Sparsified Distributed Adaptive Learning with Error Feedback

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Abstract

To be completed...

1 Introduction

Deep neural network has achieved the state-of-the-art learning performance on numerous AI applications, e.g., computer vision [18, 22, 42], Natural Language Processing [20, 48, 49], Reinforcement Learning [34, 40] and recommendation systems [12, 44]. With the increasing size of both data and deep networks, standard single machine training confronts with at least two major challenges:

- Due to the limited computing power of a single machine, it would take a long time to process the massive number of data samples—training would be slow.
- In many practical scenarios, data are typically stored in multiple servers, possibly at different locations, due to the storage constraints (massive user behavior data, Internet images, etc.) or privacy reasons [8]. Transmitting data might be costly.

Distributed learning framework [14] has been a common training strategy to tackle the above two issues. For example, in distributed stochastic gradient descent (SGD) protocol, data are located at N local nodes, at which the gradients of the model are computed in parallel. In each iteration, a central server aggregates the local gradients, updates the global model, and transmits back the updated model to the local nodes for subsequent gradient computation. As we can see, this setting naturally solves aforementioned issues: 1) We use N computing nodes to train the model, so the time per training epoch can be largely reduced; 2) There is no need to transmit the local data to central server. Besides, distributed training also provides stronger error tolerance since the training process could continue even one local machine breaks down. As a result of these advantages, there has been a surge of study and applications on distributed systems [7, 36, 15, 19, 23, 31, 30].

Among many optimization strategies, SGD is still the most popular prototype in distributed training for its simplicity and effectiveness [1, 46, 33]. Yet, when the deep learning model is very large, the communication between local nodes and central server could be expensive. Burdensome gradient transmission would slow down the whole training system, or even be impossible because of the limited bandwidth in some applications. Thus, reducing the communication cost in distributed SGD has become an active topic, and an important ingredient of large-scale distributed systems (e.g. [38]). Solutions based on quantization, sparsification and other compression techniques of the local gradients are proposed [3, 45, 26, 43, 21, 10, 25, 41, 2, 5, 13, 47, 24]. As one would expect, in most approaches, there exists a trade-off between compression and model accuracy. In particular, larger bias of the compressed gradients usually brings more significant performance downgrade. Interestingly, [28] shows that the technique of *error feedback* is able to remedy the issue of such biased compressors, achieving same convergence rate and learning performance as full-gradient SGD.

On the other hand, in recent years, adaptive optimization algorithms (e.g. AdaGrad [16], Adam [29] and AMSGrad [37]) have become popular because of their superior empirical performance. These

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methods use different implicit learning rates for different coordinates that keep changing adaptively throughout the training process, based on the learning trajectory. In many learning problems, adaptive methods have been shown to converge faster than SGD, sometimes with better generalization as well. However, the body of literature that combines adaptive methods with distributed training is still very limited.

1.1 Our contributions

2 Related Work

Sparse Optimization Methods.

Distributed Learning. When a large number of compute engines is available, being able to train global machine learning models while mutualizing the available and *decentralized* source of computation has been a growing focus for the community.

Decentralized optimization methods include methods such as ADMM [7], Distributed Subgradient Descent [36], Dual Averaging [15], Prox-PDA [23], GNSD [31], and Choco-SGD [30].

A recent work [9], which focuses on adaptive gradient methods, namely the Adam [29] annd the AMSGrad [37] optimization methods, develops a decentralized variant of gradient based and adaptive methods in the context of gossip protocols. To date, very few contributions provided attempt to efficiently run adaptive gradient method is such a distributed setting. Apart from [9], **(author?)** [35] proposes a decentralized version of AMSGrad [37] which provably satisfies some non-standard regret. Though, no sparsified variants of them have been proposed for practical purposes nor been studied in the literature.

Compression-Based Distributed Optimization. While the capabilities of the compute powers is exploding, the communication complexity between either the central server and the decentralized workers or among workers is becoming ineffectively large [11, 32]. Gradient sparsification constitutes one popular method to induce sparsity through the optimization procedure and reduce the number of bits transmitted at each iteration. Extensive works have studied this technique to improve the communication efficiency of SGD-based methods such as distributed SGD. This large class of sparsification techniques include gradient quantization leveraging quantized vector of gradients in the communication phase [3, 45, 26, 43, 21, 10, 25], gradient sparsificaiton generally selection top k components of the vector to be communicated, see [41, 2], or variants of the particular SGD algorithm such as low-precision SGD [5, 28] proposing a trade-off between communication cost and precision, and signSGD [13, 47] where only the signs of the gradient vectors are communicated. Most of these works apply to the SGD method [6] as a prototype where a novel method and some convergence results are presented with a rate of $\mathcal{O}(\frac{1}{\sqrt{T}})$ where T denotes the total number of iterations, see [4], thus achieving the same rate as plain SGD, see [17, 27].

Yet these communication reduction techniques, still presents a negative dependence on the number of workers, typically a linear dependence. Hence the need for even more efficient techniques which constitutes the object of our paper.

3 Method

Most modern machine learning tasks can be casted as a large finite-sum optimization problem written as:

$$\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \tag{1}$$

where n denotes the number of workers, f_i represents the average loss for worker i and θ the global model parameter taking value in Θ, a subset of $\mathbb{R}^d$.

Some related work:

[28] develops variant of signSGD (as a biased compression schemes) for distributed optimization. Contributions are mainly on this error feedback variant. In [39], the authors provide theoretical

results on the convergence of sparse Gradient SGD for distributed optimization (we want that for AMS here). [41] develops a variant of distributed SGD with sparse gradients too. Contributions include a memory term used while compressing the gradient (using top k for instance). Speeding up the convergence in $\frac{1}{T^3}$.

Consider standard synchronous distributed optimization setting. AMSGrad is used as the prototype, and the local workers is only in charge of gradient computation.

3.1 TopK AMSGrad with Error Feedback

The key difference (and interesting part) of our TopK AMSGrad compared with the following arxiv paper "Quantized Adam"https://arxiv.org/pdf/2004.14180.pdf is that, in our model only gradients are transmitted. In "QAdam", each local worker keeps a local copy of moment estimator m and v, and compresses and transmits m/v as a whole. Thus, that method is very much like the sparsified distributed SGD, except that g is changed into m/v. In our model, the moment estimates m and v are computed only at the central server, with the compressed gradients instead of the full gradient. This would be the key (and difficulty) in convergence analysis.

Algorithm 1 SPARS-AMS for Distributed Learning

```
Input: parameter β1, β2, learning rate η_t.
Initialize: central server parameter θ_0 ∈ Θ ⊆ R^d; e_{0,i} = 0 the error accumulator for each
    worker; sparsity parameter k; n local workers; m_0 = 0, v_0 = 0, v̂_0 = 0
for t = 1 to T do
    parallel for worker i ∈ [n] do:
        Receive model parameter θ_t from central server
        Compute stochastic gradient g_{t,i} at θ_t
        Compute g̃_{t,i} = TopK(g_{t,i} + e_{t,i}, k)
        Update the error e_{t+1,i} = e_{t,i} + g_{t,i} − g̃_{t,i}
        Send g̃_{t,i} back to central server
    end parallel
    Central server do:
    ḡ_t = (1/n) Σ_{i=1}^N g̃_{t,i}
    m_t = β1 m_{t−1} + (1 − β1) ḡ_t
    v_t = β2 v_{t−1} + (1 − β2) ḡ_t^2
    v̂_t = max(v_t, v̂_{t−1})
    Update global model θ_t = θ_{t−1} − η_t m_t / (√v̂_t + ε)
end for
```

3.2 Convergence Analysis

Several mild assumptions to make: Nonconvex and smooth loss function, unbiased stochastic gradient, bounded variance of the gradient, bounded norm of the gradient, control of the distance between the true gradient and its sparse variant.

Check [9] starting with single machine and extending to distributed settings (several machines).

Under the distributed setting, the goal is to derive an upper bound to the second order moment of the gradient of the objective function at some iteration $T_f \in [1, T]$.

3.3 Mild Assumptions

We begin by making the following assumptions.

A 1. *(Smoothness) For $i \in [\![n]\!]$, f_i is L-smooth: $\|\nabla f_i(\theta) - \nabla f_i(\vartheta)\| \leq L \|\theta - \vartheta\|$.*

A 2. *(Unbiased and Bounded gradient **per worker**) For any iteration index $t > 0$ and worker index $i \in [\![n]\!]$, the stochastic gradient is unbiased and bounded from above: $\mathbb{E}[g_{t,i}] = \nabla f_i(\theta_t)$ and $\|g_{t,i}\| \leq G_i$.*

A 3. *(Bounded variance **per worker**) For any iteration index $t > 0$ and worker index $i \in [\![n]\!]$, the variance of the noisy gradient is bounded: $\mathbb{E}[|g_{t,i} - \nabla f_i(\theta_t)|^2] < \sigma_i^2$.*

Denote by $Q(\cdot)$ the quantization operator Line 7 of Algorithm 1, which takes as input a gradient vector and returns a quantized version of it, and note $\tilde{g} := Q(g)$. Assume that

A 4. *(Bounded Quantization) For any iteration $t > 0$, there exists a constant $q > 0$ such that $\|g_{t,i} - \tilde{g}_{t,i}\| \leq q \|g_{t,i}\|$, where $g_{t,i}$ is the stochastic gradient computed at iteration t for worker i. (high q means large quantization so loss of precision on the true gradient)*

Denote for all $\theta \in \Theta$:

$$f(\theta) := \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \,, \tag{2}$$

where n denotes the number of workers.

3.4 Intermediary Lemmas

Lemma 1. *Under Assumption 2 and Assumption 4 we have for any iteration $t > 0$:*

$$\|m_t\|^2 \leq (q^2+1)G^2 \quad and \quad \hat{v}_t \leq (q^2+1)G^2 \tag{3}$$

where m_t and $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ are defined Line 15 of Algorithm 1 and $G^2 = \frac{1}{n}\sum_{i=1}^{N} G_i^2$.

Lemma 2. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} \bar{g}_t \right\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2 n^2} \tag{4}$$

*where I_d is the identity matrix, $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1 and $\bar{g}_t$ is the aggregation of all **quantized** gradients from the workers.*

Lemma 3. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2 n^2} \\
& - \eta_{t+1}\beta_1 \mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right) \|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned} \tag{5}$$

where d denotes the dimension of the parameter vector

The main theorem in the decentralized setting reads:

Theorem 1. *Under A1 to A4, with a constant stepsize $\eta_t = \eta = \frac{L}{\sqrt{T_m}}$, we have:*

$$\frac{1}{T_m} \sum_{t=0}^{T_m - 1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_m})]}{L \Delta_1 \sqrt{T_m}} + d \frac{L \Delta_3}{\Delta_1 \sqrt{T_m}} + \frac{\Delta_2}{\eta \Delta_1 T_m} + \frac{1-\beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2+1)} G^2 \tag{6}$$

where

$$\begin{aligned}
&\Delta_1 := \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \quad , \quad \Delta_2 := q^2 + \sum_{k=t+1}^{\infty} \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2} \\
&\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right)(1-\beta_2)^{-1}(1 - \frac{\beta_1^2}{\beta_2})^{-1}
\end{aligned} \tag{7}$$

We remark from this bound in Theorem 1, that the more quantization we apply to our gradient vectors ($q \uparrow$), the larger the upper bound of the stationary condition is, *i.e.,* the slower the algorithm is. This is intuitive as using compressed quantities will definitely impact the algorithm speed. We will observe in the numerical section below that a trade-off on the level of quantization q can be found to achieve similar speed of convergence with less computation resources used throughout the training.

Belhal Try for Single Machine Setting:

Define the auxiliary model

$$\begin{aligned}\theta'_{t+1} &:= \theta_{t+1} - e_{t+1} \\ &= \theta_t - \eta a_t - e_{t+1} \\ &= \theta_t - \eta a_t - e_t - g_t + \tilde{g}_t \\ &= \theta_t - \eta a_t - e_t - \Delta_t \\ &= \theta'_t - \eta a_t - \Delta_t\end{aligned}$$

where $a_t := \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$ and $\Delta_t := g_t - \tilde{g}_t$. By smoothness assumption we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \langle \nabla f(\theta'_t), \eta a_t + \Delta_t \rangle + \frac{L}{2} \|\theta'_{t+1} - \theta'_t\|^2 .$$

Thus,

$$\begin{aligned}\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq -\mathbb{E}[\langle \nabla f(\theta'_t), \eta a_t + \Delta_t \rangle] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2] \\ &\leq \eta \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), \eta a_t + \Delta_t \rangle] - \mathbb{E}[\langle \nabla f(\theta_t), \eta a_t + \Delta_t \rangle] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2]\end{aligned}$$

Using the smoothness assumption A1 we have

$$\mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), \eta a_t + \Delta_t \rangle] \leq L\mathbb{E}[\|\theta_t - \theta'_t\|]E[\|\eta a_t + \Delta_t\|]$$

Hence,

$$\begin{aligned}\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq -\mathbb{E}[\langle \nabla f(\theta'_t), \eta a_t + \Delta_t \rangle] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2] \\ &\leq -\left(\eta \frac{1}{\sqrt{G^2+\epsilon}} + q\right) \mathbb{E}\|\nabla f(\theta_t)\|^2 + L\mathbb{E}[\|\theta_t - \theta'_t\|]E[\|\eta a_t + \Delta_t\|] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2] \\ &\leq -\left(\eta \frac{1}{\sqrt{G^2+\epsilon}} + q\right) \mathbb{E}\|\nabla f(\theta_t)\|^2 + L\mathbb{E}[\|e_t\| \, \|\eta a_t + \Delta_t\|] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2]\end{aligned}$$

Summing from $t = 0$ to $t = T_{\text{m}} - 1$ and divide it by T_{m} yields:

$$\begin{aligned}&\left(\eta \frac{1}{\sqrt{G^2+\epsilon}} + q\right) \frac{1}{T_{\text{m}}} \sum_{t=0}^{T_{\text{m}}-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\leq \sum_{t=0}^{T_{\text{m}}-1} \frac{\mathbb{E}[f(\theta'_t) - f(\theta'_{t+1})]}{T_{\text{m}}} + \frac{1}{T_{\text{m}}} \sum_{t=0}^{T_{\text{m}}-1} \mathbb{E}[\|e_t\| \, \|\eta a_t + \Delta_t\|] + \frac{L}{2T_{\text{m}}} \sum_{t=0}^{T_{\text{m}}-1} \mathbb{E}[\|\eta a_t + \Delta_t\|^2]\end{aligned}$$

Bounding $\frac{1}{T_{\text{m}}} \sum_{t=0}^{T_{\text{m}}-1} \mathbb{E}[\|e_t\| \, \|\eta a_t + \Delta_t\|]$:

To begin with

$$\begin{aligned}\|e_t\| &= \|e_{t-1} + g_{t-1} - \tilde{g}_{t-1}\| \\ &\leq q \, \|g_{t-1}\| + \|e_{t-1}\|\end{aligned}$$

Bounding $\frac{L}{2T_{\text{m}}} \sum_{t=0}^{T_{\text{m}}-1} \mathbb{E}[\|\eta a_t + \Delta_t\|^2]$:

4 Sequential Model

Single machine method

Algorithm 2 SPARS-AMS : Single machine setting

Input: parameter β_1, β_2, learning rate η_t.
Initialize: central server parameter $\theta_0 \in \Theta \subseteq \mathbb{R}^d$; $e_0 = 0$ the error accumulator; sparsity parameter k; $m_0 = 0$, $v_0 = 0$, $\hat{v}_0 = 0$
for $t = 1$ to T **do**
  Compute stochastic gradient $g_t = g_{t,i_t}$ at θ_t for randomly sampled index i_t
  Compute $\tilde{g}_t = TopK(g_t + e_t, k)$
  Update the error $e_{t+1} = e_t + g_t - \tilde{g}_t$
  $m_t = \beta_1 m_{t-1} + (1 - \beta_1)\tilde{g}_t$
  $v_t = \beta_2 v_{t-1} + (1 - \beta_2)\tilde{g}_t^2$
  $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
  Update global model $\theta_t = \theta_{t-1} - \eta_t \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$
end for

Let m_t' and $\hat{v}_t'$ be the first and second moment moving average of standard AMSGrad using full gradients. Denote

$$a_t = \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}, \quad a_t' = \frac{m_t'}{\sqrt{\hat{v}_t' + \epsilon}}.$$

Define the sequence

$$\mathcal{E}_{t+1} = \mathcal{E}_t + a_t' - a_t,$$

such that the auxiliary model

$$\begin{aligned}
\theta_{t+1}' &:= \theta_{t+1} - \eta\mathcal{E}_{t+1} \\
&= \theta_t - \eta a_t - \eta\mathcal{E}_{t+1} \\
&= \theta_t - \eta a_t - \eta(\mathcal{E}_t + a_t' - a_t) \\
&= \theta_t' - \eta a_t'
\end{aligned}$$

follows the update of full-gradient AMSGrad. By smoothness assumption we have

$$f(\theta_{t+1}') \leq f(\theta_t') - \eta\langle\nabla f(\theta_t'), a_t'\rangle + \frac{L}{2}\|\theta_{t+1}' - \theta_t'\|^2.$$

Thus,

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}') - f(\theta_t')] &\leq -\eta\mathbb{E}[\langle\nabla f(\theta_t'), a_t'\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a_t'\|^2] \\
&= -\eta\mathbb{E}[\langle\nabla f(\theta_t), a_t'\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a_t'\|^2] + \eta\mathbb{E}[\langle\nabla f(\theta_t) - \nabla f(\theta_t'), a_t'\rangle] \\
&\leq -\eta\mathbb{E}[\langle\nabla f(\theta_t), a_t'\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a_t'\|^2] + \eta\mathbb{E}[\frac{\eta^2\rho}{2}\|\mathcal{E}_t\|^2 + \frac{1}{2\rho}\|a_t'\|^2] \\
&\leq -\eta\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\sqrt{G^2 + \epsilon}} + \frac{\eta}{2\rho}\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\epsilon} + \frac{\eta^2 L}{2}\mathbb{E}[\|a_t'\|^2] + \frac{\eta^3\rho}{2}\mathbb{E}\|\mathcal{E}_t\|^2,
\end{aligned}$$

when $\beta_1 = 0$ for example. We may discard this assumption and use more complicated bound on the first two terms. The third term can be bounded by constant yielding $O(1/\sqrt{T})$ rate eventually when taking decreasing learning rate. The key is to get a good bound on the cumulative error sequence, $\mathcal{E}_t$. We have the following:

$$\begin{aligned}
\mathbb{E}\|\mathcal{E}_{t+1}\|^2 &= \mathbb{E}\|\mathcal{E}_t + a_t' - a_t + TopK(\mathcal{E}_t + a_t') - TopK(\mathcal{E}_t + a_t')\|^2 \\
&\leq 2\mathbb{E}\|\mathcal{E}_t + a_t' - TopK(\mathcal{E}_t + a_t')\|^2 + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a_t')\|^2 \\
&\overset{(a)}{\leq} 2q\mathbb{E}\|\mathcal{E}_t + a_t'\| + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a_t')\|^2 \\
&\leq 2q[(1 + r)\mathbb{E}\|\mathcal{E}_t\|^2 + (1 + \frac{1}{r})\mathbb{E}\|a_t'\|^2] + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a_t')\|^2.
\end{aligned}$$

where (a) uses A3. Current try: If we can bound the last term in the same form as the first two terms, then we can use recursion to get the desired result. We can have

$$\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a_t')\|^2 = \mathbb{E}\|\frac{\tilde{m}_t}{\sqrt{\hat{v}_t + \epsilon}} - \|^2$$

5 Experiments

Our proposed TopK-EF with AMSGrad matches that of full AMSGrad, in distributed learning. Number of local workers is 20. Error feedback fixes the convergence issue of using solely the TopK gradient.

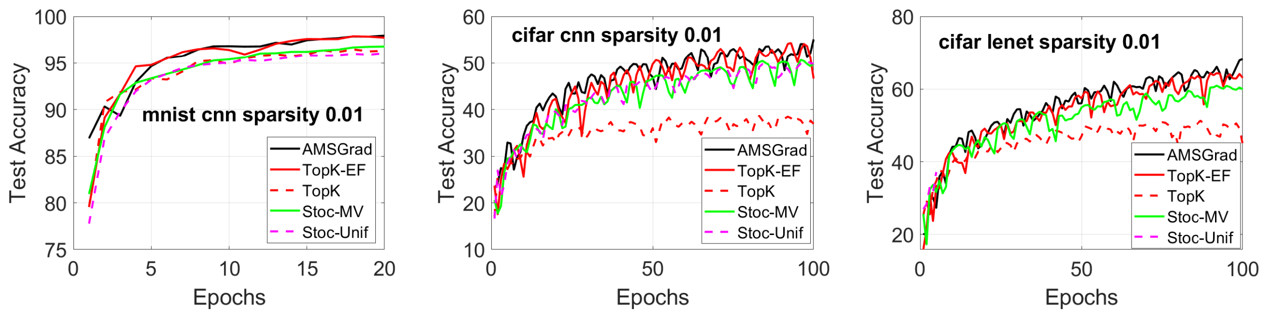

Figure 1: Test accuracy.

6 Conclusion

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A Appendix

B Proofs

B.1 Proof of Lemmas

Lemma. *Under Assumption 2 and Assumption 4 we have for any iteration $t > 0$:*

$$\|m_t\|^2 \leq (q^2+1)G^2 \quad \textit{and} \quad \hat{v}_t \leq (q^2+1)G^2 \tag{8}$$

where m_t and $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ are defined Line 15 of Algorithm 1 and $G^2 = \frac{1}{n}\sum_{i=1}^N G_i^2$.

Proof. We start by writing

$$\|\bar{g}_t\|^2 = \left\|\frac{1}{n}\sum_{i=1}^N \tilde{g}_{t,i}\right\|^2 \leq \frac{1}{n}\sum_{i=1}^N \|\tilde{g}_{t,i}\|^2 \tag{9}$$

Though, using Assumption 2 and Assumption 4 we have:

$$\|\tilde{g}_{t,i}\|^2 = \|g_{t,i} + \tilde{g}_{t,i} - g_{t,i}\|^2 \leq \|g_{t,i}\|^2 + \|\tilde{g}_{t,i} - g_{t,i}\|^2 \leq (q^2+1)G_i^2 \tag{10}$$

Hence

$$\|\bar{g}_t\|^2 \leq (q^2+1)G^2 \tag{11}$$

where $G^2 = \frac{1}{n}\sum_{i=1}^N G_i^2$. Then, by construction in Algorithm 1:

$$\|m_t\|^2 \leq \beta_1^2\|m_{t-1}\|^2 + (1-\beta_1)^2\|\bar{g}_t\|^2 \leq \beta_1^2\|m_{t-1}\|^2 + (1-\beta_1)^2(q^2+1)G^2 \tag{12}$$

Since we have by initialization that $\|m_0\|^2 \leq G^2$, then we prove by induction that $\|m_t\|^2 \leq (q^2+1)G^2$.

Similarly

$$\hat{v}_t = \max(v_t, \hat{v}_{t-1}) = \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1-\beta_2)\bar{g}_t^2) \leq max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1-\beta_2)(q^2+1)G^2) \tag{13}$$

□

Lemma. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \tag{14}$$

*where $\mathsf{I_d}$ is the identity matrix, $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1 and $\bar{g}_t$ is the aggregation of all **quantized** gradients from the workers.*

Proof. We first decompose $\bar{g}_t$ as the sum of the unbiased stochastic gradients and its quantized versions as computed Line 7 of Algorithm 1:

$$\bar{g}_t = \frac{1}{n}\sum_{i=1}^N \tilde{g}_{t,i} = \frac{1}{n}\sum_{i=1}^N [g_{t,i} + \tilde{g}_{t,i} - g_{t,i}] \tag{15}$$

Hence,

$$\begin{aligned} T_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \\ =& \underbrace{-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^N g_{t,i} \right\rangle]}_{t1} \underbrace{-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^N \tilde{g}_{t,i} - g_{t,i} \right\rangle]}_{t_2} \end{aligned} \tag{16}$$

Bounding t_1: Using the Tower rule, we have:

$$\begin{aligned} t_1 :=& -\eta_{t+1}\mathbb{E}[\Big\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \Big\rangle] \\ =& -\eta_{t+1}\mathbb{E}[\mathbb{E}[\Big\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \Big\rangle \,|\mathcal{F}_t]] \\ =& -\eta_{t+1}\mathbb{E}[\Big\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\mathbb{E}[\frac{1}{n}\sum_{i=1}^{N} g_{t,i}|\mathcal{F}_t] \Big\rangle] \end{aligned} \tag{17}$$

Using Assumption 2 and Lemma 1, we have that

$$\begin{aligned} t_1 :=& -\eta_{t+1}\mathbb{E}[\Big\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \Big\rangle] \\ \leq& -\eta_{t+1}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \end{aligned} \tag{18}$$

Bounding t_2:

We first recall Young's inequality with a constant $\delta \in (0, 1)$ as follows:

$$\langle X \,|\, Y \rangle \leq \frac{1}{\delta}\|X\|^2 + \delta\|Y\|^2 \,. \tag{19}$$

Using Young's inequality (19) with parameter equal to 1:

$$\begin{aligned} t_2 \leq& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\ \overset{(a)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\|^2\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\ \overset{(b)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\|^2]\mathbb{E}[\|\sum_{i=1}^{N}\{\tilde{g}_{t,i}-g_{t,i}\}\|^2] \\ \overset{(c)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{\epsilon 2n^2}\mathbb{E}[\|\sum_{i=1}^{N}\tilde{g}_{t,i}-g_{t,i}\|^2] \\ \overset{(d)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \end{aligned} \tag{20}$$

where (a) uses the Cauchy-Schwartz inequality, (b) is due to the non-negativeness of both $\hat{V}_{t+1}$ and $\|\sum_{i=1}^{N}\{g_{t,i}+\tilde{g}_{t,i}-g_{t,i}\}\|^2$ and (c) uses the Triangle inequality. We use Assumption 3 and Assumption 4 in (d).

Finally, combining (18) and (20) yields

$$-\eta_{t+1}\mathbb{E}[\Big\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \Big\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \tag{21}$$

$\square$

Lemma. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$\begin{aligned}\mathbb{E}[f(\theta_{t+1})-f(\theta_t)] \leq & -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon+\frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2]+q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2}\\ & -\eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_{t-1})\,|\,(\hat{V}_t+\epsilon \mathsf{I_d})^{-1/2}m_t\right\rangle]\\ & +\left(\frac{L}{2}+\beta_1 L\right)\|\theta_t-\theta_{t-1}\|^2\\ & +\eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j+\epsilon)^{-1/2}-(\hat{v}_t^j+\epsilon)^{-1/2}\right]]\end{aligned} \tag{22}$$

where d denotes the dimension of the parameter vector

Proof. Denote the following auxiliary variables at iteration $t+1$

$$z_{t+1}=\theta_{t+1}+\frac{\beta_1}{1-\beta_1}(\theta_{t+1}-\theta_t) \tag{23}$$

By assumption Assumption 1, we can write the smoothness condition on the overall objective (2), between iteration t and $t+1$:

$$f(\theta_{t+1})\leq f(\theta_t)+\langle \nabla f(\theta_t)\,|\,\theta_{t+1}-\theta_t\rangle+\frac{L}{2}\|\theta_{t+1}-\theta_t\|^2 \tag{24}$$

Denote by $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t=\max(v_t,\hat{v}_{t-1})$ defined Line 15 of Algorithm 1. Hence, we obtain,

$$f(\theta_{t+1})\leq f(\theta_t)-\eta_{t+1}\left\langle \nabla f(\theta_t)\,|\,(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle+\frac{L}{2}\|\theta_{t+1}-\theta_t\|^2 \tag{25}$$

where $\mathsf{I_d}$ denotes the identity matrix.

We now take the expectation of those various terms conditioned on the filtration $\mathcal{F}_t$ of the total randomness up to iteration t.

$$\mathbb{E}[f(\theta_{t+1})-f(\theta_t)]\leq -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t)\,|\,(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle]+\frac{L}{2}\mathbb{E}[\|\theta_{t+1}-\theta_t\|^2] \tag{26}$$

We now focus on the computation of the inner product obtained in the equation above. We have

$$\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t)\,|\,(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle] \tag{27}$$

$$\begin{aligned}&=\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t)\,|\,(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}m_{t+1}+(\hat{V}_t+\epsilon\mathsf{I_d})^{-1/2}m_{t+1}-(\hat{V}_t+\epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle]\\ &=\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t)\,|\,(\hat{V}_t+\epsilon\mathsf{I_d})^{-1/2}m_{t+1}\right\rangle]+\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t)\,|\,\left[(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}-(\hat{V}_t+\epsilon\mathsf{I_d})^{-1/2}\right]m_{t+1}\right\rangle]\\ &=\eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_t)\,|\,(\hat{V}_t+\epsilon\mathsf{I_d})^{-1/2}m_t\right\rangle]+\eta_{t+1}(1-\beta_1)\mathbb{E}[\left\langle \nabla f(\theta_t)\,|\,(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}\bar{g}_t\right\rangle]\\ &\quad+\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t)\,|\,\left[(\hat{V}_{t+1}+\epsilon\mathsf{I_d})^{-1/2}-(\hat{V}_t+\epsilon\mathsf{I_d})^{-1/2}\right]m_{t+1}\right\rangle]\end{aligned} \tag{28}$$

where $\bar{g}_t$ is the aggregated gradients from all workers.

Plugging the above in (26) yields:

$$\begin{aligned}
&\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \\
\leq &\underbrace{-\beta_1 \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle]}_{A_t} \eta_{t+1} \\
&\underbrace{-\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2}\right] m_{t+1} \right\rangle]}_{B_t} \eta_{t+1} \\
&\underbrace{-(1-\beta_1)\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} \bar{g}_t \right\rangle]}_{C_t} \eta_{t+1} + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]
\end{aligned} \tag{29}$$

To begin with, by the tower rule, we have that

$$A_t = -\beta_1 \mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle | \mathcal{F}_t]] \tag{30}$$

$$= -\beta_1 \left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] - \beta_1 \left\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] \tag{31}$$

$$\tag{32}$$

where we recognize the first term as the term in (27), at iteration $t-1$ and hence apply the same decomposition as in (28). Coupling with the smoothness of f, which gives that

$$-\beta_1 \left\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] \leq \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2$$

we obtain,

$$\begin{aligned}
A_t &= -\beta_1 \mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle | \mathcal{F}_t]] \\
&\leq \eta_{t+1}\beta_1 (A_{t-1} + B_{t-1} + C_{t-1}) + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2
\end{aligned} \tag{33}$$

Then,

$$\begin{aligned}
B_t &= -\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2}\right] m_{t+1} \right\rangle] \\
&= \mathbb{E}[\sum_{j=1}^d \nabla^j f(\theta_t) m_{t=1}^j \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]] \\
&\overset{(a)}{\leq} \mathbb{E}[\|\nabla f(\theta_t)\| \, \|m_{t=1}\| \sum_{j=1}^d \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]] \\
&\overset{(b)}{\leq} G^2 \mathbb{E}[\sum_{j=1}^d \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned} \tag{34}$$

where $\nabla^j f(\theta_t)$ denotes the j-th component of the gradient vector $\nabla f(\theta_t)$, (a) uses of the Cauchy-Schwartz inequality and (b) boils down from the norm of the gradient vector boundedness assumption 2, denoting $G := \frac{1}{n}\sum_{i=1}^n G_i$.

Plugging the above into (29) yields

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & \eta_{t+1}(A_t + B_t + C_t) + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \\
\leq & - \eta_{t+1}\beta_1 \mathbb{E}[\Big\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \Big\rangle] \\
& + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^{d} \Big[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\Big]] \\
& + \left(\frac{L}{2} + \eta_{t+1}\frac{\beta_1 L}{\eta_{t-1}}\right) \|\theta_t - \theta_{t-1}\|^2 \\
& - \eta_{t+1}(1-\beta_1)\mathbb{E}[\Big\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \Big\rangle]
\end{aligned} \tag{35}$$

We bound the last term on the RHS, $-\eta_{t+1}\mathbb{E}[\Big\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \Big\rangle]$ with Lemma 2

Under the assumption that we use a decreasing stepsize such that $\eta_{t+1} \leq \eta_t$, and given that according to Line 15 we have that $\hat{v}_{t+1} \geq \hat{v}_t$ by construction, we obtain

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & - \frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2 n^2} \\
& - \eta_{t+1}\beta_1 \mathbb{E}[\Big\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \Big\rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right) \|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^{d} \Big[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\Big]]
\end{aligned} \tag{36}$$

Finally, using Lemma 2, we obtain the desired result. □

B.2 Proof of Theorem 1

Theorem. *Under A1 to A4, with a constant stepsize $\eta_t = \eta = \frac{L}{\sqrt{T_m}}$, we have:*

$$\frac{1}{T_m}\sum_{t=0}^{T_m - 1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_m})]}{L \Delta_1 \sqrt{T_m}} + d\frac{L\Delta_3}{\Delta_1 \sqrt{T_m}} + \frac{\Delta_2}{\eta \Delta_1 T_m} + \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2 \tag{37}$$

where

$$\begin{aligned}
& \Delta_1 := \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \quad , \quad \Delta_2 := q^2 + \sum_{k=t+1}^{\infty} \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2} \\
& \Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right)(1-\beta_2)^{-1}(1 - \frac{\beta_1^2}{\beta_2})^{-1}
\end{aligned} \tag{38}$$

Proof. By Lemma 3 we have

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & - \frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2 n^2} \\
& - \eta_{t+1}\beta_1 \mathbb{E}[\Big\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \Big\rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right) \|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^{d} \Big[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\Big]]
\end{aligned} \tag{39}$$

Let us consider the following sequence, defined for all $t > 0$:

$$R_t := f(\theta_t) - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \tag{40}$$

We compute the following expectation:

$$\begin{aligned} \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] =& \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] - \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle] \\ &+ \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \end{aligned} \tag{41}$$

Using the Assumption 1, we note that:

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1} \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle] + \frac{L}{2} \left\| \theta_{t+1} - \theta_t \right\|^2 \tag{42}$$

which yields

$$\begin{aligned} \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] =& -(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle] \\ &+ \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \\ &+ \frac{L}{2} \left\| \theta_{t+1} - \theta_t \right\|^2 \\ \leq& (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \mathbb{E}[A_t + B_t + C_t] \\ &- \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\ &+ \frac{L}{2} \left\| \theta_{t+1} - \theta_t \right\|^2 \end{aligned} \tag{43}$$

where A_t, B_t, C_t are defined in (29).

We use (33) and (34) to bound A_t and B_t, and Lemma 2 to bound C_t where we precise that the learning rate η_{t+1} becomes $\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}$. Hence

$$\begin{aligned} \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] \leq& \left((\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \right) \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\ &+ (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\ &+ \left(\frac{L}{2} + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{\beta_1 L}{\eta_{t-1}} \right) \left\| \theta_{t+1} - \theta_t \right\|^2 \\ &- (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1) G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\left\| \nabla f(\theta_t) \right\|^2] \\ &+ q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2} \end{aligned} \tag{44}$$

where the last term in the LHS is due to Lemma 3.

By assumption, we have that for all $t > 0$, $\eta_{t=1} \leq \eta_t$. Also, set the tuning parameters such that

$$\eta_t + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \leq \frac{\eta_t}{1-\beta_1} \tag{45}$$

so that

$$\begin{aligned} &(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} = 0 \\ \iff &(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 = \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \end{aligned} \tag{46}$$

Note that $-(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \leq -\eta_{t+1} \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}$ since $\sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \geq 0$.

The above coupled with (44) yields

$$\begin{aligned} \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] \leq &- \eta_{t+1} \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2} \\ &- (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \right]] \\ &+ \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1} \right) \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \end{aligned} \tag{47}$$

We now sum from $t = 0$ to $t = T_{\text{m}} - 1$ the inequality in (47), and divide it by T_{m}:

$$\begin{aligned} &\eta \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \frac{1}{T_{\text{m}}} \sum_{t=0}^{T_{\text{m}}-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ \leq &\frac{\mathbb{E}[R_0] - \mathbb{E}[R_{T_{\text{m}}}]}{T_{\text{m}}} + \frac{q^2 \eta + \sum_{k=t+1}^{\infty} \eta \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}}{T_{\text{m}}} \\ &+ \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1} \right) \frac{1}{T_{\text{m}}} \sum_{t=0}^{T_{\text{m}}-1} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \end{aligned} \tag{48}$$

where we have used the fact that $(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \geq 0$ for all dimension $j \in [d]$ by construction of $\hat{v}_{t+1}^j$.

We now bound the two remaining terms:

Bounding $-\mathbb{E}[R_{T_{\text{m}}}]$:

By definition (40) of R_t we have, using Lemma 1:

$$\begin{aligned} -\mathbb{E}[R_{T_{\text{m}}}] &\leq \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] - f(\theta_{T_{\text{m}}}) \\ &\leq \| \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \| \| \nabla f(\theta_{t-1}) \| \| (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \| \\ &\leq \eta_{t+1} (1-\beta_1) \epsilon^{-\frac{1}{2}} \sqrt{(q^2+1)} G^2 - f(\theta_{T_{\text{m}}}) \end{aligned} \tag{49}$$

Bounding $\sum_{t=0}^{T_{\mathsf{m}}-1} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]$:

By definition in Algorithm 1:

$$\|\theta_{t+1} - \theta_t\|^2 = \eta_{t+1}^2 \left[(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-\frac{1}{2}} m_{t+1}\right]^2 = \eta_{t+1}^2 \sum_{j=1}^{d} \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j + \epsilon} \tag{50}$$

For any dimension $j \in [d]$,

$$\begin{aligned} |m_{t+1}^j|^2 &= |\beta_1 m_t^j + (1-\beta_1)\bar{g}_t^j|^2 \\ &\leq \beta_1(\beta_1^2 |m_{t-1}^j|^2 + (1-\beta_1)^2 |\bar{g}_{t-1}^j|^2) + |\bar{g}_t^j|^2 \\ &\leq \sum_{k=0}^{t} \beta_1^{2(t-k)} |\bar{g}_k^j|^2 \\ &\leq \sum_{k=0}^{t} \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}} \beta_2^{t-k} |\bar{g}_k^j|^2 \end{aligned} \tag{51}$$

Using Cauchy-Schwartz inequality we obtain

$$\begin{aligned} |m_{t+1}^j|^2 \leq \sum_{k=0}^{t} \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}} \beta_2^{t-k} |\bar{g}_k^j|^2 &\leq \sum_{k=0}^{t} \left(\frac{\beta_1^2}{\beta_2}\right)^{t-k} \sum_{k=0}^{t} \beta_2^{t-k} |\bar{g}_k^j|^2 \\ &\leq \frac{1}{1 - \frac{\beta_1^2}{\beta_2}} \sum_{k=0}^{t} \beta_2^{t-k} |\bar{g}_k^j|^2 \end{aligned} \tag{52}$$

On the other hand we have

$$\hat{v}_{t+1}^j \geq \beta_2 \hat{v}_t^j + (1-\beta_2)(\bar{g}_t^j)^2 \tag{53}$$

and since it is also true for iteration $t = 1$, we have by induction replacing v_t^j in the above that

$$\hat{v}_{t+1}^j \geq (1-\beta_2) \sum_{k=0}^{t} \beta_2^{t-k} |\bar{g}_k^j|^2 \iff \frac{\sum_{k=0}^{t} \beta_2^{t-k} |\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \leq (1-\beta_2)^{-1} \tag{54}$$

Hence, we can derive from (50) that

$$\begin{aligned} \|\theta_{t+1} - \theta_t\|^2 = \eta_{t+1}^2 \sum_{j=1}^{d} \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j + \epsilon} &\leq \eta_{t+1}^2 \sum_{j=1}^{d} \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j} \\ &\overset{(a)}{\leq} \eta_{t+1}^2 \sum_{j=1}^{d} \frac{1}{1 - \frac{\beta_1^2}{\beta_2}} \frac{\sum_{k=0}^{t} \beta_2^{t-k} |\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \\ &\overset{(b)}{\leq} \eta_{t+1}^2 d (1-\beta_2)^{-1} (1 - \frac{\beta_1^2}{\beta_2})^{-1} \end{aligned} \tag{55}$$

where (a) uses (52) and (b) uses (54).

Plugging the two bounds in (48), we obtain the following bound:

$$\begin{aligned} \frac{1}{T_{\mathrm{m}}} \sum_{t=0}^{T_{\mathrm{m}}-1} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq & \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_{\mathrm{m}}})]}{\eta \Delta_1 T_{\mathrm{m}}} + \frac{q^2 \eta + \sum_{k=t+1}^{\infty} \eta \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2}}{\eta \Delta_1 T_{\mathrm{m}}} \\ & + \frac{1-\beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2+1)} G^2 \\ & + \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right) \frac{1}{\eta \Delta_1} \eta^2 d (1-\beta_2)^{-1} (1 - \frac{\beta_1^2}{\beta_2})^{-1} \end{aligned} \tag{56}$$

where $\Delta_1 := \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}$

With a constant stepsize $\eta = \frac{L}{\sqrt{T_{\text{m}}}}$ we get the final convergence bound as follows:

$$\begin{aligned}\frac{1}{T_{\text{m}}}\sum_{t=0}^{T_{\text{m}}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq& \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_{\text{m}}})]}{L\Delta_1\sqrt{T_{\text{m}}}} + d\frac{L\Delta_3}{\Delta_1\sqrt{T_{\text{m}}}} \\ &+ \frac{\Delta_2}{\eta\Delta_1 T_{\text{m}}} + \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2\end{aligned} \tag{57}$$

where $\Delta_2 := q^2 + \sum_{k=t+1}^{\infty}\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}$ and $\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right)(1-\beta_2)^{-1}(1-\frac{\beta_1^2}{\beta_2})^{-1}$.

$\square$