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# Fast Two-Timescale Stochastic EM Algorithms

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## Abstract

The Expectation-Maximization (EM) algorithm is a popular choice for learning latent variable models. Variants of the EM have been initially introduced by [28], using incremental updates to scale to large datasets, and by [33, 12], using Monte Carlo (MC) approximations to bypass the intractable conditional expectation of the latent data for most nonconvex models. In this paper, we propose a general class of methods called Two-Timescale EM Methods based on a two-stage approach of stochastic updates to tackle an essential nonconvex optimization task for latent variable models. We motivate the choice of a double dynamic by invoking the variance reduction virtue of each stage of the method on both sources of noise: the index sampling for the incremental update and the MC approximation. We establish finite-time and global convergence bounds for nonconvex objective functions. Numerical applications are also presented to illustrate our findings.

## 1 Introduction

Learning latent variable models is critical for modern machine learning problems, see (e.g.,) [26] for references. We formulate the training of such model as an empirical risk minimization problem:

$$\min_{\theta \in \Theta} \bar{L}(\theta) := L(\theta) + r(\theta) \quad \text{with} \quad L(\theta) = \frac{1}{n} \sum_{i=1}^n L_i(\theta) := \frac{1}{n} \sum_{i=1}^n \{ -\log g(y_i; \theta) \}, \quad (1)$$

where  $\{y_i\}_{i=1}^n$  are observations,  $\Theta \subset \mathbb{R}^d$  is the parameters set and  $r : \Theta \rightarrow \mathbb{R}$  is a smooth regularizer. The objective function  $\bar{L}(\theta)$  is possibly *nonconvex* and is assumed to be lower bounded. In the latent variable model, the likelihood  $g(y_i; \theta)$ , is the marginal of the complete data likelihood defined as  $f(z_i, y_i; \theta)$ , i.e.,  $g(y_i; \theta) = \int_{\mathcal{Z}} f(z_i, y_i; \theta) \mu(dz_i)$ , where  $\{z_i\}_{i=1}^n$  are the latent variables. In this paper, we assume that the complete model belongs to the curved exponential family [14]:

$$f(z_i, y_i; \theta) = h(z_i, y_i) \exp \left( \langle S(z_i, y_i) | \phi(\theta) \rangle - \psi(\theta) \right), \quad (2)$$

where  $\psi(\theta)$ ,  $h(z_i, y_i)$  are scalar functions,  $\phi(\theta) \in \mathbb{R}^k$  is a vector function, and  $\{S(z_i, y_i) \in \mathbb{R}^k\}_{i=1}^n$  is the vector of sufficient statistics. Batch EM [13, 34], the method of reference for (1), is comprised of two steps. The E-step computes the conditional expectation of the sufficient statistics of (2):

$$\text{E-step: } \bar{s}(\theta) = \frac{1}{n} \sum_{i=1}^n \bar{s}_i(\theta) \quad \text{where} \quad \bar{s}_i(\theta) = \int_{\mathcal{Z}} S(z_i, y_i) p(z_i | y_i; \theta) \mu(dz_i), \quad (3)$$

and the M-step is given by

$$\text{M-step: } \hat{\theta} = \bar{\theta}(\bar{s}(\theta)) := \arg \min_{\vartheta \in \Theta} \{ r(\vartheta) + \psi(\vartheta) - \langle \bar{s}(\theta) | \phi(\vartheta) \rangle \}. \quad (4)$$

Two caveats of this method are the following: (a) with the explosion of data, the first step of the EM is computationally inefficient as it requires, at each iteration, a full pass over the dataset; and (b) the complexity of modern models makes the expectation in (3) intractable. So far, and to the best of our knowledge, both challenges have been addressed separately, as detailed in the sequel.

**Prior Work:** Inspired by stochastic optimization procedures, [28] and [8] develop respectively an incremental and an online variant of the E-step in models where the expectation is computable, and were then extensively used and studied in [30, 23, 7]. Some improvements of those methods have been provided and analyzed, globally and in finite-time, in [20] where variance reduction techniques taken from the optimization literature have been efficiently applied to scale the EM algorithm to large datasets. Regarding the computation of the expectation under the posterior distribution, the Monte Carlo EM (MCEM) has been introduced in the seminal paper [33] where an MC approximation for this expectation is computed. A variant of that algorithm is the Stochastic Approximation of the EM (SAEM) in [12] leveraging the power of Robbins-Monro update [32] to ensure pointwise convergence of the vector of estimated parameters using a decreasing stepsize rather than increasing the number of MC samples. The MCEM and the SAEM have been successfully applied in mixed effects models [25, 16, 4] or to do inference for joint modeling of time to event data coming from clinical trials in [10], unsupervised clustering in [29], variational inference of graphical models in [5] among other applications. Recently, an incremental variant of the SAEM was proposed in [22] showing positive empirical results but its analysis is limited to asymptotic consideration.

**Contributions:** This paper *introduces* and *analyzes* a new class of methods which purpose is to update two proxies for the target expected quantities in a two-timescale manner. Those approximated quantities are then used to optimize the objective function (1) for modern examples and settings using the M-step of the EM algorithm. The main contributions of the paper are:

- We propose a two-timescale method based on (i) Stochastic Approximation (SA), to alleviate the problem of computing MC approximations, and on (ii) Incremental updates, to scale to large datasets. We describe in details the edges of each level of our method based on variance reduction arguments. Such class of algorithms has two advantages. First, it naturally leverages variance reduction and Robbins-Monro type of updates to tackle large-scale and highly nonlinear learning tasks. Then, it gives a simple formulation as a *scaled-gradient method* which makes the global analysis and the implementation accessible.
- We also establish global (independent of the initialization) and finite-time (true at each iteration) upper bounds on a classical sub-optimality condition in the nonconvex literature [18, 15], *i.e.*, the second order moment of the gradient of the objective function. We discuss the double dynamic of those bounds due to the two-timescale property of our algorithm update and we theoretically stress the advantages of introducing variance reduction in a *Stochastic Approximation* [32] scheme.

In Section 2 we formalize both incremental and Monte Carlo variants of the EM. Then, we introduce our two-timescale class of EM algorithms for which we derive several global statistical guarantees in Section 3 for possibly *nonconvex* functions. Section 4 is devoted to numerical illustrations. The supplementary material of this paper includes proofs of our theoretical results.

## 2 Two-Timescale Stochastic EM Algorithms

We recall and formalize in this section the different methods found in the literature that aim at solving the intractable expectation and the large-scale problem. We then provide the general framework of our method that efficiently tackles the optimization problem (1).

### 2.1 Monte Carlo Integration and Stochastic Approximation

As mentioned in the Introduction, for complex and possibly nonconvex models, the expectation under the posterior distribution defined in (3) is not tractable. In that case, the first solution involves computing a Monte Carlo integration of that latter. For all  $i \in [n]$ , where  $[n] := \{1, \dots, n\}$ , draw  $\{z_{i,m} \sim p(z_i | y_i; \theta)\}_{m=1}^M$  samples and compute the MC integration  $\tilde{s}$  of  $\bar{s}(\theta)$  defined by (3):

$$\text{MC-step : } \tilde{s} := \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}, y_i). \quad (5)$$

Then update the parameter  $\hat{\theta} = \bar{\theta}(\tilde{s})$ . This algorithm bypasses the intractable expectation issue but is rather computationally expensive in order to reach point wise convergence ( $M$  needs to be large). An alternative to that stochastic algorithm is to use a Robbins-Monro (RM) type of update. We

denote, at iteration  $k$ , the number of samples  $M_k$  and the following MC approximation  $\tilde{S}^{(k+1)}$ :

$$\tilde{S}^{(k+1)} := \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M_k} \sum_{m=1}^{M_k} S(z_{i,m}^{(k)}, y_i) \quad \text{where} \quad z_{i,m}^{(k)} \sim p(z_i | y_i; \theta^{(k)}). \quad (6)$$

Then, the RM update of the sufficient statistics  $\hat{s}^{(k+1)}$  reads:

$$\text{SA-step : } \hat{s}^{(k+1)} = \hat{s}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{s}^{(k)}), \quad (7)$$

where  $\{\gamma_k\}_{k \geq 1} \in (0, 1)$  is a sequence of decreasing stepsizes to ensure asymptotic convergence. This is called the Stochastic Approximation of the EM (SAEM) and has been shown to converge to a maximum likelihood of the observations under very general conditions [12]. In simple scenarios, the samples  $\{z_{i,m}\}_{m=1}^{M_k}$  are conditionally independent and identically distributed with distribution  $p(z_i, \theta)$ . Nevertheless, in most cases, since the loss function between the observed data  $y_i$  and the latent variable  $z_i$  can be nonconvex, sampling exactly from this distribution is not an option and the MC batch is sampled by Markov Chain Monte Carlo (MCMC) algorithm [27, 6]. It has been proved in [21] that (7) converges almost surely when coupled with an MCMC sampling procedure.

**Role of the stepsize  $\gamma_k$ :** The sequence of decreasing positive integers  $\{\gamma_k\}_{k \geq 1}$  controls the convergence of the algorithm. It is inefficient to start with small values for the stepsize  $\gamma_k$  and large values for the number of simulations  $M_k$ . Rather, it is recommended that one decreases  $\gamma_k$ , as in  $\gamma_k = 1/k^\alpha$ , with  $\alpha \in (0, 1)$ , and keeps a constant and small number  $M_k$  bypassing the computationally involved sampling step in (5). In practice,  $\gamma_k$  is set equal to 1 during the first few iterations to let the iterates explore the parameter space without memory and converge quickly to a neighborhood of the target estimate. The Stochastic Approximation is performed during the remaining iterations ensuring the almost sure convergence of the vector of estimates.

This Robbins-Monro type of update constitutes the *first level* of our algorithm, needed to temper the variance and noise introduced by the Monte Carlo integration. In the next section, we derive variants of this algorithm to adapt to the sheer size of data of today's applications and formalize the *second level* of our class of two-timescale EM methods.

## 2.2 Incremental and Two-Stage Stochastic EM Methods

Efficient strategies to scale to large datasets include incremental [28] and variance reduced [11, 19] methods. We will explicit a general update that covers those latter variants and that represents the *second level* of our algorithm, i.e., the incremental update of the noisy statistics  $\tilde{S}^{(k+1)}$  in (7):

$$\text{Incremental-step : } \tilde{S}^{(k+1)} = \tilde{S}^{(k)} + \rho_{k+1} (\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}). \quad (8)$$

Note that  $\{\rho_k\}_{k \geq 1} \in (0, 1)$  is a sequence of stepsizes,  $\mathcal{S}^{(k)}$  is a proxy for  $\tilde{S}^{(k)}$ . If the stepsize is equal to one and the proxy  $\mathcal{S}^{(k)} = \tilde{S}^{(k)}$ , i.e., computed in a full batch manner as in (6), then we recover the SAEM algorithm. Also if  $\rho_k = 1$ ,  $\gamma_k = 1$  and  $\mathcal{S}^{(k)} = \tilde{S}^{(k)}$ , then we recover the MCEM [33]. For all methods, we define a random index drawn at iteration  $k$ , noted  $i_k \in [n]$ , and  $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$  as the iteration index where  $i \in [n]$  is last drawn prior to iteration  $k$ . The proposed fitTEM method draws two indices *independently* and uniformly as  $i_k, j_k \in [n]$ . Thus, we define  $t_j^k = \{k' : j_{k'} = j, k' < k\}$  to be the iteration index where the sample  $j \in [n]$  is last drawn as  $j_k$  prior to iteration  $k$  in addition to  $\tau_i^k$  which was defined w.r.t.  $i_k$ . Recall  $\tilde{S}_{i_k}^{(k)} = \frac{1}{M_k} \sum_{m=1}^{M_k} S(z_{i_k,m}^{(k)}, y_{i_k})$

**Table 1** Proxies for the Incremental-step (8)

1: iSAEM	$\mathcal{S}^{(k+1)} = \mathcal{S}^{(k)} + n^{-1} (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)})$
2: vrTTEM	$\mathcal{S}^{(k+1)} = \tilde{S}^{(\ell(k))} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\ell(k))})$
3: fitTEM	$\mathcal{S}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})$
	$\overline{\mathcal{S}}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + n^{-1} (\tilde{S}_{j_k}^{(k)} - \tilde{S}_{j_k}^{(t_{j_k}^k)})$

where  $z_{i_k,m}^{(k)}$  are samples drawn from  $p(z_{i_k} | y_{i_k}; \theta^{(k)})$ . The stepsize in (8) is set to  $\rho_{k+1} = 1$  for the iSAEM method and we initialize with  $\mathcal{S}^{(0)} = \tilde{S}^{(0)}$ ;  $\rho_{k+1} = \rho$  is constant for the vrTTEM and fitTEM methods. Note that we initialize as follows  $\overline{\mathcal{S}}^{(0)} = \tilde{S}^{(0)}$  for the fitTEM which can be seen as a slightly modified version of SAGA inspired by [31]. For vrTTEM we set an epoch size of  $m$  and we define  $\ell(k) := m \lfloor k/m \rfloor$  as the first iteration number in the epoch that iteration  $k$  is in.

122 **Two-Timescale Stochastic EM methods:** We now introduce the general method derived using the  
 123 two variance reduction techniques described above. Algorithm 1 leverages both levels (7) and (8) in  
 124 order to output a vector of fitted parameters  $\hat{\theta}^{(K_m)}$  where  $K_m$  is the total number of iterations.

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**Algorithm 1** Two-Timescale Stochastic EM methods.

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- 1: **Input:**  $\hat{\theta}^{(0)} \leftarrow 0, \hat{s}^{(0)} \leftarrow \tilde{S}^{(0)}, \{\gamma_k\}_{k>0}, \{\rho_k\}_{k>0}$  and  $K_m \in \mathbb{N}^*$ .
- 2: **for**  $k = 0, 1, 2, \dots, K_m - 1$  **do**
- 3:   Draw index  $i_k \in [n]$  uniformly (and  $j_k \in [n]$  for fitTEM).
- 4:   Compute  $\tilde{S}_{i_k}^{(k)}$  using the MC-step (5), for the drawn indices.
- 5:   Compute the surrogate sufficient statistics  $\mathcal{S}^{(k+1)}$  using Lines 1, 2 or 3 in Table 1.
- 6:   Compute  $\tilde{S}^{(k+1)}$  and  $\hat{s}^{(k+1)}$  using respectively (8) and (7):

$$\begin{aligned}\tilde{S}^{(k+1)} &= \tilde{S}^{(k)} + \rho_{k+1}(\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}) \\ \hat{s}^{(k+1)} &= \hat{s}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{s}^{(k)})\end{aligned}\tag{9}$$

- 7:   Compute  $\hat{\theta}^{(k+1)} = \bar{\theta}(\hat{s}^{(k+1)})$  via the M-step.
  - 8: **end for**
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125 The update in (9) is said to have a two-timescale property as the stepsizes satisfy  $\lim_{k \rightarrow \infty} \gamma_k / \rho_k < 1$   
 126 such that  $\tilde{S}^{(k+1)}$  is updated at a faster time-scale, determined by  $\rho_{k+1}$ , than  $\hat{s}^{(k+1)}$ , determined by  
 127  $\gamma_{k+1}$ . The next section introduces the main results of this paper and establishes global and finite-  
 128 time bounds for the three different updates of our scheme.

### 129 3 Finite Time Analysis of the Two-Timescale Scheme

130 Following [8], it can be shown that stationary points of the objective function (1) corresponds to the  
 131 stationary points of the following *nonconvex* Lyapunov function:

$$\min_{\mathbf{s} \in \mathcal{S}} V(\mathbf{s}) := \bar{L}(\bar{\theta}(\mathbf{s})) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\bar{\theta}(\mathbf{s})) + r(\bar{\theta}(\mathbf{s})), \tag{10}$$

132 that we propose to study in this article.

#### 133 3.1 Assumptions and Intermediate Lemmas

134 Several important assumptions required to derive convergence guarantees read as follows:

135 **A1.** *The sets  $\mathcal{Z}, \mathcal{S}$  are compact. There exist constants  $C_S, C_Z$  such that:*

$$C_S := \max_{\mathbf{s}, \mathbf{s}' \in \mathcal{S}} \|\mathbf{s} - \mathbf{s}'\| < \infty, \quad C_Z := \max_{i \in [n]} \int_{\mathcal{Z}} |S(z, y_i)| \mu(dz) < \infty. \tag{11}$$

136 **A2.** *For any  $i \in [n], z \in \mathcal{Z}, \theta, \theta' \in \text{int}(\Theta)^2$ , we have  $|p(z|y_i; \theta) - p(z|y_i; \theta')| \leq L_p \|\theta - \theta'\|$   
 137 where  $\text{int}(\Theta)$  denotes the interior of  $\Theta$ .*

138 We also recall that we consider curved exponential family models assuming the following:

139 **A3.** *For any  $\mathbf{s} \in \mathcal{S}$ , the function  $\theta \mapsto L(\mathbf{s}, \theta) := r(\theta) + \psi(\theta) - \langle \mathbf{s} | \phi(\theta) \rangle$  admits a unique global  
 140 minimum  $\bar{\theta}(\mathbf{s}) \in \text{int}(\Theta)$ . In addition,  $J_{\phi}^{\theta}(\bar{\theta}(\mathbf{s}))$  is full rank,  $L_p$ -Lipschitz and  $\bar{\theta}(\mathbf{s})$  is  $L_t$ -Lipschitz.*

141 We denote by  $H_L^{\theta}(\mathbf{s}, \theta)$  the Hessian (w.r.t to  $\theta$  for a given value of  $\mathbf{s}$ ) of the function  $\theta \mapsto L(\mathbf{s}, \theta) =$   
 142  $r(\theta) + \psi(\theta) - \langle \mathbf{s} | \phi(\theta) \rangle$ , and define  $B(\mathbf{s}) := J_{\phi}^{\theta}(\bar{\theta}(\mathbf{s})) \left( H_L^{\theta}(\mathbf{s}, \bar{\theta}(\mathbf{s})) \right)^{-1} J_{\phi}^{\theta}(\bar{\theta}(\mathbf{s}))^{\top}$ .

143 **A4.** *It holds that  $v_{\max} := \sup_{\mathbf{s} \in \mathcal{S}} \|B(\mathbf{s})\| < \infty$  and  $0 < v_{\min} := \inf_{\mathbf{s} \in \mathcal{S}} \lambda_{\min}(B(\mathbf{s}))$ . There exists  
 144 a constant  $L_b$  such that for all  $\mathbf{s}, \mathbf{s}' \in \mathcal{S}^2$ , we have  $\|B(\mathbf{s}) - B(\mathbf{s}')\| \leq L_b \|\mathbf{s} - \mathbf{s}'\|$ .*

145 The class of algorithms we develop in this paper is composed of two levels where the second stage  
 146 corresponds to the variance reduction trick used in [20] in order to accelerate incremental methods  
 147 and reduce the variance introduced by the index sampling. The first stage is the Robbins-Monro  
 148 update that aims at reducing the Monte Carlo noise of  $\tilde{S}^{(k+1)}$  at iteration  $k$  denoted as follows:

$$\eta_i^{(k)} := \tilde{S}_i^{(k)} - \bar{s}_i(\vartheta^{(k)}) \quad \text{for all } i \in [n] \quad \text{and } k > 0. \tag{12}$$

For instance, we consider that the MC approximation is unbiased if for all  $i \in [n]$  and  $m \in [M]$ , the samples  $z_{i,m} \sim p(z_i|y_i; \theta)$  are i.i.d. under the posterior distribution, i.e.,  $\mathbb{E}[\eta_i^{(k)}|\mathcal{F}_k] = 0$  where  $\mathcal{F}_k$  is the filtration up to iteration  $k$ . The following results are derived under the assumption that the fluctuations implied by the approximation are bounded:

**A5.** For all  $k > 0$ ,  $i \in [n]$ , it holds:  $\mathbb{E}[\|\eta_i^{(k)}\|^2] \leq \infty$  and  $\mathbb{E}[\|\mathbb{E}[\eta_i^{(k)}|\mathcal{F}_k]\|^2] \leq \infty$ .

Note that typically, the controls exhibited above are vanishing when the number of MC samples  $M_k$  increases with  $k$ . We now state two important results on the Lyapunov function; its smoothness:

**Lemma 1.** [20] Assume A1-A4. For all  $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$  and  $i \in [n]$ , we have

$$\|\bar{\mathbf{s}}_i(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \bar{\mathbf{s}}_i(\bar{\boldsymbol{\theta}}(\mathbf{s}'))\| \leq L_s \|\mathbf{s} - \mathbf{s}'\|, \quad \|\nabla V(\mathbf{s}) - \nabla V(\mathbf{s}')\| \leq L_V \|\mathbf{s} - \mathbf{s}'\|, \quad (13)$$

where  $L_s := C_Z L_p L_t$  and  $L_V := v_{\max}(1 + L_s) + L_b C_S$ .

We also establish a growth condition on the gradient of  $V$  related to the mean field of the algorithm:

**Lemma 2.** Assume A3 and A4. For all  $\mathbf{s} \in \mathcal{S}$ ,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2. \quad (14)$$

We present in the following sections a finite-time and global (independent of the initialization) analysis of both the incremental and two-timescale variants our method.

### 3.2 Global Convergence of Incremental Stochastic EM Algorithms

The following result for the iSAEM algorithm is derived under the control of the Monte Carlo fluctuations as described by Assumption A5 and is built upon an intermediary Lemma, characterizing the quantity of interest  $(\hat{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)})$ :

**Lemma 3.** Assume A1. The iSAEM update (1) is equivalent to the following update on the statistics  $\hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \hat{\mathbf{s}}^{(k)})$ . Also:

$$\mathbb{E}[\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] = \mathbb{E}[\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] + \frac{1}{n} \mathbb{E}[\eta_{i_k}^{(k+1)}],$$

where  $\bar{\mathbf{s}}^{(k)}$  is defined by (3) and  $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$ .

Then, the following non-asymptotic convergence rate can be derived for the iSAEM algorithm:

**Theorem 1.** Assume A1-A5. Consider the iSAEM sequence  $\{\hat{\mathbf{s}}^{(k)}\}_{k \geq 0} \in \mathcal{S}$  obtained with  $\rho_{k+1} = 1$  for any  $k \leq K_m$  where  $K_m$  is a positive integer. Let  $\{\gamma_k = 1/(k^a \alpha c_1 \bar{L})\}_{k \geq 0}$ , where  $a \in (0, 1)$ , be a sequence of stepsizes,  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\beta = c_1 \bar{L}/n$ . Then:

$$v_{\max}^{-2} \sum_{k=0}^{K_m} \tilde{\alpha}_k \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})] + \sum_{k=0}^{K_m-1} \tilde{\Gamma}_k \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2].$$

Note that, in Theorem 1, the convergence bound is composed of an initialization term  $V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})$  and suffers from the Monte Carlo noise introduced by the posterior sampling step, see the second term on the RHS of the inequality. We observe, in the next section, that when variance reduction is applied ( $\rho_k < 1$ ), a second phase of convergence will be included in our bounds.

### 3.3 Global Convergence of Two-Timescale Stochastic EM Algorithms

We now deal with the analysis of Algorithm 1 when variance reduction is applied i.e.,  $\rho < 1$ . Two important intermediate Lemmas are needed in order to establish finite-time bounds for the vrTTEM and the fitTEM methods. We first derive an identity for the drift term of the vrTTEM :

**Lemma 4.** Consider the vrTTEM update (2) with  $\rho_k = \rho$ , it holds for all  $k > 0$

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &\quad + 2(1 - \rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

where we recall that  $\ell(k)$  is the first iteration number in the epoch that iteration  $k$  is in.

183 The second one derives an identity for the quantity  $\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2]$  using the fitTEM update:

184 **Lemma 5.** Consider the fitTEM update (3) with  $\rho_k = \rho$ . It holds for all  $k > 0$  that

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + 2(1 - \rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

185 where  $L_s$  is the smoothness constant defined in Lemma 1.

186 Let  $K$  be an independent discrete r.v. drawn from  $\{1, \dots, K_m\}$  with distribution  $\{\gamma_{k+1}/P_m\}_{k=0}^{K_m-1}$ ,  
187 then, for any  $K_m > 0$ , the convergence criterion used in our study reads

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] = \frac{1}{P_m} \sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2],$$

188 where  $P_m = \sum_{\ell=0}^{K_m-1} \gamma_\ell$  and the expectation is over the stochasticity of the algorithm. Denote  
189  $\Delta V = V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})$ . We now state the main result regarding the vrTTEM method:

190 **Theorem 2.** Assume A1-A5. Consider the vrTTEM sequence  $\{\hat{\mathbf{s}}^{(k)}\}_{k>0} \in \mathcal{S}$  for any  $k \leq K_m$  where  
191  $K_m$  is a positive integer. Let  $\{\gamma_{k+1} = 1/(k^a \bar{L})\}_{k>0}$ , where  $a \in (0, 1)$ , be a sequence of stepsizes,  
192  $\bar{L} = \max\{L_s, L_V\}$ ,  $\rho = \mu/(c_1 \bar{L} n^{2/3})$ ,  $m = nc_1^2/(2\mu^2 + \mu c_1^2)$  and a constant  $\mu \in (0, 1)$ . Then:

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] \leq \frac{2n^{2/3} \bar{L}}{\mu P_m v_{\min}^2 v_{\max}^2} \left( \mathbb{E}[\Delta V] + \sum_{k=0}^{K_m-1} \tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \right).$$

193 Furthermore, the fitTEM method has the following convergence rate:

194 **Theorem 3.** Assume A1-A5. Consider the fitTEM sequence  $\{\hat{\mathbf{s}}^{(k)}\}_{k>0} \in \mathcal{S}$  for any  $k \leq K_m$  where  
195  $K_m$  be a positive integer. Let  $\{\gamma_{k+1} = 1/(k^a \alpha c_1 \bar{L})\}_{k>0}$ , where  $a \in (0, 1)$ , be a sequence of  
196 positive stepsizes,  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\beta = 1/(\alpha n)$ ,  $\rho = 1/(\alpha c_1 \bar{L} n^{2/3})$   
197 and  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ . Then:

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] \leq \frac{4\alpha \bar{L} n^{2/3}}{P_m v_{\min}^2 v_{\max}^2} \left( \mathbb{E}[\Delta V] + \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \Gamma^{(k+1)} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \right).$$

198 Note that in those two bounds, the quantities  $\tilde{\eta}^{(k+1)}$  and  $\Xi^{(k+1)}$  depend only on the Monte Carlo  
199 noises  $\mathbb{E}[\|\eta_{i_k}^{(k)}\|^2]$ ,  $\mathbb{E}[\|\mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r]\|^2]$ , bounded under Assumption A5, and some constants.

200 *Remarks:* Theorem 2 and Theorem 3 exhibit in their convergence bounds *two different phases*. The  
201 upper bounds display a *bias term* due to the initial conditions, i.e., the term  $\Delta V$ , and a *double*  
202 *dynamic* burden exemplified by the term  $\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]$ . Indeed, the following remarks are  
203 worth doing on this quantity: (i) This term is the price we pay for the two-timescale dynamic and  
204 corresponds to the gap between the two *asynchronous* updates (one on  $\hat{\mathbf{s}}^{(k)}$  and the other on  $\tilde{S}^{(k)}$ ).  
205 (ii) It is readily understood that if  $\rho = 1$ , i.e., there is no variance reduction, then for any  $k > 0$

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] = \mathbb{E}[\|\mathbf{S}^{(k+1)} - \tilde{S}^{(k+1)}\|^2] = 0 \quad \text{with} \quad \hat{\mathbf{s}}^{(0)} = \tilde{S}^{(0)} = 0,$$

206 which strengthen the fact that this quantity characterizes the impact of the variance reduction tech-  
207 nique introduced in our class of methods. The following Lemma characterizes this gap:

208 **Lemma 6.** Considering a decreasing stepsize  $\gamma_k \in (0, 1)$  and a constant  $\rho \in (0, 1)$ , we have

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \leq \frac{\rho}{1 - \rho} \sum_{\ell=0}^k (1 - \gamma_\ell)^2 (\mathbf{S}^{(\ell)} - \tilde{S}^{(\ell)}),$$

209 where  $\mathbf{S}^{(k)}$  is defined either by Line 2 (vrTTEM) or Line 3 (fitTEM).



## 4 Numerical Examples

This section presents several numerical applications for our proposed class of Algorithms 1.

### 4.1 Gaussian Mixture Models

We begin by a simple and illustrative example. The authors acknowledge that the following model can be trained using deterministic EM-type of algorithms but propose to apply stochastic methods, including theirs, and to compare their performances. Given  $n$  observations  $\{y_i\}_{i=1}^n$ , we want to fit a Gaussian Mixture Model (GMM) whose distribution is modeled as a Gaussian mixture of  $M$  components, each with a unit variance. Let  $z_i \in [M]$  be the latent labels of each component, the complete log-likelihood is defined as:

$$\log f(z_i, y_i; \theta) = \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) [\log(\omega_m) - \mu_m^2/2] + \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) \mu_m y_i + \text{constant}.$$

where  $\theta := (\omega, \mu)$  with  $\omega = \{\omega_m\}_{m=1}^{M-1}$  are the mixing weights with the convention  $\omega_M = 1 - \sum_{m=1}^{M-1} \omega_m$  and  $\mu = \{\mu_m\}_{m=1}^M$  are the means. We use the penalization  $r(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \log \text{Dir}(\omega; M, \epsilon)$  where  $\delta > 0$  and  $\text{Dir}(\cdot; M, \epsilon)$  is the  $M$  dimensional symmetric Dirichlet distribution with concentration parameter  $\epsilon > 0$ . The constraint set is given by  $\Theta = \{\omega_m, m = 1, \dots, M-1 : \omega_m \geq 0, \sum_{m=1}^{M-1} \omega_m \leq 1\} \times \{\mu_m \in \mathbb{R}, m = 1, \dots, M\}$ . In the following experiments on synthetic data, we generate 30 synthetic datasets of size  $n = 10^5$  from a GMM model with  $M = 2$  components with two mixtures with means  $\mu_1 = -\mu_2 = 0.5$ . We run the EM method until convergence (to double precision) to obtain the ML estimate  $\mu^*$  averaged on 50 datasets. We compare the EM, iEM, SAEM, iSAEM, vrTTEM and fitTTEM methods in terms of their precision measured by  $|\mu - \mu^*|^2$ . We set the stepsize of the SA-step of all method as  $\gamma_k = 1/k^\alpha$  with  $\alpha = 0.5$ , and the stepsize  $\rho_k$  for vrTTEM and the fitTTEM to a constant stepsize equal to  $1/n^{2/3}$ . The number of MC samples is fixed to  $M = 10$  chains. Figure 1 shows the precision  $|\mu - \mu^*|^2$  for the different methods against the epoch(s) elapsed (one epoch equals  $n$  iterations). vrTTEM and fitTTEM methods outperform the other stochastic methods, supporting the benefits of our scheme.

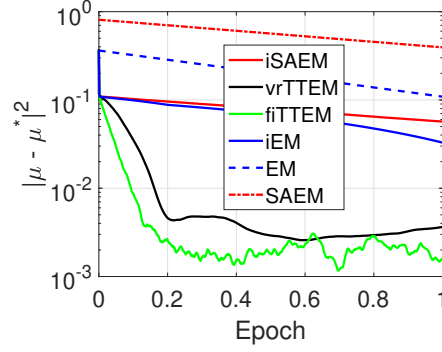


Figure 1: Precision  $|\mu^{(k)} - \mu^*|^2$  per epoch

### 4.2 Deformable Template Model for Image Analysis

Let  $(y_i, i \in [n])$  be observed gray level images defined on a grid of pixels. Let  $u \in \mathcal{U} \subset \mathbb{R}^2$  denotes the pixel index on the image and  $x_u \in \mathcal{D} \subset \mathbb{R}^2$  its location. The model used in this experiment suggests that each image  $y_i$  is a deformation of a template, noted  $I : \mathcal{D} \rightarrow \mathbb{R}$ , common to all images of the dataset:

$$y_i(u) = I(x_u - \Phi_i(x_u, z_i)) + \varepsilon_i(u) \quad (15)$$

where  $\phi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a deformation function,  $z_i$  some latent variable parameterizing this deformation and  $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$  is an observation error. The template model, given  $\{p_k\}_{k=1}^{k_p}$  landmarks on the template, a fixed known kernel  $\mathbf{K}_p$  and a vector of parameters  $\beta \in \mathbb{R}^{k_p}$  is defined as follows:

$$I_\xi = \mathbf{K}_p \beta, \quad \text{where} \quad (\mathbf{K}_p \beta)(x) = \sum_{k=1}^{k_p} \mathbf{K}_p(x, p_k) \beta_k.$$

Given a set of landmarks  $\{g_k\}_{k=1}^{k_g}$  and a fixed kernel  $\mathbf{K}_g$ , we parameterize the deformation  $\Phi_i$  as:

$$\Phi_i = \mathbf{K}_g z_i \quad \text{where} \quad (\mathbf{K}_g z_i)(x) = \sum_{k=1}^{k_g} \mathbf{K}_g(x, g_k) \left( z_i^{(1)}(k), z_i^{(2)}(k) \right),$$

where we put a Gaussian prior on the latent variables,  $z_i \sim \mathcal{N}(0, \Gamma)$  and  $z_i \in (\mathbb{R}^{k_g})^2$ . The vector of parameters we estimate is thus  $\theta = (\beta, \Gamma, \sigma)$ . The complete model (15) belongs to the curved exponential family, see [2], which vector of sufficient statistics for all  $i \in [n]$  is defined by  $S(y_i, z_i) = (\mathbf{K}_{p, z_i}^\top y_i, \mathbf{K}_{p, z_i}^\top \mathbf{K}_{p, z_i} z_i)$  where we denote  $\mathbf{K}_{p, z_i} = \mathbf{K}_{p, z_i}(x_u - \phi_i(x_u, z_i), p_j)$ . Then, the Two-Timescale M-step yields the following parameter

updates  $\bar{\theta}(\hat{s}) = (\beta(\hat{s}) = \hat{s}_2^{-1}(z)\hat{s}_1(z), \Gamma(\hat{s}) = \hat{s}_3(z)/n, \sigma(\hat{s}) = \beta(\hat{s})^\top \hat{s}_2(z)\beta(\hat{s}) - 2\beta(\hat{s})\hat{s}_1(z))$   
 where  $\hat{s} = (\hat{s}_1(z), \hat{s}_2(z), \hat{s}_3(z))$  is the vector of statistics obtained via (9) in Algorithm 1.

**Numerical Experiment:** We apply model (15) and our Algorithm 1 to a collection of handwritten digits, called the US postal database [17], featuring  $n = 1000$  ( $16 \times 16$ )-pixel images for each class of digits from 0 to 9. The main difficulty with these data comes from the geometric dispersion within each class of digit as shown Figure 2 for digit 5. We thus ought to use our deformable template model (15) in order to account for both sources of variability: the intrinsic template to each class of digit and the small and local deformation in each observed image.



Figure 2: Training set of the USPS database (20 images for digit 5)

Figure 3 shows the resulting synthetic images for digit 5 through several epochs, for the batch method, the online SAEM, the incremental SAEM and the various TTS methods. For all methods, the initialization of the template (16) is the mean of the gray level images. In our experiments, we have chosen Gaussian kernels for both,  $\mathbf{K}_p$  and  $\mathbf{K}_g$ , defined on  $\mathbb{R}^2$  and centered on the landmark points  $\{p_k\}_{k=1}^{k_p}$  and  $\{g_k\}_{k=1}^{k_g}$  with standard respective standard deviations of 0.12 and 0.3. We set  $k_p = 15$  and  $k_g = 6$  equidistributed landmarks points on the grid for the training procedure. Those hyperparameters are inspired by relevant studies [1, 3]. In particular, the choice of the geometric covariance, indexed by  $g$ , in such study is critical since it has a direct impact on the *sharpness* of the templates. As for the photometric hyperparameter, indexed by  $p$ , both the template and the geometry are impacted, in the sense that with a large photometric variance, the kernel centered on one landmark *spreads out* to many of its neighbors.

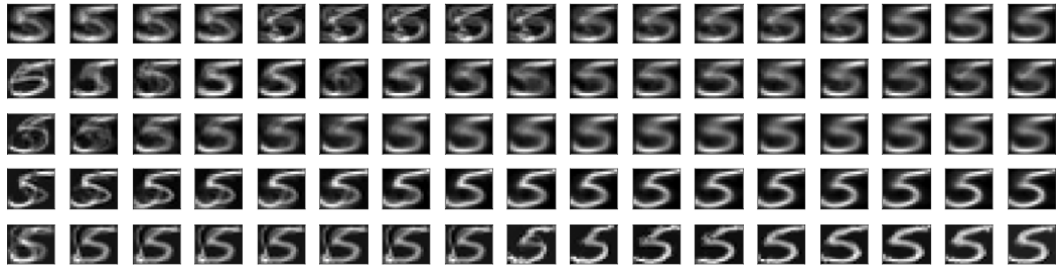


Figure 3: (USPS Digits) Estimation of the template. From top to bottom: batch, online, iSAEM, vrT-TEM and fitTEM through 7 epochs. Note that Batch method templates are replicated in-between epochs for a fair comparison with incremental variants.

As the iterations proceed, the templates become sharper. Figure 3 displays the virtue of the vrTTEM and fitTEM methods that obtain a more *contrasted* and *accurate* template estimate. The incremental and online version are looking much better on the very first epochs compared to the batch method, which is intuitive given the high computational cost of the latter. After a few epochs, the batch SAEM estimates similar template as the incremental and online methods due to their high variance. Our variance reduced and fast incremental variants are effective in the long run and sharpen the final template estimates contrasting between the background and the regions of interest in the image.

## 5 Conclusion

This paper introduces a new class of two-timescale EM methods for learning latent variable models. In particular, the models dealt with in this paper belong to the curved exponential family and are possibly nonconvex. The nonconvexity of the problem is tackled using a Robbins-Monro type of update, which represents the *first level* of our class of methods. The scalability with the number of samples is performed through a variance reduced and incremental update, the *second* and last level of our newly introduced scheme. The various algorithms are interpreted as scaled gradient methods, in the space of the sufficient statistics, and our convergence results are *global*, in the sense of independence of the initial values, and *non-asymptotic*, *i.e.*, true for any random termination number. Numerical examples illustrate the benefits of our scheme on synthetic and real tasks.



## 6 Broader Impact

Our work aims at improving training procedures for latent data models. Latent data models are particularly interesting in several impactful domains such as sociology, economy or pharmacology. Indeed, in those latter domains, a special instance of latent data models, namely mixed-effect models, can be employed. It considers a latent structure in order to take into account the variability between subjects, which can be individuals, companies or patients. In that case, our class of algorithms becomes useful as demonstrated in the additional experiment in the supplementary material. It is worth noting that other types of latent variable models could be used for impactful research, such as the missing data framework when the considered observations are sensible, yet missing.

## References

- [1] Stéphanie Allasonnière and Estelle Kuhn. Stochastic algorithm for parameter estimation for dense deformable template mixture model. *arXiv preprint arXiv:0802.1521*, 2008.
- [2] Stéphanie Allasonnière, Yali Amit, and Alain Trounev. Towards a coherent statistical framework for dense deformable template estimation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 69(1):3–29, 2007.
- [3] Stéphanie Allasonnière, Estelle Kuhn, Alain Trounev, et al. Construction of bayesian deformable models via a stochastic approximation algorithm: a convergence study. *Bernoulli*, 16(3):641–678, 2010.
- [4] Charlotte Baey, Samis Trevezas, and Paul-Henry Cournède. A non linear mixed effects model of plant growth and estimation via stochastic variants of the em algorithm. *Communications in Statistics-Theory and Methods*, 45(6):1643–1669, 2016.
- [5] David M. Blei, Alp Kucukelbir, and Jon D. McAuliffe. Variational Inference: A Review for Statisticians. *Journal of the American statistical Association*, 112(518):859–877, JUN 2017. ISSN 0162-1459. doi: {10.1080/01621459.2017.1285773}.
- [6] Steve Brooks, Andrew Gelman, Galin Jones, and Xiao-Li Meng. *Handbook of markov chain monte carlo*. CRC press, 2011.
- [7] Olivier Cappé. Online EM algorithm for hidden markov models. *Journal of Computational and Graphical Statistics*, 20(3):728–749, 2011.
- [8] Olivier Cappé and Eric Moulines. On-line expectation–maximization algorithm for latent data models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 71(3): 593–613, 2009.
- [9] Bradley P Carlin and Siddhartha Chib. Bayesian model choice via markov chain monte carlo methods. *Journal of the Royal Statistical Society: Series B (Methodological)*, 57(3):473–484, 1995.
- [10] Arindom Chakraborty and Kalyan Das. Inferences for joint modelling of repeated ordinal scores and time to event data. *Computational and mathematical methods in medicine*, 11(3): 281–295, 2010.
- [11] Jianfei Chen, Jun Zhu, Yee Whye Teh, and Tong Zhang. Stochastic expectation maximization with variance reduction. In *Advances in Neural Information Processing Systems*, pages 7978–7988, 2018.
- [12] Bernard Delyon, Marc Lavielle, and Eric Moulines. Convergence of a stochastic approximation version of the em algorithm. *Ann. Statist.*, 27(1):94–128, 03 1999. doi: 10.1214/aos/1018031103. URL <https://doi.org/10.1214/aos/1018031103>.

- [13] Arthur P Dempster, Nan M Laird, and Donald B Rubin. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the royal statistical society. Series B (methodological)*, pages 1–38, 1977.
- [14] Bradley Efron et al. Defining the curvature of a statistical problem (with applications to second order efficiency). *The Annals of Statistics*, 3(6):1189–1242, 1975.
- [15] Saeed Ghadimi and Guanghui Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013.
- [16] James P Hughes. Mixed effects models with censored data with application to hiv rna levels. *Biometrics*, 55(2):625–629, 1999.
- [17] Jonathan J. Hull. A database for handwritten text recognition research. *IEEE Transactions on pattern analysis and machine intelligence*, 16(5):550–554, 1994.
- [18] Prateek Jain and Purushottam Kar. Non-convex optimization for machine learning. *arXiv preprint arXiv:1712.07897*, 2017.
- [19] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *Advances in neural information processing systems*, pages 315–323, 2013.
- [20] Belhal Karimi, Hoi-To Wai, Éric Moulines, and Marc Lavielle. On the global convergence of (fast) incremental expectation maximization methods. In *Advances in Neural Information Processing Systems*, pages 2833–2843, 2019.
- [21] Estelle Kuhn and Marc Lavielle. Coupling a stochastic approximation version of em with an mcmc procedure. *ESAIM: Probability and Statistics*, 8:115–131, 2004.
- [22] Estelle Kuhn, Catherine Matias, and Tabea Rebafka. Properties of the stochastic approximation em algorithm with mini-batch sampling. *arXiv preprint arXiv:1907.09164*, 2019.
- [23] Percy Liang and Dan Klein. Online em for unsupervised models. In *Proceedings of human language technologies: The 2009 annual conference of the North American chapter of the association for computational linguistics*, pages 611–619, 2009.
- [24] Florian Maire, Eric Moulines, and Sidonie Lefebvre. Online em for functional data, 2016. URL <http://arxiv.org/abs/1604.00570>. cite arxiv:1604.00570v1.pdf.
- [25] Charles E McCulloch. Maximum likelihood algorithms for generalized linear mixed models. *Journal of the American statistical Association*, 92(437):162–170, 1997.
- [26] Geoffrey McLachlan and Thriyambakam Krishnan. *The EM algorithm and extensions*, volume 382. John Wiley & Sons, 2007.
- [27] Sean P Meyn and Richard L Tweedie. *Markov chains and stochastic stability*. Springer Science & Business Media, 2012.
- [28] Radford M Neal and Geoffrey E Hinton. A view of the EM algorithm that justifies incremental, sparse, and other variants. In *Learning in graphical models*, pages 355–368. Springer, 1998.
- [29] SK Ng and GJ McLachlan. On the choice of the number of blocks with the incremental EM algorithm for the fitting of normal mixtures. *Statistics and Computing*, 13(1):45–55, FEB 2003. ISSN 0960-3174. doi: {10.1023/A:1021987710829}.
- [30] Hien D Nguyen, Florence Forbes, and Geoffrey J McLachlan. Mini-batch learning of exponential family finite mixture models. *Statistics and Computing*, pages 1–18, 2020.

- 372 [31] Sashank J Reddi, Suvrit Sra, Barnabás Póczos, and Alex Smola. Fast incremental method for  
373 nonconvex optimization. *arXiv preprint arXiv:1603.06159*, 2016.
- 374 [32] Herbert Robbins and Sutton Monro. A stochastic approximation method. *The annals of math-*  
375 *ematical statistics*, pages 400–407, 1951.
- 376 [33] Greg CG Wei and Martin A Tanner. A monte carlo implementation of the em algorithm and  
377 the poor man’s data augmentation algorithms. *Journal of the American statistical Association*,  
378 85(411):699–704, 1990.
- 379 [34] CF Jeff Wu. On the convergence properties of the em algorithm. *The Annals of statistics*,  
380 pages 95–103, 1983.

## A Proofs for the iSAEM Algorithm

### A.1 Proof of Lemma 2

**Lemma.** Assume A3, A4. For all  $\mathbf{s} \in S$ ,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (16)$$

**Proof** Using A3 and the fact that we can exchange integration with differentiation and the Fisher's identity, we obtain

$$\begin{aligned} \nabla_{\mathbf{s}} V(\mathbf{s}) &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left( \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{L}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left( \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} (\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))), \end{aligned} \quad (17)$$

Consider the following vector map:

$$\mathbf{s} \rightarrow \nabla_{\boldsymbol{\theta}} L(\mathbf{s}, \boldsymbol{\theta})|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})} = \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \mathbf{s}.$$

Taking the gradient of the above map w.r.t.  $\mathbf{s}$  and using assumption A3, we show that:

$$\mathbf{0} = -\mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \underbrace{\left( \nabla_{\boldsymbol{\theta}}^2 (\psi(\boldsymbol{\theta}) + \mathbf{r}(\boldsymbol{\theta}) - \langle \phi(\boldsymbol{\theta}) | \mathbf{s} \rangle) \right)|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})}}_{=\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \boldsymbol{\theta})} \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s}).$$

The above yields

$$\nabla_{\mathbf{s}} V(\mathbf{s}) = \mathbf{B}(\mathbf{s})(\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))),$$

where we recall  $\mathbf{B}(\mathbf{s}) = \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \left( \mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \bar{\boldsymbol{\theta}}(\mathbf{s})) \right)^{-1} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top}$ . The proof of (16) follows directly from the assumption A4.  $\square$

### A.2 Proof of Theorem 1

Beforehand, We present two intermediary Lemmas important for the analysis of the incremental update of the iSAEM algorithm. The first one gives a characterization of the quantity  $\mathbb{E}[\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}]$ :

**Lemma.** Assume A1. The update (1) is equivalent to the following update on the resulting statistics

$$\hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}).$$

Also:

$$\mathbb{E}[\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] = \mathbb{E}[\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] + \frac{1}{n} \mathbb{E}[\eta_{i_k}^{(k+1)}],$$

where  $\bar{\mathbf{s}}^{(k)}$  is defined by (3) and  $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$ .

**Proof** From update (1), we have:

$$\begin{aligned} \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} + \frac{1}{n} \left( \tilde{S}_{i_k}^{(k+1)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)} \right) \\ &= \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{S}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left( \tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \tilde{S}_{i_k}^{(k+1)} \right). \end{aligned}$$

Since  $\tilde{S}_{i_k}^{(k+1)} = \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) + \eta_{i_k}^{(k+1)}$  we have

$$\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{S}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left( \tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) \right) + \frac{1}{n} \eta_{i_k}^{(k+1)}.$$

401 Taking the full expectation of both side of the equation leads to:

$$\begin{aligned}\mathbb{E}[\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] &= \mathbb{E}[\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right] \\ &\quad - \frac{1}{n} \mathbb{E}[\mathbb{E}[\tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k]] + \frac{1}{n} \mathbb{E}[\eta_{i_k}^{(k+1)}] .\end{aligned}$$

402 Since we have  $\mathbb{E}[\tilde{S}_i^{(\tau_i^k)} | \mathcal{F}_k] = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)}$  and  $\mathbb{E}[\bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k] = \bar{\mathbf{s}}^{(k)}$ , we conclude the proof  
403 of the Lemma.  $\square$

404 We also derived the following auxiliary Lemma which sets an upper bound for the quantity  
405  $\mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2]$ :

406 **Lemma 7.** For any  $k \geq 0$  and consider the iSAEM update in (1), it holds that

$$\begin{aligned}\mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] &\leq 4\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{2\mathbf{L}_s^2}{n^3} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + 2\frac{c_\eta}{M_k} + 4\mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] .\end{aligned}$$

407 **Proof** Applying the iSAEM update yields:

$$\begin{aligned}\mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] &= \mathbb{E}[\|\tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} - \frac{1}{n}(\tilde{S}_{i_k}^{(\tau_i^k)} - \tilde{S}_{i_k}^{(t_i^k)})\|^2] \\ &\leq 4\mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] + 4\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \frac{2}{n^2} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_i^k)}\|^2] + 2\frac{c_\eta}{M_k} .\end{aligned}$$

408 The last expectation can be further bounded by

$$\frac{2}{n^2} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_i^k)}\|^2] = \frac{2}{n^3} \sum_{i=1}^n \mathbb{E}[\|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2] \stackrel{(a)}{\leq} \frac{2\mathbf{L}_s^2}{n^3} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] ,$$

409 where (a) is due to Lemma 1 and which concludes the proof of the Lemma.

410  $\square$

411 **Theorem.** Assume A1-A5. Consider the iSAEM sequence  $\{\hat{\mathbf{s}}^{(k)}\}_{k \geq 0} \in \mathcal{S}$  obtained with  $\rho_{k+1} = 1$   
412 for any  $k \leq K_m$  where  $K_m$  is a positive integer. Let  $\{\gamma_k = 1/(k^a \alpha c_1 \bar{L})\}_{k \geq 0}$ , where  $a \in (0, 1)$ , be a  
413 sequence of stepsizes,  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{\mathbf{L}_s, \mathbf{L}_V\}$ ,  $\beta = c_1 \bar{L}/n$ . Then:

$$v_{\max}^{-2} \sum_{k=0}^{K_m} \tilde{\alpha}_k \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})] + \sum_{k=0}^{K_m-1} \tilde{\Gamma}_k \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] .$$

414 **Proof** Under the smoothness of the Lyapunov function  $V$  (cf. Lemma 1), we can write:

$$V(\hat{\mathbf{s}}^{(k+1)}) \leq V(\hat{\mathbf{s}}^{(k)}) + \gamma_{k+1} \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 \mathbf{L}_V}{2} \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 .$$

415 Taking the expectation on both sides yields:

$$\begin{aligned}\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] + \gamma_{k+1} \mathbb{E}[\langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad + \frac{\gamma_{k+1}^2 \mathbf{L}_V}{2} \mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] .\end{aligned}$$

416 Using Lemma 3, we obtain:

$$\begin{aligned}
& \mathbb{E} \left[ \langle \tilde{S}^{(k+1)} - \hat{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] \\
&= \mathbb{E} \left[ \langle \bar{s}^{(k)} - \hat{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \right\rangle \right] \\
&\quad + \frac{1}{n} \mathbb{E} \left[ \langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] \\
&\stackrel{(a)}{\leq} -v_{\min} \mathbb{E}[\|\bar{s}^{(k)} - \hat{s}^{(k)}\|^2] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[ \left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \right\rangle \right] \\
&\quad + \frac{1}{n} \mathbb{E} \left[ \langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{s}^{(k)}) \rangle \right] \\
&\stackrel{(b)}{\leq} -v_{\min} \mathbb{E}[\|\bar{s}^{(k)} - \hat{s}^{(k)}\|^2] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] \\
&\quad + \frac{\beta(n-1)+1}{2n} \mathbb{E} \left[ \left\| \nabla V(\hat{s}^{(k)}) \right\|^2 \right] + \frac{1}{2n} \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \\
&\stackrel{(a)}{\leq} \left( v_{\max}^2 \frac{\beta(n-1)+1}{2n} - v_{\min} \right) \mathbb{E}[\|\bar{s}^{(k)} - \hat{s}^{(k)}\|^2] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] \\
&\quad + \frac{1}{2n} \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2],
\end{aligned}$$

417 where (a) is due to the growth condition (2) and (b) is due to Young's inequality (with  $\beta \rightarrow 1$ ). Note

418  $a_k = \gamma_{k+1} \left( v_{\min} - v_{\max}^2 \frac{\beta(n-1)+1}{2n} \right)$  and

$$\begin{aligned}
a_k \mathbb{E}[\|\bar{s}^{(k)} - \hat{s}^{(k)}\|^2] &\leq \mathbb{E} \left[ V(\hat{s}^{(k)}) - V(\hat{s}^{(k+1)}) \right] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} \left[ \|\tilde{S}^{(k+1)} - \hat{s}^{(k)}\|^2 \right] \\
&\quad + \frac{\gamma_{k+1}(1 - \frac{1}{n})}{2\beta} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] + \frac{\gamma_{k+1}}{2n} \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2].
\end{aligned} \tag{18}$$

419 We now give an upper bound of  $\mathbb{E} \left[ \|\tilde{S}^{(k+1)} - \hat{s}^{(k)}\|^2 \right]$  using Lemma 7 and plug it into (18):

$$\begin{aligned}
& (a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E}[\|\bar{s}^{(k)} - \hat{s}^{(k)}\|^2] \\
&\leq \mathbb{E} \left[ V(\hat{s}^{(k)}) - V(\hat{s}^{(k+1)}) \right] \\
&\quad + \gamma_{k+1} \left( \frac{1}{2\beta} \left(1 - \frac{1}{n}\right) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{s}^{(k)} \right\|^2 \right] \\
&\quad + \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \\
&\quad + \frac{\gamma_{k+1}^2 L_V L_s^2}{n^3} \sum_{i=1}^n \mathbb{E}[\|\hat{s}^{(k)} - \hat{s}^{(\tau_i^k)}\|^2].
\end{aligned} \tag{19}$$

420 Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{s}^{(k+1)} - \hat{s}^{(\tau_i^{k+1})}\|^2] = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \mathbb{E}[\|\hat{s}^{(k+1)} - \hat{s}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E}[\|\hat{s}^{(k+1)} - \hat{s}^{(\tau_i^k)}\|^2] \right),$$



421 where the equality holds as  $i_k$  and  $j_k$  are drawn independently. For any  $\beta > 0$ , it holds

$$\begin{aligned}
& \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\
&= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\
&= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 - 2\gamma_{k+1}\langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\
&\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 + \gamma_{k+1}\beta\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right],
\end{aligned}$$

422 where the last inequality is due to Young's inequality. Subsequently, we have

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\
&\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}\left[\left(1 + \gamma_{k+1}\beta\right)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2\right].
\end{aligned}$$

423 Observe that  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)})$ . Applying Lemma 7 yields

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\
&\leq \left(\gamma_{k+1}^2 + \frac{n-1}{n} \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right] \\
&\leq 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \\
&\quad + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] \\
&\quad + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2\right].
\end{aligned}$$

424 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2].$$

425 From the above, we get

$$\begin{aligned}
\Delta^{(k+1)} &\leq \left(1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})\right) \Delta^{(k)} + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
&\quad + 2\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right].
\end{aligned}$$

426 Setting  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{8, 1 + 6v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$ ,  $\beta = \frac{c_1 \bar{L}}{n}$ ,

427  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 6$ ,  $\alpha \geq 8$ , we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \leq 1 - \frac{c_1(k\alpha - 1) - 4}{k\alpha n c_1} \leq 1 - \frac{2}{k\alpha n c_1},$$

428 which shows that  $1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \in (0, 1)$  for any  $k > 0$ . Denote  $\Lambda_{(k+1)} =$   
 429  $\frac{1}{n} - \gamma_{k+1}\beta - \frac{2\gamma_{k+1}L_{\mathbf{s}}^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})$  and note that  $\Delta^{(0)} = 0$ , thus the telescoping sum yields:

$$\begin{aligned} \Delta^{(k+1)} &\leq 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] \\ &\quad + 2 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E} \left[ \left\| \eta_{i_\ell}^{(\ell)} \right\|^2 \right] \\ &\quad + 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 \\ &\quad + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^\ell)} - \bar{\mathbf{s}}^{(\ell)} \right\|^2 \right]. \end{aligned}$$

430 Note  $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$  Summing on both sides over  $k = 0$  to  $k = K_m - 1$  yields:

$$\begin{aligned} &\sum_{k=0}^{K_m-1} \Delta^{(k+1)} \\ &= 4 \sum_{k=0}^{K_m-1} (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2 \sum_{k=0}^{K_m-1} (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E} \left[ \left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\ &\quad + \sum_{k=0}^{K_m-1} 4 (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \tag{20} \\ &\leq \sum_{k=0}^{K_m-1} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{k=0}^{K_m-1} \frac{2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E} \left[ \left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\ &\quad + \sum_{k=0}^{K_m-1} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right]. \end{aligned}$$

431 We recall (19) where we have summed on both sides from  $k = 0$  to  $k = K_m - 1$ :

$$\begin{aligned} &\sum_{k=0}^{K_m-1} (a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] \\ &\quad + \sum_{k=0}^{K_m-1} \gamma_{k+1} \left( \frac{1}{2\beta} (1 - \frac{1}{n}) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \tag{21} \\ &\quad + \sum_{k=0}^{K_m-1} \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \\ &\quad + \sum_{k=0}^{K_m-1} \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^2} \Delta^{(k)}. \end{aligned}$$

432 Plugging (20) into (21) results in:

$$\begin{aligned} & \sum_{k=0}^{K_m-1} \tilde{\alpha}_k \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{k=0}^{K_m-1} \tilde{\beta}_k \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ & \leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] + \sum_{k=0}^{K_m-1} \tilde{\Gamma}_k \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2], \end{aligned}$$

433 where

$$\begin{aligned} \tilde{\alpha}_k &= a_k - 2\gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}}, \\ \tilde{\beta}_k &= \gamma_{k+1} \left( \frac{1}{2\beta} \left(1 - \frac{1}{n}\right) + 2\gamma_{k+1} L_V \right) - \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}}, \\ \tilde{\Gamma}_k &= \gamma_{k+1} \left( \gamma_{k+1} L_V + \frac{1}{2n} \right) + \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}}, \end{aligned}$$

434 and

$$\begin{aligned} a_k &= \gamma_{k+1} \left( v_{\min} - v_{\max}^2 \frac{\beta(n-1) + 1}{2n} \right), \\ \Lambda_{(k+1)} &= \frac{1}{n} - \gamma_{k+1} \beta - \frac{2\gamma_{k+1} L_s^2}{n^2} \left( \gamma_{k+1} + \frac{1}{\beta} \right), \\ c_1 &= v_{\min}^{-1}, \alpha = \max\{8, 1 + 6v_{\min}\}, \bar{L} = \max\{L_s, L_V\}, \gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}, \beta = \frac{c_1 \bar{L}}{n}. \end{aligned}$$

435 When, for any  $k > 0$ ,  $\tilde{\alpha}_k \geq 0$ , we have by Lemma 2 that:

$$\sum_{k=0}^{K_m} \tilde{\alpha}_k \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq v_{\max}^2 \sum_{k=0}^{K_m} \tilde{\alpha}_k \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2],$$

436 which yields an upper bound of the gradient of the Lyapunov function  $V$  along the path of the  
437 iSAEM update and concludes the proof of the Theorem.  $\square$

## 438 B Proofs for the vrTTEM and the fiTTEM Algorithms

### 439 B.1 Proofs of Auxiliary Lemmas ( Lemma 4, Lemma 5 and Lemma 6)

440 **Lemma.** Consider the vrTTEM update (2) with  $\rho_k = \rho$ , it holds for all  $k > 0$

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &\quad + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

441 where we recall that  $\ell(k)$  is the first iteration number in the epoch that iteration  $k$  is in.

442 **Proof** Beforehand, we provide a rewriting of the quantity  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$  that will be useful through-  
443 out this proof:

$$\begin{aligned} \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) \\ &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathcal{S}^{(k+1)}) \\ &= -\gamma_{k+1} \left( (1-\rho) \left[ \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[ \hat{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)} \right] \right). \end{aligned} \quad (22)$$

444 We observe, using the identity (22), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)}\|^2] + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)}\|^2]. \quad (23)$$

445 For the latter term, we obtain its upper bound as

$$\begin{aligned} &\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)}\|^2] \\ &= \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \tilde{S}_i^{\ell(k)}) - (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{\ell(k)}) \right\|^2 \right] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{\ell(k)}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \stackrel{(b)}{\leq} L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

446 where (a) uses the variance inequality and (b) uses Lemma 1. Substituting into (23) proves the  
447 lemma.  $\square$

448 **Lemma.** Consider the fiTTEM update (3) with  $\rho_k = \rho$ . It holds for all  $k > 0$  that

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

449 where  $L_s$  is the smoothness constant defined in Lemma 1.

450 **Proof** Beforehand, we provide a rewriting of the quantity  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$  that will be useful through-  
451 out this proof:

$$\begin{aligned} \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) \\ &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathcal{S}^{(k+1)}) \\ &= -\gamma_{k+1} \left( (1-\rho) \left[ \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[ \hat{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)} \right] \right) \\ &= -\gamma_{k+1} \left( (1-\rho) \left[ \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[ \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} - (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \right] \right). \end{aligned} \quad (24)$$

452 We observe, using the identity (24), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)}\|^2] + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)}\|^2]. \quad (25)$$

453 For the latter term, we obtain its upper bound as

$$\begin{aligned}\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] &= \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(k)}) - (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})\right\|^2\right] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(\ell(k))}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2],\end{aligned}$$

454 where (a) uses the variance inequality. We can further bound the last expectation using Lemma 1:

$$\mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)}\|^2] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2] \stackrel{(a)}{\leq} \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2].$$

455 Substituting into (25) proves the lemma.  $\square$

456 **Lemma.** Considering a decreasing stepsize  $\gamma_k \in (0, 1)$  and a constant  $\rho \in (0, 1)$ , we have

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \leq \frac{\rho}{1-\rho} \sum_{\ell=0}^k (1-\gamma_\ell)^2 (\mathbf{S}^{(\ell)} - \tilde{S}^{(\ell)}),$$

457 where  $\mathbf{S}^{(k)}$  is defined either by Line 2 (vrTTEM) or Line 3 (fiTTEM).

458 **Proof** We begin by writing the two-timescale update:

$$\begin{aligned}\tilde{S}^{(k+1)} &= \tilde{S}^{(k)} + \rho(\mathbf{S}^{(k+1)} - \tilde{S}^{(k)}), \\ \hat{\mathbf{s}}^{(k+1)} &= \hat{\mathbf{s}}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}),\end{aligned}\tag{26}$$

459 where  $\mathbf{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(t_i^k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})$  according to (3). Denote  $\delta^{(k+1)} = \hat{\mathbf{s}}^{(k+1)} - \tilde{S}^{(k+1)}$ .  
460 Then from (26), doing the subtraction of both equations yields:

$$\delta^{(k+1)} = (1 - \gamma_{k+1})\delta^{(k)} + \frac{\rho}{1-\rho}(1 - \gamma_{k+1})(\mathbf{S}^{(k+1)} - \tilde{S}^{(k+1)}).$$

461 Using the telescoping sum and noting that  $\delta^{(0)} = 0$ , we have

$$\delta^{(k+1)} \leq \frac{\rho}{1-\rho} \sum_{\ell=0}^k (1 - \gamma_{\ell+1})^2 (\mathbf{S}^{(\ell+1)} - \tilde{S}^{(\ell+1)}).$$

462  $\square$

## 463 B.2 Additional Intermediary Result

464 **Lemma 8.** At iteration  $k + 1$ , the drift term of update (3), with  $\rho_{k+1} = \rho$ , is equivalent to the  
465 following :

$$\begin{aligned}\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} &= \rho(\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}) + \rho\eta_{i_k}^{(k+1)} + \rho\left[(\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}]\right] \\ &\quad + (1 - \rho)(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}),\end{aligned}$$

466 where we recall that  $\eta_{i_k}^{(k+1)}$ , defined in (12), which is the gap between the MC approximation and  
467 the expected statistics.

468 **Proof** Using the fitTEM update  $\tilde{S}^{(k+1)} = (1-\rho)\tilde{S}^{(k)} + \rho\mathcal{S}^{(k+1)}$  where  $\mathcal{S}^{(k+1)} = \bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})$  leads to the following decomposition:

$$\begin{aligned}
& \tilde{S}^{(k+1)} - \hat{s}^{(k)} \\
&= (1-\rho)\tilde{S}^{(k)} + \rho \left( \bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \right) - \hat{s}^{(k)} + \rho\bar{\mathcal{S}}^{(k)} - \rho\bar{\mathcal{S}}^{(k)} \\
&= \rho(\bar{\mathcal{S}}^{(k)} - \hat{s}^{(k)}) + \rho(\tilde{S}_{i_k}^{(k)} - \bar{\mathcal{S}}_{i_k}^{(k)}) + (1-\rho) \left( \tilde{S}^{(k)} - \hat{s}^{(k)} \right) + \rho \left( \bar{\mathcal{S}}^{(k)} - \bar{\mathcal{S}}^{(k)} + (\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \right) \\
&= \rho(\bar{\mathcal{S}}^{(k)} - \hat{s}^{(k)}) + \rho\eta_{i_k}^{(k+1)} - \rho \left[ (\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] \\
&+ (1-\rho) \left( \tilde{S}^{(k)} - \hat{s}^{(k)} \right),
\end{aligned}$$

470 where we observe that  $\mathbb{E}[\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] = \bar{\mathcal{S}}^{(k)} - \bar{\mathcal{S}}^{(k)}$  and which concludes the proof.

471 *Important Note:* Note that  $\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}$  is not equal to  $\eta_{i_k}^{(k+1)}$ , defined in (12), which is the gap  
472 between the MC approximation and the expected statistics. Indeed  $\tilde{S}_{i_k}^{(t_{i_k}^k)}$  is not computed under the  
473 same model as  $\bar{\mathcal{S}}_{i_k}^{(k)}$ .  $\square$

### 474 B.3 Proof of Theorem 2

475 **Theorem.** Assume A1-A5. Consider the vrTTEM sequence  $\{\hat{s}^{(k)}\}_{k \geq 0} \in \mathcal{S}$  for any  $k \leq K_m$  where  
476  $K_m$  is a positive integer. Let  $\{\gamma_{k+1} = 1/(k^a \bar{L})\}_{k \geq 0}$ , where  $a \in (0, 1)$ , be a sequence of stepsizes,  
477  $\bar{L} = \max\{L_S, L_V\}$ ,  $\rho = \mu/(c_1 \bar{L} n^{2/3})$ ,  $m = nc_1^2/(2\mu^2 + \mu c_1^2)$  and a constant  $\mu \in (0, 1)$ . Then:

$$\mathbb{E}[\|\nabla V(\hat{s}^{(K)})\|^2] \leq \frac{2n^{2/3}\bar{L}}{\mu P_m v_{\min}^2 v_{\max}^2} \left( \mathbb{E}[\Delta V] + \sum_{k=0}^{K_m-1} \tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k)}\|^2] \right).$$

478 **Proof** Using the smoothness of  $V$  and update (2), we obtain:

$$\begin{aligned}
V(\hat{s}^{(k+1)}) &\leq V(\hat{s}^{(k)}) + \langle \hat{s}^{(k+1)} - \hat{s}^{(k)} | \nabla V(\hat{s}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{s}^{(k+1)} - \hat{s}^{(k)}\|^2 \\
&\leq V(\hat{s}^{(k)}) - \gamma_{k+1} \langle \hat{s}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{s}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{s}^{(k)} - \tilde{S}^{(k+1)}\|^2.
\end{aligned} \tag{27}$$

479 Denote  $H_{k+1} := \hat{s}^{(k)} - \tilde{S}^{(k+1)}$  the drift term of the fitTEM update in (7) and  $h_k = \hat{s}^{(k)} - \bar{\mathcal{S}}^{(k)}$ .  
480 Taking expectations on both sides show that

$$\begin{aligned}
& \mathbb{E}[V(\hat{s}^{(k+1)})] \\
&\stackrel{(a)}{\leq} \mathbb{E}[V(\hat{s}^{(k)})] - \gamma_{k+1}(1-\rho)\mathbb{E}[\langle \hat{s}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{s}^{(k)}) \rangle] - \gamma_{k+1}\rho\mathbb{E}[\langle \hat{s}^{(k)} - \mathcal{S}^{(k+1)} | \nabla V(\hat{s}^{(k)}) \rangle] \\
&+ \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|H_{k+1}\|^2] \\
&\stackrel{(b)}{\leq} \mathbb{E}[V(\hat{s}^{(k)})] - \gamma_{k+1}\rho\mathbb{E}[\langle h_k | \nabla V(\hat{s}^{(k)}) \rangle] - \gamma_{k+1}(1-\rho)\mathbb{E}[\langle \hat{s}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{s}^{(k)}) \rangle] \\
&- \gamma_{k+1}\rho\mathbb{E}[\langle \eta_{i_k}^{(k+1)} | \nabla V(\hat{s}^{(k)}) \rangle] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|H_{k+1}\|^2] \\
&\stackrel{(c)}{\leq} \mathbb{E}[V(\hat{s}^{(k)})] - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2) \mathbb{E}[\|h_k\|^2] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|H_{k+1}\|^2] \\
&- \gamma_{k+1}\rho\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \gamma_{k+1}(1-\rho)\mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k)}\|^2],
\end{aligned} \tag{28}$$



481 where we have used (22) in (a) and  $\mathbb{E}[\mathcal{S}^{(k+1)}] = \bar{\mathbf{s}}^{(k)} + \mathbb{E}[\eta_{i_k}^{(k+1)}]$  in (b), the growth condition in  
 482 Lemma 2 and Young's inequality with the constant equal to 1 in (c).

483 Furthermore, for  $k+1 \leq \ell(k) + m$  (i.e.,  $k+1$  is in the same epoch as  $k$ ), we have

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] = \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} + \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \rangle\right] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2\right. \\ &\quad \left.- 2\gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \rho(\mathbf{h}_k - \eta_{i_k}^{(k+1)}) + (1-\rho)(\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}) \rangle\right] \\ &\leq \mathbb{E}\left[(1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2\right. \\ &\quad \left.+ \frac{\gamma_{k+1}\rho}{\beta} \|\eta_{i_k}^{(k+1)}\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right], \end{aligned}$$

484 where we first used (22) and the last inequality is due to Young's inequality.

485 Consider the following sequence

$$R_k := \mathbb{E}[V(\hat{\mathbf{s}}^{(k)}) + b_k \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2],$$

486 where  $b_k := \bar{b}_{k \bmod m}$  is a periodic sequence where:

$$\bar{b}_i = \bar{b}_{i+1}(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2) + \gamma_{k+1}^2 \rho^2 \mathbf{L}_V \mathbf{L}_s^2, \quad i = 0, 1, \dots, m-1 \quad \text{with } \bar{b}_m = 0.$$

487 Note that  $\bar{b}_i$  is decreasing with  $i$  and this implies

$$\bar{b}_i \leq \bar{b}_0 = \gamma_{k+1}^2 \rho^2 \mathbf{L}_V \mathbf{L}_s^2 \frac{(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2)^m - 1}{\gamma_{k+1}\beta + 2\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2}, \quad i = 1, 2, \dots, m.$$

488 For  $k+1 \leq \ell(k) + m$ , we have the following inequality

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2) \|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}^2 \mathbf{L}_V}{2} \|\mathbf{H}_{k+1}\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[\rho \|\eta_{i_k}^{(k+1)}\|^2 - (1-\rho) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\frac{\gamma_{k+1}\rho}{\beta} \|\eta_{i_k}^{(k+1)}\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right]. \end{aligned}$$

489 And using Lemma 4 we obtain:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2 - \gamma_{k+1}^2 \rho^2 \mathbf{L}_V) \|\mathbf{h}_k\|^2 + \gamma_{k+1}^2 \rho^2 \mathbf{L}_V \mathbf{L}_s^2 \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2 \rho^2\right) \|\mathbf{h}_k\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[(\rho + \rho^2 \gamma_{k+1} \mathbf{L}_V) \|\eta_{i_k}^{(k+1)}\|^2 - (1-\rho - (1-\rho)^2 \gamma_{k+1} \mathbf{L}_V) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2 \rho^2\right) \|\eta_{i_k}^{(k+1)}\|^2 + \left(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2 (1-\rho)^2\right) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right]. \end{aligned}$$

490 Rearranging the terms yields:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1} \rho^2 \mathbf{L}_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1} \rho^2)) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\quad + \underbrace{\left(b_{k+1}(1 + \gamma\beta + 2\gamma^2 \rho^2 \mathbf{L}_s^2) + \gamma^2 \rho^2 \mathbf{L}_V \mathbf{L}_s^2\right)}_{=b_k \text{ since } k+1 \leq \ell(k) + m} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] + \tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}, \end{aligned}$$

491 where

$$\begin{aligned}\tilde{\eta}^{(k+1)} &= \left( \gamma_{k+1}(\rho + \rho^2 \gamma_{k+1} L_V) + b_{k+1}(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2 \rho^2) \right) \mathbb{E} \left[ \left\| \eta_{i_k}^{(k+1)} \right\|^2 \right] \\ \chi^{(k+1)} &= \left( b_{k+1}(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2(1-\rho)^2) - \gamma_{k+1}(1-\rho - (1-\rho)^2 \gamma_{k+1} L_V) \right) \\ \tilde{\chi}^{(k+1)} &= \chi^{(k+1)} \mathbb{E} \left[ \left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right].\end{aligned}$$

492 This leads, using Lemma 2, that for any  $\gamma_{k+1}$ ,  $\rho$  and  $\beta$  such that  $\rho v_{\min} + v_{\max}^2 -$   
493  $\gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2) > 0$ ,

$$\begin{aligned}v_{\max}^2 \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] \\ &\leq \frac{R_k - R_{k+1}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))} \\ &\quad + \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))}.\end{aligned}$$

494 We first remark that

$$\begin{aligned}&\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2)) \\ &\geq \frac{\gamma_{k+1}\rho}{c_1} (1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1)),\end{aligned}$$

495 where  $c_1 = v_{\min}^{-1}$ . By setting  $\bar{L} = \max\{L_s, L_V\}$ ,  $\beta = \frac{c_1 \bar{L}}{n^{1/3}}$ ,  $\rho = \frac{\mu}{c_1 \bar{L} n^{2/3}}$ ,  $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$  and  
496  $\{\gamma_{k+1}\}$  any sequence of decreasing stepsizes in  $(0, 1)$ , it can be shown that there exists  $\mu \in (0, 1)$ ,  
497 such that the following lower bound holds

$$\begin{aligned}&1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1) \\ &\geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \bar{b}_0(\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ &\geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{L_V \mu^2}{c_1^2 n^{\frac{4}{3}}} \frac{(1 + \gamma\beta + 2\gamma^2 L_s^2)^m - 1}{\gamma\beta + 2\gamma^2 L_s^2} (\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ &\stackrel{(a)}{\geq} 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{\mu}{c_1^2} (e - 1)(1 + \frac{2\mu}{n}) \geq 1 - \mu - \mu(1 + 2\mu) \frac{e - 1}{c_1^2} \stackrel{(b)}{\geq} \frac{1}{2},\end{aligned}$$

498 where the simplification in (a) is due to

$$\frac{\mu}{n} \leq \gamma\beta + 2\gamma^2 L_s^2 \leq \frac{\mu}{n} + \frac{2\mu^2}{c_1^2 n^{\frac{4}{3}}} \leq \frac{\mu c_1^2 + 2\mu^2}{c_1^2} \frac{1}{n} \text{ and } (1 + \gamma\beta + 2\gamma^2 L_s^2)^m \leq e - 1.$$

499 and the required  $\mu$  in (b) can be found by solving the quadratic equation.

500 Finally, these results yield:

$$v_{\max}^2 \sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq \frac{2(R_0 - R_{K_m})}{v_{\min}\rho} + 2 \sum_{k=0}^{K_m-1} \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{v_{\min}\rho}.$$

501 Note that  $R_0 = \mathbb{E}[V(\hat{\mathbf{s}}^{(0)})]$  and if  $K_m$  is a multiple of  $m$ , then  $R_{K_m} = \mathbb{E}[V(\hat{\mathbf{s}}^{(K_m)})]$ . Under the latter  
502 condition, we have

$$\sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})] + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_m-1} [\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}].$$

503 This concludes our proof.

504 □

505 **B.4 Proof of Theorem 3**

506 **Theorem.** Assume A1-A5. Consider the fitTEM sequence  $\{\hat{\mathbf{s}}^{(k)}\}_{k>0} \in \mathcal{S}$  for any  $k \leq K_m$  where  
 507  $K_m$  be a positive integer. Let  $\{\gamma_{k+1} = 1/(k^a \alpha c_1 \bar{L})\}_{k>0}$ , where  $a \in (0, 1)$ , be a sequence of  
 508 positive stepsizes,  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\beta = 1/(\alpha n)$ ,  $\rho = 1/(\alpha c_1 \bar{L} n^{2/3})$   
 509 and  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ . Then:

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] \leq \frac{4\alpha \bar{L} n^{2/3}}{P_m v_{\min}^2 v_{\max}^2} \left( \mathbb{E}[\Delta V] + \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \Gamma^{(k+1)} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \right).$$

510 **Proof** Using the smoothness of  $V$  and update (3), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) + \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2. \end{aligned} \quad (29)$$

511 Denote  $H_{k+1} := \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}$  the drift term of the fitTEM update in (7) and  $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$ .  
 512 Using Lemma 8 and the additional following identity:

$$\mathbb{E} \left[ (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] = 0, \quad (30)$$

513 we have:

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] \\ &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} \rho \mathbb{E}[\langle \mathbf{h}_k | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1} \mathbb{E} \left[ \langle \rho \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] + (1 - \rho) \mathbb{E}[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}] | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2 \\ &\stackrel{(a)}{\leq} -v_{\min} \gamma_{k+1} \rho \mathbb{E}[\|\mathbf{h}_k\|^2] - \gamma_{k+1} \mathbb{E} \left[ \|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2 \right] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2 \\ &\stackrel{(b)}{\leq} -(v_{\min} \gamma_{k+1} \rho + \gamma_{k+1} v_{\max}^2) \mathbb{E}[\|\mathbf{h}_k\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2, \end{aligned}$$

514 where  $\xi^{(k+1)} = \mathbb{E}[\|\mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k]\|^2]$ .

515 **Bounding**  $\mathbb{E}[\|H_{k+1}\|^2]$  Using Lemma 5, we obtain:

$$\begin{aligned} &\gamma_{k+1} (v_{\min} \rho + v_{\max}^2 - \gamma_{k+1} \rho^2 L_V) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\leq \mathbb{E} \left[ V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)}) \right] + \tilde{\xi}^{(k+1)} + \left( (1 - \rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V \rho^2 L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2], \end{aligned} \quad (31)$$

516 where  $\tilde{\xi}^{(k+1)} = \gamma_{k+1}^2 \rho^2 L_V \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)}$ . Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \right), \quad (32)$$

517 where the equality holds as  $i_k$  and  $j_k$  are drawn independently. Then,

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle\right]. \end{aligned}$$

518 Note that  $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) = -\gamma_{k+1}\mathbf{H}_{k+1}$  and that in expectation we recall  
 519 that  $\mathbb{E}[\mathbf{H}_{k+1}|\mathcal{F}_k] = \rho\mathbf{h}_k + \rho\mathbb{E}[\eta_{i_k}^{(k+1)}|\mathcal{F}_k] + (1-\rho)\mathbb{E}[\tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)}]$  where  $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$ . Thus,  
 520 for any  $\beta > 0$ , it holds

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle\right] \\ &\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \right. \\ &\quad \left. + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]\right], \end{aligned}$$

521 where the last inequality is due to Young's inequality. Plugging this into (32) yields:

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle\right] \\ &\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \right. \\ &\quad \left. + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]\right]. \end{aligned}$$

522 Subsequently, we have

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}\left[\left(1 + \gamma_{k+1}\beta\right)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 \right. \\ &\quad \left. + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]\right]. \end{aligned}$$

523 We now use Lemma 5 on  $\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 = \gamma_{k+1}^2\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2$  and obtain:

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\ &\leq \left(2\gamma_{k+1}^2\rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \sum_{i=1}^n \left(\frac{\gamma_{k+1}^2\rho^2\mathbf{L}_s^2}{n} + \frac{(n-1)(1+\gamma_{k+1}\beta)}{n^2}\right)\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + \gamma_{k+1}(1-\rho)^2\left(2\gamma_{k+1} + \frac{1}{\beta}\right)\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left(2\gamma_{k+1}^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\leq \left(2\gamma_{k+1}^2\rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \left(\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2\rho^2\mathbf{L}_s^2}{n}\right)\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + \gamma_{k+1}(1-\rho)^2\left(2\gamma_{k+1} + \frac{1}{\beta}\right)\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left(2\gamma_{k+1}^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]. \end{aligned}$$

524 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] .$$

525 From the above, we get

$$\begin{aligned} \Delta^{(k+1)} &\leq \left(1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2\right) \Delta^{(k)} + \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] + \gamma_{k+1} \left(2\gamma_{k+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] . \end{aligned}$$

526 Setting  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{2, 1+2v_{\min}\}$ ,  $\bar{L} = \max\{\mathbf{L}_s, \mathbf{L}_V\}$ ,  $\gamma_{k+1} = \frac{1}{k}$ ,  $\beta = \frac{1}{\alpha n}$ ,  $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$ ,  
527  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ , we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2 \leq 1 - \frac{1}{n} + \frac{1}{\alpha k n} + \frac{1}{\alpha^2 c_1^2 k^2 n^{4/3}} \leq 1 - \frac{c_1(k\alpha - 1) - 1}{k\alpha n c_1} \leq 1 - \frac{1}{k\alpha n c_1}$$

528 which shows that  $1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2 \in (0, 1)$  for any  $k > 0$ . Denote  $\Lambda_{(k+1)} = \frac{1}{n} -$   
529  $\gamma_{k+1}\beta - \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2$  and note that  $\Delta^{(0)} = 0$ , thus the telescoping sum yields:

$$\begin{aligned} \Delta^{(k+1)} &\leq \sum_{\ell=0}^k \omega_{k,\ell} \left(2\gamma_{\ell+1}^2 \rho^2 + \frac{\gamma_{\ell+1}^2 \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] \\ &\quad + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1}(1-\rho)^2 \left(2\gamma_{\ell+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\tilde{\mathbf{S}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1} \tilde{\epsilon}^{(\ell+1)} , \end{aligned}$$

530 where  $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$  and  $\tilde{\epsilon}^{(\ell+1)} = \left(2\gamma_{k+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]$ .

531 Summing on both sides over  $k = 0$  to  $k = K_m - 1$  yields:

$$\begin{aligned} \sum_{k=0}^{K_m-1} \Delta^{(k+1)} &\leq \sum_{k=0}^{K_m-1} \frac{2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}^2 \rho^2}{\beta}}{\Lambda_{(k+1)}} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \sum_{k=0}^{K_m-1} \frac{\gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right)}{\Lambda_{(k+1)}} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] + \sum_{k=0}^{K_m-1} \frac{\gamma_{k+1}}{\Lambda_{(k+1)}} \tilde{\epsilon}^{(k+1)} . \end{aligned}$$

532 We recall (31) where we have summed on both sides from  $k = 0$  to  $k = K_m - 1$ :

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(K_m)}) - V(\hat{\mathbf{s}}^{(0)})] \\ &\leq \sum_{k=0}^{K_m-1} \left\{ \gamma_{k+1}(-(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}\rho^2 \mathbf{L}_V) \mathbb{E}[\|\mathbf{h}_k\|^2] + \gamma^2 \mathbf{L}_V \rho^2 \mathbf{L}_s^2 \Delta^{(k)} \right\} \\ &\quad + \sum_{k=0}^{K_m-1} \left\{ \tilde{\xi}^{(k+1)} + \left( (1-\rho)^2 \gamma_{k+1}^2 \mathbf{L}_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] \right\} \\ &\leq \sum_{k=0}^{K_m-1} \left\{ \left[ -\gamma_{k+1}(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 \mathbf{L}_V + \frac{\rho^2 \gamma_{k+1}^2 \mathbf{L}_V \mathbf{L}_s^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)}{\Lambda_{(k+1)}} \right] \mathbb{E}[\|\mathbf{h}_k\|^2] \right\} \\ &\quad + \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \sum_{k=0}^{K_m-1} \Gamma^{(k+1)} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] , \end{aligned} \tag{33}$$

where

$$\Xi^{(k+1)} = \tilde{\xi}^{(k+1)} + \frac{\gamma_{k+1}^3 \mathbf{L}_V \rho^2 \mathbf{L}_s^2}{\Lambda_{(k+1)}} \tilde{\epsilon}^{(k+1)}$$

and

$$\Gamma^{(k+1)} = \left( (1-\rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) + \frac{\gamma_{k+1}^3 L_V \rho^2 L_s^2 (1-\rho)^2 \left( 2\gamma_{k+1} + \frac{1}{\beta} \right)}{\Lambda_{(k+1)}}.$$

533 We now analyse the following quantity

$$\begin{aligned} & -\gamma_{k+1}(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 L_V + \frac{\rho^2 \gamma_{k+1}^2 L_V L_s^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \\ & = \gamma_{k+1} \left[ -(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right]. \end{aligned} \quad (34)$$

534 Furthermore, we recall that  $c_1 = v_{\min}^{-1}$ ,  $\alpha = \max\{2, 1 + 2v_{\min}\}$ ,  $\bar{L} = \max\{L_s, L_V\}$ ,  $\gamma_{k+1} = \frac{1}{k}$ ,  
535  $\beta = \frac{1}{\alpha n}$ ,  $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$ ,  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ ,  $\alpha \geq 2$ . Then,

$$\begin{aligned} & \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\frac{1}{n} - \gamma_{k+1}\beta - \gamma_{k+1}^2 \rho^2 L_s^2} \\ & \leq \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L}(k\alpha^2 c_1^2 n^{4/3})^{-1} \left( \frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k\alpha c_1^2 \bar{L}^2 n^{1/3}} \right)}{\frac{1}{n} - \frac{1}{k\alpha n} - \frac{1}{k^2 \alpha^2 c_1^2 n^{4/3}}} \\ & = \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L} \left( \frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k\alpha c_1^2 \bar{L}^2 n^{1/3}} \right)}{(k\alpha c_1 n^{1/3})(k\alpha - 1)c_1 - 1} \\ & \stackrel{(a)}{\leq} \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\frac{1}{k\alpha c_1^2 \bar{L} n^{1/3}} \left( \frac{2}{k\alpha n} + 1 \right)}{2(\alpha c_1 n^{1/3}) - 1} \\ & \leq \frac{1}{k^2 \alpha c_1^2 \bar{L} n^{4/3}} + \frac{1}{4k\alpha^2 c_1^3 \bar{L} n^{2/3}} \\ & \leq \frac{3/4}{\alpha c_1^2 \bar{L} n^{2/3}}, \end{aligned} \quad (35)$$

where (a) is due to  $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$  and  $k\alpha c_1 n^{1/3} \geq 1$ . Note also that

$$-(v_{\min}\rho + v_{\max}^2) \leq -\rho v_{\min} = -\frac{1}{\alpha c_1^2 \bar{L} n^{2/3}},$$

which yields that

$$\left[ -(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left( 2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right] \leq -\frac{1/4}{\alpha c_1^2 \bar{L} n^{2/3}}.$$

536 Using the Lemma 2, we know that  $v_{\max}^2 \|\nabla V(\hat{s}^{(k)})\|^2 \leq \|\hat{s}^{(k)} - \bar{s}^{(k)}\|^2$  and using (35) on (33)  
537 yields:

$$\begin{aligned} & v_{\max}^2 \sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \\ & \leq \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2} [V(\hat{s}^{(0)}) - V(\hat{s}^{(K_m)})] \\ & \quad + \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2} \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \sum_{k=0}^{K_m-1} \Gamma^{(k+1)} \mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k)}\|^2], \end{aligned}$$



538 proving the bound on the second order moment of the gradient of the Lyapunov function:

$$\begin{aligned} \sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2 v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})] \\ &\quad + \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \sum_{k=0}^{K_m-1} \Gamma^{(k+1)} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] . \end{aligned}$$

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□

## C Practical Implementations of Two-Timescale EM Methods

### C.1 Application on GMM

#### C.1.1 Explicit Updates

We first recognize that the constraint set for  $\theta$  is given by

$$\Theta = \Delta^M \times \mathbb{R}^M.$$

Using the partition of the sufficient statistics as  $S(y_i, z_i) = (S^{(1)}(y_i, z_i)^\top, S^{(2)}(y_i, z_i)^\top, S^{(3)}(y_i, z_i)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$ , the partition  $\phi(\theta) = (\phi^{(1)}(\theta)^\top, \phi^{(2)}(\theta)^\top, \phi^{(3)}(\theta)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$  and the fact that  $\mathbb{1}_{\{M\}}(z_i) = 1 - \sum_{m=1}^{M-1} \mathbb{1}_{\{m\}}(z_i)$ , the complete data log-likelihood can be expressed as in (2) with

$$\begin{aligned} s_{i,m}^{(1)} &= \mathbb{1}_{\{m\}}(z_i), \quad \phi_m^{(1)}(\theta) = \left\{ \log(\omega_m) - \frac{\mu_m^2}{2} \right\} - \left\{ \log(1 - \sum_{j=1}^{M-1} \omega_j) - \frac{\mu_M^2}{2} \right\}, \\ s_{i,m}^{(2)} &= \mathbb{1}_{\{m\}}(z_i)y_i, \quad \phi_m^{(2)}(\theta) = \mu_m, \quad s_i^{(3)} = y_i, \quad \phi^{(3)}(\theta) = \mu_M, \end{aligned} \quad (36)$$

and  $\psi(\theta) = -\left\{ \log(1 - \sum_{m=1}^{M-1} \omega_m) - \frac{\mu_M^2}{2\sigma^2} \right\}$ . We also define for each  $m \in \llbracket 1, M \rrbracket$ ,  $j \in \llbracket 1, 3 \rrbracket$ ,  $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$ . Consider the following latent sample used to compute an approximation of the conditional expected value  $\mathbb{E}_\theta[\mathbb{1}_{\{z_i=m\}}|y = y_i]$ :

$$z_{i,m} \sim \mathbb{P}(z_i = m | y_i; \theta) \quad (37)$$

where  $m \in \llbracket 1, M \rrbracket$ ,  $i \in [n]$  and  $\theta = (\mathbf{w}, \boldsymbol{\mu}) \in \Theta$ .

In particular, given iteration  $k + 1$ , the computation of the approximated quantity  $\tilde{S}_{i_k}^{(k)}$  during Incremental-step updates, see (8) can be written as

$$\tilde{S}_{i_k}^{(k)} = \left( \underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1}), \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})}_{:=\tilde{s}_{i_k}^{(1)}}, \underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1})y_{i_k}, \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})y_{i_k}}_{:=\tilde{s}_{i_k}^{(2)}}, \underbrace{y_{i_k}}_{:=\tilde{s}_{i_k}^{(3)}(\theta^{(k)})} \right)^\top. \quad (38)$$

Recall that we have used the following regularizer:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m), \quad (39)$$

It can be shown that the regularized M-step evaluates to

$$\bar{\theta}(\mathbf{s}) = \begin{pmatrix} (1 + \epsilon M)^{-1} (s_1^{(1)} + \epsilon, \dots, s_{M-1}^{(1)} + \epsilon)^\top \\ ((s_1^{(1)} + \delta)^{-1} s_1^{(2)}, \dots, (s_{M-1}^{(1)} + \delta)^{-1} s_{M-1}^{(2)})^\top \\ (1 - \sum_{m=1}^{M-1} s_m^{(1)} + \delta)^{-1} (s^{(3)} - \sum_{m=1}^{M-1} s_m^{(2)}) \end{pmatrix} = \begin{pmatrix} \bar{\omega}(\mathbf{s}) \\ \bar{\boldsymbol{\mu}}(\mathbf{s}) \\ \bar{\mu}_M(\mathbf{s}) \end{pmatrix}. \quad (40)$$

where we have defined for all  $m \in \llbracket 1, M \rrbracket$  and  $j \in \llbracket 1, 3 \rrbracket$ ,  $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$ .

#### C.1.2 Model Assumptions (GMM example)

We use the GMM example to illustrate the required assumptions.

Many practical models can satisfy the compactness of the sets as in Assumption A1. For instance, the GMM example satisfies (11) as the sufficient statistics are composed of indicator functions and observations as defined Section C.1 Equation (36).

Assumptions A2 and A3 are standard for the curved exponential family models. For GMM, the following (strongly convex) regularization  $\mathbf{r}(\theta)$  ensures A3:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m),$$

563 since it ensures  $\theta^{(k)}$  is unique and lies in  $\text{int}(\Delta^M) \times \mathbb{R}^M$ . We remark that for A2, it is possible to  
 564 define the Lipschitz constant  $L_p$  independently for each data  $y_i$  to yield a refined characterization.

565 Again, A4 is satisfied by practical models. For GMM, it can be verified by deriving the closed form  
 566 expression for  $B(s)$  and using A1.

567 Under A1 and A3, we have  $\|\hat{s}^{(k)}\| < \infty$  since  $S$  is compact and  $\hat{\theta}^{(k)} \in \text{int}(\Theta)$  for any  $k \geq 0$  which  
 568 thus ensure that the EM methods operate in a closed set throughout the optimization process.

### 569 C.1.3 Algorithms updates

570 In the sequel, recall that, for all  $i \in [n]$  and iteration  $k$ , the computed statistic  $\tilde{S}_{i_k}^{(k)}$  is defined by (38).  
 571 At iteration  $k$ , the several E-steps defined by (1) or (2) and (3) leads to the definition of the quantity  
 572  $\hat{s}^{(k+1)}$ . For the GMM example, after the initialization of the quantity  $\hat{s}^{(0)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(0)}$ , those  
 573 E-steps break down as follows:

574 **Batch EM (EM):** for all  $i \in [n]$ , compute  $\bar{s}_i^{(k)}$  and set

$$\hat{s}^{(k+1)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(k)}.$$

575 where  $\bar{s}_i^{(k)}$  are computed using the exact conditional expected value  $\mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i]$ :

$$\tilde{\omega}_m(y_i; \theta) := \mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i] = \frac{\omega_m \exp(-\frac{1}{2}(y_i - \mu_i)^2)}{\sum_{j=1}^M \omega_j \exp(-\frac{1}{2}(y_i - \mu_j)^2)},$$

576 **Incremental EM (iEM):** draw an index  $i_k$  uniformly at random on  $[n]$ , compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} + \frac{1}{n} (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\tau_i^k)}) = n^{-1} \sum_{i=1}^n \bar{s}_i^{(\tau_i^k)}.$$

577 **batch SAEM (SAEM):** draw an index  $i_k$  uniformly at random on  $[n]$ , compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)}(1 - \gamma_{k+1}) + \gamma_{k+1} \tilde{S}^{(k)}.$$

578 where  $= \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k)}$  with  $\tilde{S}_i^{(k)}$  defined in (38).

579 **Incremental SAEM (iSAEM):** draw an index  $i_k$  uniformly at random on  $[n]$ , compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)}(1 - \gamma_{k+1}) + \gamma_{k+1} \left( \tilde{S}^{(k)} + \frac{1}{n} (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\tau_i^k)}) \right).$$

580 **Variance Reduced Two-Timescale EM (vrTTEM):** draw an index  $i_k$  uniformly at random on  $[n]$ ,  
 581 compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)}(1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)}(1 - \rho) + \rho(\tilde{S}^{(\ell(k))} + (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\ell(k))}))).$$

582 **Fast Incremental Two-Timescale EM (fiTTEM):** draw an index  $i_k$  uniformly at random on  $[n]$ ,  
 583 compute  $\bar{s}_{i_k}^{(k)}$  and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)}(1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)}(1 - \rho) + \rho(\bar{\mathcal{S}}^{(k)} + (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(t_{i_k}^k)}))).$$

584 Finally, the  $k$ -th update reads  $\hat{\theta}^{(k+1)} = \bar{\theta}(\hat{s}^{(k+1)})$  where the function  $s \rightarrow \bar{\theta}(s)$  is defined by (40).

## 585 C.2 Deformable Template Model for Image Analysis

### 586 C.2.1 Model and Updates

587 The complete model belongs to the curved exponential family, see [2], which vector of sufficient  
 588 statistics  $S = (S_1(z), S_2(z), S_3(z))$  read:

$$\begin{aligned} S_1(z) &= \frac{1}{n} \sum_{i=1}^n S_1(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top y_i, \\ S_2(z) &= \frac{1}{n} \sum_{i=1}^n S_2(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top (\mathbf{K}_p^{z_i}), \\ S_3(z) &= \frac{1}{n} \sum_{i=1}^n S_3(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n z_i^t z_i, \end{aligned} \quad (41)$$

589 where for any pixel  $u \in \mathbb{R}^2$  and  $j \in \llbracket 1, k_g \rrbracket$  we denote:

$$\mathbf{K}_p^{z_i}(x_u, j) = \mathbf{K}_p^{z_i}(x_u - \phi_i(x_u, z_i), p_j).$$

590 Finally, the Two-Timescale M-step yields the following parameter updates:

$$\bar{\theta}(\hat{s}) = \begin{pmatrix} \beta(\hat{s}) = \hat{s}_2^{-1}(z) \hat{s}_1(z) \\ \Gamma(\hat{s}) = \frac{1}{n} \hat{s}_3(z) \\ \sigma(\hat{s}) = \beta(\hat{s})^\top \hat{s}_2(z) \beta(\hat{s}) - 2\beta(\hat{s}) \hat{s}_1(z) \end{pmatrix}, \quad (42)$$

591 where  $\hat{s} = (\hat{s}_1(z), \hat{s}_2(z), \hat{s}_3(z))$  is the vector of statistics obtained via the **SA-step** (7) and using the  
 592 MC approximation of the sufficient statistics  $(S_1(z), S_2(z), S_3(z))$  defined in (41).

### 593 C.2.2 Numerical Applications

594 For the inference of the template, we use the Matlab code (online SAEM) used in [24] and implement  
 595 our own batch, incremental, Variance reduced and Fast Incremental variants. The hyperparameters  
 596 are kept the same and reads as follows  $M = 400$ ,  $\gamma_k = 1/k^{0.6}$  and  $p = 16$ . The number of  
 597 landmarks for the template is  $k_p = 15$  points and for the deformation  $k_g = 6$  points. Both have  
 598 Gaussian kernels with respectively standard deviation of 0.12 and 0.3. The standard deviation of the  
 599 measurement errors is set to 0.1.

600 For the simulation part, we use the Carlin and Chib MCMC procedure, see [9]. Refer to [24] for  
 601 more details.

## D Additional Experiment: Pharmacokinetics (PK) Model with Absorption Lag Time

This numerical example was conducted in order to characterize the pharmacokinetics (PK) of orally administered drug to simulated patients, using a population pharmacokinetics approach.  $M = 50$  synthetic datasets were generated for  $n = 5000$  patients with 10 observations (concentration measures) per patient. The goal is to model the evolution of the concentration of the absorbed drug using a nonlinear and latent variable model.

**Model and Explicit Updates:** We consider a one-compartment PK model for oral administration with an absorption lag-time ( $T^{\text{lag}}$ ), assuming first-order absorption and linear elimination processes. The final model includes the following variables:  $ka$  the absorption rate constant,  $V$  the volume of distribution,  $k$  the elimination rate constant and  $T^{\text{lag}}$  the absorption lag-time. We also add several covariates to our model such as  $D$  the dose of drug administered,  $t$  the time at which measures are taken and the weight of the patient influencing the volume  $V$ . More precisely, the log-volume  $\log(V)$  is a linear function of the log-weight  $lw70 = \log(wt/70)$ . Let  $z_i = (T_i^{\text{lag}}, ka_i, V_i, k_i)$  be the vector of individual PK parameters, different for each individual  $i$ . The final model reads:

$$y_{ij} = f(t_{ij}, z_i) + \varepsilon_{ij} \quad \text{where} \quad f(t_{ij}, z_i) = \frac{D ka_i}{V(ka_i - k_i)} (e^{-ka_i(t_{ij} - T_i^{\text{lag}})} - e^{-k_i(t_{ij} - T_i^{\text{lag}})}) , \quad (43)$$

where  $y_{ij}$  is the  $j$ -th concentration measurement of the drug of dosage  $D$  injected at time  $t_{ij}$  for patient  $i$ . We assume in this example that the residual errors  $\varepsilon_{ij}$  are independent and normally distributed with mean 0 and variance  $\sigma^2$ . Lognormal distributions are used for the four PK parameters.

Lognormal distributions are used for the four PK parameters:

$$\begin{aligned} \log(T_i^{\text{lag}}) &\sim \mathcal{N}(\log(T_{\text{pop}}^{\text{lag}}), \omega_{T^{\text{lag}}}^2), \log(ka_i) \sim \mathcal{N}(\log(ka_{\text{pop}}), \omega_{ka}^2), \\ \log(V_i) &\sim \mathcal{N}(\log(V_{\text{pop}}), \omega_V^2), \log(k_i) \sim \mathcal{N}(\log(k_{\text{pop}}), \omega_k^2). \end{aligned}$$

We recall that the complete model  $(y, z)$  defined by (43) belongs to the curved exponential family, which vector of sufficient statistics  $S = (S_1(z), S_2(z), S_3(z))$  read:

$$S_1(z) = \frac{1}{n} \sum_{i=1}^n z_i, \quad S_2(z) = \frac{1}{n} \sum_{i=1}^n z_i^\top z_i, \quad S_3(z) = \frac{1}{n} \sum_{i=1}^n (y_i - f(t_i, z_i))^2 \quad (44)$$

where we have noted  $y_i$  and  $t_i$  the vector of observations and time for each patient  $i$ . At iteration  $k$ , and setting the number of MC samples to 1 for the sake of clarity, the MC sampling  $z_i^{(k)} \sim p(z_i | y_i, \theta^{(k)})$  is performed using a Metropolis-Hastings procedure detailed in Algorithm 2. The quantities  $\tilde{S}^{(k+1)}$  and  $\hat{s}^{(k+1)}$  are then updated according to the different methods. Finally the maximization step yields:

$$\bar{\theta}(s) = \begin{pmatrix} \hat{s}_1^{(k+1)} \\ \hat{s}_2^{(k+1)} - \hat{s}_1^{(k+1)} (\hat{s}_1^{(k+1)})^\top \\ \hat{s}_3^{(k+1)} \end{pmatrix} = \begin{pmatrix} \overline{z_{\text{pop}}}(\hat{s}^{(k+1)}) \\ \overline{\omega_z}(\hat{s}^{(k+1)}) \\ \overline{\sigma}(\hat{s}^{(k+1)}) \end{pmatrix}. \quad (45)$$

where  $z_{\text{pop}}$  denotes the vector of fixed effects  $(T_{\text{pop}}^{\text{lag}}, ka_{\text{pop}}, V_{\text{pop}}, k_{\text{pop}})$ .

**Metropolis Hastings algorithm.** During the simulation step of the MISSO method, the sampling from the target distribution  $\pi(z_i, \theta) := p(z_i | y_i, \theta)$  is performed using a Metropolis Hastings (MH) algorithm [27] with proposal distribution  $q(z_i, \delta)$  where  $\theta = (z_{\text{pop}}, \omega_z)$  and  $\delta$  is the vector of parameters of the proposal distribution. Commonly they parameterize a Gaussian proposal. The MH algorithm is summarized in 2.

---

**Algorithm 2** MH algorithm

---

```
1: Input: initialization  $z_{i,0} \sim q(z_i; \delta)$ 
2: for  $m = 1, \dots, M$  do
3:   Sample  $z_{i,m} \sim q(z_i; \delta)$ 
4:   Sample  $u \sim \mathcal{U}([0, 1])$ 
5:   Calculate the ratio  $r = \frac{\pi(z_{i,m}; \theta) / q(z_{i,m}; \delta)}{\pi(z_{i,m-1}; \theta) / q(z_{i,m-1}; \delta)}$ 
6:   if  $u < r$  then
7:     Accept  $z_{i,m}$ 
8:   else
9:      $z_{i,m} \leftarrow z_{i,m-1}$ 
10:  end if
11: end for
12: Output:  $z_{i,M}$ 
```

---

634 **Monte Carlo study:** We conduct a Monte Carlo study to showcase the benefits of our scheme.  $M =$   
635 50 datasets have been simulated using the following PK parameters values:  $T_{\text{pop}}^{\text{lag}} = 1$ ,  $ka_{\text{pop}} = 1$ ,  
636  $V_{\text{pop}} = 8$ ,  $k_{\text{pop}} = 0.1$ ,  $\omega_{T^{\text{lag}}} = 0.4$ ,  $\omega_{ka} = 0.5$ ,  $\omega_V = 0.2$ ,  $\omega_k = 0.3$  and  $\sigma^2 = 0.5$ . We define  
637 the mean square distance over the  $M$  replicates  $E_k(\ell) = \frac{1}{M} \sum_{m=1}^M \left( \theta_k^{(m)}(\ell) - \theta^* \right)^2$  and plot it  
638 against the epochs (passes over the data) Figure 4. Note that the MC-step (5) is performed using a  
639 Metropolis Hastings procedure since the posterior distribution under the model  $\theta$  noted  $p(z_i | y_i, \theta)$   
640 is intractable due to the nonlinearity of the model (43). Figure 4 shows clear advantage of variance  
641 reduced methods (vrTTEM and fiTTEM) avoiding the twists and turns displayed by the incremental  
642 and the batch methods.

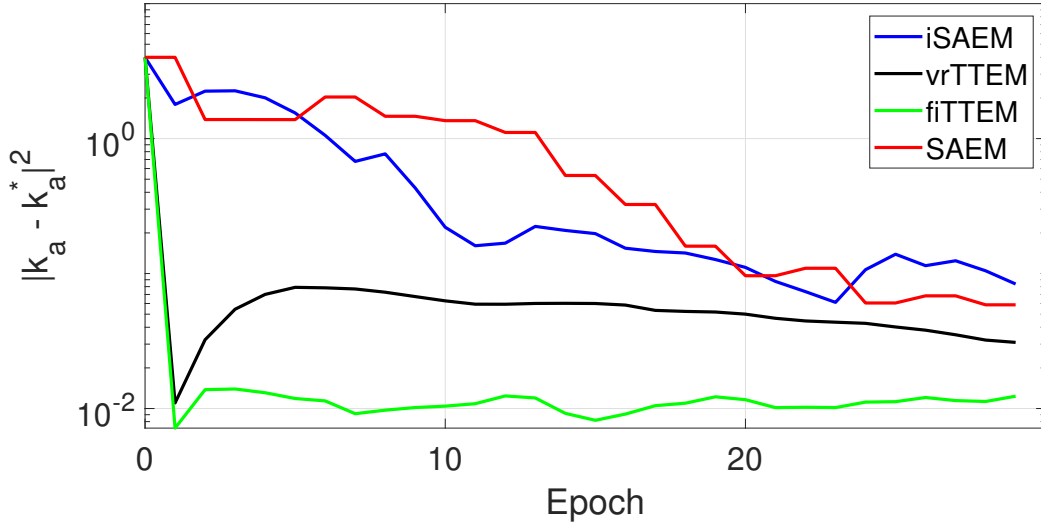


Figure 4: Precision  $|ka^{(k)} - ka^*|^2$  per epoch