Sparsified Distributed Adaptive Learning with Error Feedback

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Abstract

In this paper, we present a novel optimization algorithm for single-machine and distributed learning, based on sparsification and error feedback techniques to lighten the communications between a central server and distributed workers. The method we introduce builds on the adaptivity of the AMSGrad method for nonconvex optimization, and includes a TopK operation to alleviate any communication bottleneck between a large amount of devices and a central computing server, combined with a correction of the natural bias induced by the latter compression operator. Despite the sparsity induced by our algorithm, we show that SPARS-AMSreaches a stationary point in $\mathcal{O}(1/\sqrt{T})$ iterations, matching that of state-of-the-art single-machine methods. We illustrate on benchmark datasets the effectiveness of our method both under the single-machine and distributed settings.

1 Introduction

Deep neural network has achieved the state-of-the-art learning performance on numerous AI applications, e.g., computer vision [21, 24, 45], Natural Language Processing [23, 52, 56], Reinforcement Learning [35, 43] and recommendation systems [14, 47]. With the increasing size of both data and deep networks, standard single machine training confronts with at least two major challenges:

- Due to the limited computing power of a single machine, it would take a long time to process the massive number of data samples—training would be slow.
- In many practical scenarios, data are typically stored in multiple servers, possibly at different locations, due to the storage constraints (massive user behavior data, Internet images, etc.) or privacy reasons [9]. Transmitting data might be costly.

Distributed learning framework [16] has been a common training strategy to tackle the above two issues. For example, in centralized distributed stochastic gradient descent (SGD) protocol, data are located at N local nodes, at which the gradients of the model are computed in parallel. In each iteration, a central server aggregates the local gradients, updates the global model, and transmits back the updated model to the local nodes for subsequent gradient computation. As we can see, this setting naturally solves aforementioned issues: 1) We use N computing nodes to train the model, so the time per training epoch can be largely reduced; 2) There is no need to transmit the local data to central server. Besides, distributed training also provides stronger error tolerance since the training process could continue even one local machine breaks down. As a result of these advantages, there has been a surge of study and applications on distributed systems [8, 37, 18, 22, 25, 33, 31].

Among many optimization strategies, SGD is still the most popular prototype in distributed training for its simplicity and effectiveness [12, 1, 34]. Yet, when the deep learning model is very large, the communication between local nodes and central server could be expensive. Burdensome gradient transmission would slow down the whole training system, or even be impossible because of the limited bandwidth in some applications. Thus, reducing the communication cost in distributed SGD has become an active topic, and an important ingredient of large-scale distributed systems (e.g. [40]). Solutions based on quantization, sparsification and other compression techniques of the local gradients are proposed, e.g., [3, 48, 46, 44, 2, 6, 15, 50, 26]. As one would expect, in most approaches, there exists a trade-off between compression and model accuracy. In particular, larger bias of the compressed gradients usually brings more significant performance downgrade. Interestingly, [29] shows that the technique of *error feedback* is able to remedy the issue of such biased compressors, achieving same convergence rate and learning performance as full-gradient SGD.

On the other hand, in recent years, adaptive optimization algorithms (e.g. AdaGrad [19], Adam [30] and AMSGrad [39]) have become popular because of their superior empirical performance. These methods use different implicit learning rates for different coordinates that keep changing adaptively throughout the training process, based on the learning trajectory. In many learning problems, adaptive methods have been shown to converge faster than SGD, sometimes with better generalization as well. However, the body of literature that combines adaptive methods with distributed training is still very limited. In this papar, we propose a distributed optimization algorithm with AMSGrad as the backbone, along with Top-k sparsification to reduce the communication cost.

1.1 Our contributions

We develop a simple optimization leveraging the adaptivity of AMSGrad, and the computational virtue of TopK sparsification, for tackling a large finite-sum of nonconvex objective functions.

Our technique is shown to be both theoretically and empirically effective under *the classical centralized setting* and *the distributed setting*.

In this contribution,

- We derive a sparsified AMSGrad with error feedback, called SPARS-AMS, with a single machine and provide its decentralized counter part.
- We provide a non-asymptotic convergence rate under each setting,
- We highlight the effectiveness of both methods through several numerical experiments

2 Related Work

2.1 Communication-efficient distributed SGD

Quantization. As we mentioned before, SGD is the most commonly adopted optimization method in distributed training of deep neural nets. To reduce the expensive communication in large-scale distributed systems, extensive works have considered various compression techniques applied to the gradient transaction procedure. The first strategy is quantization. [17] condenses 32-bit floating numbers into 8-bits when representing the gradients. [40, 6, 29, 7] use the extreme 1-bit information (sign) of the gradients, combined with tricks like momentum, majority vote and memory. Other quantization-based methods include QSGD [3, 49, 55] and LPC-SVRG [53], leveraging unbiased stochastic quantization. The saving in communication of quantization methods is moderate: for example, 8-bit quantization reduces the cost to 25% (compared with 32-bit full-precision). Even in the extreme 1-bit case, the largest compression ratio is around $1/32 \approx 3.1\%$.

Sparsification. Gradient sparsification is another popular solution which may provide higher compression rate. Instead of commuting the full gradient, each local worker only passes a few coordinates to the central server and zeros out the others. Thus, we can more freely choose higher compression ratio (e.g., 1%, 0.1%), still achieving impressive performance in many applications [32]. Stochastic sparsification methods, including uniform sampling and magnitude based sampling [46], select coordinates based on some sampling probability yielding unbiased gradient compressors. Deterministic methods are simpler, e.g., Random-k, Top-k [44, 42] (selecting k elements with largest magnitude), Deep Gradient Compression [32], but usually lead to biased gradient estimation. In [26], the central server identifies heavy-hitters from the count-sketch [10] of the local gradients, which can be regarded as a noisy variant of Top-k strategy. More applications and analysis of compressed distributed SGD can be found in [28, 41, 4, 5, 27], among others.

Error Feedback. Biased gradient estimation, which is a consequence of many aforementioned methods (e.g., signSGD, Top-k), undermines the model training, both theoretically and empirically, with slower convergence and worse generalization. The technique of *error feedback* is able to "correct for the bias" and fix the problems. In this procedure, the difference between the true stochastic gradient and the compressed one is accumulated locally, which is then added back to the local gradients in later iterations. [44, 29] prove the $\mathcal{O}(\frac{1}{T})$ and $\mathcal{O}(\frac{1}{\sqrt{T}})$ convergence rate of EF-SGD in strongly convex and non-convex setting respectively, matching the rates of vanilla SGD [38, 20].

2.2 Adaptive optimization

In each SGD update, all the gradient coordinates share a same learning rate, either constant or decreasing over iterations. Adaptive optimization methods cast different learning rate on each dimension. AdaGrad [19] divides the gradient element-wisely by $\sqrt{\sum_{t=1}^{T} g_t^2} \in \mathbb{R}^d$, where $g_t \in \mathbb{R}^d$ is the gradient vector at time t and d is the model dimensionality. Thus, it intrinsically assigns different learning rates to different coordinates throughout the training—elements with smaller previous gradient magnitude tend to move a larger step. AdaGrad has been shown to perform well especially under some sparsity structure. AdaDelta [54] and Adam [30] introduce momentum and moving average of second moment estimation into AdaGrad which lead to better performance. AMSGrad [39] fixes the potential convergence issue of Adam, which will serve as the prototype in this paper. We present the psudocode in Algorithm . In general, adaptive optimization methods are easier to tune in practice, and usually exhibit faster convergence than SGD. Thus, they have been widely used in training deep learning models in language and computer vision applications, e.g., [13, 51, 57]. In distributed setting, the work [36] proposes a decentralized system in online optimization. However, communication efficiency is not considered. The recent work [11] is the most relevant to our paper. Yet, their method is based on Adam, and requires every local node to store a local estimation of first and second moment, thus being less efficient. We will present more detailed comparison in Section 3.

3 Communication-Efficient Adaptive Optimization

Most modern machine learning tasks can be casted as a large finite-sum optimization problem written as:

$$\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \tag{1}$$

where n denotes the number of workers, f_i represents the average loss for worker i and θ the global model parameter taking value in Θ, a subset of $\mathbb{R}^d$.

Some related work:

[29] develops variant of signSGD (as a biased compression schemes) for distributed optimization. Contributions are mainly on this error feedback variant. In [42], the authors provide theoretical results on the convergence of sparse Gradient SGD for distributed optimization (we want that for AMS here). [44] develops a variant of distributed SGD with sparse gradients too. Contributions include a memory term used while compressing the gradient (using top k for instance). Speeding up the convergence in $\frac{1}{T^3}$.

Consider standard synchronous distributed optimization setting. AMSGrad is used as the prototype, and the local workers is only in charge of gradient computation.

3.1 TopK AMSGrad with Error Feedback

The key difference (and interesting part) of our TopK AMSGrad compared with the following arxiv paper "Quantized Adam"`https://arxiv.org/pdf/2004.14180.pdf` is that, in our model only gradients are transmitted. In "QAdam", each local worker keeps a local copy of moment estimator m and v, and compresses and transmits m/v as a whole. Thus, that method is very much like the sparsified distributed SGD, except that g is changed into m/v. In our model, the moment estimates m and v are computed only at the central server, with the compressed gradients instead of the full gradient. This would be the key (and difficulty) in convergence analysis.

Algorithm 1 SPARS-AMS for Distributed Learning

1: **Input**: parameter β_1, β_2, learning rate η_t.
2: Initialize: central server parameter $\theta_0 \in \Theta \subseteq \mathbb{R}^d$; $e_{1,i} = 0$ the error accumulator for each worker; sparsity parameter k; n local workers; $m_0 = 0$, $v_0 = 0$, $\hat{v}_0 = 0$
3: **for** $t = 1$ to T **do**
4:     **parallel for worker** $i \in [n]$ **do**:
5:         Receive model parameter θ_t from central server
6:         Compute stochastic gradient $g_{t,i}$ at θ_t
7:         Compute $\tilde{g}_{t,i} = TopK(g_{t,i} + e_{t,i}, k)$
8:         Update the error $e_{t+1,i} = e_{t,i} + g_{t,i} - \tilde{g}_{t,i}$
9:         Send $\tilde{g}_{t,i}$ back to central server
10:     **end parallel**
11:     **Central server do:**
12:     $\bar{g}_t = \frac{1}{n} \sum_{i=1}^{N} \tilde{g}_{t,i}$
13:     $m_t = \beta_1 m_{t-1} + (1 - \beta_1)\bar{g}_t$
14:     $v_t = \beta_2 v_{t-1} + (1 - \beta_2)\bar{g}_t^2$
15:     $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
16:     Update global model $\theta_{t+1} = \theta_t - \eta_t \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$
17: **end for**

3.2 Convergence Analysis

Several mild assumptions to make: Nonconvex and smooth loss function, unbiased stochastic gradient, bounded variance of the gradient, bounded norm of the gradient, control of the distance between the true gradient and its sparse variant.

Check [11] starting with single machine and extending to distributed settings (several machines).

Under the distributed setting, the goal is to derive an upper bound to the second order moment of the gradient of the objective function at some iteration $T_f \in [1, T]$.

3.3 Mild Assumptions

We begin by making the following assumptions.

A 1. *(Smoothness) For $i \in [\![n]\!]$, f_i is L-smooth: $\|\nabla f_i(\theta) - \nabla f_i(\vartheta)\| \leq L \|\theta - \vartheta\|$.*

A 2. *(Unbiased and Bounded gradient **per worker**) For any iteration index $t > 0$ and worker index $i \in [\![n]\!]$, the stochastic gradient is unbiased and bounded from above: $\mathbb{E}[g_{t,i}] = \nabla f_i(\theta_t)$ and $\|g_{t,i}\| \leq G_i$.*

A 3. *(Bounded variance **per worker**) For any iteration index $t > 0$ and worker index $i \in [\![n]\!]$, the variance of the noisy gradient is bounded: $\mathbb{E}[|g_{t,i} - \nabla f_i(\theta_t)|^2] < \sigma_i^2$.*

Denote by $Q(\cdot)$ the quantization operator Line 7 of Algorithm 1, which takes as input a gradient vector and returns a quantized version of it, and note $\tilde{g} := Q(g)$. Assume that

A 4. *(Bounded Quantization) For any iteration $t > 0$, there exists a constant $0 < q < 1$ such that $\|g_{t,i} - \tilde{g}_{t,i}\| \leq q \|g_{t,i}\|$, where $g_{t,i}$ is the stochastic gradient computed at iteration t for worker i and $\tilde{g}_{t,i}$ is its quantized counterpart. (high q means large quantization so loss of precision on the true gradient)*

Denote for all $\theta \in \Theta$:

$$f(\theta) := \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \,, \tag{2}$$

where n denotes the number of workers.

3.4 Intermediary Lemmas

Lemma 1. *Under Assumption 2 and Assumption 4 we have for any iteration* $t > 0$*:*

$$\|m_t\|^2 \leq (q^2+1)G^2 \quad \textit{and} \quad \hat{v}_t \leq (q^2+1)G^2 \tag{3}$$

where m_t *and* $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ *are defined Line 15 of Algorithm 1 and* $G^2 = \frac{1}{n}\sum_{i=1}^{N} G_i^2$.

Lemma 2. *Under A1 to A4, with a decreasing sequence of stepsize* $\{\eta_t\}_{t>0}$*, we have:*

$$-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1}+\epsilon \mathsf{I}_\mathsf{d})^{-1/2}\bar{g}_t \right\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \tag{4}$$

where I_d *is the identity matrix,* $\hat{V}_t$ *the diagonal matrix which diagonal entries are* $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ *defined Line 15 of Algorithm 1 and* $\bar{g}_t$ *is the aggregation of all **quantized** gradients from the workers.*

Lemma 3. *Under A1 to A4, with a decreasing sequence of stepsize* $\{\eta_t\}_{t>0}$*, we have:*

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
& - \eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right)\|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j+\epsilon)^{-1/2}\right]]
\end{aligned} \tag{5}$$

where d *denotes the dimension of the parameter vector*

Decentralized Workers Setting:

The main theorem in the decentralized setting reads:

Theorem 1. *Under A1 to A4, with a constant stepsize* $\eta_t = \eta = \frac{L}{\sqrt{T_m}}$*, the sequence of iterates* $\{\theta_t\}_{t>0}$ *output from Algorithm 1 satisfies:*

$$\frac{1}{T_m}\sum_{t=0}^{T_m-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_m})]}{L\Delta_1\sqrt{T_m}} + d\frac{L\Delta_3}{\Delta_1\sqrt{T_m}} + \frac{\Delta_2}{\eta\Delta_1 T_m} + \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2 \tag{6}$$

where

$$\begin{aligned}
&\Delta_1 := \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \quad , \quad \Delta_2 := q^2 + \sum_{k=t+1}^{\infty}\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2} \\
&\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right)(1-\beta_2)^{-1}(1 - \frac{\beta_1^2}{\beta_2})^{-1}
\end{aligned} \tag{7}$$

We remark from this bound in Theorem 1, that the more quantization we apply to our gradient vectors ($q \uparrow$), the larger the upper bound of the stationary condition is, *i.e.,* the slower the algorithm is. This is intuitive as using compressed quantities will definitely impact the algorithm speed. We will observe in the numerical section below that a trade-off on the level of quantization q can be found to achieve similar speed of convergence with less computation resources used throughout the training.

Single Machine Setting:

Theorem 2. *Under A1 to A4, with a constant stepsize* $\eta_t = \eta = \frac{L}{\sqrt{T_m}}$*, the sequence of iterates* $\{\theta_t\}_{t>0}$ *output from Algorithm 2 satisfies:*

$$\frac{1}{T_m}\sum_{t=0}^{T_m-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0)-f(\theta_{T_m})]}{T_m(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)} + \eta^2 G^2 \frac{L}{2}\frac{q^2+1}{\epsilon(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)} \tag{8}$$

$$+ \eta G^2 \frac{q\sqrt{q^2+1}}{\sqrt{\epsilon}(1-q)(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)} + \frac{G^2}{(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)}\left(\frac{q}{1-q}\right)^2\left[\frac{L}{2}q^2+1\right] \tag{9}$$

4 Sequential Model

Single machine method

Algorithm 2 SPARS-AMS : Single machine setting

Input: parameter β_1, β_2, learning rate η_t.
Initialize: central server parameter $\theta_1 \in \Theta \subseteq \mathbb{R}^d$; $e_1 = 0$ the error accumulator; sparsity parameter k; $m_0 = 0$, $v_0 = 0$, $\hat{v}_0 = 0$
for $t = 1$ to T **do**
  Compute stochastic gradient $g_t = g_{t,i_t}$ at θ_t for randomly sampled index i_t
  Compute $\tilde{g}_t = TopK(g_t + e_t, k)$
  Update the error $e_{t+1} = e_t + g_t - \tilde{g}_t$
  $m_t = \beta_1 m_{t-1} + (1-\beta_1)\tilde{g}_t$
  $v_t = \beta_2 v_{t-1} + (1-\beta_2)\tilde{g}_t^2$
  $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
  Update global model $\theta_{t+1} = \theta_t - \eta_t \frac{m_t}{\sqrt{\hat{v}_t+\epsilon}}$
end for

Let m'_t be the first moment moving average of standard AMSGrad using full gradients. $m'_t = (1-\beta_1)\sum_{i=1}^{k}\beta_1^{t-i}g_t$. Denote

$$a_t = \frac{m_t}{\sqrt{\hat{v}_t+\epsilon}}, \quad a'_t = \frac{m'_t}{\sqrt{\hat{v}'_t+\epsilon}}.$$

Define the sequence

$$\mathcal{E}_{t+1} = \mathcal{E}_t + a'_t - a_t,$$

such that the auxiliary model

$$\begin{aligned}
\theta'_{t+1} &:= \theta_{t+1} - \eta\mathcal{E}_{t+1} \\
&= \theta_t - \eta a_t - \eta\mathcal{E}_{t+1} \\
&= \theta_t - \eta a_t - \eta(\mathcal{E}_t + a'_t - a_t) \\
&= \theta'_t - \eta a'_t
\end{aligned}$$

follows the update of full-gradient AMSGrad. By smoothness assumption we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \eta\langle\nabla f(\theta'_t), a'_t\rangle + \frac{L}{2}\|\theta'_{t+1} - \theta'_t\|^2.$$

Thus,

$$\begin{aligned}\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq -\eta\mathbb{E}[\langle \nabla f(\theta'_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] \\ &= -\eta\mathbb{E}[\langle \nabla f(\theta_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), a'_t\rangle] \\ &\leq -\eta\mathbb{E}[\langle \nabla f(\theta_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\frac{\eta^2\rho}{2}\|\mathcal{E}_t\|^2 + \frac{1}{2\rho}\|a'_t\|^2] \\ &\leq -\eta\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\sqrt{G^2+\epsilon}} + \frac{\eta}{2\rho}\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\epsilon} + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \frac{\eta^3\rho}{2}\mathbb{E}\|\mathcal{E}_t\|^2,\end{aligned}$$

when $\beta_1 = 0$ for example. We may discard this assumption and use more complicated bound on the first two terms. The third term can be bounded by constant yielding $O(1/\sqrt{T})$ rate eventually when taking decreasing learning rate. The key is to get a good bound on the cumulative error sequence, $\mathcal{E}_t$. We have the following:

$$\begin{aligned}\mathbb{E}\|\mathcal{E}_{t+1}\|^2 &= \mathbb{E}\|\mathcal{E}_t + a'_t - a_t + TopK(\mathcal{E}_t + a'_t) - TopK(\mathcal{E}_t + a'_t)\|^2 \\ &\leq 2\mathbb{E}\|\mathcal{E}_t + a'_t - TopK(\mathcal{E}_t + a'_t)\|^2 + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2 \\ &\overset{(a)}{\leq} 2q\mathbb{E}\|\mathcal{E}_t + a'_t\| + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2 \\ &\leq 2q[(1+r)\mathbb{E}\|\mathcal{E}_t\|^2 + (1+\frac{1}{r})\mathbb{E}\|a'_t\|^2] + 2\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2.\end{aligned}$$

where (a) uses A3. Current try: If we can bound the last term in the same form as the first two terms, then we can use recursion to get the desired result. We can have

$$\mathbb{E}\|a_t - TopK(\mathcal{E}_t + a'_t)\|^2 = \mathbb{E}\|\frac{\tilde{m}_t}{\sqrt{\hat{v}_t + \epsilon}} - \|^2$$

4.1 New

Let m'_t be the first moment moving average of standard AMSGrad using full gradients, *i.e.,* the gradient with respect to the index data point t_i computed Line 4 of Algorithm 2 before applying any compression operator. By construction we have $m'_t = (1-\beta_1)\sum_{i=1}^k \beta_1^{t-i} g_t$.

Denote the following quantities

$$\begin{aligned}\mathcal{E}_{t+1} &:= \frac{(1-\beta_1)\sum_{i=1}^{t+1}\beta_1^{t+1-i}e_i}{\sqrt{\hat{v}_t + \epsilon}} \\ \theta'_{t+1} &:= \theta_{t+1} - \eta\mathcal{E}_{t+1}\end{aligned}$$

Then,

$$\begin{aligned}\theta'_{t+1} &= \theta_{t+1} - \eta\mathcal{E}_{t+1} \\ &= \theta_t - \eta\frac{(1-\beta_1)\sum_{i=1}^t \beta_1^{t-i}\tilde{g}_i + (1-\beta_1)\sum_{i=1}^{t+1}\beta_1^{t+1-i}e_i}{\sqrt{\hat{v}_t + \epsilon}} \\ &= \theta_t - \eta\frac{(1-\beta_1)\sum_{i=1}^t \beta_1^{t-i}(\tilde{g}_i + e_{i+1}) + (1-\beta)\beta_1^t e_1}{\sqrt{\hat{v}_t + \epsilon}} \\ &= \theta_t - \eta\frac{(1-\beta_1)\sum_{i=1}^t \beta_1^{t-i}e_i}{\sqrt{\hat{v}_t + \epsilon}} - \eta\frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} \\ &\overset{(a)}{=} \theta'_t - \eta\frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} := \theta'_t - \eta a'_t,\end{aligned}$$

where (a) uses the fact that $\tilde{g}_t + e_{t+1} = g_t + e_t$, $e_1 = 0$ at initialization. By smoothness assumption A1 we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \eta\langle \nabla f(\theta'_t), a'_t\rangle + \frac{L}{2}\|\theta'_{t+1} - \theta'_t\|^2.$$

Thus,

$$\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] \le -\eta\mathbb{E}[\langle \nabla f(\theta'_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2]$$
$$= -\eta\mathbb{E}[\langle \nabla f(\theta_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), a'_t\rangle]$$

Using Young's inequality with parameter ρ and the smoothness assumption we have

$$\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] \le -\eta\mathbb{E}[\langle \nabla f(\theta_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\frac{\rho}{2}\|\nabla f(\theta_t) - \nabla f(\theta'_t)\|^2 + \frac{1}{2\rho}\|a'_t\|^2] \tag{10}$$

$$\le -\eta\mathbb{E}[\langle \nabla f(\theta_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\frac{\rho}{2}L^2\|\theta_t - \theta'_t\|^2 + \frac{1}{2\rho}\|a'_t\|^2] \tag{11}$$

$$\le -\eta\mathbb{E}[\langle \nabla f(\theta_t), a'_t\rangle] + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \eta\mathbb{E}[\frac{\eta^2 L^2 \rho}{2}\|\mathcal{E}_t\|^2 + \frac{1}{2\rho}\|a'_t\|^2] \tag{12}$$

$$\le -\eta\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\sqrt{G^2+\epsilon}} + \frac{\eta}{2\rho}\frac{\mathbb{E}\|\nabla f(\theta_t)\|^2}{\epsilon} + \frac{\eta^2 L}{2}\mathbb{E}[\|a'_t\|^2] + \frac{\eta^3\rho L^2}{2}\mathbb{E}\|\mathcal{E}_t\|^2 \tag{13}$$

where we set $\beta_1 = 0$ from (12) to (13).

We may discard this assumption and use more complicated bound on the first two terms. The third term can be bounded by constant yielding $O(1/\sqrt{T})$ rate eventually when taking decreasing learning rate.

Bounding $\mathbb{E}\|\mathcal{E}_t\|^2$. We know that $\|e_t\| \le \frac{q}{1-q}G$. So

$$\|\mathcal{E}_t\|^2 = \left\| \frac{(1-\beta_1)\sum_{i=1}^t \beta_1^{t-i} e_i}{\sqrt{\hat{v}_t + \epsilon}} \right\|^2$$
$$\le \left(\frac{(1-\beta_1)\sum_{i=1}^t \beta_1^{t-i} \|e_i\|}{\sqrt{\epsilon}} \right)^2$$
$$\le \frac{q^2 G^2}{\epsilon(1-q)^2}.$$

Bounding $\mathbb{E}\|a'_t\|^2$. We have (assuming $\mathbb{E}\|g_t\|^2 \le \sigma^2$)

$$\mathbb{E}\|a'_t\|^2 \le \frac{\sigma^2}{\epsilon}.$$

Choosing $\rho = \frac{\sqrt{G^2+\epsilon}}{\epsilon}$ and summing over $t = 1, ..., T$, we obtain

$$\frac{1}{T}\sum_{t=1}^T \mathbb{E}\|\nabla f(\theta_t)\|^2 \le \eta\frac{\sqrt{G^2+\epsilon}}{\epsilon}L\sigma^2 + \eta^2\frac{q^2 G^2\sqrt{G^2+\epsilon}}{\epsilon^2(1-q^2)},$$

first: variance, second: compression—small vanishing term. Compression with error feedback asymptotically has no impact. With decreasing learning rate $\eta = \frac{1}{\sqrt{T}}$, we have

$$\frac{1}{T}\sum_{t=1}^T \mathbb{E}\|\nabla f(\theta_t)\|^2 \le \mathcal{O}(\frac{1}{\sqrt{T}} + \frac{1}{T}),$$

matching the convergence rate of SGD with error feedback ([29] Theorem II).

Xiaoyun Note: I think we should introduce the variance in the bound $\mathbb{E}\|g_t\|^2 \le \sigma^2$? Extend to $\beta_1 > 0$?

Variance bound Yes for $\mathbb{E}\|g_t\|^2 \le \sigma^2$

For $\beta_1 > 0$: why not say:

$$\|m_t\|^2 \leq \beta_1^2 \|m_{t-1}\|^2 + (1-\beta_1)^2 \|g_t\|^2 \tag{14}$$

where g_t is the full gradient (not sparsed). Then

$$\mathbb{E}[\|m_t\|^2] \leq \beta_1^2 \mathbb{E}[\|m_{t-1}\|^2] + (1-\beta_1)^2 \mathbb{E}[\|g_t\|^2] \tag{15}$$

Since we have by initialization that $\|m_0\|^2 \leq \sigma^2$, then we prove by induction that $\|m_t\|^2 \leq \sigma^2$ since $\mathbb{E}\|g_t\|^2 \leq \sigma^2$.

Try as in "A Sufficient Condition for Convergences of Adam and RMSProp"

Bounding $-\eta\mathbb{E}[\langle\nabla f(\theta_t), a'_t\rangle] + \left(\frac{\eta^2 L}{2} + \frac{\eta}{2\rho}\right)\mathbb{E}[\|a'_t\|^2$

Using Young's inequality with parameter ρ and the smoothness assumption we have

$$
\begin{aligned}
\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq \langle\nabla f(\theta'_t), \theta'_{t+1} - \theta'_t\rangle + \frac{L}{2}\|\theta'_{t+1} - \theta'_t\|^2 \\
&\leq \langle\nabla f(\theta_t), \theta'_{t+1} - \theta'_t\rangle + \frac{L}{2}\|\theta'_{t+1} - \theta'_t\|^2 + \langle\nabla f(\theta'_t) - \nabla f(\theta_t), \theta'_{t+1} - \theta'_t\rangle \\
&\leq \mathbb{E}[\langle\nabla f(\theta_t), \delta_t\rangle] + \nu\mathbb{E}[\|\delta_t\|^2] + \frac{L^2\rho}{2}\mathbb{E}[\|\theta_t - \theta'_t\|^2]
\end{aligned}
$$

where $\delta_t = \theta'_{t+1} - \theta'_t$ and $\nu = \left(\frac{L}{2} + \frac{1}{2\rho}\right)$.

Denote $A_t = \mathbb{E}[\langle\nabla f(\theta_t), \delta_t\rangle] + \nu\mathbb{E}[\|\delta_t\|^2]$. We have

$$
\mathbb{E}[\langle\nabla f(\theta_t), \delta_t\rangle] = \beta_1\beta_2^{-1/2}\mathbb{E}[\langle\nabla f(\theta_t), \delta_{t-1}\rangle] + \mathbb{E}[\langle\nabla f(\theta_t), \delta_t - \beta_1\beta_2^{-1/2}\delta_{t-1}\rangle]
$$

Yet

$$
\begin{aligned}
\mathbb{E}[\langle\nabla f(\theta_t), \delta_{t-1}\rangle] &\leq \mathbb{E}[\langle\nabla f(\theta_{t-1}), \delta_{t-1}\rangle] + \mathbb{E}[\langle\nabla f(\theta_t) - \nabla f(\theta_{t-1}), \delta_{t-1}\rangle] \\
&\leq \mathbb{E}[\langle\nabla f(\theta_{t-1}), \delta_{t-1}\rangle] + \nu\left\|\delta_{t-1}\right\|^2
\end{aligned}
$$

Hence

$$
\mathbb{E}[\langle\nabla f(\theta_t), \delta_t\rangle] \leq \beta_1\beta_2^{-1/2}A_{t-1} + \mathbb{E}[\langle\nabla f(\theta_t), \delta_t - \beta_1\beta_2^{-1/2}\delta_{t-1}\rangle]
$$

Also

$$
\begin{aligned}
\delta_t - \beta_1\beta_2^{-1/2}\delta_{t-1} &= -\eta\frac{m'_t}{\sqrt{\hat{v}'_t + \epsilon}} + \eta\beta_1\beta_2^{-1/2}\frac{m'_{t-1}}{\sqrt{\hat{v}'_{t-1} + \epsilon}} \\
&= -\eta\left(\frac{m'_t}{\sqrt{\hat{v}'_t + \epsilon}} - \frac{\beta_1\beta_2^{-1/2}m'_{t-1}}{\sqrt{\hat{v}'_{t-1} + \epsilon}}\right) \\
&= -\frac{\eta(1-\beta_1)g_t}{\sqrt{\hat{v}'_t + \epsilon}} + \beta_1\eta m'_{t-1}\left(\frac{\beta_2^{-1/2}}{\sqrt{\hat{v}'_{t-1} + \epsilon}} - \frac{1}{\sqrt{v'_t + \epsilon}}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}[\langle\nabla f(\theta_t), \delta_t\rangle] &\leq \beta_1\beta_2^{-1/2}A_{t-1} - \mathbb{E}[\langle\nabla f(\theta_t), \frac{\eta(1-\beta_1)g_t}{\sqrt{\hat{v}'_t + \epsilon}}\rangle] + \mathbb{E}[\langle\nabla f(\theta_t), \beta_1\eta m'_{t-1}\left(\frac{\beta_2^{-1/2}}{\sqrt{\hat{v}'_{t-1} + \epsilon}} - \frac{1}{\sqrt{v'_t + \epsilon}}\right)\rangle] \\
&\leq \beta_1\beta_2^{-1/2}A_{t-1} - \eta(1-\beta_1)\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \beta_1\eta\mathbb{E}[\langle\nabla f(\theta_t), m'_{t-1}\left(\frac{\beta_2^{-1/2}}{\sqrt{\hat{v}'_{t-1} + \epsilon}} - \frac{1}{\sqrt{v'_t + \epsilon}}\right)\rangle]
\end{aligned}
$$

Note that for $\epsilon = 0$:

$$
\begin{aligned}
m'_{t-1}\left(\frac{\beta_2^{-1/2}}{\sqrt{\hat{v}'_{t-1}+\epsilon}} - \frac{1}{\sqrt{v'_t+\epsilon}}\right) &= m'_{t-1}\frac{\sqrt{v'_t} - \sqrt{\beta_2 v'_{t-1}}}{\sqrt{v'_t}\sqrt{\beta_2 v'_{t-1}}} \\
&= m'_{t-1}\frac{v'_t - \beta_2 v'_{t-1}}{\sqrt{v'_t}\sqrt{\beta_2 v'_{t-1}}(\sqrt{v'_t} + \sqrt{\beta_2 v'_{t-1}})} \\
&= m'_{t-1}\frac{(1-\beta_2)g_t}{\sqrt{v'_t}\sqrt{\beta_2 v'_{t-1}}(\sqrt{v'_t} + \sqrt{\beta_2 v'_{t-1}})} \\
&= (1-\beta_2)g_t\frac{m'_{t-1}}{\sqrt{v'_t} + \sqrt{\beta_2 v'_{t-1}}}\frac{1}{\sqrt{v'_t}\sqrt{\beta_2 v'_{t-1}}}
\end{aligned}
$$

5 Experiments

Our proposed TopK-EF with AMSGrad matches that of full AMSGrad, in distributed learning. Number of local workers is 20. Error feedback fixes the convergence issue of using solely the TopK gradient.

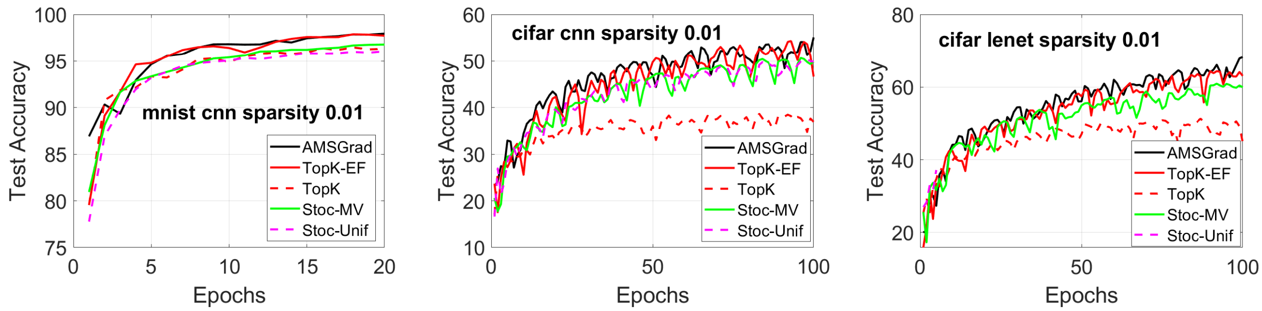

Figure 1: Test accuracy.

6 Conclusion

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A Appendix

B Proofs

B.1 Proof of Lemmas

Lemma. *Under Assumption 2 and Assumption 4 we have for any iteration $t > 0$:*

$$\|m_t\|^2 \leq (q^2+1)G^2 \quad \text{and} \quad \hat{v}_t \leq (q^2+1)G^2 \tag{16}$$

where m_t and $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ are defined Line 15 of Algorithm 1 and $G^2 = \frac{1}{n}\sum_{i=1}^{N} G_i^2$.

Proof. We start by writing

$$\|\bar{g}_t\|^2 = \left\| \frac{1}{n}\sum_{i=1}^{N} \tilde{g}_{t,i} \right\|^2 \leq \frac{1}{n}\sum_{i=1}^{N} \|\tilde{g}_{t,i}\|^2 \tag{17}$$

Though, using Assumption 2 and Assumption 4 we have:

$$\|\tilde{g}_{t,i}\|^2 = \|g_{t,i} + \tilde{g}_{t,i} - g_{t,i}\|^2 \leq \|g_{t,i}\|^2 + \|\tilde{g}_{t,i} - g_{t,i}\|^2 \leq (q^2+1)G_i^2 \tag{18}$$

Hence

$$\|\bar{g}_t\|^2 \leq (q^2+1)G^2 \tag{19}$$

where $G^2 = \frac{1}{n}\sum_{i=1}^{N} G_i^2$. Then, by construction in Algorithm 1:

$$\|m_t\|^2 \leq \beta_1^2 \|m_{t-1}\|^2 + (1-\beta_1)^2 \|\bar{g}_t\|^2 \leq \beta_1^2 \|m_{t-1}\|^2 + (1-\beta_1)^2(q^2+1)G^2 \tag{20}$$

Since we have by initialization that $\|m_0\|^2 \leq G^2$, then we prove by induction that $\|m_t\|^2 \leq (q^2+1)G^2$.

Similarly

$$\hat{v}_t = \max(v_t, \hat{v}_{t-1}) = \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1-\beta_2)\bar{g}_t^2) \leq max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1-\beta_2)(q^2+1)G^2) \tag{21}$$

□

Lemma. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \tag{22}$$

*where $\mathsf{I_d}$ is the identity matrix, $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1 and $\bar{g}_t$ is the aggregation of all **quantized** gradients from the workers.*

Proof. We first decompose $\bar{g}_t$ as the sum of the unbiased stochastic gradients and its quantized versions as computed Line 7 of Algorithm 1:

$$\bar{g}_t = \frac{1}{n}\sum_{i=1}^{N}\tilde{g}_{t,i} = \frac{1}{n}\sum_{i=1}^{N}[g_{t,i} + \tilde{g}_{t,i} - g_{t,i}] \tag{23}$$

Hence,

$$\begin{aligned} T_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \\ =& \underbrace{-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle]}_{t1} \underbrace{-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\frac{1}{n}\sum_{i=1}^{N}\tilde{g}_{t,i} - g_{t,i} \right\rangle]}_{t_2} \end{aligned} \tag{24}$$

Bounding t_1: Using the Tower rule, we have:

$$\begin{aligned}
t_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle] \\
=& -\eta_{t+1}\mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle |\mathcal{F}_t]] \\
=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \mathbb{E}[\frac{1}{n}\sum_{i=1}^{N} g_{t,i} |\mathcal{F}_t] \right\rangle]
\end{aligned} \tag{25}$$

Using Assumption 2 and Lemma 1, we have that

$$\begin{aligned}
t_1 :=& -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \frac{1}{n}\sum_{i=1}^{N} g_{t,i} \right\rangle] \\
\leq& -\eta_{t+1}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2]
\end{aligned} \tag{26}$$

Bounding t_2:

We first recall Young's inequality with a constant $\delta \in (0,1)$ as follows:

$$\langle X \,|\, Y \rangle \leq \frac{1}{\delta}\|X\|^2 + \delta\|Y\|^2 \,. \tag{27}$$

Using Young's inequality (27) with parameter equal to 1:

$$\begin{aligned}
t_2 \leq& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\sum_{i=1}^{N}\{\tilde{g}_{t,i} - g_{t,i}\}\|^2] \\
\overset{(a)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\|^2\sum_{i=1}^{N}\{\tilde{g}_{t,i} - g_{t,i}\}\|^2] \\
\overset{(b)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2}\mathbb{E}[\|(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\|^2]\mathbb{E}[\|\sum_{i=1}^{N}\{\tilde{g}_{t,i} - g_{t,i}\}\|^2] \\
\overset{(c)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{\epsilon 2n^2}\mathbb{E}[\|\sum_{i=1}^{N}\tilde{g}_{t,i} - g_{t,i}\|^2] \\
\overset{(d)}{\leq}& \frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2}
\end{aligned} \tag{28}$$

where (a) uses the Cauchy-Schwartz inequality, (b) is due to the non-negativeness of both $\hat{V}_{t+1}$ and $\|\sum_{i=1}^{N}\{g_{t,i} + \tilde{g}_{t,i} - g_{t,i}\}\|^2$ and (c) uses the Triangle inequality. We use Assumption 3 and Assumption 4 in (d).

Finally, combining (26) and (28) yields

$$-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2}\bar{g}_t \right\rangle] \leq -\frac{\eta_{t+1}}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \tag{29}$$

□

Lemma. *Under A1 to A4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:*

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & - \frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
& - \eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right)\|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned}
\tag{30}
$$

where d denotes the dimension of the parameter vector

Proof. By assumption Assumption 1, we can write the smoothness condition on the overall objective (2), between iteration t and $t+1$:

$$
f(\theta_{t+1}) \leq f(\theta_t) + \langle \nabla f(\theta_t) \,|\, \theta_{t+1} - \theta_t \rangle + \frac{L}{2}\|\theta_{t+1} - \theta_t\|^2 \tag{31}
$$

Denote by $\hat{V}_t$ the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1. Hence, we obtain,

$$
f(\theta_{t+1}) \leq f(\theta_t) - \eta_{t+1}\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle + \frac{L}{2}\|\theta_{t+1} - \theta_t\|^2 \tag{32}
$$

where $\mathsf{I_d}$ denotes the identity matrix.

We now take the expectation of those various terms conditioned on the filtration $\mathcal{F}_t$ of the total randomness up to iteration t.

$$
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle] + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \tag{33}
$$

We now focus on the computation of the inner product obtained in the equation above. We have

$$
\begin{aligned}
& \eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle] \qquad (34) \\
= & \eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} + (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} - (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle] \\
= & \eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle] + \eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2}\right] m_{t+1} \right\rangle] \\
= & \eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] + \eta_{t+1}(1-\beta_1)\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \right\rangle] \\
& + \eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2}\right] m_{t+1} \right\rangle] \qquad (35)
\end{aligned}
$$

where $\bar{g}_t$ is the aggregated gradients from all workers.

Plugging the above in (33) yields:

$$
\begin{aligned}
& \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \\
\leq & \underbrace{-\beta_1\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle]}_{A_t} \eta_{t+1} \\
& \underbrace{-\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2}\right] m_{t+1} \right\rangle]}_{B_t} \eta_{t+1} \\
& \underbrace{-(1-\beta_1)\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \right\rangle]}_{C_t} \eta_{t+1} + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]
\end{aligned}
\tag{36}
$$

To begin with, by the tower rule, we have that

$$
\begin{aligned}
A_t &= -\beta_1 \mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle |\mathcal{F}_t]] && (37)\\
&= -\beta_1 \left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] - \beta_1 \left\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] && (38)\\
& && (39)
\end{aligned}
$$

where we recognize the first term as the term in (34), at iteration $t-1$ and hence apply the same decomposition as in (35). Coupling with the smoothness of f, which gives that

$$
-\beta_1 \left\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \leq \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2
$$

we obtain,

$$
\begin{aligned}
A_t &= -\beta_1 \mathbb{E}[\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle |\mathcal{F}_t]] \\
&\leq \eta_{t+1}\beta_1 (A_{t-1} + B_{t-1} + C_{t-1}) + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2
\end{aligned} \tag{40}
$$

Then,

$$
\begin{aligned}
B_t &= -\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, \left[(\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} - (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} \right] m_{t+1} \right\rangle] \\
&= \mathbb{E}[\sum_{j=1}^{d} \nabla^j f(\theta_t) m_{t=1}^j \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
&\overset{(a)}{\leq} \mathbb{E}[\|\nabla f(\theta_t)\| \, \|m_{t=1}\| \sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
&\overset{(b)}{\leq} G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]]
\end{aligned} \tag{41}
$$

where $\nabla^j f(\theta_t)$ denotes the j-th component of the gradient vector $\nabla f(\theta_t)$, (a) uses of the Cauchy-Schwartz inequality and (b) boils down from the norm of the gradient vector boundedness assumption 2, denoting $G := \frac{1}{n}\sum_{i=1}^{n} G_i$.

Plugging the above into (36) yields

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq& \eta_{t+1}(A_t + B_t + C_t) + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \\
\leq& -\eta_{t+1}\beta_1 \mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \\
&+ \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
&+ \left(\frac{L}{2} + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_t - \theta_{t-1}\|^2 \\
&- \eta_{t+1}(1-\beta_1) \mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \right\rangle]
\end{aligned} \tag{42}
$$

We bound the last term on the RHS, $-\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} \bar{g}_t \right\rangle]$ with Lemma 2

Under the assumption that we use a decreasing stepsize such that $\eta_{t+1} \leq \eta_t$, and given that according to Line 15 we have that $\hat{v}_{t+1} \geq \hat{v}_t$ by construction, we obtain

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & - \frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
& - \eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right) \|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned}
\tag{43}
$$

Finally, using Lemma 2, we obtain the desired result. □

B.2 Proof of Theorem 1

Theorem. *Under A1 to A4, with a constant stepsize $\eta_t = \eta = \frac{L}{\sqrt{T_m}}$, we have:*

$$
\frac{1}{T_m}\sum_{t=0}^{T_m-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0) - f(\theta_{T_m})]}{L\Delta_1\sqrt{T_m}} + d\frac{L\Delta_3}{\Delta_1\sqrt{T_m}} + \frac{\Delta_2}{\eta\Delta_1 T_m} + \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2
\tag{44}
$$

where

$$
\begin{aligned}
&\Delta_1 := \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \quad , \quad \Delta_2 := q^2 + \sum_{k=t+1}^{\infty}\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2} \\
&\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right)(1-\beta_2)^{-1}(1 - \frac{\beta_1^2}{\beta_2})^{-1}
\end{aligned}
\tag{45}
$$

Proof. By Lemma 3 we have

$$
\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq & - \frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\
& - \eta_{t+1}\beta_1\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle] \\
& + \left(\frac{L}{2} + \beta_1 L\right) \|\theta_t - \theta_{t-1}\|^2 \\
& + \eta_{t+1}G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]]
\end{aligned}
\tag{46}
$$

Let us consider the following sequence, defined for all $t > 0$:

$$
R_t := f(\theta_t) - \sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle]
\tag{47}
$$

We compute the following expectation:

$$
\begin{aligned}
\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] = & \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] - \sum_{k=t+1}^{\infty}\eta_k\beta_1^{k-t+2}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I_d})^{-1/2} m_{t+1} \right\rangle] \\
& + \sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I_d})^{-1/2} m_t \right\rangle]
\end{aligned}
\tag{48}
$$

Using the Assumption 1, we note that:

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1}\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \right\rangle] + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \tag{49}$$

which yields

$$\begin{aligned}
\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] = & -(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\mathbb{E}[\left\langle \nabla f(\theta_t) \,|\, (\hat{V}_{t+1} + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_{t+1} \right\rangle] \\
& + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1}\mathbb{E}[\left\langle \nabla f(\theta_{t-1}) \,|\, (\hat{V}_t + \epsilon \mathsf{I}_\mathsf{d})^{-1/2} m_t \right\rangle] \\
& + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \\
\leq & (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\mathbb{E}[A_t + B_t + C_t] \\
& - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1}\mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\
& + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2
\end{aligned} \tag{50}$$

where A_t, B_t, C_t are defined in (36).

We use (40) and (41) to bound A_t and B_t, and Lemma 2 to bound C_t where we precise that the learning rate η_{t+1} becomes $\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}$. Hence

$$\begin{aligned}
\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] \leq & \left((\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \right) \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\
& + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E}[\sum_{j=1}^{d} \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
& + \left(\frac{L}{2} + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_{t+1} - \theta_t\|^2 \\
& - (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
& + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2}
\end{aligned} \tag{51}$$

where the last term in the LHS is due to Lemma 3.

By assumption, we have that for all $t > 0$, $\eta_{t=1} \leq \eta_t$. Also, set the tuning parameters such that

$$\eta_t + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \leq \frac{\eta_t}{1-\beta_1} \tag{52}$$

so that

$$\begin{aligned}
& (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} = 0 \\
\iff & (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2})\beta_1 = \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1}
\end{aligned} \tag{53}$$

Note that $-(\eta_{t+1}+\sum_{k=t+1}^{\infty}\eta_k\beta_1^{k-t+2})\frac{(1-\beta_1)}{2}(\epsilon+\frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \leq -\eta_{t+1}\frac{(1-\beta_1)}{2}(\epsilon+\frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}$ since $\sum_{k=t+1}^{\infty}\eta_k\beta_1^{k-t+2} \geq 0$.

The above coupled with (51) yields

$$\begin{aligned}\mathbb{E}[R_{t+1}]-\mathbb{E}[R_t] \leq & -\eta_{t+1}\frac{(1-\beta_1)}{2}(\epsilon+\frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2]+q^2\eta_{t+1}+\sum_{k=t+1}^{\infty}\eta_k\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}\\ & -(\eta_{t+1}+\sum_{k=t+1}^{\infty}\eta_k\beta_1^{k-t+2})G^2\mathbb{E}[\sum_{j=1}^{d}\left[(\hat{v}_t^j+\epsilon)^{-1/2}-(\hat{v}_{t+1}^j+\epsilon)^{-1/2}\right]]\\ & +\left(\frac{L}{2}+1+\frac{\beta_1 L}{1-\beta_1}\right)\mathbb{E}[\|\theta_{t+1}-\theta_t\|^2]\end{aligned} \tag{54}$$

We now sum from $t=0$ to $t=T_{\mathsf{m}}-1$ the inequality in (54), and divide it by T_{m}:

$$\begin{aligned}&\eta\frac{(1-\beta_1)}{2}(\epsilon+\frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\frac{1}{T_{\mathsf{m}}}\sum_{t=0}^{T_{\mathsf{m}}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2]\\ \leq&\frac{\mathbb{E}[R_0]-\mathbb{E}[R_{T_{\mathsf{m}}}]}{T_{\mathsf{m}}}+\frac{q^2\eta+\sum_{k=t+1}^{\infty}\eta\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}}{T_{\mathsf{m}}}\\ &+\left(\frac{L}{2}+1+\frac{\beta_1 L}{1-\beta_1}\right)\frac{1}{T_{\mathsf{m}}}\sum_{t=0}^{T_{\mathsf{m}}-1}\mathbb{E}[\|\theta_{t+1}-\theta_t\|^2]\end{aligned} \tag{55}$$

where we have used the fact that $(\hat{v}_t^j+\epsilon)^{-1/2}-(\hat{v}_{t+1}^j+\epsilon)^{-1/2} \geq 0$ for all dimension $j \in [d]$ by construction of $\hat{v}_{t+1}^j$.

We now bound the two remaining terms:

Bounding $-\mathbb{E}[R_{T_{\mathsf{m}}}]$:

By definition (47) of R_t we have, using Lemma 1:

$$\begin{aligned}-\mathbb{E}[R_{T_{\mathsf{m}}}] \leq & \sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\mathbb{E}[\left\langle\nabla f(\theta_{t-1})\,|\,(\hat{V}_t+\epsilon\mathsf{I}_{\mathsf{d}})^{-1/2}m_t\right\rangle]-f(\theta_{T_{\mathsf{m}}})\\ & \leq \|\sum_{k=t}^{\infty}\eta_k\beta_1^{k-t+1}\|\|\nabla f(\theta_{t-1})\|\|(\hat{V}_t+\epsilon\mathsf{I}_{\mathsf{d}})^{-1/2}m_t\|\\ & \leq \eta_{t+1}(1-\beta_1)\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2-f(\theta_{T_{\mathsf{m}}})\end{aligned} \tag{56}$$

Bounding $\sum_{t=0}^{T_{\mathsf{m}}-1}\mathbb{E}[\|\theta_{t+1}-\theta_t\|^2]$:

By definition in Algorithm 1:

$$\|\theta_{t+1}-\theta_t\|^2=\eta_{t+1}^2\left[(\hat{V}_{t+1}+\epsilon\mathsf{I}_{\mathsf{d}})^{-\frac{1}{2}}m_{t+1}\right]^2=\eta_{t+1}^2\sum_{j=1}^{d}\frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j+\epsilon} \tag{57}$$

For any dimension $j \in [d]$,

$$
\begin{aligned}
|m_{t+1}^j|^2 &= |\beta_1 m_t^j + (1-\beta_1)\bar{g}_t^j|^2 \\
&\leq \beta_1(\beta_1^2|m_{t-1}^j|^2 + (1-\beta_1)^2|\bar{g}_{t-1}^j|^2) + |\bar{g}_t^j|^2 \\
&\leq \sum_{k=0}^{t} \beta_1^{2(t-k)}|\bar{g}_k^j|^2 \\
&\leq \sum_{k=0}^{t} \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}}\beta_2^{t-k}|\bar{g}_k^j|^2
\end{aligned} \tag{58}
$$

Using Cauchy-Schwartz inequality we obtain

$$
\begin{aligned}
|m_{t+1}^j|^2 \leq \sum_{k=0}^{t} \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}}\beta_2^{t-k}|\bar{g}_k^j|^2 &\leq \sum_{k=0}^{t}\left(\frac{\beta_1^2}{\beta_2}\right)^{t-k}\sum_{k=0}^{t}\beta_2^{t-k}|\bar{g}_k^j|^2 \\
&\leq \frac{1}{1-\frac{\beta_1^2}{\beta_2}}\sum_{k=0}^{t}\beta_2^{t-k}|\bar{g}_k^j|^2
\end{aligned} \tag{59}
$$

On the other hand we have

$$
\hat{v}_{t+1}^j \geq \beta_2\hat{v}_t^j + (1-\beta_2)(\bar{g}_t^j)^2 \tag{60}
$$

and since it is also true for iteration $t = 1$, we have by induction replacing v_t^j in the above that

$$
\hat{v}_{t+1}^j \geq (1-\beta_2)\sum_{k=0}^{t}\beta_2^{t-k}|\bar{g}_k^j|^2 \iff \frac{\sum_{k=0}^{t}\beta_2^{t-k}|\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \leq (1-\beta_2)^{-1} \tag{61}
$$

Hence, we can derive from (57) that

$$
\begin{aligned}
\|\theta_{t+1}-\theta_t\|^2 = \eta_{t+1}^2\sum_{j=1}^{d}\frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j+\epsilon} &\leq \eta_{t+1}^2\sum_{j=1}^{d}\frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j} \\
&\overset{(a)}{\leq} \eta_{t+1}^2\sum_{j=1}^{d}\frac{1}{1-\frac{\beta_1^2}{\beta_2}}\frac{\sum_{k=0}^{t}\beta_2^{t-k}|\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \\
&\overset{(b)}{\leq} \eta_{t+1}^2 d(1-\beta_2)^{-1}(1-\frac{\beta_1^2}{\beta_2})^{-1}
\end{aligned} \tag{62}
$$

where (a) uses (59) and (b) uses (61).

Plugging the two bounds in (55), we obtain the following bound:

$$
\begin{aligned}
\frac{1}{T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq & \frac{\mathbb{E}[f(\theta_0)-f(\theta_{T_{\mathrm{m}}})]}{\eta\Delta_1 T_{\mathrm{m}}} + \frac{q^2\eta + \sum_{k=t+1}^{\infty}\eta\beta_1^{k-t+2}\frac{G^2}{\epsilon 2n^2}}{\eta\Delta_1 T_{\mathrm{m}}} \\
&+ \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2 \\
&+ \left(\frac{L}{2}+1+\frac{\beta_1 L}{1-\beta_1}\right)\frac{1}{\eta\Delta_1}\eta^2 d(1-\beta_2)^{-1}(1-\frac{\beta_1^2}{\beta_2})^{-1}
\end{aligned} \tag{63}
$$

where $\Delta_1 := \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}$

With a constant stepsize $\eta = \frac{L}{\sqrt{T_{\mathrm{m}}}}$ we get the final convergence bound as follows:

$$
\begin{aligned}
\frac{1}{T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq & \frac{\mathbb{E}[f(\theta_0)-f(\theta_{T_{\mathrm{m}}})]}{L\Delta_1\sqrt{T_{\mathrm{m}}}} + d\frac{L\Delta_3}{\Delta_1\sqrt{T_{\mathrm{m}}}} \\
&+ \frac{\Delta_2}{\eta\Delta_1 T_{\mathrm{m}}} + \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}G^2
\end{aligned} \tag{64}
$$

where $\Delta_2 := q^2 + \sum_{k=t+1}^{\infty} \beta_1^{k-t+2} \frac{G^2}{\epsilon 2 n^2}$ and $\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right)(1-\beta_2)^{-1}(1-\frac{\beta_1^2}{\beta_2})^{-1}$.

□

B.3 Proof of Theorem 3

Theorem 3. *Under A1 to A4, with a constant stepsize $\eta_t = \eta = \frac{L}{\sqrt{T_m}}$, the sequence of iterates $\{\theta_t\}_{t>0}$ output from Algorithm 2 satisfies:*

$$\frac{1}{T_m}\sum_{t=0}^{T_m-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_0)-f(\theta_{T_m})]}{T_m(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)} + \eta^2 G^2 \frac{L}{2}\frac{q^2+1}{\epsilon(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)} \tag{65}$$

$$+ \eta G^2 \frac{q\sqrt{q^2+1}}{\sqrt{\epsilon}(1-q)(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)} + \frac{G^2}{(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)}\left(\frac{q}{1-q}\right)^2\left[\frac{L}{2}q^2+1\right] \tag{66}$$

Proof. Define the auxiliary model

$$\begin{aligned}
\theta'_{t+1} &:= \theta_{t+1} - e_{t+1} \\
&= \theta_t - \eta a_t - e_{t+1} \\
&= \theta_t - \eta a_t - e_t - g_t + \tilde{g}_t \\
&= \theta_t - \eta a_t - e_t - \Delta_t \\
&= \theta'_t - \eta a_t - \Delta_t
\end{aligned}$$

where $a_t := \frac{m_t}{\sqrt{\hat{v}_t+\epsilon}}$ and $\Delta_t := g_t - \tilde{g}_t$. By smoothness assumption we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \langle \nabla f(\theta'_t), \eta a_t + \Delta_t\rangle + \frac{L}{2}\|\theta'_{t+1}-\theta'_t\|^2.$$

Thus,

$$\begin{aligned}
\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq -\mathbb{E}[\langle \nabla f(\theta'_t), \eta a_t + \Delta_t\rangle] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2] \\
&\leq \eta\mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), \eta a_t + \Delta_t\rangle] - \mathbb{E}[\langle \nabla f(\theta_t), \eta a_t + \Delta_t\rangle] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2]
\end{aligned}$$

Using the smoothness assumption A1 we have

$$\mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), \eta a_t + \Delta_t\rangle] \leq L\mathbb{E}[\|\theta_t - \theta'_t\|]E[\|\eta a_t + \Delta_t\|]$$

Hence,

$$\begin{aligned}
\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq -\mathbb{E}[\langle \nabla f(\theta'_t), \eta a_t + \Delta_t\rangle] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2] \\
&\leq -\left(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q\right)\mathbb{E}\|\nabla f(\theta_t)\|^2 + L\mathbb{E}[\|\theta_t - \theta'_t\|]E[\|\eta a_t + \Delta_t\|] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2] \\
&\leq -\left(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q\right)\mathbb{E}\|\nabla f(\theta_t)\|^2 + L\mathbb{E}[\|e_t\|\,\|\eta a_t + \Delta_t\|] + \frac{L}{2}\mathbb{E}[\|\eta a_t + \Delta_t\|^2]
\end{aligned}$$

Summing from $t = 0$ to $t = T_\mathrm{m} - 1$ and divide it by T_m yields:

$$\begin{aligned}
&\left(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q\right)\frac{1}{T_\mathrm{m}}\sum_{t=0}^{T_\mathrm{m}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\leq \sum_{t=0}^{T_\mathrm{m}-1}\frac{\mathbb{E}[f(\theta'_t) - f(\theta'_{t+1})]}{T_\mathrm{m}} + \frac{1}{T_\mathrm{m}}\sum_{t=0}^{T_\mathrm{m}-1}\mathbb{E}[\|e_t\|\,\|\eta a_t + \Delta_t\|] + \frac{L}{2T_\mathrm{m}}\sum_{t=0}^{T_\mathrm{m}-1}\mathbb{E}[\|\eta a_t + \Delta_t\|^2]
\end{aligned} \tag{67}$$

Bounding $\frac{1}{T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|e_t\|\,\|\eta a_t+\Delta_t\|]$:

To begin with

$$\begin{aligned}
\|e_t\| &= \|e_{t-1}+g_{t-1}-\tilde{g}_{t-1}\| \\
&= \|g_{t-1}+e_{t-1}-TopK(g_{t-1}+e_{t-1},k)\| \\
&\le q\,\|g_{t-1}+e_{t-1}\| \\
&\le q\,\|g_{t-1}\|+q\,\|e_{t-1}\| \\
&\le \sum_{k=1}^{t} q^{t-k}\,\|g_k\|
\end{aligned} \tag{68}$$

using A4.

Then we have that

$$\begin{aligned}
\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|e_t\|\,\|\eta a_t+\Delta_t\|] &\le \sum_{t=0}^{T_{\mathrm{m}}-1}\sum_{k=1}^{t} q^{t-k}\mathbb{E}[\|g_k\|\,\|\eta a_t+\Delta_t\|]] \\
&\le \frac{q}{1-q}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|\,\|\eta a_t+\Delta_t\|]] \\
&\le \frac{q}{1-q}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|\left\|\eta\frac{m_t}{\sqrt{\hat{v}_t+\epsilon}}\right\|]+\frac{q}{1-q}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|\,\|\Delta_t\|]] \\
&\le \eta\frac{q\sqrt{q^2+1}}{\sqrt{\epsilon}(1-q)}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|^2]+\frac{q}{1-q}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|\,\|g_t-\tilde{g}_t\|]]
\end{aligned}$$

where we have used Lemma 1 for the last inequality.

Note that

$$\begin{aligned}
\frac{q}{1-q}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|\,\|g_t-\tilde{g}_t\|]] &= \frac{q}{1-q}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|\,\|\tilde{g}_t-(g_t+e_t)+e_t\|]] \\
&\le \frac{q^2}{1-q}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|^2]+\left(\frac{q}{1-q}\right)^2\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|^2]
\end{aligned}$$

where we have used A3 and inequality (68)

Finally, we obtain:

$$\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|e_t\|\,\|\eta a_t+\Delta_t\|] \le \left[\eta\frac{q\sqrt{q^2+1}}{\sqrt{\epsilon}(1-q)}+\frac{q^2}{1-q}+\left(\frac{q}{1-q}\right)^2\right]\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|g_t\|^2]$$

Hence

$$\frac{1}{T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|e_t\|\,\|\eta a_t+\Delta_t\|] \le \left[\eta\frac{q\sqrt{q^2+1}}{\sqrt{\epsilon}(1-q)}+\frac{q^2}{1-q}+\left(\frac{q}{1-q}\right)^2\right]G^2$$

Bounding $\frac{L}{2T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|\eta a_t+\Delta_t\|^2]$: Similarly, we derive the following bound:

$$\frac{L}{2T_{\mathrm{m}}}\sum_{t=0}^{T_{\mathrm{m}}-1}\mathbb{E}[\|\eta a_t+\Delta_t\|^2] \le \frac{L}{2}\left[\eta^2\frac{q^2+1}{\epsilon}+\left(\frac{q}{1-q}\right)^2 q^2\right]G^2$$

Plugging the bounds of $\frac{1}{T_{\text{m}}}\sum_{t=0}^{T_{\text{m}}-1}\mathbb{E}[\|e_t\|\,\|\eta a_t+\Delta_t\|]$ and $\frac{L}{2T_{\text{m}}}\sum_{t=0}^{T_{\text{m}}-1}\mathbb{E}[\|\eta a_t+\Delta_t\|^2]$ into (67) gives:

$$
\begin{aligned}
&\left(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q\right)\frac{1}{T_{\text{m}}}\sum_{t=0}^{T_{\text{m}}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
\leq&\sum_{t=0}^{T_{\text{m}}-1}\frac{\mathbb{E}[f(\theta_t')-f(\theta_{t+1}')]}{T_{\text{m}}}+\eta G^2\left[\eta\frac{L}{2}\frac{q^2+1}{\epsilon}+\frac{q\sqrt{q^2+1}}{\sqrt{\epsilon}(1-q)}\right]+G^2\left(\frac{q}{1-q}\right)^2\left[\frac{L}{2}q^2+1\right] \\
\leq&\frac{\mathbb{E}[f(\theta_0)-f(\theta_{T_{\text{m}}})]}{T_{\text{m}}}+\eta^2G^2\frac{L}{2}\frac{q^2+1}{\epsilon}+\eta G^2\frac{q\sqrt{q^2+1}}{\sqrt{\epsilon}(1-q)}+G^2\left(\frac{q}{1-q}\right)^2\left[\frac{L}{2}q^2+1\right]
\end{aligned}
\tag{69}
$$

Finally

$$
\frac{1}{T_{\text{m}}}\sum_{t=0}^{T_{\text{m}}-1}\mathbb{E}[\|\nabla f(\theta_t)\|^2]\leq\frac{\mathbb{E}[f(\theta_0)-f(\theta_{T_{\text{m}}})]}{T_{\text{m}}(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)}+\eta^2G^2\frac{L}{2}\frac{q^2+1}{\epsilon(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)} \tag{70}
$$

$$
+\eta G^2\frac{q\sqrt{q^2+1}}{\sqrt{\epsilon}(1-q)(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)}+\frac{G^2}{(\eta\frac{1}{\sqrt{G^2+\epsilon}}+q)}\left(\frac{q}{1-q}\right)^2\left[\frac{L}{2}q^2+1\right] \tag{71}
$$

$\square$