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# Distributed Adaptive Learning with Gradient Compression

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## Abstract

1 This paper presents a new algorithm – the Sparsified AMSGrad algorithm  
2 (SPARS-AMS) – for tackling single-machine and distributed supervised learn-  
3 ing. Unlike prior works which rely on full gradient communication between the  
4 workers and the parameter-server, we design a distributed adaptive optimization  
5 method with gradient compression coupled with an error-feedback to alleviate the  
6 bias introduced by the compression. While the former allows us to only transmit  
7 fewer bits of gradient vectors to the server, we show that using the latter, which  
8 correct for the bias, SPARS-AMS reaches a stationary point in  $\mathcal{O}(1/\sqrt{T})$  itera-  
9 tions, matching that of state-of-the-art single-machine and distributed methods,  
10 without any error-feedback. We illustrate our theoretical results by showing the  
11 effectiveness of our method both under the single-machine and distributed settings  
12 on various benchmark datasets.

## 13 1 Introduction

14 Deep neural network has achieved the state-of-the-art learning performance on numerous AI appli-  
15 cations, e.g., computer vision [23, 26, 47], Natural Language Processing [25, 54, 58], Reinforcement  
16 Learning [37, 45] and recommendation systems [16, 49]. With the increasing size of both data and  
17 deep networks, standard single machine training confronts with at least two major challenges:

- 18 • Due to the limited computing power of a single machine, it would take a long time to  
19 process the massive number of data samples—training would be slow.
- 20 • In many practical scenarios, data are typically stored in multiple servers, possibly at differ-  
21 ent locations, due to the storage constraints (massive user behavior data, Internet images,  
22 etc.) or privacy reasons [11]. Transmitting data might be costly.

23 *Distributed learning* framework [18] has been a common training strategy to tackle the above two  
24 issues. For example, in centralized distributed stochastic gradient descent (SGD) protocol, data are  
25 located at  $n$  local nodes, at which the gradients of the model are computed in parallel. In each  
26 iteration, a central server aggregates the local gradients, updates the global model, and transmits  
27 back the updated model to the local nodes for subsequent gradient computation. As we can see, this  
28 setting naturally solves aforementioned issues: 1) We use  $n$  computing nodes to train the model, so  
29 the time per training epoch can be largely reduced; 2) There is no need to transmit the local data to  
30 central server. Besides, distributed training also provides stronger error tolerance since the training  
31 process could continue even one local machine breaks down. As a result of these advantages, there  
32 has been a surge of study and applications on distributed systems [10, 39, 20, 24, 27, 35, 33].

33 Among many optimization strategies, SGD is still the most popular prototype in distributed training  
34 for its simplicity and effectiveness [14, 1, 36]. Yet, when the deep learning model is very large,

the communication between local nodes and central server could be expensive. Burdensome gradient transmission would slow down the whole training system, or even be impossible because of the limited bandwidth in some applications. Thus, reducing the communication cost in distributed SGD has become an active topic, and an important ingredient of large-scale distributed systems (e.g. [42]). Solutions based on quantization, sparsification and other compression techniques of the local gradients are proposed, e.g., [4, 50, 48, 46, 3, 7, 17, 52, 28]. As one would expect, in most approaches, there exists a trade-off between compression and learning performance. In general, larger bias and variance of the compressed gradients usually bring more significant performance downgrade in terms of convergence [46, 2]. Interestingly, studies (e.g., [31]) show that the technique of *error feedback* is able to remedy the issue of such biased compressors, achieving same convergence rate as full-gradient SGD.

On the other hand, in recent years, adaptive optimization algorithms (e.g. AdaGrad [21], Adam [32] and AMSGrad [41]) have become popular because of their superior empirical performance. These methods use different implicit learning rates for different coordinates that keep changing adaptively throughout the training process, based on the learning trajectory. In many learning problems, adaptive methods have been shown to converge faster than SGD, sometimes with better generalization as well. However, the body of literature that combines adaptive methods with distributed training is still very limited. In this paper, we propose a distributed optimization algorithm with AMSGrad as the backbone, along with Top- $k$  sparsification to reduce the communication cost.

## 1.1 Our contributions

We develop a simple optimization leveraging the adaptivity of AMSGrad, and the computational virtue of TopK sparsification, for tackling a large finite-sum of nonconvex objective functions.

Our technique is shown to be both theoretically and empirically effective under *the classical centralized setting* and *the distributed setting*.

In this contribution,

- We derive a sparsified AMSGrad with error feedback, called SPARS-AMS, with a single machine and provide its decentralized counter part.
- We provide a non-asymptotic convergence rate under each setting,
- We highlight the effectiveness of both methods through several numerical experiments

## 2 Related Work

### 2.1 Distributed SGD with compressed gradients

**Quantization.** As we mentioned before, SGD is the most commonly adopted optimization method in distributed training of deep neural nets. To reduce the expensive communication in large-scale distributed systems, extensive works have considered various compression techniques applied to the gradient transaction procedure. The first strategy is quantization. [19] condenses 32-bit floating numbers into 8-bits when representing the gradients. [42, 7, 31, 8] use the extreme 1-bit information (sign) of the gradients, combined with tricks like momentum, majority vote and memory. Other quantization-based methods include QSGD [4, 51, 57] and LPC-SVRG [55], leveraging unbiased stochastic quantization. The saving in communication of quantization methods is moderate: for example, 8-bit quantization reduces the cost to 25% (compared with 32-bit full-precision). Even in the extreme 1-bit case, the largest compression ratio is around  $1/32 \approx 3.1\%$ .

**Sparsification.** Gradient sparsification is another popular solution which may provide higher compression rate. Instead of commuting the full gradient, each local worker only passes a few coordinates to the central server and zeros out the others. Thus, we can more freely choose higher compression ratio (e.g., 1%, 0.1%), still achieving impressive performance in many applications [34]. Stochastic sparsification methods, including uniform sampling and magnitude based sampling [48], select coordinates based on some sampling probability yielding unbiased gradient compressors. Deterministic methods are simpler, e.g., Random- $k$ , Top- $k$  [46, 44] (selecting  $k$  elements with largest magnitude), Deep Gradient Compression [34], but usually lead to biased gradient estima-

tion. In [28], the central server identifies heavy-hitters from the count-sketch [12] of the local gradients, which can be regarded as a noisy variant of Top- $k$  strategy. More applications and analysis of compressed distributed SGD can be found in [30, 43, 5, 6, 29], among others.

**Error Feedback.** Biased gradient estimation, which is a consequence of many aforementioned methods (e.g., signSGD, Top- $k$ ), undermines the model training, both theoretically and empirically, with slower convergence and worse generalization [2, 9]. The technique of *error feedback* is able to “correct for the bias” and fix the problems. In this procedure, the difference between the true stochastic gradient and the compressed one is accumulated locally, which is then added back to the local gradients in later iterations. [46, 31] prove the  $\mathcal{O}(\frac{1}{T})$  and  $\mathcal{O}(\frac{1}{\sqrt{T}})$  convergence rate of EF-SGD in strongly convex and non-convex setting respectively, matching the rates of vanilla SGD [40, 22].

## 2.2 Adaptive optimization

In each SGD update, all the gradient coordinates share a same learning rate, either constant or decreasing over iterations. Adaptive optimization methods cast different learning rate on each dimension. AdaGrad [21] divides the gradient element-wisely by  $\sqrt{\sum_{t=1}^T g_t^2} \in \mathbb{R}^d$ , where  $g_t \in \mathbb{R}^d$  is the gradient vector at time  $t$  and  $d$  is the model dimensionality. Thus, it intrinsically assigns different learning rates to different coordinates throughout the training—elements with smaller previous gradient magnitude tend to move a larger step. AdaGrad has been shown to perform well especially under some sparsity structure. AdaDelta [56] and Adam [32] introduce momentum and moving average of second moment estimation into AdaGrad which lead to better performance. AMSGrad [41] fixes the potential convergence issue of Adam, which will serve as the prototype in this paper. We present the pseudocode in Algorithm . In general, adaptive optimization methods are easier to tune in practice, and usually exhibit faster convergence than SGD. Thus, they have been widely used in training deep learning models in language and computer vision applications, e.g., [15, 53, 59]. In distributed setting, the work [38] proposes a decentralized system in online optimization. However, communication efficiency is not considered. The recent work [13] is the most relevant to our paper. Yet, their method is based on Adam, and requires every local node to store a local estimation of first and second moment, thus being less efficient. We will present more detailed comparison in Section 3.

## 3 Communication-Efficient Adaptive Optimization

### 3.1 Gradient Compressors

In this paper, we mainly consider deterministic  $q$ -deviate compressors defined as below.

**Assumption 1.** We say a compressor  $\mathcal{C} : \mathbb{R}^d \mapsto \mathbb{R}^d$  is  $q$ -deviate if for  $\forall x \in \mathbb{R}^d$ ,  $\exists 0 \leq q < 1$  such that  $\|\mathcal{C}(x) - x\| \leq q \|x\|$ .

Note that, smaller  $q$  indicates better approximation of the true gradient, and  $q = 0$  implies no compression, i.e.  $\mathcal{C}(x) = x$ . We give two popular and highly efficient  $q$ -deviate compressors that will be compared in this paper.

**Definition 1** (Top- $k$ ). For  $x \in \mathbb{R}^d$ , denote  $\mathcal{S}$  as the size- $k$  set of  $i \in [d]$  with largest  $k$  magnitude  $|x_i|$ . The **Top- $k$**  compressor is defined as  $\mathcal{C}(x)_i = x_i$ , if  $i \in \mathcal{S}$ ;  $\mathcal{C}(x)_i = 0$  otherwise.

**Definition 2** (SIGN). For  $x \in \mathbb{R}^d$ , define the **SIGN** compressor as  $\mathcal{C}(x) = \text{sign}(x) \times \frac{1}{d} \sum_{i=1}^d |x_i|$ .

**Remark 1.** Here the scalar, mean magnitude, multiplied to  $\text{sign}(x)$  ensures  $0 \leq q < 1$  as required by Assumption 1, which can be shown by Cauchy-Schwartz inequality. In implementation, this scalar can be arbitrary since we can offset its influence by tuning the learning rate.

Most modern machine learning tasks can be casted as a large finite-sum optimization problem written as:

$$\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n f_i(\theta) \quad (1)$$

where  $n$  denotes the number of workers,  $f_i$  represents the average loss for worker  $i$  and  $\theta$  the global model parameter taking value in  $\Theta$ , a subset of  $\mathbb{R}^d$ .

130 Some related work:

131 [31] develops variant of signSGD (as a biased compression schemes) for distributed optimization.

132 Contributions are mainly on this error feedback variant. In [44], the authors provide theoretical

133 results on the convergence of sparse Gradient SGD for distributed optimization (we want that for

134 AMS here). [46] develops a variant of distributed SGD with sparse gradients too. Contributions

135 include a memory term used while compressing the gradient (using top k for instance). Speeding up

136 the convergence in  $\frac{1}{T^3}$ .

137 Consider standard synchronous distributed optimization setting. AMSGrad is used as the prototype,

138 and the local workers is only in charge of gradient computation.

### 139 3.2 SPARS-AMS with Error Feedback

140 The key difference (and interesting part) of our TopK AMSGrad compared with the following arxiv

141 paper “Quantized Adam”<https://arxiv.org/pdf/2004.14180.pdf> is that, in our model only

142 gradients are transmitted. In “QAdam”, each local worker keeps a local copy of moment estimator

143  $m$  and  $v$ , and compresses and transmits  $m/v$  as a whole. Thus, that method is very much like the

144 sparsified distributed SGD, except that  $g$  is changed into  $m/v$ . In our model, the moment estimates

145  $m$  and  $v$  are computed only at the central server, with the compressed gradients instead of the full

146 gradient. This would be the key (and difficulty) in convergence analysis.

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#### Algorithm 1 SPARS-AMS for Distributed Learning

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1: Input: parameter  $\beta_1, \beta_2$ , learning rate  $\eta_t$ .
2: Initialize: central server parameter  $\theta_1 \in \Theta \subseteq \mathbb{R}^d$ ;  $e_{1,i} = 0$  the error accumulator for each
   worker; sparsity parameter  $k$ ;  $n$  local workers;  $m_0 = 0, v_0 = 0, \hat{v}_0 = 0$ 
3: for  $t = 1$  to  $T$  do
4:   parallel for worker  $i \in [n]$  do:
5:     Receive model parameter  $\theta_t$  from central server
6:     Compute stochastic gradient  $g_{t,i}$  at  $\theta_t$ 
7:     Compute  $\tilde{g}_{t,i} = \text{TopK}(g_{t,i} + e_{t,i}, k)$ 
8:     Update the error  $e_{t+1,i} = e_{t,i} + g_{t,i} - \tilde{g}_{t,i}$ 
9:     Send  $\tilde{g}_{t,i}$  back to central server
10:  end parallel
11:  Central server do:
12:     $\bar{g}_t = \frac{1}{n} \sum_{i=1}^n \tilde{g}_{t,i}$ 
13:     $m_t = \beta_1 m_{t-1} + (1 - \beta_1) \bar{g}_t$ 
14:     $v_t = \beta_2 v_{t-1} + (1 - \beta_2) \bar{g}_t^2$ 
15:     $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ 
16:    Update the global model  $\theta_{t+1} = \theta_t - \eta_t \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$ 
17: end for

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## 147 4 Non-Asymptotic Convergence Analysis for the Single Machine and

### 148 Decentralized settings

149 Several mild assumptions to make: Nonconvex and smooth loss function, unbiased stochastic gradi-

150 ent, bounded variance of the gradient, bounded norm of the gradient, control of the distance between

151 the true gradient and its sparse variant.

152 Check [13] starting with single machine and extending to distributed settings (several machines).

153 Under the distributed setting, the goal is to derive an upper bound to the second order moment of

154 the gradient of the objective function at some iteration  $T_f \in [1, T]$ .

155 We begin by making the following assumptions.

156 **Assumption 2. (Smoothness)** For  $i \in [n]$ ,  $f_i$  is  $L$ -smooth:  $\|\nabla f_i(\theta) - \nabla f_i(\vartheta)\| \leq L \|\theta - \vartheta\|$ .

157 **Assumption 3. (Unbiased and Bounded gradient per worker)** For any iteration index  $t > 0$  and

158 worker index  $i \in [n]$ , the stochastic gradient is unbiased and bounded from above:  $\mathbb{E}[g_{t,i}] =$

159  $\nabla f_i(\theta_t)$  and  $\|g_{t,i}\| \leq G_i$ .

160 **Assumption 4. (Bounded variance *per worker*)** For any iteration index  $t > 0$  and worker index  
 161  $i \in \llbracket n \rrbracket$ , the variance of the noisy gradient is bounded:  $\mathbb{E}[|g_{t,i} - \nabla f_i(\theta_t)|^2] < \sigma_i^2$ .

162 Denote by  $Q(\cdot)$  the quantization operator Line 7 of Algorithm 1, which takes as input a gradient  
 163 vector and returns a quantized version of it, and note  $\tilde{g} := Q(g)$ . Assume that

164 Denote for all  $\theta \in \Theta$ :

$$f(\theta) := \frac{1}{n} \sum_{i=1}^n f_i(\theta), \quad (2)$$

165 where  $n$  denotes the number of workers.

166 **Decentralized Workers Setting:** The main theorem in the decentralized setting reads:

167 **Theorem 1.** Under Assumption 2 to Assumption 4, the sequence of iterates  $\{\theta_t\}_{t>0}$  output from  
 168 Algorithm 1 satisfies:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_1) - f(\theta_{T+1})]}{\Delta_1 \eta_t T} + d \frac{\Delta_3}{\Delta_1 \eta_t T} + \frac{\Delta_2}{\Delta_1 T} + \frac{1 - \beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1)} G^2 \quad (3)$$

169 where  $\{\eta_t\}_{t>0}$  is the sequence of stepsizes and:

$$\begin{aligned} \Delta_1 &:= \frac{(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}}, \quad \Delta_2 := q^2 + \frac{G^2}{\epsilon 2n^2} \bar{\beta}_1 \\ \Delta_3 &:= \left( \frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1} \right) (1 - \beta_2)^{-1} \left( 1 - \frac{\beta_1^2}{\beta_2} \right)^{-1} \end{aligned} \quad (4)$$

170 We remark from this bound in Theorem 1, that the more quantization we apply to our gradient  
 171 vectors ( $q \uparrow$ ), the larger the upper bound of the stationary condition is, *i.e.*, the slower the algorithm  
 172 is. This is intuitive as using compressed quantities will definitely impact the algorithm speed. We  
 173 will observe in the numerical section below that a trade-off on the level of quantization  $q$  can be  
 174 found to achieve similar speed of convergence with less computation resources used throughout the  
 175 training.

176 **Corollary 1.** Under Assumption 2 to Assumption 4, setting the stepsize as  $\eta_t = L\sqrt{\frac{n}{T}}$ , the sequence  
 177 of iterates  $\{\theta_t\}_{t>0}$  output from Algorithm 1 satisfies:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathcal{O}\left(\frac{1}{L\sqrt{nT}} + d \frac{L}{\sqrt{nT}} + \frac{1}{T}\right),$$

178 **Single Machine Setting:** We first provide the formulation of our method in the single machine  
 179 settings in Algorithm 2. Here, the data and the computation are all performed on a single machine.

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**Algorithm 2** SPARS-AMS : Single machine setting

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- 1: **Input:** parameter  $\beta_1, \beta_2$ , learning rate  $\eta_t$ .
  - 2: Initialize: central server parameter  $\theta_1 \in \Theta \subseteq \mathbb{R}^d$ ;  $e_1 = 0$  the error accumulator; sparsity  
parameter  $k$ ;  $m_0 = 0, v_0 = 0, \hat{v}_0 = 0$
  - 3: **for**  $t = 1$  to  $T$  **do**
  - 4:   Compute stochastic gradient  $g_t = g_{t,i_t}$  at  $\theta_t$  for randomly sampled index  $i_t$
  - 5:   Compute  $\tilde{g}_t = \text{TopK}(g_t + e_t, k)$
  - 6:   Update the error  $e_{t+1} = e_t + g_t - \tilde{g}_t$
  - 7:    $m_t = \beta_1 m_{t-1} + (1 - \beta_1) \tilde{g}_t$
  - 8:    $v_t = \beta_2 v_{t-1} + (1 - \beta_2) \tilde{g}_t^2$
  - 9:    $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
  - 10:   Update the global model  $\theta_{t+1} = \theta_t - \eta_t \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$
  - 11: **end for**
-

180 The convergence rate of the vector of parameters estimated via Algorithm 2 is given below:

181 **Theorem 2.** *Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize*  
182  *$\{\eta_t\}_{t>0} = \frac{1}{\sqrt{t}}$ , the sequence of iterates  $\{\theta_t\}_{t>0}$  output from Algorithm 2 satisfies:*

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathcal{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{T}\right),$$

183 matching the convergence rate of SGD with error feedback [31].

## 184 5 Experiments

185 Our proposed TopK-EF with AMSGrad matches that of full AMSGrad, in distributed learning.  
186 Number of local workers is 20. Error feedback fixes the convergence issue of using solely the  
187 TopK gradient.

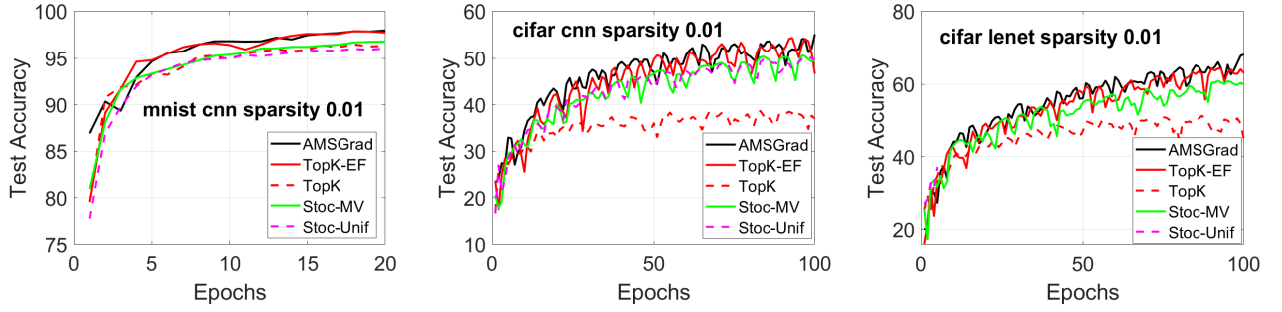


Figure 1: Test accuracy.

## 188 6 Conclusion

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## A Intermediary Lemmas

**Lemma 1.** Under Assumption 3 and Assumption 4 we have for any iteration  $t > 0$ :

$$\|m_t\|^2 \leq (q^2 + 1)G^2 \quad \text{and} \quad \hat{v}_t \leq (q^2 + 1)G^2 \quad (5)$$

where  $m_t$  and  $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$  are defined Line 15 of Algorithm 1 and  $G^2 = \frac{1}{n} \sum_{i=1}^N G_i^2$ .

**Lemma 2.** Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize  $\{\eta_t\}_{t>0}$ , we have:

$$-\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle] \leq -\frac{\eta_{t+1}}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2} \quad (6)$$

where  $\mathbf{I}_d$  is the identity matrix,  $\hat{V}_t$  the diagonal matrix which diagonal entries are  $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$  defined Line 15 of Algorithm 1 and  $\bar{g}_t$  is the aggregation of all **quantized** gradients from the workers.

**Lemma 3.** Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize  $\{\eta_t\}_{t>0}$ , we have:

$$\begin{aligned} \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq -\frac{\eta_{t+1}(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2} \\ &\quad - \eta_{t+1} \beta_1 \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\ &\quad + \left( \frac{L}{2} + \beta_1 L \right) \|\theta_t - \theta_{t-1}\|^2 \\ &\quad + \eta_{t+1} G^2 \mathbb{E} \left[ \sum_{j=1}^d \left[ (\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right] \right] \end{aligned} \quad (7)$$

where  $d$  denotes the dimension of the parameter vector

## B Proofs

### B.1 Proof of Lemmas

**Lemma.** Under Assumption 3 and Assumption 4 we have for any iteration  $t > 0$ :

$$\mathbb{E}[\|m_t\|^2] \leq (q^2 + 1)\sigma^2 \quad \text{and} \quad \mathbb{E}[\hat{v}_t] \leq (q^2 + 1)\sigma^2 \quad (8)$$

where  $m_t$  and  $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$  are defined Line 15 of Algorithm 1 and  $\sigma^2 = \frac{1}{n} \sum_{i=1}^N \sigma_i^2$ .

*Proof.* We start by writing

$$\|\bar{g}_t\|^2 = \left\| \frac{1}{n} \sum_{i=1}^N \tilde{g}_{t,i} \right\|^2 \leq \frac{1}{n} \sum_{i=1}^N \|\tilde{g}_{t,i}\|^2 \quad (9)$$

Though, using Assumption 3 and Assumption 4 we have:

$$\mathbb{E}[\|\tilde{g}_{t,i}\|^2] = \mathbb{E}[\|g_{t,i} + \tilde{g}_{t,i} - g_{t,i}\|^2] \leq \mathbb{E}[\|g_{t,i}\|^2] + \mathbb{E}[\|\tilde{g}_{t,i} - g_{t,i}\|^2] \leq (q^2 + 1)\sigma_i^2 \quad (10)$$

Hence

$$\mathbb{E}[\|\bar{g}_t\|^2] \leq (q^2 + 1)\sigma^2 \quad (11)$$

where  $G^2 = \frac{1}{n} \sum_{i=1}^N G_i^2$ . Then, by construction in Algorithm 1:

$$\mathbb{E}[\|m_t\|^2] \leq \beta_1^2 \mathbb{E}[\|m_{t-1}\|^2] + (1 - \beta_1)^2 \mathbb{E}[\|\bar{g}_t\|^2] \leq \beta_1^2 \mathbb{E}[\|m_{t-1}\|^2] + (1 - \beta_1)^2 (q^2 + 1)\sigma^2 \quad (12)$$

400 Since we have by initialization that  $\|m_0\|^2 \leq \sigma^2$ , then we prove by induction that  $\mathbb{E}[\|m_t\|^2] \leq$   
 401  $(q^2 + 1)\sigma^2$ .

402 Similarly

$$\mathbb{E}[\hat{v}_t] = \mathbb{E}[\max(v_t, \hat{v}_{t-1})] = \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1 - \beta_2) \mathbb{E}[\bar{g}_t^2]) \leq \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1 - \beta_2)(q^2 + 1)\sigma^2) \quad (13)$$

403  $\square$

404 **Lemma.** Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize  $\{\eta_t\}_{t>0}$ , we  
 405 have:

$$-\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle] \leq -\frac{\eta_{t+1}}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2} \quad (14)$$

406 where  $\mathbf{I}_d$  is the identity matrix,  $\hat{V}_t$  the diagonal matrix which diagonal entries are  $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$   
 407 defined Line 15 of Algorithm 1 and  $\bar{g}_t$  is the aggregation of all **quantized** gradients from the workers.

408 *Proof.* We first decompose  $\bar{g}_t$  as the sum of the unbiased stochastic gradients and its quantized  
 409 versions as computed Line 7 of Algorithm 1:

$$\bar{g}_t = \frac{1}{n} \sum_{i=1}^N \tilde{g}_{t,i} = \frac{1}{n} \sum_{i=1}^N [g_{t,i} + \tilde{g}_{t,i} - g_{t,i}] \quad (15)$$

410 Hence,

$$\begin{aligned} T_1 &:= -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle] \\ &= \underbrace{-\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^N g_{t,i} \rangle]}_{t_1} - \underbrace{\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^N \tilde{g}_{t,i} - g_{t,i} \rangle]}_{t_2} \end{aligned} \quad (16)$$

411 **Bounding  $t_1$ :** Using the Tower rule, we have:

$$\begin{aligned} t_1 &:= -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^N g_{t,i} \rangle] \\ &= -\eta_{t+1} \mathbb{E}[\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^N g_{t,i} \rangle | \mathcal{F}_t]] \\ &= -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \mathbb{E}[\frac{1}{n} \sum_{i=1}^N g_{t,i} | \mathcal{F}_t] \rangle] \end{aligned} \quad (17)$$

412 Using Assumption 3 and Lemma 1, we have that

$$\begin{aligned} t_1 &:= -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^N g_{t,i} \rangle] \\ &\leq -\eta_{t+1} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \end{aligned} \quad (18)$$

413 **Bounding  $t_2$ :**

414 We first recall Young's inequality with a constant  $\delta \in (0, 1)$  as follows:

$$\langle X | Y \rangle \leq \frac{1}{\delta} \|X\|^2 + \delta \|Y\|^2. \quad (19)$$

415 Using Young's inequality (19) with parameter equal to 1:

$$\begin{aligned}
t_2 &\leq \frac{\eta_{t+1}}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2} \mathbb{E}[\|(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \sum_{i=1}^N \{\tilde{g}_{t,i} - g_{t,i}\}\|^2] \\
&\stackrel{(a)}{\leq} \frac{\eta_{t+1}}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2} \mathbb{E}[\|(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2}\|^2 \sum_{i=1}^N \{\tilde{g}_{t,i} - g_{t,i}\}^2] \\
&\stackrel{(b)}{\leq} \frac{\eta_{t+1}}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2} \mathbb{E}[\|(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2}\|^2] \mathbb{E}[\sum_{i=1}^N \{\tilde{g}_{t,i} - g_{t,i}\}^2] \\
&\stackrel{(c)}{\leq} \frac{\eta_{t+1}}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{\epsilon 2n^2} \mathbb{E}[\sum_{i=1}^N \tilde{g}_{t,i}^2 - g_{t,i}^2] \\
&\stackrel{(d)}{\leq} \frac{\eta_{t+1}}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2}
\end{aligned} \tag{20}$$

416 where (a) uses the Cauchy-Schwartz inequality, (b) is due to the non-negativeness of both  $\hat{V}_{t+1}$   
417 and  $\|\sum_{i=1}^N \{g_{t,i} + \tilde{g}_{t,i} - g_{t,i}\}\|^2$  and (c) uses the Triangle inequality. We use Assumption 1 and  
418 Assumption 4 in (d).

419 Finally, combining (18) and (20) yields

$$-\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \tilde{g}_t \rangle] \leq -\frac{\eta_{t+1}}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2} \tag{21}$$

420

□

421 **Lemma.** Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize  $\{\eta_t\}_{t>0}$ , we  
422 have:

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq -\frac{\eta_{t+1}(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2} \\
&\quad - \eta_{t+1} \beta_1 \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\
&\quad + \left( \frac{L}{2} + \beta_1 L \right) \|\theta_t - \theta_{t-1}\|^2 \\
&\quad + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^d [(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}]]
\end{aligned} \tag{22}$$

423 where  $d$  denotes the dimension of the parameter vector

424 *Proof.* By assumption Assumption 2, we can write the smoothness condition on the overall objective  
425 (2), between iteration  $t$  and  $t + 1$ :

$$f(\theta_{t+1}) \leq f(\theta_t) + \langle \nabla f(\theta_t) | \theta_{t+1} - \theta_t \rangle + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \tag{23}$$

426 Denote by  $\hat{V}_t$  the diagonal matrix which diagonal entries are  $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$  defined Line 15 of  
427 Algorithm 1. Hence, we obtain,

$$f(\theta_{t+1}) \leq f(\theta_t) - \eta_{t+1} \langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \tag{24}$$

428 where  $\mathbf{I}_d$  denotes the identity matrix.

429 We now take the expectation of those various terms conditioned on the filtration  $\mathcal{F}_t$  of the total  
430 randomness up to iteration  $t$ .

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] + \frac{L}{2} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \quad (25)$$

431 We now focus on the computation of the inner product obtained in the equation above. We have

$$\begin{aligned} & \eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] \quad (26) \\ &= \eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} + (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] \\ &= \eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] + \eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | [(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2}] m_{t+1} \rangle] \\ &= \eta_{t+1} \beta_1 \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] + \eta_{t+1} (1 - \beta_1) \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle] \\ & \quad + \eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | [(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2}] m_{t+1} \rangle] \quad (27) \end{aligned}$$

432 where  $\bar{g}_t$  is the aggregated gradients from all workers.

433 Plugging the above in (25) yields:

$$\begin{aligned} & \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \\ & \leq \underbrace{-\beta_1 \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle]}_{A_t} \eta_{t+1} \\ & \quad \underbrace{-\mathbb{E}[\langle \nabla f(\theta_t) | [(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2}] m_{t+1} \rangle]}_{B_t} \eta_{t+1} \quad (28) \\ & \quad \underbrace{-(1 - \beta_1) \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle]}_{C_t} \eta_{t+1} + \frac{L}{2} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \end{aligned}$$

434 To begin with, by the tower rule, we have that

$$A_t = -\beta_1 \mathbb{E}[\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle | \mathcal{F}_t]] \quad (29)$$

$$= -\beta_1 \langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle - \beta_1 \langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle \quad (30)$$

$$(31)$$

where we recognize the first term as the term in (26), at iteration  $t - 1$  and hence apply the same decomposition as in (27). Coupling with the smoothness of  $f$ , which gives that

$$-\beta_1 \langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle \leq \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2$$

435 we obtain,

$$\begin{aligned} A_t &= -\beta_1 \mathbb{E}[\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle | \mathcal{F}_t]] \\ &\leq \eta_{t+1} \beta_1 (A_{t-1} + B_{t-1} + C_{t-1}) + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2 \quad (32) \end{aligned}$$



436 Then,

$$\begin{aligned}
B_t &= -\mathbb{E}[\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} \rangle m_{t+1}] \\
&= \mathbb{E}[\sum_{j=1}^d \nabla^j f(\theta_t) m_{t+1}^j \left[ (\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
&\stackrel{(a)}{\leq} \mathbb{E}[\|\nabla f(\theta_t)\| \|m_{t+1}\| \sum_{j=1}^d \left[ (\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
&\stackrel{(b)}{\leq} G^2 \mathbb{E}[\sum_{j=1}^d \left[ (\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]]
\end{aligned} \tag{33}$$

437 where  $\nabla^j f(\theta_t)$  denotes the  $j$ -th component of the gradient vector  $\nabla f(\theta_t)$ , (a) uses of the Cauchy-  
438 Schwartz inequality and (b) boils down from the norm of the gradient vector boundedness assump-  
439 tion 3, denoting  $G := \frac{1}{n} \sum_{i=1}^n G_i$ .

440 Plugging the above into (28) yields

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq \eta_{t+1}(A_t + B_t + C_t) + \frac{L}{2} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \\
&\leq -\eta_{t+1} \beta_1 \mathbb{E}[\langle \nabla f(\theta_{t-1}) \mid (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\
&\quad + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^d \left[ (\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]] \\
&\quad + \left( \frac{L}{2} + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_t - \theta_{t-1}\|^2 \\
&\quad - \eta_{t+1} (1 - \beta_1) \mathbb{E}[\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle]
\end{aligned} \tag{34}$$

441 We bound the last term on the RHS,  $-\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle]$  with Lemma 2

442 Under the assumption that we use a decreasing stepsize such that  $\eta_{t+1} \leq \eta_t$ , and given that according  
443 to Line 15 we have that  $\hat{v}_{t+1} \geq \hat{v}_t$  by construction, we obtain

$$\begin{aligned}
\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq -\frac{\eta_{t+1}(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2} \\
&\quad - \eta_{t+1} \beta_1 \mathbb{E}[\langle \nabla f(\theta_{t-1}) \mid (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\
&\quad + \left( \frac{L}{2} + \beta_1 L \right) \|\theta_t - \theta_{t-1}\|^2 \\
&\quad + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^d \left[ (\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right]]
\end{aligned} \tag{35}$$

444 Finally, using Lemma 2, we obtain the desired result.  $\square$

## 445 B.2 Proof of Theorem 1

446 **Theorem.** Under Assumption 2 to Assumption 4, with a constant stepsize  $\eta_t = \eta = \frac{L}{\sqrt{T}}$ , we have:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_1) - f(\theta_{T+1})]}{L \Delta_1 \sqrt{T}} + d \frac{L \Delta_3}{\Delta_1 \sqrt{T}} + \frac{\Delta_2}{\eta \Delta_1 T} + \frac{1 - \beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1)} G^2 \tag{36}$$

447 where

$$\begin{aligned}\Delta_1 &:= \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \quad , \quad \Delta_2 := q^2 + \sum_{k=t+1}^{\infty} \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2} \\ \Delta_3 &:= \left( \frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1} \right) (1-\beta_2)^{-1} (1 - \frac{\beta_1^2}{\beta_2})^{-1}\end{aligned}\tag{37}$$

448 *Proof.* By Lemma 3 we have

$$\begin{aligned}\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2 \eta_{t+1}}{\epsilon 2n^2} \\ &\quad - \eta_{t+1} \beta_1 \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\ &\quad + \left( \frac{L}{2} + \beta_1 L \right) \|\theta_t - \theta_{t-1}\|^2 \\ &\quad + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^d [(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}]]\end{aligned}\tag{38}$$

449 Let us consider the following sequence, defined for all  $t > 0$ :

$$R_t := f(\theta_t) - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle]\tag{39}$$

450 We compute the following expectation:

$$\begin{aligned}\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] &= \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] - \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] \\ &\quad + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle]\end{aligned}\tag{40}$$

451 Using the Assumption 2, we note that:

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2\tag{41}$$

452 which yields

$$\begin{aligned}\mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] &= -(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] \\ &\quad + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\ &\quad + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \\ &\leq (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \mathbb{E}[A_t + B_t + C_t] \\ &\quad - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\ &\quad + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2\end{aligned}\tag{42}$$

453 where  $A_t, B_t, C_t$  are defined in (28).

454 We use (32) and (33) to bound  $A_t$  and  $B_t$ , and Lemma 2 to bound  $C_t$  where we precise that the  
 455 learning rate  $\eta_{t+1}$  becomes  $\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}$ . Hence

$$\begin{aligned}
 \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] &\leq \left( (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \right) \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\
 &\quad + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E} \left[ \sum_{j=1}^d \left[ (\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right] \right] \\
 &\quad + \left( \frac{L}{2} + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_{t+1} - \theta_t\|^2 \\
 &\quad - (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
 &\quad + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}
 \end{aligned} \tag{43}$$

456 where the last term in the LHS is due to Lemma 3.

457 By assumption, we have that for all  $t > 0$ ,  $\eta_{t=1} \leq \eta_t$ . Also, set the tuning parameters such that

$$\eta_t + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \leq \frac{\eta_t}{1 - \beta_1} \tag{44}$$

458 so that

$$\begin{aligned}
 &(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} = 0 \\
 &\iff (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \beta_1 = \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1}
 \end{aligned} \tag{45}$$

459 Note that  $-(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \leq -\eta_{t+1} \frac{(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}}$   
 460 since  $\sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \geq 0$ .

461 The above coupled with (43) yields

$$\begin{aligned}
 \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] &\leq -\eta_{t+1} \frac{(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2} \\
 &\quad - (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E} \left[ \sum_{j=1}^d \left[ (\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \right] \right] \\
 &\quad + \left( \frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1} \right) \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]
 \end{aligned} \tag{46}$$

462 We now sum from  $t = 1$  to  $t = T$  the inequality in (46), and divide it by  $T$ :

$$\begin{aligned}
& \eta \frac{(1 - \beta_1)}{2} \left( \epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
& \leq \frac{\mathbb{E}[R_1] - \mathbb{E}[R_{T+1}]}{T} + \frac{q^2 \eta + \sum_{k=t+1}^{\infty} \eta \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}}{T} \\
& \quad + \left( \frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1} \right) \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]
\end{aligned} \tag{47}$$

463 where we have used the fact that  $(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \geq 0$  for all dimension  $j \in [d]$  by  
464 construction of  $\hat{v}_{t+1}^j$ .

465 We now bound the two remaining terms:

466 **Bounding**  $-\mathbb{E}[R_{T+1}]$ :

467 By definition (39) of  $R_t$  we have, using Lemma 1:

$$\begin{aligned}
-\mathbb{E}[R_{T+1}] & \leq \sum_{k=T+1}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] - f(\theta_{T+1}) \\
& \leq \left\| \sum_{k=T+1}^{\infty} \eta_k \beta_1^{k-t+1} \|\nabla f(\theta_{t-1})\| (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \right\| \\
& \leq \eta_{t+1} (1 - \beta_1) \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1)} G^2 - f(\theta_{T+1})
\end{aligned} \tag{48}$$

468 **Bounding**  $\sum_{t=1}^T \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]$ :

469 By definition in Algorithm 1:

$$\|\theta_{t+1} - \theta_t\|^2 = \eta_{t+1}^2 \left[ (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-\frac{1}{2}} m_{t+1} \right]^2 = \eta_{t+1}^2 \sum_{j=1}^d \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j + \epsilon} \tag{49}$$

470 For any dimension  $j \in [d]$ ,

$$\begin{aligned}
|m_{t+1}^j|^2 & = |\beta_1 m_t^j + (1 - \beta_1) \bar{g}_t^j|^2 \\
& \leq \beta_1 (\beta_1^2 |m_{t-1}^j|^2 + (1 - \beta_1)^2 |\bar{g}_{t-1}^j|^2) + |\bar{g}_t^j|^2 \\
& \leq \sum_{k=0}^t \beta_1^{2(t-k)} |\bar{g}_k^j|^2 \\
& \leq \sum_{k=0}^t \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}} \beta_2^{t-k} |\bar{g}_k^j|^2
\end{aligned} \tag{50}$$

471 Using Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
|m_{t+1}^j|^2 & \leq \sum_{k=0}^t \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}} \beta_2^{t-k} |\bar{g}_k^j|^2 \leq \sum_{k=0}^t \left( \frac{\beta_1^2}{\beta_2} \right)^{t-k} \sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2 \\
& \leq \frac{1}{1 - \frac{\beta_1^2}{\beta_2}} \sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2
\end{aligned} \tag{51}$$

472 On the other hand we have

$$\hat{v}_{t+1}^j \geq \beta_2 \hat{v}_t^j + (1 - \beta_2) (\bar{g}_t^j)^2 \tag{52}$$

473 and since it is also true for iteration  $t = 1$ , we have by induction replacing  $v_t^j$  in the above that

$$\hat{v}_{t+1}^j \geq (1 - \beta_2) \sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2 \iff \frac{\sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \leq (1 - \beta_2)^{-1} \quad (53)$$

474 Hence, we can derive from (49) that

$$\begin{aligned} \|\theta_{t+1} - \theta_t\|^2 &= \eta_{t+1}^2 \sum_{j=1}^d \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j + \epsilon} \leq \eta_{t+1}^2 \sum_{j=1}^d \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j} \\ &\stackrel{(a)}{\leq} \eta_{t+1}^2 \sum_{j=1}^d \frac{1}{1 - \frac{\beta_1^2}{\beta_2}} \frac{\sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \\ &\stackrel{(b)}{\leq} \eta_{t+1}^2 d(1 - \beta_2)^{-1} \left(1 - \frac{\beta_1^2}{\beta_2}\right)^{-1} \end{aligned} \quad (54)$$

475 where (a) uses (51) and (b) uses (53).

476 Plugging the two bounds in (47), we obtain the following bound:

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] &\leq \frac{\mathbb{E}[f(\theta_1) - f(\theta_{T+1})]}{\eta \Delta_1 T} + \frac{q^2 \eta + \sum_{k=t+1}^{\infty} \eta \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}}{\eta \Delta_1 T} \\ &\quad + \frac{1 - \beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1)G^2} \\ &\quad + \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1}\right) \frac{1}{\eta \Delta_1} \eta^2 d(1 - \beta_2)^{-1} \left(1 - \frac{\beta_1^2}{\beta_2}\right)^{-1} \end{aligned} \quad (55)$$

477 where  $\Delta_1 := \frac{(1 - \beta_1)}{2} \left(\epsilon + \frac{(q^2 + 1)G^2}{1 - \beta_2}\right)^{-\frac{1}{2}}$

478 With a constant stepsize  $\eta = \frac{L}{\sqrt{T}}$  we get the final convergence bound as follows:

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] &\leq \frac{\mathbb{E}[f(\theta_1) - f(\theta_{T+1})]}{L \Delta_1 \sqrt{T}} + d \frac{L \Delta_3}{\Delta_1 \sqrt{T}} \\ &\quad + \frac{\Delta_2}{\Delta_1 T} + \frac{1 - \beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1)G^2} \end{aligned} \quad (56)$$

479 where  $\Delta_2 := q^2 + \sum_{k=t+1}^{\infty} \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}$  and  $\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1}\right) (1 - \beta_2)^{-1} \left(1 - \frac{\beta_1^2}{\beta_2}\right)^{-1}$ .

480 □

### 481 B.3 Proof of Theorem 2

482 **Theorem.** Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize  $\{\eta_t\}_{t>0} =$   
483  $\frac{1}{\sqrt{T}}$ , the sequence of iterates  $\{\theta_t\}_{t>0}$  output from Algorithm 2 satisfies:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathcal{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{T}\right),$$

484 *Proof.* Let  $m_t'$  be the first moment moving average of standard AMSGrad using full gradients,  
485 i.e., the gradient with respect to the index data point  $i_t$  computed Line 4 of Algorithm 2 before  
486 applying any compression operator.

487 Denote

$$m_t = \beta_1 m_{t-1} + (1 - \beta_1) \tilde{g}_t \quad \text{and} \quad m'_t = \beta_1 m'_{t-1} + (1 - \beta_1) g_t$$

$$a_t = \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}, \quad \text{and} \quad a'_t = \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}}.$$

488 By construction we have  $m'_t = (1 - \beta_1) \sum_{i=1}^k \beta_1^{t-i} g_t$ .

489 Denote the following auxiliary sequences,

$$\mathcal{E}_{t+1} := \frac{(1 - \beta_1) \sum_{i=1}^{t+1} \beta_1^{t+1-i} e_i}{\sqrt{\hat{v}_t + \epsilon}}$$

$$\theta'_{t+1} := \theta_{t+1} - \eta \mathcal{E}_{t+1}.$$

490 Then,

$$\begin{aligned} \theta'_{t+1} &= \theta_{t+1} - \eta \mathcal{E}_{t+1} \\ &= \theta_t - \eta \frac{(1 - \beta_1) \sum_{i=1}^t \beta_1^{t-i} \tilde{g}_i + (1 - \beta_1) \sum_{i=1}^{t+1} \beta_1^{t+1-i} e_i}{\sqrt{\hat{v}_t + \epsilon}} \\ &= \theta_t - \eta \frac{(1 - \beta_1) \sum_{i=1}^t \beta_1^{t-i} (\tilde{g}_i + e_{i+1}) + (1 - \beta_1) \beta_1^t e_1}{\sqrt{\hat{v}_t + \epsilon}} \\ &= \theta_t - \eta \frac{(1 - \beta_1) \sum_{i=1}^t \beta_1^{t-i} e_i}{\sqrt{\hat{v}_t + \epsilon}} - \eta \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} \\ &\stackrel{(a)}{=} \theta'_t - \eta \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} := \theta'_t - \eta a'_t, \end{aligned}$$

491 where (a) uses the fact that  $\tilde{g}_t + e_{t+1} = g_t + e_t$ ,  $e_1 = 0$  at initialization. By Assumption 2 we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \eta \langle \nabla f(\theta'_t), a'_t \rangle + \frac{L}{2} \|\theta'_{t+1} - \theta'_t\|^2.$$

492 Thus,

$$\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] \leq -\eta \mathbb{E}[\langle \nabla f(\theta'_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] \quad (57)$$

$$= -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] + \eta \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), a'_t \rangle] \quad (58)$$

$$\leq -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] + \eta^2 L \mathbb{E}[\|\mathcal{E}_t\| \|a'_t\|] \quad (59)$$

493 **Bounding the first term (extracting  $\nabla f$ ).** We have

$$\begin{aligned} M_t &:= -\mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] = -\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} \rangle] \\ &= \underbrace{-\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle]}_I + \underbrace{\mathbb{E}[\langle \nabla f(\theta_t), (\frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}}) m'_t \rangle]}_{II}. \end{aligned}$$

494 To bound I, note that

$$\begin{aligned} I &= -\mathbb{E}[\langle \nabla f(\theta_t), \frac{(1 - \beta_1) g_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\ &= -\mathbb{E}[\langle \nabla f(\theta_t), \frac{(1 - \beta_1) g_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle | \mathcal{F}_{t-1}] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\ &= -(1 - \beta_1) \mathbb{E}[\frac{\|\nabla f(\theta_t)\|^2}{\sqrt{\hat{v}_{t-1} + \epsilon}}] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\ &\leq -\frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] - \beta_1 \mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle]. \end{aligned}$$

495 Regarding the second term, we have

$$\begin{aligned}
-\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] &= -\mathbb{E}[\langle \nabla f(\theta_{t-1}), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&= M_{t-1} + \eta L \mathbb{E}[\| \frac{m_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \| \| \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \|] \\
&\leq M_{t-1} + \frac{\eta L(q^2 + 1)G^4}{\epsilon}.
\end{aligned}$$

496 Putting parts together we obtain

$$I \leq \beta_1 M_{t-1} + \frac{\eta \beta_1 L(q^2 + 1)G^4}{\epsilon} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2].$$

497 For II, it holds that

$$II \leq G^2 \mathbb{E}[\sum_{i=1}^d | \frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}} |].$$

498 Thus, we arrive at

$$\begin{aligned}
M_t &\leq \beta_1 M_{t-1} + \frac{\eta \beta_1 L(q^2 + 1)G^4}{\epsilon} + G^2 \mathbb{E}[\sum_{i=1}^d | \frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}} |] - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&:= \beta_1 M_{t-1} + \frac{\eta \beta_1 L(q^2 + 1)G^4}{\epsilon} + G^2 H_t - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\leq \beta_1 M_{t-1} + \frac{\eta \beta_1 L(q^2 + 1)G^4}{\epsilon} + G^2 H_t.
\end{aligned}$$

499 By induction, we have

$$M_t \leq \beta_1^{t-1} M_1 + G^2 \sum_{i=0}^{t-2} \beta_1^i H_{t-i} + \frac{\eta \beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2],$$

500 since  $\beta_1 < 1$ . Summing over  $t = 1, \dots, T$ , we obtain

$$\begin{aligned}
\sum_{t=1}^T M_t &\leq \sum_{t=1}^T \beta_1^{t-1} M_1 + G^2 \sum_{t=2}^T \sum_{i=0}^{t-2} \beta_1^i H_{t-i} + \frac{T \eta \beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} - \sum_{t=1}^T \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\stackrel{(a)}{\leq} \frac{dG^2}{(1 - \beta_1)\sqrt{\epsilon}} + G^2 \sum_{t=2}^T (\sum_{i=0}^{T-t} \beta_1^{t-i}) H_t + \frac{T \eta \beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} - \sum_{t=1}^T \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\leq \frac{dG^2}{(1 - \beta_1)\sqrt{\epsilon}} + \frac{G^2}{1 - \beta_1} \sum_{t=2}^T \mathbb{E}[\sum_{i=1}^d | \frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}} |] \\
&\quad + \frac{T \eta \beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} - \sum_{t=1}^T \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\stackrel{(b)}{\leq} \frac{2dG^2}{(1 - \beta_1)\sqrt{\epsilon}} + \frac{T \eta \beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} - \sum_{t=1}^T \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2],
\end{aligned}$$

501 where (a) is because  $M_1 = \mathbb{E}[\langle \nabla f(\theta_1), a'_0 \rangle] \leq \beta_1 dG^2/\sqrt{\epsilon}$ , and (b) is derived by cancelling terms  
502 due to the fact that  $\{\hat{v}_t\}_{t \geq 0}$  is a non-decreasing sequence, hence  $\hat{v}_t \leq \hat{v}_{t-1}$ . It remains to bound the  
503 last two terms in (63).

504 **Bounding the variance term.** We have

$$\mathbb{E}[\|a'_t\|^2] = \mathbb{E}[\| \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} \|^2] \leq \frac{1}{\epsilon} \mathbb{E}[\|m'_t\|^2],$$

505 and by Young's inequality,

$$\begin{aligned}\mathbb{E}[\|m'_t\|^2] &= \mathbb{E}[\|\beta_1 m'_{t-1} + (1 - \beta_1)g_t\|^2] \\ &\leq (1 + \frac{\rho}{2})\beta_1^2 \mathbb{E}[\|m'_{t-1}\|^2] + (1 + \frac{1}{2\rho})(1 - \beta_1)^2 \mathbb{E}[\|g_t\|^2].\end{aligned}$$

506 Choosing  $\rho = 2(1 - \beta_1^2)$ , we derive

$$\begin{aligned}\mathbb{E}[\|m'_t\|^2] &\leq \beta_1^2(2 - \beta_1^2) \mathbb{E}[\|m'_{t-1}\|^2] + (1 - \beta_1)^2(1 + \frac{1}{4(1 - \beta_1^2)}) \mathbb{E}[\|g_t\|^2] \\ &\leq \frac{(1 - \beta_1)^2}{1 - \beta_1^2(2 - \beta_1^2)}(1 + \frac{1}{4(1 - \beta_1^2)})\sigma^2 := C\sigma^2,\end{aligned}$$

507 due to  $\beta_1 < 1$ ,  $m'_0 = 0$  and the bounded variance assumption. Hence,

$$\mathbb{E}[\|a'_t\|^2] \leq \frac{C\sigma^2}{\epsilon}.$$

508 **Bounding the compression error.** For the last term in (63), again by induction,

$$\begin{aligned}\|e_t\| &= \|e_{t-1} + g_{t-1} - \tilde{g}_{t-1}\| \\ &= \|g_{t-1} + e_{t-1} - \text{TopK}(g_{t-1} + e_{t-1}, k)\| \\ &\leq q \|g_{t-1} + e_{t-1}\| \\ &\leq q \|e_{t-1}\| + q \|g_{t-1}\| \\ &\leq \frac{q}{1 - q} G.\end{aligned}\tag{60}$$

509 Since  $\|a'_t\|^2 \leq G/\epsilon$ , we derive

$$\mathbb{E}[\|\mathcal{E}_t\| \|a'_t\|] \leq \frac{qG^2}{(1 - q)\epsilon}.$$

510 **Completing the proof.** Summing (63) over  $t = 1, \dots, T$  and integrating things together, we have

$$\begin{aligned}\mathbb{E}[f(\theta'_{T+1}) - f(\theta'_1)] &\leq \eta \sum_{t=1}^T M_t + \frac{T\eta^2 CL\sigma^2}{2\epsilon} + \frac{T\eta^2 LqG^2}{(1 - q)\epsilon} \\ &\leq - \sum_{t=1}^T \frac{\eta(1 - \beta_1)}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{2\eta dG^2}{(1 - \beta_1)\sqrt{\epsilon}} \\ &\quad + \frac{T\eta^2 \beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} + \frac{T\eta^2 CL\sigma^2}{2\epsilon} + \frac{T\eta^2 LqG^2}{(1 - q)\epsilon}.\end{aligned}$$

511 Thus,

$$\begin{aligned}\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] &\leq C' \left( \frac{\mathbb{E}[f(\theta'_1) - f(\theta'_{T+1})]}{T\eta} + \frac{2dG^2}{T(1 - \beta_1)\sqrt{\epsilon}} \right. \\ &\quad \left. + \frac{\eta\beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} + \frac{\eta CL\sigma^2}{2\epsilon} + \frac{\eta LqG^2}{(1 - q)\epsilon} \right) \\ &\leq C' \left( \frac{\mathbb{E}[f(\theta_1) - f(\theta^*)]}{T\eta} + \frac{2dG^2}{T(1 - \beta_1)\sqrt{\epsilon}} \right. \\ &\quad \left. + \frac{\eta\beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} + \frac{\eta CL\sigma^2}{2\epsilon} + \frac{\eta LqG^2}{(1 - q)\epsilon} \right).\end{aligned}$$

512 where  $C' = \frac{\sqrt{(q^2 + 1)G^2 + \epsilon}}{1 - \beta_1}$ , and  $C = \frac{(1 - \beta_1)^2}{1 - \beta_1^2(2 - \beta_1^2)}(1 + \frac{1}{4(1 - \beta_1^2)})$ . The last inequality is because

513  $\theta'_1 = \theta_1$ , and  $\theta^* = \arg \min_{\theta} f(\theta)$ .



514 Taking decreasing learning rate  $\eta = 1/\sqrt{T}$ , we obtain

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathcal{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{T}\right),$$

515 matching the convergence rate of SGD with error feedback [31].

516 □

## 517 C Distributed setting

518 Denote the average gradient as  $\bar{g}_t = \frac{1}{N} \sum_{i=1}^N \tilde{g}_{t,i}$ , and  $\bar{g}'_t = \frac{1}{N} \sum_{i=1}^N g_{t,i}$  be the average of true  
519 (uncompressed) local gradients. With a little change of notation, we denote  $\bar{m}_0 = \bar{m}'_0 = 0$ , and

$$\begin{aligned} \bar{m}_t &= \beta_1 \bar{m}_{t-1} + (1 - \beta_1) \bar{g}_t \quad \text{and} \quad \bar{m}'_t = \beta_1 \bar{m}'_{t-1} + (1 - \beta_1) \bar{g}'_t \\ a_t &= \frac{\bar{m}_t}{\sqrt{\hat{v}_t + \epsilon}}, \quad \text{and} \quad a'_t = \frac{\bar{m}'_t}{\sqrt{\hat{v}_t + \epsilon}}. \end{aligned}$$

520 By construction we have  $m'_t = (1 - \beta_1) \sum_{i=1}^k \beta_1^{t-i} \bar{g}_t$ .

521 Let  $\bar{e}_t = \frac{1}{N} \sum_{i=1}^N e_{t,i}$ . Denote the following auxiliary sequences,

$$\begin{aligned} \mathcal{E}_{t+1} &:= \frac{(1 - \beta_1) \sum_{i=1}^{t+1} \beta_1^{t+1-i} \bar{e}_i}{\sqrt{\hat{v}_t + \epsilon}} \\ \theta'_{t+1} &:= \theta_{t+1} - \eta \mathcal{E}_{t+1}. \end{aligned}$$

522 Then,

$$\begin{aligned} \theta'_{t+1} &= \theta_{t+1} - \eta \mathcal{E}_{t+1} \\ &= \theta_t - \eta \frac{(1 - \beta_1) \sum_{i=1}^t \beta_1^{t-i} \bar{g}_i + (1 - \beta_1) \sum_{i=1}^{t+1} \beta_1^{t+1-i} \bar{e}_i}{\sqrt{\hat{v}_t + \epsilon}} \\ &= \theta_t - \eta \frac{(1 - \beta_1) \sum_{i=1}^t \beta_1^{t-i} (\bar{g}_i + \bar{e}_{i+1}) + (1 - \beta) \beta_1^t \bar{e}_1}{\sqrt{\hat{v}_t + \epsilon}} \\ &= \theta_t - \eta \frac{(1 - \beta_1) \sum_{i=1}^t \beta_1^{t-i} \bar{e}_i}{\sqrt{\hat{v}_t + \epsilon}} - \eta \frac{\bar{m}'_t}{\sqrt{\hat{v}_t + \epsilon}} \\ &\stackrel{(a)}{=} \theta'_t - \eta \frac{\bar{m}'_t}{\sqrt{\hat{v}_t + \epsilon}} := \theta'_t - \eta a'_t, \end{aligned}$$

523 where (a) uses the fact that  $\tilde{g}_{t,i} + e_{t+1,i} = g_{t,i} + e_{t,i}$  for  $\forall i \in [N]$ . By Assumption 2 we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \eta \langle \nabla f(\theta'_t), a'_t \rangle + \frac{L}{2} \|\theta'_{t+1} - \theta'_t\|^2.$$

524 Thus,

$$\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] \leq -\eta \mathbb{E}[\langle \nabla f(\theta'_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] \quad (61)$$

$$= -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] + \eta \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), a'_t \rangle] \quad (62)$$

$$\leq -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] + \eta^2 L \mathbb{E}[\|\mathcal{E}_t\| \|a'_t\|] \quad (63)$$

525 **Bounding the first term (extracting  $\nabla f$ ).** We have

$$\begin{aligned} M_t &:= -\mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] = -\mathbb{E}[\langle \nabla f(\theta_t), \frac{\bar{m}'_t}{\sqrt{\hat{v}_t + \epsilon}} \rangle] \\ &= \underbrace{-\mathbb{E}[\langle \nabla f(\theta_t), \frac{\bar{m}'_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle]}_I + \underbrace{\mathbb{E}[\langle \nabla f(\theta_t), (\frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}}) \bar{m}'_t \rangle]}_{II}. \end{aligned}$$

526 To bound I, note that

$$\begin{aligned}
I &= -\mathbb{E}[\langle \nabla f(\theta_t), \frac{(1-\beta_1)g_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&= -\mathbb{E}[\langle \nabla f(\theta_t), \frac{(1-\beta_1)g_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle | \mathcal{F}_{t-1}] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&= -(1-\beta_1)\mathbb{E}[\frac{\|\nabla f(\theta_t)\|^2}{\sqrt{\hat{v}_{t-1} + \epsilon}}] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&\leq -\frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] - \beta_1\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle].
\end{aligned}$$

527 Regarding the second term, we have

$$\begin{aligned}
-\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] &= -\mathbb{E}[\langle \nabla f(\theta_{t-1}), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&= M_{t-1} + \eta L \mathbb{E}[\|\frac{m_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}}\| \|\frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}}\|] \\
&\leq M_{t-1} + \frac{\eta L(q^2+1)G^4}{\epsilon}.
\end{aligned}$$

528 Putting parts together we obtain

$$I \leq \beta_1 M_{t-1} + \frac{\eta \beta_1 L(q^2+1)G^4}{\epsilon} - \frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2].$$

529 For II, it holds that

$$II \leq G^2 \mathbb{E}[\sum_{i=1}^d |\frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}}|].$$

530 Thus, we arrive at

$$\begin{aligned}
M_t &\leq \beta_1 M_{t-1} + \frac{\eta \beta_1 L(q^2+1)G^4}{\epsilon} + G^2 \mathbb{E}[\sum_{i=1}^d |\frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}}|] - \frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&:= \beta_1 M_{t-1} + \frac{\eta \beta_1 L(q^2+1)G^4}{\epsilon} + G^2 H_t - \frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\leq \beta_1 M_{t-1} + \frac{\eta \beta_1 L(q^2+1)G^4}{\epsilon} + G^2 H_t.
\end{aligned}$$

531 By induction, we have

$$M_t \leq \beta_1^{t-1} M_1 + G^2 \sum_{i=0}^{t-2} \beta_1^i H_{t-i} + \frac{\eta \beta_1 L(q^2+1)G^4}{(1-\beta_1)\epsilon} - \frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2],$$

532 since  $\beta_1 < 1$ . Summing over  $t = 1, \dots, T$ , we obtain

$$\begin{aligned}
\sum_{t=1}^T M_t &\leq \sum_{t=1}^T \beta_1^{t-1} M_1 + G^2 \sum_{t=2}^T \sum_{i=0}^{t-2} \beta_1^i H_{t-i} + \frac{T \eta \beta_1 L(q^2+1)G^4}{(1-\beta_1)\epsilon} - \sum_{t=1}^T \frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\stackrel{(a)}{\leq} \frac{dG^2}{(1-\beta_1)\sqrt{\epsilon}} + G^2 \sum_{t=2}^T (\sum_{i=0}^{T-t} \beta_1^{t-i}) H_t + \frac{T \eta \beta_1 L(q^2+1)G^4}{(1-\beta_1)\epsilon} - \sum_{t=1}^T \frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\leq \frac{dG^2}{(1-\beta_1)\sqrt{\epsilon}} + \frac{G^2}{1-\beta_1} \sum_{t=2}^T \mathbb{E}[\sum_{i=1}^d |\frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}}|] \\
&\quad + \frac{T \eta \beta_1 L(q^2+1)G^4}{(1-\beta_1)\epsilon} - \sum_{t=1}^T \frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\stackrel{(b)}{\leq} \frac{2dG^2}{(1-\beta_1)\sqrt{\epsilon}} + \frac{T \eta \beta_1 L(q^2+1)G^4}{(1-\beta_1)\epsilon} - \sum_{t=1}^T \frac{1-\beta_1}{\sqrt{(q^2+1)G^2 + \epsilon}}\mathbb{E}[\|\nabla f(\theta_t)\|^2],
\end{aligned}$$

533 where (a) is because  $M_1 = \mathbb{E}[\langle \nabla f(\theta_1), a'_0 \rangle] \leq \beta_1 dG^2 / \sqrt{\epsilon}$ , and (b) is derived by cancelling terms  
 534 due to the fact that  $\{\hat{v}_t\}_{t>0}$  is a non-decreasing sequence, hence  $\hat{v}_t \leq \hat{v}_{t-1}$ . It remains to bound the  
 535 last two terms in (63).

536 **Bounding the variance term.** We have

$$\mathbb{E}[\|a'_t\|^2] = \mathbb{E}[\|\frac{m'_t}{\sqrt{\hat{v}_t} + \epsilon}\|^2] \leq \frac{1}{\epsilon} \mathbb{E}[\|m'_t\|^2],$$

537 and by Young's inequality,

$$\begin{aligned} \mathbb{E}[\|m'_t\|^2] &= \mathbb{E}[\|\beta_1 m'_{t-1} + (1 - \beta_1)g_t\|^2] \\ &\leq (1 + \frac{\rho}{2})\beta_1^2 \mathbb{E}[\|m'_{t-1}\|^2] + (1 + \frac{1}{2\rho})(1 - \beta_1)^2 \mathbb{E}[\|g_t\|^2]. \end{aligned}$$

538 Choosing  $\rho = 2(1 - \beta_1^2)$ , we derive

$$\begin{aligned} \mathbb{E}[\|m'_t\|^2] &\leq \beta_1^2(2 - \beta_1^2) \mathbb{E}[\|m'_{t-1}\|^2] + (1 - \beta_1)^2(1 + \frac{1}{4(1 - \beta_1^2)}) \mathbb{E}[\|g_t\|^2] \\ &\leq \frac{(1 - \beta_1)^2}{1 - \beta_1^2(2 - \beta_1^2)}(1 + \frac{1}{4(1 - \beta_1^2)})\sigma^2 := C\sigma^2, \end{aligned}$$

539 due to  $\beta_1 < 1$ ,  $m'_0 = 0$  and the bounded variance assumption. Hence,

$$\mathbb{E}[\|a'_t\|^2] \leq \frac{C\sigma^2}{\epsilon}.$$

540 **Bounding the compression error.** For the last term in (63), again by induction,

$$\begin{aligned} \|e_t\| &= \|e_{t-1} + g_{t-1} - \tilde{g}_{t-1}\| \\ &= \|g_{t-1} + e_{t-1} - \text{Top}K(g_{t-1} + e_{t-1}, k)\| \\ &\leq q \|g_{t-1} + e_{t-1}\| \\ &\leq q \|e_{t-1}\| + q \|g_{t-1}\| \\ &\leq \frac{q}{1 - q} G. \end{aligned} \tag{64}$$

541 Since  $\|a'_t\|^2 \leq G/\epsilon$ , we derive

$$\mathbb{E}[\|\mathcal{E}_t\| \|a'_t\|] \leq \frac{qG^2}{(1 - q)\epsilon}.$$

542 **Completing the proof.** Summing (63) over  $t = 1, \dots, T$  and integrating things together, we have

$$\begin{aligned} \mathbb{E}[f(\theta'_{T+1}) - f(\theta'_1)] &\leq \eta \sum_{t=1}^T M_t + \frac{T\eta^2 CL\sigma^2}{2\epsilon} + \frac{T\eta^2 LqG^2}{(1 - q)\epsilon} \\ &\leq - \sum_{t=1}^T \frac{\eta(1 - \beta_1)}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{2\eta dG^2}{(1 - \beta_1)\sqrt{\epsilon}} \\ &\quad + \frac{T\eta^2 \beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} + \frac{T\eta^2 CL\sigma^2}{2\epsilon} + \frac{T\eta^2 LqG^2}{(1 - q)\epsilon}. \end{aligned}$$

543 Thus,

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] &\leq C' \left( \frac{\mathbb{E}[f(\theta'_1) - f(\theta'_{T+1})]}{T\eta} + \frac{2dG^2}{T(1 - \beta_1)\sqrt{\epsilon}} \right. \\ &\quad \left. + \frac{\eta\beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} + \frac{\eta CL\sigma^2}{2\epsilon} + \frac{\eta LqG^2}{(1 - q)\epsilon} \right) \\ &\leq C' \left( \frac{\mathbb{E}[f(\theta_1) - f(\theta^*)]}{T\eta} + \frac{2dG^2}{T(1 - \beta_1)\sqrt{\epsilon}} \right. \\ &\quad \left. + \frac{\eta\beta_1 L(q^2 + 1)G^4}{(1 - \beta_1)\epsilon} + \frac{\eta CL\sigma^2}{2\epsilon} + \frac{\eta LqG^2}{(1 - q)\epsilon} \right). \end{aligned}$$

544 where  $C' = \frac{\sqrt{(q^2+1)G^2+\epsilon}}{1-\beta_1}$ , and  $C = \frac{(1-\beta_1)^2}{1-\beta_1^2(2-\beta_1)^2}(1 + \frac{1}{4(1-\beta_1^2)})$ . The last inequality is because  
 545  $\theta'_1 = \theta_1$ , and  $\theta^* = \arg \min_{\theta} f(\theta)$ .

546 Taking decreasing learning rate  $\eta = 1/\sqrt{T}$ , we obtain

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathcal{O}(\frac{1}{\sqrt{T}} + \frac{1}{T}),$$

547 matching the convergence rate of SGD with error feedback [31].