
Fast Two-Time-Scale Noisy EM Algorithms

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Abstract

1 Training latent data models using the EM algorithm is the most common choice
2 for current learning tasks. Variants of the EM to scale to large datasets and by-
3 pass the impossible conditional expectation of the latent data for most nonlinear
4 models have been initially introduced respectively by [Neal and Hinton, 1998],
5 using incremental updates, and [Wei and Tanner, 1990, Delyon et al., 1999], using
6 Monte-Carlo (MC) approximations. In this paper, we propose to combine those
7 both techniques in a single class of methods called Two-Time-Scale EM Methods.
8 We motivate the choice of a double dynamics by invoking the variance reduction
9 virtue of each stage of the method on both noise: the incremental update and the
10 MC approximation. We establish finite-time convergence bounds for nonconvex
11 objective function and independent of the initialization. Numerical applications
12 are also presented in this article to illustrate our findings.

1 Introduction

14 Learning latent data models is critical for modern machine learning problems, see [McLachlan and
15 Krishnan, 2007] for references. We formulate the training of such model as the following empirical
16 risk minimization problem:

$$\min_{\theta \in \Theta} \bar{L}(\theta) := r(\theta) + L(\theta) \quad \text{with} \quad L(\theta) = \frac{1}{n} \sum_{i=1}^n L_i(\theta) := \frac{1}{n} \sum_{i=1}^n \{ -\log g(y_i; \theta) \}, \quad (1)$$

17 We denote the observations by $\{y_i\}_{i=1}^n$, $\Theta \subset \mathbb{R}^d$ is the convex parameters space. We consider a
18 regularized model where $r : \Theta \rightarrow \mathbb{R}$ is a smooth convex regularization function and for $\theta \in \Theta$,
19 $g(y; \theta)$ is the (incomplete) likelihood of each individual observation. The objective function $\bar{L}(\theta)$ is
20 possibly *nonconvex* and is assumed to be lower bounded $\bar{L}(\theta) > -\infty$ for all $\theta \in \Theta$.

21 In the latent variable model, $g(y_i; \theta)$, is the marginal of the complete data likelihood defined as
22 $f(z_i, y_i; \theta)$, i.e. $g(y_i; \theta) = \int_{\mathcal{Z}} f(z_i, y_i; \theta) \mu(dz_i)$, where $\{z_i\}_{i=1}^n$ are the (unobserved) latent vari-
23 ables. In this paper, we make the assumption of a complete model belonging to the curved expo-
24 nential family, i.e.,

$$f(z_i, y_i; \theta) = h(z_i, y_i) \exp(\langle S(z_i, y_i) | \phi(\theta) \rangle - \psi(\theta)), \quad (2)$$

25 where $\psi(\theta)$, $h(z_i, y_i)$ are scalar functions, $\phi(\theta) \in \mathbb{R}^k$ is a vector function, and $S(z_i, y_i) \in \mathbb{R}^k$ is
26 the complete data sufficient statistics.

27 Full batch EM [Dempster et al., 1977] is the method of reference for that kind of task and is a two
28 steps procedure. The E-step amounts to computing the conditional expectation of the complete data
29 sufficient statistics,

$$\bar{s}(\theta) = \frac{1}{n} \sum_{i=1}^n \bar{s}_i(\theta) \quad \text{where} \quad \bar{s}_i(\theta) = \int_{\mathcal{Z}} S(z_i, y_i) p(z_i | y_i; \theta) \mu(dz_i). \quad (3)$$

30 The M-step is given by

$$\text{M-step: } \hat{\theta} = \bar{\theta}(\bar{s}(\theta)) := \arg \min_{\vartheta \in \Theta} \{r(\vartheta) + \psi(\vartheta) - \langle \bar{s}(\theta) | \phi(\vartheta) \rangle\}, \quad (4)$$

31 Two caveats of this method are the following: (a) with the explosion of data, the first step of the EM
 32 is computationally inefficient as it requires a full pass over the dataset at each iteration and (b) the
 33 complexity of modern models makes the expectation intractable. So far, both challenges have been
 34 addressed separately, to the best of our knowledge, and we give an overview of current solutions in
 35 the sequel.

36 **Prior Work** Inspired by stochastic optimization procedures, [Neal and Hinton, 1998] and [Cappé
 37 and Moulines, 2009] developed respectively an incremental and an online variant of the E-step in
 38 models where the expectation is computable then extensively used and studied in [Nguyen et al.,
 39 2020, Liang and Klein, 2009, Cappé, 2011]. Some improvements of that methods have been pro-
 40 vided and analyzed, globally and in finite-time, in [Karimi et al., 2019] where variance reduction
 41 techniques taken from the optimization literature have been efficiently applied to scale the EM algo-
 42 rithm to large datasets.

43 Regarding the computation of the expectation under the posterior distribution, the first method was
 44 the Monte-Carlo EM (MCEM) introduced in the seminal paper [Wei and Tanner, 1990] where a MC
 45 approximation of this expectation is computed. A variant of that method is the Stochastic Approx-
 46 imation of the EM (SAEM) in [Delyon et al., 1999] leveraging the power of Robbins-Monro type of
 47 update [Robbins and Monro, 1951] to ensure pointwise convergence of the vector of estimated pa-
 48 rameters rather using a decreasing stepsize than increasing the number of MC samples. The MCEM
 49 and the SAEM have been successfully applied in mixed effects models [McCulloch, 1997, Hughes,
 50 1999, Baey et al., 2016] or to do inference for joint modelling of time to event data coming from
 51 clinical trials in [Chakraborty and Das, 2010], among other applications.

52 Recently, an incremental variant of the SAEM was proposed in [Kuhn et al., 2019] showing positive
 53 empirical results but its analysis is limited to asymptotic consideration. Gradient-based methods
 54 have been developed and analyzed in [Zhu et al., 2017] but they remain out of the scope of this
 55 paper as they tackle the high-dimensionality issue.

56 **Contributions** This paper *introduces* and *analyzes* a new class of methods which purpose is to
 57 combine both solutions proposed in the past years in a two-time-scale manner in order to optimize
 58 (1) for current modern examples and settings. The main contributions of the paper are:

- 59 • We propose a two-time-scale method based on Stochastic Approximation (SA), to alleviate
 60 the problem of MC computation, and on Incremental updates, to scale to large datasets.
 61 We describe in details the edges of each level of our method based on variance reduc-
 62 tion arguments. The derivation of such class of algorithms has two advantages. First, it
 63 combines two powerful ideas, commonly used separately, to tackle large scale and highly
 64 nonlinear learning tasks. Then, it gives a simple formulation as a *scaled-gradient method*,
 65 as introduced in [Karimi et al., 2019], which makes the global analysis accessible.
- 66 • We also establish global (independent of the initialization) and finite-time (true at each
 67 iteration) upper bounds on a classical suboptimality condition in the nonconvex literature,
 68 *i.e.*, the second order moment of the gradient of the objective function.

69 In Section 2 we give rigorous mathematical definitions of the various updates used for both incre-
 70 mental and Monte-Carlo EMs and we introduce the main class of new algorithms, based on two
 71 different dynamics, we are proposing to analyze and compare to baselines algorithms. Section 3
 72 presents the main theoretical guarantees of this newly introduced two-time-scale class of algorithms.
 73 Results are given both in finite-time and in the nonconvex setting. Finally, we illustrate the advan-
 74 tages of our method in Section 4 on two numerical experiments.

75 2 Two-Time-Scale Stochastic EM Algorithms

76 We recall and formalize in this section the different methods found in the literature that aim to solv-
 77 ing the large scale problem and the intractable expectation. We then provide the general framework
 78 of our method to efficiently tackle the optimization problem (1).

79 2.1 Monte Carlo Integration and Stochastic Approximation

80 As mentioned in the introduction, for complex and possibly nonlinear models, the expectation under
 81 the posterior distribution defined in (3) is not tractable. In that case, the first solution involves
 82 computing a Monte Carlo integration of that latter term. For all $i \in \llbracket 1, n \rrbracket$, draw for $m \in \llbracket 1, M \rrbracket$,
 83 samples $z_{i,m} \sim p(z_i | y_i; \theta)$ and compute the MC integration \tilde{s} of the deterministic quantity $\bar{s}(\theta)$:

$$\text{MC-step : } \tilde{s} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}, y_i) \quad (5)$$

84 and compute $\hat{\theta} = \bar{\theta}(\tilde{s})$.

85 This algorithm bypasses the intractable expectation issue but is rather computationally expensive in
 86 order to reach point wise convergence (M needs to be large).

87 As a result, an alternative to that stochastic algorithm is to use a Robbins-Monro (RM) type of
 88 update. We denote

$$\tilde{S}^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(k+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{M} \sum_{m=1}^M S(z_{i,m}^{(k)}, y_i) \quad (6)$$

89 where $z_{i,m}^{(k)} \sim p(z_i | y_i; \theta^{(k)})$. At iteration k , the sufficient statistics $\hat{s}^{(k+1)}$ is approximated as follows:

$$\text{SA-step : } \hat{s}^{(k+1)} = \hat{s}^{(k)} + \gamma_{k+1}(\tilde{S}^{(k+1)} - \hat{s}^{(k)}) \quad (7)$$

90 where $\{\gamma_k\}_{k=1}^\infty \in [0, 1]$ is a sequence of decreasing step sizes to ensure asymptotic convergence.
 91 This is called the Stochastic Approximation of the EM (SAEM), see [Delyon et al., 1999] and allows
 92 a smooth convergence to the target parameter. It represents the *first level* of our algorithm (needed
 93 to temper the variance and noise implied by MC integration).

94 In the next section, we derive variants of this algorithm to adapt of the sheer size of data of today's
 95 applications.

96 2.2 Incremental and Bi-Level Inexact EM Methods

97 Strategies to scale to large datasets include classical incremental and variance reduced variants. We
 98 will explicit a general update that will cover those variants and that represents the *second level* of our
 99 algorithm, namely the incremental update of the noisy statistics $\hat{S}^{(k)}$ inside the RM type of update.

$$\text{Inexact-step : } \tilde{S}^{(k+1)} = \tilde{S}^{(k)} + \rho_{k+1}(\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}), \quad (8)$$

100 Note $\{\rho_k\}_{k=1}^\infty \in [0, 1]$ is a sequence of step sizes, $\mathcal{S}^{(k)}$ is a proxy for $\tilde{S}^{(k)}$, If the stepsize is equal
 101 to one and the proxy $\mathcal{S}^{(k)} = \hat{S}^{(k)}$, i.e., computed in a full batch manner as in (6), then we recover
 102 the SAEM algorithm. Also if $\rho_k = 1$, $\gamma_k = 1$ and $\mathcal{S}^{(k)} = \tilde{S}^{(k)}$, then we recover the Monte Carlo
 103 EM algorithm.

104 We now introduce three variants of the SAEM update depending on different definitions of the proxy
 105 $\mathcal{S}^{(k)}$ and the choice of the stepsize ρ_k . Let $i_k \in \llbracket 1, n \rrbracket$ be a random index drawn at iteration k and
 106 $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$ be the iteration index where $i \in \llbracket 1, n \rrbracket$ is last drawn prior to
 107 iteration k . For iteration $k \geq 0$, the fiSAEM method draws *two* indices *independently* and uniformly
 108 as $i_k, j_k \in \llbracket 1, n \rrbracket$. In addition to τ_i^k which was defined w.r.t. i_k , we define $t_j^k = \{k' : j_{k'} = j, k' <$
 109 $k\}$ to be the iteration index where the sample $j \in \llbracket 1, n \rrbracket$ is last drawn as j_k prior to iteration k . With
 110 the initialization $\bar{\mathcal{S}}^{(0)} = \bar{s}^{(0)}$, we use a slightly different update rule from SAGA inspired by [Reddi

111 et al., 2016]. Then, we obtain:

$$(iSAEM [Karimi, 2019, Kuhn et al., 2019]) \quad \mathcal{S}^{(k+1)} = \mathcal{S}^{(k)} + \frac{1}{n} (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)}) \quad (9)$$

$$(vrSAEM This paper) \quad \mathcal{S}^{(k+1)} = \tilde{S}^{(\ell(k))} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\ell(k))}) \quad (10)$$

$$(fiSAEM This paper) \quad \mathcal{S}^{(k+1)} = \bar{\mathcal{S}}^{(k)} + (\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(t_{i_k}^k)}) \quad (11)$$

$$\bar{\mathcal{S}}^{(k+1)} = \bar{\mathcal{S}}^{(k)} + n^{-1} (\tilde{S}_{j_k}^{(k)} - \tilde{S}_{j_k}^{(t_{j_k}^k)}). \quad (12)$$

112 The stepsize is set to $\rho_{k+1} = 1$ for the iSAEM method; $\rho_{k+1} = \gamma$ is constant for the vrSAEM and
 113 fiSAEM methods. Moreover, for iSAEM we initialize with $\mathcal{S}^{(0)} = \tilde{S}^{(0)}$; for vrSAEM we set an
 114 epoch size of m and define $\ell(k) := m \lfloor k/m \rfloor$ as the first iteration number in the epoch that iteration
 115 k is in.

116 2.3 Two-Time-Scale Noisy EM methods

117 We now introduce the general method derived using the two variance reduction techniques described
 118 above. Algorithm 1 leverages both levels (7) and (8) in order to output a vector of fitted parameters
 119 $\hat{\theta}^{(K)}$ where K is some randomly chosen termination point.

120 The updates in (14) is said to have two timescales as the step sizes satisfy $\lim_{k \rightarrow \infty} \gamma_k / \rho_k < 1$ such that
 121 $\tilde{S}^{(k+1)}$ is updated at a faster timescale than $\hat{s}^{(k+1)}$.

Algorithm 1 Two-Time-Scale Noisy EM methods.

- 1: **Input:** initializations $\hat{\theta}^{(0)} \leftarrow 0, \hat{s}^{(0)} \leftarrow \tilde{S}^{(0)}, K_{\max} \leftarrow \text{max. iteration number.}$
- 2: Set the terminating iteration number, $K \in \{0, \dots, K_{\max} - 1\}$, as a discrete r.v. with:

$$P(K = k) = \frac{\gamma_k}{\sum_{\ell=0}^{K_{\max}-1} \gamma_{\ell}}. \quad (13)$$

- 3: **for** $k = 0, 1, 2, \dots, K$ **do**
- 4: Draw index $i_k \in \llbracket 1, n \rrbracket$ uniformly (and $j_k \in \llbracket 1, n \rrbracket$ for fiSAEM).
- 5: Compute $\hat{S}_{i_k}^{(k)}$ using the MC-step (5), for the drawn indices.
- 6: Compute the surrogate sufficient statistics $\mathcal{S}^{(k+1)}$ using (9) or (10) or (11).
- 7: Compute $\tilde{S}^{(k+1)}$ and $\hat{s}^{(k+1)}$ using respectively (8) and (7):

$$\begin{aligned} \tilde{S}^{(k+1)} &= \tilde{S}^{(k)} + \rho_{k+1} (\mathcal{S}^{(k+1)} - \tilde{S}^{(k)}) \\ \hat{s}^{(k+1)} &= \hat{s}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{s}^{(k)}) \end{aligned} \quad (14)$$

- 8: Compute $\hat{\theta}^{(k+1)}$ via the M-step (4).
 - 9: **end for**
 - 10: **Return:** $\hat{\theta}^{(K)}$.
-

122 3 Global and Finite Time Analysis of the Scheme

123 First, we consider the following minimization problem on the statistics space:

$$\min_{\mathbf{s} \in \mathcal{S}} V(\mathbf{s}) := \bar{\mathcal{L}}(\bar{\theta}(\mathbf{s})) = \mathbf{r}(\bar{\theta}(\mathbf{s})) + \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\bar{\theta}(\mathbf{s})) \quad (15)$$

124 It has been shown that this minimization problem is equivalent to the optimization problem (1), see
 125 [Karimi et al., 2019, Lemma2]

126 **H1.** Θ is an open set of \mathbb{R}^d and the sets Z, S are measurable open sets such that:

$$S \supset \left\{ n^{-1} \sum_{i=1}^n u_i, u_i \in \text{conv}(\bar{\mathbf{s}}_i(\theta)) \right\} \quad (16)$$

127 where $\bar{\mathbf{s}}_i(\theta)$ is defined in (3).

128 **H2.** The conditional distribution is smooth on $\text{int}(\Theta)$. For any $i \in \llbracket 1, n \rrbracket$, $z \in \mathcal{Z}$, $\theta, \theta' \in \text{int}(\Theta)^2$,
 129 we have $|p(z|y_i; \theta) - p(z|y_i; \theta')| \leq L_p \|\theta - \theta'\|$.

130 We also recall from the introduction that we consider curved exponential family models. besides:

131 **H3.** For any $s \in S$, the function $\theta \mapsto L(s, \theta) := r(\theta) + \psi(\theta) - \langle s | \phi(\theta) \rangle$ admits a unique global
 132 minimum $\bar{\theta}(s) \in \text{int}(\Theta)$. In addition, $J_\phi^\theta(\bar{\theta}(s))$ is full rank and $\bar{\theta}(s)$ is L_θ -Lipschitz.

133 Similar to [Karimi et al., 2019], we denote by $H_L^\theta(s, \theta)$ the Hessian (w.r.t to θ for a given value of
 134 s) of the function $\theta \mapsto L(s, \theta) = r(\theta) + \psi(\theta) - \langle s | \phi(\theta) \rangle$, and define

$$B(s) := J_\phi^\theta(\bar{\theta}(s)) \left(H_L^\theta(s, \bar{\theta}(s)) \right)^{-1} J_\phi^\theta(\bar{\theta}(s))^\top. \quad (17)$$

135 **H4.** It holds that $v_{\max} := \sup_{s \in S} \|B(s)\| < \infty$ and $0 < v_{\min} := \inf_{s \in S} \lambda_{\min}(B(s))$. There exists
 136 a constant L_B such that for all $s, s' \in S^2$, we have $\|B(s) - B(s')\| \leq L_B \|s - s'\|$.

137 We now formulate the main difference with the work done in [Karimi et al., 2019]. The class of
 138 algorithms we develop in this paper are two time-scale where the first stage corresponds to the
 139 variance reduction trick used in [Karimi et al., 2019] in order to accelerate incremental methods and
 140 kill the variance induced by the index sampling. The second stage is the Robbins-Monro type of
 141 update that aims to kill the variance induced by the MC approximations

142 Indeed the expectations (3) are never available and requires Monte Carlo approximation. Thus, at
 143 iteration $k + 1$, we introduce the errors when approximating the quantity $\bar{s}_i(\hat{\theta}(\hat{s}^{(k-1)}))$. For all
 144 $i \in \llbracket 1, n \rrbracket$, $r > 0$ and $\vartheta \in \Theta$, define:

$$\eta_i^{(r)} := \tilde{S}_i^{(r)} - \bar{s}_i(\vartheta^{(r)}) \quad (18)$$

145 For instance, we consider that the MC approximation is unbiased if for all $i \in \llbracket 1, n \rrbracket$ and $m \in$
 146 $\llbracket 1, M \rrbracket$, the samples $z_{i,m} \sim p(z_i|y_i; \theta)$ are i.i.d. under the posterior distribution, i.e., $\mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r] = 0$
 147 where \mathcal{F}_r is the filtration up to iteration r .

148 The following results are derived under the assumption of control of the fluctuations implied by the
 149 approximation stated as follows:

150 **H5.** There exist a positive sequence of MC batch size $\{M_r\}_{r>0}$ and constants (C, C_η) such that for
 151 all $k > 0$, $i \in \llbracket 1, n \rrbracket$ and $\vartheta \in \Theta$:

$$\mathbb{E} \left[\left\| \eta_i^{(r)} \right\|^2 \right] \leq \frac{C_\eta}{M_r} \quad \text{and} \quad \mathbb{E} \left[\left\| \mathbb{E}[\eta_i^{(r)} | \mathcal{F}_r] \right\|^2 \right] \leq \frac{C}{M_r} \quad (19)$$

152 **Lemma 1.** [Karimi et al., 2019] Assume H2, H3, H4. For all $s, s' \in S$ and $i \in \llbracket 1, n \rrbracket$, we have

$$\|\bar{s}_i(\bar{\theta}(s)) - \bar{s}_i(\bar{\theta}(s'))\| \leq L_s \|s - s'\|, \quad \|\nabla V(s) - \nabla V(s')\| \leq L_V \|s - s'\|, \quad (20)$$

153 where $L_s := C_Z L_p L_\theta$ and $L_V := v_{\max}(1 + L_s) + L_B C_S$.

154 3.1 Global Convergence of Incremental Noisy EM Algorithms

155 Following the asymptotic analysis of update (9), we present a finite-time analysis of the incremental
 156 variant of the Stochastic Approximation of the EM algorithm.

157 The first intermediate result is the computation of the quantity $\hat{S}^{(k+1)} - \hat{s}^{(k)}$, which corresponds to
 158 the drift term of (7) and reads as follows:

159 **Lemma 2.** Assume H1. The update (9) is equivalent to the following update on the resulting statis-
 160 tics

$$\hat{s}^{(k+1)} = \hat{s}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{s}^{(k)}) \quad (21)$$

161 Also:

$$\mathbb{E} [\tilde{S}^{(k+1)} - \hat{s}^{(k)}] = \mathbb{E} [\bar{s}^{(k)} - \hat{s}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \hat{s}^{(k)} \right] + \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}] \quad (22)$$

162 where $\bar{s}^{(k)}$ is defined by (3) and $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$.

The following main result for the iSAEM algorithm is derived under a control of the Monte Carlo fluctuations as described by assumption H 5. Typically, the controls exhibited below are of interest when the number of MC samples M_k increase with the iteration index f .

Theorem 1. *Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of positive step sizes and consider the iSAEM sequence $\{\hat{s}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = 1$ for any $k > 0$.*

Assume that $\hat{s}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$.

TO COMPLETE WITH BOUND

See proof in Appendix C.

3.2 Global Convergence of Two-Time-Scale Noisy EM Algorithms

We now proceed by giving our main result regarding the global convergence of the fiSAEM algorithm.

Theorem 2. *Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of positive step sizes and consider the fiSAEM sequence $\{\hat{s}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = \rho$ for any $k > 0$.*

Assume that $\hat{s}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$.

TO COMPLETE WITH BOUND

See proof in Appendix D.

4 Numerical Examples

4.1 Gaussian Mixture Models

Given n observations $\{y_i\}_{i=1}^n$, we want to fit a Gaussian Mixture Model (GMM) whose distribution is modeled as a Gaussian mixture of M components, each with a unit variance. Let $z_i \in \llbracket M \rrbracket$ be the latent labels of each component, the complete log-likelihood is defined as:

$$\log f(z_i, y_i; \theta) = \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) [\log(\omega_m) - \mu_m^2/2] + \sum_{m=1}^M \mathbb{1}_{\{m\}}(z_i) \mu_m y_i + \text{constant} . \quad (23)$$

where $\theta := (\omega, \mu)$ with $\omega = \{\omega_m\}_{m=1}^{M-1}$ are the mixing weights with the convention $\omega_M = 1 - \sum_{m=1}^{M-1} \omega_m$ and $\mu = \{\mu_m\}_{m=1}^M$ are the means. We use the penalization $r(\theta) = \frac{\delta}{2} \sum_{m=1}^M \mu_m^2 - \log \text{Dir}(\omega; M, \epsilon)$ where $\delta > 0$ and $\text{Dir}(\cdot; M, \epsilon)$ is the M dimensional symmetric Dirichlet distribution with concentration parameter $\epsilon > 0$. The constraint set on θ is given by

$$\Theta = \{\omega_m, m = 1, \dots, M-1 : \omega_m \geq 0, \sum_{m=1}^{M-1} \omega_m \leq 1\} \times \{\mu_m \in \mathbb{R}, m = 1, \dots, M\}. \quad (24)$$

In the following experiments of synthetic data, we generate samples from a GMM model with $M = 2$ components with two mixtures with means $\mu_1 = -\mu_2 = 0.5$.

We use $n = 10^4$ synthetic samples and run the bEM method until convergence (to double precision) to obtain the ML estimate μ^* . We compare the bEM, SAEM, iSAEM, vrSAEM and fiSAEM methods in terms of their precision measured by $|\mu - \mu^*|^2$. The left plot of Figure ?? shows the convergence of the precision $|\mu - \mu^*|^2$ for the different methods against the epoch(s) elapsed (one epoch equals n iterations). We observe that the vrSAEM and fiSAEM methods outperform the other methods, supporting our analytical results.

4.2 Deformable Template Model for Image Analysis

We now run our different methods using an example taken from [Allasonnière et al., 2010]. Let $(y_i, i \in \llbracket 1, n \rrbracket)$ be observed images. Let $u \in \mathcal{U} \subset \mathbb{R}^2$ denote the pixel index on the image and $x_u \in \mathcal{D} \subset \mathbb{R}^2$ its location.

The model used in this experiment suggests that each image y_i is a deformation of a template, noted $I : \mathcal{D} \rightarrow \mathbb{R}$, common to all images of the dataset:

$$y_i(u) = I(x_u - \Phi_i(x_u)) + \varepsilon_i(u) \quad (25)$$

202 where $\phi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a deformation function, and $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$ is an observation error.

203 The template model, given $(p_k, k \in \llbracket 1, k_p \rrbracket)$ landmarks on the template, a fixed known kernel $\mathbf{K}_{\mathbf{p}}$
 204 and a vector of parameters $\beta \in \mathbb{R}^{k_p}$ is defined as follows:

$$I_{\xi} = \mathbf{K}_{\mathbf{p}}\beta, \quad \text{where} \quad (\mathbf{K}_{\mathbf{p}}\beta)(x) = \sum_{k=1}^{k_p} \mathbf{K}_{\mathbf{p}}(x, p_k) \beta_k \quad (26)$$

205 Besides, we parameterize the deformation model given some landmarks $(g_k, k \in \llbracket 1, k_g \rrbracket)$ and a
 206 fixed kernel $\mathbf{K}_{\mathbf{g}}$ as:

$$\Phi_i(x) = (\mathbf{K}_{\mathbf{g}}z_i)(x) = \sum_{k=1}^{k_s} \mathbf{K}_{\mathbf{g}}(x, g_k) \left(z_i^{(1)}(k), z_i^{(2)}(k) \right) \quad (27)$$

207 where $z_i \sim (0, \Gamma)$ and $z_i \in (\mathbb{R}^{k_g})^2$.

208 5 Conclusion

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257 A Proof of Lemma 2

258 **Lemma.** Assume H1. The update (9) is equivalent to the following update on the resulting statistics

$$259 \hat{\mathbf{s}}^{(k+1)} = \hat{\mathbf{s}}^{(k)} + \gamma_{k+1} (\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}) \quad (28)$$

260 Also:

$$\mathbb{E} [\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] = \mathbb{E} [\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \hat{\mathbf{s}}^{(k)} \right] + \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}] \quad (29)$$

261 where $\bar{\mathbf{s}}^{(k)}$ is defined by (3) and $\tau_i^k = \max\{k' : i_{k'} = i, k' < k\}$.

262 **Proof** From update (9), we have:

$$\begin{aligned} \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} &= \tilde{S}^{(k)} - \hat{\mathbf{s}}^{(k)} + \frac{1}{n} \left(\tilde{S}_{i_k}^{(k+1)} - \tilde{S}_{i_k}^{(\tau_{i_k}^k)} \right) \\ &= \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{S}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left(\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \tilde{S}_{i_k}^{(k+1)} \right) \end{aligned} \quad (30)$$

263 Since $\tilde{S}_{i_k}^{(k+1)} = \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) + \eta_{i_k}^{(k+1)}$ we have

$$\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} = \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} + \tilde{S}^{(k)} - \bar{\mathbf{s}}^{(k)} - \frac{1}{n} \left(\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) \right) - \frac{1}{n} \eta_{i_k}^{(k+1)} \quad (31)$$

264 Taking the full expectation of both side of the equation leads to:

$$\begin{aligned} \mathbb{E} [\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}] &= \mathbb{E} [\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}] + \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right] \\ &\quad - \frac{1}{n} \mathbb{E} \left[\mathbb{E} [\tilde{S}_{i_k}^{(\tau_{i_k}^k)} - \bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k] \right] - \frac{1}{n} \mathbb{E} [\eta_{i_k}^{(k+1)}] \end{aligned} \quad (32)$$

265 The following equalities:

$$\mathbb{E} [\tilde{S}_i^{(\tau_i^k)} | \mathcal{F}_k] = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} \quad \text{and} \quad \mathbb{E} [\bar{\mathbf{s}}_{i_k}(\boldsymbol{\theta}^{(k)}) | \mathcal{F}_k] = \bar{\mathbf{s}}^{(k)} \quad (33)$$

266 concludes the proof of the Lemma. \square

267 B Auxiliary Lemma

268 **Lemma.** Assume H3, H4. For all $\mathbf{s} \in \mathcal{S}$,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (34)$$

269 **Proof** Using H3 and the fact that we can exchange integration with differentiation and the Fisher's
270 identity, we obtain

$$\begin{aligned} \nabla_{\mathbf{s}} V(\mathbf{s}) &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left(\nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{L}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \left(\nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \right) \\ &= \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s})^{\top} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} (\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))), \end{aligned} \quad (35)$$

271 Consider the following vector map:

$$\mathbf{s} \rightarrow \nabla_{\boldsymbol{\theta}} L(\mathbf{s}, \boldsymbol{\theta})|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})} = \nabla_{\boldsymbol{\theta}} \psi(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \nabla_{\boldsymbol{\theta}} \mathbf{r}(\bar{\boldsymbol{\theta}}(\mathbf{s})) - \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top} \mathbf{s}. \quad (36)$$

272 Taking the gradient of the above map w.r.t. \mathbf{s} and using assumption H3, we show that:

$$\mathbf{0} = -\mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) + \underbrace{\left(\nabla_{\boldsymbol{\theta}}^2 (\psi(\boldsymbol{\theta}) + \mathbf{r}(\boldsymbol{\theta}) - \langle \phi(\boldsymbol{\theta}) | \mathbf{s} \rangle) \right)}_{=\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \boldsymbol{\theta})} |_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}(\mathbf{s})} \mathbf{J}_{\bar{\boldsymbol{\theta}}}^{\mathbf{s}}(\mathbf{s}). \quad (37)$$

273 The above yields

$$\nabla_{\mathbf{s}} V(\mathbf{s}) = \mathbf{B}(\mathbf{s})(\mathbf{s} - \bar{\mathbf{s}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))) \quad (38)$$

274 where we recall $\mathbf{B}(\mathbf{s}) = \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s})) \left(\mathbf{H}_L^{\boldsymbol{\theta}}(\mathbf{s}; \bar{\boldsymbol{\theta}}(\mathbf{s})) \right)^{-1} \mathbf{J}_{\phi}^{\boldsymbol{\theta}}(\bar{\boldsymbol{\theta}}(\mathbf{s}))^{\top}$. The proof of (39) follows directly
275 from the assumption H4. \square

276 C Proof of Theorem 1

277 **Theorem.** Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of positive step sizes and
 278 consider the iSAEM sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = 1$ for any $k > 0$.

279 Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$.

280 **TO COMPLETE WITH BOUND**

281 **Proof** Under the smoothness of the Lyapunov function V (cf. Lemma 1) and the following growth
 282 condition for all $\mathbf{s} \in \mathcal{S}$,

$$v_{\min}^{-1} \langle \nabla V(\mathbf{s}) | \mathbf{s} - \bar{\mathbf{s}}(\bar{\theta}(\mathbf{s})) \rangle \geq \|\mathbf{s} - \bar{\mathbf{s}}(\bar{\theta}(\mathbf{s}))\|^2 \geq v_{\max}^{-2} \|\nabla V(\mathbf{s})\|^2, \quad (39)$$

283 proven in [Karimi et al., 2019, Lemma 3], we can write:

$$V(\hat{\mathbf{s}}^{(k+1)}) \leq V(\hat{\mathbf{s}}^{(k)}) + \gamma_{k+1} \langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2 \quad (40)$$

284 Taking the expectation on both sides yields:

$$\mathbb{E} [V(\hat{\mathbf{s}}^{(k+1)})] \leq \mathbb{E} [V(\hat{\mathbf{s}}^{(k)})] + \gamma_{k+1} \mathbb{E} [\langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} [\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] \quad (41)$$

285 Using Lemma 2, we obtain:

$$\begin{aligned} \mathbb{E} [\langle \tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] &= \mathbb{E} [\langle \bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[\left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \right\rangle \right] \\ &\quad + \frac{1}{n} \mathbb{E} [\langle \eta_{i_k}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\stackrel{(a)}{\leq} -v_{\min} \mathbb{E} [\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \left(1 - \frac{1}{n}\right) \mathbb{E} \left[\left\langle \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \right\rangle \right] \\ &\quad + \frac{1}{n} \mathbb{E} [\langle \eta_{i_k}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\stackrel{(b)}{\leq} -v_{\min} \mathbb{E} [\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] \\ &\quad + \frac{\beta(n-1) + \beta'}{2n} \mathbb{E} [\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2] + \frac{1}{2\beta'n} \mathbb{E} [\|\eta_{i_k}^{(k)}\|^2] \\ &\stackrel{(a)}{\leq} \left(v_{\max}^2 \frac{\beta(n-1) + \beta'}{2n} - v_{\min} \right) \mathbb{E} [\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \frac{1 - \frac{1}{n}}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] + \frac{1}{2\beta'n} \mathbb{E} [\|\eta_{i_k}^{(k)}\|^2] \end{aligned} \quad (42)$$

286 where (a) the growth condition (39) is due to and (b) is due to Young's inequality. Note $a_k =$

287 $\gamma_{k+1} \left(v_{\min} - v_{\max}^2 \frac{\beta(n-1) + \beta'}{2n} \right)$ and

$$\begin{aligned} a_k \mathbb{E} [\|\bar{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(k)}\|^2] &\leq \mathbb{E} [V(\hat{\mathbf{s}}^{(k)}) - V(\hat{\mathbf{s}}^{(k+1)})] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E} [\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2] \\ &\quad + \frac{\gamma_{k+1}(1 - \frac{1}{n})}{2\beta} \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^{(\tau_i^k)} - \bar{\mathbf{s}}^{(k)} \right\|^2 \right] + \frac{\gamma_{k+1}}{2\beta'n} \mathbb{E} [\|\eta_{i_k}^{(k)}\|^2] \end{aligned} \quad (43)$$

288 We now give an upper bound of $\mathbb{E} [\|\tilde{S}^{(k+1)} - \hat{\mathbf{s}}^{(k)}\|^2]$.

289 \square

290 D Proof of Theorem 2

291 **Theorem.** Let K_{\max} be a positive integer. Let $\{\gamma_k, k \in \mathbb{N}\}$ be a sequence of positive step sizes and
 292 consider the fiSAEM sequence $\{\hat{\mathbf{s}}^{(k)}, k \in \mathbb{N}\}$ obtained with $\rho_{k+1} = 1$ for any $k > 0$.

293 Assume that $\hat{\mathbf{s}}^{(k)} \in \mathcal{S}$ for any $k \leq K_{\max}$.

294 **TO COMPLETE WITH BOUND**

295 Proof

296 **Lemma 3.** For any $k \geq 0$ and consider the fiSAEM update in (11) with $\rho_k = \rho$, it holds that

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 4\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + \frac{4\rho^2 L_s^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + 2\rho^2 \frac{C_\eta}{M_k} + 4(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] \end{aligned} \quad (44)$$

297 **Proof** Applying the fiSAEM update yields:

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &= \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} + \rho(\tilde{S}^{(k)} - \mathcal{S}^{(k+1)})\|^2] \\ &= \mathbb{E}[\|(1-\rho)(\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}) + \rho(\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}) + \rho[(\bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k)}) - (\tilde{S}_{i_k}^{(k)}) - \tilde{S}_{i_k}^{(\tau_i^k)}]\|^2] \end{aligned} \quad (45)$$

298 We observe that $\mathcal{S}^{(k)} = \frac{1}{n} \sum_{i=1}^n \bar{\mathbf{s}}_i^{(t_i^k)}$ and $\mathbb{E}[\tilde{S}_{i_k}^{(k)} - \tilde{S}_{i_k}^{(\tau_i^k)}] = \bar{\mathbf{s}}^{(k)} - \mathcal{S}^{(k)} + \mathbb{E}[\eta_{i_k}^{(\tau_i^k)}]$. Thus

$$\begin{aligned} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] &\leq 4(1-\rho)^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\|^2] + 4\rho^2 \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\|^2] + 2\rho^2 \mathbb{E}\left[\left\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)}\right\|^2\right] + 2\rho^2 \frac{C_\eta}{M_k} \end{aligned} \quad (46)$$

299 where we use the variance inequality. The last expectation can be further bounded by

$$\mathbb{E}[\|\bar{\mathbf{s}}_{i_k}^{(k)} - \bar{\mathbf{s}}_{i_k}^{(t_{i_k}^k)}\|^2] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\bar{\mathbf{s}}_i^{(k)} - \bar{\mathbf{s}}_i^{(t_i^k)}\|^2] \stackrel{(a)}{\leq} \frac{L_s}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2], \quad (47)$$

300 where (a) is due to Lemma 1 and which concludes the proof of the Lemma.

301 □

302 Using the smoothness of V and update (11), we obtain:

$$\begin{aligned} V(\hat{\mathbf{s}}^{(k+1)}) &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \\ &\quad - \gamma_{k+1} \rho \langle \tilde{S}^{(k)} - \mathcal{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \\ &\leq V(\hat{\mathbf{s}}^{(k)}) - \gamma_{k+1} \rho \langle \hat{\mathbf{s}}^{(k)} - \mathcal{S}^{(k+1)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle \\ &\quad - \gamma_{k+1} (1-\rho) \langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle + \frac{\gamma_{k+1}^2 L_V}{2} \|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2 \end{aligned} \quad (48)$$

303 Taking the expectaitons on both sides and noting that $\mathbb{E}[\mathcal{S}^{(k+1)}] = \mathbb{E}[\mathbb{E}[\mathcal{S}^{(k+1)}|\mathcal{F}_k]] = \bar{\mathbf{s}}^{(k)}$
 304 (independence of both indices i_k and j_k), we have:

$$\begin{aligned} \mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)}) - V(\hat{\mathbf{s}}^{(k)})] &\leq -\gamma_{k+1}\rho\mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] \\ &\quad - \gamma_{k+1}(1-\rho)\mathbb{E}[\langle \hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle] + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \\ &\stackrel{(a)}{\leq} -(\gamma_{k+1}v_{\max}^2 \frac{\beta(1-\rho)}{2} + \gamma_{k+1}\rho)\mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] - \gamma_{k+1}\frac{1-\rho}{2\beta}\mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\right\|^2\right] \\ &\quad + \frac{\gamma_{k+1}^2 L_V}{2} \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2] \end{aligned} \tag{49}$$

305 where (a) used the growth condition (39) twice (on $\langle \hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)} | \nabla V(\hat{\mathbf{s}}^{(k)}) \rangle$ and $\|\nabla V(\hat{\mathbf{s}}^{(k)})\|^2$ and
 306 Young's inequality.

307 **Bounding** $\mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k+1)}\|^2]$ Using Lemma 3, we obtain:

$$\begin{aligned} \mathbb{E}[V(\hat{\mathbf{s}}^{(k+1)}) - V(\hat{\mathbf{s}}^{(k)})] &\leq -\gamma_{k+1}\left(v_{\max}^2 \frac{\beta(1-\rho)}{2} + \rho - 2\rho^2\gamma_{k+1} L_V\right)\mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \bar{\mathbf{s}}^{(k)}\right\|^2\right] \\ &\quad + \frac{2\gamma_{k+1}^2 L_V \rho^2 L_{\mathbf{s}}^2}{n} \sum_{i=1}^n \mathbb{E}[\|\hat{\mathbf{s}}^{(k)} - \hat{\mathbf{s}}^{(t_i^k)}\|^2] \\ &\quad + \gamma_{k+1}(1-\rho)\left(2\gamma_{k+1} L_V(1-\rho) - \frac{1}{2\beta}\right)\mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\right\|^2\right] \\ &\quad + \gamma_{k+1}^2 \rho^2 L_V \frac{C_{\eta}}{M_k} \end{aligned} \tag{50}$$

308 **Bounding** $\mathbb{E}\left[\left\|\hat{\mathbf{s}}^{(k)} - \tilde{S}^{(k)}\right\|^2\right]$ Remark that this term is the price we pay for the two time scale
 309 dynamics and corresponds to the gap between the two asynchrone updates (one is on $\hat{\mathbf{s}}^{(k)}$ and the
 310 other on $\tilde{S}^{(k)}$). □

311