
Fast Two-Time-Scale Noisy EM Algorithms

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Abstract

1 T.B.C

2 1 Introduction

3 We formulate the following empirical risk minimization as:

$$\min_{\theta \in \Theta} \bar{\mathcal{L}}(\theta) := R(\theta) + \mathcal{L}(\theta) \text{ with } \mathcal{L}(\theta) = \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\theta) := \frac{1}{n} \sum_{i=1}^n \{ -\log g(y_i; \theta) \}, \quad (1)$$

4 where $\{y_i\}_{i=1}^n$ are the observations, Θ is a convex subset of \mathbb{R}^d for the parameters, $R : \Theta \rightarrow \mathbb{R}$ is a
5 smooth convex regularization function and for each $\theta \in \Theta$, $g(y; \theta)$ is the (incomplete) likelihood of
6 each individual observation. The objective function $\bar{\mathcal{L}}(\theta)$ is possibly *non-convex* and is assumed to
7 be lower bounded $\bar{\mathcal{L}}(\theta) > -\infty$ for all $\theta \in \Theta$.

8 In the latent variable model, $g(y_i; \theta)$, is the marginal of the complete data likelihood defined as
9 $f(z_i, y_i; \theta)$, i.e. $g(y_i; \theta) = \int_{\mathcal{Z}} f(z_i, y_i; \theta) \mu(dz_i)$, where $\{z_i\}_{i=1}^n$ are the (unobserved) latent vari-
10 ables. We make the assumption of a complete model belonging to the curved exponential family,
11 i.e.,

$$f(z_i, y_i; \theta) = h(z_i, y_i) \exp \left(\langle S(z_i, y_i) | \phi(\theta) \rangle - \psi(\theta) \right), \quad (2)$$

12 where $\psi(\theta)$, $h(z_i, y_i)$ are scalar functions, $\phi(\theta) \in \mathbb{R}^k$ is a vector function, and $S(z_i, y_i) \in \mathbb{R}^k$ is
13 the complete data sufficient statistics.

14 **Prior Work** Cite Kuhn [Kuhn et al., 2019] (for ISAEM) and incremental EM like papers. As well
15 as Optim papers (Variance reduction, SAGA etc.)

16 2 Expectation Maximization Algorithm

17 Full batch EM is a two steps procedure. The E-step amounts to computing the conditional expecta-
18 tion of the complete data sufficient statistics,

$$\bar{s}(\theta) = \frac{1}{n} \sum_{i=1}^n \bar{s}_i(\theta) \text{ where } \bar{s}_i(\theta) = \int_{\mathcal{Z}} S(z_i, y_i) p(z_i | y_i; \theta) \mu(dz_i). \quad (3)$$

19 The M-step is given by

$$\text{M-step: } \hat{\theta} = \bar{\theta}(\bar{s}(\theta)) := \arg \min_{\vartheta \in \Theta} \{ R(\vartheta) + \psi(\vartheta) - \langle \bar{s}(\theta) | \phi(\vartheta) \rangle \}, \quad (4)$$

20 3 Monte Carlo Integration and Stochastic Approximation

21 For complex and possibly nonlinear models, the expectation under the posterior distribution defined
 22 in (3) is not tractable. In that case, the first solution involves computing a Monte Carlo integration
 23 of that latter term. For all $i \in \llbracket 1, n \rrbracket$, draw for $m \in \llbracket 1, M \rrbracket$, samples $z_{i,m} \sim p(z_i|y_i; \theta)$ and compute
 24 the MC integration \tilde{s} of the deterministic quantity $\bar{s}(\theta)$:

$$\text{MC-step : } \tilde{s} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}, y_i) \quad (5)$$

25 and compute $\hat{\theta} = \bar{\theta}(\hat{s})$.

26 This algorithm bypasses the intractable expectation issue but is rather computationally expensive in
 27 order to reach point wise convergence (M needs to be large).

28 As a result, an alternative to that stochastic algorithm is to use a Robbins-Monro (RM) type of
 29 update. We denote

$$\tilde{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}^{(k)}, y_i) \quad (6)$$

30 where $z_{i,m}^{(k)} \sim p(z_i|y_i; \theta^{(k)})$. At iteration k , the sufficient statistics $\hat{s}^{(k+1)}$ is approximated as follows:

$$\text{SA-step : } \hat{s}^{(k+1)} = \hat{s}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{s}^{(k)}) \quad (7)$$

31 where $\{\gamma_k\}_{k=1}^\infty \in [0, 1]$ is a sequence of decreasing step sizes to ensure asymptotic convergence.
 32 This is called the Stochastic Approximation of the EM (SAEM), see [Delyon et al., 1999] and allows
 33 a smooth convergence to the target parameter. It represents the *first level* of our algorithm (needed
 34 to temper the variance and noise implied by MC integration).

35 In the next section, we derive variants of this algorithm to adapt of the sheer size of data of today's
 36 applications.

37 4 Incremental and Bi-Level Inexact EM Methods

38 Strategies to scale to large datasets include classical incremental and variance reduced variants. We
 39 will explicit a general update that will cover those variants and that represents the *second level* of our
 40 algorithm, namely the incremental update of the noisy statistics $\hat{S}^{(k)}$ inside the RM type of update.

$$\text{Inexact-step : } \tilde{S}^{(k+1)} = \tilde{S}^{(k)} + \rho_{k+1}(\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}), \quad (8)$$

41 Note $\{\rho_k\}_{k=1}^\infty \in [0, 1]$ is a sequence of step sizes, $\mathcal{S}^{(k)}$ is a proxy for $\tilde{S}^{(k)}$, If the stepsize is equal
 42 to one and the proxy $\mathcal{S}^{(k)} = \hat{S}^{(k)}$, i.e., computed in a full batch manner as in (6), then we recover
 43 the SAEM algorithm. Also if $\rho_k = 1$, $\gamma_k = 1$ and $\mathcal{S}^{(k)} = \tilde{S}^{(k)}$, then we recover the Monte Carlo
 44 EM algorithm.

45 We now introduce three variants of the SAEM update depending on different definitions of the proxy
 46 $\mathcal{S}^{(k)}$ and the choice of the stepsize ρ_k . Let $i_k \in \llbracket 1, n \rrbracket$ be a random index drawn at iteration k and
 47 $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$ be the iteration index where $i \in \llbracket 1, n \rrbracket$ is last drawn prior to
 48 iteration k . For iteration $k \geq 0$, the fiSAEM method draws *two* indices *independently* and uniformly
 49 as $i_k, j_k \in \llbracket 1, n \rrbracket$. In addition to τ_i^k which was defined *w.r.t.* i_k , we define $t_j^k = \{k' : j_{k'} = j, k' <$
 50 $k\}$ to be the iteration index where the sample $j \in \llbracket 1, n \rrbracket$ is last drawn as j_k prior to iteration k . With
 51 the initialization $\bar{\mathcal{S}}^{(0)} = \bar{s}^{(0)}$, we use a slightly different update rule from SAGA inspired by [Reddi

et al., 2016]. Then, we obtain:

$$(iSAEM [Karimi, 2019, Kuhn et al., 2019]) \quad \mathcal{S}^{(k+1)} = \mathcal{S}^{(k)} + \frac{1}{n} (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)}) \quad (9)$$

$$(vrSAEM This paper) \quad \mathcal{S}^{(k+1)} = \tilde{S}^{(\ell(k))} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\ell(k))}) \quad (10)$$

$$(fiSAEM This paper) \quad \mathcal{S}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \quad (11)$$

$$\overline{\mathcal{S}}^{(k+1)} = \overline{\mathcal{S}}^{(k)} + n^{-1} (\tilde{S}_{j_k}^{(k)} - \tilde{S}_{j_k}^{(t_{j_k}^k)}). \quad (12)$$

The stepsize is set to $\rho_{k+1} = 1$ for the iSAEM method; $\rho_{k+1} = \gamma$ is constant for the vrSAEM and fiSAEM methods. Moreover, for iSAEM we initialize with $\mathcal{S}^{(0)} = \tilde{S}^{(0)}$; for vrSAEM we set an epoch size of m and define $\ell(k) := m \lfloor k/m \rfloor$ as the first iteration number in the epoch that iteration k is in.

Algorithm 1 Two-Time-Scale Noisy EM methods.

- 1: **Input:** initializations $\hat{\theta}^{(0)} \leftarrow 0, \hat{s}^{(0)} \leftarrow \hat{S}^{(0)}, K_{\max} \leftarrow \text{max. iteration number}$.
- 2: Set the terminating iteration number, $K \in \{0, \dots, K_{\max} - 1\}$, as a discrete r.v. with:

$$P(K = k) = \frac{\gamma_k}{\sum_{\ell=0}^{K_{\max}-1} \gamma_{\ell}}. \quad (13)$$

- 3: **for** $k = 0, 1, 2, \dots, K$ **do**
 - 4: Draw index $i_k \in \llbracket 1, n \rrbracket$ uniformly (and $j_k \in \llbracket 1, n \rrbracket$ for fiSAEM).
 - 5: Compute $\hat{S}_i^{(k)}$ using the MC-step (5), for the drawn indices.
 - 6: Compute the surrogate sufficient statistics $\mathcal{S}^{(k+1)}$ using (9) or (10) or (11).
 - 7: Compute $\hat{S}^{(k+1)}$ via the Inexact-step (8).
 - 8: Compute $\hat{s}^{(k+1)}$ via the SA-step (7).
 - 9: Compute $\hat{\theta}^{(k+1)}$ via the M-step (4).
 - 10: **end for**
 - 11: **Return:** $\hat{\theta}^{(K)}$.
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5 Finite Time Analysis

First, we consider the following minimization problem on the statistics space:

$$\min_{\mathbf{s} \in \mathcal{S}} V(\mathbf{s}) := \overline{\mathcal{L}}(\overline{\theta}(\mathbf{s})) = R(\overline{\theta}(\mathbf{s})) + \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\overline{\theta}(\mathbf{s})) \quad (14)$$

It has been shown that this minimization problem is equivalent to the optimization problem (1), see [Karimi et al., 2019, Lemma2]

H1. Θ is an open set of \mathbb{R}^d and the sets \mathcal{Z}, \mathcal{S} are measurable open sets such that:

$$\mathcal{S} \supset \left\{ n^{-1} \sum_{i=1}^n u_i, u_i \in \text{conv}(\overline{\mathbf{s}}_i(\theta)) \right\} \quad (15)$$

where $\overline{\mathbf{s}}_i(\theta)$ is defined in (3).

H2. The conditional distribution is smooth on $\text{int}(\Theta)$. For any $i \in \llbracket 1, n \rrbracket, z \in \mathcal{Z}, \theta, \theta' \in \text{int}(\Theta)^2$, we have $|p(z|y_i; \theta) - p(z|y_i; \theta')| \leq L_p \|\theta - \theta'\|$.

We also recall from the introduction that we consider curved exponential family models. besides:

H3. For any $\mathbf{s} \in \mathcal{S}$, the function $\theta \mapsto L(\mathbf{s}, \theta) := R(\theta) + \psi(\theta) - \langle \mathbf{s} | \phi(\theta) \rangle$ admits a unique global minimum $\overline{\theta}(\mathbf{s}) \in \text{int}(\Theta)$. In addition, $J_{\phi}^{\theta}(\overline{\theta}(\mathbf{s}))$ is full rank and $\overline{\theta}(\mathbf{s})$ is L_{θ} -Lipschitz.

Similar to [Karimi et al., 2019], we denote by $H_L^{\theta}(\mathbf{s}, \theta)$ the Hessian (w.r.t to θ for a given value of \mathbf{s}) of the function $\theta \mapsto L(\mathbf{s}, \theta) = R(\theta) + \psi(\theta) - \langle \mathbf{s} | \phi(\theta) \rangle$, and define

$$B(\mathbf{s}) := J_{\phi}^{\theta}(\overline{\theta}(\mathbf{s})) \left(H_L^{\theta}(\mathbf{s}, \overline{\theta}(\mathbf{s})) \right)^{-1} J_{\phi}^{\theta}(\overline{\theta}(\mathbf{s}))^{\top}. \quad (16)$$

70 **H4.** It holds that $v_{\max} := \sup_{\mathbf{s} \in \mathcal{S}} \|\mathbf{B}(\mathbf{s})\| < \infty$ and $0 < v_{\min} := \inf_{\mathbf{s} \in \mathcal{S}} \lambda_{\min}(\mathbf{B}(\mathbf{s}))$. There exists
 71 a constant L_B such that for all $\mathbf{s}, \mathbf{s}' \in \mathcal{S}^2$, we have $\|\mathbf{B}(\mathbf{s}) - \mathbf{B}(\mathbf{s}')\| \leq L_B \|\mathbf{s} - \mathbf{s}'\|$.

72 We now formulate the main difference with the work done in [Karimi et al., 2019]. The class of
 73 algorithms we develop in this paper are two time-scale where the first stage corresponds to the
 74 variance reduction trick used in [Karimi et al., 2019] in order to accelerate incremental methods and
 75 kill the variance induced by the index sampling. The second stage is the Robbins-Monro type of
 76 update that aims to kill the variance induced by the MC approximations

77 Indeed the expectations (3) are never available and requires Monte Carlo approximation. Thus, at
 78 iteration $k + 1$, we introduce the errors when approximating the quantity $\bar{\mathbf{s}}_i(\hat{\boldsymbol{\theta}}(\hat{\mathbf{s}}^{(k-1)}))$. For all
 79 $i \in \llbracket 1, n \rrbracket$, $r > 0$ and $\vartheta \in \Theta$, define:

$$\eta_{i,\vartheta}^{(r)} := \tilde{S}_i^{(r)} - \bar{\mathbf{s}}_i(\vartheta) \quad (17)$$

80 For instance, we consider that the MC approximation is unbiased if for all $i \in \llbracket 1, n \rrbracket$ and $m \in$
 81 $\llbracket 1, M \rrbracket$, the samples $z_{i,m} \sim p(z_i | y_i; \theta)$ are i.i.d. under the posterior distribution, i.e., $\mathbb{E}[\eta_{i,\vartheta}^{(r)} | \mathcal{F}_r] = 0$
 82 where \mathcal{F}_r is the filtration up to iteration r .

83 The following results are derived under the assumption of control of the fluctuations implied by the
 84 approximation stated as follows:

85 **H5.** There exist a positive sequence of MC batch size $\{M_k\}_{k>0}$ and constants (C, C_η) such that for
 86 all $k > 0$, $i \in \llbracket 1, n \rrbracket$ and $\vartheta \in \Theta$:

$$\mathbb{E} \left[\left\| \eta_{i,\vartheta}^{(r)} \right\|^2 \right] \leq \frac{C_\eta}{M_r} \quad \text{and} \quad \mathbb{E} \left[\left\| \mathbb{E}[\eta_{i,\vartheta}^{(r)} | \mathcal{F}_r] \right\|^2 \right] \leq \frac{C}{M_r} \quad (18)$$

87 **Lemma 1.** [Karimi et al., 2019] Assume H2, H3, H4. For all $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$ and $i \in \llbracket 1, n \rrbracket$, we have

$$\|\bar{\mathbf{s}}_i(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \bar{\mathbf{s}}_i(\bar{\boldsymbol{\theta}}(\mathbf{s}'))\| \leq L_s \|\mathbf{s} - \mathbf{s}'\|, \quad \|\nabla V(\mathbf{s}) - \nabla V(\mathbf{s}')\| \leq L_V \|\mathbf{s} - \mathbf{s}'\|, \quad (19)$$

88 where $L_s := C_Z L_p L_\theta$ and $L_V := v_{\max}(1 + L_s) + L_B C_S$.

89 5.1 Global Convergence of Incremental Noisy EM Algorithms

90 Following the asymptotic analysis of update (9), we present a finite-time analysis of the incremental
 91 variant of the Stochastic Approximation of the EM algorithm.

92 The first intermediate result is the computation of the quantity $\hat{\mathbf{S}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}$, which corresponds to
 93 the drift term of (7) and reads as follows:

94 **Lemma 2.** Assume H1. The update (9) is equivalent to the following update on the resulting statis-
 95 tics

$$\hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} \left(n^{-1} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \hat{\mathbf{s}}^{(k)} \right) \quad (20)$$

96 where $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$. Also:

$$\mathbb{E} \left[\left\| \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \leq \mathbb{E} \left[\left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] + 2L_s^2 \left(1 - \frac{1}{n} \right)^2 n^{-1} \sum_{i=1}^n \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \right\|^2 \right] + \frac{2C}{M_k} \quad (21)$$

97 where $\bar{\mathbf{s}}^{(k)}$ is defined by (3).

98 The following main result for the iSAEM algorithm is derived under a control of the Monte Carlo
 99 fluctuations as described by assumption H5. Typically, the controls exhibited below are of interest
 100 when the number of MC samples M_k increase with the iteration index k .

101 **Theorem 1.** Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of positive step sizes
 102 and consider the iSAEM sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = 1$ for any k .

103 Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$.

104 **Proof** Under some regularity conditions of the Lyapunov function V , cf. Lemma 19, and the fol-
 105 lowing growth condition for all $\mathbf{s} \in \mathbf{S}$,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (22)$$

106 proven in [Karimi et al., 2019, Lemma 3], we can write:

$$V(\hat{\mathbf{s}}^{(k+1)}) \leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \quad (23)$$

107 Taking the expectation on both sides and using the growth condition (22), we obtain:

$$\begin{aligned} \mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} v_{\min} \mathbb{E} \left[\left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] + \mathbb{E} \left[\frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \right] \\ &\quad - \gamma_{k+1} \mathbb{E} \left[\langle \bar{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \end{aligned} \quad (24)$$

108 We then establish an auxiliary Lemma yielding an upper-bound on the quantity

109 $\mathbb{E} \left[\langle \bar{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right]$ where:

$$\bar{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} = \bar{\mathbf{s}}^{(k)} - \left(\tilde{S}^{(k)} + \frac{1}{n} (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)}) \right) \quad (25)$$

110

Lemma 3.

$$\mathbb{E} \left[\langle \bar{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \leq \quad (26)$$

111 Using Lemma 2:

$$\begin{aligned} \mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)})] &\leq \mathbb{E}[V(\hat{\mathbf{s}}^{(k)})] - \gamma_{k+1} \left(v_{\min} - \frac{\gamma_{k+1} L_V}{2} \right) \mathbb{E} \left[\left\| \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &\quad + \gamma_{k+1}^2 L_V L_s^2 \left(1 - \frac{1}{n} \right)^2 n^{-1} \sum_{i=1}^n \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(\tau_i^k)} \right\|^2 \right] + \frac{\gamma_{k+1}^2 L_V C}{M_k} \\ &\quad - \gamma_{k+1} \mathbb{E} \left[\langle \bar{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \right] \end{aligned} \quad (27)$$

112 Besides,

$$n^{-1} \sum_{i=1}^n \mathbb{E} \left[\left\| \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^{k+1})} \right\|^2 \right] = n^{-1} \sum_{i=1}^n \left(\frac{1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{n-1}{n} \mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(\tau_i^k)}\|^2] \right) \quad (28)$$

113 yielding for any numbers $\beta_k > 0$,

$$\begin{aligned} &\mathbb{E}[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &= \mathbb{E} \left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + 2 \langle \hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle \right] \\ &= \mathbb{E} \left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 - 2 \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} | \hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)} \rangle \right] \\ &\leq \mathbb{E} \left[\|\hat{\mathbf{s}}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 + \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 + \frac{\gamma_{k+1}}{\beta_{k+1}} \|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2 + \gamma_{k+1} \beta_{k+1} \|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2 \right] \end{aligned} \quad (29)$$

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□

115 5.2 Global Convergence of Two-Time-Scale Noisy EM Algorithms

116 We now proceed by giving our main result regarding the global convergence of the fiSAEM algo-
 117 rithm.

118 **6 Numerical Examples**

119 **6.1 Gaussian Mixture Models**

120 Graphs obtained and relevant

121 **6.2 Deep Latent Variable Models using noisy EM**

122 See if makes sense to use EM instead of Variational Inference

123 **6.3 Deformable Template Model for Image Analysis**

124 See Kuhn et.al. paper.

125 **7 Conclusion**

References

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