

381 A Proofs for the iSAEM Algorithm

382 A.1 Proof of Lemma 2

383 **Lemma.** Assume A3, A4. For all $\mathbf{s} \in S$,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (16)$$

384 **Proof** Using A3 and the fact that we can exchange integration with differentiation and the Fisher's
385 identity, we obtain

$$\begin{aligned} \nabla_{\mathbf{s}} V(\mathbf{s}) &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left(\nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} L(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left(\nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} (\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))), \end{aligned} \quad (17)$$

386 Consider the following vector map:

$$\mathbf{s} \rightarrow \nabla_{\boldsymbol{\theta}} L(\mathbf{s}, \boldsymbol{\theta})|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})} = \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \mathbf{s}.$$

387 Taking the gradient of the above map w.r.t. \mathbf{s} and using assumption A3, we show that:

$$\mathbf{0} = -\mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \underbrace{\left(\nabla_{\boldsymbol{\theta}}^2 (\psi(\boldsymbol{\theta}) + \mathbf{r}(\boldsymbol{\theta}) - \langle \phi(\boldsymbol{\theta}) | \mathbf{s} \rangle) \right)}_{=\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \boldsymbol{\theta})} \Big|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})} \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s}).$$

388 The above yields

$$\nabla_{\mathbf{s}} V(\mathbf{s}) = \mathbf{B}(\mathbf{s})(\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))),$$

389 where we recall $\mathbf{B}(\mathbf{s}) = \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \left(\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \bar{\boldsymbol{\theta}}(\mathbf{s})) \right)^{-1} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top}$. The proof of (16) follows directly
390 from the assumption A4. \square

391 A.2 Proof of Theorem 1

392 Beforehand, We present two intermediary Lemmas important for the analysis of the incremental
393 update of the iSAEM algorithm. The first one gives a characterization of the quantity $\mathbb{E}[\tilde{S}^{(k+1)} -$
394 $\hat{\mathbf{s}}^{(k)}]$:

395 **Lemma.** Assume A1. The update (1) is equivalent to the following update on the resulting statistics
396

$$\hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}).$$

397 Also:

$$\mathbb{E}[\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] = \mathbb{E}[\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] + \frac{1}{n} \mathbb{E}[\eta_{i_k}^{(k+1)}],$$

398 where $\bar{\mathbf{s}}^{(k)}$ is defined by (3) and $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$.

399 **Proof** From update (1), we have:

$$\begin{aligned} \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} + \frac{1}{n} \left(\tilde{S}_{i_k}^{(k+1)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)} \right) \\ &= \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{S}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left(\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \tilde{S}_{i_k}^{(k+1)} \right). \end{aligned}$$

400 Since $\tilde{S}_{i_k}^{(k+1)} = \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) + \eta_{i_k}^{(k+1)}$ we have

$$\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{S}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left(\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) \right) + \frac{1}{n} \eta_{i_k}^{(k+1)}.$$

401 Taking the full expectation of both side of the equation leads to:

$$\begin{aligned}\mathbb{E}[\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] &= \mathbb{E}[\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right] \\ &\quad - \frac{1}{n} \mathbb{E}[\mathbb{E}[\tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k]] + \frac{1}{n} \mathbb{E}[\eta_{i_k}^{(k+1)}] .\end{aligned}$$

402 Since we have $\mathbb{E}[\tilde{S}_i^{(\tau_i^k)} | \mathcal{F}_k] = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)}$ and $\mathbb{E}[\bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k] = \bar{\mathbf{s}}^{(k)}$, we conclude the proof
403 of the Lemma. \square

404 We also derived the following auxiliary Lemma which sets an upper bound for the quantity
405 $\mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2]$:

406 **Lemma 7.** For any $k \geq 0$ and consider the iSAEM update in (1), it holds that

$$\begin{aligned}\mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] &\leq 4\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{2\mathbf{L}_s^2}{n^3} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + 2\frac{c_\eta}{M_k} + 4\mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] .\end{aligned}$$

407 **Proof** Applying the iSAEM update yields:

$$\begin{aligned}\mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] &= \mathbb{E}[\|\tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} - \frac{1}{n}(\tilde{S}_{i_k}^{(\tau_i^k)} - \tilde{S}_{i_k}^{(t_i^k)})\|^2] \\ &\leq 4\mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] + 4\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \frac{2}{n^2} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_i^k)}\|^2] + 2\frac{c_\eta}{M_k} .\end{aligned}$$

408 The last expectation can be further bounded by

$$\frac{2}{n^2} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_i^k)}\|^2] = \frac{2}{n^3} \sum_{i=1}^n \mathbb{E}[\|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2] \stackrel{(a)}{\leq} \frac{2\mathbf{L}_s^2}{n^3} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] ,$$

409 where (a) is due to Lemma 1 and which concludes the proof of the Lemma.

410 \square

411 **Theorem.** Assume A1-A5. Consider the iSAEM sequence $\{\hat{\mathbf{s}}^{(k)}\}_{k \geq 0} \in \mathcal{S}$ obtained with $\rho_{k+1} = 1$
412 for any $k \leq K_m$ where K_m is a positive integer. Let $\{\gamma_k = 1/(k^a \alpha c_1 \bar{L})\}_{k \geq 0}$, where $a \in (0, 1)$, be a
413 sequence of stepsizes, $c_1 = v_{\min}^{-1}$, $\alpha = \max\{8, 1 + 6v_{\min}\}$, $\bar{L} = \max\{\mathbf{L}_s, \mathbf{L}_V\}$, $\beta = c_1 \bar{L}/n$. Then:

$$v_{\max}^{-2} \sum_{k=0}^{K_m} \tilde{\alpha}_k \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})] + \sum_{k=0}^{K_m-1} \tilde{\Gamma}_k \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] .$$

414 **Proof** Under the smoothness of the Lyapunov function V (cf. Lemma 1), we can write:

$$V(\hat{\mathbf{s}}^{(k+1)}) \leq V(\hat{\mathbf{s}}^{(k)}) + \gamma_{k+1} \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 \mathbf{L}_V}{2} \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 .$$

415 Taking the expectation on both sides yields:

$$\begin{aligned}\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] + \gamma_{k+1} \mathbb{E}[\langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad + \frac{\gamma_{k+1}^2 \mathbf{L}_V}{2} \mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] .\end{aligned}$$

416 Using Lemma 3, we obtain:

$$\begin{aligned}
& \mathbb{E} \left[\langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\
&= \mathbb{E} \left[\langle \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] + \left(1 - \frac{1}{n} \right) \mathbb{E} \left[\left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \right\rangle \right] \\
&\quad + \frac{1}{n} \mathbb{E} \left[\langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\
&\stackrel{(a)}{\leq} -v_{\min} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \left(1 - \frac{1}{n} \right) \mathbb{E} \left[\left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \right\rangle \right] \\
&\quad + \frac{1}{n} \mathbb{E} \left[\langle \eta_{i_k}^{(k)} \mid \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\
&\stackrel{(b)}{\leq} -v_{\min} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\
&\quad + \frac{\beta(n-1)+1}{2n} \mathbb{E} \left[\left\| \nabla V(\hat{\mathbf{s}}^{(k)}) \right\|^2 \right] + \frac{1}{2n} \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \\
&\stackrel{(a)}{\leq} \left(v_{\max}^2 \frac{\beta(n-1)+1}{2n} - v_{\min} \right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\
&\quad + \frac{1}{2n} \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2],
\end{aligned}$$

417 where (a) is due to the growth condition (2) and (b) is due to Young's inequality (with $\beta \rightarrow 1$). Note

418 $a_k = \gamma_{k+1} \left(v_{\min} - v_{\max}^2 \frac{\beta(n-1)+1}{2n} \right)$ and

$$\begin{aligned}
a_k \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] &\leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)}) \right] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} \left[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \right] \\
&\quad + \frac{\gamma_{k+1}(1 - \frac{1}{n})}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] + \frac{\gamma_{k+1}}{2n} \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2].
\end{aligned} \tag{18}$$

419 We now give an upper bound of $\mathbb{E} \left[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \right]$ using Lemma 7 and plug it into (18):

$$\begin{aligned}
& (a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
&\leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)}) \right] \\
&\quad + \gamma_{k+1} \left(\frac{1}{2\beta} \left(1 - \frac{1}{n} \right) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\
&\quad + \gamma_{k+1} \left(\gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \\
&\quad + \frac{\gamma_{k+1}^2 L_V L_{\mathbf{s}}^2}{n^3} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2].
\end{aligned} \tag{19}$$

420 Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2] = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2] \right),$$

421 where the equality holds as i_k and j_k are drawn independently. For any $\beta > 0$, it holds

$$\begin{aligned}
& \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\
&= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\
&= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 - 2\gamma_{k+1}\langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \rangle\right] \\
&\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 + \gamma_{k+1}\beta\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right],
\end{aligned}$$

422 where the last inequality is due to Young's inequality. Subsequently, we have

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\
&\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}\left[\left(1 + \gamma_{k+1}\beta\right)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta}\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2\right].
\end{aligned}$$

423 Observe that $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)})$. Applying Lemma 7 yields

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})}\|^2] \\
&\leq \left(\gamma_{k+1}^2 + \frac{n-1}{n} \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2\right] \\
&\leq 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \\
&\quad + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] \\
&\quad + \sum_{i=1}^n \mathbb{E}\left[\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})}{n} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2\right].
\end{aligned}$$

424 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2].$$

425 From the above, we get

$$\begin{aligned}
\Delta^{(k+1)} &\leq \left(1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})\right) \Delta^{(k)} + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\
&\quad + 2\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] + 4\left(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}\right) \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right].
\end{aligned}$$

426 Setting $c_1 = v_{\min}^{-1}$, $\alpha = \max\{8, 1 + 6v_{\min}\}$, $\bar{L} = \max\{L_s, L_V\}$, $\gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}$, $\beta = \frac{c_1 \bar{L}}{n}$,

427 $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 6$, $\alpha \geq 8$, we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \leq 1 - \frac{c_1(k\alpha - 1) - 4}{k\alpha n c_1} \leq 1 - \frac{2}{k\alpha n c_1},$$

428 which shows that $1 - \frac{1}{n} + \gamma_{k+1}\beta + \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta}) \in (0, 1)$ for any $k > 0$. Denote $\Lambda_{(k+1)} =$
 429 $\frac{1}{n} - \gamma_{k+1}\beta - \frac{2\gamma_{k+1}L_s^2}{n^2}(\gamma_{k+1} + \frac{1}{\beta})$ and note that $\Delta^{(0)} = 0$, thus the telescoping sum yields:

$$\begin{aligned} \Delta^{(k+1)} &\leq 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] \\ &\quad + 2 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E} \left[\left\| \eta_{i_\ell}^{(\ell)} \right\|^2 \right] \\ &\quad + 4 \sum_{\ell=0}^k \prod_{j=\ell+1}^k \left(1 - \Lambda_{(j)}\right) (\gamma_{\ell+1}^2 \\ &\quad + \frac{\gamma_{\ell+1}}{\beta}) \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^\ell)} - \bar{\mathbf{s}}^{(\ell)} \right\|^2 \right]. \end{aligned}$$

430 Note $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$ Summing on both sides over $k = 0$ to $k = K_m - 1$ yields:

$$\begin{aligned} &\sum_{k=0}^{K_m-1} \Delta^{(k+1)} \\ &= 4 \sum_{k=0}^{K_m-1} (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + 2 \sum_{k=0}^{K_m-1} (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E} \left[\left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\ &\quad + \sum_{k=0}^{K_m-1} 4 (\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta}) \omega_{k,1} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \tag{20} \\ &\leq \sum_{k=0}^{K_m-1} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{k=0}^{K_m-1} \frac{2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E} \left[\left\| \eta_{i_\ell}^{(k)} \right\|^2 \right] \\ &\quad + \sum_{k=0}^{K_m-1} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right]. \end{aligned}$$

431 We recall (19) where we have summed on both sides from $k = 0$ to $k = K_m - 1$:

$$\begin{aligned} &\sum_{k=0}^{K_m-1} (a_k - 2\gamma_{k+1}^2 L_V) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] \\ &\quad + \sum_{k=0}^{K_m-1} \gamma_{k+1} \left(\frac{1}{2\beta} (1 - \frac{1}{n}) + 2\gamma_{k+1} L_V \right) \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \tag{21} \\ &\quad + \sum_{k=0}^{K_m-1} \gamma_{k+1} \left(\gamma_{k+1} L_V + \frac{1}{2n} \right) \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2] \\ &\quad + \sum_{k=0}^{K_m-1} \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \Delta^{(k)}. \end{aligned}$$

432 Plugging (20) into (21) results in:

$$\begin{aligned} & \sum_{k=0}^{K_m-1} \tilde{\alpha}_k \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{k=0}^{K_m-1} \tilde{\beta}_k \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ & \leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K)}) \right] + \sum_{k=0}^{K_m-1} \tilde{\Gamma}_k \mathbb{E}[\|\eta_{i_k}^{(k)}\|^2], \end{aligned}$$

433 where

$$\begin{aligned} \tilde{\alpha}_k &= a_k - 2\gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}}, \\ \tilde{\beta}_k &= \gamma_{k+1} \left(\frac{1}{2\beta} \left(1 - \frac{1}{n}\right) + 2\gamma_{k+1} L_V \right) - \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{4(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}}, \\ \tilde{\Gamma}_k &= \gamma_{k+1} \left(\gamma_{k+1} L_V + \frac{1}{2n} \right) + \frac{\gamma_{k+1}^2 L_V L_s^2}{n^2} \frac{2(\gamma_{k+1}^2 + \frac{\gamma_{k+1}}{\beta})}{\Lambda_{(k+1)}}, \end{aligned}$$

434 and

$$\begin{aligned} a_k &= \gamma_{k+1} \left(v_{\min} - v_{\max}^2 \frac{\beta(n-1) + 1}{2n} \right), \\ \Lambda_{(k+1)} &= \frac{1}{n} - \gamma_{k+1} \beta - \frac{2\gamma_{k+1} L_s^2}{n^2} \left(\gamma_{k+1} + \frac{1}{\beta} \right), \\ c_1 &= v_{\min}^{-1}, \alpha = \max\{8, 1 + 6v_{\min}\}, \bar{L} = \max\{L_s, L_V\}, \gamma_{k+1} = \frac{1}{k\alpha c_1 \bar{L}}, \beta = \frac{c_1 \bar{L}}{n}. \end{aligned}$$

435 When, for any $k > 0$, $\tilde{\alpha}_k \geq 0$, we have by Lemma 2 that:

$$\sum_{k=0}^{K_m} \tilde{\alpha}_k \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq v_{\max}^2 \sum_{k=0}^{K_m} \tilde{\alpha}_k \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2],$$

436 which yields an upper bound of the gradient of the Lyapunov function V along the path of the
437 iSAEM update and concludes the proof of the Theorem. \square

438 B Proofs for the vrTTEM and the fiTTEM Algorithms

439 B.1 Proofs of Auxiliary Lemmas (Lemma 4, Lemma 5 and Lemma 6)

440 **Lemma.** Consider the vrTTEM update (2) with $\rho_k = \rho$, it holds for all $k > 0$

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &\quad + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

441 where we recall that $\ell(k)$ is the first iteration number in the epoch that iteration k is in.

442 **Proof** Beforehand, we provide a rewriting of the quantity $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$ that will be useful through-
443 out this proof:

$$\begin{aligned} \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) \\ &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathcal{S}^{(k+1)}) \\ &= -\gamma_{k+1} \left((1-\rho) \left[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[\hat{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)} \right] \right). \end{aligned} \tag{22}$$

444 We observe, using the identity (22), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)}\|^2] + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)}\|^2]. \tag{23}$$

445 For the latter term, we obtain its upper bound as

$$\begin{aligned} &\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)}\|^2] \\ &= \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \tilde{S}_i^{\ell(k)}) - (\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{\ell(k)}) \right\|^2 \right] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{\ell(k)}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \stackrel{(b)}{\leq} L_s^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

446 where (a) uses the variance inequality and (b) uses Lemma 1. Substituting into (23) proves the
447 lemma. \square

448 **Lemma.** Consider the fiTTEM update (3) with $\rho_k = \rho$. It holds for all $k > 0$ that

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2], \end{aligned}$$

449 where L_s is the smoothness constant defined in Lemma 1.

450 **Proof** Beforehand, we provide a rewriting of the quantity $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$ that will be useful through-
451 out this proof:

$$\begin{aligned} \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) \\ &= -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - (1-\rho)\tilde{S}^{(k)} - \rho\mathcal{S}^{(k+1)}) \\ &= -\gamma_{k+1} \left((1-\rho) \left[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[\hat{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)} \right] \right) \\ &= -\gamma_{k+1} \left((1-\rho) \left[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} \right] + \rho \left[\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} - (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \right] \right). \end{aligned} \tag{24}$$

452 We observe, using the identity (24), that

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \leq 2\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)}\|^2] + 2(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(\ell(k))} - \tilde{S}^{(k)}\|^2]. \tag{25}$$

453 For the latter term, we obtain its upper bound as

$$\begin{aligned}\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \mathbf{S}^{(k+1)}\|^2] &= \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n (\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(k)}) - (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})\right\|^2\right] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(\ell(k))}\|^2] + \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2],\end{aligned}$$

454 where (a) uses the variance inequality. We can further bound the last expectation using Lemma 1:

$$\mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)}\|^2] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2] \stackrel{(a)}{\leq} \frac{L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2].$$

455 Substituting into (25) proves the lemma. \square

456 **Lemma.** Considering a decreasing stepsize $\gamma_k \in (0, 1)$ and a constant $\rho \in (0, 1)$, we have

$$\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \leq \frac{\rho}{1-\rho} \sum_{\ell=0}^k (1-\gamma_\ell)^2 (\mathbf{S}^{(\ell)} - \tilde{S}^{(\ell)}),$$

457 where $\mathbf{S}^{(k)}$ is defined either by Line 2 (vrTTEM) or Line 3 (fiTTEM).

458 **Proof** We begin by writing the two-timescale update:

$$\begin{aligned}\tilde{S}^{(k+1)} &= \tilde{S}^{(k)} + \rho(\mathbf{S}^{(k+1)} - \tilde{S}^{(k)}), \\ \hat{\mathbf{s}}^{(k+1)} &= \hat{\mathbf{s}}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}),\end{aligned}\tag{26}$$

459 where $\mathbf{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(t_i^k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})$ according to (3). Denote $\delta^{(k+1)} = \hat{\mathbf{s}}^{(k+1)} - \tilde{S}^{(k+1)}$.
460 Then from (26), doing the subtraction of both equations yields:

$$\delta^{(k+1)} = (1 - \gamma_{k+1})\delta^{(k)} + \frac{\rho}{1-\rho}(1 - \gamma_{k+1})(\mathbf{S}^{(k+1)} - \tilde{S}^{(k+1)}).$$

461 Using the telescoping sum and noting that $\delta^{(0)} = 0$, we have

$$\delta^{(k+1)} \leq \frac{\rho}{1-\rho} \sum_{\ell=0}^k (1 - \gamma_{\ell+1})^2 (\mathbf{S}^{(\ell+1)} - \tilde{S}^{(\ell+1)}).$$

462 \square

463 B.2 Additional Intermediary Result

464 **Lemma 8.** At iteration $k + 1$, the drift term of update (3), with $\rho_{k+1} = \rho$, is equivalent to the
465 following :

$$\begin{aligned}\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} &= \rho(\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}) + \rho\eta_{i_k}^{(k+1)} + \rho\left[(\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}]\right] \\ &\quad + (1 - \rho)(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}),\end{aligned}$$

466 where we recall that $\eta_{i_k}^{(k+1)}$, defined in (12), which is the gap between the MC approximation and
467 the expected statistics.

468 **Proof** Using the fitTEM update $\tilde{S}^{(k+1)} = (1-\rho)\tilde{S}^{(k)} + \rho\mathcal{S}^{(k+1)}$ where $\mathcal{S}^{(k+1)} = \bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)})$ leads to the following decomposition:

$$\begin{aligned}
& \tilde{S}^{(k+1)} - \hat{s}^{(k)} \\
&= (1-\rho)\tilde{S}^{(k)} + \rho \left(\bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \right) - \hat{s}^{(k)} + \rho\bar{\mathcal{S}}^{(k)} - \rho\bar{\mathcal{S}}^{(k)} \\
&= \rho(\bar{\mathcal{S}}^{(k)} - \hat{s}^{(k)}) + \rho(\tilde{S}_{i_k}^{(k)} - \bar{\mathcal{S}}_{i_k}^{(k)}) + (1-\rho) \left(\tilde{S}^{(k)} - \hat{s}^{(k)} \right) + \rho \left(\bar{\mathcal{S}}^{(k)} - \bar{\mathcal{S}}^{(k)} + (\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \right) \\
&= \rho(\bar{\mathcal{S}}^{(k)} - \hat{s}^{(k)}) + \rho\eta_{i_k}^{(k+1)} - \rho \left[(\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] \\
&+ (1-\rho) \left(\tilde{S}^{(k)} - \hat{s}^{(k)} \right),
\end{aligned}$$

470 where we observe that $\mathbb{E}[\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] = \bar{\mathcal{S}}^{(k)} - \bar{\mathcal{S}}^{(k)}$ and which concludes the proof.

471 *Important Note:* Note that $\bar{\mathcal{S}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}$ is not equal to $\eta_{i_k}^{(k+1)}$, defined in (12), which is the gap
472 between the MC approximation and the expected statistics. Indeed $\tilde{S}_{i_k}^{(t_{i_k}^k)}$ is not computed under the
473 same model as $\bar{\mathcal{S}}_{i_k}^{(k)}$. \square

474 B.3 Proof of Theorem 2

475 **Theorem.** Assume A1-A5. Consider the vrTTEM sequence $\{\hat{s}^{(k)}\}_{k \geq 0} \in \mathcal{S}$ for any $k \leq K_m$ where
476 K_m is a positive integer. Let $\{\gamma_{k+1} = 1/(k^a \bar{L})\}_{k \geq 0}$, where $a \in (0, 1)$, be a sequence of stepsizes,
477 $\bar{L} = \max\{L_S, L_V\}$, $\rho = \mu/(c_1 \bar{L} n^{2/3})$, $m = nc_1^2/(2\mu^2 + \mu c_1^2)$ and a constant $\mu \in (0, 1)$. Then:

$$\mathbb{E}[\|\nabla V(\hat{s}^{(K)})\|^2] \leq \frac{2n^{2/3}\bar{L}}{\mu P_m v_{\min}^2 v_{\max}^2} \left(\mathbb{E}[\Delta V] + \sum_{k=0}^{K_m-1} \tilde{\eta}^{(k+1)} + \chi^{(k+1)} \mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k)}\|^2] \right).$$

478 **Proof** Using the smoothness of V and update (2), we obtain:

$$\begin{aligned}
V(\hat{s}^{(k+1)}) &\leq V(\hat{s}^{(k)}) + \langle \hat{s}^{(k+1)} - \hat{s}^{(k)} | \nabla V(\hat{s}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{s}^{(k+1)} - \hat{s}^{(k)}\|^2 \\
&\leq V(\hat{s}^{(k)}) - \gamma_{k+1} \langle \hat{s}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{s}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{s}^{(k)} - \tilde{S}^{(k+1)}\|^2.
\end{aligned} \tag{27}$$

479 Denote $H_{k+1} := \hat{s}^{(k)} - \tilde{S}^{(k+1)}$ the drift term of the fitTEM update in (7) and $h_k = \hat{s}^{(k)} - \bar{\mathcal{S}}^{(k)}$.
480 Taking expectations on both sides show that

$$\begin{aligned}
& \mathbb{E}[V(\hat{s}^{(k+1)})] \\
&\stackrel{(a)}{\leq} \mathbb{E}[V(\hat{s}^{(k)})] - \gamma_{k+1}(1-\rho)\mathbb{E}[\langle \hat{s}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{s}^{(k)}) \rangle] - \gamma_{k+1}\rho\mathbb{E}[\langle \hat{s}^{(k)} - \mathcal{S}^{(k+1)} | \nabla V(\hat{s}^{(k)}) \rangle] \\
&+ \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|H_{k+1}\|^2] \\
&\stackrel{(b)}{\leq} \mathbb{E}[V(\hat{s}^{(k)})] - \gamma_{k+1}\rho\mathbb{E}[\langle h_k | \nabla V(\hat{s}^{(k)}) \rangle] - \gamma_{k+1}(1-\rho)\mathbb{E}[\langle \hat{s}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{s}^{(k)}) \rangle] \\
&- \gamma_{k+1}\rho\mathbb{E}[\langle \eta_{i_k}^{(k+1)} | \nabla V(\hat{s}^{(k)}) \rangle] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|H_{k+1}\|^2] \\
&\stackrel{(c)}{\leq} \mathbb{E}[V(\hat{s}^{(k)})] - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2) \mathbb{E}[\|h_k\|^2] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|H_{k+1}\|^2] \\
&- \gamma_{k+1}\rho\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \gamma_{k+1}(1-\rho)\mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k)}\|^2],
\end{aligned} \tag{28}$$

481 where we have used (22) in (a) and $\mathbb{E}[\mathcal{S}^{(k+1)}] = \bar{\mathbf{s}}^{(k)} + \mathbb{E}[\eta_{i_k}^{(k+1)}]$ in (b), the growth condition in
 482 Lemma 2 and Young's inequality with the constant equal to 1 in (c).

483 Furthermore, for $k+1 \leq \ell(k) + m$ (i.e., $k+1$ is in the same epoch as k), we have

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] = \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} + \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \rangle\right] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2\right. \\ &\quad \left.- 2\gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))} | \rho(\mathbf{h}_k - \eta_{i_k}^{(k+1)}) + (1-\rho)(\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}) \rangle\right] \\ &\leq \mathbb{E}\left[(1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2\right. \\ &\quad \left.+ \frac{\gamma_{k+1}\rho}{\beta} \|\eta_{i_k}^{(k+1)}\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right], \end{aligned}$$

484 where we first used (22) and the last inequality is due to Young's inequality.

485 Consider the following sequence

$$R_k := \mathbb{E}[V(\hat{\mathbf{s}}^{(k)}) + b_k \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2],$$

486 where $b_k := \bar{b}_{k \bmod m}$ is a periodic sequence where:

$$\bar{b}_i = \bar{b}_{i+1}(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2) + \gamma_{k+1}^2 \rho^2 \mathbf{L}_V \mathbf{L}_s^2, \quad i = 0, 1, \dots, m-1 \quad \text{with } \bar{b}_m = 0.$$

487 Note that \bar{b}_i is decreasing with i and this implies

$$\bar{b}_i \leq \bar{b}_0 = \gamma_{k+1}^2 \rho^2 \mathbf{L}_V \mathbf{L}_s^2 \frac{(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2)^m - 1}{\gamma_{k+1}\beta + 2\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2}, \quad i = 1, 2, \dots, m.$$

488 For $k+1 \leq \ell(k) + m$, we have the following inequality

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2) \|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}^2 \mathbf{L}_V}{2} \|\mathbf{H}_{k+1}\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[\rho \|\eta_{i_k}^{(k+1)}\|^2 - (1-\rho) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \gamma_{k+1}^2 \|\mathbf{H}_{k+1}\|^2 + \frac{\gamma_{k+1}\rho}{\beta} \|\mathbf{h}_k\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\frac{\gamma_{k+1}\rho}{\beta} \|\eta_{i_k}^{(k+1)}\|^2 + \frac{\gamma_{k+1}(1-\rho)}{\beta} \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right]. \end{aligned}$$

489 And using Lemma 4 we obtain:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}\left[V(\hat{\mathbf{s}}^{(k)}) - (\gamma_{k+1}\rho v_{\min} + \gamma_{k+1}v_{\max}^2 - \gamma_{k+1}^2 \rho^2 \mathbf{L}_V) \|\mathbf{h}_k\|^2 + \gamma_{k+1}^2 \rho^2 \mathbf{L}_V \mathbf{L}_s^2 \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[(1 + \gamma_{k+1}\beta + 2\gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2) \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2 + \left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2 \rho^2\right) \|\mathbf{h}_k\|^2\right] \\ &\quad + \gamma_{k+1} \mathbb{E}\left[(\rho + \rho^2 \gamma_{k+1} \mathbf{L}_V) \|\eta_{i_k}^{(k+1)}\|^2 - (1-\rho - (1-\rho)^2 \gamma_{k+1} \mathbf{L}_V) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right] \\ &\quad + b_{k+1} \mathbb{E}\left[\left(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2 \rho^2\right) \|\eta_{i_k}^{(k+1)}\|^2 + \left(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2 (1-\rho)^2\right) \|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{s}}^{(k)}\|^2\right]. \end{aligned}$$

490 Rearranging the terms yields:

$$\begin{aligned} R_{k+1} &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1} \rho^2 \mathbf{L}_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1} \rho^2)) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\quad + \underbrace{\left(b_{k+1}(1 + \gamma\beta + 2\gamma^2 \rho^2 \mathbf{L}_s^2) + \gamma^2 \rho^2 \mathbf{L}_V \mathbf{L}_s^2\right)}_{=b_k \text{ since } k+1 \leq \ell(k) + m} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\ell(k))}\|^2] + \tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}, \end{aligned}$$

491 where

$$\begin{aligned}\tilde{\eta}^{(k+1)} &= \left(\gamma_{k+1}(\rho + \rho^2 \gamma_{k+1} L_V) + b_{k+1}(\frac{\gamma_{k+1}\rho}{\beta} + 2\gamma_{k+1}^2 \rho^2) \right) \mathbb{E} \left[\left\| \eta_{i_k}^{(k+1)} \right\|^2 \right] \\ \chi^{(k+1)} &= \left(b_{k+1}(\frac{\gamma_{k+1}(1-\rho)}{\beta} + 2\gamma_{k+1}^2(1-\rho)^2) - \gamma_{k+1}(1-\rho - (1-\rho)^2 \gamma_{k+1} L_V) \right) \\ \tilde{\chi}^{(k+1)} &= \chi^{(k+1)} \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)} \right\|^2 \right].\end{aligned}$$

492 This leads, using Lemma 2, that for any γ_{k+1} , ρ and β such that $\rho v_{\min} + v_{\max}^2 -$
493 $\gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2) > 0$,

$$\begin{aligned}v_{\max}^2 \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] \\ &\leq \frac{R_k - R_{k+1}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))} \\ &\quad + \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2))}.\end{aligned}$$

494 We first remark that

$$\begin{aligned}&\gamma_{k+1}(\rho v_{\min} + v_{\max}^2 - \gamma_{k+1}\rho^2 L_V - b_{k+1}(\frac{\rho}{\beta} + 2\gamma_{k+1}\rho^2)) \\ &\geq \frac{\gamma_{k+1}\rho}{c_1} (1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1)),\end{aligned}$$

495 where $c_1 = v_{\min}^{-1}$. By setting $\bar{L} = \max\{L_s, L_V\}$, $\beta = \frac{c_1 \bar{L}}{n^{1/3}}$, $\rho = \frac{\mu}{c_1 \bar{L} n^{2/3}}$, $m = \frac{nc_1^2}{2\mu^2 + \mu c_1^2}$ and
496 $\{\gamma_{k+1}\}$ any sequence of decreasing stepsizes in $(0, 1)$, it can be shown that there exists $\mu \in (0, 1)$,
497 such that the following lower bound holds

$$\begin{aligned}&1 - \gamma_{k+1}c_1\rho L_V - b_{k+1}(\frac{c_1}{\beta} + 2\gamma_{k+1}\rho c_1) \\ &\geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \bar{b}_0(\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ &\geq 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{L_V \mu^2}{c_1^2 n^{\frac{4}{3}}} \frac{(1 + \gamma\beta + 2\gamma^2 L_s^2)^m - 1}{\gamma\beta + 2\gamma^2 L_s^2} (\frac{n^{\frac{1}{3}}}{\bar{L}} + \frac{2\mu}{\bar{L}n^{\frac{2}{3}}}) \\ &\stackrel{(a)}{\geq} 1 - \frac{\mu}{n^{\frac{2}{3}}} - \frac{\mu}{c_1^2} (e - 1)(1 + \frac{2\mu}{n}) \geq 1 - \mu - \mu(1 + 2\mu) \frac{e - 1}{c_1^2} \stackrel{(b)}{\geq} \frac{1}{2},\end{aligned}$$

498 where the simplification in (a) is due to

$$\frac{\mu}{n} \leq \gamma\beta + 2\gamma^2 L_s^2 \leq \frac{\mu}{n} + \frac{2\mu^2}{c_1^2 n^{\frac{4}{3}}} \leq \frac{\mu c_1^2 + 2\mu^2}{c_1^2} \frac{1}{n} \text{ and } (1 + \gamma\beta + 2\gamma^2 L_s^2)^m \leq e - 1.$$

499 and the required μ in (b) can be found by solving the quadratic equation.

500 Finally, these results yield:

$$v_{\max}^2 \sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq \frac{2(R_0 - R_{K_m})}{v_{\min}\rho} + 2 \sum_{k=0}^{K_m-1} \frac{\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}}{v_{\min}\rho}.$$

501 Note that $R_0 = \mathbb{E}[V(\hat{\mathbf{s}}^{(0)})]$ and if K_m is a multiple of m , then $R_{K_m} = \mathbb{E}[V(\hat{\mathbf{s}}^{(K_m)})]$. Under the latter
502 condition, we have

$$\sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] \leq \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \mathbb{E}[V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})] + \frac{2n^{2/3}\bar{L}}{\mu v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_m-1} [\tilde{\eta}^{(k+1)} + \tilde{\chi}^{(k+1)}].$$

503 This concludes our proof.

504 □

505 **B.4 Proof of Theorem 3**

506 **Theorem.** Assume A1-A5. Consider the fitTEM sequence $\{\hat{\mathbf{s}}^{(k)}\}_{k>0} \in \mathcal{S}$ for any $k \leq K_m$ where
 507 K_m be a positive integer. Let $\{\gamma_{k+1} = 1/(k^a \alpha c_1 \bar{L})\}_{k>0}$, where $a \in (0, 1)$, be a sequence of
 508 positive stepsizes, $\alpha = \max\{2, 1 + 2v_{\min}\}$, $\bar{L} = \max\{L_s, L_V\}$, $\beta = 1/(\alpha n)$, $\rho = 1/(\alpha c_1 \bar{L} n^{2/3})$
 509 and $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$, $\alpha \geq 2$. Then:

$$\mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(K)})\|^2] \leq \frac{4\alpha \bar{L} n^{2/3}}{P_m v_{\min}^2 v_{\max}^2} \left(\mathbb{E}[\Delta V] + \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \Gamma^{(k+1)} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \right).$$

510 **Proof** Using the smoothness of V and update (3), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) + \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{L_V}{2} \|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2. \end{aligned} \quad (29)$$

511 Denote $H_{k+1} := \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}$ the drift term of the fitTEM update in (7) and $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$.
 512 Using Lemma 8 and the additional following identity:

$$\mathbb{E} \left[(\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) - \mathbb{E}[\bar{\mathbf{s}}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}] \right] = 0, \quad (30)$$

513 we have:

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] \\ &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} \rho \mathbb{E}[\langle \mathbf{h}_k | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] - \gamma_{k+1} \mathbb{E} \left[\langle \rho \mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k] + (1 - \rho) \mathbb{E}[\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}] | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2 \\ &\stackrel{(a)}{\leq} -v_{\min} \gamma_{k+1} \rho \mathbb{E}[\|\mathbf{h}_k\|^2] - \gamma_{k+1} \mathbb{E} \left[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2 \right] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2 \\ &\stackrel{(b)}{\leq} -(v_{\min} \gamma_{k+1} \rho + \gamma_{k+1} v_{\max}^2) \mathbb{E}[\|\mathbf{h}_k\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)} - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \|H_{k+1}\|^2, \end{aligned}$$

514 where $\xi^{(k+1)} = \mathbb{E}[\|\mathbb{E}[\eta_{i_k}^{(k+1)} | \mathcal{F}_k]\|^2]$.

515 **Bounding** $\mathbb{E}[\|H_{k+1}\|^2]$ Using Lemma 5, we obtain:

$$\begin{aligned} &\gamma_{k+1} (v_{\min} \rho + v_{\max}^2 - \gamma_{k+1} \rho^2 L_V) \mathbb{E}[\|\mathbf{h}_k\|^2] \\ &\leq \mathbb{E} \left[V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)}) \right] + \tilde{\xi}^{(k+1)} + \left((1 - \rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1} (1 - \rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}^2 L_V \rho^2 L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2], \end{aligned} \quad (31)$$

516 where $\tilde{\xi}^{(k+1)} = \gamma_{k+1}^2 \rho^2 L_V \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] - \frac{\gamma_{k+1} \rho^2}{2} \xi^{(k+1)}$. Next, we observe that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \right), \quad (32)$$

517 where the equality holds as i_k and j_k are drawn independently. Then,

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle\right]. \end{aligned}$$

518 Note that $\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = -\gamma_{k+1}(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}) = -\gamma_{k+1}\mathbf{H}_{k+1}$ and that in expectation we recall
 519 that $\mathbb{E}[\mathbf{H}_{k+1}|\mathcal{F}_k] = \rho\mathbf{h}_k + \rho\mathbb{E}[\eta_{i_k}^{(k+1)}|\mathcal{F}_k] + (1-\rho)\mathbb{E}[\tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)}]$ where $\mathbf{h}_k = \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}$. Thus,
 520 for any $\beta > 0$, it holds

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle\right] \\ &\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \right. \\ &\quad \left. + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]\right], \end{aligned}$$

521 where the last inequality is due to Young's inequality. Plugging this into (32) yields:

$$\begin{aligned} & \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2\langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \mid \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle\right] \\ &\leq \mathbb{E}\left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + (1 + \gamma_{k+1}\beta)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \right. \\ &\quad \left. + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]\right]. \end{aligned}$$

522 Subsequently, we have

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\ &\leq \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n^2} \sum_{i=1}^n \mathbb{E}\left[\left(1 + \gamma_{k+1}\beta\right)\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\|\mathbf{h}_k\|^2 \right. \\ &\quad \left. + \frac{\gamma_{k+1}\rho^2}{\beta}\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] + \frac{\gamma_{k+1}(1-\rho)^2}{\beta}\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2]\right]. \end{aligned}$$

523 We now use Lemma 5 on $\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 = \gamma_{k+1}^2\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2$ and obtain:

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^{k+1})}\|^2] \\ &\leq \left(2\gamma_{k+1}^2\rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \sum_{i=1}^n \left(\frac{\gamma_{k+1}^2\rho^2\mathbf{L}_s^2}{n} + \frac{(n-1)(1+\gamma_{k+1}\beta)}{n^2}\right)\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + \gamma_{k+1}(1-\rho)^2\left(2\gamma_{k+1} + \frac{1}{\beta}\right)\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left(2\gamma_{k+1}^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2] \\ &\leq \left(2\gamma_{k+1}^2\rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)\mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \sum_{i=1}^n \left(\frac{1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2\rho^2\mathbf{L}_s^2}{n}\right)\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + \gamma_{k+1}(1-\rho)^2\left(2\gamma_{k+1} + \frac{1}{\beta}\right)\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + \left(2\gamma_{k+1}^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)\mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]. \end{aligned}$$

524 Let us define

$$\Delta^{(k)} := \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2].$$

525 From the above, we get

$$\begin{aligned} \Delta^{(k+1)} &\leq \left(1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2\right) \Delta^{(k)} + \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] + \gamma_{k+1} \left(2\gamma_{k+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]. \end{aligned}$$

526 Setting $c_1 = v_{\min}^{-1}$, $\alpha = \max\{2, 1+2v_{\min}\}$, $\bar{L} = \max\{\mathbf{L}_s, \mathbf{L}_V\}$, $\gamma_{k+1} = \frac{1}{k}$, $\beta = \frac{1}{\alpha n}$, $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$,
527 $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$, $\alpha \geq 2$, we observe that

$$1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2 \leq 1 - \frac{1}{n} + \frac{1}{\alpha k n} + \frac{1}{\alpha^2 c_1^2 k^2 n^{4/3}} \leq 1 - \frac{c_1(k\alpha - 1) - 1}{k\alpha n c_1} \leq 1 - \frac{1}{k\alpha n c_1}$$

528 which shows that $1 - \frac{1}{n} + \gamma_{k+1}\beta + \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2 \in (0, 1)$ for any $k > 0$. Denote $\Lambda_{(k+1)} = \frac{1}{n} -$
529 $\gamma_{k+1}\beta - \gamma_{k+1}^2 \rho^2 \mathbf{L}_s^2$ and note that $\Delta^{(0)} = 0$, thus the telescoping sum yields:

$$\begin{aligned} \Delta^{(k+1)} &\leq \sum_{\ell=0}^k \omega_{k,\ell} \left(2\gamma_{\ell+1}^2 \rho^2 + \frac{\gamma_{\ell+1}^2 \rho^2}{\beta}\right) \mathbb{E}[\|\bar{\mathbf{s}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] \\ &\quad + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1}(1-\rho)^2 \left(2\gamma_{\ell+1} + \frac{1}{\beta}\right) \mathbb{E}[\|\tilde{\mathbf{S}}^{(\ell)} - \hat{\mathbf{s}}^{(\ell)}\|^2] + \sum_{\ell=0}^k \omega_{k,\ell} \gamma_{\ell+1} \tilde{\epsilon}^{(\ell+1)}, \end{aligned}$$

530 where $\omega_{k,\ell} = \prod_{j=\ell+1}^k (1 - \Lambda_{(j)})$ and $\tilde{\epsilon}^{(\ell+1)} = \left(2\gamma_{k+1} + \frac{\rho^2}{\beta}\right) \mathbb{E}[\|\eta_{i_k}^{(k+1)}\|^2]$.

531 Summing on both sides over $k = 0$ to $k = K_m - 1$ yields:

$$\begin{aligned} \sum_{k=0}^{K_m-1} \Delta^{(k+1)} &\leq \sum_{k=0}^{K_m-1} \frac{2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}^2 \rho^2}{\beta}}{\Lambda_{(k+1)}} \mathbb{E}[\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \sum_{k=0}^{K_m-1} \frac{\gamma_{k+1}(1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta}\right)}{\Lambda_{(k+1)}} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] + \sum_{k=0}^{K_m-1} \frac{\gamma_{k+1}}{\Lambda_{(k+1)}} \tilde{\epsilon}^{(k+1)}. \end{aligned}$$

532 We recall (31) where we have summed on both sides from $k = 0$ to $k = K_m - 1$:

$$\begin{aligned} &\mathbb{E}[V(\hat{\mathbf{s}}^{(K_m)}) - V(\hat{\mathbf{s}}^{(0)})] \\ &\leq \sum_{k=0}^{K_m-1} \left\{ \gamma_{k+1}(-(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}\rho^2 \mathbf{L}_V) \mathbb{E}[\|\mathbf{h}_k\|^2] + \gamma^2 \mathbf{L}_V \rho^2 \mathbf{L}_s^2 \Delta^{(k)} \right\} \\ &\quad + \sum_{k=0}^{K_m-1} \left\{ \tilde{\xi}^{(k+1)} + \left((1-\rho)^2 \gamma_{k+1}^2 \mathbf{L}_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2] \right\} \\ &\leq \sum_{k=0}^{K_m-1} \left\{ \left[-\gamma_{k+1}(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 \mathbf{L}_V + \frac{\rho^2 \gamma_{k+1}^2 \mathbf{L}_V \mathbf{L}_s^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta}\right)}{\Lambda_{(k+1)}} \right] \mathbb{E}[\|\mathbf{h}_k\|^2] \right\} \\ &\quad + \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \sum_{k=0}^{K_m-1} \Gamma^{(k+1)} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{\mathbf{S}}^{(k)}\|^2], \end{aligned} \tag{33}$$

where

$$\Xi^{(k+1)} = \tilde{\xi}^{(k+1)} + \frac{\gamma_{k+1}^3 \mathbf{L}_V \rho^2 \mathbf{L}_s^2}{\Lambda_{(k+1)}} \tilde{\epsilon}^{(k+1)}$$

and

$$\Gamma^{(k+1)} = \left((1-\rho)^2 \gamma_{k+1}^2 L_V - \frac{\gamma_{k+1}(1-\rho)^2}{2} \right) + \frac{\gamma_{k+1}^3 L_V \rho^2 L_s^2 (1-\rho)^2 \left(2\gamma_{k+1} + \frac{1}{\beta} \right)}{\Lambda_{(k+1)}}.$$

533 We now analyse the following quantity

$$\begin{aligned} & -\gamma_{k+1}(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1}^2 \rho^2 L_V + \frac{\rho^2 \gamma_{k+1}^2 L_V L_s^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \\ & = \gamma_{k+1} \left[-(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right]. \end{aligned} \quad (34)$$

534 Furthermore, we recall that $c_1 = v_{\min}^{-1}$, $\alpha = \max\{2, 1 + 2v_{\min}\}$, $\bar{L} = \max\{L_s, L_V\}$, $\gamma_{k+1} = \frac{1}{k}$,
535 $\beta = \frac{1}{\alpha n}$, $\rho = \frac{1}{\alpha c_1 \bar{L} n^{2/3}}$, $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$, $\alpha \geq 2$. Then,

$$\begin{aligned} & \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\frac{1}{n} - \gamma_{k+1}\beta - \gamma_{k+1}^2 \rho^2 L_s^2} \\ & \leq \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L}(k\alpha^2 c_1^2 n^{4/3})^{-1} \left(\frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k\alpha c_1^2 \bar{L}^2 n^{1/3}} \right)}{\frac{1}{n} - \frac{1}{k\alpha n} - \frac{1}{k^2 \alpha^2 c_1^2 n^{4/3}}} \\ & = \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\bar{L} \left(\frac{2}{k^2 \alpha^2 c_1^2 \bar{L}^2 n^{4/3}} + \frac{1}{k\alpha c_1^2 \bar{L}^2 n^{1/3}} \right)}{(k\alpha c_1 n^{1/3})(k\alpha - 1)c_1 - 1} \\ & \stackrel{(a)}{\leq} \frac{1}{k\alpha^2 c_1^2 \bar{L} n^{4/3}} + \frac{\frac{1}{k\alpha c_1^2 \bar{L} n^{1/3}} \left(\frac{2}{k\alpha n} + 1 \right)}{2(\alpha c_1 n^{1/3}) - 1} \\ & \leq \frac{1}{k^2 \alpha c_1^2 \bar{L} n^{4/3}} + \frac{1}{4k\alpha^2 c_1^3 \bar{L} n^{2/3}} \\ & \leq \frac{3/4}{\alpha c_1^2 \bar{L} n^{2/3}}, \end{aligned} \quad (35)$$

where (a) is due to $c_1(k\alpha - 1) \geq c_1(\alpha - 1) \geq 2$ and $k\alpha c_1 n^{1/3} \geq 1$. Note also that

$$-(v_{\min}\rho + v_{\max}^2) \leq -\rho v_{\min} = -\frac{1}{\alpha c_1^2 \bar{L} n^{2/3}},$$

which yields that

$$\left[-(v_{\min}\rho + v_{\max}^2) + \gamma_{k+1} \rho^2 L_V + \frac{\rho^2 \gamma_{k+1} L_V L_s^2 \left(2\gamma_{k+1}^2 \rho^2 + \frac{\gamma_{k+1}\rho^2}{\beta} \right)}{\Lambda_{(k+1)}} \right] \leq -\frac{1/4}{\alpha c_1^2 \bar{L} n^{2/3}}.$$

536 Using the Lemma 2, we know that $v_{\max}^2 \|\nabla V(\hat{s}^{(k)})\|^2 \leq \|\hat{s}^{(k)} - \bar{s}^{(k)}\|^2$ and using (35) on (33)
537 yields:

$$\begin{aligned} & v_{\max}^2 \sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{s}^{(k)})\|^2] \\ & \leq \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2} [V(\hat{s}^{(0)}) - V(\hat{s}^{(K_m)})] \\ & \quad + \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2} \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \sum_{k=0}^{K_m-1} \Gamma^{(k+1)} \mathbb{E}[\|\hat{s}^{(k)} - \tilde{S}^{(k)}\|^2], \end{aligned}$$

538 proving the bound on the second order moment of the gradient of the Lyapunov function:

$$\begin{aligned} \sum_{k=0}^{K_m-1} \gamma_{k+1} \mathbb{E}[\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] &\leq \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2 v_{\max}^2} [V(\hat{\mathbf{s}}^{(0)}) - V(\hat{\mathbf{s}}^{(K_m)})] \\ &\quad + \frac{4\alpha \bar{L} n^{2/3}}{v_{\min}^2 v_{\max}^2} \sum_{k=0}^{K_m-1} \Xi^{(k+1)} + \sum_{k=0}^{K_m-1} \Gamma^{(k+1)} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] . \end{aligned}$$

539

□

C Practical Implementations of Two-Timescale EM Methods

C.1 Application on GMM

C.1.1 Explicit Updates

We first recognize that the constraint set for θ is given by

$$\Theta = \Delta^M \times \mathbb{R}^M.$$

Using the partition of the sufficient statistics as $S(y_i, z_i) = (S^{(1)}(y_i, z_i)^\top, S^{(2)}(y_i, z_i)^\top, S^{(3)}(y_i, z_i)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$, the partition $\phi(\theta) = (\phi^{(1)}(\theta)^\top, \phi^{(2)}(\theta)^\top, \phi^{(3)}(\theta)^\top)^\top \in \mathbb{R}^{M-1} \times \mathbb{R}^{M-1} \times \mathbb{R}$ and the fact that $\mathbb{1}_{\{M\}}(z_i) = 1 - \sum_{m=1}^{M-1} \mathbb{1}_{\{m\}}(z_i)$, the complete data log-likelihood can be expressed as in (2) with

$$\begin{aligned} s_{i,m}^{(1)} &= \mathbb{1}_{\{m\}}(z_i), \quad \phi_m^{(1)}(\theta) = \left\{ \log(\omega_m) - \frac{\mu_m^2}{2} \right\} - \left\{ \log(1 - \sum_{j=1}^{M-1} \omega_j) - \frac{\mu_M^2}{2} \right\}, \\ s_{i,m}^{(2)} &= \mathbb{1}_{\{m\}}(z_i)y_i, \quad \phi_m^{(2)}(\theta) = \mu_m, \quad s_i^{(3)} = y_i, \quad \phi^{(3)}(\theta) = \mu_M, \end{aligned} \quad (36)$$

and $\psi(\theta) = -\left\{ \log(1 - \sum_{m=1}^{M-1} \omega_m) - \frac{\mu_M^2}{2\sigma^2} \right\}$. We also define for each $m \in \llbracket 1, M \rrbracket$, $j \in \llbracket 1, 3 \rrbracket$, $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$. Consider the following latent sample used to compute an approximation of the conditional expected value $\mathbb{E}_\theta[\mathbb{1}_{\{z_i=m\}}|y = y_i]$:

$$z_{i,m} \sim \mathbb{P}(z_i = m | y_i; \theta) \quad (37)$$

where $m \in \llbracket 1, M \rrbracket$, $i \in [n]$ and $\theta = (\mathbf{w}, \boldsymbol{\mu}) \in \Theta$.

In particular, given iteration $k + 1$, the computation of the approximated quantity $\tilde{S}_{i_k}^{(k)}$ during Incremental-step updates, see (8) can be written as

$$\tilde{S}_{i_k}^{(k)} = \left(\underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1}), \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})}_{:=\tilde{s}_{i_k}^{(1)}}, \underbrace{\mathbb{1}_{\{1\}}(z_{i_k,1})y_{i_k}, \dots, \mathbb{1}_{\{M-1\}}(z_{i_k,M-1})y_{i_k}}_{:=\tilde{s}_{i_k}^{(2)}}, \underbrace{y_{i_k}}_{:=\tilde{s}_{i_k}^{(3)}(\theta^{(k)})} \right)^\top. \quad (38)$$

Recall that we have used the following regularizer:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m), \quad (39)$$

It can be shown that the regularized M-step evaluates to

$$\bar{\theta}(\mathbf{s}) = \begin{pmatrix} (1 + \epsilon M)^{-1} (s_1^{(1)} + \epsilon, \dots, s_{M-1}^{(1)} + \epsilon)^\top \\ ((s_1^{(1)} + \delta)^{-1} s_1^{(2)}, \dots, (s_{M-1}^{(1)} + \delta)^{-1} s_{M-1}^{(2)})^\top \\ (1 - \sum_{m=1}^{M-1} s_m^{(1)} + \delta)^{-1} (s^{(3)} - \sum_{m=1}^{M-1} s_m^{(2)}) \end{pmatrix} = \begin{pmatrix} \bar{\omega}(\mathbf{s}) \\ \bar{\boldsymbol{\mu}}(\mathbf{s}) \\ \bar{\mu}_M(\mathbf{s}) \end{pmatrix}. \quad (40)$$

where we have defined for all $m \in \llbracket 1, M \rrbracket$ and $j \in \llbracket 1, 3 \rrbracket$, $s_m^{(j)} = n^{-1} \sum_{i=1}^n s_{i,m}^{(j)}$.

C.1.2 Model Assumptions (GMM example)

We use the GMM example to illustrate the required assumptions.

Many practical models can satisfy the compactness of the sets as in Assumption A1. For instance, the GMM example satisfies (11) as the sufficient statistics are composed of indicator functions and observations as defined Section C.1 Equation (36).

Assumptions A2 and A3 are standard for the curved exponential family models. For GMM, the following (strongly convex) regularization $\mathbf{r}(\theta)$ ensures A3:

$$\mathbf{r}(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \epsilon \sum_{m=1}^M \log(\omega_m) - \epsilon \log(1 - \sum_{m=1}^{M-1} \omega_m),$$

563 since it ensures $\theta^{(k)}$ is unique and lies in $\text{int}(\Delta^M) \times \mathbb{R}^M$. We remark that for A2, it is possible to
 564 define the Lipschitz constant L_p independently for each data y_i to yield a refined characterization.

565 Again, A4 is satisfied by practical models. For GMM, it can be verified by deriving the closed form
 566 expression for $B(s)$ and using A1.

567 Under A1 and A3, we have $\|\hat{s}^{(k)}\| < \infty$ since S is compact and $\hat{\theta}^{(k)} \in \text{int}(\Theta)$ for any $k \geq 0$ which
 568 thus ensure that the EM methods operate in a closed set throughout the optimization process.

569 C.1.3 Algorithms updates

570 In the sequel, recall that, for all $i \in [n]$ and iteration k , the computed statistic $\tilde{S}_{i_k}^{(k)}$ is defined by (38).
 571 At iteration k , the several E-steps defined by (1) or (2) and (3) leads to the definition of the quantity
 572 $\hat{s}^{(k+1)}$. For the GMM example, after the initialization of the quantity $\hat{s}^{(0)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(0)}$, those
 573 E-steps break down as follows:

574 **Batch EM (EM):** for all $i \in [n]$, compute $\bar{s}_i^{(k)}$ and set

$$\hat{s}^{(k+1)} = n^{-1} \sum_{i=1}^n \bar{s}_i^{(k)}.$$

575 where $\bar{s}_i^{(k)}$ are computed using the exact conditional expected value $\mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i]$:

$$\tilde{\omega}_m(y_i; \theta) := \mathbb{E}_{\theta}[\mathbb{1}_{\{z_i=m\}} | y = y_i] = \frac{\omega_m \exp(-\frac{1}{2}(y_i - \mu_i)^2)}{\sum_{j=1}^M \omega_j \exp(-\frac{1}{2}(y_i - \mu_j)^2)},$$

576 **Incremental EM (iEM):** draw an index i_k uniformly at random on $[n]$, compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} + \frac{1}{n} (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\tau_i^k)}) = n^{-1} \sum_{i=1}^n \bar{s}_i^{(\tau_i^k)}.$$

577 **batch SAEM (SAEM):** draw an index i_k uniformly at random on $[n]$, compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)}(1 - \gamma_{k+1}) + \gamma_{k+1} \tilde{S}^{(k)}.$$

578 where $= \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k)}$ with $\tilde{S}_i^{(k)}$ defined in (38).

579 **Incremental SAEM (iSAEM):** draw an index i_k uniformly at random on $[n]$, compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)}(1 - \gamma_{k+1}) + \gamma_{k+1} \left(\tilde{S}^{(k)} + \frac{1}{n} (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\tau_i^k)}) \right).$$

580 **Variance Reduced Two-Timescale EM (vrTTEM):** draw an index i_k uniformly at random on $[n]$,
 581 compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)}(1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)}(1 - \rho) + \rho(\tilde{S}^{(\ell(k))} + (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(\ell(k))}))).$$

582 **Fast Incremental Two-Timescale EM (fiTTEM):** draw an index i_k uniformly at random on $[n]$,
 583 compute $\bar{s}_{i_k}^{(k)}$ and set

$$\hat{s}^{(k+1)} = \hat{s}^{(k)}(1 - \gamma_{k+1}) + \gamma_{k+1} (\tilde{S}^{(k)}(1 - \rho) + \rho(\bar{\mathcal{S}}^{(k)} + (\bar{s}_{i_k}^{(k)} - \bar{s}_{i_k}^{(t_{i_k}^k)}))).$$

584 Finally, the k -th update reads $\hat{\theta}^{(k+1)} = \bar{\theta}(\hat{s}^{(k+1)})$ where the function $s \rightarrow \bar{\theta}(s)$ is defined by (40).

585 C.2 Deformable Template Model for Image Analysis

586 C.2.1 Model and Updates

587 The complete model belongs to the curved exponential family, see [2], which vector of sufficient
 588 statistics $S = (S_1(z), S_2(z), S_3(z))$ read:

$$\begin{aligned} S_1(z) &= \frac{1}{n} \sum_{i=1}^n S_1(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top y_i, \\ S_2(z) &= \frac{1}{n} \sum_{i=1}^n S_2(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n (\mathbf{K}_p^{z_i})^\top (\mathbf{K}_p^{z_i}), \\ S_3(z) &= \frac{1}{n} \sum_{i=1}^n S_3(y_i, z_i) = \frac{1}{n} \sum_{i=1}^n z_i^t z_i, \end{aligned} \quad (41)$$

589 where for any pixel $u \in \mathbb{R}^2$ and $j \in \llbracket 1, k_g \rrbracket$ we denote:

$$\mathbf{K}_p^{z_i}(x_u, j) = \mathbf{K}_p^{z_i}(x_u - \phi_i(x_u, z_i), p_j).$$

590 Finally, the Two-Timescale M-step yields the following parameter updates:

$$\bar{\theta}(\hat{s}) = \begin{pmatrix} \beta(\hat{s}) = \hat{s}_2^{-1}(z) \hat{s}_1(z) \\ \Gamma(\hat{s}) = \frac{1}{n} \hat{s}_3(z) \\ \sigma(\hat{s}) = \beta(\hat{s})^\top \hat{s}_2(z) \beta(\hat{s}) - 2\beta(\hat{s}) \hat{s}_1(z) \end{pmatrix}, \quad (42)$$

591 where $\hat{s} = (\hat{s}_1(z), \hat{s}_2(z), \hat{s}_3(z))$ is the vector of statistics obtained via the **SA-step** (7) and using the
 592 MC approximation of the sufficient statistics $(S_1(z), S_2(z), S_3(z))$ defined in (41).

593 C.2.2 Numerical Applications

594 For the inference of the template, we use the Matlab code (online SAEM) used in [24] and implement
 595 our own batch, incremental, Variance reduced and Fast Incremental variants. The hyperparameters
 596 are kept the same and reads as follows $M = 400$, $\gamma_k = 1/k^{0.6}$ and $p = 16$. The number of
 597 landmarks for the template is $k_p = 15$ points and for the deformation $k_g = 6$ points. Both have
 598 Gaussian kernels with respectively standard deviation of 0.12 and 0.3. The standard deviation of the
 599 measurement errors is set to 0.1.

600 For the simulation part, we use the Carlin and Chib MCMC procedure, see [9]. Refer to [24] for
 601 more details.

D Additional Experiment: Pharmacokinetics (PK) Model with Absorption Lag Time

This numerical example was conducted in order to characterize the pharmacokinetics (PK) of orally administered drug to simulated patients, using a population pharmacokinetics approach. $M = 50$ synthetic datasets were generated for $n = 5000$ patients with 10 observations (concentration measures) per patient. The goal is to model the evolution of the concentration of the absorbed drug using a nonlinear and latent variable model.

Model and Explicit Updates: We consider a one-compartment PK model for oral administration with an absorption lag-time (T^{lag}), assuming first-order absorption and linear elimination processes. The final model includes the following variables: ka the absorption rate constant, V the volume of distribution, k the elimination rate constant and T^{lag} the absorption lag-time. We also add several covariates to our model such as D the dose of drug administered, t the time at which measures are taken and the weight of the patient influencing the volume V . More precisely, the log-volume $\log(V)$ is a linear function of the log-weight $lw70 = \log(wt/70)$. Let $z_i = (T_i^{\text{lag}}, ka_i, V_i, k_i)$ be the vector of individual PK parameters, different for each individual i . The final model reads:

$$y_{ij} = f(t_{ij}, z_i) + \varepsilon_{ij} \quad \text{where} \quad f(t_{ij}, z_i) = \frac{D ka_i}{V(ka_i - k_i)} (e^{-ka_i(t_{ij} - T_i^{\text{lag}})} - e^{-k_i(t_{ij} - T_i^{\text{lag}})}) , \quad (43)$$

where y_{ij} is the j -th concentration measurement of the drug of dosage D injected at time t_{ij} for patient i . We assume in this example that the residual errors ε_{ij} are independent and normally distributed with mean 0 and variance σ^2 . Lognormal distributions are used for the four PK parameters.

Lognormal distributions are used for the four PK parameters:

$$\begin{aligned} \log(T_i^{\text{lag}}) &\sim \mathcal{N}(\log(T_{\text{pop}}^{\text{lag}}), \omega_{T^{\text{lag}}}^2), \log(ka_i) \sim \mathcal{N}(\log(ka_{\text{pop}}), \omega_{ka}^2), \\ \log(V_i) &\sim \mathcal{N}(\log(V_{\text{pop}}), \omega_V^2), \log(k_i) \sim \mathcal{N}(\log(k_{\text{pop}}), \omega_k^2). \end{aligned}$$

We recall that the complete model (y, z) defined by (43) belongs to the curved exponential family, which vector of sufficient statistics $S = (S_1(z), S_2(z), S_3(z))$ read:

$$S_1(z) = \frac{1}{n} \sum_{i=1}^n z_i, \quad S_2(z) = \frac{1}{n} \sum_{i=1}^n z_i^\top z_i, \quad S_3(z) = \frac{1}{n} \sum_{i=1}^n (y_i - f(t_i, z_i))^2 \quad (44)$$

where we have noted y_i and t_i the vector of observations and time for each patient i . At iteration k , and setting the number of MC samples to 1 for the sake of clarity, the MC sampling $z_i^{(k)} \sim p(z_i | y_i, \theta^{(k)})$ is performed using a Metropolis-Hastings procedure detailed in Algorithm 2. The quantities $\tilde{S}^{(k+1)}$ and $\hat{s}^{(k+1)}$ are then updated according to the different methods. Finally the maximization step yields:

$$\bar{\theta}(s) = \begin{pmatrix} \hat{s}_1^{(k+1)} \\ \hat{s}_2^{(k+1)} - \hat{s}_1^{(k+1)} (\hat{s}_1^{(k+1)})^\top \\ \hat{s}_3^{(k+1)} \end{pmatrix} = \begin{pmatrix} \overline{z_{\text{pop}}}(\hat{s}^{(k+1)}) \\ \overline{\omega_z}(\hat{s}^{(k+1)}) \\ \overline{\sigma}(\hat{s}^{(k+1)}) \end{pmatrix}. \quad (45)$$

where z_{pop} denotes the vector of fixed effects $(T_{\text{pop}}^{\text{lag}}, ka_{\text{pop}}, V_{\text{pop}}, k_{\text{pop}})$.

Metropolis Hastings algorithm. During the simulation step of the MISSO method, the sampling from the target distribution $\pi(z_i, \theta) := p(z_i | y_i, \theta)$ is performed using a Metropolis Hastings (MH) algorithm [27] with proposal distribution $q(z_i, \delta)$ where $\theta = (z_{\text{pop}}, \omega_z)$ and δ is the vector of parameters of the proposal distribution. Commonly they parameterize a Gaussian proposal. The MH algorithm is summarized in 2.

Algorithm 2 MH algorithm

```

1: Input: initialization  $z_{i,0} \sim q(z_i; \delta)$ 
2: for  $m = 1, \dots, M$  do
3:   Sample  $z_{i,m} \sim q(z_i; \delta)$ 
4:   Sample  $u \sim \mathcal{U}([0, 1])$ 
5:   Calculate the ratio  $r = \frac{\pi(z_{i,m}; \theta) / q(z_{i,m}; \delta)}{\pi(z_{i,m-1}; \theta) / q(z_{i,m-1}; \delta)}$ 
6:   if  $u < r$  then
7:     Accept  $z_{i,m}$ 
8:   else
9:      $z_{i,m} \leftarrow z_{i,m-1}$ 
10:  end if
11: end for
12: Output:  $z_{i,M}$ 

```

634 **Monte Carlo study:** We conduct a Monte Carlo study to showcase the benefits of our scheme. $M =$
 635 50 datasets have been simulated using the following PK parameters values: $T_{\text{pop}}^{\text{lag}} = 1$, $ka_{\text{pop}} = 1$,
 636 $V_{\text{pop}} = 8$, $k_{\text{pop}} = 0.1$, $\omega_{T^{\text{lag}}} = 0.4$, $\omega_{ka} = 0.5$, $\omega_V = 0.2$, $\omega_k = 0.3$ and $\sigma^2 = 0.5$. We define
 637 the mean square distance over the M replicates $E_k(\ell) = \frac{1}{M} \sum_{m=1}^M \left(\theta_k^{(m)}(\ell) - \theta^* \right)^2$ and plot it
 638 against the epochs (passes over the data) Figure 4. Note that the MC-step (5) is performed using a
 639 Metropolis Hastings procedure since the posterior distribution under the model θ noted $p(z_i | y_i, \theta)$
 640 is intractable due to the nonlinearity of the model (43). Figure 4 shows clear advantage of variance
 641 reduced methods (vrTTEM and fiTTEM) avoiding the twists and turns displayed by the incremental
 642 and the batch methods.

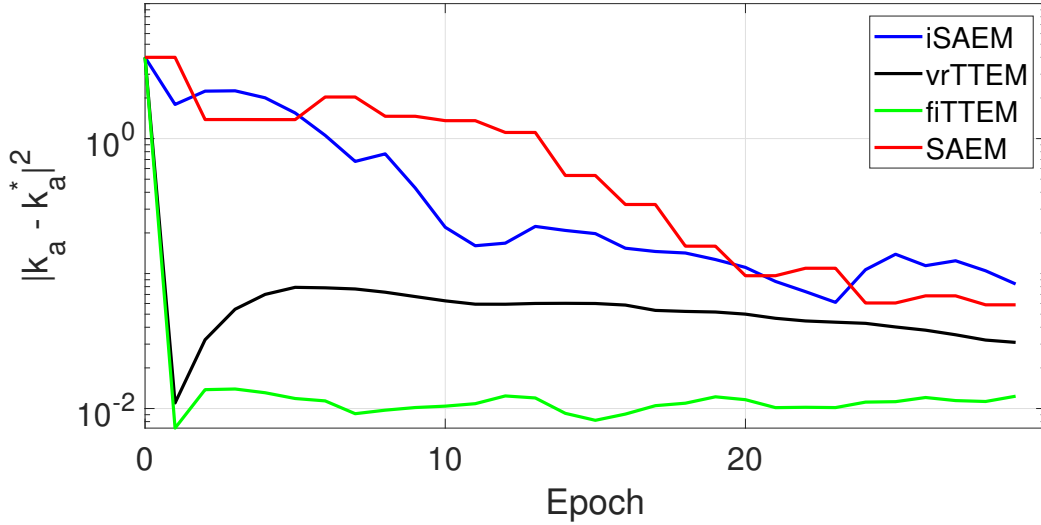


Figure 4: Precision $|ka^{(k)} - ka^*|^2$ per epoch