)}{\leq} \frac{2L_s^2}{n^3} \sum_{i=1}^n \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \right\|^2 \right], \quad (57)$$

where (a) is due to Lemma 1 and which concludes the proof of the Lemma.  $\square$

Under the smoothness of the Lyapunov function  $V$  (cf. Lemma 1), we can write:

$$V(\hat{\mathbf{s}}^{(k+1)}) \leq V(\hat{\mathbf{s}}^{(k)}) + \gamma_{k+1} \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \quad (58)$$

Taking the expectation on both sides yields:

$$\mathbb{E} \left[ V(\hat{\mathbf{s}}^{(k+1)}) \right] \leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(k)}) \right] + \gamma_{k+1} \mathbb{E} \left[ \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} \left[ \left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \quad (59)$$



400 Using Lemma 3, we obtain:

$$\begin{aligned}
& \mathbb{E} \left[ \langle \tilde{S}^{(k+1)} - \hat{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] = \\
& \mathbb{E} \left[ \langle \bar{s}^{(k)} - \hat{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] + \left( 1 - \frac{1}{n} \right) \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \right\rangle \right] + \frac{1}{n} \mathbb{E} \left[ \langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] \\
& \stackrel{(a)}{\leq} -v_{\min} \mathbb{E} \left[ \left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] + \left( 1 - \frac{1}{n} \right) \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \right\rangle \right] + \frac{1}{n} \mathbb{E} \left[ \langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] \\
& \stackrel{(b)}{\leq} -v_{\min} \mathbb{E} \left[ \left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] \\
& + \frac{\beta(n-1)+1}{2n} \mathbb{E} \left[ \left\| \nabla V(\hat{s}^{(k)}) \right\|^2 \right] + \frac{1}{2n} \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right] \\
& \stackrel{(a)}{\leq} \left( v_{\max}^2 \frac{\beta(n-1)+1}{2n} - v_{\min} \right) \mathbb{E} \left[ \left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] + \frac{1}{2n} \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right]
\end{aligned} \tag{60}$$

401 where (a) is due to the growth condition (2) and (b) is due to Young's inequality (with  $\beta \rightarrow 1$ ). Note

402  $a_k = \gamma_{k+1} \left( v_{\min} - v_{\max}^2 \frac{\beta(n-1)+1}{2n} \right)$  and

$$\begin{aligned}
a_k \mathbb{E} \left[ \left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] & \leq \mathbb{E} \left[ V(\hat{s}^{(k)}) - V(\hat{s}^{(k+1)}) \right] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} \left[ \left\| \tilde{S}^{(k+1)} - \hat{s}^{(k)} \right\|^2 \right] \\
& + \frac{\gamma_{k+1}(1 - \frac{1}{n})}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] + \frac{\gamma_{k+1}}{2n} \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right]
\end{aligned} \tag{61}$$

403 We now give an upper bound of  $\mathbb{E} \left[ \left\| \tilde{S}^{(k+1)} - \hat{s}^{(k)} \right\|^2 \right]$  using Lemma 7 and plug it into (61):

$$\begin{aligned}
(a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E} \left[ \left\| \bar{s}^{(k)} - \hat{s}^{(k)} \right\|^2 \right] & \leq \mathbb{E} \left[ V(\hat{s}^{(k)}) - V(\hat{s}^{(k+1)}) \right] \\
& + \gamma_{k+1} \left( \frac{1}{2\beta} \left( 1 - \frac{1}{n} \right) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] \\
& + \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right] \\
& + \frac{\gamma_{k+1}^2 L_V L_s^2}{n^3} \sum_{i=1}^n \mathbb{E} \left[ \left\| \hat{s}^{(k)} - \hat{s}^{(\tau_i^k)} \right\|^2 \right]
\end{aligned} \tag{62}$$

404 Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \left\| \hat{s}^{(k+1)} - \hat{s}^{(\tau_i^{k+1})} \right\|^2 \right] = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \mathbb{E} \left[ \left\| \hat{s}^{(k+1)} - \hat{s}^{(k)} \right\|^2 \right] + \frac{n-1}{n} \mathbb{E} \left[ \left\| \hat{s}^{(k+1)} - \hat{s}^{(\tau_i^k)} \right\|^2 \right] \right) \tag{63}$$

405 where the equality holds as  $i_k$  and  $j_k$  are drawn independently. For any  $\beta > 0$ , it holds

$$\begin{aligned}
& \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\
&= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\
&= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 - 2\gamma_{k+1}\langle \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\
&\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}\|^2 + \gamma_{k+1}\beta\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right]
\end{aligned} \tag{64}$$

406 where the last inequality is due to the Young's inequality. Subsequently, we have

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\
&\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}\left[(1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}\|^2\right]
\end{aligned} \tag{65}$$

407 Observe that  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)})$ . Applying Lemma 7 yields

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\
&\leq (\gamma_{k+1}^2 + \frac{n-1}{n} \frac{\gamma_{k+1}}{\beta}) \mathbb{E}[\|\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right] \\
&\leq 4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}\left[\|\eta_{i_k}^{(k)}\|^2\right] \\
&\quad + 4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{S}}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] \\
&\quad + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2\right]
\end{aligned} \tag{66}$$

408 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2] \tag{67}$$

409 From the above, we get

$$\begin{aligned}
\Delta^{(k+1)} &\leq (1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}))\Delta^{(k)} + 4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
&\quad + 2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}\left[\|\eta_{i_k}^{(k)}\|^2\right] + 4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{S}}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right]
\end{aligned} \tag{68}$$

410 Setting  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ ,

411  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 6$ ,  $\alpha \geq 8$ , we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \leq 1 - \frac{c_1(k\alpha - 1) - 4}{k\alpha n c_1} \leq 1 - \frac{2}{k\alpha n c_1} \tag{69}$$

412 which shows that  $1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \in (0, 1)$  for any  $k > 0$ . Denote  $\Lambda_{(k+1)} =$   
 413  $\frac{1}{n} - \gamma_{k+1}\beta - \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})$  and note that  $\Delta^{(0)} = 0$ , thus the telescoping sum yields:

$$\begin{aligned} \Delta^{(k+1)} \leq & 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] + 2 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E} \left[ \left\| \eta_{i_\ell}^{(\ell)} \right\|^2 \right] \\ & + 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^\ell)} - \bar{\mathbf{s}}^{(\ell)} \right\|^2 \right] \end{aligned} \quad (70)$$

414 Note  $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$  Summing on both sides over  $k = 0$  to  $k = K_{\max} - 1$  yields:

$$\begin{aligned} & \sum_{k=0}^{K_{\max}-1} \Delta^{(k+1)} \\ &= 4 \sum_{k=0}^{K_{\max}-1} (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2 \sum_{k=0}^{K_{\max}-1} (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E} \left[ \left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\ &+ \sum_{k=0}^{K_{\max}-1} 4 (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &\leq \sum_{k=0}^{K_{\max}-1} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{k=0}^{K_{\max}-1} \frac{2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E} \left[ \left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \end{aligned} \quad (71)$$

415 We recall (62) where we have summed on both sides from  $k = 0$  to  $k = K_{\max} - 1$ :

$$\begin{aligned} & \sum_{k=0}^{K_{\max}-1} (a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E} \left[ \left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \left( \frac{1}{2\beta} (1 - \frac{1}{n}) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \Delta^{(k)} \end{aligned} \quad (72)$$

416 Plugging (71) into (72) results in:

$$\begin{aligned} & \sum_{k=0}^{K_{\max}-1} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\beta}_k \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] \\ &+ \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right] \end{aligned} \quad (73)$$

417 where:

$$\begin{aligned}\tilde{\alpha}_k &= a_k - 2\gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \\ \tilde{\beta}_k &= \gamma_{k+1} \left( \frac{1}{2\beta} (1 - \frac{1}{n}) + 2\gamma_{k+1} L_V \right) - \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \\ \tilde{\Gamma}_k &= \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) + \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}}\end{aligned}$$

418 and

$$\begin{aligned}a_k &= \gamma_{k+1} \left( v_{\min} - v_{\max}^2 \frac{\beta(n-1) + 1}{2n} \right) \\ \Lambda_{(k+1)} &= \frac{1}{n} - \gamma_{k+1}\beta - \frac{2\gamma_{k+1} L_s^2}{n^2} (\gamma_{k+1} + \frac{1}{\beta}) \\ c_1 &= v_{\min}^{-1}, \alpha = \max\{8, 1 + 6v_{\min}\}, \bar{L} = \max\{L_s, L_V\}, \gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}, \beta = \frac{c_1 \bar{L}}{n}\end{aligned}$$

419 When, for any  $k > 0$ ,  $\tilde{\alpha}_k \geq 0$ , we have by Lemma 2 that:

$$\sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \nabla V(\hat{\mathbf{s}}^{(k)}) \right\|^2 \right] \leq v_{\max}^2 \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \quad (74)$$

420 which yields an upper bound of the gradient of the Lyapunov function  $V$  along the path of the  
421 iSAEM update and concludes the proof of the Theorem.  $\square$

## 422 D Proofs of Auxiliary Lemmas

### 423 D.1 Proof of Lemma 4 and Lemma 5

424 **Lemma.** For any  $k \geq 0$  and consider the vrTTSEM update in (10) with  $\rho_k = \rho$ , it holds for all  
425  $k > 0$

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &\quad + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (75)$$

426 where we recall that  $\ell(k)$  is the first iteration number in the epoch that iteration  $k$  is in.

427 **Proof** Beforehand, we provide a rewriting of the quantity  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$  that will be useful through-  
428 out this proof:

$$\begin{aligned} \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathbf{S}^{(k+1)}) \\ &= -\gamma_{k+1} \left( (1-\rho) \left[ \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[ \hat{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)} \right] \right) \end{aligned} \quad (76)$$

429 We observe, using the identity (76), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] \quad (77)$$

430 For the latter term, we obtain its upper bound as

$$\begin{aligned} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] &= \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \tilde{S}_i^{\ell(k)}) - (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{\ell(k)}) \right\|^2 \right] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{\ell(k)}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \stackrel{(b)}{\leq} L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{\ell(k)}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (78)$$

431 where (a) uses the variance inequality and (b) uses Lemma 1. Substituting into (77) proves the  
432 lemma.  $\square$

433 **Lemma.** For any  $k \geq 0$  and consider the fiTTSEM update in (11) with  $\rho_k = \rho$ , it holds for all  $k > 0$   
434

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_k)}\|^2] \\ &\quad + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned} \quad (79)$$

435 **Proof** Beforehand, we provide a rewriting of the quantity  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$  that will be useful through-  
436 out this proof:

$$\begin{aligned} \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) \\ &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathbf{S}^{(k+1)}) \\ &= -\gamma_{k+1} \left( (1-\rho) \left[ \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[ \hat{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)} \right] \right) \\ &= -\gamma_{k+1} \left( (1-\rho) \left[ \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[ \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} - (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_k)}) \right] \right) \end{aligned} \quad (80)$$

437 We observe, using the identity (80), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] \quad (81)$$

438 For the latter term, we obtain its upper bound as

$$\begin{aligned}\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] &= \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{S}}_i^{(k)}) - (\tilde{\mathbf{S}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)})\right\|^2\right] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(\ell(k))}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]\end{aligned}\quad (82)$$

439 where (a) uses the variance inequality. We can further bound the last expectation using Lemma 1:

$$\mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)}\|^2] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2] \stackrel{(a)}{\leq} \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \quad (83)$$

440 Substituting into (81) proves the lemma.  $\square$

## 441 D.2 Proof of Lemma 6

442 **Lemma.** Consider a decreasing stepsize  $\gamma_k \in (0, 1)$  and a constant  $\rho$ , then the following inequality  
443 holds:

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] \leq \frac{\rho}{1-\rho} \sum_{\ell=0}^k (1-\gamma_\ell)^2 (\mathbf{S}^{(\ell)} - \tilde{\mathbf{S}}^{(\ell)}) \quad (84)$$

444 where  $\mathbf{S}^{(k)}$  is defined either by (11) (fTTSEM) or (10) (vrTTSEM)

445 **Proof** We begin by writing the two-time-scale update:

$$\begin{aligned}\tilde{\mathbf{S}}^{(k+1)} &= \tilde{\mathbf{S}}^{(k)} + \rho(\mathbf{S}^{(k+1)} - \tilde{\mathbf{S}}^{(k)}) \\ \hat{\mathbf{s}}^{(k+1)} &= \hat{\mathbf{s}}^{(k)} + \gamma_{k+1}(\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)})\end{aligned}\quad (85)$$

446 where  $\mathbf{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{S}}_i^{(t_i^k)} + (\tilde{\mathbf{S}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)})$  according to (11). Denote  $\delta^{(k+1)} = \hat{\mathbf{s}}^{(k+1)} -$   
447  $\tilde{\mathbf{S}}^{(k+1)}$ . Then from (85), doing the subtraction of both equations yields:

$$\delta^{(k+1)} = (1 - \gamma_{k+1})\delta^{(k)} + \frac{\rho}{1-\rho} (1 - \gamma_{k+1})(\mathbf{S}^{(k+1)} - \tilde{\mathbf{S}}^{(k+1)}) \quad (86)$$

448 Using the telescoping sum and noting that  $\delta^{(0)} = 0$ , we have

$$\delta^{(k+1)} \leq \frac{\rho}{1-\rho} \sum_{\ell=0}^k (1 - \gamma_{\ell+1})^2 (\mathbf{S}^{(\ell+1)} - \tilde{\mathbf{S}}^{(\ell+1)}) \quad (87)$$

449  $\square$

## 450 D.3 Additional Intermediary Result

451 **Lemma 8.** At iteration  $k + 1$ , the drift term of update (11), with  $\rho_{k+1} = \rho$ , is equivalent to the  
452 following :

$$\begin{aligned}\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)} &= \rho(\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}) + \rho\eta_{i_k}^{(k+1)} + \rho \left[ (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)}] \right] \\ &\quad + (1 - \rho) (\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)})\end{aligned}\quad (88)$$

453 where we recall that  $\eta_{i_k}^{(k+1)}$ , defined in (19), which is the gap between the MC approximation and  
454 the expected statistics.



455 **Proof** Using the fTTSEM update  $\tilde{S}^{(k+1)} = (1-\rho)\tilde{S}^{(k)} + \rho\mathcal{S}^{(k+1)}$  where  $\mathcal{S}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})$  leads to the following decomposition:

$$\begin{aligned}
& \tilde{S}^{(k+1)} - \hat{s}^{(k)} \\
&= (1-\rho)\tilde{S}^{(k)} + \rho\left(\overline{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})\right) - \hat{s}^{(k)} + \rho\overline{\mathcal{S}}^{(k)} - \rho\overline{\mathcal{S}}^{(k)} \\
&= \rho(\overline{\mathcal{S}}^{(k)} - \hat{s}^{(k)}) + \rho(\tilde{S}_{i_k}^{(k)} - \overline{\mathcal{S}}_{i_k}^{(k)}) + (1-\rho)\left(\tilde{S}^{(k)} - \hat{s}^{(k)}\right) + \rho\left(\overline{\mathcal{S}}^{(k)} - \overline{\mathcal{S}}^{(k)} + (\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})\right) \\
&= \rho(\overline{\mathcal{S}}^{(k)} - \hat{s}^{(k)}) + \rho\eta_{i_k}^{(k+1)} - \rho\left[(\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}]\right] \\
&+ (1-\rho)\left(\tilde{S}^{(k)} - \hat{s}^{(k)}\right)
\end{aligned}$$

457 where we observe that  $\mathbb{E}[\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] = \overline{\mathcal{S}}^{(k)} - \overline{\mathcal{S}}^{(k)}$  and which concludes the proof.

458 *Important Note:* Note that  $\overline{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}$  is not equal to  $\eta_{i_k}^{(k+1)}$ , defined in (19), which is the gap  
459 between the MC approximation and the expected statistics. Indeed  $\tilde{S}_{i_k}^{(t_{i_k}^k)}$  is not computed under the  
460 same model as  $\overline{\mathcal{S}}_{i_k}^{(k)}$ .  $\square$

## 461 E Proof of Theorem 2

462 **Theorem.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
 463 positive step sizes and consider the vrTTSEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = \rho$  for  
 464 any  $k > 0$ .

465 Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ . By setting  $\bar{L} = \max\{L_S, L_V\}$ ,  $\rho = \frac{\mu}{c_1 \bar{L} n^{2/3}}$ ,  $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$   
 466 and a constant  $\mu \in (0, 1)$ , we have the following bound:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[ \tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right] \end{aligned} \quad (89)$$

467 **Proof** Using the smoothness of  $V$  and update (10), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) + \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \end{aligned} \quad (90)$$

468 Denote  $\mathbf{H}_{k+1} := \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}$  the drift term of the fiTTSEM update in (7) and  $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$ .  
 469 Taking expectations on both sides show that

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] \\ &\stackrel{(a)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}(1-\rho) \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1}\rho \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\stackrel{(b)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}\rho \mathbb{E}[\langle \mathbf{h}_k | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1}(1-\rho) \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad - \gamma_{k+1}\rho \mathbb{E}[\langle \eta_{i_k}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\stackrel{(c)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2) \mathbb{E}[\|\mathbf{h}_k\|^2] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\quad - \gamma_{k+1}\rho \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \gamma_{k+1}(1-\rho) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \end{aligned} \quad (91)$$

470 where we have used (76) in (a) and  $\mathbb{E}[\mathbf{S}^{(k+1)}] = \bar{\mathbf{s}}^{(k)} + \mathbb{E}[\eta_{i_k}^{(k+1)}]$  in (b), the growth condition in  
 471 Lemma 2 and the Young's inequality with the constant equal to 1 in (c).

472 Furthermore, for  $k+1 \leq \ell(k) + m$  (i.e.,  $k+1$  is in the same epoch as  $k$ ), we have

$$\begin{aligned} &\mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] = \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} + \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \rangle] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 \\ &\quad - 2\gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \rho(\mathbf{h}_k - \eta_{i_k}^{(k+1)}) + (1-\rho)(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}) \rangle] \\ &\leq \mathbb{E}[(1 + \gamma_{k+1}\beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2 \\ &\quad + \frac{\gamma_{k+1}\rho}{\beta} \|\eta_{i_k}^{(k+1)}\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2], \end{aligned} \quad (92)$$

473 where we first used (76) and the last inequality is due to the Young's inequality.

474 Consider the following sequence

$$R_k := \mathbb{E}[V(\hat{\mathbf{s}}^{(k)}) + b_k \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \quad (93)$$

475 where  $b_k := \bar{b}_{k \bmod m}$  is a periodic sequence where:

$$\bar{b}_i = \bar{b}_{i+1}(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2) + \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2, \quad i = 0, 1, \dots, m-1 \quad \text{with } \bar{b}_m = 0. \quad (94)$$

476 Note that  $\bar{b}_i$  is decreasing with  $i$  and this implies

$$\bar{b}_i \leq \bar{b}_0 = \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2 \frac{(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2)^m - 1}{\gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2}, \quad i = 1, 2, \dots, m. \quad (95)$$

477 For  $k+1 \leq \ell(k) + m$ , we have the following inequality

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2) \|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[\rho \left\|\eta_{i_k}^{(k+1)}\right\|^2 - (1-\rho) \left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\frac{\gamma_{k+1}\rho}{\beta} \left\|\eta_{i_k}^{(k+1)}\right\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \end{aligned} \quad (96)$$

478 And using Lemma 4 we obtain:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2 - \gamma_{k+1}^2\rho^2 L_V) \|\mathbf{h}_k\|^2 + \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2 \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2\right) \|\mathbf{h}_k\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[(\rho + \rho^2\gamma_{k+1} L_V) \left\|\eta_{i_k}^{(k+1)}\right\|^2 - (1-\rho - (1-\rho)^2\gamma_{k+1} L_V) \left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2\right) \left\|\eta_{i_k}^{(k+1)}\right\|^2 + \left(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2(1-\rho)^2\right) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2\right] \end{aligned} \quad (97)$$

479 Rearranging the terms yields:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2)) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\quad + \underbrace{\left(b_{k+1}(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2) + \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2\right)}_{=b_k \text{ since } k+1 \leq \ell(k) + m} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] + \tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)} \end{aligned} \quad (98)$$

480 where

$$\begin{aligned} \tilde{\eta}^{(k+1)} &= \left(\gamma_{k+1}(\rho + \rho^2\gamma_{k+1} L_V) + b_{k+1}(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2)\right) \mathbb{E}\left[\left\|\eta_{i_k}^{(k+1)}\right\|^2\right] \\ \chi^{(k+1)} &= \left(b_{k+1}(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2(1-\rho)^2) - \gamma_{k+1}(1-\rho - (1-\rho)^2\gamma_{k+1} L_V)\right) \\ \tilde{\chi}^{(k+1)} &= \chi^{(k+1)} \mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \end{aligned} \quad (99)$$

481 This leads, using Lemma 2, that for any  $\gamma_{k+1}$ ,  $\rho$  and  $\beta$  such that  $\rho v_{\min} + v_{\max}^2 -$   
482  $\gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2) > 0$ ,

$$\begin{aligned} v_{\max}^2 \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] \leq \frac{R_k - R_{k+1}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))} \\ &\quad + \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))} \end{aligned} \quad (100)$$

483 We first remark that

$$\begin{aligned} & \gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2)) \\ & \geq \frac{\gamma_{k+1}\rho}{c_1}(1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1)) \end{aligned} \quad (101)$$

484 where  $c_1 = v_{\min}^{-1}$ . By setting  $\bar{L} = \max\{L_s, L_V\}$ ,  $\beta = \frac{c_1\bar{L}}{n^{1/3}}$ ,  $\rho = \frac{\mu}{c_1\bar{L}n^{2/3}}$ ,  $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$  and  
 485  $\{\gamma_{k+1}\}$  any sequence of decreasing stepsizes in  $(0, 1)$ , it can be shown that there exists  $\mu \in (0, 1)$ ,  
 486 such that the following lower bound holds

$$\begin{aligned} & 1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1) \geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \bar{b}_0(\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ & \geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{L_V\mu^2}{c_1^2n^{\frac{4}{3}}}\frac{(1 + \gamma\beta + 2\gamma^2L_s^2)^m - 1}{\gamma\beta + 2\gamma^2L_s^2}(\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ & \stackrel{(a)}{\geq} 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{\mu}{c_1^2}(e - 1)(1 + \frac{2\mu}{n}) \geq 1 - \mu - \mu(1 + 2\mu)\frac{e - 1}{c_1^2} \stackrel{(b)}{\geq} \frac{1}{2} \end{aligned} \quad (102)$$

487 where the simplification in (a) is due to

$$\frac{\mu}{n} \leq \gamma\beta + 2\gamma^2L_s^2 \leq \frac{\mu}{n} + \frac{2\mu^2}{c_1^2n^{\frac{4}{3}}} \leq \frac{\mu c_1^2 + 2\mu^2}{c_1^2} \frac{1}{n} \text{ and } (1 + \gamma\beta + 2\gamma^2L_s^2)^m \leq e - 1. \quad (103)$$

488 and the required  $\mu$  in (b) can be found by solving the quadratic equation.

489 Finally, these results yield:

$$v_{\max}^2 \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \leq \frac{2(R_0 - R_{K_{\max}})}{v_{\min}\rho} + 2 \sum_{k=0}^{K_{\max}-1} \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{v_{\min}\rho} \quad (104)$$

490 Note that  $R_0 = \mathbb{E}[V(\hat{s}^{(0)})]$  and if  $K_{\max}$  is a multiple of  $m$ , then  $R_{\max} = \mathbb{E}[V(\hat{s}^{(K_{\max})})]$ . Under the  
 491 latter condition, we have

$$\sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{s}^{(0)}) - V(\hat{s}^{(K_{\max})})] + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} [\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}] \quad (105)$$

492 This concludes our proof.

493 □

494 **F Proof of Theorem 3**

495 **Theorem.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
 496 positive step sizes and consider the fTTSEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = \rho$  for  
 497 any  $k > 0$ .

498 Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ . By setting  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$ ,  
 499  $\beta = \frac{c_1 \bar{L}}{n}$ ,  $\rho = \frac{1}{n^{2/3}}$ ,  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ , we have the following bound:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[ \Xi^{(k+1)} + \Gamma_{k+1} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right] \end{aligned} \quad (106)$$

500 **Proof** Using the smoothness of  $V$  and update (11), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) + \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \end{aligned} \quad (107)$$

501 Denote  $\mathbf{H}_{k+1} := \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}$  the drift term of the fTTSEM update in (7) and  $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$ .  
 502 Using Lemma 8 and the additional following identity:

$$\mathbb{E} \left[ (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] = 0 \quad (108)$$

503 we have:

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] \\ &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} \rho \mathbb{E}[\langle \mathbf{h}_k | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1} \mathbb{E} \left[ \langle \rho \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] + (1 - \rho) \mathbb{E}[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}] | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2 \\ &\stackrel{(a)}{\leq} -v_{\min} \gamma_{k+1} \rho \mathbb{E}[\|\mathbf{h}_k\|^2] - \gamma_{k+1} \mathbb{E} \left[ \|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2 \right] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2 \\ &\stackrel{(b)}{\leq} -(v_{\min} \gamma_{k+1} \rho + \gamma_{k+1} v_{\max}^2) \mathbb{E}[\|\mathbf{h}_k\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2 \end{aligned} \quad (109)$$

504 where  $\xi^{(k+1)} = \mathbb{E} \left[ \left\| \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] \right\|^2 \right]$ . **Bounding  $\mathbb{E}[\|\mathbf{H}_{k+1}\|^2]$**  Using Lemma 5, we obtain:

$$\begin{aligned} &\gamma_{k+1} (v_{\min} \rho + v_{\max}^2 - \gamma_{k+1} \rho^2 L_V) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\leq \mathbb{E} [V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)})] + \tilde{\xi}^{(k+1)} + \left( (1 - \rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V \rho^2 L_{\mathbf{s}}^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \end{aligned} \quad (110)$$

505 where  $\tilde{\xi}^{(k+1)} = \gamma_{k+1}^2 \rho^2 \mathbb{L}_V \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \frac{\gamma_{k+1}\rho^2}{2} \xi^{(k+1)}$ . Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \right) \quad (111)$$

506 where the equality holds as  $i_k$  and  $j_k$  are drawn independently. Next,

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \end{aligned} \quad (112)$$

507 Note that  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) = -\gamma_{k+1}\mathbf{H}_{k+1}$  and that in expectation we recall  
 508 that  $\mathbb{E}[\mathbf{H}_{k+1}|\mathcal{F}_k] = \rho\mathbf{h}_k + \rho\mathbb{E}[\eta_{i_k}^{(k+1)}|\mathcal{F}_k] + (1-\rho)\mathbb{E}[\tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)}]$  where  $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$ . Thus,  
 509 for any  $\beta > 0$ , it holds

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\quad + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned} \quad (113)$$

510 where the last inequality is due to the Young's inequality. Plugging this into (111) yields:

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\quad + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned} \quad (114)$$

511 Subsequently, we have

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}[(1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2] \\ &\quad + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned} \quad (115)$$



512 We now use Lemma 5 on  $\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 = \gamma_{k+1}^2 \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2$  and obtain:

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\
& \leq \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \left(\frac{\gamma_{k+1}^2 \rho^2 L_s^2}{n} + \frac{(n-1)(1+\gamma_{k+1}\beta)}{n^2}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\
& \quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left(2\gamma_{k+1}^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\
& \leq \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \left(\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 L_s^2}{n}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\
& \quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left(2\gamma_{k+1}^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]
\end{aligned} \tag{116}$$

513 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \tag{117}$$

514 From the above, we get

$$\begin{aligned}
\Delta^{(k+1)} & \leq \left(1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 L_s^2\right) \Delta^{(k)} + \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
& \quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \gamma_{k+1} \left(2\gamma_{k+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]
\end{aligned} \tag{118}$$

515 Setting  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ ,  
516  $\rho = \frac{1}{n^{2/3}}$ ,  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ , we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 L_s^2 \leq 1 - \frac{1}{n} + \frac{1}{\alpha k n} + \frac{1}{\alpha^2 c_1^2 k^2 n^{4/3}} \leq 1 - \frac{c_1(k\alpha - 1) - 1}{k\alpha n c_1} \leq 1 - \frac{1}{k\alpha n c_1} \tag{119}$$

517 which shows that  $1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 L_s^2 \in (0, 1)$  for any  $k > 0$ . Denote  $\Lambda_{(k+1)} = \frac{1}{n} -$   
518  $\gamma_{k+1}\beta - \gamma_{k+1}^2 \rho^2 L_s^2$  and note that  $\Delta^{(0)} = 0$ , thus the telescoping sum yields:

$$\begin{aligned}
\Delta^{(k+1)} & \leq \sum_{\ell=0}^k \omega_{k,\ell} \left(2\gamma_{\ell+1}^2 \rho^2 + \frac{\gamma_{\ell+1} \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] \\
& \quad + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1} (1-\rho)^2 \left(2\gamma_{\ell+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\tilde{S}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1} \tilde{\epsilon}^{(\ell+1)}
\end{aligned} \tag{120}$$

519 where  $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$  and  $\tilde{\epsilon}^{(\ell+1)} = \left(2\gamma_{\ell+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(\ell+1)}\|^2]$ .

520 Summing on both sides over  $k = 0$  to  $k = K_{\max} - 1$  yields:

$$\begin{aligned}
\sum_{k=0}^{K_{\max}-1} \Delta^{(k+1)} & \leq \sum_{k=0}^{K_{\max}-1} \frac{2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta}}{\Lambda_{(k+1)}} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
& \quad + \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right)}{\Lambda_{(k+1)}} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}}{\Lambda_{(k+1)}} \tilde{\epsilon}^{(k+1)}
\end{aligned} \tag{121}$$

521 We recall (110) where we have summed on both sides from  $k = 0$  to  $k = K_{\max} - 1$ :

$$\begin{aligned}
& \mathbb{E}[V(\hat{\mathbf{s}}^{(K_{\max})}) - V(\hat{\mathbf{s}}^{(0)})] \\
& \leq \sum_{k=0}^{K_{\max}-1} \left\{ \gamma_{k+1}(-(\nu_{\min}\rho + v_{\max}^2) + \gamma_{k+1}\rho^2 L_V) \mathbb{E}[\|\mathbf{h}_k\|^2] + \gamma^2 L_V \rho^2 L_{\mathbf{s}}^2 \Delta^{(k)} \right\} \\
& + \sum_{k=0}^{K_{\max}-1} \left\{ \tilde{\xi}^{(k+1)} + \left( (1-\rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \right\} \\
& \leq \sum_{k=0}^{K_{\max}-1} \left\{ -\gamma_{k+1}(\nu_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 L_V + \frac{\rho^2 \gamma_{k+1}^2 L_V L_{\mathbf{s}}^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right\} \mathbb{E}[\|\mathbf{h}_k\|^2] \\
& + \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma_{k+1} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]
\end{aligned} \tag{122}$$

where

$$\Xi^{(k+1)} = \tilde{\xi}^{(k+1)} + \frac{\gamma_{k+1}^3 L_V \rho^2 L_{\mathbf{s}}^2}{\Lambda_{(k+1)}} \epsilon^{(k+1)}$$

and

$$\Gamma_{k+1} = \left( (1-\rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) + \frac{\gamma_{k+1}^3 L_V \rho^2 L_{\mathbf{s}}^2 (1-\rho)^2 \left( 2\gamma_{k+1} + \frac{1}{\beta} \right)}{\Lambda_{(k+1)}}$$

522 We now analyse the following quantity

$$\begin{aligned}
& -\gamma_{k+1}(\nu_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 L_V + \frac{\rho^2 \gamma_{k+1}^2 L_V L_{\mathbf{s}}^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \\
& = -\gamma_{k+1} \left[ (\nu_{\min}\rho + v_{\max}^2) + \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_{\mathbf{s}}^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right]
\end{aligned} \tag{123}$$

523 Furthermore, we recall that  $c_1 = \nu_{\min}^{-1}$ ,  $\alpha = \max\{2, 1+2\nu_{\min}\}$ ,  $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$ ,

524  $\beta = \frac{c_1 \bar{L}}{n}$ ,  $\rho = \frac{1}{n^{2/3}}$ ,  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ . Then,

$$\begin{aligned}
& \gamma_{k+1}^2 \rho^2 L_V + \frac{\rho^2 \gamma_{k+1}^2 L_V L_{\mathbf{s}}^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \\
& \leq \frac{1}{k^2 \alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L} (k^2 \alpha^2 c_1^2 n^{4/3})^{-1} \left( \frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k \alpha c_1^2 \bar{L}^2 n^{1/3}} \right)}{\frac{1}{n} - \frac{1}{k \alpha n} - \frac{1}{k^2 \alpha^2 c_1^2 n^{4/3}}} \\
& = \frac{1}{k^2 \alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L} \left( \frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k \alpha c_1^2 \bar{L}^2 n^{1/3}} \right)}{(k \alpha c_1 n^{1/3})(k \alpha - 1) c_1 - 1} \\
& \stackrel{(a)}{\leq} \frac{1}{k^2 \alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\frac{1}{k \alpha c_1^2 \bar{L} n^{1/3}} \left( \frac{2}{k \alpha n} + 1 \right)}{2(\alpha c_1 n^{1/3}) - 1} \\
& \leq \frac{1}{k^2 \alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{1}{k \alpha^2 c_1^3 \bar{L} n^{2/3}} \\
& \leq \frac{1}{\alpha c_1 \bar{L} n^{2/3}}
\end{aligned} \tag{124}$$

525 where (a) is due to  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$  and  $k \alpha c_1 n^{1/3} \geq 1$ . Also, since  $-\gamma_{k+1}(\nu_{\min}\rho +$   
526  $v_{\max}^2) \leq -\gamma_{k+1}\rho \nu_{\min} = -$

527 Using the Lemma 2, we know that  $v_{\max}^2 \|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2 \leq \|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2$  and using (124) on (122)  
 528 yields:

$$\begin{aligned} v_{\max}^2 \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{\alpha \bar{L} n^{2/3}}{v_{\min}} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{\alpha \bar{L} n^{2/3}}{v_{\min}} \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma_{k+1} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \end{aligned} \quad (125)$$

529 proving the final bound on the gradient of the Lyapunov function:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma_{k+1} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \end{aligned} \quad (126)$$

530

□

## G Practical Implementations of Two-Time-Scale EM Methods

### G.1 Application on GMM

We first recognize that the constraint set for  $\theta$  is given by

$$\Theta = \Delta^M \times \mathbb{R}^M. \quad (127)$$

Using the partition of the sufficient statistics as  $S(y_i, z_i) = (S^{(1)}(y_i, z_i)^\top, S^{(2)}(y_i, z_i)^\top, S^{(3)}(y_i, z_i)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$ , the partition  $\phi(\theta) = (\phi^{(1)}(\theta)^\top, \phi^{(2)}(\theta)^\top, \phi^{(3)}(\theta)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$  and the fact that  $\mathbb{1}_{\{M\}}(z_i) = 1 - \sum_{m=1}^{M-1} \mathbb{1}_{\{m\}}(z_i)$ , the complete data log-likelihood can be expressed as in (2) with

$$\begin{aligned} s_{i,m}^{(1)} &= \mathbb{1}_{\{m\}}(z_i), \quad \phi_m^{(1)}(\theta) = \left\{ \log(\omega_m) - \frac{\mu_m^2}{2} \right\} - \left\{ \log(1 - \sum_{j=1}^{M-1} \omega_j) - \frac{\mu_M^2}{2} \right\}, \\ s_{i,m}^{(2)} &= \mathbb{1}_{\{m\}}(z_i) y_i, \quad \phi_m^{(2)}(\theta) = \mu_m, \quad s_i^{(3)} = y_i, \quad \phi^{(3)}(\theta) = \mu_M, \end{aligned} \quad (128)$$

and  $\psi(\theta) = -\left\{ \log(1 - \sum_{m=1}^{M-1} \omega_m) - \frac{\mu_M^2}{2\sigma^2} \right\}$ . We also define for each  $m \in \llbracket 1, M \rrbracket$ ,  $j \in \llbracket 1, 3 \rrbracket$ ,  $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$ . Consider the following latent sample used to compute an approximation of the conditional expected value  $\mathbb{E}_\theta[\mathbb{1}_{\{z_i=m\}} | y = y_i]$ :

$$z_{i,m} \sim \mathbb{P}(z_i = m | y_i; \theta) \quad (129)$$

where  $m \in \llbracket 1, M \rrbracket$ ,  $i \in \llbracket 1, n \rrbracket$  and  $\theta = (\mathbf{w}, \boldsymbol{\mu}) \in \Theta$ .

In particular, given iteration  $k + 1$ , the computation of the approximated quantity  $\tilde{S}_{i_k}^{(k)}$  during Incremental-step updates, see (8) can be written as

$$\tilde{S}_{i_k}^{(k)} = \left( \underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1}), \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})}_{:=\tilde{s}_{i_k}^{(1)}}, \underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1})y_{i_k}, \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})y_{i_k}}_{:=\tilde{s}_{i_k}^{(2)}}, \underbrace{y_{i_k}}_{:=\tilde{s}_{i_k}^{(3)}(\theta^{(k)})} \right)^\top. \quad (130)$$

Recall that we have used the following regularizer:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m), \quad (131)$$

It can be shown that the regularized M-step in (4) evaluates to

$$\bar{\theta}(\mathbf{s}) = \begin{pmatrix} (1 + \epsilon M)^{-1} (s_1^{(1)} + \epsilon, \dots, s_{M-1}^{(1)} + \epsilon)^\top \\ ((s_1^{(1)} + \delta)^{-1} s_1^{(2)}, \dots, (s_{M-1}^{(1)} + \delta)^{-1} s_{M-1}^{(2)})^\top \\ (1 - \sum_{m=1}^{M-1} s_m^{(1)} + \delta)^{-1} (s^{(3)} - \sum_{m=1}^{M-1} s_m^{(2)}) \end{pmatrix} = \begin{pmatrix} \bar{\omega}(\mathbf{s}) \\ \bar{\boldsymbol{\mu}}(\mathbf{s}) \\ \bar{\mu}_M(\mathbf{s}) \end{pmatrix}. \quad (132)$$

where we have defined for all  $m \in \llbracket 1, M \rrbracket$  and  $j \in \llbracket 1, 3 \rrbracket$ ,  $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$ .

### G.2 Model Assumptions (GMM example)

We use the GMM example to illustrate the required assumptions.

Many practical models can satisfy the compactness of the sets as in Assumption H1. For instance, the GMM example satisfies (16) as the sufficient statistics are composed of indicator functions and observations as defined Section G.1 Equation (128).

Assumptions H2 and H3 are standard for the curved exponential family models. For GMM, the following (strongly convex) regularization  $\mathbf{r}(\theta)$  ensures H3:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m)$$

553 since it ensures  $\theta^{(k)}$  is unique and lies in  $\text{int}(\Delta^M) \times \mathbb{R}^M$ . We remark that for H2, it is possible to  
 554 define the Lipschitz constant  $L_p$  independently for each data  $y_i$  to yield a refined characterization.

555 Again, H4 is satisfied by practical models. For GMM, it can be verified by deriving the closed form  
 556 expression for  $B(s)$  and using H1.

557 Under H1 and H3, we have  $\|\hat{s}^{(k)}\| < \infty$  since  $S$  is compact and  $\hat{\theta}^{(k)} \in \text{int}(\Theta)$  for any  $k \geq 0$  which  
 558 thus ensure that the EM methods operate in a closed set throughout the optimization process.

### 559 G.3 Algorithms updates

560 In the sequel, recall that, for all  $i \in \llbracket n \rrbracket$  and iteration  $k$ , the computed statistic  $\tilde{S}_{i_k}^{(k)}$  is defined by  
 561 (130). At iteration  $k$ , the several E-steps defined by (9) or (10) and (11) leads to the definition of the  
 562 quantity  $\hat{s}^{(k+1)}$ . For the GMM example, after the initialization of the quantity  $\hat{s}^{(0)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(0)}$ ,  
 563 those E-steps break down as follows:

564 **Batch EM (EM):** for all  $i \in \llbracket 1, n \rrbracket$ , compute  $\bar{s}_i^{(k)}$  and set

$$\hat{s}^{(k+1)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(k)}. \quad (133)$$

565 where  $\bar{s}_i^{(k)}$  are computed using the exact conditional expected value  $\mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i]$ :

$$\tilde{\omega}_m(y_i; \theta) := \mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i] = \frac{\omega_m \exp(-\frac{1}{2}(y_i - \mu_i)^2)}{\sum_{j=1}^M \omega_j \exp(-\frac{1}{2}(y_i - \mu_j)^2)}, \quad (134)$$

566 **Incremental EM (iEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ , compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} + \frac{1}{n} (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\tau_i^k)}) = n^{-1} \sum_{i=1}^n \bar{s}_i^{(\tau_i^k)}. \quad (135)$$

567 **batch SAEM (SAEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ , compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} \tilde{S}^{(k)}. \quad (136)$$

568 where  $\tilde{S} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k)}$  with  $\tilde{S}_i^{(k)}$  defined in (130).

569 **Incremental SAEM (iSAEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ , compute  $\bar{s}_{i_k}^{(k)}$  and set  
 570

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} + \frac{1}{n} (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_i^k)})). \quad (137)$$

571 **Variance Reduced Two-Time-Scale EM (vrTTSEM):** draw an index  $i_k$  uniformly at random on  
 572  $\llbracket n \rrbracket$ , compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} (1 - \rho) + \rho (\tilde{S}^{(\ell(k))} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\ell(k))}))). \quad (138)$$

573 **Fast Incremental Two-Time-Scale EM (fiTTSEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ ,  
 574 compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} (1 - \rho) + \rho (\bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}))). \quad (139)$$

575 Finally, the  $k$ -th update reads  $\hat{\theta}^{(k+1)} = \bar{\theta}(\hat{s}^{(k+1)})$  where the function  $s \rightarrow \bar{\theta}(s)$  is defined by (132).