Sparsified Distributed Adaptive Learning with Error Feedback

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Abstract

To be completed...

1 Introduction

Most modern machine learning tasks can be casted as a large finite-sum optimization problem written as:

$$\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \tag{1}$$

where n denotes the number of workers, f_i represents the average loss for worker i and θ the global model parameter taking value in Θ, a subset of $\mathbb{R}^d$.

Some related work:

[18] develops variant of signSGD (as a biased compression schemes) for distributed optimization. Contributions are mainly on this error feedback variant. In [26], the authors provide theoretical results on the convergence of sparse Gradient SGD for distributed optimization (we want that for AMS here). [27] develops a variant of distributed SGD with sparse gradients too. Contributions include a memory term used while compressing the gradient (using top k for instance). Speeding up the convergence in $\frac{1}{T^3}$.

2 Preliminaries

Sparse Optimization Methods.

Distributed Learning. When a large number of compute engines is available, being able to train global machine learning models while mutualizing the available and *decentralized* source of computation has been a growing focus for the community.

Decentralized optimization methods include methods such as ADMM [6], Distributed Subgradient Descent [24], Dual Averaging [11], Prox-PDA [14], GNSD [21], and Choco-SGD [20].

A recent work [7], which focuses on adaptive gradient methods, namely the Adam [19] annd the AMSGrad [25] optimization methods, develops a decentralized variant of gradient based and adaptive methods in the context of gossip protocols. To date, very few contributions provided attempt to efficiently run adaptive gradient method is such a distributed setting. Apart from [7], **(author?)** [23] proposes a decentralized version of AMSGrad [25] which provably satisfies some non-standard regret. Though, no sparsified variants of them have been proposed for practical purposes nor been studied in the literature.

Submitted to 34th Conference on Neural Information Processing Systems (NeurIPS 2020). Do not distribute.

Compression-Based Distributed Optimization. While the capabilities of the compute powers is exploding, the communication complexity between either the central server and the decentralized workers or among workers is becoming ineffectively large [9, 22]. Gradient sparsification constitutes one popular method to induce sparsity through the optimization procedure and reduce the number of bits transmitted at each iteration. Extensive works have studied this technique to improve the communication efficiency of SGD-based methods such as distributed SGD. This large class of sparsification techniques include gradient quantization leveraging quantized vector of gradients in the communication phase [2, 29, 16, 28, 13, 8, 15], gradient sparsificaiton generally selection top k components of the vector to be communicated, see [27, 1], or variants of the particular SGD algorithm such as low-precision SGD [4, 18] proposing a trade-off between communication cost and precision, and signSGD [10, 30] where only the signs of the gradient vectors are communicated. Most of these works apply to the SGD method [5] as a prototype where a novel method and some convergence results are presented with a rate of $\mathcal{O}(\frac{1}{\sqrt{T}})$ where T denotes the total number of iterations, see [3], thus achieving the same rate as plain SGD, see [12, 17].

Yet these communication reduction techniques, still presents a negative dependence on the number of workers, typically a linear dependence. Hence the need for even more efficient techniques which constitutes the object of our paper.

3 Method

Consider standard synchronous distributed optimization setting. AMSGrad is used as the prototype, and the local workers is only in charge of gradient computation.

3.1 TopK AMSGrad with Error Feedback

The key difference (and interesting part) of our TopK AMSGrad compared with the following arxiv paper "Quantized Adam"https://arxiv.org/pdf/2004.14180.pdf is that, in our model only gradients are transmitted. In "QAdam", each local worker keeps a local copy of moment estimator m and v, and compresses and transmits m/v as a whole. Thus, that method is very much like the sparsified distributed SGD, except that g is changed into m/v. In our model, the moment estimates m and v are computed only at the central server, with the compressed gradients instead of the full gradient. This would be the key (and difficulty) in convergence analysis.

Algorithm 1 SPARS-AMS for Distributed Learning

1: **Input**: parameter β_1, β_2, learning rate η_t.
2: Initialize: central server parameter $\theta_0 \in \Theta \subseteq \mathbb{R}^d$; $e_{0,i} = 0$ the error accumulator for each worker; sparsity parameter k; n local workers; $m_0 = 0$, $v_0 = 0$, $\hat{v}_0 = 0$
3: **for** $t = 1$ to T **do**
4: **parallel for worker** $i \in [n]$ **do**:
5: Receive model parameter θ_t from central server
6: Compute stochastic gradient $g_{t,i}$ at θ_t
7: Compute $\tilde{g}_{t,i} = TopK(g_{t,i} + e_{t,i}, k)$
8: Update the error $e_{t+1,i} = e_{t,i} + g_{t,i} - \tilde{g}_{t,i}$
9: Send $\tilde{g}_{t,i}$ back to central server
10: **end parallel**
11: **Central server do:**
12: $\bar{g}_t = \frac{1}{n}\sum_{i=1}^{N} \tilde{g}_{t,i}$
13: $m_t = \beta_1 m_{t-1} + (1-\beta_1)\bar{g}_t$
14: $v_t = \beta_2 v_{t-1} + (1-\beta_2)\bar{g}_t^2$
15: $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
16: Update global model $\theta_t = \theta_{t-1} - \eta_t \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$
17: **end for**

3.2 Convergence Analysis

Several mild assumptions to make: Nonconvex and smooth loss function, unbiased stochastic gradient, bounded variance of the gradient, bounded norm of the gradient, control of the distance between the true gradient and its sparse variant.

Check [7] starting with single machine and extending to distributed settings (several machines).

Under the distributed setting, the goal is to derive an upper bound to the second order moment of the gradient of the objective function at some iteration $T_f \in [1, T]$.

3.3 Mild Assumptions

We begin by making the following assumptions.

Assumption 1. *(Smoothness) For $i \in [\![n]\!]$, f_i is L-smooth: $\|\nabla f_i(\theta) - \nabla f_i(\vartheta)\| \leq L \|\theta - \vartheta\|$.*

Assumption 2. *(Unbiased and Bounded gradient **per worker**) For any iteration index $t > 0$ and worker index $i \in [\![n]\!]$, the stochastic gradient is unbiased and bounded from above: $\mathbb{E}[g_{t,i}] = \nabla f_i(\theta_t)$ and $\|g_{t,i}\| \leq G_i$.*

Assumption 3. *(Bounded variance **per worker**) For any iteration index $t > 0$ and worker index $i \in [\![n]\!]$, the variance of the noisy gradient is bounded: $\mathbb{E}[|g_{t,i} - \nabla f_i(\theta_t)|^2] < \sigma_i^2$.*

Denote by $Q(\cdot)$ the quantization operator Line 7 of Algorithm 1, which takes as input a gradient vector and returns a quantized version of it, and note $\tilde{g} := Q(g)$. Assume that

Assumption 4. *(Bounded Quantization) For any iteration $t > 0$, there exists a constant $q > 0$ such that $\|g_{t,i} - \tilde{g}_{t,i}\| \leq q \|g_{t,i}\|$, where $g_{t,i}$ is the stochastic gradient computed at iteration t for worker i.*

Denote for all $\theta \in \Theta$:

$$f(\theta) := \frac{1}{n} \sum_{i=1}^{n} f_i(\theta)\,, \tag{2}$$

where n denotes the number of workers.

3.4 Intermediary Lemmas

Lemma 1. *Under Assumption 2 and Assumption 4 we have for any iteration $t > 0$:*

$$\|m_t\|^2 \leq (q^2+1)G^2 \quad \textit{and} \quad \hat{v}_t \leq (q^2+1)G^2 \tag{3}$$

where m_t and $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ are defined Line 15 of Algorithm 1 and $G^2 = \frac{1}{n}\sum_{i=1}^{N} G_i^2$.

Proof. We start by writing

$$\|\bar{g}_t\|^2 = \left\| \frac{1}{n} \sum_{i=1}^{N} \tilde{g}_{t,i} \right\|^2 \leq \frac{1}{n} \sum_{i=1}^{N} \|\tilde{g}_{t,i}\|^2 \tag{4}$$

Though, using Assumption 2 and Assumption 4 we have:

$$\|\tilde{g}_{t,i}\|^2 = \|g_{t,i} + \tilde{g}_{t,i} - g_{t,i}\|^2 \leq \|g_{t,i}\|^2 + \|\tilde{g}_{t,i} - g_{t,i}\|^2 \leq (q^2+1)G_i^2 \tag{5}$$

Hence

$$\|\bar{g}_t\|^2 \leq (q^2+1)G^2 \tag{6}$$

where $G^2 = \frac{1}{n}\sum_{i=1}^{N} G_i^2$. Then, by construction in Algorithm 1:

$$\|m_t\|^2 \leq \beta_1^2 \|m_{t-1}\|^2 + (1-\beta_1)^2 \|\bar{g}_t\|^2 \leq \beta_1^2 \|m_{t-1}\|^2 + (1-\beta_1)^2 (q^2+1)G^2 \tag{7}$$

Since we have by initialization that $\|m_0\|^2 \leq G^2$, then we prove by induction that $\|m_t\|^2 \leq (q^2+1)G^2$.

Similarly

$$\hat{v}_t = \max(v_t, \hat{v}_{t-1}) = \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1-\beta_2)\bar{g}_t^2) \leq max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1-\beta_2)(q^2+1)G^2) \tag{8}$$

□

Lemma 2. *Under Assumption1 to Assumption4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \tag{9}$$

*where $\mathsf{I_d}$ is the identity matrix, $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1 and $\bar{g}_t$ is the aggregation of all **quantized** gradients from the workers.*

Proof. We first decompose $\bar{g}_t$ as the sum of the unbiased stochastic gradients and its quantized versions as computed Line 7 of Algorithm 1:

$$\bar{g}_t = \frac{1}{n}\sum_{i=1}^{N}\tilde{g}_{t,i} = \frac{1}{n}\sum_{i=1}^{N}[g_{t,i} + \tilde{g}_{t,i} - g_{t,i}] \tag{10}$$

Hence,

$$\begin{aligned} T_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \\ =& \underbrace{-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle]}_{t1} \underbrace{-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} \tilde{g}_{t,i} - g_{t,i} \right\rangle]}_{t_2} \end{aligned} \tag{11}$$

Bounding t_1: Using the Tower rule, we have:

$$\begin{aligned} t_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle] \\ =& -\eta_{t+1}\mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle |\mathcal{F}_t]] \\ =& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\mathbb{E}[\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle |\mathcal{F}_t]] \end{aligned} \tag{12}$$

Using Assumption 2 and Lemma 1, we have that

$$\begin{aligned} t_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle] \\ \leq& -\eta_{t+1}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \end{aligned} \tag{13}$$

Bounding t_2:

We first recall Young's inequality with a constant $\delta \in (0,1)$ as follows:

$$\langle X \,|\, Y \rangle \leq \frac{1}{\delta}\|X\|^2 + \delta\|Y\|^2 \,. \tag{14}$$

Using Young's inequality (14) with parameter equal to 1:

$$
\begin{aligned}
t_2 \leq & \frac{\eta_{t+1}}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\
\overset{(a)}{\leq} & \frac{\eta_{t+1}}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\|^2\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\
\overset{(b)}{\leq} & \frac{\eta_{t+1}}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\|^2]\mathbb{E}[\|\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\
\overset{(c)}{\leq} & \frac{\eta_{t+1}}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{\epsilon 2n^2}\mathbb{E}[\|\sum_{i=1}^{N}\tilde{g}_{t,i}-g_{t,i}\|^2] \\
\overset{(d)}{\leq} & \frac{\eta_{t+1}}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2}
\end{aligned}
\tag{15}
$$

where (a) uses the Cauchy-Schwartz inequality, (b) is due to the non-negativeness of both $\hat{V}_{t+1}$ and $\|\sum_{i=1}^{N}\{g_{t,i}+\tilde{g}_{t,i}-g_{t,i}\}\|^2$ and (c) uses the Triangle inequality. We use Assumption 3 and Assumption 4 in (d).

Finally, combining (13) and (15) yields

$$
-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \tag{16}
$$

□

Lemma 3. *Under Assumption1 to Assumption4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1})-f(\theta_t)] \leq & -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
& - \eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t+\epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \\
& + \left(\frac{L}{2}+\beta_1 L\right)\|\theta_t-\theta_{t-1}\|^2 \\
& + \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j+\epsilon)^{-1/2}-(\hat{v}_t^j+\epsilon)^{-1/2}\right]]
\end{aligned}
\tag{17}
$$

where d denotes the dimension of the parameter vector

Proof. Denote the following auxiliary variables at iteration $t+1$

$$
z_{t+1} = \theta_{t+1} + \frac{\beta_1}{1-\beta_1}(\theta_{t+1}-\theta_t) \tag{18}
$$

By assumption Assumption 1, we can write the smoothness condition on the overall objective (2), between iteration t and $t+1$:

$$
f(\theta_{t+1}) \leq f(\theta_t) + \langle \nabla f(\theta_t) \,|\, \theta_{t+1}-\theta_t \rangle + \frac{L}{2}\|\theta_{t+1}-\theta_t\|^2 \tag{19}
$$

Denote by $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1. Hence, we obtain,

$$f(\theta_{t+1}) \leq f(\theta_t) - \eta_{t+1} \left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \right\rangle + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \tag{20}$$

where I_d denotes the identity matrix.

We now take the expectation of those various terms conditioned on the filtration $\mathcal{F}_t$ of the total randomness up to iteration t.

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1} \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \right\rangle] + \frac{L}{2} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \tag{21}$$

We now focus on the computation of the inner product obtained in the equation above. We have

$$\begin{aligned}
&\eta_{t+1} \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \right\rangle] && (22)\\
=&\eta_{t+1} \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} + (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} - (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \right\rangle] \\
=&\eta_{t+1} \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \right\rangle] + \eta_{t+1} \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} \right] m_{t+1} \right\rangle] \\
=&\eta_{t+1} \beta_1 \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] + \eta_{t+1} (1 - \beta_1) \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} \bar{g}_t \right\rangle] \\
&+ \eta_{t+1} \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} \right] m_{t+1} \right\rangle] && (23)
\end{aligned}$$

where $\bar{g}_t$ is the aggregated gradients from all workers.

Plugging the above in (21) yields:

$$\begin{aligned}
&\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \\
\leq &\underbrace{-\beta_1 \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle]}_{A_t} \eta_{t+1} \\
&\underbrace{-\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} \right] m_{t+1} \right\rangle]}_{B_t} \eta_{t+1} \\
&\underbrace{-(1 - \beta_1) \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} \bar{g}_t \right\rangle]}_{C_t} \eta_{t+1} + \frac{L}{2} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]
\end{aligned} \tag{24}$$

To begin with, by the tower rule, we have that

$$\begin{aligned}
A_t &= -\beta_1 \mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle | \mathcal{F}_t]] && (25)\\
&= -\beta_1 \left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] - \beta_1 \left\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] && (26)\\
& && (27)
\end{aligned}$$

where we recognize the first term as the term in (22), at iteration $t-1$ and hence apply the same decomposition as in (23). Coupling with the smoothness of f, which gives that

$$-\beta_1 \left\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] \leq \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2$$

we obtain,

$$\begin{aligned}
A_t &= -\beta_1 \mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle | \mathcal{F}_t]] \\
&\leq \eta_{t+1} \beta_1 (A_{t-1} + B_{t-1} + C_{t-1}) + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2
\end{aligned} \tag{28}$$

Then,

$$
\begin{aligned}
B_t &= -\mathbb{E}[\left\langle \nabla f(\theta_t) \mid \left[(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} \right] m_{t+1} \right\rangle] \\
&= \mathbb{E}[\sum_{j=1}^{d} \nabla^j f(\theta_t) m_{t=1}^j \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
&\overset{(a)}{\leq} \mathbb{E}[\|\nabla f(\theta_t)\| \, \|m_{t=1}\| \sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
&\overset{(b)}{\leq} G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]]
\end{aligned}
\tag{29}
$$

where $\nabla^j f(\theta_t)$ denotes the j-th component of the gradient vector $\nabla f(\theta_t)$, (a) uses of the Cauchy-Schwartz inequality and (b) boils down from the norm of the gradient vector boundedness assumption 2, denoting $G := \frac{1}{n}\sum_{i=1}^{n} G_i$.

Plugging the above into (24) yields

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & \eta_{t+1}(A_t + B_t + C_t) + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \\
\leq & - \eta_{t+1}\beta_1 \mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \mid (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \\
& + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
& + \left(\frac{L}{2} + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_t - \theta_{t-1}\|^2 \\
& - \eta_{t+1}(1-\beta_1)\mathbb{E}[\left\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \right\rangle]
\end{aligned}
\tag{30}
$$

We bound the last term on the RHS, $-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \right\rangle]$ with Lemma 2

Under the assumption that we use a decreasing stepsize such that $\eta_{t+1} \leq \eta_t$, and given that according to Line 15 we have that $\hat{v}_{t+1} \geq \hat{v}_t$ by construction, we obtain

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & - \frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2 n^2} \\
& - \eta_{t+1}\beta_1 \mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \mid (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \\
& + \left(\frac{L}{2} + \beta_1 L \right) \|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]]
\end{aligned}
\tag{31}
$$

Finally, using Lemma 2, we obtain the desired result. □

The main theorem in the decentralized setting reads:

Theorem 1. *Under Assumption 1 to Assumption 4, with a decreasing sequence of stepsize* $\{\eta_t\}_{t>0}$, *we have:*

$$
\frac{1}{T_m} \sum_{t=0}^{T_m - 1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathbb{E}[f(\theta_0) - f(\theta_{T_m})] + TBD \tag{32}
$$

Proof. By Lemma 3 we have

$$\begin{aligned}\mathbb{E}[f(\theta_{t+1})-f(\theta_t)] \leq & -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon+\frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2]+q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\ & -\eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t+\epsilon \mathsf{I_d})^{-1/2}m_t \right\rangle] \\ & +\left(\frac{L}{2}+\beta_1 L\right)\|\theta_t-\theta_{t-1}\|^2 \\ & +\eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j+\epsilon)^{-1/2}-(\hat{v}_t^j+\epsilon)^{-1/2}\right]]\end{aligned} \tag{33}$$

Let us consider the following sequence, defined for all $t>0$:

$$R_t := f(\theta_t)-\sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t+\epsilon \mathsf{I_d})^{-1/2}m_t \right\rangle] \tag{34}$$

We compute the following expectation:

$$\begin{aligned}\mathbb{E}[R_{t+1}]-\mathbb{E}[R_t] = & \mathbb{E}[f(\theta_{t+1})-f(\theta_t)]-\sum_{k=t+1}^{\infty}\eta_k\beta_1^{k-t+2}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}m_{t+1} \right\rangle] \\ & +\sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t+\epsilon \mathsf{I_d})^{-1/2}m_t \right\rangle]\end{aligned} \tag{35}$$

Using the Assumption 1, we note that:

$$\mathbb{E}[f(\theta_{t+1})-f(\theta_t)] \leq -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}m_{t+1} \right\rangle]+\frac{L}{2}\|\theta_{t+1}-\theta_t\|^2 \tag{36}$$

which yields

$$\begin{aligned}\mathbb{E}[R_{t+1}]-\mathbb{E}[R_t] = & -(\eta_{t+1}+\sum_{k=t+1}^{\infty}\eta_k\beta_1^{k-t+2})\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}m_{t+1} \right\rangle] \\ & +\sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t+\epsilon \mathsf{I_d})^{-1/2}m_t \right\rangle] \\ & +\frac{L}{2}\|\theta_{t+1}-\theta_t\|^2 \\ \leq & (\eta_{t+1}+\sum_{k=t+1}^{\infty}\eta_k\beta_1^{k-t+2})\mathbb{E}[A_t+B_t+C_t] \\ & -\sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\mathbb{E}[A_{t-1}+B_{t-1}+C_{t-1}] \\ & +\frac{L}{2}\|\theta_{t+1}-\theta_t\|^2\end{aligned} \tag{37}$$

where A_t, B_t, C_t are defined in (24).

We use (28) and (29) to bound A_t and B_t, and Lemma 2 to bound C_t where we precise that the learning rate η_{t+1} becomes $\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}$. Hence

$$
\begin{aligned}
\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] \leq & \left((\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \right) \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\
& + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
& + \left(\frac{L}{2} + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_{t+1} - \theta_t\|^2 \\
& - (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1-\beta_1)}{2} (\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
& + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2}
\end{aligned} \tag{38}
$$

where the last term in the LHS is due to Lemma 3.

By assumption, we have that for all $t > 0$, $\eta_{t=1} \leq \eta_t$. Also, set the tuning parameters such that

$$
\eta_t + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \leq \frac{\eta_t}{1-\beta_1} \tag{39}
$$

so that

$$
\begin{aligned}
& (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} = 0 \\
\iff & (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 = \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1}
\end{aligned} \tag{40}
$$

Note that $-(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1-\beta_1)}{2} (\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}} \leq -\eta_{t+1} \frac{(1-\beta_1)}{2} (\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}}$ since $\sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \geq 0$.

The above coupled with (38) yields

$$
\begin{aligned}
\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] \leq & - \eta_{t+1} \frac{(1-\beta_1)}{2} (\epsilon + \frac{G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2} \\
& - (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \right]] \\
& + \left(\frac{L}{2} + \eta_{t+1} + \frac{\beta_1 L}{1-\beta_1} \right) \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]
\end{aligned} \tag{41}
$$

We now sum from $t=0$ to $t=T_{\mathrm{m}}-1$ the inequality in (41), and divide it by T_{m}:

$$
\begin{aligned}
&\eta\frac{(1-\beta_1)}{2}(\epsilon+\frac{G^2}{1-\beta_2})^{-\frac{1}{2}}\frac{1}{T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
\leq&\frac{\mathbb{E}[R_0]-\mathbb{E}[R_{T_{\mathrm{m}}}]}{T_{\mathrm{m}}}+\frac{q^2\eta+\sum_{k=t+1}^{\infty}\eta\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}}{T_{\mathrm{m}}} \\
&+\left(\frac{L}{2}+\eta_{t+1}+\frac{\beta_1 L}{1-\beta_1}\right)\frac{1}{T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|\theta_{t+1}-\theta_t\|^2]
\end{aligned}
\tag{42}
$$

where we have used the fact that $(\hat{v}_t^j+\epsilon)^{-1/2}-(\hat{v}_{t+1}^j+\epsilon)^{-1/2}\geq 0$ for all dimension $j\in[d]$ by construction of $\hat{v}_{t+1}^j$.

We now bound the two remaining terms:

Bounding $-\mathbb{E}[R_{T_{\mathrm{m}}}]$:

By definition (34) of R_t we have, using Lemma 1:

$$
\begin{aligned}
-\mathbb{E}[R_{T_{\mathrm{m}}}]&\leq\sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\mathbb{E}[\left\langle\nabla f(\theta_{t-1})\,|\,(\hat{V}_t+\epsilon\mathsf{I_d})^{-1/2}m_t\right\rangle]-f(\theta_{T_{\mathrm{m}}}) \\
&\leq\|\sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\|\|\nabla f(\theta_{t-1})\|\|(\hat{V}_t+\epsilon\mathsf{I_d})^{-1/2}m_t\| \\
&\leq\eta_{t+1}(1-\beta_1)(\epsilon+\frac{G^2}{1-\beta_2})^{-\frac{1}{2}}-f(\theta_{T_{\mathrm{m}}})
\end{aligned}
\tag{43}
$$

Bounding $\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|\theta_{t+1}-\theta_t\|^2]$:

TBD

Plugging the two bounds in (42), we obtain the following bound:

$$
\begin{aligned}
\frac{1}{T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2]\leq&\frac{\mathbb{E}[f(\theta_0)-f(\theta_{T_{\mathrm{m}}})]}{\eta\Delta_1 T_{\mathrm{m}}}+\frac{q^2\eta+\sum_{k=t+1}^{\infty}\eta\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}}{\eta\Delta_1 T_{\mathrm{m}}} \\
&+\frac{1-\beta_1}{\Delta_1}(\epsilon+\frac{G^2}{1-\beta_2})^{-\frac{1}{2}} \\
&+\left(\frac{L}{2}+\eta_{t+1}+\frac{\beta_1 L}{1-\beta_1}\right)\frac{1}{\eta\Delta_1 T_{\mathrm{m}}}Boundofgap
\end{aligned}
\tag{44}
$$

where $\Delta_1:=\frac{(1-\beta_1)}{2}(\epsilon+\frac{G^2}{1-\beta_2})^{-\frac{1}{2}}$ □

4 Sequential Model

Single machine method

Algorithm 2 SPARS-AMS : Single machine setting

Input: parameter β_1, β_2, learning rate η_t.
Initialize: central server parameter $\theta_0 \in \Theta \subseteq \mathbb{R}^d$; $e_0 = 0$ the error accumulator; sparsity parameter k; $m_0 = 0$, $v_0 = 0$, $\hat{v}_0 = 0$
for $t = 1$ to T **do**
  Compute stochastic gradient $g_t = g_{t,i_t}$ at θ_t for randomly sampled index i_t
  Compute $\tilde{g}_t = TopK(g_t + e_t, k)$
  Update the error $e_{t+1} = e_t + g_t - \tilde{g}_t$
  $m_t = \beta_1 m_{t-1} + (1 - \beta_1)\tilde{g}_t$
  $v_t = \beta_2 v_{t-1} + (1 - \beta_2)\tilde{g}_t^2$
  $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
  Update global model $\theta_t = \theta_{t-1} - \eta_t \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$
end for

Let m'_t and $\hat{v}'_t$ be the first and second moment moving average of standard AMSGrad using full gradients. Denote

$$a_t = \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}, \quad a'_t = \frac{m'_t}{\sqrt{\hat{v}'_t + \epsilon}}.$$

Define the sequence

$$\mathcal{E}_{t+1} = \mathcal{E}_t + a'_t - a_t,$$

such that the auxiliary model

$$\begin{aligned}
\theta'_{t+1} &:= \theta_{t+1} - \eta\mathcal{E}_{t+1} \\
&= \theta_t - \eta a_t - \eta\mathcal{E}_{t+1} \\
&= \theta_t - \eta a_t - \eta(\mathcal{E}_t + a'_t - a_t) \\
&= \theta'_t - \eta a'_t
\end{aligned}$$

follows the update of full-gradient AMSGrad. By smoothness assumption we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \eta\langle\nabla f(\theta'_t), a'_t\rangle + \frac{L}{2}\|\theta'_{t+1} - \theta'_t\|^2.$$

Thus,

$$\begin{aligned}
\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq -\eta\mathbb{E}[\langle\nabla f(\theta'_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] \\
&= -\eta\mathbb{E}[\langle\nabla f(\theta'_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\langle\nabla f(\theta_t) - \nabla f(\theta'_t), a'_t\rangle] \\
&\leq -\eta\mathbb{E}[\langle\nabla f(\theta'_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\frac{\eta^2\rho}{2}\|\mathcal{E}_t\|^2 + \frac{1}{2\rho}\|a'_t\|^2] \\
&\leq -\eta\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\sqrt{G^2 + \epsilon}} + \frac{\eta}{2\rho}\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\epsilon} + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \frac{\eta^3\rho}{2}\mathbb{E}\|\mathcal{E}_t\|^2,
\end{aligned}$$

when $\beta_1 = 0$ for example. We may discard this assumption and use more complicated bound on the first two terms. The third term can be bounded by constant yielding $O(1/\sqrt{T})$ rate eventually when taking decreasing learning rate. The key is to get a good bound on the cumulative error sequence, $\mathcal{E}_t$. We have the following:

$$\begin{aligned}
\mathbb{E}\|\mathcal{E}_{t+1}\|^2 &= \mathbb{E}\|\mathcal{E}_t + a'_t - a_t + TopK(\mathcal{E}_t + a'_t) - TopK(\mathcal{E}_t + a'_t)\|^2 \\
&\leq 2\mathbb{E}\|\mathcal{E}_t + a'_t - TopK(\mathcal{E}_t + a'_t)\|^2 + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2 \\
&\leq 2q\mathbb{E}\|\mathcal{E}_t + a'_t\| + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2 \\
&\leq 2q[(1+r)\mathbb{E}\|\mathcal{E}_t\|^2 + (1 + \frac{1}{r})\mathbb{E}\|a'_t\|^2] + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2.
\end{aligned}$$

Current try: If we can bound the last term in the same form as the first two terms, then we can use recursion to get the desired result. We can have

$$\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2 = \mathbb{E}\|\frac{\tilde{m}_t}{\sqrt{\hat{v}_t + \epsilon}} - \|^2$$

5 Experiments

Our proposed TopK-EF with AMSGrad matches that of full AMSGrad, in distributed learning. Number of local workers is 20. Error feedback fixes the convergence issue of using solely the TopK gradient.

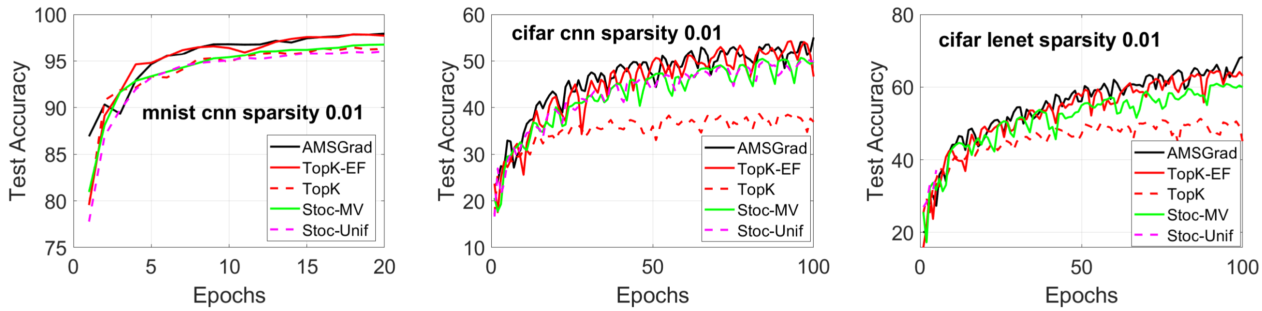

Figure 1: Test accuracy.

6 Conclusion

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A Appendix