
Optimistic Acceleration of AMSGrad for Nonconvex Optimization.

Anonymous Author(s)

Affiliation

Address

email

1 Nonconvex Analysis

We tackle the following classical optimization problem:

$$\min_{w \in \Theta} f(w) := \mathbb{E}[f(w, \xi)] \quad (1)$$

where ξ is some random noise and only noisy versions of the objective function are accessible in this work. The objective function $f(w)$ is (potentially) nonconvex and has Lipschitz gradients.

Optimistic Algorithm We present here the algorithm studied in this paper to tackle problem (1). Set the terminating iteration number, $K \in \{0, \dots, K_{\max} - 1\}$, as a discrete r.v. with:

$$P(K = \ell) = \frac{\eta_\ell}{\sum_{j=0}^{K_{\max}-1} \eta_j}. \quad (2)$$

where $K_{\max} \leftarrow$ is the maximum number of iteration. The random termination number (2) is inspired by [Ghadimi and Lan, 2013] which enables one to show non-asymptotic convergence to stationary point for non-convex optimization. Consider constants $(\beta_1, \beta_2) \in [0, 1]$, a sequence of decreasing stepsizes $\{\eta_k\}_{k \geq 0}$, Algorithm 1 introduces the new optimistic AMSGrad method.

Algorithm 1 OPTIMISTIC-AMSGRAD

```
1: Input: Parameters  $\beta_1, \beta_2, \epsilon, \eta_k$ 
2: Init.:  $w_1 = w_{-1/2} \in \mathcal{K} \subseteq \mathbb{R}^d$  and  $v_0 = \epsilon \mathbf{1} \in \mathbb{R}^d$ 
3: for  $k = 0, 1, 2, \dots, K$  do
4:   Get mini-batch stochastic gradient  $g_k$  at  $w_k$ 
5:    $\theta_k = \beta_1 \theta_{k-1} + (1 - \beta_1) g_k$ 
6:    $v_k = \beta_2 v_{k-1} + (1 - \beta_2) g_k^2$ 
7:    $\hat{v}_k = \max(\hat{v}_{k-1}, v_k)$ 
8:    $w_{k+\frac{1}{2}} = \Pi_{\mathcal{K}} \left[ w_k - \eta_k \frac{\theta_k}{\sqrt{\hat{v}_k}} \right]$ 
9:    $w_{k+1} = \Pi_{\mathcal{K}} \left[ w_{k+\frac{1}{2}} - \eta_k \frac{h_{k+1}}{\sqrt{\hat{v}_k}} \right]$ 
10:   where  $h_{k+1} := \beta_1 \theta_{k-1} + (1 - \beta_1) m_{k+1}$ 
11:   and  $m_{k+1}$  is a guess of  $g_{k+1}$ 
12: end for
13: Return:  $w_{K+1}$ .
```

The final update at iteration k can be summarized as:

$$w_{k+1} = w_k - \eta_k \frac{\theta_k}{\sqrt{\hat{v}_k}} - \eta_k \frac{h_{k+1}}{\sqrt{\hat{v}_k}} \quad (3)$$

We make the following assumptions:

13 **H1.** The loss function $f(w)$ is nonconvex w.r.t. the parameter w .

14 **H2.** For any $k > 0$, the estimated weight w_k stays within a ℓ_∞ -ball. There exists a constant $W > 0$
 15 such that:

$$\|w_k\| \leq W \quad \text{almost surely} \quad (4)$$

16 **H3.** The function $f(w)$ is L -smooth w.r.t. the parameter w . There exist some constant $L > 0$ such
 17 that for $(w, \vartheta) \in \Theta^2$:

$$f(w) - f(\vartheta) - \nabla f(\vartheta)^\top (w - \vartheta) \leq \frac{L}{2} \|w - \vartheta\|^2. \quad (5)$$

H4. There exists a constant $a > 0$ such that for any $k > 0$:

$$\|m_{k+1}\| \leq a \|g_{k+1}\|$$

18 Classically (see [Ghadimi and Lan, 2013]) in nonconvex optimization, we make an assumption on
 19 the magnitude of the gradient:

H5. There exists a constant $M > 0$ such that

$$\|\nabla f(w, \xi)\| < M \quad \text{for any } w \text{ and } \xi$$

20 We begin with some auxiliary Lemmas important for the analysis. The first one ensures bounded
 21 norms of various quantities of interests (boiling down from the classical stochastic gradient bound-
 22 edness assumption):

Lemma 1. Assume assumption H 5, then the quantities defined in Algorithm 1 satisfy for any $w \in \Theta$
 and $k > 0$:

$$\|\nabla f(w_k)\| < M, \quad \|\theta_k\| < M, \quad \|\hat{v}_k\| < M^2.$$

Proof Assume assumption H 5 we have:

$$\|\nabla f(w)\| = \|\mathbb{E}[\nabla f(w, \xi)]\| \leq \mathbb{E}[\|\nabla f(w, \xi)\|] \leq M$$

23 By induction reasoning, since $\|\theta_0\| = 0 \leq M$ and suppose that for $\|\theta_k\| \leq M$ then we have

$$\|\theta_{k+1}\| = \|\beta_1 \theta_k + (1 - \beta_1) g_{k+1}\| \leq \beta_1 \|\theta_k\| + (1 - \beta_1) \|g_{k+1}\| \leq M \quad (6)$$

24 Using the same induction reasoning we prove that

$$\|\hat{v}_{k+1}\| = \|\beta_2 \hat{v}_k + (1 - \beta_2) g_{k+1}^2\| \leq \beta_2 \|\hat{v}_k\| + (1 - \beta_1) \|g_{k+1}^2\| \leq M^2 \quad (7)$$

25 □

26 Then, following [Yan et al., 2018] and their study of the SGD with Momentum (not AMSGrad but
 27 simple momentum) we denote for any $k > 0$:

$$\bar{w}_k = w_k + \frac{\beta_1}{1 - \beta_1} (w_k - w_{k-1}) = \frac{1}{1 - \beta_1} w_k - \frac{\beta_1}{1 - \beta_1} w_{k-1}, \quad (8)$$

28 and derive an important Lemma:

29 **Lemma 2.** Assume a strictly positive and non increasing sequence of stepsizes $\{\eta_k\}_{k>0}$, $\beta \in [0, 1]$,
 30 then the following holds:

$$\bar{w}_{k+1} - \bar{w}_k \leq \frac{\beta_1}{1 - \beta_1} \tilde{\theta}_{k-1} \left[\eta_{k-1} \hat{v}_{k-1}^{-1/2} - \eta_k \hat{v}_k^{-1/2} \right] - \eta_k \hat{v}_k^{-1/2} \tilde{g}_k, \quad (9)$$

31 where $\tilde{\theta}_k = \theta_k + \beta_1 \theta_{k-1}$ and $\tilde{g}_k = g_k - \beta_1 m_k + \beta_1 g_{k-1} + m_{k+1}$.

32 **Proof** By definition (8) and using the Algorithm updates, we have:

$$\begin{aligned} \bar{w}_{k+1} - \bar{w}_k &= \frac{1}{1 - \beta_1} (w_{k+1} - w_k) - \frac{\beta_1}{1 - \beta_1} (w_k - w_{k-1}) \\ &= -\frac{1}{1 - \beta_1} \eta_k \hat{v}_k^{-1/2} (\theta_k + h_{k+1}) + \frac{\beta_1}{1 - \beta_1} \eta_{k-1} \hat{v}_{k-1}^{-1/2} (\theta_{k-1} + h_k) \\ &= -\frac{1}{1 - \beta_1} \eta_k \hat{v}_k^{-1/2} (\theta_k + \beta_1 \theta_{k-1}) - \frac{1}{1 - \beta_1} \eta_k \hat{v}_k^{-1/2} (1 - \beta_1) m_{k+1} \\ &\quad + \frac{\beta_1}{1 - \beta_1} \eta_{k-1} \hat{v}_{k-1}^{-1/2} (\theta_{k-1} + \beta_1 \theta_{k-2}) + \frac{\beta_1}{1 - \beta_1} \eta_{k-1} \hat{v}_{k-1}^{-1/2} (1 - \beta_1) m_k \end{aligned} \quad (10)$$

33 Denote $\tilde{\theta}_k = \theta_k + \beta_1 \theta_{k-1}$ and $\tilde{g}_k = g_k - \beta_1 m_k + \beta_1 g_{k-1} + m_{k+1}$. Notice that $\tilde{\theta}_k = \beta_1 \tilde{\theta}_{k-1} +$
 34 $(1 - \beta_1)(g_k + \beta_1 g_{k-1})$.

$$\bar{w}_{k+1} - \bar{w}_k \leq \frac{\beta_1}{1 - \beta_1} \tilde{\theta}_{k-1} \left[\eta_{k-1} \hat{v}_{k-1}^{-1/2} - \eta_k \hat{v}_k^{-1/2} \right] - \eta_k \hat{v}_k^{-1/2} \tilde{g}_k \quad (11)$$

35 □

36 **Lemma 3.** Assume H 5, a strictly positive and non increasing sequence of stepsizes $\{\eta_k\}_{k>0}$,
 37 $\beta \in [0, 1]$, then the following holds:

$$\sum_{k=1}^{K_{\max}} \eta_k^2 \mathbb{E} \left[\left\| \hat{v}_k^{-1/2} \theta_k \right\|_2^2 \right] \leq \frac{\eta^2 d K (1 - \beta_1)}{(1 - \beta_2)(1 - \gamma)} \quad (12)$$

38 **Proof** We denote by index $p \in [1, d]$ the dimension of each component of vectors of interest. Noting
 39 that for any $k > 0$ and dimension p we have $\hat{v}_{k,p} \geq v_{k,p}$, then:

$$\begin{aligned} \eta_k^2 \mathbb{E} \left[\left\| \hat{v}_k^{-1/2} \theta_k \right\|_2^2 \right] &= \eta_k^2 \mathbb{E} \left[\sum_{p=1}^d \frac{\theta_{k,p}^2}{\hat{v}_{k,p}} \right] \\ &\leq \eta_k^2 \mathbb{E} \left[\sum_{p=1}^d \frac{\theta_{k,p}^2}{v_{k,p}} \right] \\ &\leq \eta_k^2 \mathbb{E} \left[\sum_{p=1}^d \frac{(\sum_{t=1}^k (1 - \beta_1) \beta_1^{k-t} g_{t,p})^2}{\sum_{t=1}^k (1 - \beta_2) \beta_2^{k-t} g_{t,p}^2} \right] \end{aligned} \quad (13)$$

40 where the last inequality is due to initializations. Denote $\gamma = \frac{\beta_1}{\beta_2}$. Then,

$$\begin{aligned} \eta_k^2 \mathbb{E} \left[\left\| \hat{v}_k^{-1/2} \theta_k \right\|_2^2 \right] &\leq \frac{\eta_k^2 (1 - \beta_1)^2}{1 - \beta_2} \mathbb{E} \left[\sum_{p=1}^d \frac{(\sum_{t=1}^k \beta_1^{k-t} g_{t,p})^2}{\sum_{t=1}^k \beta_2^{k-t} g_{t,p}^2} \right] \\ &\stackrel{(a)}{\leq} \frac{\eta_k^2 (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[\sum_{p=1}^d \frac{\sum_{t=1}^k \beta_1^{k-t} g_{t,p}^2}{\sum_{t=1}^k \beta_2^{k-t} g_{t,p}^2} \right] \\ &\leq \frac{\eta_k^2 (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[\sum_{p=1}^d \sum_{t=1}^k \gamma^{k-t} \right] = \frac{\eta_k^2 d (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[\sum_{t=1}^k \gamma^{k-t} \right] \end{aligned} \quad (14)$$

41 where (a) is due to $\sum_{t=1}^k \beta_1^{k-t} \leq \frac{1}{1 - \beta_1}$. Summing from $k = 1$ to $k = K_{\max}$ on both sides yields:

$$\begin{aligned} \sum_{k=1}^{K_{\max}} \eta_k^2 \mathbb{E} \left[\left\| \hat{v}_k^{-1/2} \theta_k \right\|_2^2 \right] &\leq \frac{\eta^2 d (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[\sum_{k=1}^{K_{\max}} \sum_{t=1}^k \gamma^{k-t} \right] \\ &\leq \frac{\eta^2 d K (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[\sum_{t=1}^k \gamma^{k-t} \right] \\ &\leq \frac{\eta^2 d K (1 - \beta_1)}{(1 - \beta_2)(1 - \gamma)} \end{aligned} \quad (15)$$

42 where the last inequality is due to $\sum_{t=1}^k \gamma^{k-t} \leq \frac{1}{1 - \gamma}$ by definition of γ . □

43 We now formulate the main result of our paper giving a finite-time upper bound of the quantity
 44 $\mathbb{E} [\|\nabla f(w_K)\|^2]$ where K is a random termination number distributed according to 2, see [Ghadimi
 45 and Lan, 2013].

46 **Theorem 1.** Assume H 3-H 5, $(\beta_1, \beta_2) \in [0, 1]$ and a sequence of decreasing stepsizes $\{\eta_k\}_{k>0}$,
 47 then the following result holds:

$$\mathbb{E} [\|\nabla f(w_K)\|^2] \leq \text{tocomplete} \quad (16)$$

48 where K is a random termination number distributed according (2).

49 **Proof** Using H 3 and the iterate \bar{w}_k we have:

$$\begin{aligned} f(\bar{w}_{k+1}) &\leq f(\bar{w}_k) + \nabla f(\bar{w}_k)^\top (\bar{w}_{k+1} - \bar{w}_k) + \frac{L}{2} \|\bar{w}_{k+1} - \bar{w}_k\|^2 \\ &\leq f(\bar{w}_k) + \underbrace{\nabla f(w_k)^\top (\bar{w}_{k+1} - \bar{w}_k)}_A + \underbrace{(\nabla f(\bar{w}_k) - \nabla f(w_k))^\top (\bar{w}_{k+1} - \bar{w}_k)}_B + \frac{L}{2} \|\bar{w}_{k+1} - \bar{w}_k\| \end{aligned} \quad (17)$$

50 **Term A.** Using Lemma 2, we have that:

$$\begin{aligned} \nabla f(w_k)^\top (\bar{w}_{k+1} - \bar{w}_k) &\leq \nabla f(w_k)^\top \left[\frac{\beta_1}{1 - \beta_1} \tilde{\theta}_{k-1} \left[\eta_{k-1} \hat{v}_{k-1}^{-1/2} - \eta_k \hat{v}_k^{-1/2} \right] - \eta_k \hat{v}_k^{-1/2} \tilde{g}_k \right] \\ &\leq \frac{\beta_1}{1 - \beta_1} \|\nabla f(w_k)\| \left\| \eta_{k-1} \hat{v}_{k-1}^{-1/2} - \eta_k \hat{v}_k^{-1/2} \right\| \left\| \tilde{\theta}_{k-1} \right\| - \nabla f(w_k)^\top \eta_k \hat{v}_k^{-1/2} \tilde{g}_k \end{aligned} \quad (18)$$

51 where the inequality is due to trivial inequality for positive diagonal matrix. Using Lemma 1 and
52 assumption H4 we obtain:

$$\nabla f(w_k)^\top (\bar{w}_{k+1} - \bar{w}_k) \leq \frac{\beta_1(1 + \beta_1)}{1 - \beta_1} \mathbf{M}^2 \left[\left\| \eta_{k-1} \hat{v}_{k-1}^{-1/2} \right\| - \left\| \eta_k \hat{v}_k^{-1/2} \right\| \right] - \nabla f(w_k)^\top \eta_k \hat{v}_k^{-1/2} \tilde{g}_k \quad (19)$$

53 where we have used the fact that $\eta_k \hat{v}_k^{-1/2}$ is a diagonal matrix such that $\eta_{k-1} \hat{v}_{k-1}^{-1/2} \succcurlyeq \eta_k \hat{v}_k^{-1/2} \succcurlyeq 0$
54 (decreasing stepsize and max operator). Also note that:

$$\begin{aligned} -\nabla f(w_k)^\top \eta_k \hat{v}_k^{-1/2} \tilde{g}_k &= -\nabla f(w_k)^\top \eta_{k-1} \hat{v}_{k-1}^{-1/2} \bar{g}_k - \nabla f(w_k)^\top \left[\eta_k \hat{v}_k^{-1/2} - \eta_{k-1} \hat{v}_{k-1}^{-1/2} \right] \bar{g}_k \\ &\quad - \nabla f(w_k)^\top \eta_{k-1} \hat{v}_{k-1}^{-1/2} (\beta_1 g_{k-1} + m_{k+1}) \\ &\leq -\nabla f(w_k)^\top \eta_{k-1} \hat{v}_{k-1}^{-1/2} \bar{g}_k + (1 - a\beta_1) \mathbf{M}^2 \left[\left\| \eta_{k-1} \hat{v}_{k-1}^{-1/2} \right\| - \left\| \eta_k \hat{v}_k^{-1/2} \right\| \right] \\ &\quad - \nabla f(w_k)^\top \eta_k \hat{v}_k^{-1/2} (\beta_1 g_{k-1} + m_{k+1}) \end{aligned} \quad (20)$$

55 using Lemma 1 on $\|g_k\|$ and where that $\tilde{g}_k = \bar{g}_k + \beta_1 g_{k-1} + m_{k+1} = g_k - \beta_1 m_k + \beta_1 g_{k-1} + m_{k+1}$.
56 Plugging (20) into (19) yields:

$$\begin{aligned} &\nabla f(w_k)^\top (\bar{w}_{k+1} - \bar{w}_k) \\ &\leq -\nabla f(w_k)^\top \eta_{k-1} \hat{v}_{k-1}^{-1/2} \bar{g}_k + \frac{1}{1 - \beta_1} (a\beta_1^2 - 2a\beta_1 + \beta_1) \mathbf{M}^2 \left[\left\| \eta_{k-1} \hat{v}_{k-1}^{-1/2} \right\| - \left\| \eta_k \hat{v}_k^{-1/2} \right\| \right] \\ &\quad - \nabla f(w_k)^\top \eta_k \hat{v}_k^{-1/2} (\beta_1 g_{k-1} + m_{k+1}) \end{aligned} \quad (21)$$

57 **Term B.** By Cauchy-Schwarz (CS) inequality we have:

$$(\nabla f(\bar{w}_k) - \nabla f(w_k))^\top (\bar{w}_{k+1} - \bar{w}_k) \leq \|\nabla f(\bar{w}_k) - \nabla f(w_k)\| \|\bar{w}_{k+1} - \bar{w}_k\| \quad (22)$$

58 Using smoothness assumption H 3:

$$\begin{aligned} \|\nabla f(\bar{w}_k) - \nabla f(w_k)\| &\leq L \|\bar{w}_k - w_k\| \\ &\leq L \frac{\beta_1}{1 - \beta_1} \|w_k - w_{k-1}\| \end{aligned} \quad (23)$$

59 By Lemma 2 we also have:

$$\begin{aligned} \bar{w}_{k+1} - \bar{w}_k &= \frac{\beta_1}{1 - \beta_1} \tilde{\theta}_{k-1} \left[\eta_{k-1} \hat{v}_{k-1}^{-1/2} - \eta_k \hat{v}_k^{-1/2} \right] - \eta_k \hat{v}_k^{-1/2} \tilde{g}_k \\ &= \frac{\beta_1}{1 - \beta_1} \tilde{\theta}_{k-1} \eta_{k-1} \hat{v}_{k-1}^{-1/2} \left[I - (\eta_k \hat{v}_k^{-1/2})(\eta_{k-1} \hat{v}_{k-1}^{-1/2})^{-1} \right] - \eta_k \hat{v}_k^{-1/2} \tilde{g}_k \\ &= \frac{\beta_1}{1 - \beta_1} \left[I - (\eta_k \hat{v}_k^{-1/2})(\eta_{k-1} \hat{v}_{k-1}^{-1/2})^{-1} \right] (w_{k-1} - w_k) - \eta_k \hat{v}_k^{-1/2} \tilde{g}_k \end{aligned} \quad (24)$$

60 where the last equality is due to $\tilde{\theta}_{k-1}\eta_{k-1}\hat{v}_{k-1}^{-1/2} = w_{k-1} - w_k$ by construction of $\tilde{\theta}_k$. Taking the
61 norms on both sides, observing $\left\|I - (\eta_k\hat{v}_k^{-1/2})(\eta_{k-1}\hat{v}_{k-1}^{-1/2})^{-1}\right\| \leq 1$ due to the decreasing stepsize
62 and the construction of \hat{v}_k and using CS inequality yield:

$$\|\bar{w}_{k+1} - \bar{w}_k\| \leq \frac{\beta_1}{1 - \beta_1} \|w_{k-1} - w_k\| + \left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\| \quad (25)$$

We recall Young's inequality with a constant $\delta \in (0, 1)$ as follows:

$$\langle X | Y \rangle \leq \frac{1}{\delta} \|X\|^2 + \delta \|Y\|^2$$

63 Plugging (23) and (25) into (22) returns:

$$\begin{aligned} (\nabla f(\bar{w}_k) - \nabla f(w_k))^\top (\bar{w}_{k+1} - \bar{w}_k) &\leq L \frac{\beta_1}{1 - \beta_1} \left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\| \|w_k - w_{k-1}\| \\ &\quad + L \left(\frac{\beta_1}{1 - \beta_1}\right)^2 \|w_{k-1} - w_k\|^2 \end{aligned} \quad (26)$$

64 Applying Young's inequality with $\delta \rightarrow \frac{\beta_1}{1 - \beta_1}$ on the product $\left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\| \|w_k - w_{k-1}\|$ yields:

$$(\nabla f(\bar{w}_k) - \nabla f(w_k))^\top (\bar{w}_{k+1} - \bar{w}_k) \leq L \left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\|^2 + 2L \left(\frac{\beta_1}{1 - \beta_1}\right)^2 \|w_{k-1} - w_k\|^2 \quad (27)$$

65 The last term $\frac{L}{2} \|\bar{w}_{k+1} - \bar{w}_k\|^2$ can be upper bounded using (25):

$$\begin{aligned} \frac{L}{2} \|\bar{w}_{k+1} - \bar{w}_k\|^2 &\leq \frac{L}{2} \left[\frac{\beta_1}{1 - \beta_1} \|w_{k-1} - w_k\| + \left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\| \right]^2 \\ &\leq L \left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\|^2 + 2L \left(\frac{\beta_1}{1 - \beta_1}\right)^2 \|w_{k-1} - w_k\|^2 \end{aligned} \quad (28)$$

66 Plugging (21), (27) and (28) into (17) and taking the expectations on both sides give:

$$\begin{aligned} &\mathbb{E} \left[f(\bar{w}_{k+1}) + \frac{1}{1 - \beta_1} \tilde{M}^2 \left\|\eta_k\hat{v}_k^{-1/2}\right\| - \left(f(\bar{w}_k) + \frac{1}{1 - \beta_1} \tilde{M}^2 \left\|\eta_{k-1}\hat{v}_{k-1}^{-1/2}\right\| \right) \right] \\ &\leq \mathbb{E} \left[-\nabla f(w_k)^\top \eta_{k-1}\hat{v}_{k-1}^{-1/2}\tilde{g}_k - \nabla f(w_k)^\top \eta_k\hat{v}_k^{-1/2}(\beta_1 g_{k-1} + m_{k+1}) \right] \\ &\quad + \mathbb{E} \left[2L \left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\|^2 + 4L \left(\frac{\beta_1}{1 - \beta_1}\right)^2 \|w_{k-1} - w_k\|^2 \right] \end{aligned} \quad (29)$$

67 where $\tilde{M}^2 = (a\beta_1^2 - 2a\beta_1 + \beta_1)M^2$. Note that the expectation of \tilde{g}_k conditioned on the filtration
68 \mathcal{F}_k reads as follows

$$\begin{aligned} \mathbb{E}[\tilde{g}_k] &= \mathbb{E}[g_k - \beta_1 m_k] \\ &= (1 - a\beta_1)\nabla f(w_k) \end{aligned} \quad (30)$$

69 Summing from $k = 1$ to $k = K$ leads to

$$\begin{aligned} &M^2 \sum_{k=1}^{K_{\max}} ((1 - a\beta_1)\eta_{k-1} + (\beta_1 + a)\eta_k) \|\nabla f(w_k)\|^2 \leq \\ &\mathbb{E} \left[f(\bar{w}_1) + \frac{1}{1 - \beta_1} \tilde{M}^2 \left\|\eta_0\hat{v}_0^{-1/2}\right\| - \left(f(\bar{w}_{K_{\max}+1}) + \frac{1}{1 - \beta_1} \tilde{M}^2 \left\|\eta_{K_{\max}}\hat{v}_{K_{\max}}^{-1/2}\right\| \right) \right] \\ &\quad + 2L \sum_{k=1}^{K_{\max}} \mathbb{E} \left[\left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\|^2 \right] + 4L \left(\frac{\beta_1}{1 - \beta_1}\right)^2 \sum_{k=1}^{K_{\max}} \mathbb{E} [\|w_{k-1} - w_k\|^2] \\ &\leq \mathbb{E} \left[\Delta f + \frac{1}{1 - \beta_1} \tilde{M}^2 \left\|\eta_0\hat{v}_0^{-1/2}\right\| \right] + 2L \sum_{k=1}^{K_{\max}} \mathbb{E} \left[\left\|\eta_k\hat{v}_k^{-1/2}\tilde{g}_k\right\|^2 \right] + 4L \left(\frac{\beta_1}{1 - \beta_1}\right)^2 \sum_{k=1}^{K_{\max}} \mathbb{E} [\|w_{k-1} - w_k\|^2] \end{aligned} \quad (31)$$

70 where $\Delta f = f(\bar{w}_1) - f(\bar{w}_{K_{\max}+1})$. We note that

$$\begin{aligned} w_{k-1} - w_k &= \eta_{k-1} \hat{v}_{k-1}^{-1/2} (\theta_{k-1} + h_k) \\ &= \eta_{k-1} \hat{v}_{k-1}^{-1/2} (\theta_{k-1} + \beta_1 \theta_{k-2} + (1 - \beta_1) m_k) \end{aligned} \quad (32)$$

71 Using Lemma 3 we have

$$\mathbb{E} \left[\|w_{k-1} - w_k\|^2 \right] \quad (33)$$

72 And thus, injecting in (31) yields:

$$\begin{aligned} \mathbf{M}^2 \sum_{k=1}^{K_{\max}} ((1 - a\beta_1)\eta_{k-1} + (\beta_1 + a)\eta_k) \|\nabla f(w_k)\|^2 &\leq \\ &\leq \mathbb{E} \left[\Delta f + \frac{1}{1 - \beta_1} \tilde{\mathbf{M}}^2 \left\| \eta_0 \hat{v}_0^{-1/2} \right\| \right] + 2L \sum_{k=1}^{K_{\max}} \mathbb{E} \left[\left\| \eta_k \hat{v}_k^{-1/2} \tilde{g}_k \right\|^2 \right] + 4L \left(\frac{\beta_1}{1 - \beta_1} \right)^2 \sum_{k=1}^{K_{\max}} \mathbb{E} \left[\|w_{k-1} - w_k\|^2 \right] \end{aligned} \quad (34)$$

73

□

74 2 Checking H 2 for a Deep Neural Network

75 We show in this section that the weights satisfy assumption H 2 and stay in a bounded set when
 76 the model we are fitting, using our method, is a fully connected feed forward neural network. The
 77 activation function for this section will be sigmoid function and we add a ℓ_2 regularization.

78 For the sake of notation, we assume $\beta_1 = 0$. We consider a fully connected feed forward neural
 79 network with L layers modeled by the function $\text{MLN}(w, \xi) : \mathbb{R}^l \rightarrow \mathbb{R}$:

$$\text{MLN}(w, \xi) = \sigma \left(w^{(L)} \sigma \left(w^{(L-1)} \dots \sigma \left(w^{(1)} \xi \right) \right) \right) \quad (35)$$

80 where $w = [w^{(1)}, w^{(2)}, \dots, w^{(L)}]$ is the vector of parameters, $\xi \in \mathbb{R}^l$ is the input data and σ is the
 81 sigmoid activation function. We assume a l dimension input data and a scalar output for simplicity.
 82 The stochastic objective function (1) reads:

$$f(w, \xi) = \mathcal{L}(\text{MLN}(w, \xi), y) + \frac{\lambda}{2} \|w\|^2 \quad (36)$$

83 where $\mathcal{L}(\cdot, y)$ is the loss function (can be Huber loss or cross entropy), y are the true labels and $\lambda > 0$
 84 is the regularization parameter. Beforehand, two following mild conditions on the boundedness of
 85 the input data and of the loss function should be verified. For any $\xi \in \mathbb{R}^l$ and $y \in \mathbb{R}$ there is a constant
 86 $T > 0$ such that:

$$\|\xi\| \leq 1 \quad \text{a.s.} \quad \text{and} \quad |\mathcal{L}'(\cdot, y)| \leq T \quad (37)$$

where $\mathcal{L}'(\cdot, y)$ denotes its derivative w.r.t. the paramer. For any layer index $\ell \in [1, L]$ we denote the
 output of layer ℓ by $h^{(\ell)}(w, \xi)$:

$$h^{(\ell)}(w, \xi) = \sigma \left(w^{(\ell)} \sigma \left(w^{(\ell-1)} \dots \sigma \left(w^{(1)} \xi \right) \right) \right)$$

87 Given the sigmoid assumption we have $\|h^{(\ell)}(w, \xi)\| \leq 1$ for any $\ell \in [1, L]$ and any $(w, \xi) \in$
 88 $\mathbb{R}^d \times \mathbb{R}^l$.

89 Observe that at the last layer L :

$$\begin{aligned} \|\nabla_{w^{(L)}} \mathcal{L}(\text{MLN}(w, \xi), y)\| &= \|\mathcal{L}'(\text{MLN}(w, \xi), y) \nabla_{w^{(L)}} \text{MLN}(w, \xi)\| \\ &= \left\| \mathcal{L}'(\text{MLN}(w, \xi), y) \sigma'(w^{(L)} h^{(L-1)}(w, \xi)) h^{(L-1)}(w, \xi) \right\| \\ &\leq \frac{T}{4} \end{aligned} \quad (38)$$

90 where the last equality is due to mild assumptions (37) and to the fact that the norm of the derivative
 91 of the sigmoid function is upperbounded by 1/4.

92 From Algorithm 1, with $\beta_1 = 0$ we have for iteration index $k > 0$:

$$\begin{aligned} \|w_k - w_{k-1}\| &= \left\| -\eta_k \hat{v}_k^{-1/2} (\theta_k + h_{k+1}) \right\| \\ &= \left\| \eta_k \hat{v}_k^{-1/2} (g_k + m_{k+1}) \right\| \\ &\leq \hat{\eta} \left\| \hat{v}_k^{-1/2} g_k \right\| + \hat{\eta} a \left\| \hat{v}_k^{-1/2} g_{k+1} \right\| \end{aligned} \quad (39)$$

where $\hat{\eta} = \max_{k>0} \eta_k$. For any dimension $p \in [1, d]$, using assumption H 4, we note that

$$\sqrt{\hat{v}_{k,p}} \geq \sqrt{1 - \beta_2} g_{k,p} \quad \text{and} \quad m_{k+1} \leq a \|g_{k+1}\|$$

93 . Thus:

$$\begin{aligned} \|w_k - w_{k-1}\| &\leq \hat{\eta} \left(\left\| \hat{v}_k^{-1/2} g_k \right\| + a \left\| \hat{v}_k^{-1/2} g_{k+1} \right\| \right) \\ &\leq \hat{\eta} \frac{a+1}{\sqrt{1 - \beta_2}} \end{aligned} \quad (40)$$

94 In short there exist a constant B such that $\|w_k - w_{k-1}\| \leq B$.

Proof by induction: As in [Défossez et al., 2020], we will prove the containment of the weights by induction. Suppose an iteration index K and a coordinate i of the last layer L such that $w_{K,i}^{(L)} \geq \frac{T}{4\lambda} + B$. Using (38), we have

$$\nabla_i f(w_K^{(L)}) \geq -\frac{T}{4} + \lambda \frac{T}{\lambda 4} \geq 0$$

95 where $f(\cdot)$ is defined by (36) and is the loss of our MLN. This last equation yields $\theta_{K,i}^{(L)} \geq 0$ (given
96 the algorithm and $\beta_1 = 0$) and using the fact that $\|w_k - w_{k-1}\| \leq B$ we have

$$0 \leq w_{K-1,i}^{(L)} - B \leq w_{K,i}^{(L)} \leq w_{K-1,i}^{(L)} \quad (41)$$

which means that $|w_{K,i}^{(L)}| \leq w_{K-1,i}^{(L)}$. So if the first assumption of that induction reasoning holds, i.e., $w_{K-1,i}^{(L)} \geq \frac{T}{4\lambda} + B$, then the next iterates $w_{K,i}^{(L)}$ decreases, see (41) and go below $\frac{T}{4\lambda} + B$. This yields that for any iteration index $k > 0$ we have

$$w_{K,i}^{(L)} \leq \frac{T}{4\lambda} + 2B$$

since B is the biggest jump an iterate can do since $\|w_k - w_{k-1}\| \leq B$. Likewise we can end up showing that

$$|w_{K,i}^{(L)}| \leq \frac{T}{4\lambda} + 2B$$

97 meaning that the weights of the last layer at any iteration is bounded in some matrix norm.

98 Now that we have shown this boundedness property for the last layer L , we will do the same for the
99 previous layers and conclude the verification of assumption H 2 by induction.

100 For any layer $\ell \in [1, L - 1]$, we have:

$$\nabla_{w^{(\ell)}} \mathcal{L}(\text{MLN}(w, \xi), y) = \mathcal{L}'(\text{MLN}(w, \xi), y) \left(\prod_{j=1}^{\ell+1} \sigma' \left(w^{(j)} h^{(j-1)}(w, \xi) \right) \right) h^{(\ell-1)}(w, \xi) \quad (42)$$

This last quantity is bounded as long as we can prove that for any layer ℓ the weights $w^{(\ell)}$ are bounded in some matrix norm as $\|w^{(\ell)}\|_F \leq F_\ell$ with the Frobenius norm. Suppose we have shown $\|w^{(r)}\|_F \leq F_r$ for any layer $r > \ell$. Then having this gradient (42) bounded we can use the same lines of proof for the last layer L and show that the norm of the weights at the selected layer ℓ satisfy

$$\|w^{(\ell)}\| \leq \frac{T \prod_{k>\ell} F_k}{4^{L-\ell+1}} + 2B$$

101 Showing that the weights of the previous layers $\ell \in [1, L - 1]$ as well as for the last layer L of our
102 fully connected feed forward neural network are bounded at each iteration, leads by induction, to
103 the boundedness (at each iteration) assumption we want to check.

104 **References**

- 105 A. Défossez, L. Bottou, F. Bach, and N. Usunier. On the convergence of adam and adagrad. *arXiv*
106 *preprint arXiv:2003.02395*, 2020.
- 107 S. Ghadimi and G. Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic pro-
108 gramming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013.
- 109 Y. Yan, T. Yang, Z. Li, Q. Lin, and Y. Yang. A unified analysis of stochastic momentum methods
110 for deep learning. *arXiv preprint arXiv:1808.10396*, 2018.