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# An Optimistic Acceleration of AMSGrad for Nonconvex Optimization

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## Abstract

We propose a new variant of AMSGrad [30], a popular adaptive gradient based optimization algorithm widely used for training deep neural networks. Our algorithm adds prior knowledge about the sequence of consecutive mini-batch gradients and leverages its underlying structure making the gradients sequentially predictable. By exploiting the predictability and ideas from optimistic online learning, the proposed algorithm can accelerate the convergence and increase sample efficiency. After establishing a tighter upper bound under some convexity conditions on the regret, we offer a complimentary view of our algorithm which generalizes the offline and stochastic version of nonconvex optimization. In the nonconvex case, we establish a non-asymptotic convergence bound independent of the initialization. We illustrate, via numerical experiments, the practical speedup on several deep learning models and benchmark datasets.

## 1 Introduction

Deep learning models have been successful in several applications, from robotics (e.g., [21]), computer vision (e.g [18, 15]), reinforcement learning (e.g., [25]) and natural language processing (e.g., [16]). With the sheer size of modern datasets and the dimension of neural networks, speeding up training is of utmost importance. To do so, several algorithms have been proposed in recent years, such as AMSGRAD [30], ADAM [19], RMSPROP [34], ADADELTA [40], and NADAM [10]. All the prevalent algorithms for training deep networks mentioned above combine two ideas: the idea of adaptivity from ADAGRAD [11, 23] and the idea of momentum from NESTEROV’S METHOD [27] or HEAVY BALL method [28]. ADAGRAD is an online learning algorithm that works well compared to the standard online gradient descent when the gradient is sparse. Its update has a notable feature: it leverages an anisotropic learning rate depending on the magnitude of the gradient for each dimension which helps in exploiting the geometry of the data. On the other hand, NESTEROV’S METHOD or HEAVY BALL Method [28] is an accelerated optimization algorithm which update not only depends on the current iterate and gradient but also depends on the past gradients (i.e. momentum). State-of-the-art algorithms such as AMSGRAD [30] and ADAM [19] leverage these ideas to accelerate the training of nonconvex objective functions, for instance deep neural networks losses.

In this paper, we propose an algorithm that goes beyond the hybrid of the adaptivity and momentum approach. Our algorithm is inspired by OPTIMISTIC ONLINE LEARNING [7, 29, 33, 1, 24], which assumes that, in each round of online learning, a *predictable process* of the gradient of the loss function is available. Then, an action is played exploiting these predictors. By capitalizing on this (possibly) arbitrary process, algorithms in OPTIMISTIC ONLINE LEARNING enjoy smaller regret than the ones without gradient predictions. We combine the OPTIMISTIC ONLINE LEARNING idea with the adaptivity and the momentum ideas to design a new algorithm — OPT-AMSGRAD.

A single work along that direction stands out. [8] develop OPTIMISTIC-ADAM leveraging optimistic online mirror descent [29]. Yet, OPTIMISTIC-ADAM is specifically designed to optimize two-player

games, e.g., GANs [15] which is in particular a two-player zero-sum game. There have been some related works in OPTIMISTIC ONLINE LEARNING [7, 29, 33] showing that if both players use an OPTIMISTIC type of update, then accelerating the convergence to the equilibrium of the game is possible. [8] build on these related works and show that OPTIMISTIC-MIRROR-DESCENT can avoid the cycle behavior in a bilinear zero-sum game accelerating the convergence. In contrast, in this paper, the proposed algorithm is designed to accelerate nonconvex optimization (e.g., empirical risk minimization). To the best of our knowledge, this is the first work exploring towards this direction and bridging the unfilled *theoretical* gap at the crossroads of online learning and stochastic optimization.

The **contributions** of our paper are as follows:

- We derive an optimistic variant of AMSGRAD borrowing techniques from online learning procedures. Our method relies on (I) the addition of *prior knowledge* in the sequence of model parameter estimates leveraging a predictable process able to provide guesses of gradients through the iterations; (II) the construction of a *double update* algorithm done sequentially. We interpret this two-projection step as the learning of the global parameter and of an underlying scheme which makes the gradients sequentially predictable.
- We focus on the *theoretical* justifications of our method by establishing novel *non-asymptotic* and *global* convergence rates in both convex and nonconvex cases. Based on *convex regret minimization* and *nonconvex stochastic optimization* views, we prove, respectively, that our algorithm suffers regret of  $\mathcal{O}(\sqrt{\sum_{t=1}^T \|g_t - m_t\|_{\psi_{t-1}}^2})$  and achieves a convergence rate  $\mathcal{O}(\sqrt{d/T} + d/T)$ , where  $g_t$  is the gradient and  $m_t$  is its prediction.

The proposed algorithm adapts to the informative dimensions, exhibits momentum, and also exploits a good guess of the next gradient to facilitate acceleration. Besides the complete convergence analysis of OPT-AMSGRAD, we conduct numerical experiments and show that the proposed algorithm not only accelerates the training procedure, but also leads to better empirical generalization performance.

**Notations:** We follow the notations of adaptive optimization [19, 30]. For any  $u, v \in \mathbb{R}^d$ ,  $u/v$  represents the element-wise division,  $u^2$  the element-wise square,  $\sqrt{u}$  the element-wise square-root. We denote  $g_{1:T}[i]$  as the sum of the  $i$ th element of  $g_1, \dots, g_T \in \mathbb{R}^d$  and  $\|\cdot\|$  as the Euclidean norm.

## 2 Preliminaries

**Optimistic Online learning.** The standard setup of ONLINE LEARNING is that, in each round  $t$ , an online learner selects an action  $w_t \in \Theta \subseteq \mathbb{R}^d$ , observes  $\ell_t(\cdot)$  and suffers the associated loss  $\ell_t(w_t)$  after the action is committed. The goal is to minimize the regret,

$$\mathcal{R}_T(\{w_t\}) := \sum_{t=1}^T \ell_t(w_t) - \sum_{t=1}^T \ell_t(w^*),$$

which is the cumulative loss of the learner minus the cumulative loss of some benchmark  $w^* \in \Theta$ . The idea of OPTIMISTIC ONLINE LEARNING (e.g., [7, 29, 33, 1]) is as follows. In each round  $t$ , the learner exploits a guess  $m_t(\cdot)$  of the gradient  $\nabla \ell_t(\cdot)$  to choose an action  $w_t$ <sup>1</sup>. Consider the FOLLOW-THE-REGULARIZED-LEADER (FTRL, [17]) online learning algorithm which update reads

$$w_t = \arg \min_{w \in \Theta} \langle w, L_{t-1} \rangle + \frac{1}{\eta} \mathbf{R}(w),$$

where  $\eta$  is a parameter,  $\mathbf{R}(\cdot)$  is a 1-strongly convex function with respect to a given norm on the constraint set  $\Theta$ , and  $L_{t-1} := \sum_{s=1}^{t-1} g_s$  is the cumulative sum of gradient vectors of the loss functions up to round  $t-1$ . It has been shown that FTRL has regret at most  $\mathcal{O}(\sqrt{\sum_{t=1}^T \|g_t\|_*^2})$ . The update of its optimistic variant, called OPTIMISTIC-FTRL and developed in [33] reads

$$w_t = \arg \min_{w \in \Theta} \langle w, L_{t-1} + m_t \rangle + \frac{1}{\eta} \mathbf{R}(w), \quad (1)$$

<sup>1</sup>Imagine that if the learner would have known  $\nabla \ell_t(\cdot)$  (i.e., exact guess) before committing its action, then it would exploit the knowledge to determine its action and consequently minimize the regret.

where  $\{m_t\}_{t>0}$  is a predictable process incorporating (possibly arbitrary) knowledge about the sequence of gradients  $\{g_t := \nabla \ell_t(w_t)\}_{t>0}$ . Under the assumption that the loss functions are convex, it has been shown in [33] that the regret of OPTIMISTIC-FTRL is at most  $\mathcal{O}(\sqrt{\sum_{t=1}^T \|g_t - m_t\|_*^2})$ .

*Remark:* Note that the usual worst-case bound is preserved even when the predictors  $\{m_t\}_{t>0}$  do not predict well the gradients. Indeed, if we take the example of OPTIMISTIC-FTRL, the bound reads  $\sqrt{\sum_{t=1}^T \|g_t - m_t\|_*^2} \leq 2 \max_{w \in \Theta} \|\nabla \ell_t(w)\| \sqrt{T}$  which is equal to the usual bound up to a factor 2 [29], under certain boundedness assumptions on  $\Theta$  detailed below. Yet, when the predictors  $\{m_t\}_{t>0}$  are well designed, the resulting regret will be lower. We will have a similar argument when comparing OPT-AMSGRAD and AMSGRAD regret bounds in Section 4.1.

We emphasize, in Section 3, the importance of leveraging a good guess  $m_t$  for updating  $w_t$  in order to get a fast convergence rate (or equivalently, small regret) and introduce in Section 5 a simple predictable process  $\{m_t\}_{t>0}$  leading to empirical acceleration on various applications.

**Adaptive optimization methods.** Adaptive optimization has been popular in various deep learning applications due to their superior empirical performance. ADAM [19], a popular adaptive algorithm, combines momentum [28] and anisotropic learning rate of ADAGRAD [11]. More specifically, the learning rate of ADAGRAD at time  $T$  for dimension  $j$  is proportional to the inverse of  $\sqrt{\sum_{t=1}^T g_t[j]^2}$ , where  $g_t[j]$  is the  $j$ -th element of the gradient vector  $g_t$  at time  $t$ .

This adaptive learning rate helps accelerating the convergence when the gradient vector is sparse [11], yet, when applying ADAGRAD to train deep neural networks, it is observed that the learning rate might decay too fast, see [19] for more details. Therefore, [19] put forward ADAM that uses a moving average of the gradients divided by the square root of the second moment of this moving average (element-wise multiplication), for updating the model parameter  $w$ . A variant, called AMSGRAD and detailed in Algorithm 1, has been developed in [30] to fix ADAM failures. The difference between ADAM and AMSGRAD lies in Line 7 of Algorithm 1. The AMSGRAD algorithm [30] applies the  $\max$  operation on the second moment to guarantee a non-increasing learning rate  $\eta_t/\sqrt{\hat{v}_t}$ , which helps for the convergence (i.e. average regret  $\mathcal{R}_T/T \rightarrow 0$ ).

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#### Algorithm 1 AMSGRAD [30]

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1: Required: parameter  $\beta_1, \beta_2$ , and  $\eta_t$ .
2: Init:  $w_1 \in \Theta \subseteq \mathbb{R}^d$  and  $v_0 = \epsilon \mathbf{1} \in \mathbb{R}^d$ .
3: for  $t = 1$  to  $T$  do
4:   Get mini-batch stochastic gradient  $g_t$  at  $w_t$ .
5:    $\theta_t = \beta_1 \theta_{t-1} + (1 - \beta_1) g_t$ .
6:    $v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2$ .
7:    $\hat{v}_t = \max(\hat{v}_{t-1}, v_t)$ .
8:    $w_{t+1} = w_t - \eta_t \frac{\theta_t}{\sqrt{\hat{v}_t}}$ . (element-wise division)
9: end for

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### 3 OPT-AMSGRAD Algorithm

We formulate in this section the proposed optimistic acceleration of AMSGrad, namely OPT-AMSGRAD, and detailed in Algorithm 2. It combines the idea of *adaptive optimization* with *optimistic learning*. At each iteration, the learner computes a gradient vector  $g_t := \nabla \ell_t(w_t)$  at  $w_t$  (line 4), then maintains an exponential moving average of  $\theta_t \in \mathbb{R}^d$  (line 5) and  $v_t \in \mathbb{R}^d$  (line 6), which is followed by the  $\max$  operation to get  $\hat{v}_t \in \mathbb{R}^d$  (line 7). The learner first updates an auxiliary variable  $\tilde{w}_{t+1} \in \Theta$  (line 8), then computes the next model parameter  $w_{t+1}$  (line 9). Observe that the proposed algorithm does not reduce to AMSGRAD when  $m_t = 0$ , contrary to the optimistic variant of FTRL. Furthermore, combining line 8 and line 9 yields the following single step  $w_{t+1} = \tilde{w}_t - \eta_t(\theta_t + h_{t+1})/\sqrt{\hat{v}_t}$ .

Compared to AMSGRAD, the algorithm is characterized by a *two-level* update that interlinks some *auxiliary state*  $\tilde{w}_t$  and the model parameter state,  $w_t$ , similarly to the OPTIMISTIC MIRROR DESCENT algorithm developed in [29]. It leverages the auxiliary variable (hidden model) to update and commit  $w_{t+1}$ , which exploits the guess  $m_{t+1}$ , see Figure 1.

In the following analysis, we show that this interleaving actually leads to some cancellation in the regret bound. Such two-levels method where the guess  $m_t$  is equal to the last known gradient  $g_{t-1}$  has been exhibited recently in [7]. The gradient prediction process plays an important role as discussed in Section 5. The proposed OPT-AMSGRAD algorithm inherits three properties: (i) Adaptive learn-

ing rate of each dimension as ADAGRAD [11] (line 6, line 8 and line 9). (ii) Exponential moving average of the past gradients as NESTEROV'S METHOD [27] and the HEAVY-BALL method [28] (line 5). (iii) Optimistic update that exploits *prior knowledge* of the next gradient vector as in optimistic online learning algorithms [7, 29, 33] (line 9). The first property helps for acceleration when the gradient has a sparse structure. The second one is from the long-established idea of momentum which can also help for acceleration. The last property can lead to an acceleration when the prediction of the next gradient is good as mentioned above when introducing the regret bound for the OPTIMISTIC-FTRL algorithm. This property will be elaborated whilst establishing the theoretical analysis of OPT-AMSGRAD.

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#### Algorithm 2 OPT-AMSGRAD

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1: **Required:** parameter  $\beta_1, \beta_2, \epsilon$ , and  $\eta_t$ .  
2: Init:  $w_1 = w_{-1/2} \in \Theta \subseteq \mathbb{R}^d$  and  $v_0 = \epsilon \mathbf{1} \in \mathbb{R}^d$ .  
3: **for**  $t = 1$  to  $T$  **do**  
4:   Get mini-batch stochastic gradient  $g_t$  at  $w_t$ .  
5:    $\theta_t = \beta_1 \theta_{t-1} + (1 - \beta_1) g_t$ .  
6:    $v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2$ .  
7:    $\hat{v}_t = \max(\hat{v}_{t-1}, v_t)$ .  
8:    $\tilde{w}_{t+1} = \tilde{w}_t - \eta_t \frac{\theta_t}{\sqrt{\hat{v}_t}}$ .  
9:    $w_{t+1} = \tilde{w}_{t+1} - \eta_t \frac{h_{t+1}}{\sqrt{\hat{v}_t}}$ ,  
     where  $h_{t+1} := \beta_1 \theta_{t-1} + (1 - \beta_1) m_{t+1}$  with  
      $m_{t+1}$  the guess of  $g_{t+1}$ .  
10: **end for**

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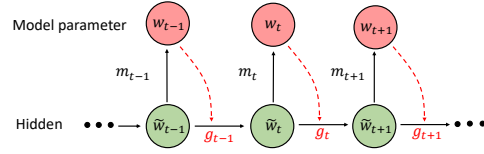


Figure 1: OPT-AMSGRAD underlying structure.

## 4 Non-Asymptotic Convergence Analysis

**More notations.** We denote the Mahalanobis norm by  $\|\cdot\|_H := \sqrt{\langle \cdot, H \cdot \rangle}$  for some positive semidefinite (PSD) matrix  $H$ . We let  $\psi_t(x) := \langle x, \text{diag}\{\hat{v}_t\}^{1/2} x \rangle$  for a PSD matrix  $H_t^{1/2} := \text{diag}\{\hat{v}_t\}^{1/2}$ , where  $\text{diag}\{\hat{v}_t\}$  represents the diagonal matrix which  $i_{th}$  diagonal element is  $\hat{v}_t[i]$  defined in Algorithm 2. We define its corresponding Mahalanobis norm by  $\|\cdot\|_{\psi_t} := \sqrt{\langle \cdot, \text{diag}\{\hat{v}_t\}^{1/2} \cdot \rangle}$ , where we abuse the notation  $\psi_t$  to represent the PSD matrix  $H_t^{1/2} := \text{diag}\{\hat{v}_t\}^{1/2}$ . Note that  $\psi_t(\cdot)$  is 1-strongly convex with respect to the norm  $\|\cdot\|_{\psi_t}$ , i.e.,  $\psi_t(\cdot)$  satisfies  $\psi_t(u) \geq \psi_t(v) + \langle \psi_t(v), u - v \rangle + \frac{1}{2} \|u - v\|_{\psi_t}^2$  for any point  $(u, v) \in \Theta^2$ . A consequence of 1-strong convexity of  $\psi_t(\cdot)$  is that  $B_{\psi_t}(u, v) \geq \frac{1}{2} \|u - v\|_{\psi_t}^2$ , where the Bregman divergence  $B_{\psi_t}(u, v)$  is defined as  $B_{\psi_t}(u, v) := \psi_t(u) - \psi_t(v) - \langle \psi_t(v), u - v \rangle$  with  $\psi_t(\cdot)$  as the distance generating function. We also define the corresponding dual norm  $\|\cdot\|_{\psi_t^*} := \sqrt{\langle \cdot, \text{diag}\{\hat{v}_t\}^{-1/2} \cdot \rangle}$ . The proofs of the results are deferred to the Appendix.

### 4.1 Convex Regret Analysis

In the following, we assume convexity of  $\{\ell_t\}_{t \geq 0}$  and that  $\Theta$  has a bounded diameter  $D_\infty$ , which is a standard assumption for adaptive methods [30, 19] and is necessary in regret analysis.

**Theorem 1.** Suppose the learner incurs a sequence of convex loss functions  $\{\ell_t(\cdot)\}$ . Then, OPT-AMSGRAD (Algorithm 2) has regret

$$\mathcal{R}_T \leq \frac{B_{\psi_1}(w^*, \tilde{w}_1)}{\eta_1} + \sum_{t=1}^T \frac{\eta_t}{2} \|g_t - \tilde{m}_t\|_{\psi_{t-1}^*}^2 + \frac{D_\infty^2}{\eta_{\min}} \sum_{i=1}^d \hat{v}_T^{1/2}[i] + D_\infty^2 \beta_1^2 \sum_{t=1}^T \|g_t - \theta_{t-1}\|_{\psi_{t-1}^*},$$

where  $\tilde{m}_{t+1} = \beta_1 \theta_{t-1} + (1 - \beta_1) m_{t+1}$ ,  $g_t := \nabla \ell_t(w_t)$ ,  $\eta_{\min} := \min_t \eta_t$  and  $D_\infty^2$  is the diameter of the bounded set  $\Theta$ . The result holds for any benchmark  $w^* \in \Theta$  and any step size sequence  $\{\eta_t\}_{t \geq 0}$ .

**Corollary 1.** Suppose  $\beta_1 = 0$  and  $\{v_t\}_{t \geq 0}$  is a monotonically increasing sequence, then we obtain the following regret bound for any  $w^* \in \Theta$  and sequence of stepsizes  $\{\eta_t = \eta/\sqrt{t}\}_{t \geq 0}$ :

$$\mathcal{R}_T \leq \frac{B_{\psi_1}}{\eta_1} + \frac{\eta \sqrt{1 + \log T}}{\sqrt{1 - \beta_2}} \sum_{i=1}^d \|(g - m)_{1:T}[i]\|_2 + \frac{D_\infty^2}{\eta_{\min}} \sum_{i=1}^d \left[ (1 - \beta_2) \sum_{s=1}^T \beta_2^{T-s} g_s^2[i] \right]^{1/2},$$

158 where  $B_{\psi_1} := B_{\psi_1}(w^*, \tilde{w}_1)$ ,  $g_t := \nabla \ell_t(w_t)$  and  $\eta_{\min} := \min_t \eta_t$ .

159 We can compare the bound of Corollary 1 with that of AMSGRAD [30] with  $\eta_t = \eta/\sqrt{t}$ :

$$\mathcal{R}_T \leq \frac{\eta\sqrt{1+\log T}}{\sqrt{1-\beta_2}} \sum_{i=1}^d \|g_{1:T}[i]\|_2 + \frac{\sqrt{T}}{2\eta} D_\infty^2 \sum_{i=1}^d \hat{v}_T[i]^2. \quad (2)$$

160 For convex regret minimization, Corollary 1 yields a regret of  $\mathcal{O}(\sqrt{\sum_{t=1}^T \|g_t - m_t\|_{\psi_{t-1}^*}^2})$  with an  
 161 access to an arbitrary predictable process  $\{m_t\}_{t>0}$  of the mini-batch gradients. We notice from  
 162 the second term in Corollary 1 compared to the first term in (2) that better predictors lead to lower  
 163 regret. The construction of the predictions  $\{m_t\}_{t>0}$  is thus of utmost importance for achieving  
 164 optimal acceleration and can be learned through the iterations [29]. In Section 5, we derive a basic,  
 165 yet effective, gradient prediction algorithm, see Algorithm 3, embedded in OPT-AMSGRAD.

## 166 4.2 Finite-Time Analysis in the Nonconvex Case

167 We discuss the offline and stochastic nonconvex optimization properties of our online framework.  
 168 As stated in the introduction, this paper is about solving optimization problems instead of solving  
 169 zero-sum games. Classically, the optimization problem we are tackling reads:

$$\min_{w \in \Theta} f(w) := \mathbb{E}[f(w, \xi)] = n^{-1} \sum_{i=1}^n \mathbb{E}[f(w, \xi_i)], \quad (3)$$

170 for a fixed batch of  $n$  samples  $\{\xi_i\}_{i=1}^n$ . Set the terminating number,  $T \in \{0, \dots, T_M - 1\}$ , as a  
 171 discrete r.v. with:

$$P(T = \ell) = \frac{\eta_\ell}{\sum_{j=0}^{T_M-1} \eta_j}, \quad (4)$$

172 where  $T_M$  is the maximum number of iteration. The random termination number (4) is inspired by  
 173 [14] and is widely used for nonconvex optimization. Assume the following:

174 **H1.** For any  $t > 0$ , the estimated parameter  $w_t$  stays within a  $\ell_\infty$ -ball. There exist a constant  
 175  $W > 0$  such that  $\|w_t\|_\infty \leq W$  almost surely.

176 **H2.** The function  $f$  is  $L$ -smooth (has  $L$ -Lipschitz gradients) w.r.t. the parameter  $w$ . There exists  
 177 some constant  $L > 0$  such that for  $(w, \vartheta) \in \Theta^2$ ,  $f(w) - f(\vartheta) - \nabla f(\vartheta)^\top (w - \vartheta) \leq \frac{L}{2} \|w - \vartheta\|^2$ .

178 We assume that the optimistic guess  $m_t$  at iteration  $t$  and the true gradient  $g_t$  are correlated:

179 **H3.** For any  $t > 0$ ,  $0 < \langle m_t | g_t \rangle = a_t \|g_t\|^2$  with some  $0 < a_t \leq 1$ , and  $\|m_t\| \leq \|g_t\|$ , where  $\langle \cdot | \cdot \rangle$   
 180 denotes the inner product.

181 Lastly, We make a classical assumption in nonconvex optimization on the magnitude of the gradient:

182 **H4.** There exist a constant  $M > 0$  such that for any  $w$  and  $\xi$ , it holds that  $\|\nabla f(w, \xi)\| < M$ .

183 We now derive important results for our global analysis. The first one ensures bounded norms of  
 184 quantities of interests (resulting from the bounded stochastic gradient assumption):

185 **Lemma 1.** Assume H4, then the quantities defined in Algorithm 2 satisfy for any  $w \in \Theta$  and  $t > 0$ ,  
 186  $\|\nabla f(w_t)\| < M$ ,  $\|\theta_t\| < M$  and  $\|\hat{v}_t\| < M^2$ .

187 We now formulate the main result of our paper yielding a finite-time upper bound of the subopti-  
 188 mality condition defined as  $\mathbb{E}[\|\nabla f(w_T)\|^2]$  (set as the convergence criterion of interest, see [14]):

189 **Theorem 2.** Assume H1-H4,  $\beta_1 < \beta_2 \in [0, 1)$  and a sequence of decreasing stepsizes  $\{\eta_t\}_{t>0}$ , then  
 190 the following result holds:

$$\mathbb{E}[\|\nabla f(w_T)\|_2^2] \leq \tilde{C}_1 \sqrt{\frac{d}{T_M}} + \tilde{C}_2 \frac{1}{T_M},$$

191 where  $T$  is a random termination number distributed according (4). The constants are defined as:

$$\begin{aligned} \tilde{C}_1 &= \frac{M}{(1 - a\beta_1) + (\beta_1 + a)} \left[ \frac{a(1 - \beta_1)^2}{1 - \beta_2} + 2L \frac{1}{1 - \beta_2} + \Delta f + \frac{4L\beta_1^2(1 + \beta_1^2)}{(1 - \beta_1)(1 - \beta_2)(1 - \gamma)} \right] \\ \tilde{C}_2 &= \frac{(a_m\beta_1^2 - 2a_m\beta_1 + \beta_1)M^2}{(1 - \beta_1)((1 - a_m\beta_1) + (\beta_1 + a_m))} \mathbb{E}[\|\hat{v}_0^{-1/2}\|], \end{aligned}$$

192 where  $\Delta f = f(\bar{w}_1) - f(\bar{w}_{T_M+1})$  and  $a_m = \min_{t=1, \dots, T} a_t$ .



193 Firstly, the bound for our OPT-AMSGrad method matches the complexity bound of  $\mathcal{O}(\sqrt{d/T_M} +$   
 194  $1/T_M)$  of [14] for SGD considering the dependence of  $T$  only, and of [41] for AMSGrad method. To  
 195 see the influence of prediction quality, we can show that when  $(1 - \beta_1)(\beta_2 - \beta_1^2 - 2L(1 - \beta_1)) -$   
 196  $\frac{4L\beta_1^2(1+\beta_1^2)}{1-\gamma} < 0$ ,  $\tilde{C}_1$  and  $\tilde{C}_2$  both decrease as  $a_m$  approaches 1, i.e. as the prediction gets more accu-  
 197 rate. Therefore, similar to the convex case, our bound also improves with better gradient prediction.

### 198 4.3 Checking H1 for a Deep Neural Network

199 As boundedness assumption H1 is generally hard to verify, we now show, for illustrative purposes,  
 200 that the weights of a fully connected feed forward neural network stay in a bounded set when being  
 201 trained using our method. The activation function for this section will be sigmoid function and we  
 202 use a  $\ell_2$  regularization. We consider a fully connected feed forward neural network with  $L$  layers  
 203 modeled by the function  $\text{MLN}(w, \xi) : \Theta^d \times \mathbb{R}^p \rightarrow \mathbb{R}$  defined as:

$$\text{MLN}(w, \xi) = \sigma \left( w^{(L)} \sigma \left( w^{(L-1)} \dots \sigma \left( w^{(1)} \xi \right) \right) \right), \quad (5)$$

where  $w = [w^{(1)}, w^{(2)}, \dots, w^{(L)}]$  is the vector of parameters,  $\xi \in \mathbb{R}^p$  is the input data and  $\sigma$  is the  
 sigmoid activation function. We assume a  $p$  dimension input data and a scalar output for simplicity.  
 In this setting, the stochastic objective function (3) reads

$$f(w, \xi) = \mathcal{L}(\text{MLN}(w, \xi), y) + \frac{\lambda}{2} \|w\|^2,$$

204 where  $\mathcal{L}(\cdot, y)$  is the loss function (e.g., cross-entropy),  $y$  are the true labels and  $\lambda > 0$  is the regular-  
 205 ization parameter. We establish that the boundedness assumption H1 is satisfied with model (5):

206 **Lemma 2.** *Given the multilayer model (5), assume the boundedness of the input data and of the*  
 207 *loss function, i.e., for any  $\xi \in \mathbb{R}^p$  and  $y \in \mathbb{R}$  there is a constant  $T > 0$  such that  $\|\xi\| \leq 1$  a.s.*  
 208 *and  $|\mathcal{L}'(\cdot, y)| \leq T$  where  $\mathcal{L}'(\cdot, y)$  denotes its derivative w.r.t. the parameter. Then for each layer*  
 209  *$\ell \in [1, L]$ , there exist a constant  $A_{(\ell)}$  such that  $\|w^{(\ell)}\| \leq A_{(\ell)}$*

## 210 5 Numerical Experiments

### 211 5.1 Gradient Estimation

212 Based on the analysis in the previous section, we understand that the choice of the prediction  $m_t$   
 213 plays an important role in the convergence of OPTIMISTIC-AMSGRAD. Some classical works in  
 214 gradient prediction methods include ANDERSON acceleration [36], MINIMAL POLYNOMIAL EX-  
 215 TRAPOLATION [4] and REDUCED RANK EXTRAPOLATION [12]. These methods aim at finding a  
 216 fixed point  $g^*$  and assume that  $\{g_t \in \mathbb{R}^d\}_{t>0}$  has the following linear relation:

$$g_t - g^* = A(g_{t-1} - g^*) + e_t, \quad (6)$$

217 where  $e_t$  is a second order term satisfying  $\|e_t\|_2 = \mathcal{O}(\|g_{t-1} - g^*\|_2^2)$  and  $A \in \mathbb{R}^{d \times d}$  is an unknown  
 218 matrix, see [31] for details and results. For our numerical experiments, we run OPT-AMSGRAD  
 219 using Algorithm 3 to construct the sequence  $\{m_t\}_{t>0}$  which is based on estimating the limit of a  
 220 sequence using the last iterates [3].

221 Specifically, at iteration  $t$ ,  $m_t$  is ob-  
 222 tained by (a) calling Algorithm 3  
 223 with a sequence of  $r$  past gradi-  
 224 ents,  $\{g_{t-1}, g_{t-2}, \dots, g_{t-r}\}$  as input  
 225 yielding the vector  $c = [c_0, \dots, c_{r-1}]$   
 226 and (b) setting  $m_t := \sum_{i=0}^{r-1} c_i g_{t-r+i}$ .  
 227 To understand why the output from  
 228 the extrapolation method may be a  
 229 reasonable estimation, assume that

---

**Algorithm 3** Regularized Approximated Minimal Poly-  
 nomial Extrapolation [31]

---

- 1: **Input:** sequence  $\{g_s \in \mathbb{R}^d\}_{s=0}^{r-1}$ , parameter  $\lambda > 0$ .
  - 2: Compute matrix  $U = [g_1 - g_0, \dots, g_r - g_{r-1}] \in \mathbb{R}^{d \times r}$ .
  - 3: Obtain  $z$  by solving  $(U^\top U + \lambda I)z = \mathbf{1}$ .
  - 4: Get  $c = z / (z^\top \mathbf{1})$ .
  - 5: **Output:**  $\sum_{i=0}^{r-1} c_i g_i$ , the approximation of the fixed point  $g^*$ .
- 

230 the update converges to a stationary point (i.e.  $g^* := \nabla f(w^*) = 0$  for the underlying function  
 231  $f$ ). Then, we might rewrite (6) as  $g_t = Ag_{t-1} + \mathcal{O}(\|g_{t-1}\|_2^2)u_{t-1}$ , for some unit vector  $u_{t-1}$ . This  
 232 equation suggests that the next gradient vector  $g_t$  is a linear transform of  $g_{t-1}$  plus an error vector  
 233 that may not be in the span of  $A$ . If the algorithm converges to a stationary point, the magnitude of  
 234 the error will converge to zero. We note that prior known gradient prediction methods are mainly

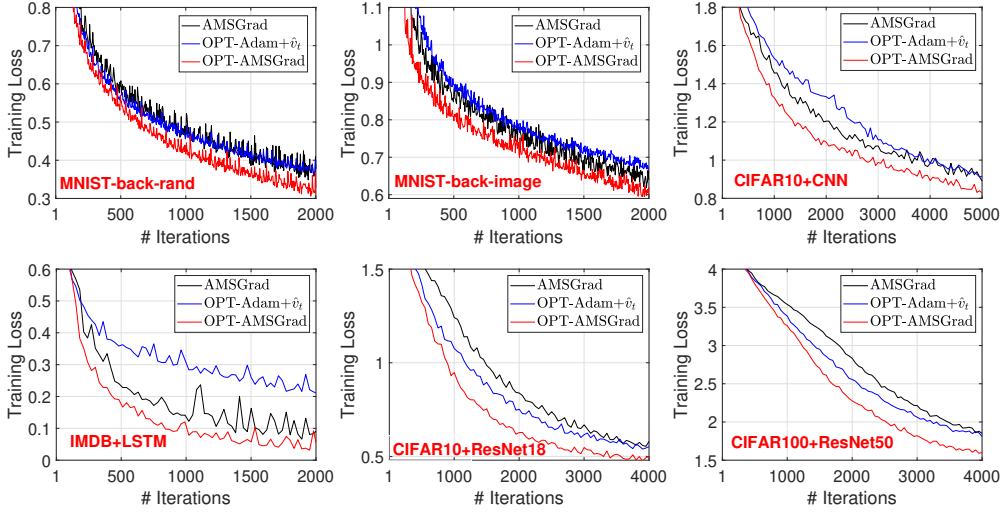


Figure 2: Training loss vs. Number of iterations for fully connected NN, CNN, LSTM and ResNet.

designed for convex functions. Algorithm 3 is employed in our following numerical applications given its empirical success in Deep Learning, see [32], nevertheless, any gradient prediction method can be embedded in our OPT-AMSGRAD framework. The search for the optimal prediction process in order to accelerate even more OPT-AMSGRAD is an interesting research direction, which is left as future work.

**Computational cost:** This extrapolation step consists in: (a) Constructing the linear system ( $U^\top U$ ) which cost can be optimized to  $\mathcal{O}(d)$ , since the matrix  $U$  only changes one column at a time. (b) Solving the linear system which cost is  $\mathcal{O}(r^3)$ , and is negligible for a small  $r$  used in practice. (c) Outputting a weighted average of previous gradients which cost is  $\mathcal{O}(r \times d)$  yielding a computational overhead of  $\mathcal{O}((r+1)d + r^3)$ . Yet, steps (a) and (c) can be parallelized in the final implementation.

## 5.2 Classification Experiments

**Methods.** We consider two baselines. The first one is the original AMSGRAD. The hyperparameters are set to be  $\beta_1 = 0.9$  and  $\beta_2 = 0.999$ , see [30]. The other benchmark method is the OPTIMISTIC-ADAM+ $\hat{v}_t$  [8], which described Algorithm 4. We use cross-entropy loss, a mini-batch size of 128 and tune the learning rates over a fine grid and report the best result for all methods. For OPT-AMSGRAD, we use  $\beta_1 = 0.9$  and  $\beta_2 = 0.999$  and the best step size  $\eta$  of AMSGRAD for a fair evaluation of the optimistic step. In our implementation, OPT-AMSGRAD has an additional parameter  $r$  that controls the number of previous gradients used for gradient prediction. We use  $r = 5$  past gradient for empirical reasons, see Section 5.3. The algorithms are initialized at the same point and the results are averaged over 5 repetitions.

**Datasets.** Following [30] and [19], we compare different algorithms on *MNIST*, *CIFAR10*, *CIFAR100*, and *IMDB* datasets. For *MNIST*, we use two noisy variants namely *MNIST-back-rand* and *MNIST-back-image* from [20]. They both have 12 000 training samples and 50 000 test samples, where random background is inserted to the original *MNIST* hand-written digit images. For *MNIST-back-rand*, each image is inserted with a random background, which pixel values are generated uniformly from 0 to 255, while *MNIST-back-image* takes random patches from a black and white noisy background. The input dimension is 784 ( $28 \times 28$ ) and the number of classes is 10. *CIFAR10* and *CIFAR100* are popular computer-vision datasets of 50 000 training images and 10 000 test images, of size  $32 \times 32$ . The *IMDB* movie review dataset, popular for text classification, is a binary dataset with 25 000 training and testing samples respectively.

**Network architectures.** We adopt a multi-layer fully connected neural network with hidden layers of 200 connected to another layer with 100 neurons (using ReLU activations and Softmax output). This network is tested on *MNIST* variants. For convolutional networks, we adopt a simple four layer CNN which has 2 convolutional layers following by a fully connected layer. In addition, we also

270 apply residual networks, Resnet-18 and Resnet-50 [18], which have achieved state-of-the-art results.  
 271 For the texture *IMDB* dataset, we consider a Long-Short Term Memory (LSTM) network [13]. The  
 272 latter network includes a word embedding layer with 5 000 input entries representing most frequent  
 273 words embedded into a 32 dimensional space. The output of the embedding layer is passed to 100  
 274 LSTM units then connected to 100 fully connected ReLU layers.

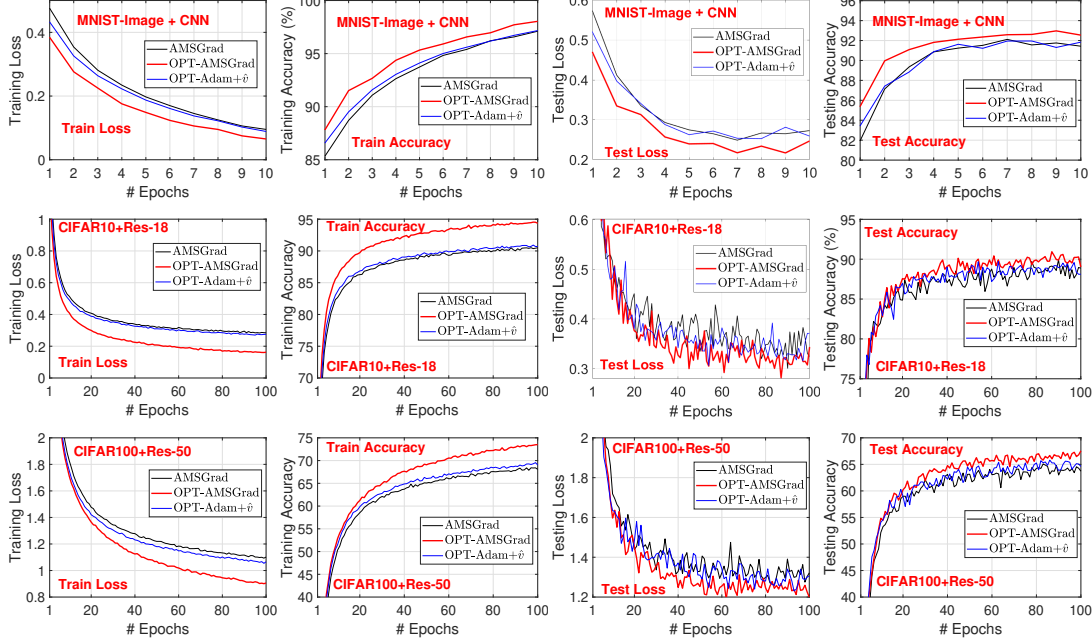


Figure 3: *MNIST-back-image* + CNN, *CIFAR10* + Res-18 and *CIFAR100* + Res-50 . We compare three methods in terms of training (cross-entropy) loss and accuracy, testing loss and accuracy.

275 **Results.** Firstly, to illustrate the acceleration effect of OPT-AMSGRAD at early stage, we pro-  
 276 vide the training loss against number of iterations in Figure 2. We clearly observe that on all  
 277 datasets, the proposed OPT-AMSGRAD converges faster than the other competing methods since  
 278 fewer iterations are required to achieve the same precision, validating one of the main edges of  
 279 OPT-AMSGRAD. We are also curious about the long-term performance and generalization of the  
 280 proposed method in test phase. In Figure 3, we plot the results when the model is trained until the test  
 281 accuracy stabilizes. We observe: (1) in the long term, OPT-AMSGRAD algorithm may converge to  
 282 a better point with smaller loss value, and (2) in these applications, our proposed OPT-AMSGRAD  
 283 also outperforms the competing methods in terms of test accuracy.

### 284 5.3 Choice of parameter $r$

285 Since the number of past gradients  $r$  is im-  
 286 portant in gradient prediction (Algorithm  
 287 3), we compare Figure 4 the performance  
 288 under different values  $r = 3, 5, 10$  on two  
 289 datasets. From the results we see that, tak-  
 290 ing into consideration both quality of gra-  
 291 dient prediction and computational cost,  
 292  $r = 5$  is a good choice for most applica-  
 293 tions. We remark that, empirically, the per-  
 294 formance comparison among  $r = 3, 5, 10$   
 295 is not absolutely consistent (i.e. more means better) in all cases. We suspect one possible reason  
 296 is that for deep neural networks, the diversity of computed gradients through the iterations, due to  
 297 the highly nonconvex loss, makes them inefficient for sequentially building the predictable process  
 298  $\{m_t\}_{t>0}$ . Thus, sometimes, the recent gradient vectors (e.g.  $r \leq 5$ ) can be more informative.  
 299 Yet, in some sense, this characteristic, very specific to deep neural networks, is itself a fundamental  
 300 problem of gradient prediction methods.

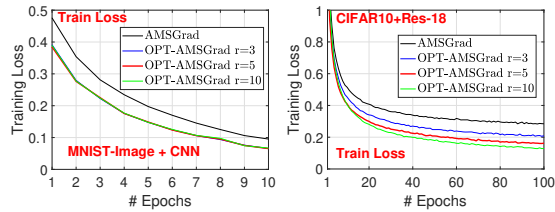


Figure 4: Training loss w.r.t.  $r$ .



## 301 6 Conclusion

302 In this paper, we propose OPT-AMSGRAD, which combines optimistic online learning and AMS-  
303 GRAD to improve sample efficiency and accelerate the training process, in particular for fitting deep  
304 neural networks on a finite batch of observations. Given a well-designed gradient prediction pro-  
305 cess, we theoretically show that the regret, through the iterations, can be smaller than that of standard  
306 AMSGRAD. We also establish a finite-time convergence bound on the second order moment of the  
307 gradient of the objective loss function matching that of state-of-the-art adaptive gradient methods.  
308 Experiments on several benchmark datasets using various deep learning models demonstrate the ef-  
309 fectiveness of the proposed algorithm in accelerating the empirical risk minimization procedure and  
310 empirically show better generalization properties of our method OPT-AMSGRAD.

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## Checklist

### 1. For all authors...

- (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? **[TODO]**
- (b) Did you describe the limitations of your work? **[TODO]**
- (c) Did you discuss any potential negative societal impacts of your work? **[TODO]**
- (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? **[TODO]**

### 2. If you are including theoretical results...

- (a) Did you state the full set of assumptions of all theoretical results? **[TODO]**
- (b) Did you include complete proofs of all theoretical results? **[TODO]**

### 3. If you ran experiments...

- (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? **[TODO]**
- (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? **[TODO]**
- (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? **[TODO]**
- (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? **[TODO]**

### 4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

- (a) If your work uses existing assets, did you cite the creators? **[TODO]**
- (b) Did you mention the license of the assets? **[TODO]**
- (c) Did you include any new assets either in the supplemental material or as a URL? **[TODO]**
- (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? **[TODO]**
- (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? **[TODO]**

### 5. If you used crowdsourcing or conducted research with human subjects...

- (a) Did you include the full text of instructions given to participants and screenshots, if applicable? **[TODO]**
- (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? **[TODO]**
- (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? **[TODO]**

## Appendix: An Optimistic Acceleration of AMSGrad for Nonconvex Optimization

### A Proof of Theorem 1

**Theorem.** Suppose the learner incurs a sequence of convex loss functions  $\{\ell_t(\cdot)\}$ . Then, OPT-AMSGRAD (Algorithm 2) has regret

$$\mathcal{R}_T \leq \frac{B_{\psi_1}(w^*, \tilde{w}_1)}{\eta_1} + \sum_{t=1}^T \frac{\eta_t}{2} \|g_t - \tilde{m}_t\|_{\psi_{t-1}^*}^2 + \frac{D_\infty^2}{\eta_{\min}} \sum_{i=1}^d \hat{v}_T^{1/2}[i] + D_\infty^2 \beta_1^2 \sum_{t=1}^T \|g_t - \theta_{t-1}\|_{\psi_{t-1}^*},$$

where  $\tilde{m}_{t+1} = \beta_1 \theta_{t-1} + (1 - \beta_1) m_{t+1}$ ,  $g_t := \nabla \ell_t(w_t)$ ,  $\eta_{\min} := \min_t \eta_t$  and  $D_\infty^2$  is the diameter of the bounded set  $\Theta$ . The result holds for any benchmark  $w^* \in \Theta$  and any step size sequence  $\{\eta_t\}_{t>0}$ .

**Proof** Beforehand, we denote:

$$\begin{aligned} \tilde{g}_t &= \beta_1 \theta_{t-1} + (1 - \beta_1) g_t, \\ \tilde{m}_{t+1} &= \beta_1 \theta_{t-1} + (1 - \beta_1) m_{t+1}, \end{aligned} \tag{7}$$

where we recall that  $g_t$  and  $m_{t+1}$  are respectively the gradient  $\nabla \ell_t(w_t)$  and the predictable guess. By regret decomposition, we have that

$$\begin{aligned} \mathcal{R}_T &:= \sum_{t=1}^T \ell_t(w_t) - \min_{w \in \Theta} \sum_{t=1}^T \ell_t(w) \\ &\leq \sum_{t=1}^T \langle w_t - w^*, \nabla \ell_t(w_t) \rangle \\ &= \sum_{t=1}^T \langle w_t - \tilde{w}_{t+1}, g_t - \tilde{m}_t \rangle + \langle w_t - \tilde{w}_{t+1}, \tilde{m}_t \rangle + \langle \tilde{w}_{t+1} - w^*, \tilde{g}_t \rangle + \langle \tilde{w}_{t+1} - w^*, g_t - \tilde{g}_t \rangle. \end{aligned} \tag{8}$$

Recall the notation  $\psi_t(x)$  and the Bregman divergence  $B_{\psi_t}(u, v)$  defined Section 4. We exploit a useful inequality (which appears in e.g., [35]). For any update of the form  $\hat{w} = \arg \min_{w \in \Theta} \langle w, \theta \rangle + B_{\psi_t}(w, v)$ , it holds that

$$\langle \hat{w} - u, \theta \rangle \leq B_{\psi_t}(u, v) - B_{\psi_t}(u, \hat{w}) - B_{\psi_t}(\hat{w}, v) \quad \text{for any } u \in \Theta. \tag{9}$$

For  $\beta_1 = 0$ , we can rewrite the update on line 8 of (Algorithm 2) as

$$\tilde{w}_{t+1} = \arg \min_{w \in \Theta} \eta_t \langle w, \tilde{g}_t \rangle + B_{\psi_t}(w, \tilde{w}_t). \tag{10}$$

By using (9) for (10) with  $\hat{w} = \tilde{w}_{t+1}$  (the output of the minimization problem),  $u = w^*$  and  $v = \tilde{w}_t$ , we have

$$\langle \tilde{w}_{t+1} - w^*, \tilde{g}_t \rangle \leq \frac{1}{\eta_t} [B_{\psi_t}(w^*, \tilde{w}_t) - B_{\psi_t}(w^*, \tilde{w}_{t+1}) - B_{\psi_t}(\tilde{w}_{t+1}, \tilde{w}_t)]. \tag{11}$$

We can also rewrite the update on line 9 of (Algorithm 2) at time  $t$  as

$$w_{t+1} = \arg \min_{w \in \Theta} \eta_{t+1} \langle w, \tilde{m}_{t+1} \rangle + B_{\psi_t}(w, \tilde{w}_{t+1}). \tag{12}$$

and, by using (9) for (12) (written at iteration  $t$ ), with  $\hat{w} = w_t$  (the output of the minimization problem),  $u = \tilde{w}_{t+1}$  and  $v = \tilde{w}_t$ , we have

$$\langle w_t - \tilde{w}_{t+1}, \tilde{m}_t \rangle \leq \frac{1}{\eta_t} [B_{\psi_{t-1}}(\tilde{w}_{t+1}, \tilde{w}_t) - B_{\psi_{t-1}}(\tilde{w}_{t+1}, w_t) - B_{\psi_{t-1}}(w_t, \tilde{w}_t)]. \tag{13}$$



By (8), (11), and (13), we obtain

$$\begin{aligned}
\mathcal{R}_T &\stackrel{(8)}{\leq} \sum_{t=1}^T \langle w_t - \tilde{w}_{t+1}, g_t - \tilde{m}_t \rangle + \langle w_t - \tilde{w}_{t+1}, \tilde{m}_t \rangle + \langle \tilde{w}_{t+1} - w^*, \tilde{g}_t \rangle + \langle \tilde{w}_{t+1} - w^*, g_t - \tilde{g}_t \rangle \\
&\stackrel{(11),(13)}{\leq} \sum_{t=1}^T \|w_t - \tilde{w}_{t+1}\|_{\psi_{t-1}} \|g_t - \tilde{m}_t\|_{\psi_{t-1}^*} + \|\tilde{w}_{t+1} - w^*\|_{\psi_{t-1}} \|g_t - \tilde{g}_t\|_{\psi_{t-1}^*} \\
&\quad + \frac{1}{\eta_t} [B_{\psi_{t-1}}(\tilde{w}_{t+1}, \tilde{w}_t) - B_{\psi_{t-1}}(\tilde{w}_{t+1}, w_t) - B_{\psi_{t-1}}(w_t, \tilde{w}_t) \\
&\quad + B_{\psi_t}(w^*, \tilde{w}_t) - B_{\psi_t}(w^*, \tilde{w}_{t+1}) - B_{\psi_t}(\tilde{w}_{t+1}, \tilde{w}_t)] ,
\end{aligned} \tag{14}$$

which is further bounded by

$$\begin{aligned}
\mathcal{R}_T &\leq \sum_{t=1}^T \left\{ \frac{1}{2\eta_t} \|w_t - \tilde{w}_{t+1}\|_{\psi_{t-1}}^2 + \frac{\eta_t}{2} \|g_t - m_t\|_{\psi_{t-1}^*}^2 + \|\tilde{w}_{t+1} - w^*\|_{\psi_{t-1}} \|g_t - \tilde{g}_t\|_{\psi_{t-1}^*} \right. \\
&\quad + \frac{1}{\eta_t} \underbrace{(B_{\psi_{t-1}}(\tilde{w}_{t+1}, \tilde{w}_t) - B_{\psi_t}(\tilde{w}_{t+1}, \tilde{w}_t))}_{A_1} - \frac{1}{2} \|\tilde{w}_{t+1} - w_t\|_{\psi_{t-1}}^2 \\
&\quad \left. + \underbrace{(B_{\psi_t}(w^*, \tilde{w}_t) - B_{\psi_t}(w^*, \tilde{w}_{t+1}))}_{A_2} \right\} ,
\end{aligned} \tag{15}$$

where the inequality is due to  $\|w_t - \tilde{w}_{t+1}\|_{\psi_{t-1}} \|g_t - m_t\|_{\psi_{t-1}^*} = \inf_{\beta > 0} \frac{1}{2\beta} \|w_t - \tilde{w}_{t+1}\|_{\psi_{t-1}}^2 + \frac{\beta}{2} \|g_t - m_t\|_{\psi_{t-1}^*}^2$  by Young's inequality and the 1-strongly convex of  $\psi_{t-1}(\cdot)$  with respect to  $\|\cdot\|_{\psi_{t-1}}$  which yields that  $B_{\psi_{t-1}}(\tilde{w}_{t+1}, w_t) \geq \frac{1}{2} \|\tilde{w}_{t+1} - w_t\|_{\psi_t}^2 \geq 0$ .

To proceed, notice that

$$\begin{aligned}
A_1 &:= B_{\psi_{t-1}}(\tilde{w}_{t+1}, \tilde{w}_t) - B_{\psi_t}(\tilde{w}_{t+1}, \tilde{w}_t) \\
&= \langle \tilde{w}_{t+1} - \tilde{w}_t, \text{diag}(\hat{v}_{t-1}^{1/2} - \hat{v}_t^{1/2})(\tilde{w}_{t+1} - \tilde{w}_t) \rangle \leq 0 ,
\end{aligned} \tag{16}$$

as the sequence  $\{\hat{v}_t\}$  is non-decreasing. And that

$$\begin{aligned}
A_2 &:= B_{\psi_t}(w^*, \tilde{w}_t) - B_{\psi_t}(w^*, \tilde{w}_{t+1}) = \langle w^* - \tilde{w}_{t+1}, \text{diag}(\hat{v}_{t+1}^{1/2} - \hat{v}_t^{1/2})(w^* - \tilde{w}_{t+1}) \rangle \\
&\leq (\max_i (w^*[i] - \tilde{w}_{t+1}[i])^2) \cdot \left( \sum_{i=1}^d \hat{v}_{t+1}^{1/2}[i] - \hat{v}_t^{1/2}[i] \right) .
\end{aligned} \tag{17}$$

Therefore, by (15), (17), (16), we have

$$\mathcal{R}_T \leq \frac{D_\infty^2}{\eta_{\min}} \sum_{i=1}^d \hat{v}_T^{1/2}[i] + \frac{B_{\psi_1}(w^*, \tilde{w}_1)}{\eta_1} + \sum_{t=1}^T \frac{\eta_t}{2} \|g_t - \tilde{m}_t\|_{\psi_{t-1}^*}^2 + D_\infty^2 \beta_1^2 \sum_{t=1}^T \|g_t - \theta_{t-1}\|_{\psi_{t-1}^*} ,$$

since  $\|g_t - \tilde{g}_t\|_{\psi_{t-1}^*} = \|g_t - \beta_1 \theta_{t-1} - (1 - \beta_1) g_t\|_{\psi_{t-1}^*} = \beta^2 \|g_t - \theta_{t-1}\|_{\psi_{t-1}^*}$ . This completes the proof.

□

## B Proof of Corollary 1

**Corollary.** Suppose  $\beta_1 = 0$  and  $\{v_t\}_{t>0}$  is a monotonically increasing sequence, then we obtain the following regret bound for any  $w^* \in \Theta$  and sequence of stepsizes  $\{\eta_t = \eta/\sqrt{t}\}_{t>0}$ :

$$\mathcal{R}_T \leq \frac{B_{\psi_1}}{\eta_1} + \frac{\eta \sqrt{1 + \log T}}{\sqrt{1 - \beta_2}} \sum_{i=1}^d \|(g - m)_{1:T}[i]\|_2 + \frac{D_\infty^2}{\eta_{\min}} \sum_{i=1}^d \left[ (1 - \beta_2) \sum_{s=1}^T \beta_2^{T-s} g_s^2[i] \right]^{1/2} ,$$

where  $B_{\psi_1} := B_{\psi_1}(w^*, \tilde{w}_1)$ ,  $g_t := \nabla \ell_t(w_t)$  and  $\eta_{\min} := \min_t \eta_t$ .

465 **Proof** Recall the bound in Theorem 1:

$$\mathcal{R}_T \leq \frac{B_{\psi_1}(w^*, \tilde{w}_1)}{\eta_1} + \sum_{t=1}^T \frac{\eta_t}{2} \|g_t - \tilde{m}_t\|_{\psi_{t-1}^*}^2 + \frac{D_\infty^2}{\eta_{\min}} \sum_{i=1}^d \hat{v}_T^{1/2}[i] + D_\infty^2 \beta_1^2 \sum_{t=1}^T \|g_t - \theta_{t-1}\|_{\psi_{t-1}^*}.$$

466 The second term reads:

$$\begin{aligned} & \sum_{t=1}^T \frac{\eta_t}{2} \|g_t - m_t\|_{\psi_{t-1}^*}^2 \\ &= \sum_{t=1}^{T-1} \frac{\eta_t}{2} \|g_t - m_t\|_{\psi_{t-1}^*}^2 + \eta_T \sum_{i=1}^d \frac{(g_T[i] - m_T[i])^2}{\sqrt{v_{T-1}[i]}} \\ &= \sum_{t=1}^{T-1} \frac{\eta_t}{2} \|g_t - m_t\|_{\psi_{t-1}^*}^2 + \eta \sum_{i=1}^d \frac{(g_T[i] - m_T[i])^2}{\sqrt{T((1-\beta_2) \sum_{s=1}^{T-1} \beta_2^{T-1-s} (g_s[i] - m_s[i])^2)}} \\ &\leq \eta \sum_{i=1}^d \sum_{t=1}^T \frac{(g_t[i] - m_t[i])^2}{\sqrt{t((1-\beta_2) \sum_{s=1}^{t-1} \beta_2^{t-1-s} (g_s[i] - m_s[i])^2)}}. \end{aligned}$$

467 To interpret the bound, let us make a rough approximation such that  $\sum_{s=1}^{t-1} \beta_2^{t-1-s} (g_s[i] - m_s[i])^2 \simeq$   
 468  $(g_t[i] - m_t[i])^2$ . Then, we can further get an upper-bound as

$$\sum_{t=1}^T \frac{\eta_t}{2} \|g_t - m_t\|_{\psi_{t-1}^*}^2 \leq \frac{\eta}{\sqrt{1-\beta_2}} \sum_{i=1}^d \sum_{t=1}^T \frac{|g_t[i] - m_t[i]|}{\sqrt{t}} \leq \frac{\eta \sqrt{1+\log T}}{\sqrt{1-\beta_2}} \sum_{i=1}^d \|(g - m)_{1:T}[i]\|_2,$$

469 where the last inequality is due to Cauchy-Schwarz.

470 □

## 471 C Proofs of Auxiliary Lemmas

472 Following [38] and their study of the SGD with Momentum we denote for any  $t > 0$ :

$$\bar{w}_t = w_t + \frac{\beta_1}{1-\beta_1} (w_t - \tilde{w}_{t-1}) = \frac{1}{1-\beta_1} w_t - \frac{\beta_1}{1-\beta_1} \tilde{w}_{t-1}. \quad (18)$$

473 **Lemma 3.** Assume a strictly positive and non increasing sequence of stepsizes  $\{\eta_t\}_{t>0}$ ,  $\beta_1 < \beta_2 \in$   
 474  $[0, 1)$ , then the following holds:

$$\bar{w}_{t+1} - \bar{w}_t \leq \frac{\beta_1}{1-\beta_1} \tilde{\theta}_{t-1} \left[ \eta_{t-1} \hat{v}_{t-1}^{-1/2} - \eta_t \hat{v}_t^{-1/2} \right] - \eta_t \hat{v}_t^{-1/2} \tilde{g}_t,$$

475 where  $\tilde{\theta}_t = \theta_t + \beta_1 \theta_{t-1}$  and  $\tilde{g}_t = g_t - \beta_1 m_t + \beta_1 g_{t-1} + m_{t+1}$ .

476 **Proof** By definition (18) and using the Algorithm updates, we have:

$$\begin{aligned} \bar{w}_{t+1} - \bar{w}_t &= \frac{1}{1-\beta_1} (w_{t+1} - \tilde{w}_t) - \frac{\beta_1}{1-\beta_1} (w_t - \tilde{w}_{t-1}) \\ &= -\frac{1}{1-\beta_1} \eta_t \hat{v}_t^{-1/2} (\theta_t + h_{t+1}) + \frac{\beta_1}{1-\beta_1} \eta_{t-1} \hat{v}_{t-1}^{-1/2} (\theta_{t-1} + h_t) \\ &= -\frac{1}{1-\beta_1} \eta_t \hat{v}_t^{-1/2} (\theta_t + \beta_1 \theta_{t-1}) - \frac{1}{1-\beta_1} \eta_t \hat{v}_t^{-1/2} (1-\beta_1) m_{t+1} \\ &\quad + \frac{\beta_1}{1-\beta_1} \eta_{t-1} \hat{v}_{t-1}^{-1/2} (\theta_{t-1} + \beta_1 \theta_{t-2}) + \frac{\beta_1}{1-\beta_1} \eta_{t-1} \hat{v}_{t-1}^{-1/2} (1-\beta_1) m_t. \end{aligned} \quad (19)$$

477 Denote  $\tilde{\theta}_t = \theta_t + \beta_1 \theta_{t-1}$  and  $\tilde{g}_t = g_t - \beta_1 m_t + \beta_1 g_{t-1} + m_{t+1}$ . Notice that  $\tilde{\theta}_t = \beta_1 \tilde{\theta}_{t-1} + (1 -$   
 478  $\beta_1)(g_t + \beta_1 g_{t-1})$ .

$$\bar{w}_{t+1} - \bar{w}_t \leq \frac{\beta_1}{1 - \beta_1} \tilde{\theta}_{t-1} \left[ \eta_{t-1} \hat{v}_{t-1}^{-1/2} - \eta_t \hat{v}_t^{-1/2} \right] - \eta_t \hat{v}_t^{-1/2} \tilde{g}_t. \quad (20)$$

479 □

480 **Lemma 4.** Assume H4, a strictly positive and a sequence of constant stepsizes  $\{\eta_t\}_{t>0}$ ,  $(\beta_1, \beta_2) \in$   
 481  $[0, 1]$ , then the following holds:

$$\sum_{t=1}^{T_M} \eta_t^2 \mathbb{E} \left[ \left\| \hat{v}_t^{-1/2} \theta_t \right\|_2^2 \right] \leq \frac{\eta^2 d T_M (1 - \beta_1)}{(1 - \beta_2)(1 - \gamma)}. \quad (21)$$

482 **Proof** We denote by index  $p \in [1, d]$  the dimension of each component of vectors of interest. Noting  
 483 that for any  $t > 0$  and dimension  $p$  we have  $\hat{v}_{t,p} \geq v_{t,p}$ , then:

$$\begin{aligned} \eta_t^2 \mathbb{E} \left[ \left\| \hat{v}_t^{-1/2} \theta_t \right\|_2^2 \right] &= \eta_t^2 \mathbb{E} \left[ \sum_{p=1}^d \frac{\theta_{t,p}^2}{\hat{v}_{t,p}} \right] \\ &\leq \eta_t^2 \mathbb{E} \left[ \sum_{i=1}^d \frac{\theta_{t,p}^2}{v_{t,p}} \right] \\ &\leq \eta_t^2 \mathbb{E} \left[ \sum_{i=1}^d \frac{(\sum_{r=1}^t (1 - \beta_1) \beta_1^{t-r} g_{r,p})^2}{\sum_{r=1}^t (1 - \beta_2) \beta_2^{t-r} g_{r,p}^2} \right], \end{aligned} \quad (22)$$

484 where the last inequality is due to initializations. Denote  $\gamma = \frac{\beta_1}{\beta_2}$ . Then,

$$\begin{aligned} \eta_t^2 \mathbb{E} \left[ \left\| \hat{v}_t^{-1/2} \theta_t \right\|_2^2 \right] &\leq \frac{\eta_t^2 (1 - \beta_1)^2}{1 - \beta_2} \mathbb{E} \left[ \sum_{i=1}^d \frac{(\sum_{r=1}^t \beta_1^{t-r} g_{r,p})^2}{\sum_{r=1}^t \beta_2^{t-r} g_{r,p}^2} \right] \\ &\stackrel{(a)}{\leq} \frac{\eta_t^2 (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[ \sum_{i=1}^d \frac{\sum_{r=1}^t \beta_1^{t-r} g_{r,p}^2}{\sum_{r=1}^t \beta_2^{t-r} g_{r,p}^2} \right] \\ &\leq \frac{\eta_t^2 (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[ \sum_{i=1}^d \sum_{r=1}^t \gamma^{t-r} \right] = \frac{\eta_t^2 d (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[ \sum_{r=1}^t \gamma^{t-r} \right], \end{aligned} \quad (23)$$

485 where (a) is due to  $\sum_{r=1}^t \beta_1^{t-r} \leq \frac{1}{1 - \beta_1}$ . Summing from  $t = 1$  to  $t = T_M$  on both sides yields:

$$\begin{aligned} \sum_{t=1}^{T_M} \eta_t^2 \mathbb{E} \left[ \left\| \hat{v}_t^{-1/2} \theta_t \right\|_2^2 \right] &\leq \frac{\eta^2 d (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[ \sum_{t=1}^{T_M} \sum_{r=1}^t \gamma^{t-r} \right] \\ &\leq \frac{\eta^2 d T (1 - \beta_1)}{1 - \beta_2} \mathbb{E} \left[ \sum_{t=t}^t \gamma^{t-r} \right] \\ &\leq \frac{\eta^2 d T (1 - \beta_1)}{(1 - \beta_2)(1 - \gamma)}, \end{aligned} \quad (24)$$

486 where the last inequality is due to  $\sum_{r=1}^t \gamma^{t-r} \leq \frac{1}{1 - \gamma}$  by definition of  $\gamma$ . □

### 487 C.1 Proof of Lemma 1

**Lemma.** Assume assumption H4, then the quantities defined in Algorithm 2 satisfy for any  $w \in \Theta$   
 and  $t > 0$ :

$$\|\nabla f(w_t)\| < M, \quad \|\theta_t\| < M, \quad \|\hat{v}_t\| < M^2.$$

**Proof** Assume assumption H4 we have:

$$\|\nabla f(w)\| = \|\mathbb{E}[\nabla f(w, \xi)]\| \leq \mathbb{E}[\|\nabla f(w, \xi)\|] \leq M.$$

488 By induction reasoning, since  $\|\theta_0\| = 0 \leq M$  and suppose that for  $\|\theta_t\| \leq M$  then we have

$$\|\theta_{t+1}\| = \|\beta_1 \theta_t + (1 - \beta_1) g_{t+1}\| \leq \beta_1 \|\theta_t\| + (1 - \beta_1) \|g_{t+1}\| \leq M. \quad (25)$$

489 Using the same induction reasoning we prove that

$$\|\hat{v}_{t+1}\| = \|\beta_2 \hat{v}_t + (1 - \beta_2) g_{t+1}^2\| \leq \beta_2 \|\hat{v}_t\| + (1 - \beta_2) \|g_{t+1}^2\| \leq M^2. \quad (26)$$

490

□

## 491 D Proof of Theorem 2

492 **Theorem.** Assume H1-H4,  $\beta_1 < \beta_2 \in [0, 1)$  and a sequence of decreasing stepsizes  $\{\eta_t\}_{t>0}$ , then  
493 the following result holds:

$$\mathbb{E} [\|\nabla f(w_T)\|_2^2] \leq \tilde{C}_1 \sqrt{\frac{d}{T_M}} + \tilde{C}_2 \frac{1}{T_M},$$

494 where  $T$  is a random termination number distributed according (4). The constants are defined as:

$$\begin{aligned} \tilde{C}_1 &= \frac{M}{(1 - a_m \beta_1) + (\beta_1 + a_m)} \left[ \frac{a_m(1 - \beta_1)^2}{1 - \beta_2} + 2L \frac{1}{1 - \beta_2} + \Delta f + \frac{4L\beta_1^2(1 + \beta_1^2)}{(1 - \beta_1)(1 - \beta_2)(1 - \gamma)} \right], \\ \tilde{C}_2 &= \frac{(a_m \beta_1^2 - 2a_m \beta_1 + \beta_1)M^2}{(1 - \beta_1)((1 - a_m \beta_1) + (\beta_1 + a_m))} \mathbb{E} [\|\hat{v}_0^{-1/2}\|], \end{aligned}$$

495 where  $\Delta f = f(\bar{w}_1) - f(\bar{w}_{T_M+1})$  and  $a_m = \min_{t=1, \dots, T} a_t$ .

496 **Proof** Using H2 and the iterate  $\bar{w}_t$  we have:

$$\begin{aligned} f(\bar{w}_{t+1}) &\leq f(\bar{w}_t) + \nabla f(\bar{w}_t)^\top (\bar{w}_{t+1} - \bar{w}_t) + \frac{L}{2} \|\bar{w}_{t+1} - \bar{w}_t\|^2 \\ &\leq f(\bar{w}_t) + \underbrace{\nabla f(w_t)^\top (\bar{w}_{t+1} - \bar{w}_t)}_A \\ &\quad + \underbrace{(\nabla f(\bar{w}_t) - \nabla f(w_t))^\top (\bar{w}_{t+1} - \bar{w}_t)}_B + \frac{L}{2} \|\bar{w}_{t+1} - \bar{w}_t\|. \end{aligned} \quad (27)$$

497 **Term A.** Using Lemma 3, we have that:

$$\begin{aligned} \nabla f(w_t)^\top (\bar{w}_{t+1} - \bar{w}_t) &\leq \nabla f(w_t)^\top \left[ \frac{\beta_1}{1 - \beta_1} \tilde{\theta}_{t-1} [\eta_{t-1} \hat{v}_{t-1}^{-1/2} - \eta_t \hat{v}_t^{-1/2}] - \eta_t \hat{v}_t^{-1/2} \tilde{g}_t \right] \\ &\leq \frac{\beta_1}{1 - \beta_1} \|\nabla f(w_t)\| \|\eta_{t-1} \hat{v}_{t-1}^{-1/2} - \eta_t \hat{v}_t^{-1/2}\| \|\tilde{\theta}_{t-1}\| - \nabla f(w_t)^\top \eta_t \hat{v}_t^{-1/2} \tilde{g}_t, \end{aligned}$$

498 where the inequality is due to trivial inequality for positive diagonal matrix. Using Lemma 1 and  
499 assumption H3 we obtain:

$$\nabla f(w_t)^\top (\bar{w}_{t+1} - \bar{w}_t) \leq \frac{\beta_1(1 + \beta_1)}{1 - \beta_1} M^2 [\|\eta_{t-1} \hat{v}_{t-1}^{-1/2}\| - \|\eta_t \hat{v}_t^{-1/2}\|] - \nabla f(w_t)^\top \eta_t \hat{v}_t^{-1/2} \tilde{g}_t, \quad (28)$$

500 where we have used the fact that  $\eta_t \hat{v}_t^{-1/2}$  is a diagonal matrix such that  $\eta_{t-1} \hat{v}_{t-1}^{-1/2} \succcurlyeq \eta_t \hat{v}_t^{-1/2} \succcurlyeq 0$   
501 (decreasing stepsize and max operator). Also note that:

$$\begin{aligned} -\nabla f(w_t)^\top \eta_t \hat{v}_t^{-1/2} \tilde{g}_t &= -\nabla f(w_t)^\top \eta_{t-1} \hat{v}_{t-1}^{-1/2} \tilde{g}_t - \nabla f(w_t)^\top [\eta_t \hat{v}_t^{-1/2} - \eta_{t-1} \hat{v}_{t-1}^{-1/2}] \tilde{g}_t \\ &\quad - \nabla f(w_t)^\top \eta_{t-1} \hat{v}_{t-1}^{-1/2} (\beta_1 g_{t-1} + m_{t+1}) \\ &\leq -\nabla f(w_t)^\top \eta_{t-1} \hat{v}_{t-1}^{-1/2} \tilde{g}_t + (1 - a_t \beta_1) M^2 [\|\eta_{t-1} \hat{v}_{t-1}^{-1/2}\| - \|\eta_t \hat{v}_t^{-1/2}\|] \\ &\quad - \nabla f(w_t)^\top \eta_t \hat{v}_t^{-1/2} (\beta_1 g_{t-1} + m_{t+1}), \end{aligned} \quad (29)$$

where we have used Lemma 1 on  $\|g_t\|$  and where that  $\tilde{g}_t = \bar{g}_t + \beta_1 g_{t-1} + m_{t+1} = g_t - \beta_1 m_t + \beta_1 g_{t-1} + m_{t+1}$ . Plugging (29) into (28) yields:

$$\begin{aligned} & \nabla f(w_t)^\top (\bar{w}_{t+1} - \bar{w}_t) \\ & \leq -\nabla f(w_t)^\top \eta_{t-1} \hat{v}_{t-1}^{-1/2} \bar{g}_t + \frac{1}{1-\beta_1} (a_t \beta_1^2 - 2a_t \beta_1 + \beta_1) \mathbf{M}^2 [\|\eta_{t-1} \hat{v}_{t-1}^{-1/2}\| - \|\eta_t \hat{v}_t^{-1/2}\|] \\ & \quad - \nabla f(w_t)^\top \eta_t \hat{v}_t^{-1/2} (\beta_1 g_{t-1} + m_{t+1}). \end{aligned} \quad (30)$$

**Term B.** By Cauchy-Schwarz (CS) inequality we have:

$$(\nabla f(\bar{w}_t) - \nabla f(w_t))^\top (\bar{w}_{t+1} - \bar{w}_t) \leq \|\nabla f(\bar{w}_t) - \nabla f(w_t)\| \|\bar{w}_{t+1} - \bar{w}_t\|. \quad (31)$$

Using smoothness assumption H2:

$$\begin{aligned} \|\nabla f(\bar{w}_t) - \nabla f(w_t)\| & \leq L \|\bar{w}_t - w_t\| \\ & \leq L \frac{\beta_1}{1-\beta_1} \|w_t - \tilde{w}_{t-1}\|. \end{aligned} \quad (32)$$

By Lemma 3 we also have:

$$\begin{aligned} \bar{w}_{t+1} - \bar{w}_t & = \frac{\beta_1}{1-\beta_1} \tilde{\theta}_{t-1} \left[ \eta_{t-1} \hat{v}_{t-1}^{-1/2} - \eta_t \hat{v}_t^{-1/2} \right] - \eta_t \hat{v}_t^{-1/2} \tilde{g}_t \\ & = \frac{\beta_1}{1-\beta_1} \tilde{\theta}_{t-1} \eta_{t-1} \hat{v}_{t-1}^{-1/2} \left[ I - (\eta_t \hat{v}_t^{-1/2})(\eta_{t-1} \hat{v}_{t-1}^{-1/2})^{-1} \right] - \eta_t \hat{v}_t^{-1/2} \tilde{g}_t \\ & = \frac{\beta_1}{1-\beta_1} \left[ I - (\eta_t \hat{v}_t^{-1/2})(\eta_{t-1} \hat{v}_{t-1}^{-1/2})^{-1} \right] (\tilde{w}_{t-1} - w_t) - \eta_t \hat{v}_t^{-1/2} \tilde{g}_t, \end{aligned} \quad (33)$$

where the last equality is due to  $\tilde{\theta}_{t-1} \eta_{t-1} \hat{v}_{t-1}^{-1/2} = \tilde{w}_{t-1} - w_t$  by construction of  $\tilde{\theta}_t$ . Taking the norms on both sides, observing  $\|I - (\eta_t \hat{v}_t^{-1/2})(\eta_{t-1} \hat{v}_{t-1}^{-1/2})^{-1}\| \leq 1$  due to the decreasing stepsize and the construction of  $\hat{v}_t$  and using CS inequality yield:

$$\|\bar{w}_{t+1} - \bar{w}_t\| \leq \frac{\beta_1}{1-\beta_1} \|\tilde{w}_{t-1} - w_t\| + \|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\|. \quad (34)$$

We recall Young's inequality with a constant  $\delta \in (0, 1)$  as follows:

$$\langle X | Y \rangle \leq \frac{1}{\delta} \|X\|^2 + \delta \|Y\|^2.$$

Plugging (32) and (34) into (31) returns:

$$\begin{aligned} (\nabla f(\bar{w}_t) - \nabla f(w_t))^\top (\bar{w}_{t+1} - \bar{w}_t) & \leq L \frac{\beta_1}{1-\beta_1} \|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\| \|w_t - \tilde{w}_{t-1}\| \\ & \quad + L \left( \frac{\beta_1}{1-\beta_1} \right)^2 \|\tilde{w}_{t-1} - w_t\|^2. \end{aligned}$$

Applying Young's inequality with  $\delta \rightarrow \frac{\beta_1}{1-\beta_1}$  on the product  $\|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\| \|w_t - \tilde{w}_{t-1}\|$  yields:

$$(\nabla f(\bar{w}_t) - \nabla f(w_t))^\top (\bar{w}_{t+1} - \bar{w}_t) \leq L \|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\|^2 + 2L \left( \frac{\beta_1}{1-\beta_1} \right)^2 \|\tilde{w}_{t-1} - w_t\|^2. \quad (35)$$

The last term  $\frac{L}{2} \|\bar{w}_{t+1} - \bar{w}_t\|^2$  can be upper bounded using (34):

$$\begin{aligned} \frac{L}{2} \|\bar{w}_{t+1} - \bar{w}_t\|^2 & \leq \frac{L}{2} \left[ \frac{\beta_1}{1-\beta_1} \|\tilde{w}_{t-1} - w_t\| + \|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\| \right]^2 \\ & \leq L \|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\|^2 + 2L \left( \frac{\beta_1}{1-\beta_1} \right)^2 \|\tilde{w}_{t-1} - w_t\|^2. \end{aligned} \quad (36)$$



513 Plugging (30), (35) and (36) into (27) and taking the expectations on both sides give:

$$\begin{aligned}
& \mathbb{E} \left[ f(\bar{w}_{t+1}) + \frac{1}{1-\beta_1} \tilde{M}_t^2 \|\eta_t \hat{v}_t^{-1/2}\| - \left( f(\bar{w}_t) + \frac{1}{1-\beta_1} \tilde{M}_t^2 \|\eta_{t-1} \hat{v}_{t-1}^{-1/2}\| \right) \right] \\
& \leq \mathbb{E} \left[ -\nabla f(w_t)^\top \eta_{t-1} \hat{v}_{t-1}^{-1/2} \tilde{g}_t - \nabla f(w_t)^\top \eta_t \hat{v}_t^{-1/2} (\beta_1 g_{t-1} + m_{t+1}) \right] \\
& + \mathbb{E} \left[ 2L \|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\|^2 + 4L \left( \frac{\beta_1}{1-\beta_1} \right)^2 \|\tilde{w}_{t-1} - w_t\|^2 \right],
\end{aligned}$$

514 where  $\tilde{M}_t^2 = (a_t \beta_1^2 + \beta_1) M^2$ . Note that the expectation of  $\tilde{g}_t$  conditioned on the filtration  $\mathcal{F}_t$  reads  
515 as follows

$$\mathbb{E} [\nabla f(w_t)^\top \tilde{g}_t] = \mathbb{E} [\nabla f(w_t)^\top (g_t - \beta_1 m_t)] = (1 - a_t \beta_1) \|\nabla f(w_t)\|^2. \quad (37)$$

516 Summing from  $t = 1$  to  $t = T$  leads to

$$\begin{aligned}
& \frac{1}{M} \sum_{t=1}^{T_M} ((1 - a_t \beta_1) \eta_{t-1} + (\beta_1 + a_t) \eta_t) \|\nabla f(w_t)\|^2 \leq \\
& \mathbb{E} \left[ f(\bar{w}_1) + \frac{1}{1-\beta_1} \tilde{M}_t^2 \|\eta_0 \hat{v}_0^{-1/2}\| - \left( f(\bar{w}_{T_M+1}) + \frac{1}{1-\beta_1} \tilde{M}_t^2 \|\eta_{T_M} \hat{v}_{T_M}^{-1/2}\| \right) \right] \\
& + 2L \sum_{t=1}^{T_M} \mathbb{E} [\|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\|^2] + 4L \left( \frac{\beta_1}{1-\beta_1} \right)^2 \sum_{t=1}^{T_M} \mathbb{E} [\|\tilde{w}_{t-1} - w_t\|^2] \quad (38) \\
& \leq \mathbb{E} \left[ \Delta f + \frac{1}{1-\beta_1} \tilde{M}_t^2 \|\eta_0 \hat{v}_0^{-1/2}\| \right] + 2L \sum_{t=1}^{T_M} \mathbb{E} [\|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\|^2] \\
& + 4L \left( \frac{\beta_1}{1-\beta_1} \right)^2 \sum_{t=1}^{T_M} \mathbb{E} [\|\tilde{w}_{t-1} - w_t\|^2],
\end{aligned}$$

517 where we denote  $\Delta f := f(\bar{w}_1) - f(\bar{w}_{T_M+1})$ . We note that by definition of  $\hat{v}_t$ , and a constant  
518 learning rate  $\eta_t$ , we have

$$\begin{aligned}
\|\tilde{w}_{t-1} - w_t\|^2 &= \|\eta_{t-1} \hat{v}_{t-1}^{-1/2} (\theta_{t-1} + h_t)\|^2 \\
&= \|\eta_{t-1} \hat{v}_{t-1}^{-1/2} (\theta_{t-1} + \beta_1 \theta_{t-2} + (1 - \beta_1) m_t)\|^2 \\
&\leq \|\eta_{t-1} \hat{v}_{t-1}^{-1/2} \theta_{t-1}\|^2 + \|\eta_{t-2} \hat{v}_{t-2}^{-1/2} \beta_1 \theta_{t-2}\|^2 + (1 - \beta_1)^2 \|\eta_{t-1} \hat{v}_{t-1}^{-1/2} m_t\|^2.
\end{aligned}$$

519 Using Lemma 4 we have

$$\begin{aligned}
& \sum_{t=1}^{T_M} \mathbb{E} [\|\tilde{w}_{t-1} - w_t\|^2] \\
& \leq (1 + \beta_1^2) \frac{\eta^2 d T_M (1 - \beta_1)}{(1 - \beta_2)(1 - \gamma)} + (1 - \beta_1)^2 \sum_{t=1}^{T_M} \mathbb{E} [\|\eta_{t-1} \hat{v}_{t-1}^{-1/2} m_t\|].
\end{aligned}$$

520 Assume  $a_m = \min_{1, \dots, T_M} a_t$  and denote  $\tilde{M}_m^2 = (a_m \beta_1^2 + \beta_1) M^2$ . Setting a constant learning rate  
 521  $\eta_t = \eta$  and plugging in (38) yields:

$$\begin{aligned} \mathbb{E}[\|\nabla f(w_T)\|^2] &= \frac{1}{\sum_{j=1}^{T_M} \eta_j} \sum_{t=1}^{T_M} \eta_t \|\nabla f(w_t)\|^2 = \frac{\sum_{t=1}^{T_M} \|\nabla f(w_t)\|^2}{T_M} \\ &\leq \frac{M}{T_M \eta ((1 - a_m \beta_1) + (\beta_1 + a_m))} \mathbb{E} \left[ \Delta f + \frac{1}{1 - \beta_1} \tilde{M}_m^2 \|\eta_0 \hat{v}_0^{-1/2}\| \right] \\ &\quad + \frac{4L \left( \frac{\beta_1}{1 - \beta_1} \right)^2 M}{T_M \eta ((1 - a_m \beta_1) + (\beta_1 + a_m))} (1 + \beta_1^2) \frac{\eta^2 d T_M (1 - \beta_1)}{(1 - \beta_2)(1 - \gamma)} \\ &\quad + \frac{M}{T_M \eta ((1 - a_m \beta_1) + (\beta_1 + a_m))} (1 - \beta_1)^2 \sum_{t=1}^{T_M} \mathbb{E}[\|\eta_{t-1} \hat{v}_{t-1}^{-1/2} m_t\|] \\ &\quad + \frac{2LM}{T_M \eta ((1 - a_m \beta_1) + (\beta_1 + a_m))} \sum_{t=1}^{T_M} \mathbb{E}[\|\eta_t \hat{v}_t^{-1/2} \tilde{g}_t\|^2], \end{aligned}$$

522 where  $T$  is a random termination number distributed according (4) and  $T_M$  is the maximum number  
 523 of iteration. Setting the stepsize to  $\eta = \frac{1}{\sqrt{dT_M}}$  yields :

$$\mathbb{E}[\|\nabla f(w_T)\|^2] \leq C_{1,m} \sqrt{\frac{d}{T_M}} + C_{2,m} \frac{1}{T_M} + \frac{\eta}{T_M} D_{1,m} \mathbb{E}[\|\hat{v}_{t-1}^{-1/2} m_t\|] + \frac{\eta}{T_M} D_{2,m} \mathbb{E}[\|\hat{v}_{t-1}^{-1/2} \tilde{g}_t\|],$$

524 where

$$\begin{aligned} C_{1,m} &= \frac{M}{(1 - a_m \beta_1) + (\beta_1 + a_m)} \Delta f + \frac{4L \left( \frac{\beta_1}{1 - \beta_1} \right)^2 M}{(1 - a_m \beta_1) + (\beta_1 + a_m)} \frac{(1 + \beta_1^2)(1 - \beta_1)}{(1 - \beta_2)(1 - \gamma)}, \\ C_{2,m} &= \frac{M}{(1 - \beta_1)((1 - a_m \beta_1) + (\beta_1 + a_m))} (a_m \beta_1^2 + \beta_1) M^2 \mathbb{E}[\|\hat{v}_0^{-1/2}\|]. \end{aligned}$$

525 **Simple case as in [41]:** if  $\beta_1 = 0$  then  $\tilde{g}_t = g_t + m_{t+1}$  and  $g_t = \theta_t$ . Also using Lemma 4 we have  
 526 that:

$$\sum_{t=1}^{T_M} \eta_t^2 \mathbb{E} \left[ \left\| \hat{v}_t^{-1/2} g_t \right\|_2^2 \right] \leq \frac{\eta^2 d T_M}{(1 - \beta_2)};$$

527 which leads to the final bound:

$$\mathbb{E}[\|\nabla f(w_T)\|^2] \leq \sqrt{\frac{d}{T_M}} \tilde{C}_{1,m} + \frac{1}{T_M} \tilde{C}_{2,m},$$

528 where

$$\begin{aligned} \tilde{C}_{1,m} &= C_{1,m} + \frac{M}{(1 - a_m \beta_1) + (\beta_1 + a_m)} \left[ \frac{a_m (1 - \beta_1)^2}{1 - \beta_2} + 2L \frac{1}{1 - \beta_2} \right], \\ \tilde{C}_{2,m} &= C_{2,m} = \frac{M}{(1 - \beta_1)((1 - a_m \beta_1) + (\beta_1 + a_m))} \tilde{M}_m^2 \mathbb{E}[\|\hat{v}_0^{-1/2}\|]. \end{aligned}$$

529

□

## 530 E Proof of Lemma 2 (Boundedness of the iterates H1)

531 **Lemma.** Given the multilayer model (5), assume the boundedness of the input data and of the loss  
 532 function, i.e., for any  $\xi \in \mathbb{R}^p$  and  $y \in \mathbb{R}$  there is a constant  $T > 0$  such that:

$$\|\xi\| \leq 1 \quad \text{a.s.} \quad \text{and} \quad |\mathcal{L}'(\cdot, y)| \leq T, \quad (39)$$

where  $\mathcal{L}'(\cdot, y)$  denotes its derivative w.r.t. the parameter. Then for each layer  $\ell \in [1, L]$ , there exist  
 a constant  $A_{(\ell)}$  such that:

$$\|w^{(\ell)}\| \leq A_{(\ell)}.$$

**Proof** For any index  $\ell \in [1, L]$  we denote the output of layer  $\ell$  by

$$h^{(\ell)}(w, \xi) = \sigma \left( w^{(\ell)} \sigma \left( w^{(\ell-1)} \dots \sigma \left( w^{(1)} \xi \right) \right) \right) .$$

533 Given the sigmoid assumption we have  $\|h^{(\ell)}(w, \xi)\| \leq 1$  for any  $\ell \in [1, L]$  and any  $(w, \xi) \in$   
 534  $\mathbb{R}^d \times \mathbb{R}^p$ . We also recall that  $\mathcal{L}(\cdot, y)$  is the loss function, which can be Huber loss or cross entropy.  
 535 Observe that at the last layer  $L$ :

$$\begin{aligned} \|\nabla_{w^{(L)}} \mathcal{L}(\text{MLN}(w, \xi), y)\| &= \|\mathcal{L}'(\text{MLN}(w, \xi), y) \nabla_{w^{(L)}} \text{MLN}(w, \xi)\| \\ &= \|\mathcal{L}'(\text{MLN}(w, \xi), y) \sigma'(w^{(L)} h^{(L-1)}(w, \xi)) h^{(L-1)}(w, \xi)\| \\ &\leq \frac{T}{4} , \end{aligned} \quad (40)$$

536 where the last equality is due to mild assumptions (39) and to the fact that the norm of the derivative  
 537 of the sigmoid function is upperbounded by  $1/4$ .

538 From Algorithm 2, and with  $\beta_1 = 0$  for the sake of notation, we have for iteration index  $t > 0$ :

$$\begin{aligned} \|w_t - \tilde{w}_{t-1}\| &= \|\eta_t \hat{v}_t^{-1/2} (\theta_t + h_{t+1})\| = \|\eta_t \hat{v}_t^{-1/2} (g_t + m_{t+1})\| \\ &\leq \hat{\eta} \|\hat{v}_t^{-1/2} g_t\| + \hat{\eta} a \|\hat{v}_t^{-1/2} g_{t+1}\| , \end{aligned}$$

where  $\hat{\eta} = \max_{t \geq 0} \eta_t$ . For any dimension  $p \in [1, d]$ , using assumption H3, we note that

$$\sqrt{\hat{v}_{t,p}} \geq \sqrt{1 - \beta_2} g_{t,p} \quad \text{and} \quad m_{t+1} \leq a \|g_{t+1}\| .$$

539 Thus:

$$\|w_t - \tilde{w}_{t-1}\| \leq \hat{\eta} \left( \|\hat{v}_t^{-1/2} g_t\| + a \|\hat{v}_t^{-1/2} g_{t+1}\| \right) \leq \hat{\eta} \frac{a+1}{\sqrt{1-\beta_2}} .$$

540 In short there exist a constant  $B$  such that  $\|w_t - \tilde{w}_{t-1}\| \leq B$ .

**Proof by induction:** As in [9], we will prove the containment of the weights by induction. Suppose an iteration index  $T$  and a coordinate  $i$  of the last layer  $L$  such that  $w_{T,i}^{(L)} \geq \frac{T}{4\lambda} + B$ . Using (40), we have

$$\nabla_i f(w_t^{(L)}, \xi) \geq -\frac{T}{4} + \lambda \frac{T}{\lambda 4} \geq 0 ,$$

541 where  $f(w, \xi) = \mathcal{L}(\text{MLN}(w, \xi), y) + \frac{\lambda}{2} \|w\|^2$  and is the loss of our MLN. This last equation yields  
 542  $\theta_{T,i}^{(L)} \geq 0$  (given the algorithm and  $\beta_1 = 0$ ) and using the fact that  $\|w_t - \tilde{w}_{t-1}\| \leq B$  we have

$$0 \leq w_{T-1,i}^{(L)} - B \leq w_{T,i}^{(L)} \leq w_{T-1,i}^{(L)} , \quad (41)$$

which means that  $|w_{T,i}^{(L)}| \leq w_{T-1,i}^{(L)}$ . So if the first assumption of that induction reasoning holds, i.e.,  $w_{T-1,i}^{(L)} \geq \frac{T}{4\lambda} + B$ , then the next iterates  $w_{T,i}^{(L)}$  decreases, see (41) and go below  $\frac{T}{4\lambda} + B$ . This yields that for any iteration index  $t > 0$  we have

$$w_{T,i}^{(L)} \leq \frac{T}{4\lambda} + 2B ,$$

since  $B$  is the biggest jump an iterate can do since  $\|w_t - \tilde{w}_{t-1}\| \leq B$ . Likewise we can end up showing that

$$|w_{T,i}^{(L)}| \leq \frac{T}{4\lambda} + 2B ,$$

543 meaning that the weights of the last layer at any iteration is bounded in some matrix norm.

544 Now that we have shown this boundedness property for the last layer  $L$ , we will do the same for the  
 545 previous layers and conclude the verification of assumption H1 by induction.

546 For any layer  $\ell \in [1, L-1]$ , we have:

$$\nabla_{w^{(\ell)}} \mathcal{L}(\text{MLN}(w, \xi), y) = \mathcal{L}'(\text{MLN}(w, \xi), y) \left( \prod_{j=1}^{\ell+1} \sigma' \left( w^{(j)} h^{(j-1)}(w, \xi) \right) \right) h^{(\ell-1)}(w, \xi) . \quad (42)$$

This last quantity is bounded as long as we can prove that for any layer  $\ell$  the weights  $w^{(\ell)}$  are bounded in some matrix norm as  $\|w^{(\ell)}\|_F \leq F_\ell$  with the Frobenius norm. Suppose we have shown  $\|w^{(r)}\|_F \leq F_r$  for any layer  $r > \ell$ . Then having this gradient (42) bounded we can use the same lines of proof for the last layer  $L$  and show that the norm of the weights at the selected layer  $\ell$  satisfy

$$\|w^{(\ell)}\| \leq \frac{T \prod_{t>\ell} F_t}{4^{L-\ell+1}} + 2B.$$

547 Showing that the weights of the previous layers  $\ell \in [1, L-1]$  as well as for the last layer  $L$  of our  
 548 fully connected feed forward neural network are bounded at each iteration, leads by induction, to  
 549 the boundedness (at each iteration) assumption we want to check, thus proving Lemma 2.  $\square$

## 550 F Comparison to related methods

551 **Comparison to nonconvex optimization methods.** Recently, [39, 5, 37, 41, 42, 22] provide  
 552 some theoretical analysis of ADAM-type algorithms when applying them to smooth nonconvex op-  
 553 timization problems. For example, [5] provide the following bound  $\min_{t \in [T]} \mathbb{E}[\|\nabla f(w_t)\|^2] =$   
 554  $\mathcal{O}(\log T / \sqrt{T})$ . Yet, this data independent bound does not show any advantage over standard  
 555 stochastic gradient descent. Similar concerns appear in other related works. To get some adap-  
 556 tive data dependent bound written in terms of the gradient norms observed along the trajectory when  
 557 applying OPT-AMSGRAD to nonconvex optimization, one can follow the approach of [2] or [6].  
 558 They provide a modular approach to convert algorithms with adaptive data dependent regret bound  
 559 for convex loss functions (e.g., ADAGRAD) to algorithms that can find an approximate stationary  
 560 point of nonconvex objectives. These variants can outperform the ones instantiated by other ADAM-  
 561 type algorithms when the gradient prediction  $m_t$  is close to the true gradient  $g_t$ .

562 **Comparison to AO-FTRL [26].** In [26], the authors propose AO-FTRL, which update reads  
 563  $w_{t+1} = \arg \min_{w \in \Theta} (\sum_{s=1}^t g_s)^\top w + m_{t+1}^\top w + r_{0:t}(w)$ , where  $r_{0:t}(\cdot)$  is a 1-strongly convex loss  
 564 function with respect to some norm  $\|\cdot\|_{(t)}$  that may be different for different iteration  $t$ . Data depen-  
 565 dent regret bound provided in [26] reads  $r_{0:T}(w^*) + \sum_{t=1}^T \|g_t - m_t\|_{(t)}^*$  for any benchmark  $w^* \in \Theta$ .  
 566 We remark that if one selects  $r_{0:t}(w) := \langle w, \text{diag}\{\hat{v}_t\}^{1/2} w \rangle$  and  $\|\cdot\|_{(t)} := \sqrt{\langle \cdot, \text{diag}\{\hat{v}_t\}^{1/2} \cdot \rangle}$ , then  
 567 the update might be viewed as an optimistic variant of ADAGRAD. However, no experiments were  
 568 provided in [26] to back those findings.

569 **Comparison to OPTIMISTIC-ADAM [8].** This is an optimistic variant of ADAM,  
 570 namely OPTIMISTIC-ADAM. A slightly modified version is summarized in Algorithm 4.  
 571 Here, OPTIMISTIC-ADAM+ $\hat{v}_t$  corresponds to OPTIMISTIC-ADAM with the additional max  
 572 operation  $\hat{v}_t = \max(\hat{v}_{t-1}, v_t)$  to guarantee that the weighted second moment is monotone  
 573 increasing.

---

### Algorithm 4 OPTIMISTIC-ADAM [8]+ $\hat{v}_t$ .

---

- 1: Required: parameter  $\beta_1, \beta_2$ , and  $\eta_t$ .
  - 2: Init:  $w_1 \in \Theta$  and  $\hat{v}_0 = v_0 = \epsilon 1 \in \mathbb{R}^d$ .
  - 3: **for**  $t = 1$  to  $T$  **do**
  - 4:   Get mini-batch stochastic gradient vector  $g_t \in \mathbb{R}^d$  at  $w_t$ .
  - 5:    $\theta_t = \beta_1 \theta_{t-1} + (1 - \beta_1) g_t$ .
  - 6:    $v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2$ .
  - 7:    $\hat{v}_t = \max(\hat{v}_{t-1}, v_t)$ .
  - 8:    $w_{t+1} = \Pi_k[w_t - 2\eta_t \frac{\theta_t}{\sqrt{\hat{v}_t}} + \eta_t \frac{\theta_{t-1}}{\sqrt{\hat{v}_{t-1}}}]$ .
  - 9: **end for**
- 

574 We want to emphasize that the motivations of our optimistic algorithm are different. OPTIMISTIC-  
 575 ADAM is designed to optimize two-player games (e.g., GANs [15]), while our proposed algorithm  
 576 OPT-AMSGRAD is designed to accelerate optimization (e.g., solving empirical risk minimization).  
 577 [8] focuses on training GANs [15] as a two-player zero-sum game. [8] is inspired by these related  
 578 works and shows that OPTIMISTIC-MIRROR-DESCENT can avoid the cycle behavior in a bilinear  
 579 zero-sum game thus accelerating convergence.

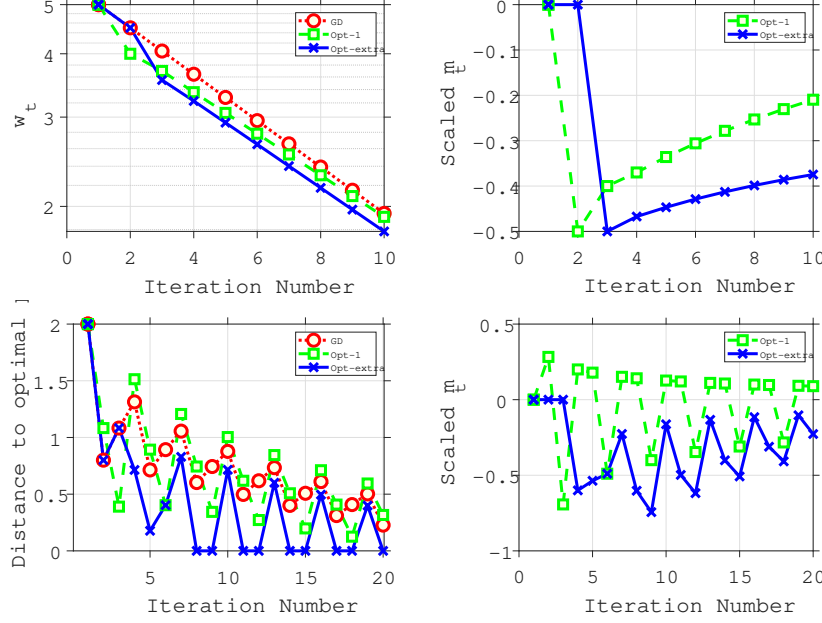


Figure 5: (a): The iterate  $w_t$ ; the closer to the optimal point 0 the better. (b): A scaled and clipped version of  $m_t$ :  $w_t - w_{t-1/2}$ , which measures how the prediction of  $m_t$  drives the update towards the optimal point. In this scenario, the more negative the better. (c): Distance to the optimal point  $-1$ . The smaller the better. (d): A scaled and clipped version of  $m_t$ :  $w_t - w_{t-1/2}$ , which measures how the prediction of  $m_t$  drives the update towards the optimal point. In this scenario, the more negative the better.

## 580 G Additional Remarks and Runs on the Gradient Prediction Process

**Two illustrative examples.** We provide two toy examples to demonstrate how OPT-AMSGRAD works with the chosen extrapolation method. First, consider minimizing a quadratic function  $H(w) := \frac{b}{2}w^2$  with vanilla gradient descent method  $w_{t+1} = w_t - \eta_t \nabla H(w_t)$ . The gradient  $g_t := \nabla H(w_t)$  can be recursively expressed as  $g_{t+1} = bw_{t+1} = b(w_t - \eta_t g_t) = g_t - b\eta_t g_t$ . Thus, the update can be written in the form of

$$g_t = Ag_{t-1} + \mathcal{O}(\|g_{t-1}\|_2^2)u_{t-1},$$

581 where  $A = (1 - b\eta)$  and  $u_{t-1} = 0$  by setting  $\eta_t = \eta$  (constant step size). Therefore, the extrapolation  
 582 method should predict well. Specifically, consider optimizing  $H(w) := w^2/2$  by the following  
 583 three algorithms with the same step size. One is Gradient Descent (GD):  $w_{t+1} = w_t - \eta_t g_t$ , while  
 584 the other two are OPT-AMSGRAD with  $\beta_1 = 0$  and the second moment term  $\hat{v}_t$  being dropped:  
 585  $w_{t+\frac{1}{2}} = \Pi_{\Theta}[w_{t-\frac{1}{2}} - \eta_t g_t]$ ,  $w_{t+1} = \Pi_{\Theta}[w_{t+\frac{1}{2}} - \eta_{t+1} m_{t+1}]$ . We denote the algorithm that sets  
 586  $m_{t+1} = g_t$  as OPT-1, and denote the algorithm that uses the extrapolation method to get  $m_{t+1}$  as  
 587 OPT-EXTRA. We let  $\eta_t = 0.1$  and the initial point  $w_0 = 5$  for all three methods. The simulation  
 588 results are on Figure 5 (a) and (b). Sub-figure (a) plots update  $w_t$  over iteration, where the updates  
 589 should go towards the optimal point 0. Sub-figure (b) displays a scaled and clipped version of  $m_t$ ,  
 590 defined as  $w_t - w_{t-1/2}$ , which can be viewed as  $-\eta_t m_t$  if the projection (if exists) is lifted. Sub-  
 591 figure (a) shows that OPT-EXTRA converges faster than the other methods. Furthermore, sub-figure  
 592 (b) shows that the prediction by the extrapolation method is better than the prediction by simply  
 593 using the previous gradient. The sub-figure shows that  $-m_t$  from both methods points to 0 for each  
 594 iteration and the magnitude is larger for the one produced by the extrapolation method after iteration  
 595 2.<sup>2</sup>

596 Now let us consider another problem: an online learning problem proposed in [30]<sup>3</sup>. Assume the  
 597 learner's decision space is  $\Theta = [-1, 1]$ , and the loss function is  $\ell_t(w) = 3w$  if  $t \bmod 3 = 1$ , and  
 598  $\ell_t(w) = -w$  otherwise. The optimal point to minimize the cumulative loss is  $w^* = -1$ . We let

<sup>2</sup>The extrapolation needs at least two gradients for prediction. Thus, in the first two iterations,  $m_t = 0$ .

<sup>3</sup>[30] uses this example to show that ADAM [19] fails to converge.



599  $\eta_t = 0.1/\sqrt{t}$  and the initial point  $w_0 = 1$  for all three methods. The parameter  $\lambda$  of the extrapolation  
 600 method is set to  $\lambda = 10^{-3} > 0$ . The results are reported Figure 5 (c) and (d). Sub-figure (c) shows  
 601 that OPT-EXTRA converges faster than the other methods while OPT-1 is not performing better than  
 602 GD. The reason is that the gradient changes from  $-1$  to  $3$  at  $t \bmod 3 = 1$  and it changes from  $3$  to  $-1$   
 603 at  $t \bmod 3 = 2$ . Consequently, using the current gradient as the guess for the next is empirically not  
 604 a good choice, since the next gradient is in the opposite direction of the current one, according to our  
 605 experiments. Sub-figure (d) shows that  $-m_t$ , obtained with the extrapolation method, always points  
 606 to  $w^* = -1$ , while the one obtained by using the previous negative direction points to the opposite  
 607 direction in two thirds of rounds. It shows that the extrapolation method is much less affected by the  
 608 gradient oscillation and always makes the prediction in the right direction, which suggests that the  
 609 method can capture the aggregate effect.