Sparsified Distributed Adaptive Learning with Error Feedback

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Abstract

1   To be completed...

1 Introduction

3 Most modern machine learning tasks can be casted as a large finite-sum optimization problem writ-
4 ten as:

$$\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \tag{1}$$

5 where n denotes the number of workers, f_i represents the average loss for worker i and θ the global
6 model parameter taking value in Θ, a subset of $\mathbb{R}^d$.

7 Some related work:

8 [18] develops variant of signSGD (as a biased compression schemes) for distributed optimization.
9 Contributions are mainly on this error feedback variant. In [26], the authors provide theoretical
10 results on the convergence of sparse Gradient SGD for distributed optimization (we want that for
11 AMS here). [27] develops a variant of distributed SGD with sparse gradients too. Contributions
12 include a memory term used while compressing the gradient (using top k for instance). Speeding up
13 the convergence in $\frac{1}{T^3}$.

2 Preliminaries

15 **Sparse Optimization Methods.**

16 **Distributed Learning.**   When a large number of compute engines is available, being able to
17 train global machine learning models while mutualizing the available and *decentralized* source of
18 computation has been a growing focus for the community.

19 Decentralized optimization methods include methods such as ADMM [6], Distributed Subgradient
20 Descent [24], Dual Averaging [11], Prox-PDA [14], GNSD [21], and Choco-SGD [20].

21 A recent work [7], which focuses on adaptive gradient methods, namely the Adam [19] annd the
22 AMSGrad [25] optimization methods, develops a decentralized variant of gradient based and adap-
23 tive methods in the context of gossip protocols. To date, very few contributions provided attempt
24 to efficiently run adaptive gradient method is such a distributed setting. Apart from [7], **(author?)**
25 [23] proposes a decentralized version of AMSGrad [25] which provably satisfies some non-standard
26 regret. Though, no sparsified variants of them have been proposed for practical purposes nor been
27 studied in the literature.

Compression-Based Distributed Optimization. While the capabilities of the compute powers is exploding, the communication complexity between either the central server and the decentralized workers or among workers is becoming ineffectively large [9, 22]. Gradient sparsification constitutes one popular method to induce sparsity through the optimization procedure and reduce the number of bits transmitted at each iteration. Extensive works have studied this technique to improve the communication efficiency of SGD-based methods such as distributed SGD. This large class of sparsification techniques include gradient quantization leveraging quantized vector of gradients in the communication phase [2, 29, 16, 28, 13, 8, 15], gradient sparsificaiton generally selection top k components of the vector to be communicated, see [27, 1], or variants of the particular SGD algorithm such as low-precision SGD [4, 18] proposing a trade-off between communication cost and precision, and signSGD [10, 30] where only the signs of the gradient vectors are communicated. Most of these works apply to the SGD method [5] as a prototype where a novel method and some convergence results are presented with a rate of $\mathcal{O}(\frac{1}{\sqrt{T}})$ where T denotes the total number of iterations, see [3], thus achieving the same rate as plain SGD, see [12, 17].

Yet these communication reduction techniques, still presents a negative dependence on the number of workers, typically a linear dependence. Hence the need for even more efficient techniques which constitutes the object of our paper.

3 Method

Consider standard synchronous distributed optimization setting. AMSGrad is used as the prototype, and the local workers is only in charge of gradient computation.

3.1 TopK AMSGrad with Error Feedback

The key difference (and interesting part) of our TopK AMSGrad compared with the following arxiv paper "Quantized Adam" https://arxiv.org/pdf/2004.14180.pdf is that, in our model only gradients are transmitted. In "QAdam", each local worker keeps a local copy of moment estimator m and v, and compresses and transmits m/v as a whole. Thus, that method is very much like the sparsified distributed SGD, except that g is changed into m/v. In our model, the moment estimates m and v are computed only at the central server, with the compressed gradients instead of the full gradient. This would be the key (and difficulty) in convergence analysis.

Algorithm 1 SPARS-AMS for Distributed Learning

1: **Input**: parameter β_1, β_2, learning rate η_t.
2: Initialize: central server parameter $\theta_0 \in \Theta \subseteq \mathbb{R}^d$; $e_{0,i} = 0$ the error accumulator for each worker; sparsity parameter k; n local workers; $m_0 = 0$, $v_0 = 0$, $\hat{v}_0 = 0$
3: **for** $t = 1$ to T **do**
4:     **parallel for worker** $i \in [n]$ **do**:
5:         Receive model parameter θ_t from central server
6:         Compute stochastic gradient $g_{t,i}$ at θ_t
7:         Compute $\tilde{g}_{t,i} = TopK(g_{t,i} + e_{t,i}, k)$
8:         Update the error $e_{t+1,i} = e_{t,i} + g_{t,i} - \tilde{g}_{t,i}$
9:         Send $\tilde{g}_{t,i}$ back to central server
10:     **end parallel**
11:     **Central server do:**
12:         $\bar{g}_t = \frac{1}{n} \sum_{i=1}^{N} \tilde{g}_{t,i}$
13:         $m_t = \beta_1 m_{t-1} + (1 - \beta_1)\bar{g}_t$
14:         $v_t = \beta_2 v_{t-1} + (1 - \beta_2)\bar{g}_t^2$
15:         $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
16:         Update global model $\theta_t = \theta_{t-1} - \eta_t \frac{m_t}{\sqrt{\hat{v}_t} + \epsilon}$
17: **end for**

3.2 Convergence Analysis

Several mild assumptions to make: Nonconvex and smooth loss function, unbiased stochastic gradient, bounded variance of the gradient, bounded norm of the gradient, control of the distance between the true gradient and its sparse variant.

Check [7] starting with single machine and extending to distributed settings (several machines).

Under the distributed setting, the goal is to derive an upper bound to the second order moment of the gradient of the objective function at some iteration $T_f \in [1, T]$.

3.3 Mild Assumptions

We begin by making the following assumptions.

A 1. *(Smoothness) For $i \in [\![n]\!]$, f_i is L-smooth: $\|\nabla f_i(\theta) - \nabla f_i(\vartheta)\| \leq L \|\theta - \vartheta\|$.*

A 2. *(Unbiased and Bounded gradient **per worker**) For any iteration index $t > 0$ and worker index $i \in [\![n]\!]$, the stochastic gradient is unbiased and bounded from above: $\mathbb{E}[g_{t,i}] = \nabla f_i(\theta_t)$ and $\|g_{t,i}\| \leq G_i$.*

A 3. *(Bounded variance **per worker**) For any iteration index $t > 0$ and worker index $i \in [\![n]\!]$, the variance of the noisy gradient is bounded: $\mathbb{E}[|g_{t,i} - \nabla f_i(\theta_t)|^2] < \sigma_i^2$.*

Denote by $Q(\cdot)$ the quantization operator Line 7 of Algorithm 1, which takes as input a gradient vector and returns a quantized version of it, and note $\tilde{g} := Q(g)$. Assume that

A 4. *(Bounded Quantization) For any iteration $t > 0$, there exists a constant $q > 0$ such that $\|g_{t,i} - \tilde{g}_{t,i}\| \leq q \|g_{t,i}\|$, where $g_{t,i}$ is the stochastic gradient computed at iteration t for worker i.*
(high q means large quantization so loss of precision on the true gradient)

Denote for all $\theta \in \Theta$:

$$f(\theta) := \frac{1}{n} \sum_{i=1}^{n} f_i(\theta), \tag{2}$$

where n denotes the number of workers.

3.4 Intermediary Lemmas

Lemma 1. *Under Assumption 2 and Assumption 4 we have for any iteration $t > 0$:*

$$\|m_t\|^2 \leq (q^2 + 1)G^2 \quad \text{and} \quad \hat{v}_t \leq (q^2 + 1)G^2 \tag{3}$$

where m_t and $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ are defined Line 15 of Algorithm 1 and $G^2 = \frac{1}{n} \sum_{i=1}^{N} G_i^2$.

Lemma 2. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$-\eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2} \bar{g}_t \rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2} \tag{4}$$

*where $\mathsf{I_d}$ is the identity matrix, $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1 and $\bar{g}_t$ is the aggregation of all **quantized** gradients from the workers.*

Lemma 3. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq &- \frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
&- \eta_{t+1}\beta_1\mathbb{E}[\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2} m_t \rangle] \\
&+ \left(\frac{L}{2} + \beta_1 L\right)\|\theta_t - \theta_{t-1}\|^2 \\
&+ \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned} \tag{5}$$

3

85 *where d denotes the dimension of the parameter vector*

86 The main theorem in the decentralized setting reads:

87 **Theorem 1.** *Under A1 to A4, with a constant stepsize $\eta_t = \eta = \frac{L}{\sqrt{T_m}}$, we have:*

$$\frac{1}{T_m}\sum_{t=0}^{T_m-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0)-f(\theta_{T_m})]}{L\Delta_1\sqrt{T_m}} + d\frac{L\Delta_3}{\Delta_1\sqrt{T_m}} + \frac{\Delta_2}{\eta\Delta_1 T_m} + \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2$$

(6)

88 *where*

$$\Delta_1 := \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \quad, \quad \Delta_2 := q^2 + \sum_{k=t+1}^{\infty}\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}$$

$$\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right)(1-\beta_2)^{-1}(1-\frac{\beta_1^2}{\beta_2})^{-1}$$

(7)

4   Sequential Model

90 Single machine method

Algorithm 2 SPARS-AMS : Single machine setting

1: **Input**: parameter β_1, β_2, learning rate η_t.
2: Initialize: central server parameter $\theta_0 \in \Theta \subseteq \mathbb{R}^d$; $e_0 = 0$ the error accumulator; sparsity parameter k; $m_0 = 0, v_0 = 0, \hat{v}_0 = 0$
3: **for** $t = 1$ to T **do**
4:     Compute stochastic gradient $g_t = g_{t,i_t}$ at θ_t for randomly sampled index i_t
5:     Compute $\tilde{g}_t = TopK(g_t + e_t, k)$
6:     Update the error $e_{t+1} = e_t + g_t - \tilde{g}_t$
7:     $m_t = \beta_1 m_{t-1} + (1-\beta_1)\tilde{g}_t$
8:     $v_t = \beta_2 v_{t-1} + (1-\beta_2)\tilde{g}_t^2$
9:     $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
10:    Update global model $\theta_t = \theta_{t-1} - \eta_t\frac{m_t}{\sqrt{\hat{v}_t}+\epsilon}$
11: **end for**

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91 Let m'_t and $\hat{v}'_t$ be the first and second moment moving average of standard AMSGrad using full
92 gradients. Denote

$$a_t = \frac{m_t}{\sqrt{\hat{v}_t}+\epsilon}, \quad a'_t = \frac{m'_t}{\sqrt{\hat{v}'_t}+\epsilon}.$$

93 Define the sequence

$$\mathcal{E}_{t+1} = \mathcal{E}_t + a'_t - a_t,$$

94 such that the auxiliary model

$$\begin{aligned}\theta'_{t+1} &:= \theta_{t+1} - \eta\mathcal{E}_{t+1}\\ &= \theta_t - \eta a_t - \eta\mathcal{E}_{t+1}\\ &= \theta_t - \eta a_t - \eta(\mathcal{E}_t + a'_t - a_t)\\ &= \theta'_t - \eta a'_t\end{aligned}$$

95 follows the update of full-gradient AMSGrad. By smoothness assumption we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \eta\langle\nabla f(\theta'_t), a'_t\rangle + \frac{L}{2}\|\theta'_{t+1} - \theta'_t\|^2.$$

96   Thus,

$$\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] \leq -\eta\mathbb{E}[\langle\nabla f(\theta'_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2]$$

$$= -\eta\mathbb{E}[\langle\nabla f(\theta'_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\langle\nabla f(\theta_t) - \nabla f(\theta'_t), a'_t\rangle]$$

$$\leq -\eta\mathbb{E}[\langle\nabla f(\theta'_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\frac{\eta^2\rho}{2}\|\mathcal{E}_t\|^2 + \frac{1}{2\rho}\|a'_t\|^2]$$

$$\leq -\eta\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\sqrt{G^2 + \epsilon}} + \frac{\eta}{2\rho}\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\epsilon} + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \frac{\eta^3\rho}{2}\mathbb{E}\|\mathcal{E}_t\|^2,$$

97   when $\beta_1 = 0$ for example. We may discard this assumption and use more complicated bound on the
98   first two terms. The third term can be bounded by constant yielding $O(1/\sqrt{T})$ rate eventually when
99   taking decreasing learning rate. The key is to get a good bound on the cumulative error sequence,
100   $\mathcal{E}_t$. We have the following:

$$\mathbb{E}\|\mathcal{E}_{t+1}\|^2 = \mathbb{E}\|\mathcal{E}_t + a'_t - a_t + TopK(\mathcal{E}_t + a'_t) - TopK(\mathcal{E}_t + a'_t)\|^2$$

$$\leq 2\mathbb{E}\|\mathcal{E}_t + a'_t - TopK(\mathcal{E}_t + a'_t)\|^2 + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2$$

$$\leq 2q\mathbb{E}\|\mathcal{E}_t + a'_t\| + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2$$

$$\leq 2q[(1 + r)\mathbb{E}\|\mathcal{E}_t\|^2 + (1 + \frac{1}{r})\mathbb{E}\|a'_t\|^2] + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2.$$

101   Current try: If we can bound the last term in the same form as the first two terms, then we can use
102   recursion to get the desired result. We can have

$$\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2 = \mathbb{E}\|\frac{\tilde{m}_t}{\sqrt{\hat{v}_t + \epsilon}} - \|^2$$

5

5 Experiments

Our proposed TopK-EF with AMSGrad matches that of full AMSGrad, in distributed learning. Number of local workers is 20. Error feedback fixes the convergence issue of using solely the TopK gradient.
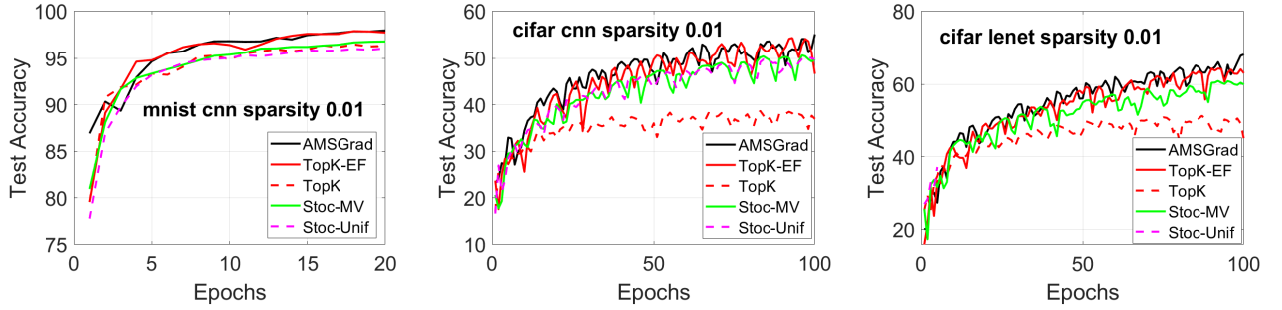

Figure 1: Test accuracy.

6 Conclusion

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A   Appendix

B   Proofs

B.1   Proof of Lemmas

Lemma. *Under Assumption 2 and Assumption 4 we have for any iteration $t > 0$:*

$$\|m_t\|^2 \le (q^2+1)G^2 \quad and \quad \hat{v}_t \le (q^2+1)G^2 \tag{8}$$

where m_t and $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ are defined Line 15 of Algorithm 1 and $G^2 = \frac{1}{n}\sum_{i=1}^{N} G_i^2$.

Proof. We start by writing

$$\|\bar{g}_t\|^2 = \left\|\frac{1}{n}\sum_{i=1}^{N}\tilde{g}_{t,i}\right\|^2 \le \frac{1}{n}\sum_{i=1}^{N}\|\tilde{g}_{t,i}\|^2 \tag{9}$$

Though, using Assumption 2 and Assumption 4 we have:

$$\|\tilde{g}_{t,i}\|^2 = \|g_{t,i} + \tilde{g}_{t,i} - g_{t,i}\|^2 \le \|g_{t,i}\|^2 + \|\tilde{g}_{t,i} - g_{t,i}\|^2 \le (q^2+1)G_i^2 \tag{10}$$

Hence

$$\|\bar{g}_t\|^2 \le (q^2+1)G^2 \tag{11}$$

where $G^2 = \frac{1}{n}\sum_{i=1}^{N} G_i^2$. Then, by construction in Algorithm 1:

$$\|m_t\|^2 \le \beta_1^2\|m_{t-1}\|^2 + (1-\beta_1)^2\|\bar{g}_t\|^2 \le \beta_1^2\|m_{t-1}\|^2 + (1-\beta_1)^2(q^2+1)G^2 \tag{12}$$

Since we have by initialization that $\|m_0\|^2 \le G^2$, then we prove by induction that $\|m_t\|^2 \le (q^2 + 1)G^2$.

Similarly

$$\hat{v}_t = \max(v_t, \hat{v}_{t-1}) = \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1-\beta_2)\bar{g}_t^2) \le \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1-\beta_2)(q^2+1)G^2) \tag{13}$$

$\square$

Lemma. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$-\eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}\bar{g}_t \rangle] \le -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \tag{14}$$

*where $\mathsf{I_d}$ is the identity matrix, $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1 and $\bar{g}_t$ is the aggregation of all **quantized** gradients from the workers.*

Proof. We first decompose $\bar{g}_t$ as the sum of the unbiased stochastic gradients and its quantized versions as computed Line 7 of Algorithm 1:

$$\bar{g}_t = \frac{1}{n}\sum_{i=1}^{N}\tilde{g}_{t,i} = \frac{1}{n}\sum_{i=1}^{N}[g_{t,i} + \tilde{g}_{t,i} - g_{t,i}] \tag{15}$$

Hence,

$$T_1 := -\eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}\bar{g}_t \rangle]$$

$$= \underbrace{-\eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \rangle]}_{t1} \underbrace{-\eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} \tilde{g}_{t,i} - g_{t,i} \rangle]}_{t_2} \tag{16}$$

Bounding t_1: Using the Tower rule, we have:

$$
\begin{aligned}
t_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle] \\
=& -\eta_{t+1}\mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle |\mathcal{F}_t]] \\
=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\mathbb{E}[\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle |\mathcal{F}_t]]
\end{aligned}
\tag{17}
$$

Using Assumption 2 and Lemma 1, we have that

$$
\begin{aligned}
t_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle] \\
\leq& -\eta_{t+1}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2]
\end{aligned}
\tag{18}
$$

Bounding t_2:

We first recall Young's inequality with a constant $\delta \in (0,1)$ as follows:

$$
\langle X \,|\, Y \rangle \leq \frac{1}{\delta}\|X\|^2 + \delta\|Y\|^2 \ .
\tag{19}
$$

Using Young's inequality (19) with parameter equal to 1:

$$
\begin{aligned}
t_2 \leq& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\
\overset{(a)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\|^2\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\
\overset{(b)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\|^2]\mathbb{E}[\|\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\
\overset{(c)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{\epsilon 2n^2}\mathbb{E}[\|\sum_{i=1}^{N}\tilde{g}_{t,i}-g_{t,i}\|^2] \\
\overset{(d)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2}
\end{aligned}
\tag{20}
$$

where (a) uses the Cauchy-Schwartz inequality, (b) is due to the non-negativeness of both $\hat{V}_{t+1}$ and $\|\sum_{i=1}^{N}\{g_{t,i}+\tilde{g}_{t,i}-g_{t,i}\}\|^2$ and (c) uses the Triangle inequality. We use Assumption 3 and Assumption 4 in (d).

Finally, combining (18) and (20) yields

$$
-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2]+q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2}
\tag{21}
$$

$\square$

218 **Lemma.** *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq &-\frac{\eta_{t+1}(1-\beta_1)}{2}\left(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2}\right)^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
&- \eta_{t+1}\beta_1\mathbb{E}[\left\langle\nabla f(\theta_{t-1})\,|\,(\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}m_t\right\rangle] \\
&+ \left(\frac{L}{2} + \beta_1 L\right)\|\theta_t - \theta_{t-1}\|^2 \\
&+ \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned}
\tag{22}
$$

219 *where d denotes the dimension of the parameter vector*

220 *Proof.* Denote the following auxiliary variables at iteration $t+1$

$$
z_{t+1} = \theta_{t+1} + \frac{\beta_1}{1-\beta_1}(\theta_{t+1} - \theta_t)
\tag{23}
$$

221 By assumption Assumption 1, we can write the smoothness condition on the overall objective (2),
222 between iteration t and $t+1$:

$$
f(\theta_{t+1}) \leq f(\theta_t) + \langle\nabla f(\theta_t)\,|\,\theta_{t+1} - \theta_t\rangle + \frac{L}{2}\|\theta_{t+1} - \theta_t\|^2
\tag{24}
$$

223 Denote by $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of
224 Algorithm 1. Hence, we obtain,

$$
f(\theta_{t+1}) \leq f(\theta_t) - \eta_{t+1}\left\langle\nabla f(\theta_t)\,|\,(\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle + \frac{L}{2}\|\theta_{t+1} - \theta_t\|^2
\tag{25}
$$

225 where $\mathsf{I_d}$ denotes the identity matrix.

226 We now take the expectation of those various terms conditioned on the filtration $\mathcal{F}_t$ of the total
227 randomness up to iteration t.

$$
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1}\mathbb{E}[\left\langle\nabla f(\theta_t)\,|\,(\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle] + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]
\tag{26}
$$

228 We now focus on the computation of the inner product obtained in the equation above. We have

$$
\begin{aligned}
&\eta_{t+1}\mathbb{E}[\left\langle\nabla f(\theta_t)\,|\,(\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle] \\
=&\eta_{t+1}\mathbb{E}[\left\langle\nabla f(\theta_t)\,|\,(\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}m_{t+1} + (\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}m_{t+1} - (\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle] \\
=&\eta_{t+1}\mathbb{E}[\left\langle\nabla f(\theta_t)\,|\,(\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle] + \eta_{t+1}\mathbb{E}[\left\langle\nabla f(\theta_t)\,|\,\left[(\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}\right]m_{t+1}\right\rangle] \\
=&\eta_{t+1}\beta_1\mathbb{E}[\left\langle\nabla f(\theta_t)\,|\,(\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}m_t\right\rangle] + \eta_{t+1}(1-\beta_1)\mathbb{E}[\left\langle\nabla f(\theta_t)\,|\,(\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}\bar{g}_t\right\rangle] \\
&+ \eta_{t+1}\mathbb{E}[\left\langle\nabla f(\theta_t)\,|\,\left[(\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}\right]m_{t+1}\right\rangle]
\end{aligned}
\tag{28}
$$

229 where $\bar{g}_t$ is the aggregated gradients from all workers.

Plugging the above in (26) yields:

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)]$$

$$\leq \underbrace{-\beta_1 \mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \rangle] \eta_{t+1}}_{A_t}$$

$$\underbrace{-\mathbb{E}[\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} \right] m_{t+1} \rangle] \eta_{t+1}}_{B_t} \quad (29)$$

$$\underbrace{-(1-\beta_1)\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \rangle] \eta_{t+1}}_{C_t} + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]$$

To begin with, by the tower rule, we have that

$$A_t = -\beta_1 \mathbb{E}[\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \rangle \,|\, \mathcal{F}_t]] \quad (30)$$

$$= -\beta_1 \langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \rangle] - \beta_1 \langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \rangle]$$

$$\quad (31)$$

$$\quad (32)$$

where we recognize the first term as the term in (27), at iteration $t-1$ and hence apply the same decomposition as in (28). Coupling with the smoothness of f, which gives that

$$-\beta_1 \langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \rangle] \leq \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2$$

we obtain,

$$A_t = -\beta_1 \mathbb{E}[\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \rangle \,|\, \mathcal{F}_t]]$$

$$\leq \eta_{t+1} \beta_1 (A_{t-1} + B_{t-1} + C_{t-1}) + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2 \quad (33)$$

Then,

$$B_t = -\mathbb{E}[\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} \right] m_{t+1} \rangle]$$

$$= \mathbb{E}[\sum_{j=1}^d \nabla^j f(\theta_t) m_{t=1}^j \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]]$$

$$\overset{(a)}{\leq} \mathbb{E}[\|\nabla f(\theta_t)\| \|m_{t=1}\| \sum_{j=1}^d \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \quad (34)$$

$$\overset{(b)}{\leq} G^2 \mathbb{E}[\sum_{j=1}^d \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]]$$

where $\nabla^j f(\theta_t)$ denotes the j-th component of the gradient vector $\nabla f(\theta_t)$, (a) uses of the Cauchy-Schwartz inequality and (b) boils down from the norm of the gradient vector boundedness assumption 2, denoting $G := \frac{1}{n} \sum_{i=1}^n G_i$.

Plugging the above into (29) yields

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & \eta_{t+1}(A_t + B_t + C_t) + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \\
\leq & -\eta_{t+1}\beta_1\mathbb{E}[\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}m_t \rangle] \\
& + \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]] \\
& + \left(\frac{L}{2} + \eta_{t+1}\frac{\beta_1 L}{\eta_{t-1}}\right)\|\theta_t - \theta_{t-1}\|^2 \\
& - \eta_{t+1}(1 - \beta_1)\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}\bar{g}_t \rangle]
\end{aligned}
\tag{35}
$$

We bound the last term on the RHS, $-\eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon\mathsf{I_d})^{-1/2}\bar{g}_t \rangle]$ with Lemma 2

Under the assumption that we use a decreasing stepsize such that $\eta_{t+1} \leq \eta_t$, and given that according to Line 15 we have that $\hat{v}_{t+1} \geq \hat{v}_t$ by construction, we obtain

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & -\frac{\eta_{t+1}(1 - \beta_1)}{2}(\epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
& - \eta_{t+1}\beta_1\mathbb{E}[\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}m_t \rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right)\|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned}
\tag{36}
$$

Finally, using Lemma 2, we obtain the desired result. $\qquad\square$

B.2 Proof of Theorem 1

Theorem. *Under A1 to A4, with a constant stepsize $\eta_t = \eta = \frac{L}{\sqrt{T_m}}$, we have:*

$$
\frac{1}{T_m}\sum_{t=0}^{T_m-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_m})]}{L\Delta_1\sqrt{T_m}} + d\frac{L\Delta_3}{\Delta_1\sqrt{T_m}} + \frac{\Delta_2}{\eta\Delta_1 T_m} + \frac{1 - \beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2 + 1)}G^2
\tag{37}
$$

where

$$
\Delta_1 := \frac{(1 - \beta_1)}{2}(\epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2})^{-\frac{1}{2}} \quad, \quad \Delta_2 := q^2 + \sum_{k=t+1}^{\infty}\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}
$$
$$
\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1}\right)(1 - \beta_2)^{-1}(1 - \frac{\beta_1^2}{\beta_2})^{-1}
\tag{38}
$$

Proof. By Lemma 3 we have

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & -\frac{\eta_{t+1}(1 - \beta_1)}{2}(\epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
& - \eta_{t+1}\beta_1\mathbb{E}[\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon\mathsf{I_d})^{-1/2}m_t \rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right)\|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned}
\tag{39}
$$

13

246   Let us consider the following sequence, defined for all $t > 0$:

$$R_t := f(\theta_t) - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \rangle] \tag{40}$$

247   We compute the following expectation:

$$\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] = \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] - \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \rangle]$$
$$+ \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \rangle] \tag{41}$$

248   Using the Assumption 1, we note that:

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \le -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \rangle] + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \tag{42}$$

249   which yields

$$\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] = -(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \mathbb{E}[\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \rangle]$$
$$+ \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \rangle]$$
$$+ \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2$$
$$\le (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \mathbb{E}[A_t + B_t + C_t] \tag{43}$$
$$- \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}]$$
$$+ \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2$$

250   where A_t, B_t, C_t are defined in (29).

251   We use (33) and (34) to bound A_t and B_t, and Lemma 2 to bound C_t where we precise that the
252   learning rate η_{t+1} becomes $\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}$. Hence

$$\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] \le \left((\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \right) \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}]$$
$$+ (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]]$$
$$+ \left(\frac{L}{2} + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_{t+1} - \theta_t\|^2$$
$$- (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1 - \beta_1)}{2} (\epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2]$$
$$+ q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2} \tag{44}$$

253 where the last term in the LHS is due to Lemma 3.

254 By assumption, we have that for all $t > 0$, $\eta_{t=1} \leq \eta_t$. Also, set the tuning parameters such that

$$\eta_t + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \leq \frac{\eta_t}{1 - \beta_1} \tag{45}$$

255 so that

$$(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} = 0$$

$$\iff (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 = \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \tag{46}$$

256 Note that $-(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \leq -\eta_{t+1} \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}$
257 since $\sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \geq 0$.

258 The above coupled with (44) yields

$$\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] \leq - \eta_{t+1} \frac{(1 - \beta_1)}{2} (\epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}$$

$$- (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \right]]$$

$$+ \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1} \right) \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \tag{47}$$

259 We now sum from $t = 0$ to $t = T_m - 1$ the inequality in (47), and divide it by T_m:

$$\eta \frac{(1 - \beta_1)}{2} (\epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2})^{-\frac{1}{2}} \frac{1}{T_m} \sum_{t=0}^{T_m-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2]$$

$$\leq \frac{\mathbb{E}[R_0] - \mathbb{E}[R_{T_m}]}{T_m} + \frac{q^2 \eta + \sum_{k=t+1}^{\infty} \eta \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}}{T_m}$$

$$+ \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1} \right) \frac{1}{T_m} \sum_{t=0}^{T_m-1} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \tag{48}$$

260 where we have used the fact that $(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \geq 0$ for all dimension $j \in [d]$ by
261 construction of $\hat{v}_{t+1}^j$.

262 We now bound the two remaining terms:

263 **Bounding $-\mathbb{E}[R_{T_m}]$:**

264 By definition (40) of R_t we have, using Lemma 1:

$$-\mathbb{E}[R_{T_m}] \leq \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \rangle] - f(\theta_{T_m})$$

$$\leq \| \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \| \| \nabla f(\theta_{t-1}) \| \| (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \|$$

$$\leq \eta_{t+1} (1 - \beta_1) \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1)} G^2 - f(\theta_{T_m}) \tag{49}$$

15

265 **Bounding** $\sum_{t=0}^{T_{\mathtt{m}}-1} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]$**:**

266 By definition in Algorithm 1:

$$\|\theta_{t+1} - \theta_t\|^2 = \eta_{t+1}^2 \left[(\hat{V}_{t+1} + \epsilon\mathsf{I}_{\mathsf{d}})^{-\frac{1}{2}} m_{t+1}\right]^2 = \eta_{t+1}^2 \sum_{j=1}^{d} \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j + \epsilon} \tag{50}$$

267 For any dimension $j \in [d]$,

$$\begin{aligned}
|m_{t+1}^j|^2 &= |\beta_1 m_t^j + (1-\beta_1)\bar{g}_t^j|^2 \\
&\leq \beta_1(\beta_1^2|m_{t-1}^j|^2 + (1-\beta_1)^2|\bar{g}_{t-1}^j|^2) + |\bar{g}_t^j|^2 \\
&\leq \sum_{k=0}^{t} \beta_1^{2(t-k)}|\bar{g}_k^j|^2 \\
&\leq \sum_{k=0}^{t} \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}} \beta_2^{t-k}|\bar{g}_k^j|^2
\end{aligned} \tag{51}$$

268 Using Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
|m_{t+1}^j|^2 \leq \sum_{k=0}^{t} \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}} \beta_2^{t-k}|\bar{g}_k^j|^2 &\leq \sum_{k=0}^{t} \left(\frac{\beta_1^2}{\beta_2}\right)^{t-k} \sum_{k=0}^{t} \beta_2^{t-k}|\bar{g}_k^j|^2 \\
&\leq \frac{1}{1 - \frac{\beta_1^2}{\beta_2}} \sum_{k=0}^{t} \beta_2^{t-k}|\bar{g}_k^j|^2
\end{aligned} \tag{52}$$

269 On the other hand we have

$$\hat{v}_{t+1}^j \geq \beta_2 \hat{v}_t^j + (1-\beta_2)(\bar{g}_t^j)^2 \tag{53}$$

270 and since it is also true for iteration $t=1$, we have by induction replacing v_t^j in the above that

$$\hat{v}_{t+1}^j \geq (1-\beta_2) \sum_{k=0}^{t} \beta_2^{t-k}|\bar{g}_k^j|^2 \iff \frac{\sum_{k=0}^{t} \beta_2^{t-k}|\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \leq (1-\beta_2)^{-1} \tag{54}$$

271 Hence, we can derive from (50) that

$$\begin{aligned}
\|\theta_{t+1} - \theta_t\|^2 = \eta_{t+1}^2 \sum_{j=1}^{d} \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j + \epsilon} &\leq \eta_{t+1}^2 \sum_{j=1}^{d} \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j} \\
&\stackrel{(a)}{\leq} \eta_{t+1}^2 \sum_{j=1}^{d} \frac{1}{1 - \frac{\beta_1^2}{\beta_2}} \frac{\sum_{k=0}^{t} \beta_2^{t-k}|\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \\
&\stackrel{(b)}{\leq} \eta_{t+1}^2 d(1-\beta_2)^{-1}(1 - \frac{\beta_1^2}{\beta_2})^{-1}
\end{aligned} \tag{55}$$

272 where (a) uses (52) and (b) uses (54).

273 Plugging the two bounds in (48), we obtain the following bound:

$$\begin{aligned}
\frac{1}{T_{\mathtt{m}}} \sum_{t=0}^{T_{\mathtt{m}}-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq{}& \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_{\mathtt{m}}})]}{\eta\Delta_1 T_{\mathtt{m}}} + \frac{q^2\eta + \sum_{k=t+1}^{\infty} \eta\beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}}{\eta\Delta_1 T_{\mathtt{m}}} \\
&+ \frac{1-\beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2+1)} G^2 \\
&+ \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right) \frac{1}{\eta\Delta_1} \eta^2 d(1-\beta_2)^{-1}(1 - \frac{\beta_1^2}{\beta_2})^{-1}
\end{aligned} \tag{56}$$

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274 where $\Delta_1 := \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}$

275 With a constant stepsize $\eta = \frac{L}{\sqrt{T_{\mathrm{m}}}}$ we get the final convergence bound as follows:

$$
\frac{1}{T_{\mathrm{m}}} \sum_{t=0}^{T_{\mathrm{m}}-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_{\mathrm{m}}})]}{L\Delta_1\sqrt{T_{\mathrm{m}}}} + d\frac{L\Delta_3}{\Delta_1\sqrt{T_{\mathrm{m}}}}
$$
$$
+ \frac{\Delta_2}{\eta\Delta_1 T_{\mathrm{m}}} + \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2 \tag{57}
$$

276 where $\Delta_2 := q^2 + \sum_{k=t+1}^{\infty} \beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}$ and $\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right)(1-\beta_2)^{-1}(1-\frac{\beta_1^2}{\beta_2})^{-1}$.

277 $\square$