Minimization by Incremental Stochastic Surrogate Optimization for Large Scale Nonconvex Problems

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Stochastic Approximation

- **Objective:** Find a *stationary point* of smooth Lyapunov function $V(\boldsymbol{\eta})$.
- SA scheme (Robbins and Monro, 1951) is a stochastic process:

$$\boldsymbol{\eta}_{n+1} = \boldsymbol{\eta}_n - \gamma_{n+1} H_{\boldsymbol{\eta}_n}(X_{n+1}), \quad n \in \mathbb{N} \tag{1}$$

where $\boldsymbol{\eta}_n \in \mathcal{H} \subseteq \mathbb{R}^d$ is the nth state, $\gamma_n > 0$ is the step size.

- The *drift term* $H_{\boldsymbol{\eta}_n}(X_{n+1})$ depends on an **i.i.d. random element** X_{n+1} and

$$h(\boldsymbol{\eta}_n) = \mathbb{E}\big[H_{\boldsymbol{\eta}_n}(X_{n+1}) | \mathcal{F}_n\big] = \nabla V(\boldsymbol{\eta}_n),$$

where $\mathcal{F}_n = \sigma(\boldsymbol{\eta}_0, \{X_m\}_{m \leq n})$. In this case, SA is better known as the SGD method.

Biased SA Scheme

- The **mean field** is **biased** $\Leftarrow$ gradient is sometimes difficult to compute...

  We have $h(\boldsymbol{\eta}) \neq \nabla V(\boldsymbol{\eta})$ and for some $c_0 \geq 0$, $c_1 > 0$,

$$c_0 + c_1 \langle \nabla V(\boldsymbol{\eta}) \,|\, h(\boldsymbol{\eta}) \rangle \geq \|h(\boldsymbol{\eta})\|^2, \ \forall\, \boldsymbol{\eta} \in \mathcal{H}$$

- The **drift term** $\{H_{\boldsymbol{\eta}_n}(X_{n+1})\}_{n \geq 1}$ is **not i.i.d.**. For example, in reinforcement learning, $\boldsymbol{\eta}_n$ controls the policy in a MDP & $H_{\boldsymbol{\eta}_n}(X_{n+1})$ is computed from the MDP's state.

  The random elements $\{X_n\}_{n \geq 1}$ form a **state-dependent Markov chain**:

$$\mathbb{E}[H_{\boldsymbol{\eta}_n}(X_{n+1}) | \mathcal{F}_n] = P_{\boldsymbol{\eta}_n} H_{\boldsymbol{\eta}_n}(X_n) = \int H_{\boldsymbol{\eta}_n}(x) P_{\boldsymbol{\eta}_n}(X_n, \mathrm{d}x),$$

  where $P_{\boldsymbol{\eta}_n} : \mathsf{X} \times \mathcal{X} \to \mathbb{R}_+$ is Markov kernel with a unique stationary distribution $\pi_{\boldsymbol{\eta}_n}$.
- In the latter case, the mean field is given by $h(\boldsymbol{\eta}) = \int H_{\boldsymbol{\eta}}(x) \pi_{\boldsymbol{\eta}}(\mathrm{d}x)$.
- **Stopping criterion**: fix any $n \geq 1$, we stop the SA at a random iteration N with

$$\mathbb{P}(N = \ell) = \left(\textstyle\sum_{k=0}^{n} \gamma_{k+1}\right)^{-1} \gamma_{\ell+1}, \quad \text{with} \quad N \in \{1, \ldots, n\}.$$

Prior Work

- We focus on the **non-asymptotic convergence** analysis of SA scheme, where the relevant results are rare. Define:

$$\boldsymbol{e}_{n+1} := H_{\boldsymbol{\eta}_n}(X_{n+1}) - h(\boldsymbol{\eta}_n) \tag{2}$$

Case 1: When $\{\boldsymbol{e}_n\}_{n \geq 1}$ is Martingale difference — $\mathbb{E}[\boldsymbol{e}_{n+1} | \mathcal{F}_n] = 0$

- *Asymptotic analysis*: (Robbins and Monro, 1951); *Non-asymptotic analysis*: (Ghadimi and Lan, 2013).

Case 2: When $\{\boldsymbol{e}_n\}_{n \geq 1}$ is state-controlled Markov noise

$$\mathbb{E}[\boldsymbol{e}_{n+1} | \mathcal{F}_n] = P_{\boldsymbol{\eta}_n} H_{\boldsymbol{\eta}_n}(X_n) - h(\boldsymbol{\eta}_n) \neq 0.$$

- *Asymptotic analysis*: (Tadić and Doucet, 2017); *Non-asymptotic analysis:* (Sun et al., 2018), (Duchi et al., 2012), (Bhandari et al., 2018)

Analysis For Martingale Difference Noise (Case 1)

Assumption: $\mathbb{E}\left[\boldsymbol{e}_{n+1} \,|\, \mathcal{F}_n\right] = \mathbf{0}$, $\mathbb{E}\left[\|\boldsymbol{e}_{n+1}\|^2 \,\middle|\, \mathcal{F}_n\right] \leq \sigma_0^2 + \sigma_1^2 \|h(\boldsymbol{\eta}_n)\|^2$. (*e.g.*, when *$X_n$ is i.i.d.* similar to the SGD setting).

> **Theorem 1.** *Let* $\gamma_{n+1} \leq (2c_1 L(1 + \sigma_1^2))^{-1}$ *and* $V_{0,n} := \mathbb{E}[V(\boldsymbol{\eta}_0) - V(\boldsymbol{\eta}_{n+1})]$,
>
> $$\mathbb{E}[\|h(\boldsymbol{\eta}_N)\|^2] \leq \frac{2c_1 \left(V_{0,n} + \sigma_0^2 L \sum_{k=0}^{n} \gamma_{k+1}^2\right)}{\sum_{k=0}^{n} \gamma_{k+1}} + 2c_0 \,,$$

Set $\gamma_k = (2c_1 L(1 + \sigma_1^2)\sqrt{k})^{-1} \Longrightarrow \mathbb{E}[\|h(\boldsymbol{\eta}_N)\|^2] = \mathcal{O}(c_0 + \log n / \sqrt{n})$. *Remark*: if $h(\boldsymbol{\eta}) = \nabla V(\boldsymbol{\eta})$ (with $c_0 = d_0 = 0$), it recovers *(Ghadimi and Lan, 2013, Theorem 2.1)*.

Analysis For State-dependent Markov Noise (Case 2)

Assumptions: we need a few regularity conditions in this case,

1. There exists a Borel measurable function $\hat{H} : \mathcal{H} \times \mathsf{X} \to \mathcal{H}$,

$$\hat{H}_{\boldsymbol{\eta}}(x) - P_{\boldsymbol{\eta}} \hat{H}_{\boldsymbol{\eta}}(x) = H_{\boldsymbol{\eta}}(x) - h(\boldsymbol{\eta}), \ \forall\, \boldsymbol{\eta} \in \mathcal{H}, x \in \mathsf{X}.$$

   $\Longrightarrow$ existence of solution to the *Poisson equation*.

2. For all $\boldsymbol{\eta} \in \mathcal{H}$ and $x \in \mathsf{X}$, $\|\hat{H}_{\boldsymbol{\eta}}(x)\| \leq L_{PH}^{(0)}$, $\|P_{\boldsymbol{\eta}} \hat{H}_{\boldsymbol{\eta}}(x)\| \leq L_{PH}^{(0)}$, and

$$\sup_{x \in \mathsf{X}} \|P_{\boldsymbol{\eta}} \hat{H}_{\boldsymbol{\eta}}(x) - P_{\boldsymbol{\eta}'} \hat{H}_{\boldsymbol{\eta}'}(x)\| \leq L_{PH}^{(1)} \|\boldsymbol{\eta} - \boldsymbol{\eta}'\|, \ \forall\, (\boldsymbol{\eta}, \boldsymbol{\eta}') \in \mathcal{H}^2.$$

   $\Longrightarrow$ *smoothness* of $\hat{H}_{\boldsymbol{\eta}}(x)$, satisfied if $P_{\boldsymbol{\eta}}$, $H_{\boldsymbol{\eta}}(X)$ are smooth *w.r.t.* $\boldsymbol{\eta}$.

3. It holds that $\sup_{\boldsymbol{\eta} \in \mathcal{H}, x \in \mathsf{X}} \|H_{\boldsymbol{\eta}}(x) - h(\boldsymbol{\eta})\| \leq \sigma$.

   $\Longrightarrow$ requires the noise is *uniformly bounded* for all $x \in \mathsf{X}$.

Example: assumptions 1 & 2 are satisfied if the Markov kernel $P_{\boldsymbol{\eta}_\eta}$ is geometrically ergodic + smooth, and the drift term is smooth *w.r.t.* $\boldsymbol{\eta}$.

> **Theorem 2.** *Suppose that the step sizes are decreasing and* $\gamma_1 \leq 0.5\big(c_1(L + C_h)\big)^{-1}$ *(+other conditions). Let* $V_{0,n} := \mathbb{E}[V(\boldsymbol{\eta}_0) - V(\boldsymbol{\eta}_{n+1})]$,
>
> $$\mathbb{E}[\|h(\boldsymbol{\eta}_N)\|^2] \leq \frac{2c_1 \left(V_{0,n} + C_{0,n} + \left(\sigma^2 L + C_\gamma\right) \sum_{k=0}^{n} \gamma_{k+1}^2\right)}{\sum_{k=0}^{n} \gamma_{k+1}} + 2c_0 \,.$$

- Set $\gamma_k = (2c_1 L(1 + C_h)\sqrt{k})^{-1} \Longrightarrow \mathbb{E}[\|h(\boldsymbol{\eta}_N)\|^2] = \mathcal{O}(c_0 + \log n / \sqrt{n})$ (same as Case 1).
- **Proof idea:** challenge is that $\boldsymbol{e}_{n+1}$ is not zero-mean $\Longrightarrow$ bound the sum of $\mathbb{E}[\langle \nabla V(\boldsymbol{\eta}_n) \,|\, \boldsymbol{e}_{n+1} \rangle]$ w/ Poisson equation + a novel decomposition (cf. *Lemma 2*).

Regularized Online EM Algorithm

- **Special Case of GMM**: we fit the data $\{Y_n\}_{n \geq 1}$, $Y_n \sim \pi$ into the parametric model with $\boldsymbol{\theta} = (\{\omega_m\}_{m=1}^{M-1}, \{\mu_m\}_{m=1}^{M})$

$$g(y; \boldsymbol{\theta}) \propto \left(1 - \textstyle\sum_{m=1}^{M-1} \omega_m\right) \exp\left(-\frac{(y - \mu_M)^2}{2}\right) + \textstyle\sum_{m=1}^{M-1} \omega_m \exp\left(-\frac{(y - \mu_m)^2}{2}\right),$$

- Data arrives in a streaming fashion, Cappé and Moulines (2009) does:

$$\begin{aligned} \text{E-step:} \quad & \hat{\boldsymbol{s}}_{n+1} = \hat{\boldsymbol{s}}_n + \gamma_{n+1} \left\{ \overline{\boldsymbol{s}}(Y_{n+1}; \hat{\boldsymbol{\theta}}_n) - \hat{\boldsymbol{s}}_n \right\}, \\ \text{M-step:} \quad & \hat{\boldsymbol{\theta}}_{n+1} = \overline{\boldsymbol{\theta}}(\hat{\boldsymbol{s}}_{n+1}). \end{aligned}$$

- The **E-step** is a biased SA step on $\boldsymbol{s}$ with the drift term & mean field

$$H_{\hat{\boldsymbol{s}}_n}(Y_{n+1}) = \hat{\boldsymbol{s}}_n - \overline{\boldsymbol{s}}(Y_{n+1}; \overline{\boldsymbol{\theta}}(\hat{\boldsymbol{s}}_n)), \quad h(\hat{\boldsymbol{s}}_n) = \hat{\boldsymbol{s}}_n - \mathbb{E}_\pi\left[\overline{\boldsymbol{s}}(Y_{n+1}; \overline{\boldsymbol{\theta}}(\hat{\boldsymbol{s}}_n))\right]$$

Analysis of the ro-EM Algorithm (Application of Case 1)

Consider the KL divergence as a function of sufficient statistics $\boldsymbol{s}$:

$$V(\boldsymbol{s}) := \mathrm{KL}(\pi | g(\cdot; \overline{\boldsymbol{\theta}}(\boldsymbol{s}))) + \mathrm{R}(\overline{\boldsymbol{\theta}}(\boldsymbol{s})) = \mathbb{E}_\pi\left[\log\left(\pi(Y) / g(Y; \overline{\boldsymbol{\theta}}(\boldsymbol{s}))\right)\right] + \mathrm{R}(\overline{\boldsymbol{\theta}}(\boldsymbol{s})).$$

> **Corollary 1.** *Set* $\gamma_k = (2c_1 L(1 + \sigma_1^2)\sqrt{k})^{-1}$. *Ro-EM method for GMM finds* $\hat{\boldsymbol{s}}_N$ *such that*
>
> $$\mathbb{E}[\|\nabla V(\hat{\boldsymbol{s}}_N)\|^2] = \mathcal{O}(\log n / \sqrt{n})$$
>
> *The expectation is taken* w.r.t. *N and the observation law π.*

- First *explicit non-asymptotic* rate given for online EM method.
- Consider a slightly modified/regularized M-step update for satisfaction of the technical conditions.

(Online) Policy Gradient Method

- Consider a Markov Decision Process (MDP) $(\mathsf{S}, \mathsf{A}, \mathsf{R}, \mathsf{P})$:
  - S, A is the finite set of state/action.
  - $\mathsf{R} : \mathsf{S} \times \mathsf{A} \to [0, \mathsf{R}_{\max}]$ is a reward function; P is the transition model.
- A **policy** is parameterized by $\boldsymbol{\eta} \in \mathbb{R}^d$ as (e.g., soft-max):

$$\Pi_{\boldsymbol{\eta}}(a'; s') = \text{probability of taking action } a' \text{ in state } s'$$

- Update $\boldsymbol{\eta}$ in an online fashion (Tadić and Doucet, 2017) using observed state-action pair:

$$\begin{aligned} G_{n+1} &= \lambda G_n + \nabla \log \Pi_{\boldsymbol{\eta}_n}(A_{n+1}; S_{n+1}) \,, \\ \boldsymbol{\eta}_{n+1} &= \boldsymbol{\eta}_n + \gamma_{n+1} G_{n+1} \mathrm{R}(S_{n+1}, A_{n+1}) \end{aligned}$$

  where $\lambda \in (0, 1)$ is a parameter for the variance-bias trade-off.
- The $\boldsymbol{\eta}$-update is an biased SA step with the drift term:

$$H_{\boldsymbol{\eta}_n}(X_{n+1}) = G_{n+1} \mathrm{R}(S_{n+1}, A_{n+1})$$

Analysis of Policy Gradient Method (Application of Case 2)

Let $\upsilon_{\boldsymbol{\eta}}(s, a)$ be the invariant distribution of $\{(S_t, A_t)\}_{t \geq 1}$, we consider:

$$J(\boldsymbol{\eta}) := \textstyle\sum_{s \in \mathsf{S}, a \in \mathsf{A}} \upsilon_{\boldsymbol{\eta}}(s, a) \mathrm{R}(s, a) \,.$$

> **Corollary 2.** *Set* $\gamma_k = (2c_1 L(1 + C_h)\sqrt{k})^{-1}$. *For any* $n \in \mathbb{N}$, *the policy gradient algorithm* (3) *finds a policy that*
>
> $$\mathbb{E}\left[\|\nabla J(\boldsymbol{\eta}_N)\|^2\right] = \mathcal{O}\Big((1 - \lambda)^2 \Gamma^2 + c(\lambda) \log n / \sqrt{n}\Big),$$
>
> *where* $c(\lambda) = \mathcal{O}(\frac{1}{1-\lambda})$. *Expectation is taken* w.r.t. *N and (A_n, S_n).*

- It shows the *first convergence rate* for the online PG method.
- Our result shows the *variance-bias trade-off* with $\lambda \in (0, 1)$.
- Setting $\lambda \to 1$ reduces the bias, but decreases the convergence speed.

Conclusion

- Theorem 1 & 2 show the non-asymptotic convergence rate of biased SA scheme with smooth (possibly non-convex) Lyapunov function.
- With appropriate step size, in n iterations the SA scheme finds $\mathbb{E}[\|h(\boldsymbol{\eta}_N)\|^2] = \mathcal{O}(c_0 + \log n / \sqrt{n})$, where c_0 is the bias and $h(\cdot)$ is the mean field.
- Applications to online EM and online policy gradient.

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