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# Fast Two-Timescale Stochastic EM Algorithms

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## Abstract

Using the Expectation-Maximization (EM) algorithm is a popular choice for latent data model learning tasks. For modern and complex models, variants of the EM have been initially introduced by [20], using incremental updates to scale to large datasets, and by [24, 8], using Monte-Carlo (MC) approximations to bypass the impossible conditional expectation of the latent data for most nonconvex models. In this paper, we propose a general class of methods called Two-Timescale EM Methods based on double stages of stochastic updates to tackle an essential non-convex optimization task for latent data models. We motivate the choice of a double dynamic by invoking the variance reduction virtue of each stage of the method on both sources of noise: the incremental update and the MC approximation. We establish finite-time and global convergence bounds for nonconvex objective functions. Numerical applications are also presented to illustrate our findings.

## 1 Introduction

Learning latent data models is critical for modern machine learning problems, see [18] for references. We formulate the training of such model as an empirical risk minimization problem:

$$\min_{\theta \in \Theta} \bar{L}(\theta) := r(\theta) + L(\theta) \quad \text{with} \quad L(\theta) = \frac{1}{n} \sum_{i=1}^n L_i(\theta) := \frac{1}{n} \sum_{i=1}^n \{ -\log g(y_i; \theta) \}, \quad (1)$$

We denote the observations by  $\{y_i\}_{i=1}^n$ ,  $\Theta \subset \mathbb{R}^d$  is the convex parameters space. We consider a smooth convex regularization noted  $r : \Theta \rightarrow \mathbb{R}$  and  $g(y; \theta)$  is the (incomplete) likelihood of each observation. The objective function  $\bar{L}(\theta)$  is possibly *nonconvex* and is assumed to be lower bounded.

In the latent variable model,  $g(y_i; \theta)$ , is the marginal of the complete data likelihood defined as  $f(z_i, y_i; \theta)$ , i.e.  $g(y_i; \theta) = \int_{\mathcal{Z}} f(z_i, y_i; \theta) \mu(dz_i)$ , where  $\{z_i\}_{i=1}^n$  are the latent variables. In this paper, we make the assumption of a complete model belonging to the curved exponential family:

$$f(z_i, y_i; \theta) = h(z_i, y_i) \exp \left( \langle S(z_i, y_i) | \phi(\theta) \rangle - \psi(\theta) \right), \quad (2)$$

where  $\psi(\theta)$ ,  $h(z_i, y_i)$  are scalar functions,  $\phi(\theta) \in \mathbb{R}^k$  is a vector function, and  $S(z_i, y_i) \in \mathbb{R}^k$  is the complete data sufficient statistics. Full batch EM [9] is the method of reference for that kind of task and is a two steps procedure. The **E-step** amounts to computing the conditional expectation of the complete data sufficient statistics,

$$\text{E-step: } \bar{s}(\theta) = \frac{1}{n} \sum_{i=1}^n \bar{s}_i(\theta) \quad \text{where} \quad \bar{s}_i(\theta) = \int_{\mathcal{Z}} S(z_i, y_i) p(z_i | y_i; \theta) \mu(dz_i), \quad (3)$$

and the **M-step** is given by

$$\text{M-step: } \hat{\theta} = \bar{\theta}(\bar{s}(\theta)) := \arg \min_{\vartheta \in \Theta} \{ r(\vartheta) + \psi(\vartheta) - \langle \bar{s}(\theta) | \phi(\vartheta) \rangle \}. \quad (4)$$

Two caveats of this method are the following: (a) with the explosion of data, the first step of the EM is computationally inefficient as it requires a full pass over the dataset at each iteration and (b) the

complexity of modern models makes the expectation in (3) intractable. So far, both challenges have been addressed separately, to the best of our knowledge, as detailed the sequel.

**Prior Work** Inspired by stochastic optimization procedures, [20] and [5] developed respectively an incremental and an online variant of the **E-step** in models where the expectation is computable then extensively used and studied in [21, 15, 4]. Some improvements of that methods have been provided and analyzed, globally and in finite-time, in [12] where variance reduction techniques taken from the optimization literature have been efficiently applied to scale the EM algorithm to large datasets.

Regarding the computation of the expectation under the posterior distribution, the first method was the Monte-Carlo EM (MCEM) introduced in the seminal paper [24] where a MC approximation for this expectation is computed. A variant of that method is the Stochastic Approximation of the EM (SAEM) in [8] leveraging the power of Robbins-Monro type of update [23] to ensure pointwise convergence of the vector of estimated parameters rather using a decreasing stepsize than increasing the number of MC samples. The MCEM and the SAEM have been successfully applied in mixed effects models [17, 10, 3] or to do inference for joint modeling of time to event data coming from clinical trials in [7], among other applications. Recently, an incremental variant of the SAEM was proposed in [14] showing positive empirical results but its analysis is limited to asymptotic consideration. Gradient-based methods have been developed and analyzed in [25] but they remain out of the scope of this paper as they tackle the high-dimensionality issue.

**Contributions** This paper *introduces* and *analyzes*<sup>1</sup> a new class of methods which purpose is to update two proxies for target expected quantities in a two-timescale manner. Those approximated quantities are then used to optimize (1) for modern examples and settings using EM Maximization step. The main contributions of the paper are:

- We propose a two-timescale method based on Stochastic Approximation (SA), to alleviate the problem of MC computation, and on Incremental updates, to scale to large datasets. We describe in details the edges of each level of our method based on variance reduction arguments. The derivation of such class of algorithms has two advantages. First, it naturally leverages variance reduction and Robbins-Monro type of updates to tackle large-scale and highly nonlinear learning tasks. Then, it gives a simple formulation as a *scaled-gradient method* which makes the global analysis and the implementation accessible.
- We also establish global (independent of the initialization) and finite-time (true at each iteration) upper bounds on a classical sub-optimality condition in the nonconvex literature, *i.e.*, the second order moment of the gradient of the objective function.

In Section 2 we formalize both incremental and Monte-Carlo variants of the EM. Then, we introduce our two-timescale class of EM algorithms for which we derive several global statistical guarantees in Section 3 for possibly *nonconvex* functions. Section 4 is devoted to numerical illustrations.

## 2 Two-Timescale Stochastic EM Algorithms

We recall and formalize in this section the different methods found in the literature that aim to solving the large-scale problem and the intractable expectation. We then provide the general framework of our method that efficiently tackles the optimization problem (1).

### 2.1 Monte Carlo Integration and Stochastic Approximation

As mentioned in the introduction, for complex and possibly nonlinear models, the expectation under the posterior distribution defined in (3) is not tractable. In that case, the first solution involves computing a Monte Carlo integration of that latter term. For all  $i \in \llbracket 1, n \rrbracket$ , draw for  $m \in \llbracket 1, M \rrbracket$ , samples  $z_{i,m} \sim p(z_i | y_i; \theta)$  and compute the MC integration  $\tilde{s}$  of the deterministic quantity  $\bar{s}(\theta)$ :

$$\text{MC-step : } \tilde{s} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}, y_i), \quad (5)$$

and then update the parameter  $\hat{\theta} = \bar{\theta}(\tilde{s})$ . This algorithm bypasses the intractable expectation issue but is rather computationally expensive in order to reach point wise convergence ( $M$  needs to be

<sup>1</sup>Proofs of this paper are moved to the Supplementary Material

large). An alternative to that stochastic algorithm is to use a Robbins-Monro (RM) type of update. We denote, at iteration  $k$ , the following quantity

$$\tilde{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}^{(k)}, y_i) \quad \text{where} \quad z_{i,m}^{(k)} \sim p(z_i | y_i; \theta^{(k)}) . \quad (6)$$

Then, the RM update of the sufficient statistics  $\hat{s}^{(k+1)}$  reads:

$$\text{SA-step : } \hat{s}^{(k+1)} = \hat{s}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{s}^{(k)}) , \quad (7)$$

where  $\{\gamma_k\}_{k \geq 1} \in (0, 1)$  is a sequence of decreasing step sizes to ensure asymptotic convergence. This is called the Stochastic Approximation of the EM (SAEM) and has been shown to converge to a maximum likelihood of the observations under very general conditions [8]. In the simulation step (6), since the loss function between the observed data  $y_i$  and the latent variable  $z_i$  can be nonconvex, sampling from the posterior distribution  $p(z_i | y_i; \theta)$ , under the current model  $\theta$ , requires using an inference algorithm. [13] proved almost sure convergence of the sequence of parameters obtained by this algorithm coupled with an MCMC procedure during the simulation step. In simple scenarios, the samples  $\{z_{i,m}\}_{m=0}^{M-1}$  are conditionally independent and identically distributed with distribution  $p(z_i, \theta)$ . Nevertheless, in most cases, sampling exactly from this distribution is not an option and the Monte Carlo batch is sampled by Monte Carlo Markov Chains (MCMC) algorithm. The Stochastic Approximation is performed during the remaining iterations where  $\gamma_k = 1/k^\alpha$ , where  $\alpha \in (0, 1)$ , ensuring the almost sure convergence of the estimate. It is inappropriate to start with small values for step size  $\gamma_k$  and large values for the number of simulations  $M_k$ . Rather, it is recommended that one decreases  $\gamma_k$ , as in  $\gamma_k = 1/k^\alpha$ , where  $\alpha \in (0, 1)$ , and keeps a constant and small number  $M_k$  bypassing the expensive sampling step in (5). This Robbins-Monro type of update represents the *first level* of our algorithm, needed to temper the variance and noise introduced by MC integration. In the next section, we derive variants of this algorithm to adapt to the sheer size of data of today's applications and formalize the *second level* of our class of two-timescale EM methods.

## 2.2 Incremental and Bi-Level Stochastic EM Methods

Efficient strategies to scale to large datasets include incremental and variance reduced methods. We will explicit a general update that covers those latter variants and that represents the *second level* of our algorithm, namely the incremental update of the approached statistics  $\hat{S}^{(k)}$  inside the SA-Step.

$$\text{Incremental-step : } \tilde{S}^{(k+1)} = \tilde{S}^{(k)} + \rho_{k+1} (\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}) . \quad (8)$$

Note  $\{\rho_k\}_{k \geq 1} \in (0, 1)$  is a sequence of step sizes,  $\mathcal{S}^{(k)}$  is a proxy for  $\tilde{S}^{(k)}$ . If the stepsize is equal to one and the proxy  $\mathcal{S}^{(k)} = \tilde{S}^{(k)}$ , i.e., computed in a full batch manner as in (6), then we recover the SAEM algorithm. Also if  $\rho_k = 1$ ,  $\gamma_k = 1$  and  $\mathcal{S}^{(k)} = \tilde{S}^{(k)}$ , then we recover the MCEM [24]. For all methods, we define a random index drawn at iteration  $k$ , noted  $i_k \in \llbracket 1, n \rrbracket$ , and  $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$  as the iteration index where  $i \in \llbracket 1, n \rrbracket$  is last drawn prior to iteration  $k$ . The proposed fitTEM method draws *two* indices *independently* and uniformly as  $i_k, j_k \in \llbracket 1, n \rrbracket$ . Thus, we define  $t_j^k = \{k' : j_{k'} = j, k' < k\}$  to be the iteration index where the sample  $j \in \llbracket 1, n \rrbracket$  is last drawn as  $j_k$  prior to iteration  $k$  in addition to  $\tau_i^k$  which was defined w.r.t.  $i_k$ .

$$\text{iSAEM} \quad \mathcal{S}^{(k+1)} = \mathcal{S}^{(k)} + \frac{1}{n} (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)}) \quad (9)$$

$$\text{vrTTEM} \quad \mathcal{S}^{(k+1)} = \tilde{S}^{(\ell(k))} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\ell(k))}) \quad (10)$$

$$\text{fitTEM} \quad \mathcal{S}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)}), \quad \overline{\mathcal{S}}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + n^{-1} (\tilde{S}_{j_k}^{(k)} - \tilde{S}_{j_k}^{(t_{j_k}^k)}) \quad (11)$$

where  $\tilde{S}_{i_k}^{(k)} = \frac{1}{M_k} \sum_{m=1}^{M_k} S(z_{i_k,m}^{(k)}, y_{i_k})$  and  $z_{i_k,m}^{(k)} \sim p(z_{i_k} | y_{i_k}; \theta^{(k)})$ . The stepsize is set to  $\rho_{k+1} = 1$  for the iSAEM method and we initialize with  $\mathcal{S}^{(0)} = \tilde{S}^{(0)}$ ;  $\rho_{k+1} = \rho$  is constant for the vrTTEM and fitTEM methods. Note that we initialize as follows  $\overline{\mathcal{S}}^{(0)} = \overline{s}^{(0)}$  for the fitTEM which can be seen as a slightly modified version of SAGA inspired by [22].

115 **Two-Timescale Stochastic EM methods:** We now introduce the general method derived using the  
 116 two variance reduction techniques described above. Algorithm 1 leverages both levels (7) and (8) in  
 117 order to output a vector of fitted parameters  $\hat{\theta}^{(K)}$  where  $K$  is a randomly chosen termination point.

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**Algorithm 1** Two-Timescale Stochastic EM methods.

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- 1: **Input:** initializations  $\hat{\theta}^{(0)} \leftarrow 0, \hat{s}^{(0)} \leftarrow \tilde{S}^{(0)}, K_{\max} \leftarrow \text{max. iteration number}$ .  
 2: Set the terminating iteration number,  $K \in \{0, \dots, K_{\max} - 1\}$ , as a discrete r.v. with:

$$P(K = k) = \frac{\gamma_k}{\sum_{\ell=0}^{K_{\max}-1} \gamma_\ell} = \frac{\gamma_k}{P_{\max}}. \quad (12)$$

- 3: **for**  $k = 0, 1, 2, \dots, K$  **do**  
 4:   Draw index  $i_k \in \llbracket 1, n \rrbracket$  uniformly (and  $j_k \in \llbracket 1, n \rrbracket$  for fitTEM).  
 5:   Compute  $\hat{S}_{i_k}^{(k)}$  using the MC-step (5), for the drawn indices.  
 6:   Compute the surrogate sufficient statistics  $\mathcal{S}^{(k+1)}$  using (9) or (10) or (11).  
 7:   Compute  $\hat{S}^{(k+1)}$  and  $\hat{s}^{(k+1)}$  using respectively (8) and (7):

$$\begin{aligned} \tilde{S}^{(k+1)} &= \tilde{S}^{(k)} + \rho_{k+1}(\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}) \\ \hat{s}^{(k+1)} &= \hat{s}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{s}^{(k)}) \end{aligned} \quad (13)$$

- 8:   Compute  $\hat{\theta}^{(k+1)} = \bar{\theta}(\hat{s}^{(k+1)})$  via the M-step.  
 9: **end for**  
 10: **Return:**  $\hat{\theta}^{(K)}$ .
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118 The update in (13) is said to have two-timescale as the step sizes satisfy  $\lim_{k \rightarrow \infty} \gamma_k / \rho_k < 1$  such that  
 119  $\tilde{S}^{(k+1)}$  is updated at a faster time-scale, determined by  $\rho_k$ , than  $\hat{s}^{(k+1)}$ , determined by  $\gamma_k$ . The next  
 120 section presents the main results of this paper and establishes global and finite-time bounds for the  
 121 three different updates of our two-timescale scheme.

### 122 3 Finite Time Analysis of the Two-Timescale Scheme

123 Following [5], it can be shown that stationary points of the objective function (1) corresponds to the  
 124 stationary points of the following *nonconvex* Lyapunov function:

$$\min_{\mathbf{s} \in \mathcal{S}} V(\mathbf{s}) := \bar{L}(\bar{\theta}(\mathbf{s})) = r(\bar{\theta}(\mathbf{s})) + \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\bar{\theta}(\mathbf{s})), \quad (14)$$

125 that we propose to study in this article. Several critical assumptions required to derive convergence  
 126 guarantees read as follows:

127 **H1.** *The sets  $\mathcal{Z}, \mathcal{S}$  are compact. There exists constants  $C_S, C_Z$  such that:*

$$C_S := \max_{\mathbf{s}, \mathbf{s}' \in \mathcal{S}} \|\mathbf{s} - \mathbf{s}'\| < \infty, \quad C_Z := \max_{i \in \llbracket 1, n \rrbracket} \int_{\mathcal{Z}} |S(z, y_i)| \mu(dz) < \infty. \quad (15)$$

128 **H2.** *The conditional distribution is smooth on  $\text{int}(\Theta)$ . For any  $i \in \llbracket 1, n \rrbracket$ ,  $z \in \mathcal{Z}$ ,  $\theta, \theta' \in \text{int}(\Theta)^2$ ,  
 129 we have  $|p(z|y_i; \theta) - p(z|y_i; \theta')| \leq L_p \|\theta - \theta'\|$ .*

130 We also recall from the introduction that we consider curved exponential family models. besides:

131 **H3.** *For any  $\mathbf{s} \in \mathcal{S}$ , the function  $\theta \mapsto L(\mathbf{s}, \theta) := r(\theta) + \psi(\theta) - \langle \mathbf{s} | \phi(\theta) \rangle$  admits a unique global  
 132 minimum  $\bar{\theta}(\mathbf{s}) \in \text{int}(\Theta)$ . In addition,  $J_\phi^\theta(\bar{\theta}(\mathbf{s}))$  is full rank,  $L_\phi$ -Lipschitz and  $\bar{\theta}(\mathbf{s})$  is  $L_\theta$ -Lipschitz.*

133 We denote by  $H_L^\theta(\mathbf{s}, \theta)$  the Hessian (w.r.t to  $\theta$  for a given value of  $\mathbf{s}$ ) of the function  $\theta \mapsto L(\mathbf{s}, \theta) =$   
 134  $r(\theta) + \psi(\theta) - \langle \mathbf{s} | \phi(\theta) \rangle$ , and define

$$\mathbf{B}(\mathbf{s}) := J_\phi^\theta(\bar{\theta}(\mathbf{s})) \left( H_L^\theta(\mathbf{s}, \bar{\theta}(\mathbf{s})) \right)^{-1} J_\phi^\theta(\bar{\theta}(\mathbf{s}))^\top. \quad (16)$$

135 **H4.** *It holds that  $v_{\max} := \sup_{\mathbf{s} \in \mathcal{S}} \|\mathbf{B}(\mathbf{s})\| < \infty$  and  $0 < v_{\min} := \inf_{\mathbf{s} \in \mathcal{S}} \lambda_{\min}(\mathbf{B}(\mathbf{s}))$ . There exists  
 136 a constant  $L_B$  such that for all  $\mathbf{s}, \mathbf{s}' \in \mathcal{S}^2$ , we have  $\|\mathbf{B}(\mathbf{s}) - \mathbf{B}(\mathbf{s}')\| \leq L_B \|\mathbf{s} - \mathbf{s}'\|$ .*

The class of algorithms we develop in this paper are two-timescale where the first stage corresponds to the variance reduction trick used in [12] in order to accelerate incremental methods and reduce the variance induced by the index sampling. The second stage is the Robbins-Monro type of update that aims to reduce the variance induced by the MC approximations. As the expectations (3) are never available, we introduce the errors when approximating the quantity  $\bar{s}_i(\hat{\theta}(\hat{s}^{(k-1)}))$  at iteration  $k+1$ :

$$\eta_i^{(r)} := \tilde{S}_i^{(r)} - \bar{s}_i(\vartheta^{(r)}) \quad \text{for all } i \in \llbracket 1, n \rrbracket, r > 0 \quad \text{and } \vartheta \in \Theta. \quad (17)$$

For instance, we consider that the MC approximation is unbiased if for all  $i \in \llbracket 1, n \rrbracket$  and  $m \in \llbracket 1, M \rrbracket$ , the samples  $z_{i,m} \sim p(z_i | y_i; \theta)$  are i.i.d. under the posterior distribution, i.e.,  $\mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r] = 0$  where  $\mathcal{F}_r$  is the filtration up to iteration  $r$ . The following results are derived under the assumption of control of the fluctuations implied by the approximation stated as follows:

**H5.** *There exist a positive sequence of MC batch size  $\{M_r\}_{r>0}$  and constants  $(C, C_\eta)$  such that for all  $k > 0$ ,  $i \in \llbracket 1, n \rrbracket$  and  $\vartheta \in \Theta$ :*

$$\mathbb{E} \left[ \left\| \eta_i^{(r)} \right\|^2 \right] \leq \frac{C_\eta}{M_r} \quad \text{and} \quad \mathbb{E} \left[ \left\| \mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r] \right\|^2 \right] \leq \frac{C}{M_r}. \quad (18)$$

We can prove two important results on the Lyapunov function. The first one suggests smoothness:

**Lemma 1.** [12] *Assume H1-H4. For all  $s, s' \in \mathcal{S}$  and  $i \in \llbracket 1, n \rrbracket$ , we have*

$$\|\bar{s}_i(\bar{\theta}(s)) - \bar{s}_i(\bar{\theta}(s'))\| \leq L_s \|s - s'\|, \quad \|\nabla V(s) - \nabla V(s')\| \leq L_V \|s - s'\|, \quad (19)$$

where  $L_s := C_Z L_p L_\theta$  and  $L_V := v_{\max}(1 + L_s) + L_B C_S$ .

and the second one suggests a growth condition on the gradient of  $V$  depending on the mean field of the algorithm:

**Lemma 2.** *Assume H3, H4. For all  $s \in \mathcal{S}$ ,*

$$v_{\min}^{-1} \langle \nabla V(s) | s - \bar{s}(\bar{\theta}(s)) \rangle \geq \|s - \bar{s}(\bar{\theta}(s))\|^2 \geq v_{\max}^{-2} \|\nabla V(s)\|^2, \quad (20)$$

### 3.1 Global Convergence of Incremental Stochastic EM Algorithms

We present in this section a finite-time analysis of the incremental variant of the Stochastic Approximation of the EM algorithm. We want to draw the attention of the readers that the word "global" here does not mean for a global optimum of the nonconvex function, but of the independence of our analysis on the initialization and the iteration  $k$  (finite time).

The following Theorem for the iSAEM algorithm is derived under a control of the Monte Carlo fluctuations as described by Assumption H5 and is built upon an intermediary Lemma, detailed in the Supplementary Material, characterizing the quantity of interest  $\hat{S}^{(k+1)} - \hat{s}^{(k)}$ . Typically, the controls exhibited above are of interest when the number of MC samples  $M_k$  increase with  $k$ .

**Theorem 1.** *Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of positive step sizes and consider the iSAEM sequence  $\{\hat{s}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = 1$  for any  $k > 0$ . We also set  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k^\alpha \alpha c_1 \bar{L}}$  where  $a \in (0, 1)$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ . Assume that  $\hat{s}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ , then it holds:*

$$v_{\max}^{-2} \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \nabla V(\hat{s}^{(k)}) \right\|^2 \right] \leq \mathbb{E} \left[ V(\hat{s}^{(0)}) - V(\hat{s}^{(K)}) \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right].$$

### 3.2 Global Convergence of Two-Timescale Stochastic EM Algorithms

We now proceed by giving our main result regarding the global convergence of the fitTEM algorithm. Two important auxiliary Lemmas are needed in order to derive our finite-time bound. The first one derives an identity for the quantity  $\mathbb{E}[\|\hat{s}^{(k)} - \hat{S}^{(k+1)}\|^2]$  using the vrTTEM update:

**Lemma 3.** *Consider the vrTTEM update in (10) with  $\rho_k = \rho$ , it holds for all  $k > 0$*

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{s}^{(k)} - \tilde{S}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\|\hat{s}^{(k)} - \bar{s}^{(k)}\|^2] + 2\rho^2 L_s^2 \mathbb{E}[\|\hat{s}^{(k)} - \hat{s}^{(\ell(k))}\|^2] \\ &\quad + 2(1 - \rho)^2 \mathbb{E}[\|\hat{s}^{((k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

173 where we recall that  $\ell(k)$  is the first iteration number in the epoch that iteration  $k$  is in.

174 The second one derives an identity for the quantity  $\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2]$  using the fitTEM update:

175 **Lemma 4.** Consider the fitTEM update in (11) with  $\rho_k = \rho$ , it holds for all  $k > 0$

$$\begin{aligned} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + 2(1 - \rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{((k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]. \end{aligned}$$

176 Recalling that  $K$  is an independent discrete r.v. drawn from  $\{1, \dots, K_{\max}\}$  with distribution  
177  $\{\gamma_k/P_{\max}, 0 \leq k \leq K_{\max} - 1\}$ , as in (12), we have

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] = \frac{1}{P_{\max}} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2].$$

178

179 We now state the main result regarding the vrTTEM method:

180 **Theorem 2.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
181 positive step sizes and consider the vrTTEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ . Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for  
182 any  $k \leq K_{\max}$ . Setting  $\bar{L} = \max\{L_s, L_V\}$ ,  $\rho = \frac{\mu}{c_1 \bar{L} n^{2/3}}$ ,  $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$ , a constant  $\mu \in (0, 1)$ ,  
183  $\gamma_{k+1} = \frac{1}{k^a \bar{L}}$  where  $a \in (0, 1)$ , it holds:

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] \leq \frac{2n^{2/3} \bar{L}}{\mu P_{\max} v_{\min}^2 v_{\max}^2} \left[ \mathbb{E}[\Delta V] + \sum_{k=0}^{K_{\max}-1} \tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right],$$

184 where  $\Delta V = V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})$ . We now state the main result regarding the fitTEM method:

185 **Theorem 3.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
186 positive step sizes and consider the fitTEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ . Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for  
187 any  $k \leq K_{\max}$ . Setting  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\beta = \frac{1}{\alpha n}$ ,  $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$ ,  
188  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$  and  $\gamma_{k+1} = \frac{1}{k^a \alpha c_1 \bar{L}}$  where  $a \in (0, 1)$ , it holds:

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] \leq \frac{4\alpha \bar{L} n^{2/3}}{P_{\max} v_{\min}^2 v_{\max}^2} \left[ \mathbb{E}[\Delta V] + \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \Gamma^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right].$$

189 Note that in those two bounds, the quantities  $\tilde{\eta}^{(k+1)}$  and  $\Xi^{(k+1)}$  depend only on the MC fluctu-  
190 ations  $\mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right]$  and some constants. While Theorem 1 suffers only from the MC noise in-  
191 duced by the latent data sampling step, Theorem 2 and Theorem 3 exhibit in their convergence  
192 bounds two different phases. The upper bounds display a bias term due to the initial conditions,  
193 i.e., the term  $\Delta V = V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})$ , and a double dynamic burden exemplified by the term  
194  $\mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right]$ . Indeed, the following remarks are worth doing on this quantity :

- 195 • This term is the price we pay for the two-timescale dynamics and corresponds to the gap  
196 between the two asynchronous updates (one is on  $\hat{\mathbf{s}}^{(k)}$  and the other on  $\tilde{S}^{(k)}$ ).
- 197 • It is trivial to see that if  $\rho = 1$ , i.e., there is no variance reduction, then for any  $k > 0$

$$\mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] = \mathbb{E} \left[ \left\| \mathcal{S}^{(k+1)} - \tilde{S}^{(k+1)} \right\|^2 \right] = 0 \quad \text{with} \quad \hat{\mathbf{s}}^{(0)} = \tilde{S}^{(0)} = 0$$

198 which strengthen the fact that this quantity characterizes the impact of the variance reduc-  
199 tion technique introduced in our two stages class of methods.

200 The following Lemma characterizes this gap:

201 **Lemma 5.** Considering a decreasing stepsize  $\gamma_k \in (0, 1)$  and a constant  $\rho \in (0, 1)$ , we have

$$\mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \leq \frac{\rho}{1 - \rho} \sum_{\ell=0}^k (1 - \gamma_{\ell})^2 (\mathcal{S}^{(\ell)} - \tilde{S}^{(\ell)}),$$

202 where  $\mathcal{S}^{(k)}$  is defined either by (10) (vrTTEM) or (11) (fitTEM).



## 4 Numerical Examples

This section presents several numerical applications for our proposed algorithm.

### 4.1 Gaussian Mixture Models

We begin by a simple and illustrative example. The authors acknowledge that the following model can be trained using deterministic EM-type of algorithms but propose to apply stochastic methods, including theirs, and to compare their performances. Given  $n$  observations  $\{y_i\}_{i=1}^n$ , we want to fit a Gaussian Mixture Model (GMM) whose distribution is modeled as a Gaussian mixture of  $M$  components, each with a unit variance. Let  $z_i \in \llbracket M \rrbracket$  be the latent labels of each component, the complete log-likelihood is defined as:

$$\log f(z_i, y_i; \theta) = \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) [\log(\omega_m) - \mu_m^2/2] + \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) \mu_m y_i + \text{constant} . \quad (21)$$

where  $\theta := (\omega, \mu)$  with  $\omega = \{\omega_m\}_{m=1}^{M-1}$  are the mixing weights with the convention  $\omega_M = 1 - \sum_{m=1}^{M-1} \omega_m$  and  $\mu = \{\mu_m\}_{m=1}^M$  are the means. We use the penalization  $r(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \log \text{Dir}(\omega; M, \epsilon)$  where  $\delta > 0$  and  $\text{Dir}(\cdot; M, \epsilon)$  is the  $M$  dimensional symmetric Dirichlet distribution with concentration parameter  $\epsilon > 0$ . The constraint set on  $\Theta$  is given by  $\Theta = \{\omega_m, m = 1, \dots, M-1 : \omega_m \geq 0, \sum_{m=1}^{M-1} \omega_m \leq 1\} \times \{\mu_m \in \mathbb{R}, m = 1, \dots, M\}$ . In the following experiments on synthetic data, we generate 30 synthetic datasets of size  $n = 10^5$  from a GMM model with  $M = 2$  components with two mixtures with means  $\mu_1 = -\mu_2 = 0.5$ . We run the bEM method until convergence (to double precision) to obtain the ML estimate  $\mu^*$  averaged on 50 datasets. We compare the bEM, iEM (incremental EM), SAEM, iSAEM, vrTTEM and fitTTEM methods in terms of their precision measured by  $|\mu - \mu^*|^2$ . We set the stepsize of the SA-step of all method as  $\gamma_k = 1/k^\alpha$  with  $\alpha = 0.5$ , and the stepsizes of the Incremental-step for vrTTEM and the fitTTEM to a constant stepsize equal to  $1/n^{2/3}$ . The number of MC samples is fixed to  $M = 10$  chains. Figure 1 shows the precision  $|\mu - \mu^*|^2$  for the different methods against the epoch(s) elapsed (one epoch equals  $n$  iterations). Besides, vrTTEM and fitTTEM methods outperform the other stochastic methods, supporting the benefits of our scheme.

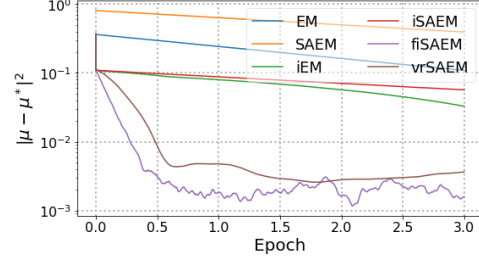


Figure 1: Precision  $|\mu^{(k)} - \mu^*|^2$  per epoch

### 4.2 Deformable Template Model for Image Analysis

Let  $(y_i, i \in \llbracket 1, n \rrbracket)$  be observed gray level images defined on a grid of pixels. Let  $u \in \mathcal{U} \subset \mathbb{R}^2$  denotes the pixel index on the image and  $x_u \in \mathcal{D} \subset \mathbb{R}^2$  its location. The model used in this experiment suggests that each image  $y_i$  is a deformation of a template, noted  $I : \mathcal{D} \rightarrow \mathbb{R}$ , common to all images of the dataset:

$$y_i(u) = I(x_u - \Phi_i(x_u, z_i)) + \varepsilon_i(u) \quad (22)$$

where  $\phi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a deformation function,  $z_i$  some latent variable parameterizing this deformation and  $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$  is an observation error. The template model, given  $\{p_k\}_{k=1}^{k_p}$  landmarks on the template, a fixed known kernel  $\mathbf{K}_p$  and a vector of parameters  $\beta \in \mathbb{R}^{k_p}$  is defined as follows:

$$I_\xi = \mathbf{K}_p \beta, \quad \text{where} \quad (\mathbf{K}_p \beta)(x) = \sum_{k=1}^{k_p} \mathbf{K}_p(x, p_k) \beta_k .$$

Given a set of landmarks  $\{g_k\}_{k=1}^{k_g}$  and a fixed kernel  $\mathbf{K}_g$ , we parameterize the deformation  $\Phi_i$  as:

$$\Phi_i = \mathbf{K}_g z_i \quad \text{where} \quad (\mathbf{K}_g z_i)(x) = \sum_{k=1}^{k_g} \mathbf{K}_g(x, g_k) \begin{pmatrix} z_i^{(1)}(k) \\ z_i^{(2)}(k) \end{pmatrix} ,$$

where we put a Gaussian prior on the latent variables,  $z_i \sim \mathcal{N}(0, \Gamma)$  and  $z_i \in (\mathbb{R}^{k_g})^2$ . The vector of parameters we estimate is thus  $\theta = (\beta, \Gamma, \sigma)$ .

244 **Numerical Experiment:** We apply model (22) and our algorithms to a collection of handwritten  
 245 digits, called the US postal database [11], featuring  $n = 1\,000$  ( $16 \times 16$ )-pixel images for each  
 246 class of digits from 0 to 9. The main difficulty with these data comes from the geometric dispersion  
 247 within each class of digit as shown Figure 2 for digit 5. We thus ought to use our deformable  
 248 template model (22) in order to account for both sources of variability: the intrinsic template to each  
 249 class of digit and the small and local deformation in each observed image.



Figure 2: Training set of the USPS database (20 images for figit 5)

250 Figure 3 shows the resulting synthetic images for digit 5 through several epochs and for the batch  
 251 method, the online SAEM, the incremental SAEM and the various TTS methods. We choose Gaus-  
 252 sian kernels for both,  $\mathbf{K}_p$  and  $\mathbf{K}_g$ , defined on  $\mathbb{R}^2$  and centered on the landmark points  $\{p_k\}_{k=1}^{k_p}$   
 253 and  $\{g_k\}_{k=1}^{k_g}$  with standard respective standard deviations of 0.12 and 0.3.  $k_p = 15$  and  $k_g = 6$   
 254 equidistributed landmarks points are chosen on the grid for the training. Those hyperparameters are  
 255 inspired by a relevant study in [2]. The kernel covariance matrices are important hyperparameters  
 256 in such study since they have a direct impact on the sharpness of the templates. Intuitively, if those  
 257 variances are large, the kernels centered arounds the equidistributed landmark spread out on too  
 258 many of its neighbors. Bad choices of such hyperparameters can lead to thicker shapes.



Figure 3: Estimation of the template: from top to bottom: batch, online, iSAEM, vrTTEM and fiTTEM through 7 epochs. Note that Batch method templates are replicated in-between epochs for a fair comparison with incremental variants.

259 Figure 3 displays the virtue of the vrTTEM and fiTTEM methods that obtain a more *contrasted* and  
 260 *accurate* template estimate. The incremental and online version are looking much better on the very  
 261 first epochs compared to the batch method, which is intuitive given the high computational cost of  
 262 the latter. After a few epochs, the batch SAEM seem to estimate similar template as the incremental  
 263 an online methods due to their high variance. Our variance reduced and fast incremental variants are  
 264 effective in the long run and sharpen the final template estimates contrasting between the background  
 265 and the regions of interest in the image.

## 266 5 Conclusion

267 This paper introduces a new class of two-timescale EM methods for learning latent data models.  
 268 In particular, the models dealt with in this paper belong to the curved exponential family and are  
 269 possibly nonconvex. The nonconvexity of the problem is tackled using a Robbins-Monro type of  
 270 update, which represent the *first* level of our class of methods and the scalability with the number  
 271 of samples is performed through a variance reduced and incremental type of update, the *second* and  
 272 last level of our newly introduced scheme. The various methods are interpreted as scaled gradient  
 273 methods, in the space of the sufficient statistics, and our convergence results are *global*, in the sense  
 274 of independent of the initial values, and *non-asymptotic*, true for any random termination number.



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## 335 A Proof of Lemma 2

336 **Lemma.** Assume H3, H4. For all  $\mathbf{s} \in S$ ,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (23)$$

337 **Proof** Using H3 and the fact that we can exchange integration with differentiation and the Fisher's  
338 identity, we obtain

$$\begin{aligned} \nabla_{\mathbf{s}} V(\mathbf{s}) &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left( \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{L}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left( \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} (\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))), \end{aligned} \quad (24)$$

339 Consider the following vector map:

$$\mathbf{s} \rightarrow \nabla_{\boldsymbol{\theta}} L(\mathbf{s}, \boldsymbol{\theta})|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})} = \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \mathbf{s}.$$

340 Taking the gradient of the above map w.r.t.  $\mathbf{s}$  and using assumption H3, we show that:

$$\mathbf{0} = -\mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \underbrace{\left( \nabla_{\boldsymbol{\theta}}^2 (\psi(\boldsymbol{\theta}) + \mathbf{r}(\boldsymbol{\theta}) - \langle \phi(\boldsymbol{\theta}) | \mathbf{s} \rangle) \right)|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})}}_{=\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \bar{\boldsymbol{\theta}})} \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s}).$$

341 The above yields

$$\nabla_{\mathbf{s}} V(\mathbf{s}) = \mathbf{B}(\mathbf{s})(\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})))$$

342 where we recall  $\mathbf{B}(\mathbf{s}) = \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \left( \mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \bar{\boldsymbol{\theta}}(\mathbf{s})) \right)^{-1} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top}$ . The proof of (23) follows directly  
343 from the assumption H4.  $\square$

## 344 B Proof of Theorem 1

345 Beforehand, We present two intermediary Lemmas important for the analysis of the incremen-  
346 tal update of the iSAEM algorithm. The first one gives a characterization of the quantity  
347  $\mathbb{E} [\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}]$ :

348 **Lemma 6.** Assume H1. The update (9) is equivalent to the following update on the resulting statis-  
349 tics

$$\hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)})$$

350 Also:

$$\mathbb{E} [\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] = \mathbb{E} [\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \left( 1 - \frac{1}{n} \right) \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] + \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}]$$

351 where  $\bar{\mathbf{s}}^{(k)}$  is defined by (3) and  $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$ .

352 **Proof** From update (9), we have:

$$\begin{aligned} \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} + \frac{1}{n} \left( \tilde{S}_{i_k}^{(k+1)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)} \right) \\ &= \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{S}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left( \tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \tilde{S}_{i_k}^{(k+1)} \right) \end{aligned}$$

353 Since  $\tilde{S}_{i_k}^{(k+1)} = \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) + \eta_{i_k}^{(k+1)}$  we have

$$\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{S}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left( \tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) \right) + \frac{1}{n} \eta_{i_k}^{(k+1)}$$

354 Taking the full expectation of both side of the equation leads to:

$$\begin{aligned}\mathbb{E} \left[ \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right] &= \mathbb{E} \left[ \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right] + \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] \\ &\quad - \frac{1}{n} \mathbb{E} \left[ \mathbb{E} \left[ \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k \right] \right] + \frac{1}{n} \mathbb{E} \left[ \eta_{i_k}^{(k+1)} \right]\end{aligned}$$

355 The following equalities:

$$\mathbb{E} \left[ \tilde{S}_i^{(\tau_i^k)} | \mathcal{F}_k \right] = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} \quad \text{and} \quad \mathbb{E} \left[ \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k \right] = \bar{\mathbf{s}}^{(k)}$$

356 concludes the proof of the Lemma.  $\square$

357 And the following auxiliary Lemma setting an upper bound for the quantity  $\mathbb{E} \left[ \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \right]$

358 **Lemma 7.** For any  $k \geq 0$  and consider the iSAEM update in (9), it holds that

$$\begin{aligned}\mathbb{E} \left[ \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] &\leq 4\mathbb{E} \left[ \|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] + \frac{2\mathbf{L}_s^2}{n^3} \sum_{i=1}^n \mathbb{E} \left[ \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 \right] \\ &\quad + 2\frac{C_\eta}{M_k} + 4\mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right]\end{aligned}$$

359 **Proof** Applying the iSAEM update yields:

$$\begin{aligned}\mathbb{E} \left[ \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] &= \mathbb{E} \left[ \left\| \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} - \frac{1}{n} (\tilde{S}_i^{(\tau_i^k)} - \tilde{S}_{i_k}^{(k)}) \right\|^2 \right] \\ &\leq 4\mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] + 4\mathbb{E} \left[ \|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] \\ &\quad + \frac{2}{n^2} \mathbb{E} \left[ \left\| \bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_i^k)} \right\|^2 \right] + 2\frac{C_\eta}{M_k}\end{aligned}$$

360 The last expectation can be further bounded by

$$\frac{2}{n^2} \mathbb{E} \left[ \|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_i^k)}\|^2 \right] = \frac{2}{n^3} \sum_{i=1}^n \mathbb{E} \left[ \|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2 \right] \stackrel{(a)}{\leq} \frac{2\mathbf{L}_s^2}{n^3} \sum_{i=1}^n \mathbb{E} \left[ \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 \right],$$

361 where (a) is due to Lemma 1 and which concludes the proof of the Lemma.

362  $\square$

363 **Theorem.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
364 positive step sizes and consider the iSAEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = 1$  for any  
365  $k > 0$ . We also set  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{\mathbf{L}_s, \mathbf{L}_V\}$ ,  $\gamma_{k+1} = \frac{1}{k^\alpha \alpha c_1 \bar{L}}$  where  
366  $a \in (0, 1)$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ . Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ .

$$v_{\max}^{-2} \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \nabla V(\hat{\mathbf{s}}^{(k)}) \right\|^2 \right] \leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right]$$

367 **Proof** Under the smoothness of the Lyapunov function  $V$  (cf. Lemma 1), we can write:

$$V(\hat{\mathbf{s}}^{(k+1)}) \leq V(\hat{\mathbf{s}}^{(k)}) + \gamma_{k+1} \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 \mathbf{L}_V}{2} \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2$$

368 Taking the expectation on both sides yields:

$$\mathbb{E} \left[ V(\hat{\mathbf{s}}^{(k+1)}) \right] \leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(k)}) \right] + \gamma_{k+1} \mathbb{E} \left[ \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} \left[ \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \right]$$

369 Using Lemma 6, we obtain:

$$\begin{aligned} & \mathbb{E} \left[ \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\ &= \mathbb{E} \left[ \langle \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] + \left( 1 - \frac{1}{n} \right) \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \right\rangle \right] + \frac{1}{n} \mathbb{E} \left[ \langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\ &\stackrel{(a)}{\leq} -v_{\min} \mathbb{E} \left[ \|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] + \left( 1 - \frac{1}{n} \right) \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \right\rangle \right] + \frac{1}{n} \mathbb{E} \left[ \langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\ &\stackrel{(b)}{\leq} -v_{\min} \mathbb{E} \left[ \|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &\quad + \frac{\beta(n-1)+1}{2n} \mathbb{E} \left[ \|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2 \right] + \frac{1}{2n} \mathbb{E} \left[ \|\eta_{i_k}^{(k)}\|^2 \right] \\ &\stackrel{(a)}{\leq} \left( v_{\max}^2 \frac{\beta(n-1)+1}{2n} - v_{\min} \right) \mathbb{E} \left[ \|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] + \frac{1}{2n} \mathbb{E} \left[ \|\eta_{i_k}^{(k)}\|^2 \right] \end{aligned}$$

370 where (a) is due to the growth condition (2) and (b) is due to Young's inequality (with  $\beta \rightarrow 1$ ). Note

371  $a_k = \gamma_{k+1} \left( v_{\min} - v_{\max}^2 \frac{\beta(n-1)+1}{2n} \right)$  and

$$\begin{aligned} a_k \mathbb{E} \left[ \|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] &\leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)}) \right] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} \left[ \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] \\ &\quad + \frac{\gamma_{k+1}(1 - \frac{1}{n})}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] + \frac{\gamma_{k+1}}{2n} \mathbb{E} \left[ \|\eta_{i_k}^{(k)}\|^2 \right] \end{aligned} \quad (25)$$

372 We now give an upper bound of  $\mathbb{E} \left[ \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \right]$  using Lemma 7 and plug it into (25):

$$\begin{aligned} (a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E} \left[ \|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] &\leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)}) \right] \\ &\quad + \gamma_{k+1} \left( \frac{1}{2\beta} \left( 1 - \frac{1}{n} \right) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &\quad + \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E} \left[ \|\eta_{i_k}^{(k)}\|^2 \right] \\ &\quad + \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^3} \sum_{i=1}^n \mathbb{E} [\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2] \end{aligned} \quad (26)$$

373 Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} [\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2] = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \mathbb{E} [\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E} [\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2] \right)$$

where the equality holds as  $i_k$  and  $j_k$  are drawn independently. For any  $\beta > 0$ , it holds

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 - 2\gamma_{k+1}\langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\ &\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 + \gamma_{k+1}\beta\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right] \end{aligned}$$

where the last inequality is due to the Young's inequality. Subsequently, we have

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}\left[\left(1 + \gamma_{k+1}\beta\right)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2\right] \end{aligned}$$

Observe that  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)})$ . Applying Lemma 7 yields

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\ &\leq \left(\gamma_{k+1}^2 + \frac{n-1}{n} \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right] \\ &\leq 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}\left[\left\|\eta_{i_k}^{(k)}\right\|^2\right] \\ &\quad + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] \\ &\quad + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2\right] \end{aligned}$$

Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2]$$

From the above, we get

$$\begin{aligned} \Delta^{(k+1)} &\leq \left(1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})\right) \Delta^{(k)} + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + 2\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}\left[\left\|\eta_{i_k}^{(k)}\right\|^2\right] + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] \end{aligned}$$

Setting  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ ,

$c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 6$ ,  $\alpha \geq 8$ , we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \leq 1 - \frac{c_1(k\alpha - 1) - 4}{k\alpha n c_1} \leq 1 - \frac{2}{k\alpha n c_1}$$

which shows that  $1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \in (0, 1)$  for any  $k > 0$ . Denote  $\Lambda_{(k+1)} =$

$\frac{1}{n} - \gamma_{k+1}\beta - \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})$  and note that  $\Delta^{(0)} = 0$ , thus the telescoping sum yields:

$$\begin{aligned} \Delta^{(k+1)} &\leq 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) \left(\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] + 2 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) \left(\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}\right) \mathbb{E}\left[\left\|\eta_{i_\ell}^{(\ell)}\right\|^2\right] \\ &\quad + 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) \left(\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}\right) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^\ell)} - \bar{\mathbf{s}}^{(\ell)}\right\|^2\right] \end{aligned}$$



383 Note  $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$  Summing on both sides over  $k = 0$  to  $k = K_{\max} - 1$  yields:

$$\begin{aligned}
& \sum_{k=0}^{K_{\max}-1} \Delta^{(k+1)} \\
&= 4 \sum_{k=0}^{K_{\max}-1} \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right) \omega_{k,1} \mathbb{E} [\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2 \sum_{k=0}^{K_{\max}-1} \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right) \omega_{k,1} \mathbb{E} \left[ \left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\
&+ \sum_{k=0}^{K_{\max}-1} 4 \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right) \omega_{k,1} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\
&\leq \sum_{k=0}^{K_{\max}-1} \frac{4 \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right)}{\Lambda_{(k+1)}} \mathbb{E} [\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{k=0}^{K_{\max}-1} \frac{2 \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right)}{\Lambda_{(k+1)}} \mathbb{E} \left[ \left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\
&+ \sum_{k=0}^{K_{\max}-1} \frac{4 \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right)}{\Lambda_{(k+1)}} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right]
\end{aligned} \tag{27}$$

384 We recall (26) where we have summed on both sides from  $k = 0$  to  $k = K_{\max} - 1$ :

$$\begin{aligned}
& \sum_{k=0}^{K_{\max}-1} \left( a_k - 2\gamma_{k+1}^2 L_V \right) \mathbb{E} \left[ \left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] \\
&+ \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \left( \frac{1}{2\beta} \left( 1 - \frac{1}{n} \right) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\
&+ \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right] \\
&+ \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^2} \Delta^{(k)}
\end{aligned} \tag{28}$$

385 Plugging (27) into (28) results in:

$$\begin{aligned}
& \sum_{k=0}^{K_{\max}-1} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\beta}_k \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\
&\leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] + \sum_{k=0}^{K_{\max}-1} \tilde{\Gamma}_k \mathbb{E} \left[ \left\| \eta_{i_k}^{(k)} \right\|^2 \right]
\end{aligned}$$

386 where

$$\begin{aligned}
\tilde{\alpha}_k &= a_k - 2\gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^2} \frac{4 \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right)}{\Lambda_{(k+1)}} \\
\tilde{\beta}_k &= \gamma_{k+1} \left( \frac{1}{2\beta} \left( 1 - \frac{1}{n} \right) + 2\gamma_{k+1} L_V \right) - \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^2} \frac{4 \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right)}{\Lambda_{(k+1)}} \\
\tilde{\Gamma}_k &= \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) + \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^2} \frac{2 \left( \gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta} \right)}{\Lambda_{(k+1)}}
\end{aligned}$$

387 and

$$\begin{aligned}
a_k &= \gamma_{k+1} \left( v_{\min} - v_{\max}^2 \frac{\beta(n-1)+1}{2n} \right) \\
\Lambda_{(k+1)} &= \frac{1}{n} - \gamma_{k+1} \beta - \frac{2\gamma_{k+1} L_{\mathbf{s}}^2}{n^2} (\gamma_{k+1} + \frac{1}{\beta}) \\
c_1 &= v_{\min}^{-1}, \alpha = \max\{8, 1 + 6v_{\min}\}, \bar{L} = \max\{L_{\mathbf{s}}, L_V\}, \gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}, \beta = \frac{c_1 \bar{L}}{n}
\end{aligned}$$

388 When, for any  $k > 0$ ,  $\tilde{\alpha}_k \geq 0$ , we have by Lemma 2 that:

$$\sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \nabla V(\hat{\mathbf{s}}^{(k)}) \right\|^2 \right] \leq v_{\max}^2 \sum_{k=0}^{K_{\max}} \tilde{\alpha}_k \mathbb{E} \left[ \left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right]$$

389 which yields an upper bound of the gradient of the Lyapunov function  $V$  along the path of the  
390 iSAEM update and concludes the proof of the Theorem.  $\square$

## 391 C Proofs of Auxiliary Lemmas

### 392 C.1 Proof of Lemma 3 and Lemma 4

393 **Lemma.** For any  $k \geq 0$  and consider the vrTTEM update in (10) with  $\rho_k = \rho$ , it holds for all  $k > 0$   
394

$$\begin{aligned}
\mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\left\| \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} \right\|^2] + 2\rho^2 L_{\mathbf{s}}^2 \mathbb{E}[\left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} \right\|^2] \\
&\quad + 2(1-\rho)^2 \mathbb{E}[\left\| \hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)} \right\|^2] + 2\rho^2 \mathbb{E}[\left\| \eta_{i_k}^{(k+1)} \right\|^2]
\end{aligned}$$

395 where we recall that  $\ell(k)$  is the first iteration number in the epoch that iteration  $k$  is in.

396 **Proof** Beforehand, we provide a rewriting of the quantity  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$  that will be useful through-  
397 out this proof:

$$\begin{aligned}
\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathcal{S}^{(k+1)}) \\
&= -\gamma_{k+1} \left( (1-\rho) \left[ \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[ \hat{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)} \right] \right)
\end{aligned} \tag{29}$$

398 We observe, using the identity (29), that

$$\mathbb{E}[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \right\|^2] \leq 2\rho^2 \mathbb{E}[\left\| \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} \right\|^2] + 2\rho^2 \mathbb{E}[\left\| \bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)} \right\|^2] + 2(1-\rho)^2 \mathbb{E}[\left\| \hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)} \right\|^2] \tag{30}$$

399 For the latter term, we obtain its upper bound as

$$\begin{aligned}
\mathbb{E}[\left\| \bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)} \right\|^2] &= \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \tilde{S}_i^{\ell(k)}) - (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{\ell(k)}) \right\|^2 \right] \\
&\stackrel{(a)}{\leq} \mathbb{E}[\left\| \bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{\ell(k)} \right\|^2] + \mathbb{E}[\left\| \eta_{i_k}^{(k+1)} \right\|^2] \stackrel{(b)}{\leq} L_{\mathbf{s}}^2 \mathbb{E}[\left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} \right\|^2] + \mathbb{E}[\left\| \eta_{i_k}^{(k+1)} \right\|^2]
\end{aligned}$$

400 where (a) uses the variance inequality and (b) uses Lemma 1. Substituting into (30) proves the  
401 lemma.  $\square$

402 **Lemma.** For any  $k \geq 0$  and consider the fitTEM update in (11) with  $\rho_k = \rho$ , it holds for all  $k > 0$

$$\begin{aligned}
\mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \right\|^2 \right] &\leq 2\rho^2 \mathbb{E}[\left\| \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} \right\|^2] + 2\rho^2 \frac{L_{\mathbf{s}}^2}{n} \sum_{i=1}^n \mathbb{E}[\left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell_i^k)} \right\|^2] \\
&\quad + 2(1-\rho)^2 \mathbb{E}[\left\| \hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)} \right\|^2] + 2\rho^2 \mathbb{E}[\left\| \eta_{i_k}^{(k+1)} \right\|^2]
\end{aligned}$$

403 **Proof** Beforehand, we provide a rewriting of the quantity  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$  that will be useful through-  
 404 out this proof:

$$\begin{aligned}
 \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}) \\
 &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1 - \rho)\tilde{\mathbf{S}}^{(k)} - \rho\mathbf{S}^{(k+1)}) \\
 &= -\gamma_{k+1}\left((1 - \rho)\left[\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right] + \rho\left[\hat{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\right]\right) \\
 &= -\gamma_{k+1}\left((1 - \rho)\left[\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right] + \rho\left[\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{S}}^{(k)} - (\tilde{\mathbf{S}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)})\right]\right)
 \end{aligned} \tag{31}$$

405 We observe, using the identity (31), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k+1)}\|^2] \leq 2\rho^2\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{S}}^{(k)}\|^2] + 2\rho^2\mathbb{E}[\|\bar{\mathbf{S}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] + 2(1 - \rho)^2\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] \tag{32}$$

406 For the latter term, we obtain its upper bound as

$$\begin{aligned}
 \mathbb{E}[\|\bar{\mathbf{S}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] &= \mathbb{E}\left[\left\|\frac{1}{n}\sum_{i=1}^n(\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{S}}_i^{(k)}) - (\tilde{\mathbf{S}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)})\right\|^2\right] \\
 &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]
 \end{aligned}$$

407 where (a) uses the variance inequality. We can further bound the last expectation using Lemma 1:

$$\mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)}\|^2] = \frac{1}{n}\sum_{i=1}^n\mathbb{E}[\|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2] \stackrel{(a)}{\leq} \frac{L_{\mathbf{S}}^2}{n}\sum_{i=1}^n\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2]$$

408 Substituting into (32) proves the lemma.  $\square$

## 409 C.2 Proof of Lemma 5

410 **Lemma.** Consider a decreasing stepsize  $\gamma_k \in (0, 1)$  and a constant  $\rho$ , then the following inequality  
 411 holds:

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] \leq \frac{\rho}{1 - \rho}\sum_{\ell=0}^k(1 - \gamma_{\ell})^2(\mathbf{S}^{(\ell)} - \tilde{\mathbf{S}}^{(\ell)})$$

412 where  $\mathbf{S}^{(k)}$  is defined either by (11) (fiTTEM) or (10) (vrTTEM)

413 **Proof** We begin by writing the two-timescale update:

$$\begin{aligned}
 \tilde{\mathbf{S}}^{(k+1)} &= \tilde{\mathbf{S}}^{(k)} + \rho(\mathbf{S}^{(k+1)} - \tilde{\mathbf{S}}^{(k)}) \\
 \hat{\mathbf{s}}^{(k+1)} &= \hat{\mathbf{s}}^{(k)} + \gamma_{k+1}(\tilde{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)})
 \end{aligned} \tag{33}$$

414 where  $\mathbf{S}^{(k+1)} = \frac{1}{n}\sum_{i=1}^n\tilde{\mathbf{S}}_i^{(t_i^k)} + (\tilde{\mathbf{S}}_{i_k}^{(k)} - \tilde{\mathbf{S}}_{i_k}^{(t_{i_k}^k)})$  according to (11). Denote  $\delta^{(k+1)} = \hat{\mathbf{s}}^{(k+1)} -$   
 415  $\tilde{\mathbf{S}}^{(k+1)}$ . Then from (33), doing the subtraction of both equations yields:

$$\delta^{(k+1)} = (1 - \gamma_{k+1})\delta^{(k)} + \frac{\rho}{1 - \rho}(1 - \gamma_{k+1})(\mathbf{S}^{(k+1)} - \tilde{\mathbf{S}}^{(k+1)})$$

416 Using the telescoping sum and noting that  $\delta^{(0)} = 0$ , we have

$$\delta^{(k+1)} \leq \frac{\rho}{1 - \rho}\sum_{\ell=0}^k(1 - \gamma_{\ell+1})^2(\mathbf{S}^{(\ell+1)} - \tilde{\mathbf{S}}^{(\ell+1)})$$

417  $\square$

418 **C.3 Additional Intermediary Result**

419 **Lemma 8.** *At iteration  $k + 1$ , the drift term of update (11), with  $\rho_{k+1} = \rho$ , is equivalent to the*  
 420 *following :*

$$\begin{aligned} \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} &= \rho(\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}) + \rho\eta_{i_k}^{(k+1)} + \rho \left[ (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] \\ &\quad + (1 - \rho) \left( \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right) \end{aligned}$$

421 *where we recall that  $\eta_{i_k}^{(k+1)}$ , defined in (18), which is the gap between the MC approximation and*  
 422 *the expected statistics.*

423 **Proof** Using the fitTEM update  $\tilde{S}^{(k+1)} = (1 - \rho)\tilde{S}^{(k)} + \rho\mathcal{S}^{(k+1)}$  where  $\mathcal{S}^{(k+1)} = \bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} -$   
 424  $\tilde{S}_{i_k}^{(t_{i_k}^k)})$  leads to the following decomposition:

$$\begin{aligned} &\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \\ &= (1 - \rho)\tilde{S}^{(k)} + \rho \left( \bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \right) - \hat{\mathbf{s}}^{(k)} + \rho\bar{\mathbf{s}}^{(k)} - \rho\bar{\mathbf{s}}^{(k)} \\ &= \rho(\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}) + \rho(\tilde{S}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(k)}) + (1 - \rho) \left( \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} \right) + \rho \left( \bar{\mathcal{S}}^{(k)} - \bar{\mathbf{s}}^{(k)} + (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \right) \\ &= \rho(\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}) + \rho\eta_{i_k}^{(k+1)} - \rho \left[ (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] \\ &\quad + (1 - \rho) \left( \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} \right) \end{aligned}$$

425 *where we observe that  $\mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] = \bar{\mathbf{s}}^{(k)} - \bar{\mathcal{S}}^{(k)}$  and which concludes the proof.*

426 *Important Note:* Note that  $\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}$  is not equal to  $\eta_{i_k}^{(k+1)}$ , defined in (18), which is the gap  
 427 *between the MC approximation and the expected statistics. Indeed  $\tilde{S}_{i_k}^{(t_{i_k}^k)}$  is not computed under the*  
 428 *same model as  $\bar{\mathbf{s}}_{i_k}^{(k)}$ .  $\square$*

## 429 D Proof of Theorem 2

430 **Theorem.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
 431 positive step sizes and consider the vrTTEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = \rho$  for  
 432 any  $k > 0$ .

433 Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ . By setting  $\bar{L} = \max\{L_S, L_V\}$ ,  $\rho = \frac{\mu}{c_1 \bar{L} n^{2/3}}$ ,  $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$   
 434 and a constant  $\mu \in (0, 1)$  and  $\gamma_{k+1} = \frac{1}{k^a \bar{L}}$  where  $a \in (0, 1)$ , we have the following bound:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[ \tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right] \end{aligned}$$

435 **Proof** Using the smoothness of  $V$  and update (10), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) + \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \end{aligned} \quad (34)$$

436 Denote  $\mathbf{H}_{k+1} := \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}$  the drift term of the fitTEM update in (7) and  $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$ .  
 437 Taking expectations on both sides show that

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] \\ &\stackrel{(a)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}(1-\rho) \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1} \rho \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\stackrel{(b)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} \rho \mathbb{E}[\langle \mathbf{h}_k | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1}(1-\rho) \mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad - \gamma_{k+1} \rho \mathbb{E}[\langle \eta_{i_k}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\stackrel{(c)}{\leq} \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - (\gamma_{k+1} \rho v_{\min} + \gamma_{k+1} v_{\max}^2) \mathbb{E}[\|\mathbf{h}_k\|^2] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\mathbf{H}_{k+1}\|^2] \\ &\quad - \gamma_{k+1} \rho \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \gamma_{k+1}(1-\rho) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \end{aligned} \quad (35)$$

438 where we have used (29) in (a) and  $\mathbb{E}[\mathbf{S}^{(k+1)}] = \bar{\mathbf{s}}^{(k)} + \mathbb{E}[\eta_{i_k}^{(k+1)}]$  in (b), the growth condition in  
 439 Lemma 2 and the Young's inequality with the constant equal to 1 in (c).

440 Furthermore, for  $k+1 \leq \ell(k) + m$  (i.e.,  $k+1$  is in the same epoch as  $k$ ), we have

$$\begin{aligned} &\mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] = \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} + \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \rangle] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 \\ &\quad - 2\gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \rho(\mathbf{h}_k - \eta_{i_k}^{(k+1)}) + (1-\rho)(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}) \rangle] \\ &\leq \mathbb{E}[(1 + \gamma_{k+1}\beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2 \\ &\quad + \frac{\gamma_{k+1}\rho}{\beta} \|\eta_{i_k}^{(k+1)}\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2], \end{aligned}$$

441 where we first used (29) and the last inequality is due to the Young's inequality.

442 Consider the following sequence

$$R_k := \mathbb{E}[V(\hat{\mathbf{s}}^{(k)}) + b_k \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2]$$

443 where  $b_k := \bar{b}_{k \bmod m}$  is a periodic sequence where:

$$\bar{b}_i = \bar{b}_{i+1}(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2) + \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2, \quad i = 0, 1, \dots, m-1 \quad \text{with } \bar{b}_m = 0.$$

444 Note that  $\bar{b}_i$  is decreasing with  $i$  and this implies

$$\bar{b}_i \leq \bar{b}_0 = \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2 \frac{(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2)^m - 1}{\gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2}, \quad i = 1, 2, \dots, m.$$

445 For  $k+1 \leq \ell(k) + m$ , we have the following inequality

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2) \|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}^2 L_V}{2} \|\mathbf{H}_{k+1}\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[\rho \left\|\eta_{i_k}^{(k+1)}\right\|^2 - (1-\rho) \left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\frac{\gamma_{k+1}\rho}{\beta} \left\|\eta_{i_k}^{(k+1)}\right\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \end{aligned}$$

446 And using Lemma 3 we obtain:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2 - \gamma_{k+1}^2\rho^2 L_V) \|\mathbf{h}_k\|^2 + \gamma_{k+1}^2\rho^2 L_V L_{\mathbf{s}}^2 \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2\rho^2 L_{\mathbf{s}}^2) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2\right) \|\mathbf{h}_k\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[(\rho + \rho^2\gamma_{k+1} L_V) \left\|\eta_{i_k}^{(k+1)}\right\|^2 - (1-\rho - (1-\rho)^2\gamma_{k+1} L_V) \left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2\right) \left\|\eta_{i_k}^{(k+1)}\right\|^2 + \left(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2(1-\rho)^2\right) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2\right] \end{aligned}$$

447 Rearranging the terms yields:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2)) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\quad + \underbrace{\left(b_{k+1}(1 + \gamma\beta + 2\gamma^2\rho^2 L_{\mathbf{s}}^2) + \gamma^2\rho^2 L_V L_{\mathbf{s}}^2\right)}_{=b_k \text{ since } k+1 \leq \ell(k) + m} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] + \tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)} \end{aligned}$$

448 where

$$\begin{aligned} \tilde{\eta}^{(k+1)} &= \left(\gamma_{k+1}(\rho + \rho^2\gamma_{k+1} L_V) + b_{k+1}(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2\rho^2)\right) \mathbb{E}\left[\left\|\eta_{i_k}^{(k+1)}\right\|^2\right] \\ \chi^{(k+1)} &= \left(b_{k+1}(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2(1-\rho)^2) - \gamma_{k+1}(1-\rho - (1-\rho)^2\gamma_{k+1} L_V)\right) \\ \tilde{\chi}^{(k+1)} &= \chi^{(k+1)} \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2\right] \end{aligned}$$

449 This leads, using Lemma 2, that for any  $\gamma_{k+1}$ ,  $\rho$  and  $\beta$  such that  $\rho v_{\min} + v_{\max}^2 -$

450  $\gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2) > 0$ ,

$$\begin{aligned} v_{\max}^2 \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] \leq \frac{R_k - R_{k+1}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))} \\ &\quad + \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))} \end{aligned}$$



451 We first remark that

$$\begin{aligned} & \gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2)) \\ & \geq \frac{\gamma_{k+1}\rho}{c_1}(1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1)) \end{aligned}$$

452 where  $c_1 = v_{\min}^{-1}$ . By setting  $\bar{L} = \max\{L_s, L_V\}$ ,  $\beta = \frac{c_1\bar{L}}{n^{1/3}}$ ,  $\rho = \frac{\mu}{c_1\bar{L}n^{2/3}}$ ,  $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$  and  
 453  $\{\gamma_{k+1}\}$  any sequence of decreasing stepsizes in  $(0, 1)$ , it can be shown that there exists  $\mu \in (0, 1)$ ,  
 454 such that the following lower bound holds

$$\begin{aligned} & 1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1) \geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \bar{b}_0(\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ & \geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{L_V\mu^2}{c_1^2n^{\frac{4}{3}}}\frac{(1 + \gamma\beta + 2\gamma^2L_s^2)^m - 1}{\gamma\beta + 2\gamma^2L_s^2}(\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ & \stackrel{(a)}{\geq} 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{\mu}{c_1^2}(e - 1)(1 + \frac{2\mu}{n}) \geq 1 - \mu - \mu(1 + 2\mu)\frac{e - 1}{c_1^2} \stackrel{(b)}{\geq} \frac{1}{2} \end{aligned}$$

455 where the simplification in (a) is due to

$$\frac{\mu}{n} \leq \gamma\beta + 2\gamma^2L_s^2 \leq \frac{\mu}{n} + \frac{2\mu^2}{c_1^2n^{\frac{4}{3}}} \leq \frac{\mu c_1^2 + 2\mu^2}{c_1^2} \frac{1}{n} \text{ and } (1 + \gamma\beta + 2\gamma^2L_s^2)^m \leq e - 1.$$

456 and the required  $\mu$  in (b) can be found by solving the quadratic equation.

457 Finally, these results yield:

$$v_{\max}^2 \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \leq \frac{2(R_0 - R_{K_{\max}})}{v_{\min}\rho} + 2 \sum_{k=0}^{K_{\max}-1} \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{v_{\min}\rho}$$

458 Note that  $R_0 = \mathbb{E}[V(\hat{s}^{(0)})]$  and if  $K_{\max}$  is a multiple of  $m$ , then  $R_{K_{\max}} = \mathbb{E}[V(\hat{s}^{(K_{\max})})]$ . Under the  
 459 latter condition, we have

$$\sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{s}^{(0)}) - V(\hat{s}^{(K_{\max})})] + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} [\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}]$$

460 This concludes our proof.

461

□

## 462 E Proof of Theorem 3

463 **Theorem.** Assume H1-H5. Let  $K_{\max}$  be a positive integer. Let  $\{\gamma_k, k \in \mathbb{N}\}$  be a sequence of  
 464 positive step sizes and consider the fitTEM sequence  $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$  obtained with  $\rho_{k+1} = \rho$  for any  
 465  $k > 0$ .

466 Assume that  $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$  for any  $k \leq K_{\max}$ . By setting  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  $\bar{L} = \max\{L_{\mathbf{s}}, L_V\}$ ,  
 467  $\beta = \frac{c_1 \bar{L}}{n}$ ,  $\rho = \frac{1}{n^{2/3}}$ ,  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$  and  $\gamma_{k+1} = \frac{1}{k^a \alpha c_1 \bar{L}}$  where  $a \in (0, 1)$ , we  
 468 have the following bound:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ &\quad + \frac{\alpha \bar{L} n^{2/3}}{v_{\min} v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \left[ \Xi^{(k+1)} + \Gamma^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right\|^2 \right] \right] \end{aligned}$$

469 **Proof** Using the smoothness of  $V$  and update (11), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) + \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \end{aligned} \quad (36)$$

470 Denote  $H_{k+1} := \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}$  the drift term of the fitTEM update in (7) and  $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$ .  
 471 Using Lemma 8 and the additional following identity:

$$\mathbb{E} \left[ (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] = 0 \quad (37)$$

472 we have:

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] \\ &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} \rho \mathbb{E}[\langle \mathbf{h}_k | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1} \mathbb{E} \left[ \langle \rho \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] + (1 - \rho) \mathbb{E}[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}] | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2 \\ &\stackrel{(a)}{\leq} -v_{\min} \gamma_{k+1} \rho \mathbb{E}[\|\mathbf{h}_k\|^2] - \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2 \\ &\stackrel{(b)}{\leq} -(v_{\min} \gamma_{k+1} \rho + \gamma_{k+1} v_{\max}^2) \mathbb{E}[\|\mathbf{h}_k\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2 \end{aligned}$$

473 where  $\xi^{(k+1)} = \mathbb{E} \left[ \left\| \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] \right\|^2 \right]$ . **Bounding  $\mathbb{E}[\|H_{k+1}\|^2]$**  Using Lemma 4, we obtain:

$$\begin{aligned} &\gamma_{k+1} (v_{\min} \rho + v_{\max}^2 - \gamma_{k+1} \rho^2 L_V) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\leq \mathbb{E} [V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)})] + \tilde{\xi}^{(k+1)} + \left( (1 - \rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V \rho^2 L_{\mathbf{s}}^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \end{aligned} \quad (38)$$

474 where  $\tilde{\xi}^{(k+1)} = \gamma_{k+1}^2 \rho^2 \mathbf{L}_V \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)}$ . Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \right) \quad (39)$$

475 where the equality holds as  $i_k$  and  $j_k$  are drawn independently. Next,

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \end{aligned}$$

476 Note that  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) = -\gamma_{k+1} \mathbf{H}_{k+1}$  and that in expectation we recall  
477 that  $\mathbb{E}[\mathbf{H}_{k+1} | \mathcal{F}_k] = \rho \mathbf{h}_k + \rho \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] + (1-\rho) \mathbb{E}[\tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)}]$  where  $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$ . Thus,  
478 for any  $\beta > 0$ , it holds

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1} \beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\quad + \frac{\gamma_{k+1} (1-\rho)^2}{\beta} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned}$$

479 where the last inequality is due to the Young's inequality. Plugging this into (39) yields:

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1} \beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\quad + \frac{\gamma_{k+1} (1-\rho)^2}{\beta} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned}$$

480 Subsequently, we have

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}[(1 + \gamma_{k+1} \beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \|\mathbf{h}_k\|^2 \\ &\quad + \frac{\gamma_{k+1} \rho^2}{\beta} \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] + \frac{\gamma_{k+1} (1-\rho)^2}{\beta} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]] \end{aligned}$$

481 We now use Lemma 4 on  $\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 = \gamma_{k+1}^2 \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2$  and obtain:

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\ &\leq \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \left( \frac{\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2}{n} + \frac{(n-1)(1 + \gamma_{k+1} \beta)}{n^2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + \gamma_{k+1} (1-\rho)^2 \left( 2\gamma_{k+1} + \frac{1}{\beta} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left( 2\gamma_{k+1}^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\leq \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \left( \frac{1 - \frac{1}{n} + \gamma_{k+1} \beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2}{n} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + \gamma_{k+1} (1-\rho)^2 \left( 2\gamma_{k+1} + \frac{1}{\beta} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left( 2\gamma_{k+1}^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \end{aligned}$$

482 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2]$$

483 From the above, we get

$$\begin{aligned} \Delta^{(k+1)} &\leq \left(1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2\right) \Delta^{(k)} + \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right) \mathbb{E}\left[\left\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\right\|^2\right] \\ &\quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] + \gamma_{k+1} \left(2\gamma_{k+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}\left[\left\|\eta_{i_k}^{(k+1)}\right\|^2\right] \end{aligned}$$

484 Setting  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{2, 1+2v_{\min}\}$ ,  $\bar{L} = \max\{\mathbf{L}_s, \mathbf{L}_V\}$ ,  $\gamma_{k+1} = \frac{1}{k}$ ,  $\beta = \frac{1}{\alpha n}$ ,  $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$ ,

485  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ , we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2 \leq 1 - \frac{1}{n} + \frac{1}{\alpha k n} + \frac{1}{\alpha^2 c_1^2 k^2 n^{4/3}} \leq 1 - \frac{c_1(k\alpha - 1) - 1}{k\alpha n c_1} \leq 1 - \frac{1}{k\alpha n c_1}$$

486 which shows that  $1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2 \in (0, 1)$  for any  $k > 0$ . Denote  $\Lambda_{(k+1)} = \frac{1}{n} -$

487  $\gamma_{k+1}\beta - \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2$  and note that  $\Delta^{(0)} = 0$ , thus the telescoping sum yields:

$$\begin{aligned} \Delta^{(k+1)} &\leq \sum_{\ell=0}^k \omega_{k,\ell} \left(2\gamma_{\ell+1}^2 \rho^2 + \frac{\gamma_{\ell+1}^2 \rho^2}{\beta}\right) \mathbb{E}\left[\left\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\right\|^2\right] \\ &\quad + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1} (1-\rho)^2 \left(2\gamma_{\ell+1} + \frac{1}{\beta}\right) \mathbb{E}\left[\left\|\tilde{\mathbf{S}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\right\|^2\right] + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1} \tilde{\epsilon}^{(\ell+1)} \end{aligned}$$

488 where  $\omega_{k,\ell} = \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right)$  and  $\tilde{\epsilon}^{(\ell+1)} = \left(2\gamma_{\ell+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}\left[\left\|\eta_{i_k}^{(k+1)}\right\|^2\right]$ .

489 Summing on both sides over  $k = 0$  to  $k = K_{\max} - 1$  yields:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \Delta^{(k+1)} &\leq \sum_{k=0}^{K_{\max}-1} \frac{2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}^2 \rho^2}{\beta}}{\Lambda_{(k+1)}} \mathbb{E}\left[\left\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\right\|^2\right] \\ &\quad + \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right)}{\Lambda_{(k+1)}} \mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] + \sum_{k=0}^{K_{\max}-1} \frac{\gamma_{k+1}}{\Lambda_{(k+1)}} \tilde{\epsilon}^{(k+1)} \end{aligned}$$

490 We recall (38) where we have summed on both sides from  $k = 0$  to  $k = K_{\max} - 1$ :

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(K_{\max})}) - V(\hat{\mathbf{s}}^{(0)})] \\ &\leq \sum_{k=0}^{K_{\max}-1} \left\{ \gamma_{k+1}(-v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}\rho^2 \mathbf{L}_V \mathbb{E}\left[\|\mathbf{h}_k\|^2\right] + \gamma^2 \mathbf{L}_V \rho^2 \mathbf{L}_s^2 \Delta^{(k)} \right\} \\ &\quad + \sum_{k=0}^{K_{\max}-1} \left\{ \tilde{\epsilon}^{(k+1)} + \left( (1-\rho)^2 \gamma_{k+1}^2 \mathbf{L}_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) \mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \right\} \\ &\leq \sum_{k=0}^{K_{\max}-1} \left\{ \left[ -\gamma_{k+1}(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 \mathbf{L}_V + \frac{\rho^2 \gamma_{k+1}^2 \mathbf{L}_V \mathbf{L}_s^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)}{\Lambda_{(k+1)}} \right] \mathbb{E}\left[\|\mathbf{h}_k\|^2\right] \right\} \\ &\quad + \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma^{(k+1)} \mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\right\|^2\right] \end{aligned} \tag{40}$$

where

$$\Xi^{(k+1)} = \tilde{\epsilon}^{(k+1)} + \frac{\gamma_{k+1}^3 \mathbf{L}_V \rho^2 \mathbf{L}_s^2}{\Lambda_{(k+1)}} \tilde{\epsilon}^{(k+1)}$$

and

$$\Gamma^{(k+1)} = \left( (1-\rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) + \frac{\gamma_{k+1}^3 L_V \rho^2 L_s^2 (1-\rho)^2 \left( 2\gamma_{k+1} + \frac{1}{\beta} \right)}{\Lambda_{(k+1)}}$$

491 We now analyse the following quantity

$$\begin{aligned} & -\gamma_{k+1}(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 L_V + \frac{\rho^2 \gamma_{k+1}^2 L_V L_s^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \\ & = \gamma_{k+1} \left[ -(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right] \end{aligned} \quad (41)$$

492 Furthermore, we recall that  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k}$ ,

493  $\beta = \frac{1}{\alpha n}$ ,  $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$ ,  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ . Then,

$$\begin{aligned} & \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \right)}{\frac{1}{n} - \gamma_{k+1} \beta - \gamma_{k+1}^2 \rho^2 L_s^2} \\ & \leq \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L}(k\alpha^2 c_1^2 n^{4/3})^{-1} \left( \frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k\alpha c_1^2 \bar{L}^2 n^{1/3}} \right)}{\frac{1}{n} - \frac{1}{k\alpha n} - \frac{1}{k^2 \alpha^2 c_1^2 n^{4/3}}} \\ & = \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L} \left( \frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k\alpha c_1^2 \bar{L}^2 n^{1/3}} \right)}{(k\alpha c_1 n^{1/3})(k\alpha - 1)c_1 - 1} \\ & \stackrel{(a)}{\leq} \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\frac{1}{k\alpha c_1^2 \bar{L} n^{1/3}} \left( \frac{2}{k\alpha n} + 1 \right)}{2(\alpha c_1 n^{1/3}) - 1} \\ & \leq \frac{1}{k^2 \alpha c_1^2 \bar{L} n^{4/3}} + \frac{1}{4k\alpha^2 c_1^3 \bar{L} n^{2/3}} \\ & \leq \frac{3/4}{\alpha c_1^2 \bar{L} n^{2/3}} \end{aligned} \quad (42)$$

where (a) is due to  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$  and  $k\alpha c_1 n^{1/3} \geq 1$ . Note also that

$$-(v_{\min}\rho + v_{\max}^2) \leq -\rho v_{\min} = -\frac{1}{\alpha c_1^2 \bar{L} n^{2/3}}$$

which yields that

$$\left[ -(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1} \rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right] \leq -\frac{1/4}{\alpha c_1^2 \bar{L} n^{2/3}}$$

494 Using the Lemma 2, we know that  $v_{\max}^2 \|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2 \leq \|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2$  and using (42) on (40)  
495 yields:

$$\begin{aligned} v_{\max}^2 \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] & \leq \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ & \quad + \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2} \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right] \end{aligned}$$

496 proving the final bound on the gradient of the Lyapunov function:

$$\begin{aligned} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] & \leq \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2 v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_{\max})})] \\ & \quad + \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_{\max}-1} \Xi^{(k+1)} + \sum_{k=0}^{K_{\max}-1} \Gamma^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right] \end{aligned}$$





## F Practical Implementations of Two-Timescale EM Methods

### F.1 Application on GMM

#### F.1.1 Explicit Updates

We first recognize that the constraint set for  $\theta$  is given by

$$\Theta = \Delta^M \times \mathbb{R}^M.$$

Using the partition of the sufficient statistics as  $S(y_i, z_i) = (S^{(1)}(y_i, z_i)^\top, S^{(2)}(y_i, z_i)^\top, S^{(3)}(y_i, z_i)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$ , the partition  $\phi(\theta) = (\phi^{(1)}(\theta)^\top, \phi^{(2)}(\theta)^\top, \phi^{(3)}(\theta)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$  and the fact that  $\mathbb{1}_{\{M\}}(z_i) = 1 - \sum_{m=1}^{M-1} \mathbb{1}_{\{m\}}(z_i)$ , the complete data log-likelihood can be expressed as in (2) with

$$\begin{aligned} s_{i,m}^{(1)} &= \mathbb{1}_{\{m\}}(z_i), \quad \phi_m^{(1)}(\theta) = \left\{ \log(\omega_m) - \frac{\mu_m^2}{2} \right\} - \left\{ \log(1 - \sum_{j=1}^{M-1} \omega_j) - \frac{\mu_M^2}{2} \right\}, \\ s_{i,m}^{(2)} &= \mathbb{1}_{\{m\}}(z_i)y_i, \quad \phi_m^{(2)}(\theta) = \mu_m, \quad s_i^{(3)} = y_i, \quad \phi^{(3)}(\theta) = \mu_M, \end{aligned} \quad (43)$$

and  $\psi(\theta) = -\left\{ \log(1 - \sum_{m=1}^{M-1} \omega_m) - \frac{\mu_M^2}{2\sigma^2} \right\}$ . We also define for each  $m \in \llbracket 1, M \rrbracket$ ,  $j \in \llbracket 1, 3 \rrbracket$ ,  $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$ . Consider the following latent sample used to compute an approximation of the conditional expected value  $\mathbb{E}_\theta[\mathbb{1}_{\{z_i=m\}}|y = y_i]$ :

$$z_{i,m} \sim \mathbb{P}(z_i = m|y_i; \theta) \quad (44)$$

where  $m \in \llbracket 1, M \rrbracket$ ,  $i \in \llbracket 1, n \rrbracket$  and  $\theta = (\mathbf{w}, \boldsymbol{\mu}) \in \Theta$ .

In particular, given iteration  $k + 1$ , the computation of the approximated quantity  $\tilde{S}_{i_k}^{(k)}$  during Incremental-step updates, see (8) can be written as

$$\tilde{S}_{i_k}^{(k)} = \left( \underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1}), \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})}_{:=\tilde{s}_{i_k}^{(1)}}, \underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1})y_{i_k}, \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})y_{i_k}}_{:=\tilde{s}_{i_k}^{(2)}}, \underbrace{y_{i_k}}_{:=\tilde{s}_{i_k}^{(3)}(\theta^{(k)})} \right)^\top. \quad (45)$$

Recall that we have used the following regularizer:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m), \quad (46)$$

It can be shown that the regularized M-step evaluates to

$$\bar{\theta}(\mathbf{s}) = \begin{pmatrix} (1 + \epsilon M)^{-1} (s_1^{(1)} + \epsilon, \dots, s_{M-1}^{(1)} + \epsilon)^\top \\ ((s_1^{(1)} + \delta)^{-1} s_1^{(2)}, \dots, (s_{M-1}^{(1)} + \delta)^{-1} s_{M-1}^{(2)})^\top \\ (1 - \sum_{m=1}^{M-1} s_m^{(1)} + \delta)^{-1} (s^{(3)} - \sum_{m=1}^{M-1} s_m^{(2)}) \end{pmatrix} = \begin{pmatrix} \bar{\omega}(\mathbf{s}) \\ \bar{\boldsymbol{\mu}}(\mathbf{s}) \\ \bar{\mu}_M(\mathbf{s}) \end{pmatrix}. \quad (47)$$

where we have defined for all  $m \in \llbracket 1, M \rrbracket$  and  $j \in \llbracket 1, 3 \rrbracket$ ,  $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$ .

#### F.1.2 Model Assumptions (GMM example)

We use the GMM example to illustrate the required assumptions.

Many practical models can satisfy the compactness of the sets as in Assumption H1. For instance, the GMM example satisfies (15) as the sufficient statistics are composed of indicator functions and observations as defined Section F.1 Equation (43).

Assumptions H2 and H3 are standard for the curved exponential family models. For GMM, the following (strongly convex) regularization  $\mathbf{r}(\theta)$  ensures H3:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m)$$

521 since it ensures  $\theta^{(k)}$  is unique and lies in  $\text{int}(\Delta^M) \times \mathbb{R}^M$ . We remark that for H2, it is possible to  
 522 define the Lipschitz constant  $L_p$  independently for each data  $y_i$  to yield a refined characterization.

523 Again, H4 is satisfied by practical models. For GMM, it can be verified by deriving the closed form  
 524 expression for  $B(s)$  and using H1.

525 Under H1 and H3, we have  $\|\hat{s}^{(k)}\| < \infty$  since  $S$  is compact and  $\hat{\theta}^{(k)} \in \text{int}(\Theta)$  for any  $k \geq 0$  which  
 526 thus ensure that the EM methods operate in a closed set throughout the optimization process.

### 527 F.1.3 Algorithms updates

528 In the sequel, recall that, for all  $i \in \llbracket n \rrbracket$  and iteration  $k$ , the computed statistic  $\tilde{S}_{i_k}^{(k)}$  is defined by  
 529 (45). At iteration  $k$ , the several E-steps defined by (9) or (10) and (11) leads to the definition of the  
 530 quantity  $\hat{s}^{(k+1)}$ . For the GMM example, after the initialization of the quantity  $\hat{s}^{(0)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(0)}$ ,  
 531 those E-steps break down as follows:

532 **Batch EM (EM):** for all  $i \in \llbracket 1, n \rrbracket$ , compute  $\bar{s}_i^{(k)}$  and set

$$\hat{s}^{(k+1)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(k)}.$$

533 where  $\bar{s}_i^{(k)}$  are computed using the exact conditional expected value  $\mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i]$ :

$$\tilde{\omega}_m(y_i; \theta) := \mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i] = \frac{\omega_m \exp(-\frac{1}{2}(y_i - \mu_i)^2)}{\sum_{j=1}^M \omega_j \exp(-\frac{1}{2}(y_i - \mu_j)^2)},$$

534 **Incremental EM (iEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ , compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} + \frac{1}{n} (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\tau_i^k)}) = n^{-1} \sum_{i=1}^n \bar{s}_i^{(\tau_i^k)}.$$

535 **batch SAEM (SAEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ , compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} \tilde{S}^{(k)}.$$

536 where  $= \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k)}$  with  $\tilde{S}_i^{(k)}$  defined in (45).

537 **Incremental SAEM (iSAEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ , compute  $\bar{s}_{i_k}^{(k)}$  and set  
 538

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} + \frac{1}{n} (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\tau_i^k)})).$$

539 **Variance Reduced Two-Timescale EM (vrTTEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ ,  
 540 compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} (1 - \rho) + \rho (\tilde{S}^{(\ell(k))} + (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\ell(k))}))).$$

541 **Fast Incremental Two-Timescale EM (fiTTEM):** draw an index  $i_k$  uniformly at random on  $\llbracket n \rrbracket$ ,  
 542 compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} (1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)} (1 - \rho) + \rho (\bar{\mathcal{S}}^{(k)} + (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(t_{i_k}^k)}))).$$

543 Finally, the  $k$ -th update reads  $\hat{\theta}^{(k+1)} = \bar{\theta}(\hat{s}^{(k+1)})$  where the function  $s \rightarrow \bar{\theta}(s)$  is defined by (47).

## 544 F.2 Deformable Template Model for Image Analysis

### 545 F.2.1 Model and Updates

546 The complete model belongs to the curved exponential family, see [1], which vector of sufficient  
547 statistics  $S = (S_1(z), S_2(z), S_3(z))$  read:

$$\begin{aligned} S_1(z) &= \frac{1}{n} \sum_{i=1}^n S_1(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top y_i \\ S_2(z) &= \frac{1}{n} \sum_{i=1}^n S_2(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top (\mathbf{K}_p^{z_i}) \\ S_3(z) &= \frac{1}{n} \sum_{i=1}^n S_3(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n z_i^t z_i \end{aligned} \quad (48)$$

548 where for any pixel  $u \in \mathbb{R}^2$  and  $j \in \llbracket 1, k_g \rrbracket$  we noted:

$$\mathbf{K}_p^{z_i}(x_u, j) = \mathbf{K}_p^{z_i}(x_u - \phi_i(x_u, z_i), p_j)$$

549 Finally, the Two-Timescale M-step yields the following parameter updates:

$$\bar{\theta}(\hat{s}) = \begin{pmatrix} \beta(\hat{s}) = \hat{s}_2^{-1}(z) \hat{s}_1(z) \\ \Gamma(\hat{s}) = \frac{1}{n} \hat{s}_3(z) \\ \sigma(\hat{s}) = \beta(\hat{s})^\top \hat{s}_2(z) \beta(\hat{s}) - 2\beta(\hat{s}) \hat{s}_1(z) \end{pmatrix} \quad (49)$$

550 where  $\hat{s} = (\hat{s}_1(z), \hat{s}_2(z), \hat{s}_3(z))$  is the vector of statistics obtained via the SA-step (7) and using the  
551 MC approximation of the sufficient statistics  $(S_1(z), S_2(z), S_3(z))$  defined in (53).

### 552 F.2.2 Numerical Applications

553 For the inference of the template, we use the Matlab code (online SAEM) used in [16] and implement  
554 our own batch, incremental, Variance reduced and Fast Incremental variants. The hyperparameters  
555 are kept the same and reads as follows  $M = 400$ ,  $\gamma_k = 1/k^{0.6}$  and  $p = 16$ . The number of  
556 landmarks for the template is  $k_p = 15$  points and for the deformation  $k_g = 6$  points. Both have  
557 Gaussian kernels with respectively standard deviation of 0.08 and 0.16. The standard deviation of  
558 the measurement errors is set to 0.1.

559 For the simulation part, we use the Carlin and Chib MCMC procedure, see [6]. Refer to [16] for  
560 more details.

## 561 G Additional Experiment: Pharmacokinetics (PK) Model with Absorption 562 Lag Time

563 This numerical example was conducted in order to characterize the pharmacokinetics (PK) of orally  
564 administered drug to simulated patients, using a population pharmacokinetic approach.  $M = 50$   
565 synthetic datasets were generated for  $n = 5000$  patients with 10 observations (concentration mea-  
566 sures) per patient. The goal is to model the evolution of the concentration of the absorbed drug  
567 using a nonlinear and latent data model.

568 **Model and Explicit Updates:** We consider a one-compartment PK model for oral administration  
569 with an absorption lag-time ( $T^{\text{lag}}$ ), assuming first-order absorption and linear elimination processes.  
570 The final model includes the following variables:  $ka$  the absorption rate constant,  $V$  the volume of  
571 distribution,  $k$  the elimination rate constant and  $T^{\text{lag}}$  the absorption lag-time. We also add several  
572 covariates to our model such as  $D$  the dose of drug administered,  $t$  the time at which measures  
573 are taken and the weight of the patient influencing the volume  $V$ . More precisely, the log-volume  
574  $\log(V)$  is a linear function of the log-weight  $lw70 = \log(wt/70)$ . Let  $z_i = (T_i^{\text{lag}}, ka_i, V_i, k_i)$  be the  
575 vector of individual PK parameters, different for each individual  $i$ . The final model reads:

$$y_{ij} = f(t_{ij}, z_i) + \varepsilon_{ij} \quad \text{where} \quad f(t_{ij}, z_i) = \frac{D ka_i}{V(ka_i - k_i)} (e^{-ka_i(t_{ij} - T_i^{\text{lag}})} - e^{-k_i(t_{ij} - T_i^{\text{lag}})}), \quad (50)$$

where  $y_{ij}$  is the  $j$ -th concentration measurement of the drug of dosage  $D$  injected at time  $t_{ij}$  for patient  $i$ . We assume in this example that the residual errors  $\varepsilon_{ij}$  are independent and normally distributed with mean 0 and variance  $\sigma^2$ . Lognormal distributions are used for the four PK parameters.

Lognormal distributions are used for the four PK parameters:

$$\log(T_i^{\text{lag}}) \sim \mathcal{N}(\log(T_{\text{pop}}^{\text{lag}}), \omega_{T^{\text{lag}}}^2), \log(ka_i) \sim \mathcal{N}(\log(ka_{\text{pop}}), \omega_{ka}^2), \quad (51)$$

$$\log(V_i) \sim \mathcal{N}(\log(V_{\text{pop}}), \omega_V^2), \log(k_i) \sim \mathcal{N}(\log(k_{\text{pop}}), \omega_k^2). \quad (52)$$

We recall that the complete model  $(y, z)$  defined by (50) belongs to the curved exponential family, which vector of sufficient statistics  $S = (S_1(z), S_2(z), S_3(z))$  read:

$$S_1(z) = \frac{1}{n} \sum_{i=1}^n z_i, \quad S_2(z) = \frac{1}{n} \sum_{i=1}^n z_i^\top z_i, \quad S_3(z) = \frac{1}{n} \sum_{i=1}^n (y_i - f(t_i, z_i))^2 \quad (53)$$

where we have noted  $y_i$  and  $t_i$  the vector of observations and time for each patient  $i$ . At iteration  $k$ , and setting the number of MC samples to 1 for the sake of clarity, the MC sampling  $z_i^{(k)} \sim p(z_i|y_i, \theta^{(k)})$  is performed using a Metropolis-Hastings procedure detailed in Algorithm 2. The quantities  $\hat{S}^{(k+1)}$  and  $\hat{s}^{(k+1)}$  are then updated according to the different methods. Finally the maximization step yields:

$$\bar{\theta}(s) = \begin{pmatrix} \hat{s}_1^{(k+1)} \\ \hat{s}_2^{(k+1)} - \hat{s}_1^{(k+1)} (\hat{s}_1^{(k+1)})^\top \\ \hat{s}_3^{(k+1)} \end{pmatrix} = \begin{pmatrix} \overline{z_{\text{pop}}}(\hat{s}^{(k+1)}) \\ \overline{\omega_z}(\hat{s}^{(k+1)}) \\ \overline{\sigma}(\hat{s}^{(k+1)}) \end{pmatrix}. \quad (54)$$

**Metropolis Hastings algorithm** During the simulation step of the MISSO method, the sampling from the target distribution  $\pi(z_i, \theta) := p(z_i|y_i, \theta)$  is performed using a Metropolis Hastings (MH) algorithm [19] with proposal distribution  $q(z_i, \delta)$  where  $\theta = (z_{\text{pop}}, \omega_z)$  and  $\delta$  is the vector of parameters of the proposal distribution. Commonly they parameterize a Gaussian proposal. The MH algorithm is summarized in 2.

---

**Algorithm 2** MH algorithm

---

```

1: Input: initialization  $z_{i,0} \sim q(z_i; \delta)$ 
2: for  $m = 1, \dots, M$  do
3:   Sample  $z_{i,m} \sim q(z_i; \delta)$ 
4:   Sample  $u \sim \mathcal{U}([0, 1])$ 
5:   Calculate the ratio  $r = \frac{\pi(z_{i,m}; \theta) / q(z_{i,m}; \delta)}{\pi(z_{i,m-1}; \theta) / q(z_{i,m-1}; \delta)}$ 
6:   if  $u < r$  then
7:     Accept  $z_{i,m}$ 
8:   else
9:      $z_{i,m} \leftarrow z_{i,m-1}$ 
10:  end if
11: end for
12: Output:  $z_{i,M}$ 

```

---

**Monte Carlo study:** We conduct a Monte Carlo study to showcase the benefits of our scheme.  $M = 50$  datasets have been simulated using the following PK parameters values:  $T_{\text{pop}}^{\text{lag}} = 1$ ,  $ka_{\text{pop}} = 1$ ,  $V_{\text{pop}} = 8$ ,  $k_{\text{pop}} = 0.1$ ,  $\omega_{T^{\text{lag}}} = 0.4$ ,  $\omega_{ka} = 0.5$ ,  $\omega_V = 0.2$ ,  $\omega_k = 0.3$  and  $\sigma^2 = 0.5$ . We define the mean square distance over the  $M$  replicates  $E_k(\ell) = \frac{1}{M} \sum_{m=1}^M (\theta_k^{(m)}(\ell) - \theta^*)^2$  and plot it against the epochs (passes over the data) Figure 4. Note that the MC-step (5) is performed using a Metropolis Hastings procedure since the posterior distribution under the model  $\theta$  noted  $p(z_i|y_i, \theta)$  is intractable due to the nonlinearity of the model (50). Figure 4 shows clear advantage of variance reduced methods (vrTTEM and fitTEM) avoiding the twists and turns displayed by the incremental and the batch methods.

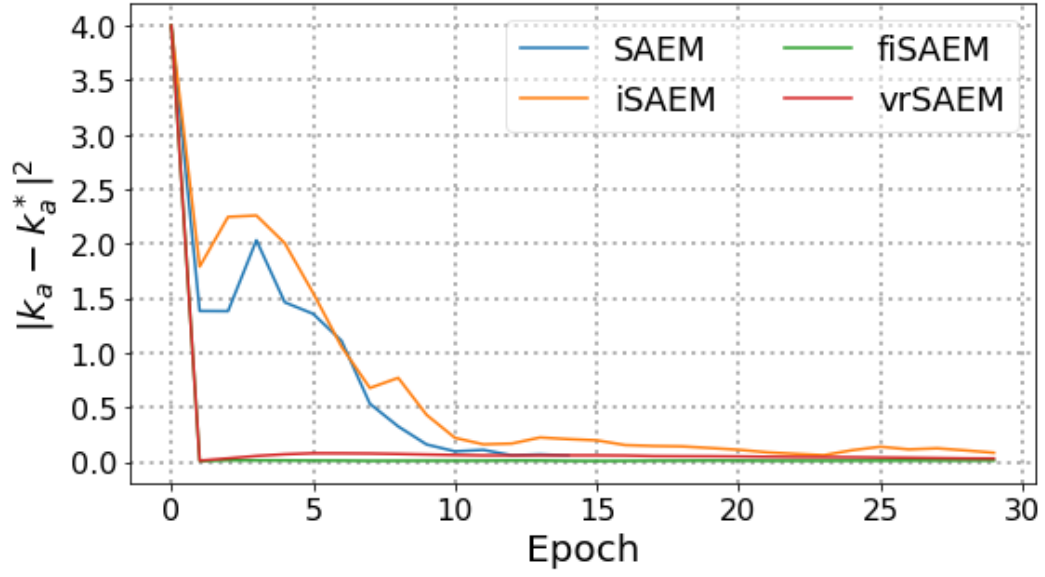


Figure 4: Precision  $|ka^{(k)} - ka^*|^2$  per epoch