Positivity of Hadamard Powers of Random Matrices

Tiefeng Jiang and Ping Li

***Abstract*—The paper studies**

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MSC(2010): Primary .

I. Introduction

Random matrices

II. Main results and discussions

Let $f(x)$ be a real-valued function defined on $\mathbb{R}$. Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix, where a_{ij}'s are real numbers. Define $f : \mathbf{A} \to f(\mathbf{A}) = (f(a_{ij}))$. We call $f(\mathbf{A})$ a Hadamard functions to distinguish it from the usual notion of matrix functions. In particular, if $\alpha > 0$ and $f(x) = x^\alpha$, then we call $\mathbf{A}^{(\alpha)} := f(\mathbf{A})$ the Hadamard power of α. Here we need to pay attention to the domain of the function $f(x) = x^\alpha$. If $\alpha > 0$ is an integer, the function is defined for every $x \in \mathbb{R}$. If $\alpha > 0$ is not an integer, the function $f(x) = x^\alpha$ is defined only on $[0, \infty)$. By the Schur product theorem, it is known that $\mathbf{A}^{(\alpha)}$ is a positive definite matrix if $\mathbf{A} = (a_{ij})$ is a positive definite matrix and $\alpha = 1, 2, \cdots$; see, for example, Theorem 5.2.1 from Horn and Johnson (1991). In fact, we know more about this conclusion. For positive definite matrices $\mathbf{U} = (u_{ij})_{n\times n}$ and $\mathbf{V} = (v_{ij})_{n\times}$, set $\mathbf{U} \circ \mathbf{V} = (u_{ij}v_{ij})_{n\times n}$. Then $\lambda_{min}(\mathbf{U}) \cdot \min_{1\le i\le n} v_{ii} \le \lambda_i(\mathbf{U} \circ \mathbf{V}) \le \lambda_{max}(\mathbf{U}) \cdot \max_{1\le i\le n} v_{ii}$ for each $1 \le i \le n$; see, for example, Schur (1911) or Theorem 5.3.4 from Horn and Johnson (1991). If α is a positive integer, then $\mathbf{A}^{(\alpha)} = \mathbf{A} \circ \cdots \circ \mathbf{A}$ from which there are α many $\mathbf{A}$ in the product. Thus, $\lambda_{min}(\mathbf{A}) > 0$ by induction if $\mathbf{A}$ is positive definite.

A. Some Known Results

Let $a \in (0, \infty]$ and $f(x) : (0, a) \to \mathbb{R}$. We say $f(x)$ is *absolutely monotonic* on $(0, a)$ if $f^{(k)}(x) \ge 0$ for every $x \in (0, \alpha)$ and $k = 0, 1, 2, \cdots$. The following general conclusion can be seen from, for example, Schoenberg (1942), Vasudeva (1979) and Hiai (2009).

THEOREM *2.1:* Assume $a \in (0, \infty]$ and $f(x)$ is a real function defined on $(-a, a)$. Then $f(\mathbf{A})$ is non-negative definite for every non-negative definite matrix $\mathbf{A}$ with entries in $(-a, a)$ if and only if $f(x)$ is analytic and absolutely monotonic on $(0, a)$.

THEOREM *2.2:* (Theorem 6.3.7 from Horn and Johnson, 1991) Let $f(\cdot)$ be an $(n-1)$-times continuously differentiable real valued function on $(0, \infty)$, and suppose that the Hadamard function $f(\mathbf{A}) = (f(a_{ij}))$ is non-negative definite for every non-negative definite matrix $\mathbf{A}$ that has positive entries. Then $f^{(k)}(t) \ge 0$ for all $t \in (0, \infty)$ and all $k = 0, 1, \cdots, n-1$.

COROLLARY *2.1:* (Corollary 6.3.8 from Horn and Johnson, 1991) Let $0 < \alpha < n-2$, α not an integer. There is some $n \times n$ non-negative definite matrix $\mathbf{A}$ with positive entries such that the Hadamard power $\mathbf{A}^{(\alpha)} = (a_{ij}^\alpha)$ is not non-negative definite.

THEOREM *2.3:* (Theorem 6.3.9 from Horn and Johnson, 1991) Let $\mathbf{A} = (a_{ij})$ be a non-negative definite matrix with nonnegative entries. If $\alpha \ge n-2$, then the Hadamard power $\mathbf{A}^{(\alpha)}$ is non-negative definite. Furthermore, the lower bound $n = 2$ is, in general, the best possible.

B. New Results

To make statement of our results clearer, we will use the following notation. For a matrix $\mathbf{M}$, we write $\mathbf{M} > 0$ if $\mathbf{M}$ is positive definite; $\mathbf{M} \ge 0$ if $\mathbf{M}$ is non-negative definite; $\mathbf{M} \not\geq 0$ if $\mathbf{M}$ is not non-negative definite.

EXAMPLE *2.1:* Consider 3×3 matrix

$$\mathbf{M} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

Its Hadamard power of α is given by

$$\mathbf{M}^{(\alpha)} = \begin{pmatrix} 1 & \frac{1}{2^\alpha} & 0 \\ \frac{1}{2^\alpha} & 1 & \frac{1}{2^\alpha} \\ 0 & \frac{1}{2^\alpha} & 1 \end{pmatrix}.$$

Then

$$\det(\mathbf{M}^{(\alpha)}) = 1 - \frac{2}{4^\alpha}.$$

Therefore, $\mathbf{M} = \mathbf{M}^{(1)} > 0$. However, $\mathbf{M}^{(\alpha)} \not\geq 0$ if $\alpha \in (0, \frac{1}{2})$.

For any $n \geq 3$, define $\mathbf{M}_n = \mathbf{M}$ for $n = 3$ and

$$\mathbf{M}_n = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-3} \end{pmatrix}, \quad n \geq 4.$$

Then, the matrix $\mathbf{M}_n$ is positive definite as $n \geq 3$. However, the Hadamard power matrix $\mathbf{M}_n^{(\alpha)} \ngeq 0$ as $\alpha \in (0, \frac{1}{2})$.

EXAMPLE *2.2:* Consider 4×4 matrix $\mathbf{M} = \mathbf{aa}' + \mathbf{bb}' + 10^{-4}\mathbf{I}_4$, where $\mathbf{a}^T = (1,1,1,1)$ and $\mathbf{b}^T = (0,1,2,3)$. The term $10^{-4}\mathbf{I}_4$ purely ensures the positivity of $\mathbf{M}$. The matrix $\mathbf{aa}' + \mathbf{bb}'$ is of rank 2. Obviously, $\mathbf{M} > 0$. It is easy to check that

$$\det(\mathbf{M}^{1.1}) = -0.000118654.$$

Hence, $\mathbf{M}^{1.1} \ngeq 0$. Define $\mathbf{M}_n = \mathbf{M}$ for $n = 4$ and

$$\mathbf{M}_n = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-4} \end{pmatrix}$$

for $n \geq 5$. Then, $\mathbf{M}_n > 0$ if $n \geq 4$. However, the Hadamard power matrix $\mathbf{M}_n^{1.1} \ngeq 0$.

THEOREM *2.4:* Assume $n \geq 4$. Let $\mathbf{M} = (\xi_{ij})$ be an $n \times n$ symmetric matrix, where $\{\xi_{ij};\, 1 \leq i \leq j \leq n\}$ are independent random variables. Suppose all of the supports of ξ_{ij}'s contain a common interval $[u, v]$ for some $v > u > 0$. Then there exists $\alpha \in (1, 2)$ for which

$$P\big(\mathbf{M} \geq 0 \text{ and } \mathbf{M}^{(\alpha)} \ngeq 0\big) > 0.$$

THEOREM *2.5:* Assume $n \geq 4$. Let $\mathbf{X} = (x_{ij})$ be an $n \times p$ matrix, where $\{x_{ij};\, 1 \leq i \leq n, 1 \leq j \leq p\}$ are independent random variables. Suppose all of the supports of ξ_{ij}'s contain a common interval $[u, v]$ for some $v > u > 0$. Then there exists $\alpha \in (1, 2)$ such that

$$P\big(\mathbf{X}^T\mathbf{X} > 0 \text{ and } (\mathbf{X}^T\mathbf{X})^{(\alpha)} \ngeq 0\big) > 0.$$

THEOREM *2.6:* Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix of which the entries are non-negative. Assume $a_{ii} \geq \sum_{j \neq i} a_{ij}$ for each $1 \leq i \leq n$. Then $\mathbf{A} \geq 0$ and $\mathbf{A}^{(\alpha)} \geq 0$ for all $\alpha \geq 1$. The conclusion still holds if all three "$\geq$" are replaced by "$>$", respectively.

C. *Proofs*

LEMMA *2.1:* For any $n \geq 4$, there exist $\alpha \in (1, 2)$, $\delta > 0$ and $n \times n$ symmetric matrix $\mathbf{M} = (m_{ij})$ with $m_{ij} \geq 0$ for all $1 \leq i, j \leq n$ such that the following holds.

(i) $\mathbf{M} = (m_{ij}) > 0$ for every $m_{ij} \in [a_{ij}, a_{ij} + \delta]$ and $1 \leq i, j \leq n$.

(ii) $\mathbf{M}^{(\alpha)} = (m_{ij}^{\alpha}) \ngeq 0$ for any $m_{ij} \in [a_{ij}, a_{ij} + \delta]$ and any $1 \leq i, j \leq n$.

Proof of Lemma 2.1. For any $n \times n$ symmetric matrix $\mathbf{M} = (m_{ij})$, let $\|\mathbf{M}\|$ be the spectral norm of $\mathbf{M}$. We use $\lambda_1(\mathbf{M}) \geq \lambda_2(\mathbf{M}) \geq \cdots \geq \lambda_n(\mathbf{M})$ to denote the eigenvalues of $\mathbf{M}$. Evidently, $\|\mathbf{M}\| \leq (\sum_{1 \leq i,j \leq n} |m_{ij}|^2)^{1/2}$. Let $\mathbf{M}_1 = (m_{ij})$ and $\mathbf{M}_2 = (\tilde{m}_{ij})$ be $n \times n$ symmetric matrices. The Weyl's perturbation theorem [see, e.g., Horn and Johnson (1985)] says that $\max_{1 \leq i \leq n} |\lambda_i(\mathbf{M}_1) - \lambda_i(\mathbf{M}_2)| \leq \|\mathbf{M}_1 - \mathbf{M}_2\|$. Therefore,

$$\max_{1 \leq i \leq n} |\lambda_i(\mathbf{M}_1) - \lambda_i(\mathbf{M}_2)| \leq \Big(\sum_{1 \leq i,j \leq n} |m_{ij} - \tilde{m}_{ij}|^2 \Big)^{1/2}. \quad (2.1)$$

This concludes that the eigenvalues of a matrix are continuous functions of its entries. This is particularly true for smallest eigenvalues.

According to Example 2.2, there exists $\alpha \in (1, 2)$ and an $n \times n$ symmetric matrix $\mathbf{A} = (a_{ij})$ such that $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$, $\mathbf{A} > 0$ and the Hadamard power matrix $\mathbf{A}^{(\alpha)} \ngeq 0$. For any $n \times n$ symmetric matrix $\mathbf{M} = (m_{ij})$, define

$$f(\mathbf{M}) := \min\big\{\lambda_n(\mathbf{M}),\, -\lambda_n(\mathbf{M}^{(\alpha)})\big\}.$$

As explained earlier, $f(\mathbf{M})$ is a continuous function in $\{m_{ij};\, 1 \leq i \leq j \leq n\}$. Since $f(\mathbf{A}) > 0$, there exist $\{\delta_{ij} > 0;\, 1 \leq i, j \leq n\}$ with $\delta_{ij} = \delta_{ji}$ for all $1 \leq i, j \leq n$ such that $f(\mathbf{M}) > 0$ for any $m_{ij} \in [a_{ij}, a_{ij} + \delta_{ij}]$ with $1 \leq i, j \leq n$. Set $\delta = \min\{\delta_{ij};\, 1 \leq i \leq j \leq n\}$. Then, $\delta > 0$. Also, $\lambda_n(\mathbf{M}) > 0$ and $\lambda_n(\mathbf{M}^{(\alpha)}) < 0$ for every $m_{ij} \in [a_{ij}, a_{ij} + \delta]$ and every $1 \leq i, j \leq n$. That is, $\mathbf{M} > 0$ and $\mathbf{M}^{(\alpha)} \ngeq 0$ for any $m_{ij} \in [a_{ij}, a_{ij} + \delta]$ and any $1 \leq i, j \leq n$. $\square$

LEMMA *2.2:* Let $\mathbf{X} = (x_{ij})_{n \times p}$ be an $n \times p$ matrix. For any n and p with $n \geq p \geq 4$, there exist $\alpha \in (1, 2)$, $\delta > 0$ and $n \times p$ matrix $\mathbf{A} = (a_{ij})$ with $a_{ij} \geq 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq q$ such that the following holds.

(i) The matrix $\mathbf{X}'\mathbf{X}$ is positive definite for every $x_{ij} \in [a_{ij}, a_{ij} + \delta]$ and $1 \leq i \leq n$ and $1 \leq j \leq p$.

(ii) The Hadamard power $(\mathbf{X}'\mathbf{X})^{(\alpha)} \ngeq 0$ for any $x_{ij} \in [a_{ij}, a_{ij} + \delta]$, $1 \leq i \leq n$ and $1 \leq j \leq p$.

Proof of Lemma 2.2. Let $\mathbf{a}^T = (1,1,1,1)$ and $\mathbf{b}^T = (0,1,2,3)$ be as in Example 2.2. It is checked that the Hadamard power $(\mathbf{aa}^T + \mathbf{bb}^T)^{(1.1)}$ has determinant -1.1856×10^{-4}. Set

$$\mathbf{A}(\epsilon) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \quad \epsilon > 0.$$

It is easy to see $\lim_{\epsilon \to 0^+} [\mathbf{A}(\epsilon)^T\mathbf{A}(\epsilon)]^{(1.1)} = (\mathbf{aa}^T + \mathbf{bb}^T)^{1.1}$ with the entrywise convergence. By continuity of determinants, there exists $\epsilon_0 > 0$ such that the determinant of $[\mathbf{A}(\epsilon_0)^T\mathbf{A}(\epsilon_0)]^{(1.1)}$ is negative. That is, $\mathbf{A}(\epsilon_0)^T\mathbf{A}(\epsilon_0) > 0$ but the Hadmard power

$[\mathbf{A}(\epsilon_0)^T\mathbf{A}(\epsilon_0)]^{(1.1)} \not\geq 0$. Now we define an $n \times p$ matrix $\mathbf{A}$ such that

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}(\epsilon_0) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-4} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{n\times p},$$

where the size of each submatrix $\mathbf{0}$ appeared in $\mathbf{A}$ can be seen from those of $\mathbf{A}(\epsilon_0)$ and $\mathbf{I}_{p-4}$. In particular, the size of the "**0**" in the bottom-right of $\mathbf{A}$ is $(n-p)\times(p-4)$. In case $n = p$, there is no third row of submatrices in $\mathbf{A}$; in case $p = 4$, there is no second row of submatrices of $\mathbf{A}$. Since

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} \mathbf{A}(\epsilon_0)^T\mathbf{A}(\epsilon_0) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-4} \end{pmatrix}.$$

Hence, $\mathbf{A}^T\mathbf{A} > 0$ but the Hadamard power $(\mathbf{A}^T\mathbf{A})^{(1.1)} \not\geq 0$.

The inequality (2.1) shows that the smallest eigenvalue $\lambda_n(\mathbf{M})$ of $\mathbf{M} = \mathbf{X}\mathbf{X}^T$ is a continuous function of the entries of $\mathbf{M}$, which in turn are the continuous functions of the entries of $\mathbf{X}$. Write $\mathbf{X} = (x_{ij})_{n\times p}$. Hence, $\lambda_n(\mathbf{M})$ is a continuous function of x_{ij}'s. Set

$$f(\mathbf{M}) := \min\left\{\lambda_n(\mathbf{M}),\, -\lambda_n(\mathbf{M}^{(\alpha)})\right\}.$$

Then $f(\mathbf{M})$ is a continuous function of x_{ij}'s and $f(\mathbf{A}\mathbf{A}^T) > 0$. Write $\mathbf{A} = (a_{ij})_{n\times p}$. Then there exist $\delta_{ij} > 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq p$ such that $f(\mathbf{M}) > 0$ for all $x_{ij} \in [a_{ij}, a_{ij} + \delta_{ij}]$ with $1 \leq i \leq n$ and $1 \leq j \leq p$. Denote $\delta = \min\{\delta_{ij};\, 1 \leq i \leq n, 1 \leq j \leq p\}$. Then $\delta > 0$ and $f(\mathbf{X}\mathbf{X}^T) > 0$ for all $x_{ij} \in [a_{ij}, a_{ij} + \delta]$ with $1 \leq i \leq n$ and $1 \leq j \leq p$. Hence, under these restrictions of x_{ij}'s, we have $\lambda_n(\mathbf{X}\mathbf{X}^T) > 0$ and $\lambda_n((\mathbf{X}\mathbf{X}^T)^{(\alpha)}) < 0$. This yields (i) and (ii). $\square$

Proof of Theorem 2.4. Review Lemma 2.1. Let $\delta > 0$ be as in the lemma. Since $[a_{ij} + \frac{1}{2}\delta, a_{ij} + \delta] \subset [a_{ij}, a_{ij} + \delta]$ for each pair of (i, j) with $1 \leq i \leq j \leq n$. Then Lemma 2.1 still holds if we strengthen the conclusion by requiring that $a_{ij} > 0$ for all $1 \leq i \leq j \leq n$. Therefore,

$$0 \tag{2.2}$$
$$< \alpha := \min\{a_{ij};\, 1 \leq i \leq j \leq n\} \tag{2.3}$$
$$< \beta := \max\{a_{ij};\, 1 \leq i \leq j \leq n\} + \delta. \tag{2.4}$$

For a random variable ξ, we use support(ξ) to denote its support. In particular, $P(a \leq \xi \leq b) > 0$ provided $[a, b] \subset$ support(ξ). Notice support$(\lambda\xi_{ij}) = \lambda\cdot$support$(\xi_{ij})$ for each i, j. Choose $\lambda > 0$ such that $\lambda[u, v] \supset [\alpha, \beta]$. It follows that

$$\textstyle\bigcup_{1\leq i\leq j\leq n}[a_{ij}, a_{ij} + \delta] \subset [\alpha, \beta] \tag{2.5}$$
$$\subset \textstyle\bigcap_{1\leq i\leq j\leq n} \text{support}(\lambda\xi_{ij}). \tag{2.6}$$

Observe

$$\begin{aligned} &P\big(\mathbf{M} > 0 \text{ and } \mathbf{M}^{(\alpha)} \not\geq 0\big) \\ &= P\big(\lambda\mathbf{M} > 0 \text{ and } (\lambda\mathbf{M})^{(\alpha)} \not\geq 0\big). \end{aligned}$$

By Lemma 2.1 and independence, the last probability above is at least

$$\begin{aligned} & P(\lambda\xi_{ij} \in [a_{ij}, a_{ij} + \delta] \text{ for each } 1 \leq i \leq j \leq n) \\ = & \prod_{1\leq i\leq j\leq n} P(\lambda\xi_{ij} \in [a_{ij}, a_{ij} + \delta]) \\ > & \ 0, \end{aligned}$$

where the last inequality comes from (2.5). The proof is complete. $\square$

Proof of Theorem 2.5. Recall Lemma 2.2. Let $\delta > 0$ be as in the lemma. By the same argument as in (2.2), without loss of generality, we assume $a_{ij} > 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq p$. Therefore,

$$\begin{aligned} 0 &< \alpha := \min\{a_{ij};\, 1 \leq i \leq n, 1 \leq j \leq p\} \\ &< \beta := \max\{a_{ij};\, 1 \leq i \leq n, 1 \leq j \leq p\} + \delta. \end{aligned}$$

By choosing $\lambda > 0$ such that $\lambda[u, v] \supset [\alpha, \beta]$, we then have

$$\bigcup [a_{ij}, a_{ij} + \delta] \subset [\alpha, \beta] \subset \bigcap \text{support}(\lambda x_{ij}), \tag{2.7}$$

where the union and the intersection are taken over $1 \leq i \leq n$ and $1 \leq j \leq p$. Let $\mathbf{X} = (x_{ij})$ be an $n \times p$ matrix. By setting $\mathbf{Y} = \lambda\mathbf{X}$, we have

$$\begin{aligned} &P\big(\mathbf{X}^T\mathbf{X} > 0 \text{ and } (\mathbf{X}^T\mathbf{X})^{(\alpha)} \not\geq 0\big) \\ &= P\big(\mathbf{Y}^T\mathbf{Y} > 0 \text{ and } (\mathbf{Y}^T\mathbf{Y})^{(\alpha)} \not\geq 0\big). \end{aligned}$$

From Lemma 2.2, the above is at least

$$\begin{aligned} & P\big(\lambda x_{ij} \in [a_{ij}, a_{ij} + \delta] \text{ for each } 1 \leq i \leq n \text{ and } 1 \leq j \leq p\big) \\ = & \prod_{1\leq i\leq n, 1\leq j\leq p} P\big(\lambda x_{ij} \in [a_{ij}, a_{ij} + \delta]\big) \\ > & \ 0 \end{aligned}$$

where the last step follows from (2.7) and independence. The proof is completed. $\square$

Proof of Theorem 2.6. By the Gershgorin disc theorem [see e.g., Horn and Johnson (1985)], all eigenvalues of $\mathbf{A}$ are in the set

$$\bigcup_{1\leq i\leq n} \Big(a_{ii} - \sum_{j\neq i} a_{ij},\, a_{ii} + \sum_{j\neq i} a_{ij}\Big). \tag{2.8}$$

By assumption, all eigenvalues are non-negative, hence $\mathbf{A} \geq 0$. On the other hand,

$$a_{ii}^{\alpha} \geq \Big(\sum_{j\neq i} a_{ij}\Big)^{\alpha} \geq \sum_{j\neq i} a_{ij}^{\alpha}$$

for all $\alpha \geq 1$ by the given condition. By the Gershgorin disc theorem again, all of the eigenvalues of the Hadamard power matrix $\mathbf{A}^{(\alpha)}$ are non-negative. Therefore, $\mathbf{A}^{(\alpha)} \geq 0$.

Evidently, if $a_{ii} > \sum_{j \neq i} a_{ij}$ for each $1 \leq i \leq n$ then all of the eigenvalues of $\mathbf{A}$ and $\mathbf{A}^{(\alpha)}$ are positive by (2.8) with "a_{ij}" being replaced by a_{ij}^{α} for all i and j. Hence $\mathbf{A} > 0$ and $\mathbf{A}^{(\alpha)} > 0$ for all $\alpha \geq 1$. □

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