
Convergent Adaptive Gradient Methods in Decentralized Optimization

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Abstract

Adaptive gradient methods including Adam, AdaGrad, and their variants have been very successful for training deep learning models, such as neural networks, in the past few years. Meanwhile, given the need for distributed training procedure, the realm of distributed optimization algorithms is at the center of attention. With the growth of computing power and the need for using machine learning models on mobile devices, the communication cost of distributed training algorithms needs careful consideration. In response to this, more and more attention is shifted from the traditional parameter server training paradigm to the decentralized one, which usually requires lower communication costs. In this paper, we try to rigorously incorporate adaptive gradient methods into decentralized training procedures, coming up with convergent decentralized adaptive gradient methods. Specifically, we propose a general algorithmic framework that can convert existing adaptive gradient methods to their decentralized counterparts. In addition, we rigorously analyze the convergence behavior of the proposed algorithmic framework and show that if a given adaptive gradient method converges, under some specific conditions, then its decentralized counterpart is also convergent.

1 Introduction

Distributed training of machine learning models is drawing growing attention in the past few years due to its practical benefits and necessities. Given the evolution of computing capabilities of CPUs and GPUs, computation time in distributed settings is gradually dominated by the communication time in many circumstances [Chilimbi et al., 2014, McMahan et al., 2016]. As a result, a large amount of recent works has been focussing on reducing communication cost for distributed learning [Alistarh et al., 2017, Lin et al., 2017, Wangni et al., 2018, Stich et al., 2018, Wang et al., 2018, Tang et al., 2019]. In the traditional parameter (central) server setting, where a parameter server is employed to manage communication in the whole network, many effective communication reductions have been proposed based on gradient compression [Aji and Heafield, 2017] and quantization [Chen et al., 2010, Ge et al., 2013, Jegou et al., 2010] techniques. Despite these communication reduction techniques, its cost still, usually, scales linearly with the number of workers. Due to this limitation and with the sheer size of decentralized devices, the *decentralized training paradigm* [Duchi et al., 2011b], where the parameter server is removed and each node only communicates with its neighbors, is drawing attention. It has been shown in Lian et al. [2017] that decentralized training algorithms can outperform parameter server-based algorithms when the training bottleneck is the communication cost. The decentralized paradigm is also preferred when a central parameter server is not available.

In light of recent advances in nonconvex optimization, an effective way to accelerate training is by using adaptive gradient methods like AdaGrad [Duchi et al., 2011a], Adam [Kingma and Ba, 2014] or AMSGrad [Reddi et al., 2019]. Their popularity are due to their practical benefits in training neural networks, featured by faster convergence and ease of parameter tuning compared with Stochastic Gradient Descent (SGD) [Robbins and Monro, 1951]. Despite a large amount of studies within the

distributed optimization literature, there have been few works considering bringing adaptive gradient methods into distributed training, largely due to the lack of understanding of adaptive gradient methods convergence behaviors. Notably, Reddi et al. [2020] develop the first decentralized ADAM method for distributed optimization problems with a direct application to federated learning. An inner loop is employed to compute mini-batch gradients on each worker nodes and a global adaptive step is performed to update the global parameter at each central-server iteration. Yet, in the settings of our paper, nodes can only communicate with their neighbors while a server/worker communication is needed in [Reddi et al., 2020]. Designing adaptive methods in such settings is highly non-trivial due to the already complicated update rules and to the interaction between the effect of using adaptive learning rates and the decentralized communication protocols. This paper is an attempt at bridging the gap between both realms in nonconvex optimization. Our contributions are summarized as follows:

- In this paper, we investigate the possibility of using any adaptive gradient methods in the decentralized training paradigm. We develop a general technique that converts an adaptive gradient method from a centralized method to its decentralized variant.
- By using our proposed technique, we present a new decentralized optimization algorithm, called decentralized AMSGrad, as the decentralized counterpart of AMSGrad.
- We provide a theoretical verification interface for analyzing the behavior of decentralized adaptive gradient methods obtained as a result of our technique. Built upon our proposed analysis framework for that type of decentralized algorithms, we can characterize the convergence rate of decentralized AMSGrad, which is the first convergent decentralized adaptive gradient method, to the best of our knowledge.

A *novel technique* in our framework is a mechanism to enforce a consensus on adaptive learning rates at different nodes. We show the importance of consensus on adaptive learning rates by proving a divergent problem instance for a recently proposed decentralized adaptive gradient method, namely, DADAM [Nazari et al., 2019], a decentralized version of ADAM, which lacks of consensus mechanisms on those nodes adaptive learning rates.

After having presented existing related work and important concepts of decentralized adaptive methods in Section 2, we develop our general framework for converting any adaptive gradient algorithm in its decentralized counterpart along with their rigorous finite-time convergence analysis in Section 3 concluded by some illustrative examples of our framework’s behavior in practice.

Notations: $x_{t,i}$ denotes variable x at node i and iteration t . $\|\cdot\|_{abs}$ denotes the entry-wise L_1 norm of a matrix, i.e. $\|A\|_{abs} = \sum_{i,j} A_{i,j}$. We introduce important notations used throughout the paper: for any $t > 0$, $G_t := [g_{t,N}]$ where $[g_{t,N}]$ denotes the vector $[g_{t,1}, g_{t,2}, \dots, g_{t,N}]$, $M_t := [m_{t,N}]$, $X_t := [x_{t,N}]$, $\bar{\nabla}f(X_t) := \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i})$, $U_t := [u_{t,N}]$, $\tilde{U}_t := [\tilde{u}_{t,N}]$, $V_t := [v_{t,N}]$, $\hat{V}_t := [\hat{v}_{t,N}]$, $\bar{X}_t := \frac{1}{N} \sum_{i=1}^N x_{t,i}$, $\bar{U}_t := \frac{1}{N} \sum_{i=1}^N u_{t,i}$ and $\bar{\tilde{U}}_t := \frac{1}{N} \sum_{i=1}^N \tilde{u}_{t,i}$.

2 Decentralized Adaptive Training and Divergence of DADAM

2.1 Related Work

Decentralized optimization: Traditional decentralized optimization methods include well-know algorithms such as ADMM [Boyd et al., 2011], Dual Averaging [Duchi et al., 2011b], Distributed Subgradient Descent [Nedic and Ozdaglar, 2009]. More recent algorithms include Extra [Shi et al., 2015], Next [Di Lorenzo and Scutari, 2016] and Prox-PDA [Hong et al., 2017]. While these algorithms are commonly used in applications other than deep learning, recent algorithmic advances in the machine learning community have shown that decentralized optimization can be useful for training deep models such as neural networks as well. Lian et al. [2017] demonstrate that a stochastic version of Decentralized Subgradient Descent can outperform parameter server-based algorithms when the communication cost is high. Tang et al. [2018] propose the D^2 algorithm improving the convergence rate over Stochastic Subgradient Descent. Assran et al. [2018] propose the Stochastic Gradient Push that is more robust to network failures for training neural networks. The study of decentralized training algorithms in the machine learning community is only at its initial stage. No existing work, to our knowledge, has seriously considered designing adaptive gradient methods in the setting of decentralized learning. One noteworthy work [Nazari et al., 2019] propose a decentralized version of AMSGrad [Reddi et al., 2019] and is proven to satisfy some non-standard regret.

Adaptive gradient methods: Adaptive gradient methods have been popular in recent years due to their superior performance in training neural networks. Most commonly used adaptive methods include AdaGrad [Duchi et al., 2011a] or Adam [Kingma and Ba, 2014] and their variants. Key features of such methods lie in the use of momentum and adaptive learning rates (which means that the learning rate is changing during the optimization and is anisotropic, i.e. depends on the dimension). The method of reference, Adam, has been analyzed in [Reddi et al., 2019] where the authors point out an error in previous convergence analyses. Since then, a variety of papers have been focussing on analyzing the convergence behavior of the numerous existing adaptive gradient methods. Ward et al. [2018] and Li and Orabona [2018] derive convergence guarantees for a variant of AdaGrad without coordinate-wise learning rates. Chen et al. [2018] analyze the convergence behavior of a broad class of algorithms including AMSGrad and AdaGrad. Zou and Shen [2018] provide a unified convergence analysis for AdaGrad with momentum. Noticeable recent works on adaptive gradient methods can be found in [Agarwal et al., 2018, Luo et al., 2019, Zaheer et al., 2018].

2.2 Decentralized Optimization

In distributed optimization (with N nodes), we aim at solving the following problem

$$\min_{x \in \mathbb{R}^d} \frac{1}{N} \sum_{i=1}^N f_i(x), \quad (1)$$

where x is the vector of parameters and f_i is only accessible by the i th node. Through the prism of empirical risk minimization procedures, f_i can be viewed as the average loss of the data samples located at node i . Throughout the paper, we make the following assumptions for analyzing the convergence behavior of the different algorithms:

A1. For all $i \in [N]$, f_i is differentiable and the gradients is L -Lipschitz, i.e., for all $(x, y) \in \mathbb{R}^d$, $\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|$.

A2. We assume that, at iteration t , node i accesses a stochastic gradient $g_{t,i}$. The stochastic gradients and the gradients of f_i have bounded L_∞ norms, i.e. $\|g_{t,i}\| \leq G_\infty$, $\|\nabla f_i(x)\|_\infty \leq G_\infty$.

A3. The gradient estimators are unbiased and each coordinate have bounded variance, i.e. $\mathbb{E}[g_{t,i}] = \nabla f_i(x_{t,i})$ and $\mathbb{E}[(g_{t,i} - \nabla f_i(x_{t,i}))_j^2] \leq \sigma^2, \forall t, i, j$.

Assumptions A1 and A3 are standard in distributed optimization literature. A2 is slightly stronger than the traditional assumption that the estimator has bounded variance, but is commonly used for the analysis of adaptive gradient methods [Chen et al., 2018, Ward et al., 2018]. Note that the bounded gradient estimator assumption in A2 implies the bounded variance assumption in A3. In decentralized optimization, the nodes are connected as a graph and each node only communicates to its neighbors. In such case, one usually constructs a $N \times N$ matrix W for information sharing when designing new algorithms. We denote λ_i to be its i th largest eigenvalue and define $\lambda \triangleq \max(|\lambda_2|, |\lambda_N|)$. As can be expected, W cannot be arbitrary, the key properties required for W are listed in A4.

A4. The matrix W satisfies: (I) $\sum_{j=1}^N W_{i,j} = 1$, $\sum_{i=1}^N W_{i,j} = 1$, $W_{i,j} \geq 0$, (II) $\lambda_1 = 1$, $|\lambda_2| < 1$, $|\lambda_N| < 1$ and (III) $W_{i,j} = 0$ if node i and node j are not neighbors.

We now present the failure to converge of current decentralized adaptive method before introducing our proposed framework.

2.3 Divergence of DADAM

Recently, Nazari et al. [2019] initiated a trial to bring adaptive gradient methods into decentralized optimization, the resulting algorithm is DADAM, which is shown in Algorithm 1. DADAM is essentially a decentralized version of AMSGrad and the key modification is the use of a consensus step on optimization variable x to transmit information across the network, encouraging convergence. The matrix W is a doubly stochastic matrix (which satisfies A4) for achieving average consensus of x . Introducing such mixing matrix is a standard approach for decentralizing an algorithm, such as distributed gradient descent [Nedic and Ozdaglar, 2009, Yuan et al., 2016]. It is proven in Nazari et al. [2019] that DADAM admits a non-standard regret bound in the online setting, however, whether the algorithm can converge to stationary points in standard offline settings such training neural networks is still unknown.

138 In the following, we show the DADAM may fail to
 139 converge in the offline optimization settings.

140 **Theorem 1.** *There exist a problem satisfying*
 141 *A1-A4 where DADAM fail to converge.*

142 *Proof.* Consider a 1 dimensional optimiza-
 143 tion problem distributed on two nodes
 144 $\min_x \frac{1}{2} \sum_{i=1}^2 f_i(x)$ where $f_i(x) = \frac{1}{2}(x - a_i)^2$
 145 and $a_1 = 0, a_2 = 1$. The network contains
 146 only two nodes and the matrix W satisfies
 147 $W_{ij} = \frac{1}{2}$ for all i, j . For simplicity, we consider
 148 running DADAM with $\beta_1 = \beta_2 = \beta_3 = 0$ and
 149 $\epsilon = 0.6$. Suppose we initialize DADAM at
 150 $x_{1,i} = 0$ for all $i \in [N]$ and use the following
 151 learning rate $\alpha = 0.001$. We have at $x_{1,i} = 0$,
 152 $\nabla f_1(x_{1,1}) = 0, \nabla f_2(x_{1,2}) = 1$, leading to $\hat{v}_{1,1} = 0.6$ and $\hat{v}_{1,2} = 1$. Thus, from step 1, we will have
 153 $\hat{v}_{1,2} \geq 1$. In addition, it is can be easily proved that, with the stepsize selection, we always have
 154 $\hat{v}_{1,1} < 1$, in fact, it will not reach 0.6. Thus, in the next iterations, the gradient of losses on node 1
 155 and 2 will be scaled differently. This scaling is equivalent to running gradient descent on a objective
 156 where the losses of the two nodes are scaled by different factors. In such case, the algorithm will
 157 converge to a stationary point of a weighted average of the loss on node 1. Recall that the problem we
 158 tackle to illustrate Theorem 1 is a quadratic problem with only one minimizer. Then, since the weight
 159 of the losses on the two nodes are different and that the unbalanced weights on the two functions
 160 yields a different minimizer, the algorithm will not converge to the unique stationary point of the
 161 original loss (which is $x = 0.5$). \square

162 Theorem 1 claims that even though DADAM is proven to satisfy some regret bounds, see [Nazari
 163 et al., 2019], it can fail to converge to stationary points in the nonconvex offline setting, which is
 164 a common setting for training neural networks. We conjecture that this inconsistency is due to the
 165 definition of the regret in [Nazari et al., 2019]. In the next section, we design decentralized adaptive
 166 gradient methods that are guaranteed to converge to stationary points of some defined objective and
 167 provide a characterization of that convergence in finite-time and independently of the initialization.

168 3 Decentralized Adaptive Gradient Methods and their Convergence

169 In this section, we discuss the difficulties of designing adaptive gradient methods in decentralized
 170 optimization and introduce an algorithmic framework that converts existing convergent adaptive gra-
 171 dient methods to their decentralized counterparts. We also develop the first convergent decentralized
 172 adaptive gradient method, converted from AMSGrad, as an instance of this proposed framework.

173 3.1 Importance and Difficulties of Consensus on Adaptive Learning Rates

174 The divergent example in the previous section implies that we should synchronize the adaptive
 175 learning rates on different nodes. This can be easily achieved in the parameter server setting where
 176 all the nodes are sending their gradients to a central server at each iteration. The parameter server
 177 can then exploit the received gradients to maintain a sequence of synchronized adaptive learning
 178 rates when updating the parameters, see [Reddi et al., 2020]. However, in our setting of decentralized
 179 training, every node can only communicate with its neighbors and such central parameter server does
 180 not exist. Under that setting, the information for updating the adaptive learning rates can only be
 181 shared locally instead of broadcasted over the whole network. This makes it impossible to obtain, in a
 182 single iteration, a synchronized adaptive learning rate update using all the information in the network.

183 *Systemic Approach:* On a systemic level, one way to alleviate this bottleneck is to design communi-
 184 cation protocols in order to give each node access to the same aggregated gradients over the whole
 185 network at least periodically if not at every iteration. Therefore, the nodes can update their individual
 186 adaptive learning rates based on the same shared information. However, such solution introduces an
 187 extra communication cost since it involves broadcasting over the network.

188 *Algorithmic Approach:* Our contributions being on an algorithmic level, another way to solve the
 189 aforementioned problem is by letting the sequences of adaptive learning rates, present on different

Algorithm 1 DADAM (with N nodes)

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1: Input:  $\alpha$ , current point  $X_t, u_{\frac{1}{2},i} = \hat{v}_{0,i} = \epsilon \mathbf{1}$ ,  

    $m_0 = 0$  and mixing matrix  $W$   

2: for  $t = 1, 2, \dots, T$  do  

3:   for all  $i \in [N]$  do in parallel  

4:      $g_{t,i} \leftarrow \nabla f_i(x_{t,i}) + \xi_{t,i}$   

5:      $m_{t,i} = \beta_1 m_{t-1,i} + (1 - \beta_1) g_{t,i}$   

6:      $v_{t,i} = \beta_2 v_{t-1,i} + (1 - \beta_2) g_{t,i}^2$   

7:      $\hat{v}_{t,i} = \beta_3 \hat{v}_{t-1,i} + (1 - \beta_3) \max(\hat{v}_{t-1,i}, v_{t,i})$   

8:      $x_{t+\frac{1}{2},i} = \sum_{j=1}^N W_{ij} x_{t,j}$   

9:      $x_{t+1,i} = x_{t+\frac{1}{2},i} - \alpha \frac{m_{t,i}}{\sqrt{\hat{v}_{t,i}}}$   

10:  end for

```

nodes, to gradually *consent*, through the iterations. Intuitively, if the adaptive learning rates can consent fast enough, the difference among the adaptive learning rates on different nodes will not affect the convergence of the algorithm. Consequently, no extra communication costs are introduced. We now develop this idea within the adaptive methods stressing on the need for a relatively low-cost and easy-to-implement consensus of adaptive learning rates.

3.2 Decentralized Adaptive Gradient Unifying Framework

As mentioned before, we need to choose a method to implement consensus of adaptive learning rates. While each node can have different $\hat{v}_{t,i}$ in DADAM, one can keep track of the min/max/average of these adaptive learning rates and use this quantity to update the adaptive learning rates. The predefinition of some convergent lower and upper bounds may also lead to a gradual synchronization of the adaptive learning rates on different nodes as developed for AdaBound in [Luo et al., 2019]. In this paper, we opt for the average consensus on $\hat{v}_{t,i}$, see operator r_t line 6 of Algorithm 2. Since for adaptive gradient methods such as AdaGrad or Adam, $\hat{v}_{t,i}$ approximates the second moment of the gradient estimator, the average of the estimations of those second moments from different nodes is an estimation of second moment on the whole network. Also, this design will not introduce any extra hyperparameters that can potentially complicate the tuning process. Our method is presented Algorithm 2. We now present the main convergence result for our class of methods:

Theorem 2. Assume A1-A4. Set $\alpha = 1/\sqrt{Td}$. When $\alpha \leq \frac{\epsilon^{0.5}}{16L}$, Algorithm 2 yields the following regret bound

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \frac{\sqrt{d}}{\sqrt{T}} \left(\mathbb{E}[f(Z_1)] - \min_z f(z) + \frac{\sigma^2}{N} \right) + \frac{C_2}{T} + \frac{C_3}{T^{1.5}d^{0.5}} \\ &\quad + \left(\frac{C_4}{TN^{0.5}} + \frac{C_5}{T^{1.5}d^{0.5}N^{0.5}} \right) \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right], \end{aligned} \quad (2)$$

where $\|\cdot\|_{abs}$ denotes the entry-wise L_1 norm of a matrix (i.e. $\|A\|_{abs} = \sum_{i,j} |A_{i,j}|$). The constants $C_1 = \max(4, 4L/\epsilon)$, $C_2 = 6((\beta_1/(1-\beta_1))^2 + 1/(1-\lambda)^2)LG_\infty^2/\epsilon^{1.5}$, $C_3 = 16L^2(1-\lambda)G_\infty^2/\epsilon^2$, $C_4 = 2/(\epsilon^{1.5}(1-\lambda))(\lambda + \beta_1/(1-\beta_1))G_\infty^2$, $C_5 = 2/(\epsilon^2(1-\lambda))L(\lambda + \beta_1/(1-\beta_1))G_\infty^2 + 4/(\epsilon^2(1-\lambda))LG_\infty^2$ are independent of d , T and N . In addition, $\frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \leq \alpha^2 \left(\frac{1}{1-\lambda} \right)^2 dG_\infty^2 \frac{1}{\epsilon}$ which quantifies the consensus error.

Theorem 2 shows that if $\mathbb{E}[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}] = o(T)$ and \bar{U}_t is upper bounded, then Algorithm 2 is guaranteed to converge to stationary points of the regret function. Intuitively, this means that if the adaptive learning rates on different nodes do not change too fast, the algorithm can converge. This is true as stated in [Chen et al., 2018] where it is shown that if such condition is violated, the algorithm can diverge. Furthermore, Theorem 2 conveys the benefits of using more nodes. As N becomes larger, the term σ^2/N will be small. This is also strengthened by the fact that with the growth of N , the training process tends to be more stable.

We now present, in Algorithm 3, a notable special case of our algorithmic framework, namely Decentralized AMSGrad, which is a decentralized variant of AMSGrad. Compared with DADAM, the above algorithm exhibits a dynamic average consensus mechanism to keep track of the average of $\{\hat{v}_{t,i}\}_{i=1}^N$, stored as $\tilde{u}_{t,i}$ on i th node, and uses $u_{t,i} = \max(\tilde{u}_{t,i}, \epsilon)$ for updating the adaptive learning rate for i th node. As the number of iteration grows, even though $\hat{v}_{t,i}$ on different nodes can converge to different constants, all the $u_{t,i}$ will converge to the same number $\lim_{t \rightarrow \infty} 1/N \sum_{i=1}^N \hat{v}_{t,i}$ if the limit exists.

Algorithm 2 Decentralized Adaptive Gradient Method (with N nodes)

```

1: Input:  $\alpha$ , initial point  $x_{1,i} = x_{init}, u_{\frac{1}{2},i} = \hat{v}_{0,i}, m_{0,i} = 0$ , mixing matrix  $W$ 
2: for  $t = 1, 2, \dots, T$  do
3:   for all  $i \in [N]$  do in parallel
4:      $g_{t,i} \leftarrow \nabla f_i(x_{t,i}) + \xi_{t,i}$ 
5:      $m_{t,i} = \beta_1 m_{t-1,i} + (1 - \beta_1)g_{t,i}$ 
6:      $\hat{v}_{t,i} = r_t(g_{1,i}, \dots, g_{t-1,i})$ 
7:      $x_{t+\frac{1}{2},i} = \sum_{j=1}^N W_{ij}x_{t,j}$ 
8:      $\tilde{u}_{t,i} = \sum_{j=1}^N W_{ij}\tilde{u}_{t-\frac{1}{2},j}$ 
9:      $u_{t,i} = \max(\tilde{u}_{t,i}, \epsilon)$ 
10:     $x_{t+1,i} = x_{t+\frac{1}{2},i} - \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}}$ 
11:     $\tilde{u}_{t+\frac{1}{2},i} = \tilde{u}_{t,i} - \hat{v}_{t-1,i} + \hat{v}_{t,i}$ 
12:  end for

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Using this average consensus mechanism enables the consensus of adaptive learning rates on different nodes, which consequentially guarantees convergence to stationary points. The consensus of adaptive learning rates is the key difference between decentralized AMSGrad and DADAM and is the reason why decentralized AMSGrad is a convergent algorithm while DADAM is not. One may noticed that decentralized AMSGrad does not deduce to AMSGrad because $u_{t,i}$ in line 10 is calculated based on $v_{t-1,i}$ instead of $v_{t,i}$. This encourages parallel execution of gradient computation and communication. Specifically, line 4-7 in Algorithm 3 and Algorithm 2 can be executed in parallel with line 8-9 to overlap communication and computation time. If $u_{t,i}$ depends on $v_{t,i}$ which in turn depends on $g_{t,i}$, the gradient computation must finish before the consensus step of adaptive learning rate in line 9. This can slow down per-iteration running time of the algorithm. To avoid such delayed adaptive learning, adding $\tilde{u}_{t-\frac{1}{2},i} = \tilde{u}_{t,i} - \hat{v}_{t-1,i} + \hat{v}_{t,i}$ before line 9 and get rid of line 12 in Algorithm 2 is an option. Similar convergence guarantees will hold since one can easily modify our proof of Theorem 2 for such update rule. As stated above, Algorithm 3 converges, with the following rate:

Theorem 3. Assume A1-A4. Set $\alpha = 1/\sqrt{Td}$. When $\alpha \leq \frac{\epsilon^{0.5}}{16L}$, Algorithm 3 yields the following regret bound

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \leq C'_1 \frac{\sqrt{d}}{\sqrt{T}} \left(\mathbb{E}[f(Z_1)] - \min_z f(z) + \frac{\sigma^2}{N} \right) + \frac{C'_2}{T} + \frac{d}{T} \sqrt{N} C'_4 + \frac{\sqrt{d}}{T^{1.5}} \sqrt{N} C'_5,$$

where $C'_1 = C_1$, $C'_2 = C_2$, $C'_3 = C_3$, $C'_4 = C_4 G_\infty^2$ and $C'_5 = C_5 G_\infty^2$. C_1, C_2, C_3, C_4, C_5 are constants independent of d, T and N defined in Theorem 2. In addition, the consensus of variables at different nodes is given by $\frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \leq \frac{1}{T} \left(\frac{1}{1-\lambda} \right)^2 G_\infty^2 \frac{1}{\epsilon}$.

Theorem 3 shows that Algorithm 3 converges with a rate of $\mathcal{O}(\sqrt{d}/\sqrt{T})$ when T is large, which is the best known convergence rate under the given assumptions. Note that in some related works, SGD admits a convergence rate of $\mathcal{O}(1/\sqrt{T})$ without any dependence on the dimension of the problem. Such improved convergence rate is derived under the assumption that the gradient estimator have a bounded L_2 norm, which can thus hide a dependency of \sqrt{d} in the final convergence rate.

3.3 Convergence Analysis

Proof of Theorem 2. The detailed proof of this section is reported in the supplementary material. We now present a proof sketch for our main convergence result of Algorithm 2.

Step 1: Reparameterization. Similarly to [Yan et al., 2018, Chen et al., 2018] with SGD (with momentum) and centralized adaptive gradient methods, define the following auxiliary sequence:

$$Z_t = \bar{X}_t + \frac{\beta_1}{1-\beta_1} (\bar{X}_t - \bar{X}_{t-1}), \quad (3)$$

with $\bar{X}_0 \triangleq \bar{X}_1$. Such an auxiliary sequence can help us deal with the bias brought by the momentum and simplifies the convergence analysis. An intermediary result needed to conduct our proof reads:

Lemma 1. For the sequence defined in (3), we have

$$Z_{t+1} - Z_t = \alpha \frac{\beta_1}{1-\beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}}$$

Algorithm 3 Decentralized AMSGrad (with N nodes)

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1: Input: learning rate  $\alpha$ , initial point  $x_{1,i} = x_{init}$ ,  $u_{\frac{1}{2},i} = \hat{v}_{0,i} = \epsilon \mathbf{1}$  (with  $\epsilon \geq 0$ ),  $m_{0,i} = 0$ , mixing matrix  $W$ 
2: for  $t = 1, 2, \dots, T$  do
3:   for all  $i \in [N]$  do in parallel
4:      $g_{t,i} \leftarrow \nabla f_i(x_{t,i}) + \xi_{t,i}$ 
5:      $m_{t,i} = \beta_1 m_{t-1,i} + (1 - \beta_1) g_{t,i}$ 
6:      $v_{t,i} = \beta_2 v_{t-1,i} + (1 - \beta_2) g_{t,i}^2$ 
7:      $\hat{v}_{t,i} = \max(\hat{v}_{t-1,i}, v_{t,i})$ 
8:      $x_{t+\frac{1}{2},i} = \sum_{j=1}^N W_{ij} x_{t,j}$ 
9:      $\tilde{u}_{t,i} = \sum_{j=1}^N W_{ij} \tilde{u}_{t-\frac{1}{2},j}$ 
10:     $u_{t,i} = \max(\tilde{u}_{t,i}, \epsilon)$ 
11:     $x_{t+1,i} = x_{t+\frac{1}{2},i} - \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}}$ 
12:     $\tilde{u}_{t+\frac{1}{2},i} = \tilde{u}_{t,i} - \hat{v}_{t-1,i} + \hat{v}_{t,i}$ 
13:  end for

```

Lemma 1 does not display any momentum term in $\frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}}$. This simplification is convenient since it is directly related to the current gradients instead of the exponential average of past gradients. *Step 2: Smoothness.* Using smoothness assumption A1 involves the following scalar product term: $\kappa_t := \langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) / \sqrt{\bar{U}_t} \rangle$ which can be lower bounded by:

$$\kappa_t \geq \frac{1}{2} \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{3}{2} \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{3}{2} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2.$$

The above inequality substituted in the smoothness condition $f(Z_{t+1}) \leq f(Z_t) + \langle \nabla f(Z_t), Z_{t+1} - Z_t \rangle + \frac{L}{2} \|Z_{t+1} - Z_t\|^2$ yields:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \leq \frac{2}{T\alpha} (\mathbb{E}[\Delta_f]) + \frac{L}{T\alpha} \sum_{t=1}^T \mathbb{E} [\|Z_{t+1} - Z_t\|^2] + \frac{2}{T} \frac{\beta_1}{1-\beta_1} T_1 + \frac{2}{T} T_2 + \frac{3}{T} T_3, \quad (4)$$

where $\Delta_f := \mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]$. T_1, T_2 and T_3 are three terms, defined in the supplementary material, and which can be tightly bounded from above. We first bound T_3 using the following quantities of interest:

$$\sum_{t=1}^T \|Z_t - \bar{X}_t\|^2 \leq T \left(\frac{\beta_1}{1-\beta_1} \right)^2 \alpha^2 d \frac{G_\infty^2}{\epsilon} \quad \text{and} \quad \sum_{t=1}^T \frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \leq T \alpha^2 \left(\frac{1}{1-\lambda} \right)^2 d G_\infty^2 \frac{1}{\epsilon}.$$

where $\lambda = \max(|\lambda_2|, |\lambda_N|)$ and recall that λ_i is i th largest eigenvalue of W .

Then, concerning the term T_2 , few derivations, not detailed here for simplicity, yields:

$$T_2 \leq \frac{G_\infty^2}{N} \mathbb{E} \left[\sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| -\sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs} \right]$$

where q_l is the eigenvector corresponding to l th largest eigenvalue of W and $\|\cdot\|_{abs}$ is the entry-wise L_1 norm of matrices. We can also show that

$$\sum_{t=1}^T \left\| -\sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs} \leq \sqrt{N} \sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs}$$

resulting in an upper bound for T_2 proportional to $\sum_{o=0}^{T-1} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs}$. Similarly:

$$T_1 \leq G_\infty^2 \frac{1}{2\epsilon^{1.5}} \frac{1}{\sqrt{N}} \mathbb{E} \left[\frac{1}{1-\lambda} \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right]$$

Step 3: Bounding the drift term variance. An important term that needs upper bounding in our proof is the variance of the gradients multiplied (element-wise) by the adaptive learning rate:

$$\mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \leq \mathbb{E}[\|\Gamma_u^f\|^2] + \frac{d}{N} \frac{\sigma^2}{\epsilon}$$

where $\Gamma_u^f := 1/N \sum_{i=1}^N \nabla f_i(x_{t,i}) / \sqrt{u_{t,i}}$. Two consecutive and simple bounding of the above yields:

$$\sum_{t=1}^T \mathbb{E}[\|\Gamma_u^f\|^2] \leq 2 \sum_{t=1}^T \mathbb{E}[\|\Gamma_{\bar{U}}^f\|^2] + 2 \sum_{t=1}^T \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \left\| \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right\| \right]$$

and

$$\sum_{t=1}^T \mathbb{E}[\|\Gamma_{\bar{U}}^f\|^2] \leq 2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t) - \nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right]. \quad (5)$$

Then, by plugging the LHS of (5) in (4), and further bounding as operated for T_2, T_3 (see supplement), we obtain the bound in Theorem 2.

Proof of Theorem 3 Recall the bound in (2) of Theorem 2. Since Algorithm 3 is a special case of Algorithm 2, the remaining of the proof consists in characterizing the growth rate of $\mathbb{E}[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}]$. By construction, \hat{V}_t is non decreasing, then it can be shown that $\mathbb{E}[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}] = \mathbb{E}[\sum_{i=1}^N \sum_{j=1}^d (-[\hat{v}_{0,i}]_j + [\hat{v}_{T-1,i}]_j)]$. Besides, since for all t, i , $\|g_{t,i}\|_\infty \leq G_\infty$ and $v_{t,i}$ is an exponential moving average of $g_{k,i}^2, k = 1, 2, \dots, t$, we have $|\hat{v}_{t,i}|_j \leq G_\infty^2$ for all t, i, j . By construction of \hat{V}_t , we also observe that each element of \hat{V}_t cannot be greater than G_∞^2 , i.e. $|\hat{v}_{t,i}|_j \leq G_\infty^2$ for all t, i, j . Given that $[\hat{v}_{0,i}]_j \geq 0$, we have

$$\mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] = \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^d (-[\hat{v}_{0,i}]_j + [\hat{v}_{T-1,i}]_j) \right] \leq \sum_{i=1}^N \sum_{j=1}^d \mathbb{E}[G_\infty^2] = NdG_\infty^2.$$

Substituting into (2) yields the desired convergence bound for Algorithm 3.

3.4 Illustrative Numerical Experiments

In this section, we conduct some experiments to test the performance of Decentralized AMSGrad, developed in Algorithm 3, on both *homogeneous* data and *heterogeneous* data distribution (i.e. the data generating distribution on different nodes are assumed to be different). Comparison with DADAM and the decentralized stochastic gradient descent (DGD) developed in [Lian et al., 2017] are conducted. We train a Convolutional Neural Network (CNN) with 3 convolution layers followed by a fully connected layer on MNIST [LeCun, 1998]. We set $\epsilon = 10^{-6}$ for both Decentralized AMSGrad and DADAM. The learning rate is chosen from the grid $[10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}]$ based on validation accuracy for all algorithms. In the following experiments, the graph contains 5 nodes and each node can only communicate with its two adjacent neighbors forming a cycle. Regarding the mixing matrix W , we set $W_{ij} = 1/3$ if nodes i and j are neighbors and $W_{ij} = 0$ otherwise. More details and experiments can be found in the supplementary material of our paper.

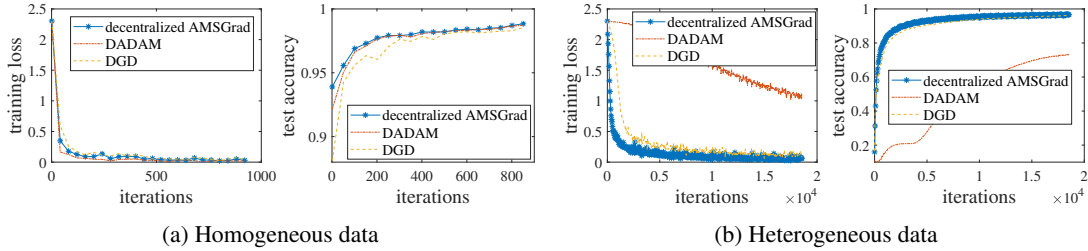


Figure 1: Training loss and Testing accuracy for homogeneous and heterogeneous data

Homogeneous data: The whole dataset is shuffled and evenly split into different nodes. We see, Figure 1(a), that decentralized AMSGrad and DADAM perform quite similarly while DGD is much slower both in terms of training loss and test accuracy. Though the (possible) non convergence of DADAM, mentioned in this paper, its performance are empirically good on homogeneous data. The reason is that the adaptive learning rates tend to be similar on different nodes in presence of homogeneous data distribution. We thus compare these algorithms under the heterogeneous regime.

Heterogeneous data: Here, each node only contains training data with two labels out of ten. We can see that each algorithm converges significantly slower than with homogeneous data. Especially, the performance of DADAM deteriorates significantly. Decentralized AMSGrad achieves the best training and testing performance in that setting as observed Figure 1(b).

4 Conclusion

This paper studies the problem of designing adaptive gradient methods for decentralized training. We propose a unifying algorithmic framework that can convert existing adaptive gradient methods to decentralized settings. With rigorous convergence analysis, we show that if the original algorithm satisfies converges under some minor conditions, the converted algorithm obtained using our proposed framework is guaranteed to converge to stationary points of the regret function. By applying our framework to AMSGrad, we propose the first convergent adaptive gradient methods, namely Decentralized AMSGrad. Experiments show that the proposed algorithm achieves better performance than the baselines.

5 Broader Impact of Our Work

We hope that efforts towards developing decentralized optimization methods can be put to good use for practical applications where data can not be shared in a central server for privacy reasons. Indeed, when the data is sensible and captured on several devices (nodes), we must come up with efficient and low-cost optimization methods for fitting complex models. We believe our work is a step forward leveraging current state-of-the-art optimization methods for decentralized optimization.

References

- Naman Agarwal, Brian Bullins, Xinyi Chen, Elad Hazan, Karan Singh, Cyril Zhang, and Yi Zhang. The case for full-matrix adaptive regularization. *arXiv preprint arXiv:1806.02958*, 2018.
- Alham Fikri Aji and Kenneth Heafield. Sparse communication for distributed gradient descent. *arXiv preprint arXiv:1704.05021*, 2017.
- Dan Alistarh, Demjan Grubic, Jerry Li, Ryota Tomioka, and Milan Vojnovic. Qsgd: Communication-efficient sgd via gradient quantization and encoding. In *Advances in Neural Information Processing Systems*, pages 1709–1720, 2017.
- Mahmoud Assran, Nicolas Loizou, Nicolas Ballas, and Michael Rabbat. Stochastic gradient push for distributed deep learning. *arXiv preprint arXiv:1811.10792*, 2018.
- Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato, Jonathan Eckstein, et al. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends® in Machine learning*, 3(1):1–122, 2011.
- Xiangyi Chen, Sijia Liu, Ruoyu Sun, and Mingyi Hong. On the convergence of a class of adam-type algorithms for non-convex optimization. *arXiv preprint arXiv:1808.02941*, 2018.
- Yongjian Chen, Tao Guan, and Cheng Wang. Approximate nearest neighbor search by residual vector quantization. *Sensors*, 10(12):11259–11273, 2010.
- Trishul Chilimbi, Yutaka Suzue, Johnson Apacible, and Karthik Kalyanaraman. Project adam: Building an efficient and scalable deep learning training system. In *11th {USENIX} Symposium on Operating Systems Design and Implementation ({OSDI} 14)*, pages 571–582, 2014.
- Paolo Di Lorenzo and Gesualdo Scutari. Next: In-network nonconvex optimization. *IEEE Transactions on Signal and Information Processing over Networks*, 2(2):120–136, 2016.
- John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of Machine Learning Research*, 12(Jul):2121–2159, 2011a.
- John C Duchi, Alekh Agarwal, and Martin J Wainwright. Dual averaging for distributed optimization: Convergence analysis and network scaling. *IEEE Transactions on Automatic control*, 57(3):592–606, 2011b.
- Tiezheng Ge, Kaiming He, Qifa Ke, and Jian Sun. Optimized product quantization for approximate nearest neighbor search. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 2946–2953, 2013.
- Mingyi Hong, Davood Hajinezhad, and Ming-Min Zhao. Prox-pda: The proximal primal-dual algorithm for fast distributed nonconvex optimization and learning over networks. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 1529–1538. JMLR.org, 2017.
- Herve Jegou, Matthijs Douze, and Cordelia Schmid. Product quantization for nearest neighbor search. *IEEE transactions on pattern analysis and machine intelligence*, 33(1):117–128, 2010.
- Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.
- Yann LeCun. The mnist database of handwritten digits. <http://yann.lecun.com/exdb/mnist/>, 1998.

384 Xiaoyu Li and Francesco Orabona. On the convergence of stochastic gradient descent with adaptive
385 stepsizes. *arXiv preprint arXiv:1805.08114*, 2018.

386 Xiangru Lian, Ce Zhang, Huan Zhang, Cho-Jui Hsieh, Wei Zhang, and Ji Liu. Can decentralized
387 algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic
388 gradient descent. In *Advances in Neural Information Processing Systems*, pages 5330–5340, 2017.

389 Yujun Lin, Song Han, Huizi Mao, Yu Wang, and William J Dally. Deep gradient compression:
390 Reducing the communication bandwidth for distributed training. *arXiv preprint arXiv:1712.01887*,
391 2017.

392 Liangchen Luo, Yuanhao Xiong, Yan Liu, and Xu Sun. Adaptive gradient methods with dynamic
393 bound of learning rate. *arXiv preprint arXiv:1902.09843*, 2019.

394 H Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, et al. Communication-efficient
395 learning of deep networks from decentralized data. *arXiv preprint arXiv:1602.05629*, 2016.

396 Parvin Nazari, Davoud Ataee Tarzanagh, and George Michailidis. Dadam: A consensus-based
397 distributed adaptive gradient method for online optimization. *arXiv preprint arXiv:1901.09109*,
398 2019.

399 Angelia Nedic and Asuman Ozdaglar. Distributed subgradient methods for multi-agent optimization.
400 *IEEE Transactions on Automatic Control*, 54(1):48, 2009.

401 Sashank Reddi, Zachary Charles, Manzil Zaheer, Zachary Garrett, Keith Rush, Jakub Konečný,
402 Sanjiv Kumar, and H Brendan McMahan. Adaptive federated optimization. *arXiv preprint*
403 *arXiv:2003.00295*, 2020.

404 Sashank J Reddi, Satyen Kale, and Sanjiv Kumar. On the convergence of adam and beyond. *arXiv*
405 *preprint arXiv:1904.09237*, 2019.

406 Herbert Robbins and Sutton Monro. A stochastic approximation method. *The annals of mathematical*
407 *statistics*, pages 400–407, 1951.

408 Wei Shi, Qing Ling, Gang Wu, and Wotao Yin. Extra: An exact first-order algorithm for decentralized
409 consensus optimization. *SIAM Journal on Optimization*, 25(2):944–966, 2015.

410 Sebastian U Stich, Jean-Baptiste Cordonnier, and Martin Jaggi. Sparsified sgd with memory. In
411 *Advances in Neural Information Processing Systems*, pages 4447–4458, 2018.

412 Hanlin Tang, Xiangru Lian, Ming Yan, Ce Zhang, and Ji Liu. D²: Decentralized training over
413 decentralized data. *arXiv preprint arXiv:1803.07068*, 2018.

414 Hanlin Tang, Xiangru Lian, Tong Zhang, and Ji Liu. Doublesqueeze: Parallel stochastic gradient
415 descent with double-pass error-compensated compression. *arXiv preprint arXiv:1905.05957*, 2019.

416 Hongyi Wang, Scott Sievert, Shengchao Liu, Zachary Charles, Dimitris Papailiopoulos, and Stephen
417 Wright. Atomo: Communication-efficient learning via atomic sparsification. In *Advances in*
418 *Neural Information Processing Systems*, pages 9850–9861, 2018.

419 Jianqiao Wangni, Jialei Wang, Ji Liu, and Tong Zhang. Gradient sparsification for communication-
420 efficient distributed optimization. In *Advances in Neural Information Processing Systems*, pages
421 1299–1309, 2018.

422 Rachel Ward, Xiaoxia Wu, and Leon Bottou. Adagrad stepsizes: Sharp convergence over nonconvex
423 landscapes, from any initialization. *arXiv preprint arXiv:1806.01811*, 2018.

424 Yan Yan, Tianbao Yang, Zhe Li, Qihang Lin, and Yi Yang. A unified analysis of stochastic momentum
425 methods for deep learning. *arXiv preprint arXiv:1808.10396*, 2018.

426 Kun Yuan, Qing Ling, and Wotao Yin. On the convergence of decentralized gradient descent. *SIAM*
427 *Journal on Optimization*, 26(3):1835–1854, 2016.

- 428 Manzil Zaheer, Sashank Reddi, Devendra Sachan, Satyen Kale, and Sanjiv Kumar. Adaptive methods
429 for nonconvex optimization. In *Advances in Neural Information Processing Systems*, pages
430 9793–9803, 2018.
- 431 Fangyu Zou and Li Shen. On the convergence of weighted adagrad with momentum for training deep
432 neural networks. *arXiv preprint arXiv:1808.03408*, 2018.

433 A Proof of Theorem 2

434 To prove convergence of the algorithm, we first define an auxiliary sequence

$$Z_t = \bar{X}_t + \frac{\beta_1}{1 - \beta_1}(\bar{X}_t - \bar{X}_{t-1}) \quad (6)$$

435 with $\bar{X}_0 \triangleq \bar{X}_1$.

436 Then we have the following Lemma to characterize the difference of iterations of sequence Z_t .

437 **Lemma.** *For the sequence defined in (6), we have*

$$Z_{t+1} - Z_t = \alpha \frac{\beta_1}{1 - \beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \quad (7)$$

438 **Proof:** See Appendix C. □

439 Since $\mathbb{E}[g_{t,i}] = \nabla f(x_{t,i})$ and $u_{t,i}$ is a function of $G_{1:t-1}$ (which denotes G_1, G_2, \dots, G_{t-1}), we
440 have

$$\mathbb{E}_{G_t|G_{1:t-1}} \left[\frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right] = \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \quad (8)$$

441 By assuming smoothness (A1) we have

$$f(Z_{t+1}) \leq f(Z_t) + \langle \nabla f(Z_t), Z_{t+1} - Z_t \rangle + \frac{L}{2} \|Z_{t+1} - Z_t\|^2 \quad (9)$$

442 Substitute (54) into the above inequality and take expectation over G_t given $G_{1:t-1}$, we have

$$\begin{aligned} \mathbb{E}_{G_t|G_{1:t-1}}[f(Z_{t+1})] &\leq f(Z_t) - \alpha \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\rangle + \frac{L}{2} \mathbb{E}_{G_t|G_{1:t-1}}[\|Z_{t+1} - Z_t\|^2] \\ &\quad + \alpha \frac{\beta_1}{1 - \beta_1} \mathbb{E}_{G_t|G_{1:t-1}} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right] \end{aligned} \quad (10)$$

443 Then take expectation over $G_{1:t-1}$ and rearrange, we have

$$\begin{aligned} \alpha \mathbb{E} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\rangle \right] &\leq \mathbb{E}[f(Z_t)] - \mathbb{E}[f(Z_{t+1})] + \frac{L}{2} \mathbb{E}[\|Z_{t+1} - Z_t\|^2] \\ &\quad + \alpha \frac{\beta_1}{1 - \beta_1} \mathbb{E} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right] \end{aligned} \quad (11)$$

444 In addition, we have

$$\begin{aligned} &\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\rangle \\ &= \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{U_t}} \right\rangle + \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left(\frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{U_t}} \right) \right\rangle \end{aligned} \quad (12)$$

445 and the first term on RHS of the equality can be lower bounded as

$$\begin{aligned}
& \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\rangle \\
&= \frac{1}{2} \left\| \frac{\nabla f(Z_t)}{\bar{U}_t^{1/4}} \right\|^2 + \frac{1}{2} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i})}{\bar{U}_t^{1/4}} \right\|^2 - \frac{1}{2} \left\| \frac{\nabla f(Z_t) - \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i})}{\bar{U}_t^{1/4}} \right\|^2 \\
&\geq \frac{1}{4} \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 + \frac{1}{4} \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{1}{2} \left\| \frac{\nabla f(Z_t) - \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i})}{\bar{U}_t^{1/4}} \right\|^2 \\
&\quad - \frac{1}{2} \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{1}{2} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \\
&\geq \frac{1}{2} \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{3}{2} \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{3}{2} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \quad (13)
\end{aligned}$$

446 where the inequalities are all due to Cauchy-Schwartz.

447 Substituting (13) and (12) into (11), we get

$$\begin{aligned}
\frac{1}{2} \alpha \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq \mathbb{E}[f(Z_t)] - \mathbb{E}[f(Z_{t+1})] + \frac{L}{2} \mathbb{E}[\|Z_{t+1} - Z_t\|^2] \\
&\quad + \alpha \frac{\beta_1}{1 - \beta_1} \mathbb{E} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right] \\
&\quad - \alpha \mathbb{E} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left(\frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right) \right\rangle \right] \\
&\quad + \frac{3}{2} \alpha \mathbb{E} \left[\left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 + \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \quad (14)
\end{aligned}$$

448 Then sum over the above inequality from $t = 1$ to T and divide both sides by $T\alpha/2$, we have

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq \frac{2}{T\alpha} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) + \frac{L}{T\alpha} \sum_{t=1}^T \mathbb{E}[\|Z_{t+1} - Z_t\|^2] \\
&\quad + \frac{2}{T} \frac{\beta_1}{1 - \beta_1} \underbrace{\sum_{t=1}^T \mathbb{E} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right]}_{T_1} \\
&\quad + \frac{2}{T} \sum_{t=1}^T \underbrace{\mathbb{E} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left(\frac{1}{\sqrt{\bar{U}_t}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right]}_{T_2} \\
&\quad + \frac{3}{T} \sum_{t=1}^T \underbrace{\mathbb{E} \left[\left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 + \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right]}_{T_3} \quad (15)
\end{aligned}$$

449 Now we need to upper bound all the terms on RHS of the above inequality to get the convergence
450 rate.

451 For terms in T_3 in (15), we can upper bound them by

$$\left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \leq \frac{1}{\min_{j \in [d]} [\bar{U}_t^{1/2}]_j} \|\nabla f(Z_t) - \nabla f(\bar{X}_t)\|^2 \leq L \frac{1}{\min_{j \in [d]} [\bar{U}_t^{1/2}]_j} \underbrace{\|Z_t - \bar{X}_t\|^2}_{T_4} \quad (16)$$

452 and

$$\begin{aligned} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 &\leq \frac{1}{\min_{j \in [d]} [\bar{U}_t^{1/2}]_j} \frac{1}{N} \sum_{i=1}^N \|\nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)\|^2 \\ &\leq L \frac{1}{\min_{j \in [d]} [\bar{U}_t^{1/2}]_j} \frac{1}{N} \underbrace{\sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2}_{T_5} \end{aligned} \quad (17)$$

453 using Jensen's inequality, Lipschitz continuity of f_i , and the fact that $f = \frac{1}{N} \sum_{i=1}^N f_i$.

454 What we need to do next is to bound T_4 and T_5 and we will bound T_5 first.

455 Before we proceed into bounding T_5 , we need some preparations. Let's recall the update rule of X_t ,
456 we have

$$X_t = X_{t-1}W - \alpha \frac{M_{t-1}}{\sqrt{U_{t-1}}} = X_1 W^{t-1} - \alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} W^k \quad (18)$$

457 where we define $W^0 = \mathbf{I}$.

458 Since W is a symmetric matrix, we can decompose it as $W = Q\Lambda Q^T$ where Q is a orthonormal
459 matrix and Λ is a diagonal matrix whose diagonal elements correspond to eigenvalues of W in an
460 descending order, i.e. $\Lambda_{ii} = \lambda_i$ with λ_i being i th largest eigenvalue of W . In addition, because W is
461 a doubly stochastic matrix, we know $\lambda_1 = 1$ and $q_1 = \frac{1}{\sqrt{N}}$

462 With eigen-decomposition of W , we can rewrite T_5 as

$$\sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 = \|X_t - \bar{X}_t \mathbf{1}_N^T\|_F^2 = \|X_t Q Q^T - X_t \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T\|_F^2 = \sum_{l=2}^N \|X_t q_l\|^2 \quad (19)$$

463 In addition, we can rewrite (18) as

$$X_t = X_1 W^{t-1} - \alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} W^k = X_1 - \alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} Q \Lambda^k Q^T \quad (20)$$

464 where the last equality is because $x_{1,i} = x_{1,j}$, $\forall i, j$ and thus $X_1 W = X_1$.

465 Then we have when $l > 1$,

$$X_t q_l = (X_1 - \alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} Q \Lambda^k Q^T) q_l = -\alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} q_l \lambda_l^k \quad (21)$$

466 because Q is orthonormal and $X_1 q_l = x_{1,1} \mathbf{1}_N^T q_l = x_{1,1} \sqrt{N} q_1^T q_l = 0, \forall l \neq 1$.

467 Combining (19) and (21), we have

$$T_5 = \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 = \sum_{l=2}^N \|X_t q_l\|^2 = \sum_{l=2}^N \alpha^2 \left\| \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} \lambda_l^k q_l \right\|^2 \leq \alpha^2 \left(\frac{1}{1-\lambda} \right)^2 N d G_\infty^2 \frac{1}{\epsilon} \quad (22)$$

468 where the last inequality follows from the fact that $g_{t,i} \leq G_\infty$, $\|q_l\| = 1$, and $|\lambda_l| \leq \lambda < 1$.

469 Now let us turn to T_4 , it can be rewritten as

$$\|Z_t - \bar{X}_t\|^2 = \left\| \frac{\beta_1}{1 - \beta_1} (\bar{X}_t - \bar{X}_{t-1}) \right\|^2 = \left(\frac{\beta_1}{1 - \beta_1} \right)^2 \alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{m_{t-1,i}}{\sqrt{u_{t-1,i}}} \right\|^2 \leq \left(\frac{\beta_1}{1 - \beta_1} \right)^2 \alpha^2 d \frac{G_\infty^2}{\epsilon} \quad (23)$$

470 Now we know both T_4 and T_5 are in the order of $\mathcal{O}(\alpha^2)$ and thus T_3 is in the order of $\mathcal{O}(\alpha^2)$.

471 Next we will bound T_2 and T_1 . Define $G_1 \triangleq \max_{t \in [T]} \max_{i \in [N]} \|\nabla f_i(x_{t,i})\|_\infty$, $G_2 \triangleq$
 472 $\max_{t \in [T]} \|\nabla f(Z_t)\|_\infty$, $G_3 \triangleq \max_{t \in [T]} \max_{i \in [N]} \|g_{t,i}\|_\infty$ and $G_\infty = \max(G_1, G_2, G_3)$

473 Then we have

$$\begin{aligned} T_2 &= \sum_{t=1}^T \mathbb{E} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left(\frac{1}{\sqrt{\bar{U}_t}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right] \\ &\leq \sum_{t=1}^T \mathbb{E} \left[G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{1}{\sqrt{[\bar{U}_t]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right| \right] \\ &= \sum_{t=1}^T \mathbb{E} \left[G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{1}{\sqrt{[\bar{U}_t]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right| \frac{\sqrt{[\bar{U}_t]_j} + \sqrt{[u_{t,i}]_j}}{\sqrt{[\bar{U}_t]_j} + \sqrt{[u_{t,i}]_j}} \right] \\ &= \sum_{t=1}^T \mathbb{E} \left[G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{[\bar{U}_t]_j - [u_{t,i}]_j}{[\bar{U}_t]_j \sqrt{[u_{t,i}]_j} + \sqrt{[\bar{U}_t]_j} [u_{t,i}]_j} \right| \right] \\ &\leq \underbrace{\mathbb{E} \left[\sum_{t=1}^T G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{[\bar{U}_t]_j - [u_{t,i}]_j}{2\epsilon^{1.5}} \right| \right]}_{T_6} \end{aligned} \quad (24)$$

474 where the last inequality is due to $[u_{t,i}]_j \geq \epsilon$, $\forall t, i, j$.

475 To simplify notations, let's define $\|A\|_{abs} = \sum_{i,j} |A_{ij}|$ to be the entry-wise L_1 norm of a matrix A ,
 476 then we have

$$\begin{aligned} T_6 &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \|\bar{U}_t \mathbf{1}^T - U_t\|_{abs} \\ &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \|\tilde{U}_t \mathbf{1}^T - \tilde{U}_t\|_{abs} \\ &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \|\tilde{U}_t \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T - \tilde{U}_t Q Q^T\|_{abs} \\ &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| - \tilde{U}_t \sum_{l=2}^N q_l q_l^T \right\|_{abs} \\ &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| - \sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs} \end{aligned}$$

477 where the second inequality is due to Lemma 2 and the fact that $U_t = \max(\tilde{U}_t, \epsilon)$ element-wisely.

478 **Lemma 2.** Given a set of numbers a_1, \dots, a_n and denote their mean to be $\bar{a} = \frac{1}{n} \sum_{i=1}^n a_i$. In
 479 addition, define $b_i(r) \triangleq \max(a_i, r)$ and $\bar{b}(r) = \frac{1}{n} \sum_{i=1}^n b_i(r)$. For any r and r' with $r' \geq r$ we
 480 have

$$\sum_{i=1}^n |b_i(r) - \bar{b}(r)| \geq \sum_{i=1}^n |b_i(r') - \bar{b}(r')| \quad (25)$$

481 and when $r \leq \min_{i \in [n]} a_i$, we have

$$\sum_{i=1}^n |b_i(r) - \bar{b}(r)| = \sum_{i=1}^n |a_i - \bar{a}| \quad (26)$$

482 **Proof:** See Appendix C. □

483 Recall from update rule of U_t , by defining $\hat{V}_{-1} \triangleq \hat{V}_0$ and $U_0 \triangleq U_{1/2}$, we have $\forall t \geq 0$

$$\tilde{U}_{t+1} = (\tilde{U}_t - \hat{V}_{t-1} + \hat{V}_t)W \quad (27)$$

484 and thus

$$\tilde{U}_t = \tilde{U}_0 W^t + \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) W^k = \tilde{U}_0 + \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) Q \Lambda^k Q^T \quad (28)$$

485 Then we further have when $l \neq 1$,

$$\tilde{U}_{tql} = (\tilde{U}_0 + \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) Q \Lambda^k Q^T) q_l = \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) q_l \lambda_l^k \quad (29)$$

486 where the last equality is due to the definition $\tilde{U}_0 \triangleq U_{1/2} = \epsilon \mathbf{1}_d \mathbf{1}_N^T = \sqrt{N} \epsilon \mathbf{1}_d \mathbf{1}_N^T$ (recall that
487 $q_1 = \frac{1}{\sqrt{N}} \mathbf{1}_N^T$) and $q_i^T q_j = 0$ when $i \neq j$.

488 Note by definition of $\|\cdot\|_{abs}$, we have $\forall A, B, \|A + B\|_{abs} \leq \|A\|_{abs} + \|B\|_{abs}$, then we have

$$\begin{aligned} T_6 &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| - \sum_{l=2}^N \tilde{U}_{tql} q_l^T \right\|_{abs} \\ &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| - \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) \sum_{l=2}^N q_l \lambda_l^k q_l^T \right\|_{abs} \\ &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \left\| (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) \sum_{l=2}^N q_l \lambda_l^k q_l^T \right\|_{abs} \\ &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \sum_{j=1}^d \left\| \sum_{l=2}^N q_l \lambda_l^k q_l^T (-\hat{V}_{t-1-k} + \hat{V}_{t-k})^T e_j \right\|_1 \\ &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \sum_{j=1}^d \left\| \sum_{l=2}^N q_l \lambda_l^k q_l^T \right\|_1 \left\| (-\hat{V}_{t-1-k} + \hat{V}_{t-k})^T e_j \right\|_1 \\ &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \sum_{j=1}^d \sqrt{N} \left\| \sum_{l=2}^N q_l \lambda_l^k q_l^T \right\|_2 \left\| (-\hat{V}_{t-1-k} + \hat{V}_{t-k})^T e_j \right\|_1 \\ &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \sum_{j=1}^d \left\| (-\hat{V}_{t-1-k} + \hat{V}_{t-k})^T e_j \right\|_1 \sqrt{N} \lambda^k \\ &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \left\| (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) \right\|_{abs} \sqrt{N} \lambda^k \\ &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{o=0}^{t-1} \left\| (-\hat{V}_{o-1} + \hat{V}_o) \right\|_{abs} \sqrt{N} \lambda^{t-o} \\ &= \frac{G_\infty^2}{N} \frac{1}{2\epsilon^{1.5}} \sum_{o=0}^{T-1} \sum_{t=o+1}^T \left\| (-\hat{V}_{o-1} + \hat{V}_o) \right\|_{abs} \sqrt{N} \lambda^{t-o} \\ &\leq \frac{G_\infty^2}{\sqrt{N}} \frac{1}{2\epsilon^{1.5}} \sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \left\| (-\hat{V}_{o-1} + \hat{V}_o) \right\|_{abs} \end{aligned} \quad (30)$$

489 where $\lambda = \max(|\lambda_2|, |\lambda_N|)$.

490 Combining (24) and (30), we have

$$T_2 \leq \frac{G_\infty^2}{\sqrt{N}} \frac{1}{2\epsilon^{1.5}} \frac{\lambda}{1-\lambda} \mathbb{E} \left[\sum_{o=0}^{T-1} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right] \quad (31)$$

491 Now we need to bound T_1 , we have

$$\begin{aligned} T_1 &= \sum_{t=1}^T \mathbb{E} \left[\left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right] \\ &\leq \sum_{t=1}^T \mathbb{E} \left[G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{1}{\sqrt{[u_{t-1,i}]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right| \right] \\ &= \sum_{t=1}^T \mathbb{E} \left[G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \left(\frac{1}{\sqrt{[u_{t-1,i}]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right) \frac{\sqrt{[u_{t,i}]_j} + \sqrt{[u_{t-1,i}]_j}}{\sqrt{[u_{t,i}]_j} + \sqrt{[u_{t-1,i}]_j}} \right| \right] \\ &\leq \sum_{t=1}^T \mathbb{E} \left[G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{1}{2\epsilon^{1.5}} ([u_{t-1,i}]_j - [u_{t,i}]_j) \right| \right] \\ &\stackrel{(a)}{\leq} \sum_{t=1}^T \mathbb{E} \left[G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \frac{1}{2\epsilon^{1.5}} |([\tilde{u}_{t-1,i}]_j - [\tilde{u}_{t,i}]_j)| \right] \\ &= G_\infty^2 \frac{1}{2\epsilon^{1.5}} \frac{1}{N} \mathbb{E} \left[\sum_{t=1}^T \|\tilde{U}_{t-1} - \tilde{U}_t\|_{abs} \right] \end{aligned} \quad (32)$$

492 where (a) is due to $[\tilde{u}_{t-1,i}]_j = \max([u_{t-1,i}]_j, \epsilon)$ and the function $\max(\cdot, \epsilon)$ is 1-Lipschitz.

493 In addition, by update rule of U_t , we have

$$\begin{aligned} &\sum_{t=1}^T \|\tilde{U}_{t-1} - \tilde{U}_t\|_{abs} \\ &= \sum_{t=1}^T \|\tilde{U}_{t-1} - (\tilde{U}_{t-1} - \hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\ &= \sum_{t=1}^T \|\tilde{U}_{t-1}(I - W) + (-\hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\ &= \sum_{t=1}^T \|\tilde{U}_{t-1}(QQ^T - Q\Lambda Q^T) + (-\hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\ &= \sum_{t=1}^T \|\tilde{U}_{t-1}(\sum_{l=2}^N q_l(1 - \lambda_l)q_l^T) + (-\hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\ &\leq \sum_{t=1}^T \left\| \sum_{k=1}^{t-1} (-\hat{V}_{t-2-k} + \hat{V}_{t-1-k}) \sum_{l=2}^N q_l \lambda_l^k (1 - \lambda_l) q_l^T \right\|_{abs} + \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\ &\leq \sum_{t=1}^T \left(\sum_{k=1}^{t-1} \|-\hat{V}_{t-2-k} + \hat{V}_{t-1-k}\|_{abs} \sqrt{N} \lambda^k \right) + \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \\ &= \sum_{t=1}^T \left(\sum_{o=1}^{t-1} \|-\hat{V}_{o-2} + \hat{V}_{o-1}\|_{abs} \sqrt{N} \lambda^{t-o} \right) + \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \end{aligned}$$

$$\begin{aligned}
&= \sum_{o=1}^{T-1} \sum_{t=o+1}^T \left(\| -\hat{V}_{o-2} + \hat{V}_{o-1} \|_{abs} \sqrt{N} \lambda^{t-o} \right) + \sum_{t=1}^T \| (-\hat{V}_{t-2} + \hat{V}_{t-1}) \|_{abs} \\
&\leq \sum_{o=1}^{T-1} \frac{\lambda}{1-\lambda} \left(\| -\hat{V}_{o-2} + \hat{V}_{o-1} \|_{abs} \sqrt{N} \right) + \sum_{t=1}^T \| (-\hat{V}_{t-2} + \hat{V}_{t-1}) \|_{abs} \\
&\leq \frac{1}{1-\lambda} \sum_{t=1}^T \| (-\hat{V}_{t-2} + \hat{V}_{t-1}) \|_{abs} \sqrt{N}
\end{aligned} \tag{33}$$

494 Combining (32) and (33), we have

$$T_1 \leq G_\infty^2 \frac{1}{2\epsilon^{1.5}} \frac{1}{N} \mathbb{E} \left[\frac{1}{1-\lambda} \sum_{t=1}^T \| (-\hat{V}_{t-2} + \hat{V}_{t-1}) \|_{abs} \sqrt{N} \right] \tag{34}$$

495 What remains is to bound $\sum_{t=1}^T \mathbb{E} [\|Z_{t+1} - Z_t\|^2]$. By update rule of Z_t , we have

$$\begin{aligned}
&\|Z_{t+1} - Z_t\|^2 \\
&= \left\| \alpha \frac{\beta_1}{1-\beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&\leq 2\alpha^2 \left\| \frac{\beta_1}{1-\beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\|^2 + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&\leq 2\alpha^2 \left(\frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \frac{1}{\sqrt{\epsilon}} \left| \frac{1}{\sqrt{[u_{t-1,i}]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right| + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&\leq 2\alpha^2 \left(\frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \frac{1}{\sqrt{\epsilon}} \left| \frac{[u_{t,i}]_j - [u_{t-1,i}]_j}{2\epsilon^{1.5}} \right| + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&\leq 2\alpha^2 \left(\frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \frac{1}{2\epsilon^2} |\tilde{u}_{t,i,j} - \tilde{u}_{t-1,i,j}| + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&= 2\alpha^2 \left(\frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \frac{1}{2\epsilon^2} \|\tilde{U}_t - \tilde{U}_{t-1}\|_{abs} + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2
\end{aligned} \tag{35}$$

496 where the last inequality is again due to the definition that $[\tilde{u}_{t,i}]_j = \max([u_{t,i}]_j, \epsilon)$ and the fact that
497 $\max(\cdot, \epsilon)$ is 1-Lipschitz.

498 Then, we have

$$\begin{aligned}
&\sum_{t=1}^T \mathbb{E} [\|Z_{t+1} - Z_t\|^2] \\
&\leq 2\alpha^2 \left(\frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \frac{1}{2\epsilon^2} \mathbb{E} \left[\sum_{t=1}^T \|\tilde{U}_t - \tilde{U}_{t-1}\|_{abs} \right] + 2\alpha^2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \\
&\leq \alpha^2 \left(\frac{\beta_1}{1-\beta_1} \right)^2 \frac{G_\infty^2}{\sqrt{N}} \frac{1}{\epsilon^2} \frac{1}{1-\lambda} \mathbb{E} \left[\sum_{t=1}^T \| (-\hat{V}_{t-2} + \hat{V}_{t-1}) \|_{abs} \right] + 2\alpha^2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right]
\end{aligned} \tag{36}$$

499 where the last inequality is due to (33).

500 Now let's bound the last term on RHS of the above inequality. A trivial bound can be

$$\sum_{t=1}^T \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \leq \sum_{t=1}^T d G_\infty^2 \frac{1}{\epsilon}$$

501 due to $\|g_{t,i}\| \leq G_\infty$ and $[u_{t,i}]_j \geq \epsilon, \forall j$ (this is easy to verify from update rule of $u_{t,i}$ and the
 502 assumption that $[v_{t,i}]_j \geq \epsilon, \forall i$). However, the above bound is independent of N , to get a better bound,
 503 we need a more involved analysis to show its dependency on N . To do this, we first notice that

$$\begin{aligned}
 & \mathbb{E}_{G_t|G_{1:t-1}} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \\
 &= \mathbb{E}_{G_t|G_{1:t-1}} \left[\frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left\langle \frac{\nabla f_i(x_{t,i}) + \xi_{t,i}}{\sqrt{u_{t,i}}}, \frac{\nabla f_j(x_{t,j}) + \xi_{t,j}}{\sqrt{u_{t,j}}} \right\rangle \right] \\
 &\stackrel{(a)}{=} \mathbb{E}_{G_t|G_{1:t-1}} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right] + \mathbb{E}_{G_t|G_{1:t-1}} \left[\frac{1}{N^2} \sum_{i=1}^N \left\| \frac{\xi_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \\
 &\stackrel{(b)}{=} \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 + \frac{1}{N^2} \sum_{i=1}^N \sum_{l=1}^d \frac{\mathbb{E}_{G_t|G_{1:t-1}} [[\xi_{t,i}]_l^2]}{[u_{t,i}]_l} \\
 &\stackrel{(c)}{\leq} \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 + \frac{d}{N} \frac{\sigma^2}{\epsilon}
 \end{aligned} \tag{37}$$

504 where (a) is due to $\mathbb{E}_{G_t|G_{1:t-1}} [\xi_{t,i}] = 0$ and $\xi_{t,i}$ is independent of $x_{t,j}, \forall j, u_{t,j}, \forall j$, and $\xi_j, \forall j \neq i$,
 505 (b) comes from the fact that $x_{t,i}, u_{t,i}$ are fixed given $G_{1:t}$, (c) is due to $\mathbb{E}_{G_t|G_{1:t-1}} [[\xi_{t,i}]_l^2] \leq \sigma^2$ and
 506 $[u_{t,i}]_l \geq \epsilon$ by definition.

507 Then we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] &= \mathbb{E}_{G_{1:t-1}} \left[\mathbb{E}_{G_t|G_{1:t-1}} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \right] \\
 &\leq \mathbb{E}_{G_{1:t-1}} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 + \frac{d}{N} \frac{\sigma^2}{\epsilon} \right] \\
 &= \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right] + \frac{d}{N} \frac{\sigma^2}{\epsilon}
 \end{aligned} \tag{38}$$

508 In traditional analysis of SGD-like distributed algorithms, the term corresponding to
 509 $\mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right]$ will be merged with the first order descent when the stepsize is cho-
 510 sen to be small enough. However, in our case, the term cannot be merged because it is different from
 511 the first order descent in our algorithm. A brute-force upper bound is possible but this will lead to a
 512 worse convergence rate in terms of N . Thus, we need a more detailed analysis for the term in the
 513 following.

$$\begin{aligned}
 & \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right] \\
 &= \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{U_t}} + \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left(\frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{U_t}} \right) \right\|^2 \right] \\
 &\leq 2\mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{U_t}} \right\|^2 \right] + 2\mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left(\frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{U_t}} \right) \right\|^2 \right]
 \end{aligned} \tag{39}$$

$$\begin{aligned}
&\leq 2\mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2\mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left(\frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right) \right\|^2 \right] \\
&\leq 2\mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2\mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \left\| \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right\|_1 \right] \tag{40}
\end{aligned}$$

514 Summing over T , we have

$$\begin{aligned}
&\sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right] \\
&\leq 2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2 \sum_{t=1}^T \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \left\| \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right\|_1 \right] \tag{41}
\end{aligned}$$

515 For the last term on RHS of (41), we can bound it similarly as what we did for T_2 from (24) to (30),
516 which yields

$$\begin{aligned}
\sum_{t=1}^T \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \left\| \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right\|_1 \right] &\leq \sum_{t=1}^T \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \frac{1}{2\epsilon^{1.5}} \|u_{t,i} - \bar{U}_t\|_1 \right] \\
&= \sum_{t=1}^T \mathbb{E} \left[\frac{1}{N} G_\infty^2 \frac{1}{2\epsilon^2} \|\bar{U}_t \mathbf{1}^T - U_t\|_{abs} \right] \\
&\leq \sum_{t=1}^T \mathbb{E} \left[\frac{1}{N} G_\infty^2 \frac{1}{2\epsilon^2} \left\| - \sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs} \right] \\
&\leq \frac{1}{\sqrt{N}} G_\infty^2 \frac{1}{2\epsilon^2} \mathbb{E} \left[\sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right] \tag{42}
\end{aligned}$$

517 Further, we have

$$\begin{aligned}
&\sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] \\
&\leq 2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t)}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t) - \nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] \\
&= 2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t) - \nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] \tag{43}
\end{aligned}$$

518 and the last term on RHS of the above inequality can be bounded following similar procedures from
519 (17) to (22), as what we did for T_3 . Completing the procedures yields

$$\begin{aligned}
&\sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t) - \nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] \\
&\leq \sum_{t=1}^T \mathbb{E} \left[L \frac{1}{\epsilon} \frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \right] \\
&\leq \sum_{t=1}^T \mathbb{E} \left[L \frac{1}{\epsilon} \frac{1}{N} \alpha^2 \left(\frac{1}{1-\lambda} \right) N d G_\infty^2 \frac{1}{\epsilon} \right] \\
&= T L \frac{1}{\epsilon^2} \alpha^2 \left(\frac{1}{1-\lambda} \right) d G_\infty^2 \tag{44}
\end{aligned}$$

520 Finally, combining (38) to (44), we get

$$\begin{aligned}
& \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \\
& \leq 4 \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 4TL \frac{1}{\epsilon^2} \alpha^2 \left(\frac{1}{1-\lambda} \right) dG_\infty^2 \\
& \quad + 2 \frac{1}{\sqrt{N}} G_\infty^2 \frac{1}{2\epsilon^2} \mathbb{E} \left[\sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right] + T \frac{d}{N} \frac{\sigma^2}{\epsilon} \\
& \leq 4 \frac{1}{\sqrt{\epsilon}} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] + 4TL \frac{1}{\epsilon^2} \alpha^2 \left(\frac{1}{1-\lambda} \right) dG_\infty^2 \\
& \quad + 2 \frac{1}{\sqrt{N}} G_\infty^2 \frac{1}{2\epsilon^2} \mathbb{E} \left[\sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right] + T \frac{d}{N} \frac{\sigma^2}{\epsilon}. \tag{45}
\end{aligned}$$

521 where the last inequality is due to each element of \bar{U}_t is lower bounded by ϵ by definition.

522 Combining all above, we can have

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \\
& \leq \frac{2}{T\alpha} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) \\
& \quad + \frac{L}{T} \alpha \left(\frac{\beta_1}{1-\beta_1} \right)^2 \frac{G_\infty^2}{\sqrt{N}} \frac{1}{\epsilon^2} \frac{1}{1-\lambda} \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \\
& \quad + \frac{8L}{T} \alpha \frac{1}{\sqrt{\epsilon}} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] + 8L^2 \alpha \frac{1}{\epsilon^2} \alpha^2 \left(\frac{1}{1-\lambda} \right) dG_\infty^2 \\
& \quad + \frac{4L}{T} \alpha \frac{1}{\sqrt{N}} G_\infty^2 \frac{1}{2\epsilon^2} \mathbb{E} \left[\sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right] + 2L\alpha \frac{d}{N} \frac{\sigma^2}{\epsilon} \\
& \quad + \frac{2}{T} \frac{\beta_1}{1-\beta_1} G_\infty^2 \frac{1}{2\epsilon^{1.5}} \frac{1}{\sqrt{N}} \mathbb{E} \left[\frac{1}{1-\lambda} \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \\
& \quad + \frac{2}{T} \frac{G_\infty^2}{\sqrt{N}} \frac{1}{2\epsilon^{1.5}} \frac{\lambda}{1-\lambda} \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \\
& \quad + \frac{3}{T} \left(\sum_{t=1}^T L \left(\frac{1}{1-\lambda} \right)^2 \alpha^2 dG_\infty^2 \frac{1}{\epsilon^{1.5}} + \sum_{t=1}^T L \left(\frac{\beta_1}{1-\beta_1} \right)^2 \alpha^2 d \frac{G_\infty^2}{\epsilon^{1.5}} \right) \\
& = \frac{2}{T\alpha} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) + 2L\alpha \frac{d}{N} \frac{\sigma^2}{\epsilon} + 8L\alpha \frac{1}{\sqrt{\epsilon}} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \\
& \quad + 3\alpha^2 d \left(\left(\frac{\beta_1}{1-\beta_1} \right)^2 + \left(\frac{1}{1-\lambda} \right)^2 \right) L \frac{G_\infty^2}{\epsilon^{1.5}} + 8\alpha^3 L^2 \left(\frac{1}{1-\lambda} \right) d \frac{G_\infty^2}{\epsilon^2} \\
& \quad + \frac{1}{T\epsilon^{1.5}} \frac{G_\infty^2}{\sqrt{N}} \frac{1}{1-\lambda} \left(L\alpha \left(\frac{\beta_1}{1-\beta_1} \right)^2 \frac{1}{\epsilon^{0.5}} + \lambda + \frac{\beta_1}{1-\beta_1} + 2L\alpha \frac{1}{\epsilon^{0.5}} \lambda \right) \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right]. \tag{46}
\end{aligned}$$

523 Set $\alpha = \frac{1}{\sqrt{dT}}$ and when $\alpha \leq \frac{\epsilon^{0.5}}{16L}$, we further have

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \\
& \leq \frac{4}{T\alpha} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) + 4L\alpha \frac{d}{N} \frac{\sigma^2}{\epsilon} \\
& \quad + 6\alpha^2 d \left(\left(\frac{\beta_1}{1-\beta_1} \right)^2 + \left(\frac{1}{1-\lambda} \right)^2 \right) L \frac{G_\infty^2}{\epsilon^{1.5}} + 16\alpha^3 L^2 \left(\frac{1}{1-\lambda} \right) d \frac{G_\infty^2}{\epsilon^2} \\
& \quad + \frac{2}{T\epsilon^{1.5}} \frac{G_\infty^2}{\sqrt{N}} \frac{1}{1-\lambda} \left(L\alpha \left(\frac{\beta_1}{1-\beta_1} \right)^2 \frac{1}{\epsilon^{0.5}} + \lambda + \frac{\beta_1}{1-\beta_1} + 2L\alpha \frac{1}{\epsilon^{0.5}} \lambda \right) \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \\
& = \frac{4\sqrt{d}}{\sqrt{T}} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) + 4L \frac{\sqrt{d}}{\sqrt{T}} \frac{1}{N} \frac{\sigma^2}{\epsilon} \\
& \quad + 6 \frac{1}{T} \left(\left(\frac{\beta_1}{1-\beta_1} \right)^2 + \left(\frac{1}{1-\lambda} \right)^2 \right) L \frac{G_\infty^2}{\epsilon^{1.5}} + 16 \frac{1}{T^{1.5} d^{0.5}} L^2 \left(\frac{1}{1-\lambda} \right) \frac{G_\infty^2}{\epsilon^2} \\
& \quad + \frac{2}{T\epsilon^{1.5}} \frac{G_\infty^2}{\sqrt{N}} \frac{1}{1-\lambda} \left(\frac{L}{\sqrt{Td}} \left(\frac{\beta_1}{1-\beta_1} \right)^2 \frac{1}{\epsilon^{0.5}} + \lambda + \frac{\beta_1}{1-\beta_1} + 2 \frac{L}{\sqrt{Td}} \frac{1}{\epsilon^{0.5}} \lambda \right) \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \\
& \leq C_1 \frac{\sqrt{d}}{\sqrt{T}} \left(\mathbb{E}[f(Z_1)] - \min_z f(z) + \frac{\sigma^2}{N} \right) + \frac{1}{T} C_2 + \frac{1}{T^{1.5} d^{0.5}} C_3 \\
& \quad + \left(\frac{1}{TN^{0.5}} C_4 + \frac{1}{T^{1.5} d^{0.5} N^{0.5}} C_5 \right) \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \tag{47}
\end{aligned}$$

524 where the first inequality is obtained by moving the term $8L\alpha \frac{1}{\sqrt{\epsilon}} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right]$ on the
525 RHS of (46) to the LHS to cancel it using the assumption $8L\alpha \frac{1}{\sqrt{\epsilon}} \leq \frac{1}{2}$ followed by multiplying both
526 sides by 2, and the constants introduced in the last step are defined as following

$$\begin{aligned}
C_1 &= \max(4, 4L/\epsilon) \\
C_2 &= 6 \left(\left(\frac{\beta_1}{1-\beta_1} \right)^2 + \left(\frac{1}{1-\lambda} \right)^2 \right) L \frac{G_\infty^2}{\epsilon^{1.5}} \\
C_3 &= 16L^2 \left(\frac{1}{1-\lambda} \right) \frac{G_\infty^2}{\epsilon^2} \\
C_4 &= \frac{2}{\epsilon^{1.5}} \frac{1}{1-\lambda} \left(\lambda + \frac{\beta_1}{1-\beta_1} \right) G_\infty^2 \\
C_5 &= \frac{2}{\epsilon^2} \frac{1}{1-\lambda} L \left(\frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 + \frac{4}{\epsilon^2} \frac{\lambda}{1-\lambda} L G_\infty^2. \tag{48}
\end{aligned}$$

527 Substituting into $Z_1 = \bar{X}_1$ completes the proof □

528 B Proof of Theorem 3

529 By Theorem 2, we know under the assumptions of the theorem, we have

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \frac{\sqrt{d}}{\sqrt{T}} \left(\mathbb{E}[f(\bar{X}_1)] - \min_z f(z) + \frac{\sigma^2}{N} \right) + \frac{1}{T} C_2 + \frac{1}{T^{1.5} d^{0.5}} C_3 \\ &\quad + \left(\frac{1}{TN^{0.5}} C_4 + \frac{1}{T^{1.5} d^{0.5} N^{0.5}} C_5 \right) \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \end{aligned} \quad (49)$$

530 where $\|\cdot\|_{abs}$ denotes the entry-wise L_1 norm of a matrix (i.e. $\|A\|_{abs} = \sum_{i,j} |A_{ij}|$) and
531 C_1, C_2, C_3, C_4, C_5 are defined in Theorem 2.

532 Since Algorithm 3 is a special case of 2, building on result of Theorem 2, we just need to characterize
533 the growth speed of $\mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right]$ to prove convergence of Algorithm 3. By the
534 update rule of Algorithm 3, we know \hat{V}_t is non decreasing and thus

$$\begin{aligned} \mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] &= \mathbb{E} \left[\sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^d |-\hat{v}_{t-2,i,j} + \hat{v}_{t-1,i,j}| \right] \\ &= \mathbb{E} \left[\sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^d (-\hat{v}_{t-2,i,j} + \hat{v}_{t-1,i,j}) \right] \\ &= \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^d (-\hat{v}_{-1,i,j} + \hat{v}_{T-1,i,j}) \right] \\ &= \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^d (-\hat{v}_{0,i,j} + \hat{v}_{T-1,i,j}) \right] \end{aligned} \quad (50)$$

535 where the last equality is because we defined $\hat{V}_{-1} \triangleq \hat{V}_0$ previously.

536 Further, because $\|g_{t,i}\|_\infty \leq G_\infty, \forall t, i$ and $v_{t,i}$ is a exponential moving average of $g_{k,i}^2, k =$
537 $1, 2, \dots, t$, we know $|[v_{t,i}]_j| \leq G_\infty^2, \forall t, i, j$. In addition, by update rule of \hat{V}_t , we also know
538 each element of \hat{V}_t also cannot be greater than G_∞^2 , i.e. $|\hat{v}_{t,i,j}| \leq G_\infty^2, \forall t, i, j$.

539 Given the fact that $[\hat{v}_{0,i}]_j \geq 0$, we have

$$\mathbb{E} \left[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] = \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^d (-\hat{v}_{0,i,j} + \hat{v}_{T-1,i,j}) \right] \leq \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^d G_\infty^2 \right] = NdG_\infty^2 \quad (51)$$

540 Substituting the above into (49), we have

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \frac{\sqrt{d}}{\sqrt{T}} \left(\mathbb{E}[f(\bar{X}_1)] - \min_z f(z) + \frac{\sigma^2}{N} \right) + \frac{1}{T} C_2 + \frac{1}{T^{1.5} d^{0.5}} C_3 \\ &\quad + \frac{d}{T} C_4 \sqrt{N} G_\infty^2 + \frac{\sqrt{d}}{T^{1.5}} C_5 \sqrt{N} G_\infty^2 \\ &= C'_1 \frac{\sqrt{d}}{\sqrt{T}} \left(\mathbb{E}[f(\bar{X}_1)] - \min_z f(z) + \frac{\sigma^2}{N} \right) + \frac{1}{T} C'_2 + \frac{1}{T^{1.5} d^{0.5}} C'_3 \\ &\quad + \frac{d}{T} \sqrt{N} C'_4 + \frac{\sqrt{d}}{T^{1.5}} \sqrt{N} C'_5 \end{aligned} \quad (52)$$

541 where we have

$$C'_1 = C_1 \quad C'_2 = C_2 \quad C'_3 = C_3 \quad C'_4 = C_4 G_\infty^2 \quad C'_5 = C_5 G_\infty^2 \quad (53)$$

542 This completes our proof.

□

543 C Proof of Lemmas

544 **Lemma 1.** *For the sequence defined in (6), we have*

$$Z_{t+1} - Z_t = \alpha \frac{\beta_1}{1 - \beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \quad (54)$$

545 **Proof:** By update rule of Algorithm 2, we first have

$$\begin{aligned} \bar{X}_{t+1} &= \frac{1}{N} \sum_{i=1}^N x_{t+1,i} \\ &= \frac{1}{N} \sum_{i=1}^N \left(x_{t+0.5,i} - \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \right) \\ &= \frac{1}{N} \sum_{i=1}^N \left(\sum_{j=1}^N W_{ij} x_{t,j} - \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \right) \\ &\stackrel{(i)}{=} \left(\frac{1}{N} \sum_{j=1}^N x_{t,j} \right) - \frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \\ &= \bar{X}_t - \frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \end{aligned} \quad (55)$$

546 where (i) is due to an interchange of summation and $\sum_{i=1} W_{ij} = 1$.

547 Then, we have

$$\begin{aligned} Z_{t+1} - Z_t &= \bar{X}_{t+1} - \bar{X}_t + \frac{\beta_1}{1 - \beta_1} (\bar{X}_{t+1} - \bar{X}_t) - \frac{\beta_1}{1 - \beta_1} (\bar{X}_{t+1} - \bar{X}_t) \\ &= \frac{1}{1 - \beta_1} (\bar{X}_{t+1} - \bar{X}_t) - \frac{\beta_1}{1 - \beta_1} (\bar{X}_{t+1} - \bar{X}_t) \\ &= \frac{1}{1 - \beta_1} \left(-\frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \right) - \frac{\beta_1}{1 - \beta_1} \left(-\frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t-1,i}}{\sqrt{u_{t-1,i}}} \right) \\ &= \frac{1}{1 - \beta_1} \left(-\frac{1}{N} \sum_{i=1}^N \alpha \frac{\beta_1 m_{t-1,i} + (1 - \beta_1) g_{t,i}}{\sqrt{u_{t,i}}} \right) - \frac{\beta_1}{1 - \beta_1} \left(-\frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t-1,i}}{\sqrt{u_{t-1,i}}} \right) \\ &= \alpha \frac{\beta_1}{1 - \beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left(\frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \end{aligned} \quad (56)$$

548 which is the desired result. \square

549 **Lemma 2.** *Given a set of numbers a_1, \dots, a_n and denote their mean to be $\bar{a} = \frac{1}{n} \sum_{i=1}^n a_i$. In*
 550 *addition, define $b_i(r) \triangleq \max(a_i, r)$ and $\bar{b}(r) = \frac{1}{n} \sum_{i=1}^n b_i(r)$. For any r and r' with $r' \geq r$ we*
 551 *have*

$$\sum_{i=1}^n |b_i(r) - \bar{b}(r)| \geq \sum_{i=1}^n |b_i(r') - \bar{b}(r')| \quad (57)$$

552 and when $r \leq \min_{i \in [n]} a_i$, we have

$$\sum_{i=1}^n |b_i(r) - \bar{b}(r)| = \sum_{i=1}^n |a_i - \bar{a}| \quad (58)$$

553 **Proof:** Without loss of generality, let's assume $a_i \leq a_j$ when $i < j$, i.e. a_i is a non-decreasing
 554 sequence. Define

$$h(r) = \sum_{i=1}^n |b_i(r) - \bar{b}(r)| = \sum_{i=1}^n \left| \max(a_i, r) - \frac{1}{n} \sum_{j=1}^n \max(a_j, r) \right|, \quad (59)$$

555 we need to prove that h is a non-increasing function of r . First, it is easy to see that h is a continuous
 556 function of r with non-differentiable points $r = a_i, i \in [n]$, thus h is a piece-wise linear function.

557 Next, we will prove that $h(r)$ is non-increasing in each piece. Define $l(r)$ to be the largest index with
 558 $a(l(r)) < r$, and $s(r)$ to be the largest index with $a_{s(r)} < \bar{b}(r)$. Note that we have $b_i(r) = r, \forall i \leq l(r)$
 559 and $b_i(r) - \bar{b}(r) \leq 0, \forall i \leq s(r)$ because a_i is a non-decreasing sequence. Therefore, we have

$$h(r) = \sum_{i=1}^{l(r)} (\bar{b}(r) - r) + \sum_{i=l(r)+1}^{s(r)} (\bar{b}(r) - a_i) + \sum_{i=s(r)+1}^n (a_i - \bar{b}(r)). \quad (60)$$

560 and

$$\bar{b}(r) = \frac{1}{n} \left(l(r)r + \sum_{i=l(r)+1}^n a_i \right) \quad (61)$$

561 Taking derivative of the above form, we know the derivative of $h(r)$ at differentiable points is

$$\begin{aligned} h'(r) &= l(r) \left(\frac{l(r)}{n} - 1 \right) + (s(r) - l(r)) \frac{l(r)}{n} - (n - s(r)) \frac{l(r)}{n} \\ &= \frac{l(r)}{n} ((l(r) - n) + (s(r) - l(r)) - (n - s(r))) \end{aligned} \quad (62)$$

562 Since we have $s(r) \leq n$ we know $(l(r) - n) + (s(r) - l(r)) - (n - s(r)) \leq 0$ and thus

$$h'(r) \leq 0 \quad (63)$$

563 which means $h(r)$ is non-increasing in each piece. Combining with the fact that $h(r)$ is continuous,
 564 (57) is proven.

565 When $r \leq a(i)$, we have $b(i) = \max(a_i, r) = r, \forall r \in [n]$ and $\bar{b}(r) = \frac{1}{n} \sum_{i=1}^n a_i = \bar{a}$ which proves
 566 (58). \square

567 D Additional experiments and details

568 In this section, we compare the learning curves of different algorithms with different stepsizes on
 569 heterogeneous data distribution. We use 5 nodes and the heterogeneous data distribution is created
 570 by assigning each node with data of only two labels and there are no overlapping labels between
 571 different nodes. For all algorithms, we compare stepsizes in the set $[1e-1, 1e-2, 1e-3, 1e-4, 1e-5, 1e-6]$.
 572 $1e-6]$.

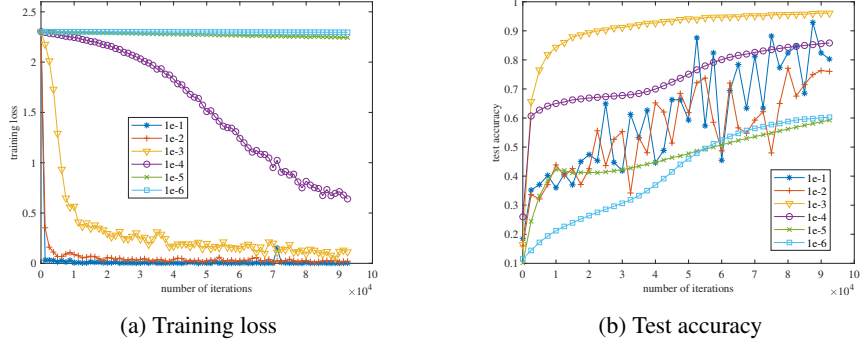


Figure 2: Performance comparison of different stepsizes for DGD

573 Figure 2 shows the training loss and test accuracy of DGD, it can be seen that the stepsize $1e-3$ works
 574 best for DGD in terms of test accuracy and $1e-1$ works best in terms of training loss. The difference
 575 is caused by the inconsistency among the value of parameters on different nodes when the stepsize
 576 is large. The training loss is calculated as the average of the loss value of different local models
 577 evaluated on their local training batch. Thus, though the training loss is small evaluated at a particular
 578 node, the test accuracy will be low when evaluating data with labels not seen by the node (recall that
 579 each node contains data with different labels).

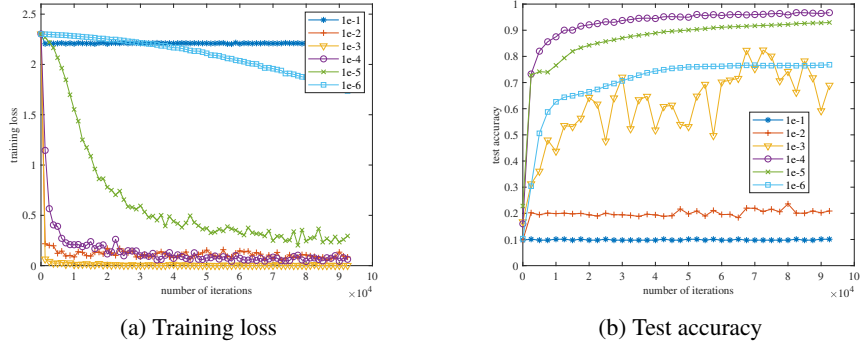
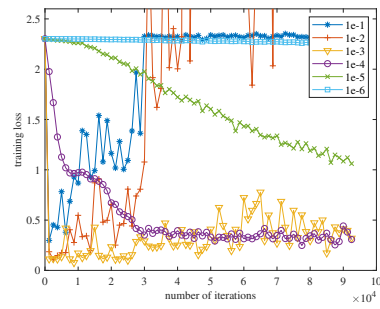


Figure 3: Performance comparison of different stepsizes for decentralized AMSGrad

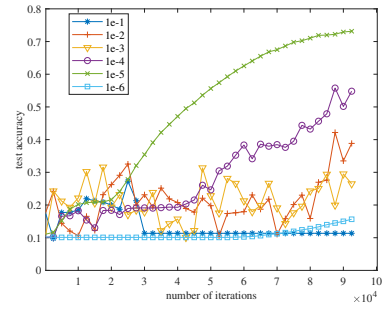
580 Figure 3 shows the performance of decentralized AMSGrad with different stepsizes, we can see its
 581 best performance is better than DGD and the performance is stabler (the test performance is less
 582 sensitive to stepsize choice).

583 Figure 4 shows the performance of DADM, as it can be expected, the performance of DADAM is
 584 not as good as DGD and decentralized AMSGrad since it is not a convergent algorithm and the
 585 heterogeneity in data amplified the non-convergence issue of DADAM.

586 From the experiments above, we can see the advantages of decentralized AMSGrad in terms of both
 587 performance and ease of parameter tuning, and the importance of ensuring the theoretical convergence
 588 of algorithms.



(a) Training loss



(b) Test accuracy

Figure 4: Performance comparison of different stepsizes for DADAM