
Distributed Adaptive Learning with Gradient Compression

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Abstract

1 This paper presents a new algorithm – the Sparsified AMSGrad algorithm
2 (SPARS-AMS) – for tackling single-machine and distributed supervised learn-
3 ing. Unlike prior works which rely on full gradient communication between the
4 workers and the parameter-server, we design a distributed adaptive optimization
5 method with gradient compression coupled with an error-feedback to alleviate the
6 bias introduced by the compression. While the former allows us to only transmit
7 fewer bits of gradient vectors to the server, we show that using the latter, which
8 correct for the bias, SPARS-AMS reaches a stationary point in $\mathcal{O}(1/\sqrt{T})$ itera-
9 tions, matching that of state-of-the-art single-machine and distributed methods,
10 without any error-feedback. We illustrate our theoretical results by showing the
11 effectiveness of our method both under the single-machine and distributed settings
12 on various benchmark datasets.

13 1 Introduction

14 Deep neural network has achieved the state-of-the-art learning performance on numerous AI appli-
15 cations, e.g., computer vision [23, 26, 47], Natural Language Processing [25, 54, 58], Reinforcement
16 Learning [37, 45] and recommendation systems [16, 49]. With the increasing size of both data and
17 deep networks, standard single machine training confronts with at least two major challenges:

- 18 • Due to the limited computing power of a single machine, it would take a long time to
19 process the massive number of data samples—training would be slow.
- 20 • In many practical scenarios, data are typically stored in multiple servers, possibly at differ-
21 ent locations, due to the storage constraints (massive user behavior data, Internet images,
22 etc.) or privacy reasons [11]. Transmitting data might be costly.

23 *Distributed learning* framework [18] has been a common training strategy to tackle the above two
24 issues. For example, in centralized distributed stochastic gradient descent (SGD) protocol, data are
25 located at n local nodes, at which the gradients of the model are computed in parallel. In each
26 iteration, a central server aggregates the local gradients, updates the global model, and transmits
27 back the updated model to the local nodes for subsequent gradient computation. As we can see, this
28 setting naturally solves aforementioned issues: 1) We use n computing nodes to train the model, so
29 the time per training epoch can be largely reduced; 2) There is no need to transmit the local data to
30 central server. Besides, distributed training also provides stronger error tolerance since the training
31 process could continue even one local machine breaks down. As a result of these advantages, there
32 has been a surge of study and applications on distributed systems [10, 39, 20, 24, 27, 35, 33].

33 Among many optimization strategies, SGD is still the most popular prototype in distributed training
34 for its simplicity and effectiveness [14, 1, 36]. Yet, when the deep learning model is very large,

the communication between local nodes and central server could be expensive. Burdensome gradient transmission would slow down the whole training system, or even be impossible because of the limited bandwidth in some applications. Thus, reducing the communication cost in distributed SGD has become an active topic, and an important ingredient of large-scale distributed systems (e.g. [42]). Solutions based on quantization, sparsification and other compression techniques of the local gradients are proposed, e.g., [4, 50, 48, 46, 3, 7, 17, 52, 28]. As one would expect, in most approaches, there exists a trade-off between compression and learning performance. In general, larger bias and variance of the compressed gradients usually bring more significant performance downgrade in terms of convergence [46, 2]. Interestingly, studies (e.g., [31]) show that the technique of *error feedback* is able to remedy the issue of such biased compressors, achieving same convergence rate as full-gradient SGD.

On the other hand, in recent years, adaptive optimization algorithms (e.g. AdaGrad [21], Adam [32] and AMSGrad [41]) have become popular because of their superior empirical performance. These methods use different implicit learning rates for different coordinates that keep changing adaptively throughout the training process, based on the learning trajectory. In many learning problems, adaptive methods have been shown to converge faster than SGD, sometimes with better generalization as well. However, the body of literature that combines adaptive methods with distributed training is still very limited. In this paper, we propose a distributed optimization algorithm with AMSGrad as the backbone, along with Top- k sparsification to reduce the communication cost.

1.1 Our contributions

We develop a simple optimization leveraging the adaptivity of AMSGrad, and the computational virtue of TopK sparsification, for tackling a large finite-sum of nonconvex objective functions.

Our technique is shown to be both theoretically and empirically effective under *the classical centralized setting* and *the distributed setting*.

In this contribution,

- We derive a sparsified AMSGrad with error feedback, called SPARS-AMS, with a single machine and provide its decentralized counter part.
- We provide a non-asymptotic convergence rate under each setting,
- We highlight the effectiveness of both methods through several numerical experiments

2 Related Work

2.1 Distributed SGD with compressed gradients

Quantization. As we mentioned before, SGD is the most commonly adopted optimization method in distributed training of deep neural nets. To reduce the expensive communication in large-scale distributed systems, extensive works have considered various compression techniques applied to the gradient transaction procedure. The first strategy is quantization. [19] condenses 32-bit floating numbers into 8-bits when representing the gradients. [42, 7, 31, 8] use the extreme 1-bit information (sign) of the gradients, combined with tricks like momentum, majority vote and memory. Other quantization-based methods include QSGD [4, 51, 57] and LPC-SVRG [55], leveraging unbiased stochastic quantization. The saving in communication of quantization methods is moderate: for example, 8-bit quantization reduces the cost to 25% (compared with 32-bit full-precision). Even in the extreme 1-bit case, the largest compression ratio is around $1/32 \approx 3.1\%$.

Sparsification. Gradient sparsification is another popular solution which may provide higher compression rate. Instead of commuting the full gradient, each local worker only passes a few coordinates to the central server and zeros out the others. Thus, we can more freely choose higher compression ratio (e.g., 1%, 0.1%), still achieving impressive performance in many applications [34]. Stochastic sparsification methods, including uniform sampling and magnitude based sampling [48], select coordinates based on some sampling probability yielding unbiased gradient compressors. Deterministic methods are simpler, e.g., Random- k , Top- k [46, 44] (selecting k elements with largest magnitude), Deep Gradient Compression [34], but usually lead to biased gradient estima-

tion. In [28], the central server identifies heavy-hitters from the count-sketch [12] of the local gradients, which can be regarded as a noisy variant of Top- k strategy. More applications and analysis of compressed distributed SGD can be found in [30, 43, 5, 6, 29], among others.

Error Feedback. Biased gradient estimation, which is a consequence of many aforementioned methods (e.g., signSGD, Top- k), undermines the model training, both theoretically and empirically, with slower convergence and worse generalization [2, 9]. The technique of *error feedback* is able to “correct for the bias” and fix the problems. In this procedure, the difference between the true stochastic gradient and the compressed one is accumulated locally, which is then added back to the local gradients in later iterations. [46, 31] prove the $\mathcal{O}(\frac{1}{T})$ and $\mathcal{O}(\frac{1}{\sqrt{T}})$ convergence rate of EF-SGD in strongly convex and non-convex setting respectively, matching the rates of vanilla SGD [40, 22].

2.2 Adaptive optimization

In each SGD update, all the gradient coordinates share a same learning rate, either constant or decreasing over iterations. Adaptive optimization methods cast different learning rate on each dimension. AdaGrad [21] divides the gradient element-wisely by $\sqrt{\sum_{t=1}^T g_t^2} \in \mathbb{R}^d$, where $g_t \in \mathbb{R}^d$ is the gradient vector at time t and d is the model dimensionality. Thus, it intrinsically assigns different learning rates to different coordinates throughout the training—elements with smaller previous gradient magnitude tend to move a larger step. AdaGrad has been shown to perform well especially under some sparsity structure. AdaDelta [56] and Adam [32] introduce momentum and moving average of second moment estimation into AdaGrad which lead to better performance. AMSGrad [41] fixes the potential convergence issue of Adam, which will serve as the prototype in this paper. We present the pseudocode in Algorithm . In general, adaptive optimization methods are easier to tune in practice, and usually exhibit faster convergence than SGD. Thus, they have been widely used in training deep learning models in language and computer vision applications, e.g., [15, 53, 59]. In distributed setting, the work [38] proposes a decentralized system in online optimization. However, communication efficiency is not considered. The recent work [13] is the most relevant to our paper. Yet, their method is based on Adam, and requires every local node to store a local estimation of first and second moment, thus being less efficient. We will present more detailed comparison in Section 3.

3 Communication-Efficient Adaptive Optimization

3.1 Gradient Compressors

In this paper, we mainly consider deterministic q -deviate compressors defined as below.

Assumption 1. We say a compressor $\mathcal{C} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is q -deviate if for $\forall x \in \mathbb{R}^d$, $\exists 0 \leq q < 1$ such that $\|\mathcal{C}(x) - x\| \leq q \|x\|$.

Note that, smaller q indicates better approximation of the true gradient, and $q = 0$ implies no compression, i.e. $\mathcal{C}(x) = x$. We give two popular and highly efficient q -deviate compressors that will be compared in this paper.

Definition 1 (Top- k). For $x \in \mathbb{R}^d$, denote \mathcal{S} as the size- k set of $i \in [d]$ with largest k magnitude $|x_i|$. The **Top- k** compressor is defined as $\mathcal{C}(x)_i = x_i$, if $i \in \mathcal{S}$; $\mathcal{C}(x)_i = 0$ otherwise.

Definition 2 (SIGN). For $x \in \mathbb{R}^d$, define the **SIGN** compressor as $\mathcal{C}(x) = \text{sign}(x) \times \frac{1}{d} \sum_{i=1}^d |x_i|$.

Remark 1. Here the scalar, mean magnitude, multiplied to $\text{sign}(x)$ ensures $0 \leq q < 1$ as required by Assumption 1, which can be shown by Cauchy-Schwartz inequality. In implementation, this scalar can be arbitrary since we can offset its influence by tuning the learning rate.

Most modern machine learning tasks can be casted as a large finite-sum optimization problem written as:

$$\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n f_i(\theta) \quad (1)$$

where n denotes the number of workers, f_i represents the average loss for worker i and θ the global model parameter taking value in Θ , a subset of \mathbb{R}^d .

130 Some related work:

131 [31] develops variant of signSGD (as a biased compression schemes) for distributed optimization.
 132 Contributions are mainly on this error feedback variant. In [44], the authors provide theoretical
 133 results on the convergence of sparse Gradient SGD for distributed optimization (we want that for
 134 AMS here). [46] develops a variant of distributed SGD with sparse gradients too. Contributions
 135 include a memory term used while compressing the gradient (using top k for instance). Speeding up
 136 the convergence in $\frac{1}{T^3}$.

137 Consider standard synchronous distributed optimization setting. AMSGrad is used as the prototype,
 138 and the local workers is only in charge of gradient computation.

139 3.2 SPARS-AMS with Error Feedback

140 The key difference (and interesting part) of our TopK AMSGrad compared with the following arxiv
 141 paper “Quantized Adam”<https://arxiv.org/pdf/2004.14180.pdf> is that, in our model only
 142 gradients are transmitted. In “QAdam”, each local worker keeps a local copy of moment estimator
 143 m and v , and compresses and transmits m/v as a whole. Thus, that method is very much like the
 144 sparsified distributed SGD, except that g is changed into m/v . In our model, the moment estimates
 145 m and v are computed only at the central server, with the compressed gradients instead of the full
 146 gradient. This would be the key (and difficulty) in convergence analysis.

Algorithm 1 Distributed SPARS-AMS with error-feedback

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1: Input: parameter  $\beta_1, \beta_2$ , learning rate  $\eta_t$ .
2: Initialize: central server parameter  $\theta_1 \in \Theta \subseteq \mathbb{R}^d$ ;  $e_{1,i} = 0$  the error accumulator for each
   worker; sparsity parameter  $k$ ;  $n$  local workers;  $m_0 = 0, v_0 = 0, \hat{v}_0 = 0$ 
3: for  $t = 1$  to  $T$  do
4:   parallel for worker  $i \in [n]$  do:
5:     Receive model parameter  $\theta_t$  from central server
6:     Compute stochastic gradient  $g_{t,i}$  at  $\theta_t$ 
7:     Compute  $\tilde{g}_{t,i} = \text{TopK}(g_{t,i} + e_{t,i}, k)$ 
8:     Update the error  $e_{t+1,i} = e_{t,i} + g_{t,i} - \tilde{g}_{t,i}$ 
9:     Send  $\tilde{g}_{t,i}$  back to central server
10:  end parallel
11:  Central server do:
12:     $\bar{g}_t = \frac{1}{n} \sum_{i=1}^n \tilde{g}_{t,i}$ 
13:     $m_t = \beta_1 m_{t-1} + (1 - \beta_1) \bar{g}_t$ 
14:     $v_t = \beta_2 v_{t-1} + (1 - \beta_2) \bar{g}_t^2$ 
15:     $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ 
16:    Update the global model  $\theta_{t+1} = \theta_t - \eta_t \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$ 
17: end for

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147 4 Non-Asymptotic Convergence Analysis for the Single Machine and 148 Decentralized settings

149 Several mild assumptions to make: Nonconvex and smooth loss function, unbiased stochastic gradi-
 150 ent, bounded variance of the gradient, bounded norm of the gradient, control of the distance between
 151 the true gradient and its sparse variant.

152 Check [13] starting with single machine and extending to distributed settings (several machines).

153 Under the distributed setting, the goal is to derive an upper bound to the second order moment of
 154 the gradient of the objective function at some iteration $T_f \in [1, T]$.

155 We begin by making the following assumptions.

156 **Assumption 2. (Smoothness)** For $i \in [n]$, f_i is L -smooth: $\|\nabla f_i(\theta) - \nabla f_i(\vartheta)\| \leq L \|\theta - \vartheta\|$.

157 **Assumption 3. (Unbiased and Bounded gradient per worker)** For any iteration index $t > 0$ and
 158 worker index $i \in [n]$, the stochastic gradient is unbiased and bounded from above: $\mathbb{E}[g_{t,i}] =$
 159 $\nabla f_i(\theta_t)$ and $\|g_{t,i}\| \leq G_i$.

160 **Assumption 4.** (*Bounded variance per worker*) For any iteration index $t > 0$ and worker index
 161 $i \in \llbracket n \rrbracket$, the variance of the noisy gradient is bounded: $\mathbb{E}[|g_{t,i} - \nabla f_i(\theta_t)|^2] < \sigma_i^2$.

162 Denote by $Q(\cdot)$ the quantization operator Line 7 of Algorithm 1, which takes as input a gradient
 163 vector and returns a quantized version of it, and note $\tilde{g} := Q(g)$. Assume that

164 Denote for all $\theta \in \Theta$:

$$f(\theta) := \frac{1}{n} \sum_{i=1}^n f_i(\theta), \quad (2)$$

165 where n denotes the number of workers.

166 **Decentralized Workers Setting:** The main theorem in the decentralized setting reads:

167 **Theorem 1.** Under Assumption 2 to Assumption 4, the sequence of iterates $\{\theta_t\}_{t>0}$ output from
 168 Algorithm 1 satisfies:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_1) - f(\theta_{T+1})]}{\Delta_1 \eta_t T} + d \frac{\Delta_3}{\Delta_1 \eta_t T} + \frac{\Delta_2}{\Delta_1 T} + \frac{1 - \beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1)} \sigma^2 \quad (3)$$

169 where $\{\eta_t\}_{t>0}$ is the sequence of stepsizes and:

$$\begin{aligned} \Delta_1 &:= \frac{(1 - \beta_1)}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}}, \quad \Delta_2 := q^2 + \frac{G^2}{\epsilon 2n^2} \bar{\beta}_1 \\ \Delta_3 &:= \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1} \right) (1 - \beta_2)^{-1} \left(1 - \frac{\beta_1^2}{\beta_2} \right)^{-1} \end{aligned} \quad (4)$$

170 We remark from this bound in Theorem 1, that the more quantization we apply to our gradient
 171 vectors ($q \uparrow$), the larger the upper bound of the stationary condition is, *i.e.*, the slower the algorithm
 172 is. This is intuitive as using compressed quantities will definitely impact the algorithm speed. We
 173 will observe in the numerical section below that a trade-off on the level of quantization q can be
 174 found to achieve similar speed of convergence with less computation resources used throughout the
 175 training.

176 **Corollary 1.** Under Assumption 2 to Assumption 4, setting the stepsize as $\eta_t = L\sqrt{\frac{n}{T}}$, the sequence
 177 of iterates $\{\theta_t\}_{t>0}$ output from Algorithm 1 satisfies:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathcal{O}\left(\frac{1}{L\sqrt{nT}} + d \frac{L}{\sqrt{nT}} + \frac{1}{T}\right),$$

178 **Single Machine Setting:** We first provide the formulation of our method in the single machine
 179 settings in Algorithm 2. Here, the data and the computation are all performed on a single machine.

Algorithm 2 SPARS-AMS with error-feedback for a single machine

- 1: **Input:** parameter β_1, β_2 , learning rate η_t .
 - 2: Initialize: central server parameter $\theta_1 \in \Theta \subseteq \mathbb{R}^d$; $e_1 = 0$ the error accumulator; sparsity
parameter k ; $m_0 = 0, v_0 = 0, \hat{v}_0 = 0$
 - 3: **for** $t = 1$ to T **do**
 - 4: Compute stochastic gradient $g_t = g_{t,i_t}$ at θ_t for randomly sampled index i_t
 - 5: Compute $\tilde{g}_t = \text{TopK}(g_t + e_t, k)$
 - 6: Update the error $e_{t+1} = e_t + g_t - \tilde{g}_t$
 - 7: $m_t = \beta_1 m_{t-1} + (1 - \beta_1) \tilde{g}_t$
 - 8: $v_t = \beta_2 v_{t-1} + (1 - \beta_2) \tilde{g}_t^2$
 - 9: $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$
 - 10: Update the global model $\theta_{t+1} = \theta_t - \eta_t \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}$
 - 11: **end for**
-

180 The convergence rate of the vector of parameters estimated via Algorithm 2 is given below:

181 **Theorem 2.** *Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize*
182 *$\{\eta_t\}_{t>0} = \frac{1}{\sqrt{t}}$, the sequence of iterates $\{\theta_t\}_{t>0}$ output from Algorithm 2 satisfies:*

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathcal{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{T}\right),$$

183 matching the convergence rate of SGD with error feedback [31].

184 5 Experiments

185 Our proposed TopK-EF with AMSGrad matches that of full AMSGrad, in distributed learning.
186 Number of local workers is 20. Error feedback fixes the convergence issue of using solely the
187 TopK gradient.

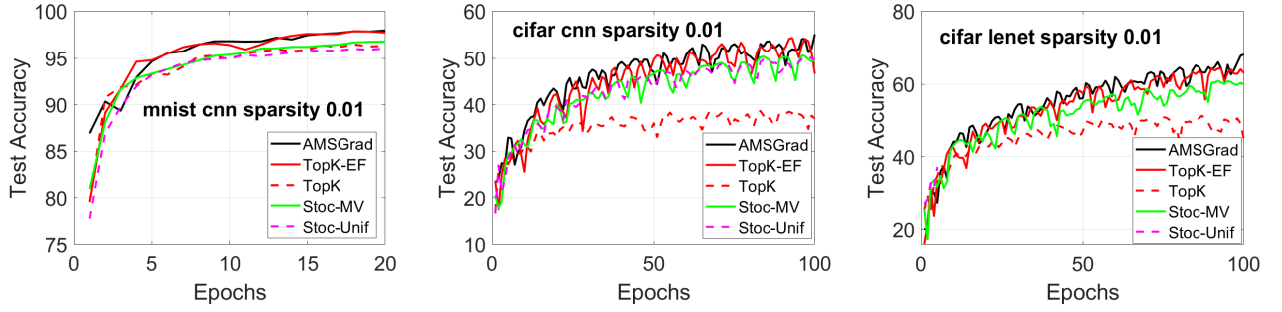


Figure 1: Test accuracy.

188 6 Conclusion

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382 A Single Machine Setting

383 A.1 Intermediary Lemmas

384 **Lemma 1.** *Under Assumption 1 to Assumption 4 we have:*

$$\begin{aligned}\mathbb{E}\|m'_t\|^2 &\leq C\sigma^2 + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2], \\ \mathbb{E}[\|m_t\|^2] &\leq (3q^2 + \frac{4q^2(6q^2 + 3)}{(1 - q^2)^2} + 1)C\sigma^2 + (6q^2 + 3)C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2],\end{aligned}$$

385 where $C_1 = (1 - \beta_1^2)(1 + \frac{1}{4(1 - \beta_1^2)})$ and $C = \frac{C_1}{1 - \beta_1^2(2 - \beta_1^2)}$.

386 *Proof.* We have by Young's inequality

$$\begin{aligned}\mathbb{E}[\|m'_t\|^2] &= \mathbb{E}[\|\beta_1 m'_{t-1} + (1 - \beta_1)g_t\|^2] \\ &\leq (1 + \rho)\beta_1^2 \mathbb{E}[\|m'_{t-1}\|^2] + (1 + \frac{1}{\rho})(1 - \beta_1)^2 \mathbb{E}[\|g_t\|^2].\end{aligned}$$

387 Since $\mathbb{E}[\|g_t\|^2] \leq \sigma^2 + \mathbb{E}[\|\nabla f(\theta_t)\|^2]$, by choosing $\rho = 1 - \beta_1^2$, we derive

$$\mathbb{E}[\|m'_t\|^2] \leq \beta_1^2(2 - \beta_1^2) \mathbb{E}[\|m'_{t-1}\|^2] + (1 - \beta_1)^2(1 + \frac{1}{4(1 - \beta_1^2)}) \mathbb{E}[\|g_t\|^2] \quad (5)$$

$$\leq \frac{(1 - \beta_1)^2}{1 - \beta_1^2(2 - \beta_1^2)}(1 + \frac{1}{4(1 - \beta_1^2)})\sigma^2 + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2] \quad (6)$$

$$:= C\sigma^2 + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2], \quad (7)$$

388 due to $\beta_1 < 1$, $m'_0 = 0$ and the bounded variance assumption. Here $C_1 = (1 - \beta_1^2)(1 + \frac{1}{4(1 - \beta_1^2)})$

389 and $C = \frac{C_1}{1 - \beta_1^2(2 - \beta_1^2)}$.

390 For m_t which consists of the compressed stochastic gradients, first note that

$$\begin{aligned}\mathbb{E}[\|\tilde{g}_t\|^2] &= \mathbb{E}[\|\mathcal{C}(g_t + e_t) - (g_t + e_t) + g_t + e_t - \nabla f(\theta_t) + \nabla f(\theta_t)\|^2] \\ &\leq \sigma^2 + 3\mathbb{E}[q^2\|g_t + e_t - \nabla f(\theta_t) + \nabla f(\theta_t)\|^2 + \|e_t\|^2 + \|\nabla f(\theta_t)\|^2] \\ &\leq (3q^2 + 1)\sigma^2 + (6q^2 + 3)\mathbb{E}[\|e_t\|^2 + \|\nabla f(\theta_t)\|^2] \\ &\leq (3q^2 + \frac{4q^2(6q^2 + 3)}{(1 - q^2)^2} + 1)\sigma^2 + (6q^2 + 3)\mathbb{E}[\|\nabla f(\theta_t)\|^2],\end{aligned}$$

391 where the first inequality is because of Assumption 1 and that the stochastic error $(g_t - \nabla f(\theta_t))$
392 is mean-zero and independent of other terms. The bound on $\|e_t\|^2$ in the last inequality is due to
393 Lemma 3 of [31]. Then by similar induction we can obtain

$$\mathbb{E}[\|m_t\|^2] \leq (3q^2 + \frac{4q^2(6q^2 + 3)}{(1 - q^2)^2} + 1)C\sigma^2 + (6q^2 + 3)C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2].$$

394 **Lemma 2.** *Suppose $\gamma = \beta_1/\beta_2 < 1$. Then, for $\forall t$,*

$$\|a_t\|^2 := \|\frac{m_t}{\sqrt{\hat{v}_t} + \epsilon}\|^2 \leq \frac{(1 - \beta_1)d}{(1 - \beta_2)(1 - \gamma)}.$$

395 *Proof.* We have

$$\begin{aligned}
\left\| \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}} \right\|^2 &= \sum_{i=1}^d \frac{m_{t,i}^2}{\hat{v}_{t,i} + \epsilon} \\
&\leq \frac{(1 - \beta_1)^2}{1 - \beta_2} \sum_{i=1}^d \frac{(\sum_{\tau=1}^t \beta_1^{t-\tau} \tilde{g}_{\tau,i})^2}{\sum_{\tau=1}^t \beta_2^{t-\tau} \tilde{g}_{\tau,i}^2} \\
&\stackrel{(a)}{\leq} \frac{(1 - \beta_1)^2}{1 - \beta_2} \sum_{i=1}^d \frac{(\sum_{\tau=1}^t \beta_1^{t-\tau})(\sum_{\tau=1}^t \beta_1^{t-\tau} \tilde{g}_{\tau,i}^2)}{\sum_{\tau=1}^t \beta_2^{t-\tau} \tilde{g}_{\tau,i}^2} \\
&\leq \frac{1 - \beta_1}{1 - \beta_2} \sum_{i=1}^d \frac{\sum_{\tau=1}^t \beta_1^{t-\tau} \tilde{g}_{\tau,i}^2}{\sum_{\tau=1}^t \beta_2^{t-\tau} \tilde{g}_{\tau,i}^2} \\
&\leq \frac{(1 - \beta_1)d}{1 - \beta_2} \sum_{\tau=1}^t \gamma^\tau \\
&\leq \frac{(1 - \beta_1)d}{(1 - \beta_2)(1 - \gamma)},
\end{aligned}$$

396 where (a) is a consequence of Cauchy-Schwartz inequality. \square

397 **Lemma 3.** *Define*

$$\begin{aligned}
H_t &:= \mathbb{E} \left[\sum_{i=1}^d \left| \frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}} \right| \right] \\
S_t &:= \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2]
\end{aligned}$$

398 *then the following inequalities hold:*

$$\begin{aligned}
\sum_{t=2}^T \sum_{\tau=0}^{t-2} \beta_1^\tau S_{t-\tau} &\leq \frac{1}{(1 - \beta_1)(1 - \beta_1^2(2 - \beta_1^2))} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
\sum_{t=2}^T \sum_{\tau=0}^{t-2} \beta_1^\tau H_{t-\tau} &\leq \frac{d}{(1 - \beta)\sqrt{\epsilon}}.
\end{aligned}$$

399 *Proof.* By arranging terms, it holds that

$$\begin{aligned}
\sum_{t=2}^T \sum_{\tau=0}^{t-2} \beta_1^\tau S_{t-\tau} &\leq \sum_{t=2}^T \left(\sum_{\tau=0}^{T-t} \beta_1^{T-t-\tau} \right) S_t \\
&\leq \frac{1}{1 - \beta_1} \sum_{t=2}^T \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2] \\
&\leq \frac{1}{1 - \beta_1} \sum_{t=1}^T \left(\sum_{\tau=0}^{T-t-1} (\beta_1^2(2 - \beta_1^2))^{T-t-\tau} \right) \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
&\leq \frac{1}{(1 - \beta_1)(1 - \beta_1^2(2 - \beta_1^2))} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2].
\end{aligned}$$

400 Using similar strategy, we can write

$$\begin{aligned}
\sum_{t=2}^T \sum_{\tau=0}^{t-2} \beta_1^\tau H_{t-\tau} &\leq \sum_{t=2}^T \left(\sum_{\tau=0}^{T-t} \beta_1^{T-t-\tau} \right) H_t \\
&\leq \frac{1}{1-\beta} \sum_{t=2}^T \mathbb{E} \left[\sum_{i=1}^d \left| \frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}} \right| \right] \\
&\leq \frac{d}{(1-\beta)\sqrt{\epsilon}},
\end{aligned}$$

401 where the last inequality is derived by cancelling terms due to the fact that $\{\hat{v}_t\}_{t \geq 0}$ is a non-
402 decreasing sequence, hence $\hat{v}_t \leq \hat{v}_{t-1}$. This completes the proof of the lemma. \square

403 **Lemma 4.** For the error sequence e_t in SPARS-AMS, under Assumption 4, we have for $\forall t$,

$$\mathbb{E}[\|e_{t+1}\|^2] \leq \frac{4q^2}{(1-q^2)^2} \sigma^2 + \frac{2q^2}{1-q^2} \sum_{\tau=1}^t \left(\frac{1+q^2}{2} \right)^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2].$$

404 *Proof.* We start by using Assumption 1 and Young's inequality to get

$$\begin{aligned}
\|e_{t+1}\|^2 &= \|g_t + e_t - \mathcal{C}(g_t + e_t)\|^2 \\
&\leq q^2 \|g_t + e_t\|^2 \\
&\leq q^2(1+\rho) \|e_t\|^2 + q^2(1+\frac{1}{\rho}) \|g_t\|^2 \\
&\leq \frac{1+q^2}{2} \|e_t\|^2 + \frac{2q^2}{1-q^2} \|g_t\|^2,
\end{aligned}$$

405 by choosing $\rho = \frac{1-q^2}{2q^2}$. Now by recursion and the initialization $e_1 = 0$, we have

$$\begin{aligned}
\mathbb{E}[\|e_{t+1}\|^2] &\leq \frac{2q^2}{1-q^2} \sum_{\tau=1}^t \left(\frac{1+q^2}{2} \right)^{t-\tau} \mathbb{E}[\|g_\tau\|^2] \\
&\leq \frac{4q^2}{(1-q^2)^2} \sigma^2 + \frac{2q^2}{1-q^2} \sum_{\tau=1}^t \left(\frac{1+q^2}{2} \right)^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2],
\end{aligned}$$

406 which proves the lemma. \square

407 **Lemma 5.** For the moving average error sequence \mathcal{E}_t , it holds that

$$\sum_{t=1}^T \mathbb{E}[\|\mathcal{E}_t\|^2] \leq \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{4q^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2].$$

408 *Proof.* Denote $K_t := \sum_{\tau=1}^t \left(\frac{1+q^2}{2} \right)^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2]$ and $K_0 = 0$. We have

$$\begin{aligned}
\mathbb{E}[\|\mathcal{E}_t\|^2] &= \mathbb{E} \left[\left\| \frac{(1-\beta_1) \sum_{\tau=1}^t \beta_1^{t-\tau} e_\tau}{\sqrt{\hat{v}_t + \epsilon}} \right\|^2 \right] \\
&\leq \frac{(1-\beta_1)^2}{\epsilon} \sum_{i=1}^d \mathbb{E} \left[\left(\sum_{\tau=1}^t \beta_1^{t-\tau} e_{\tau,i} \right)^2 \right] \\
&\stackrel{(a)}{\leq} \frac{(1-\beta_1)^2}{\epsilon} \sum_{i=1}^d \mathbb{E} \left[\left(\sum_{\tau=1}^t \beta_1^{t-\tau} \right) \left(\sum_{\tau=1}^t \beta_1^{t-\tau} e_{\tau,i}^2 \right) \right] \\
&\leq \frac{1-\beta_1}{\epsilon} \sum_{\tau=1}^t \beta_1^{t-\tau} \mathbb{E}[\|e_\tau\|^2] \\
&\stackrel{(b)}{\leq} \frac{4q^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{2q^2(1-\beta_1)}{(1-q^2)\epsilon} \sum_{\tau=1}^t \beta_1^{t-\tau} K_\tau,
\end{aligned}$$

where (a) is due to Cauchy-Schwartz and (b) is a result of Lemma 4. Summing over $t = 1, \dots, T$ and using the similar technique as in Lemma 3 leads to

$$\begin{aligned} \sum_{t=1}^T \mathbb{E}[\|\mathcal{E}_t\|^2] &= \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{2q^2(1-\beta_1)}{(1-q^2)\epsilon} \sum_{t=1}^T \sum_{\tau=1}^t \beta_1^{t-\tau} K_\tau \\ &\leq \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{2q^2}{(1-q^2)\epsilon} \sum_{t=1}^T \sum_{\tau=1}^t \left(\frac{1+q^2}{2}\right)^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2] \\ &\leq \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{4q^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2], \end{aligned}$$

which gives the desired result. \square

A.2 Proof of Theorem 3

Theorem 3. Denote $C' = \frac{4\sqrt{(q^2+1)G^2+\epsilon}}{1-\beta_1}$, $C = \frac{(1-\beta_1)^2}{1-\beta_1^2(2-\beta_1)^2} (1 + \frac{1}{4(1-\beta_1^2)})$, and $\gamma = \beta_1/\beta_2 < 1$. Under Assumption 1 to Assumption 4, with $\eta_t = \eta \leq \min\{\frac{1-\beta_1}{C}, \frac{(1-q^2)^2}{2q^2}\} \frac{(1-\beta_1)\epsilon}{4L\sqrt{(q^2+1)G^2+\epsilon}}$, SPARS-AMS satisfies

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] &\leq C' \left(\frac{\mathbb{E}[f(\theta_1) - f(\theta^*)]}{T\eta} + \frac{2dG^2}{T(1-\beta_1)\sqrt{\epsilon}} + \frac{\eta LC\sigma^2}{(1-\beta_1)\epsilon} \right. \\ &\quad \left. + \frac{\eta L\beta_1 d}{(1-\beta_2)(1-\gamma)} + \frac{2\eta Lq^2\sigma^2}{(1-q^2)^2\epsilon} \right). \end{aligned}$$

Proof. Let m'_t be the first moment moving average of standard AMSGrad using full gradients, i.e., the gradient with respect to the index data point i_t computed Line 4 of Algorithm 2 before applying any compression operator.

Denote

$$\begin{aligned} m_t &= \beta_1 m_{t-1} + (1-\beta_1) \tilde{g}_t \quad \text{and} \quad m'_t = \beta_1 m'_{t-1} + (1-\beta_1) g_t \\ a_t &= \frac{m_t}{\sqrt{\hat{v}_t + \epsilon}}, \quad \text{and} \quad a'_t = \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}}. \end{aligned}$$

By construction we have $m'_t = (1-\beta_1) \sum_{i=1}^k \beta_1^{t-i} g_t$.

Denote the following auxiliary sequences,

$$\begin{aligned} \mathcal{E}_{t+1} &:= \frac{(1-\beta_1) \sum_{\tau=1}^{t+1} \beta_1^{t+1-\tau} e_\tau}{\sqrt{\hat{v}_t + \epsilon}} \\ \theta'_{t+1} &:= \theta_{t+1} - \eta \mathcal{E}_{t+1}. \end{aligned}$$

Then,

$$\begin{aligned} \theta'_{t+1} &= \theta_{t+1} - \eta \mathcal{E}_{t+1} \\ &= \theta_t - \eta \frac{(1-\beta_1) \sum_{\tau=1}^t \beta_1^{t-\tau} \tilde{g}_\tau + (1-\beta_1) \sum_{\tau=1}^{t+1} \beta_1^{t+1-\tau} e_\tau}{\sqrt{\hat{v}_t + \epsilon}} \\ &= \theta_t - \eta \frac{(1-\beta_1) \sum_{\tau=1}^t \beta_1^{t-\tau} (\tilde{g}_\tau + e_{\tau+1}) + (1-\beta) \beta_1^t e_1}{\sqrt{\hat{v}_t + \epsilon}} \\ &= \theta_t - \eta \frac{(1-\beta_1) \sum_{\tau=1}^t \beta_1^{t-\tau} e_\tau}{\sqrt{\hat{v}_t + \epsilon}} - \eta \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} \\ &\stackrel{(a)}{=} \theta'_t - \eta \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} := \theta'_t - \eta a'_t, \end{aligned}$$

424 where (a) uses the fact that $\tilde{g}_t + e_{t+1} = g_t + e_t$, $e_1 = 0$ at initialization. By Assumption 2 we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \eta \langle \nabla f(\theta'_t), a'_t \rangle + \frac{L}{2} \|\theta'_{t+1} - \theta'_t\|^2.$$

425 Thus,

$$\begin{aligned} \mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq -\eta \mathbb{E}[\langle \nabla f(\theta'_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] \\ &= -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] + \eta \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), a'_t \rangle] \\ &\leq -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] + \eta^2 L \mathbb{E}[\|\mathcal{E}_t\| \|a'_t\|] \\ &\leq -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \eta^2 L \mathbb{E}[\|a'_t\|^2] + \frac{\eta^2 L}{2} \mathbb{E}[\|\mathcal{E}_t\|^2]. \end{aligned} \quad (8)$$

426 **Bounding the first term in (61).** We have

$$\begin{aligned} M_t &:= -\mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] = -\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_t}{\sqrt{\hat{v}_t} + \epsilon} \rangle] \\ &= \underbrace{-\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_t}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle]}_I + \underbrace{\mathbb{E}[\langle \nabla f(\theta_t), (\frac{1}{\sqrt{\hat{v}_{t-1}} + \epsilon} - \frac{1}{\sqrt{\hat{v}_t} + \epsilon}) m'_t \rangle]}_{II}. \end{aligned}$$

427 To bound I, note that

$$\begin{aligned} I &= -\mathbb{E}[\langle \nabla f(\theta_t), \frac{(1 - \beta_1)g_t}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle] \\ &= -\mathbb{E}[\langle \nabla f(\theta_t), \frac{(1 - \beta_1)g_t}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle | \mathcal{F}_{t-1}] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle] \\ &= -(1 - \beta_1) \mathbb{E}[\frac{\|\nabla f(\theta_t)\|^2}{\sqrt{\hat{v}_{t-1}} + \epsilon}] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle] \\ &\leq -\frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] - \beta_1 \mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle]. \end{aligned}$$

428 Regarding the second term, we have

$$\begin{aligned} &-\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle] \\ &= -\mathbb{E}[\langle \nabla f(\theta_{t-1}), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon} \rangle] \\ &= M_{t-1} + \eta L \mathbb{E}[\|\frac{m_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon}\| \|\frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1}} + \epsilon}\|] \\ &\leq M_{t-1} + \frac{\eta L}{\epsilon} \mathbb{E}[\|m'_{t-1}\|^2] + \eta L \mathbb{E}[\|a_{t-1}\|^2] \end{aligned} \quad (9)$$

$$\leq M_{t-1} + \frac{\eta L}{\epsilon} (C\sigma^2 + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2]) + \frac{\eta L(1 - \beta_1)d}{(1 - \beta_2)(1 - \gamma)}, \quad (10)$$

429 where Lemma 1 and Lemma 2 are used, with $C_1 = (1 - \beta_1^2)(1 + \frac{1}{4(1 - \beta_1^2)})$ and $C = \frac{C_1}{1 - \beta_1^2(2 - \beta_1^2)}$.

430 Putting parts together we obtain

$$\begin{aligned} I &\leq \beta_1 M_{t-1} + \frac{\eta \beta_1 L C \sigma^2}{\epsilon} + \frac{\eta \beta_1 L C_1}{\epsilon} \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2] \\ &\quad + \frac{\eta L \beta_1(1 - \beta_1)d}{(1 - \beta_2)(1 - \gamma)} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2]. \end{aligned}$$

431 For II, it holds that

$$II \leq G^2 \mathbb{E} \left[\sum_{i=1}^d \left| \frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}} \right| \right].$$

432 Denoting $H_t := \mathbb{E}[\sum_{i=1}^d \left| \frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}} \right|]$, $S_t := \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2]$. We
433 arrive at

$$\begin{aligned} M_t &\leq \beta_1 M_{t-1} + \frac{\eta \beta_1 LC \sigma^2}{\epsilon} + \frac{\eta \beta_1 LC_1}{\epsilon} S_t + G^2 H_t \\ &\quad + \frac{\eta L \beta_1 (1 - \beta_1) d}{(1 - \beta_2)(1 - \gamma)} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\leq \beta_1 M_{t-1} + \frac{\eta \beta_1 LC \sigma^2}{\epsilon} + \frac{\eta \beta_1 LC_1}{\epsilon} S_t + G^2 H_t + \frac{\eta L \beta_1 (1 - \beta_1) d}{(1 - \beta_2)(1 - \gamma)}. \end{aligned}$$

434 By induction, we have

$$\begin{aligned} M_t &\leq \beta_1^{t-1} M_1 + G^2 \sum_{\tau=0}^{t-2} \beta_1^\tau H_{t-\tau} + \frac{\eta \beta_1 LC_1}{\epsilon} \sum_{\tau=0}^{t-2} \beta_1^\tau S_{t-\tau} + \frac{\eta \beta_1 LC \sigma^2}{(1 - \beta_1) \epsilon} \\ &\quad + \frac{\eta L \beta_1 d}{(1 - \beta_2)(1 - \gamma)} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2], \end{aligned}$$

435 since $\beta_1 < 1$. Summing over $t = 1, \dots, T$, we obtain

$$\begin{aligned} \sum_{t=1}^T M_t &\leq \sum_{t=1}^T \beta_1^{t-1} M_1 + G^2 \sum_{t=2}^T \sum_{\tau=0}^{t-2} \beta_1^\tau H_{t-\tau} + \frac{\eta \beta_1 LC_1}{\epsilon} \sum_{t=2}^T \sum_{\tau=0}^{t-2} \beta_1^\tau S_{t-\tau} \\ &\quad + \frac{T \eta \beta_1 LC \sigma^2}{(1 - \beta_1) \epsilon} + \frac{T \eta L \beta_1 d}{(1 - \beta_2)(1 - \gamma)} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\stackrel{(a)}{\leq} \frac{2dG^2}{(1 - \beta_1)\sqrt{\epsilon}} + \frac{T \eta \beta_1 LC \sigma^2}{(1 - \beta_1) \epsilon} + \frac{T \eta L \beta_1 d}{(1 - \beta_2)(1 - \gamma)} \\ &\quad + \left[\frac{\eta LC}{(1 - \beta_1) \epsilon} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \right] \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\leq \frac{2dG^2}{(1 - \beta_1)\sqrt{\epsilon}} + \frac{T \eta \beta_1 LC \sigma^2}{(1 - \beta_1) \epsilon} + \frac{T \eta L \beta_1 d}{(1 - \beta_2)(1 - \gamma)} - \frac{3(1 - \beta_1)}{4\sqrt{(q^2 + 1)G^2 + \epsilon}} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2], \end{aligned}$$

436 when η is chosen to be $\eta \leq \frac{(1 - \beta_1)^2 \epsilon}{4LC\sqrt{(q^2 + 1)G^2 + \epsilon}}$. Here, (a) is due to $M_1 = \mathbb{E}[\langle \nabla f(\theta_1), a'_0 \rangle] \leq$
437 $\beta_1 d G^2 / \sqrt{\epsilon}$ and Lemma 3. It remains to bound the last two terms in (61).

438 **Bounding the last two terms in in (61).** We have

$$\mathbb{E}[\|a'_t\|^2] = \mathbb{E}[\|\frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}}\|^2] \leq \frac{1}{\epsilon} \mathbb{E}[\|m'_t\|^2].$$

439 By Lemma 1, it follows that

$$\mathbb{E}[\|a'_t\|^2] \leq \frac{1}{\epsilon} (C \sigma^2 + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2]).$$

440 Summing over $t = 1, \dots, T$, we obtain

$$\sum_{t=1}^T \|a'_t\|^2 \leq \frac{TC \sigma^2}{\epsilon} + \frac{C}{\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2],$$

441 where the last inequality can be derived similar to Lemma 3.

442 For the last term in (61), we have by Lemma 5

$$\sum_{t=1}^T \mathbb{E}[\|\mathcal{E}_t\|^2] \leq \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{4q^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2].$$

443 **Completing the proof.** Summing (61) over $t = 1, \dots, T$ and integrating things together, we have

$$\begin{aligned} & \mathbb{E}[f(\theta'_{T+1}) - f(\theta'_1)] \\ & \leq \eta \sum_{t=1}^T M_t + \frac{T\eta^2 CL\sigma^2}{\epsilon} + \frac{C\eta^2 L}{\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ & \quad + \frac{2T\eta^2 Lq^2\sigma^2}{(1-q^2)^2\epsilon} + \frac{2\eta^2 Lq^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ & \leq \frac{2\eta dG^2}{(1-\beta_1)\sqrt{\epsilon}} + \frac{T\eta^2\beta_1 LC\sigma^2}{(1-\beta_1)\epsilon} + \frac{T\eta^2 L\beta_1 d}{(1-\beta_2)(1-\gamma)} - \frac{3\eta(1-\beta_1)}{4\sqrt{(q^2+1)G^2+\epsilon}} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ & \quad + \frac{T\eta^2 CL\sigma^2}{\epsilon} + \left[\frac{C\eta^2 L}{\epsilon} + \frac{2\eta^2 Lq^2}{(1-q^2)^2\epsilon} \right] \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{2T\eta^2 Lq^2\sigma^2}{(1-q^2)^2\epsilon} \\ & \leq -\frac{\eta(1-\beta_1)}{4\sqrt{(q^2+1)G^2+\epsilon}} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{2\eta dG^2}{(1-\beta_1)\sqrt{\epsilon}} + \frac{T\eta^2 LC\sigma^2}{(1-\beta_1)\epsilon} \\ & \quad + \frac{T\eta^2 L\beta_1 d}{(1-\beta_2)(1-\gamma)} + \frac{2T\eta^2 Lq^2\sigma^2}{(1-q^2)^2\epsilon}, \end{aligned}$$

444 when $\eta \leq \frac{(1-q^2)^2(1-\beta_1)\epsilon}{8Lq^2\sqrt{(q^2+1)G^2+\epsilon}}$, where the last line is because $C\eta L \leq \frac{(1-\beta_1)\epsilon}{4\sqrt{(q^2+1)G^2+\epsilon}}$ also holds.

445 Re-arranging terms, we get that when $\eta \leq \min\{\frac{1-\beta_1}{C}, \frac{(1-q^2)^2}{2q^2}\} \frac{(1-\beta_1)\epsilon}{4L\sqrt{(q^2+1)G^2+\epsilon}}$,

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] & \leq C' \left(\frac{\mathbb{E}[f(\theta'_1) - f(\theta'_{T+1})]}{T\eta} + \frac{2dG^2}{T(1-\beta_1)\sqrt{\epsilon}} + \frac{\eta LC\sigma^2}{(1-\beta_1)\epsilon} \right. \\ & \quad \left. + \frac{\eta L\beta_1 d}{(1-\beta_2)(1-\gamma)} + \frac{2\eta Lq^2\sigma^2}{(1-q^2)^2\epsilon} \right) \\ & \leq C' \left(\frac{\mathbb{E}[f(\theta_1) - f(\theta^*)]}{T\eta} + \frac{2dG^2}{T(1-\beta_1)\sqrt{\epsilon}} + \frac{\eta LC\sigma^2}{(1-\beta_1)\epsilon} \right. \\ & \quad \left. + \frac{\eta L\beta_1 d}{(1-\beta_2)(1-\gamma)} + \frac{2\eta Lq^2\sigma^2}{(1-q^2)^2\epsilon} \right). \end{aligned}$$

446 where $C' = \frac{4\sqrt{(q^2+1)G^2+\epsilon}}{1-\beta_1}$, and $C = \frac{(1-\beta_1)^2}{1-\beta_1^2(2-\beta_1)^2} (1 + \frac{1}{4(1-\beta_1^2)})$. The last inequality is because
447 $\theta'_1 = \theta_1$, and $\theta^* = \arg \min_{\theta} f(\theta)$. The proof is complete.

448 □

449 **Corollary 2.** Under the setting in Theorem 3, if the learning rate is chosen to be $\eta \leq$
450 $\min\{\min\{\frac{1-\beta_1}{C}, \frac{(1-q^2)^2}{2q^2}\} \frac{(1-\beta_1)\epsilon}{4L\sqrt{(q^2+1)G^2+\epsilon}}, \frac{1}{\sqrt{T}}\}$, then the convergence rate of SPARS-AMS admits

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \mathcal{O}\left(\frac{1}{\sqrt{T}} + \frac{1}{T}\right).$$

B Distributed setting Belhal

B.1 Intermediary Lemmas

Lemma 6. Under Assumption 3 and Assumption 4 we have for any iteration $t > 0$:

$$\mathbb{E}[\|m_t\|^2] \leq (q^2 + 1)\sigma^2 \quad \text{and} \quad \mathbb{E}[\hat{v}_t] \leq (q^2 + 1)\sigma^2 \quad (11)$$

where m_t and $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ are defined Line 15 of Algorithm 1 and $\sigma^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2$.

Proof. We start by writing

$$\|\bar{g}_t\|^2 = \left\| \frac{1}{n} \sum_{i=1}^n \tilde{g}_{t,i} \right\|^2 \leq \frac{1}{n} \sum_{i=1}^n \|\tilde{g}_{t,i}\|^2 \quad (12)$$

Though, using Assumption 3 and Assumption 4 we have:

$$\mathbb{E}[\|\tilde{g}_{t,i}\|^2] = \mathbb{E}[\|g_{t,i} + \tilde{g}_{t,i} - g_{t,i}\|^2] \leq \mathbb{E}[\|g_{t,i}\|^2] + \mathbb{E}[\|\tilde{g}_{t,i} - g_{t,i}\|^2] \leq (q^2 + 1)\sigma_i^2 \quad (13)$$

Hence

$$\mathbb{E}[\|\bar{g}_t\|^2] \leq (q^2 + 1)\sigma^2 \quad (14)$$

where $\sigma^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2$. Then, by construction in Algorithm 1:

$$\mathbb{E}[\|m_t\|^2] \leq \beta_1^2 \mathbb{E}[\|m_{t-1}\|^2] + (1 - \beta_1)^2 \mathbb{E}[\|\bar{g}_t\|^2] \leq \beta_1^2 \mathbb{E}[\|m_{t-1}\|^2] + (1 - \beta_1)^2 (q^2 + 1)\sigma^2 \quad (15)$$

Since we have by initialization that $\|m_0\|^2 \leq \sigma^2$, then we prove by induction that $\mathbb{E}[\|m_t\|^2] \leq (q^2 + 1)\sigma^2$.

Similarly

$$\mathbb{E}[\hat{v}_t] = \mathbb{E}[\max(v_t, \hat{v}_{t-1})] = \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1 - \beta_2) \mathbb{E}[\bar{g}_t^2]) \leq \max(\hat{v}_{t-1}, \beta_2 v_{t-1} + (1 - \beta_2)(q^2 + 1)\sigma^2) \quad (16)$$

□

Lemma 7. Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$, we have:

$$-\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle] \leq -\frac{\eta_{t+1}}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{\sigma^2 \eta_{t+1}}{\epsilon 2n^2} \quad (17)$$

where \mathbf{I}_d is the identity matrix, \hat{V}_t the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of Algorithm 1 and \bar{g}_t is the aggregation of all **quantized** gradients from the workers.

Proof. We first decompose \bar{g}_t as the sum of the unbiased stochastic gradients and its quantized versions as computed Line 7 of Algorithm 1:

$$\bar{g}_t = \frac{1}{n} \sum_{i=1}^n \tilde{g}_{t,i} = \frac{1}{n} \sum_{i=1}^n [g_{t,i} + \tilde{g}_{t,i} - g_{t,i}] \quad (18)$$

Hence,

$$\begin{aligned} T_1 &:= -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle] \\ &= \underbrace{-\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^n g_{t,i} \rangle]}_{t_1} - \underbrace{\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^n \tilde{g}_{t,i} - g_{t,i} \rangle]}_{t_2} \end{aligned} \quad (19)$$

470 **Bounding t_1 :** Using the Tower rule, we have:

$$\begin{aligned}
t_1 &:= -\eta_{t+1} \mathbb{E} \left[\left\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^n g_{t,i} \right\rangle \right] \\
&= -\eta_{t+1} \mathbb{E} [\mathbb{E} \left[\left\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^n g_{t,i} \right\rangle \mid \mathcal{F}_t \right]] \\
&= -\eta_{t+1} \mathbb{E} \left[\left\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n g_{t,i} \mid \mathcal{F}_t \right] \right\rangle \right]
\end{aligned} \tag{20}$$

471 Using Assumption 3 and Lemma 6, we have that

$$\begin{aligned}
t_1 &:= -\eta_{t+1} \mathbb{E} \left[\left\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \frac{1}{n} \sum_{i=1}^n g_{t,i} \right\rangle \right] \\
&\leq -\eta_{t+1} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E} [\|\nabla f(\theta_t)\|^2]
\end{aligned} \tag{21}$$

472 **Bounding t_2 :**

473 We first recall Young's inequality with a constant $\delta \in (0, 1)$ as follows:

$$\langle X \mid Y \rangle \leq \frac{1}{\delta} \|X\|^2 + \delta \|Y\|^2. \tag{22}$$

474 Using Young's inequality (22) with parameter equal to 1:

$$\begin{aligned}
t_2 &\leq \frac{\eta_{t+1}}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E} [\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2} \mathbb{E} [\|(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \sum_{i=1}^n \{\tilde{g}_{t,i} - g_{t,i}\}\|^2] \\
&\stackrel{(a)}{\leq} \frac{\eta_{t+1}}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E} [\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2} \mathbb{E} [\|(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2}\|^2 \sum_{i=1}^n \|\tilde{g}_{t,i} - g_{t,i}\|^2] \\
&\stackrel{(b)}{\leq} \frac{\eta_{t+1}}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E} [\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{2n^2} \mathbb{E} [\|(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2}\|^2] \mathbb{E} [\sum_{i=1}^n \|\tilde{g}_{t,i} - g_{t,i}\|^2] \\
&\stackrel{(c)}{\leq} \frac{\eta_{t+1}}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E} [\|\nabla f(\theta_t)\|^2] + \frac{\eta_{t+1}}{\epsilon 2n^2} \mathbb{E} [\sum_{i=1}^n \|\tilde{g}_{t,i} - g_{t,i}\|^2] \\
&\stackrel{(d)}{\leq} \frac{\eta_{t+1}}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E} [\|\nabla f(\theta_t)\|^2] + q^2 \frac{\sigma^2 \eta_{t+1}}{\epsilon 2n^2}
\end{aligned} \tag{23}$$

475 where (a) uses the Cauchy-Schwartz inequality, (b) is due to the non-negativeness of both \hat{V}_{t+1}
476 and $\|\sum_{i=1}^n \{\tilde{g}_{t,i} - g_{t,i}\}\|^2$ and (c) uses the Triangle inequality. We use Assumption 1 and
477 Assumption 4 in (d).

478 Finally, combining (21) and (23) yields

$$-\eta_{t+1} \mathbb{E} \left[\left\langle \nabla f(\theta_t) \mid (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \right\rangle \right] \leq -\frac{\eta_{t+1}}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E} [\|\nabla f(\theta_t)\|^2] + q^2 \frac{\sigma^2 \eta_{t+1}}{\epsilon 2n^2} \tag{24}$$

479 \square

480 **Lemma 8.** Under Assumption 2 to Assumption 4, with a decreasing sequence of stepsize $\{\eta_t\}_{t>0}$,
 481 we have:

$$\begin{aligned} \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)\sigma^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2\frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\ &\quad - \eta_{t+1}\beta_1\mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\ &\quad + \left(\frac{L}{2} + \beta_1 L\right) \|\theta_t - \theta_{t-1}\|^2 \\ &\quad + \eta_{t+1}G^2\mathbb{E}\left[\sum_{j=1}^d \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]\right] \end{aligned} \quad (25)$$

482 where d denotes the dimension of the parameter vector

483 *Proof.* By assumption Assumption 2, we can write the smoothness condition on the overall objective
 484 (2), between iteration t and $t+1$:

$$f(\theta_{t+1}) \leq f(\theta_t) + \langle \nabla f(\theta_t) | \theta_{t+1} - \theta_t \rangle + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \quad (26)$$

485 Denote by \hat{V}_t the diagonal matrix which diagonal entries are $\hat{v}_t = \max(v_t, \hat{v}_{t-1})$ defined Line 15 of
 486 Algorithm 1. Hence, we obtain,

$$f(\theta_{t+1}) \leq f(\theta_t) - \eta_{t+1} \langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \quad (27)$$

487 where \mathbf{I}_d denotes the identity matrix.

488 We now take the expectation of those various terms conditioned on the filtration \mathcal{F}_t of the total
 489 randomness up to iteration t .

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \quad (28)$$

490 We now focus on the computation of the inner product obtained in the equation above. We have

$$\begin{aligned} &\eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] \\ &= \eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} + (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] \\ &= \eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] + \eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) | [(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2}] m_{t+1} \rangle] \\ &= \eta_{t+1}\beta_1\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] + \eta_{t+1}(1-\beta_1)\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle] \\ &\quad + \eta_{t+1}\mathbb{E}[\langle \nabla f(\theta_t) | [(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2}] m_{t+1} \rangle] \end{aligned} \quad (30)$$

491 where \bar{g}_t is the aggregated gradients from all workers.

492 Plugging the above in (28) yields:

$$\begin{aligned} &\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \\ &\leq \underbrace{-\beta_1\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle]}_{A_t} \eta_{t+1} \\ &\quad \underbrace{-\mathbb{E}[\langle \nabla f(\theta_t) | [(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2}] m_{t+1} \rangle]}_{B_t} \eta_{t+1} \\ &\quad \underbrace{-(1-\beta_1)\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle]}_{C_t} \eta_{t+1} + \frac{L}{2}\mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \end{aligned} \quad (31)$$

493 To begin with, by the tower rule, we have that

$$A_t = -\beta_1 \mathbb{E}[\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle | \mathcal{F}_t]] \quad (32)$$

$$= -\beta_1 \langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle - \beta_1 \langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle \quad (33)$$

$$(34)$$

where we recognize the first term as the term in (29), at iteration $t - 1$ and hence apply the same decomposition as in (30). Coupling with the smoothness of f , which gives that

$$-\beta_1 \langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle \leq \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2$$

494 we obtain,

$$\begin{aligned} A_t &= -\beta_1 \mathbb{E}[\mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle | \mathcal{F}_t]] \\ &\leq \eta_{t+1} \beta_1 (A_{t-1} + B_{t-1} + C_{t-1}) + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \|\theta_t - \theta_{t-1}\|^2 \end{aligned} \quad (35)$$

495 Then,

$$\begin{aligned} B_t &= -\mathbb{E}[\langle \nabla f(\theta_t) | [(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} - (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2}] m_{t+1} \rangle] \\ &= \mathbb{E}[\sum_{j=1}^d \nabla^j f(\theta_t) m_{t+1}^j [(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2}]] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\|\nabla f(\theta_t)\| \|m_{t+1}\| \sum_{j=1}^d [(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2}]] \\ &\stackrel{(b)}{\leq} G^2 \mathbb{E}[\sum_{j=1}^d [(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2}]] \end{aligned} \quad (36)$$

496 where $\nabla^j f(\theta_t)$ denotes the j -th component of the gradient vector $\nabla f(\theta_t)$, (a) uses of the Cauchy-
497 Schwartz inequality and (b) boils down from the norm of the gradient vector boundedness assump-
498 tion 3, denoting $G^2 := \frac{1}{n} \sum_{i=1}^n G_i^2$.

499 Plugging the above into (31) yields

$$\begin{aligned} \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq \eta_{t+1} (A_t + B_t + C_t) + \frac{L}{2} \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \\ &\leq -\eta_{t+1} \beta_1 \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\ &\quad + \eta_{t+1} G^2 \mathbb{E}[\sum_{j=1}^d [(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}]] \\ &\quad + \left(\frac{L}{2} + \eta_{t+1} \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_t - \theta_{t-1}\|^2 \\ &\quad - \eta_{t+1} (1 - \beta_1) \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle] \end{aligned} \quad (37)$$

500 We bound the last term on the RHS, $-\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} \bar{g}_t \rangle]$ with Lemma 7

501 Under the assumption that we use a decreasing stepsize such that $\eta_{t+1} \leq \eta_t$, and given that according
 502 to Line 15 we have that $\hat{v}_{t+1} \geq \hat{v}_t$ by construction, we obtain

$$\begin{aligned} \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\ &\quad - \eta_{t+1}\beta_1\mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\ &\quad + \left(\frac{L}{2} + \beta_1 L\right) \|\theta_t - \theta_{t-1}\|^2 \\ &\quad + \eta_{t+1}G^2\mathbb{E}\left[\sum_{j=1}^d \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]\right] \end{aligned} \quad (38)$$

503 Finally, using Lemma 7, we obtain the desired result. \square

504 B.2 Proof of Theorem 1

505 **Theorem.** Under Assumption 2 to Assumption 4, with a constant stepsize $\eta_t = \eta = \frac{L}{\sqrt{T}}$, we have:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \leq \frac{\mathbb{E}[f(\theta_1) - f(\theta_{T+1})]}{L\Delta_1\sqrt{T}} + d\frac{L\Delta_3}{\Delta_1\sqrt{T}} + \frac{\Delta_2}{\eta\Delta_1 T} + \frac{1-\beta_1}{\Delta_1}\epsilon^{-\frac{1}{2}}\sqrt{(q^2+1)}\sigma^2 \quad (39)$$

506 where

$$\begin{aligned} \Delta_1 &:= \frac{(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)\sigma^2}{1-\beta_2})^{-\frac{1}{2}} \quad , \quad \Delta_2 := q^2 + \sum_{k=t+1}^{\infty} \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2} \\ \Delta_3 &:= \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1}\right) (1-\beta_2)^{-1} (1 - \frac{\beta_1^2}{\beta_2})^{-1} \end{aligned} \quad (40)$$

507 *Proof.* By Lemma 8 we have

$$\begin{aligned} \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] &\leq -\frac{\eta_{t+1}(1-\beta_1)}{2}(\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}\mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \frac{G^2\eta_{t+1}}{\epsilon 2n^2} \\ &\quad - \eta_{t+1}\beta_1\mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\ &\quad + \left(\frac{L}{2} + \beta_1 L\right) \|\theta_t - \theta_{t-1}\|^2 \\ &\quad + \eta_{t+1}G^2\mathbb{E}\left[\sum_{j=1}^d \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2}\right]\right] \end{aligned} \quad (41)$$

508 Let us consider the following sequence, defined for all $t > 0$:

$$R_t := f(\theta_t) - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \quad (42)$$

509 We compute the following expectation:

$$\begin{aligned} \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] &= \mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] - \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] \\ &\quad + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \end{aligned} \quad (43)$$

510 Using the Assumption 2, we note that:

$$\mathbb{E}[f(\theta_{t+1}) - f(\theta_t)] \leq -\eta_{t+1} \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \quad (44)$$

511 which yields

$$\begin{aligned} \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] &= -(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \mathbb{E}[\langle \nabla f(\theta_t) | (\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-1/2} m_{t+1} \rangle] \\ &\quad + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] \\ &\quad + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \\ &\leq (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \mathbb{E}[A_t + B_t + C_t] \\ &\quad - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\ &\quad + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2 \end{aligned} \quad (45)$$

512 where A_t, B_t, C_t are defined in (31).

513 We use (35) and (36) to bound A_t and B_t , and Lemma 7 to bound C_t where we precise that the
514 learning rate η_{t+1} becomes $\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}$. Hence

$$\begin{aligned} \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] &\leq \left((\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \right) \mathbb{E}[A_{t-1} + B_{t-1} + C_{t-1}] \\ &\quad + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E} \left[\sum_{j=1}^d \left[(\hat{v}_{t+1}^j + \epsilon)^{-1/2} - (\hat{v}_t^j + \epsilon)^{-1/2} \right] \right] \\ &\quad + \left(\frac{L}{2} + (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{\beta_1 L}{\eta_{t-1}} \right) \|\theta_{t+1} - \theta_t\|^2 \\ &\quad - (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1 - \beta_1)}{2} \left(\epsilon + \frac{(q^2 + 1)\sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\quad + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2} \end{aligned} \quad (46)$$

515 where the last term in the LHS is due to Lemma 8.

516 By assumption, we have that for all $t > 0$, $\eta_{t+1} \leq \eta_t$. Also, set the tuning parameters such that

$$\eta_t + \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \leq \frac{\eta_t}{1 - \beta_1} \quad (47)$$

517 so that

$$\begin{aligned} &(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \beta_1 - \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} = 0 \\ \iff &(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \beta_1 = \sum_{k=t}^{\infty} \eta_k \beta_1^{k-t+1} \end{aligned} \quad (48)$$

518 Note that $-(\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \leq -\eta_{t+1} \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}}$
519 since $\sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \geq 0$.
520 The above coupled with (46) yields

$$\begin{aligned} \mathbb{E}[R_{t+1}] - \mathbb{E}[R_t] &\leq -\eta_{t+1} \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] + q^2 \eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2} \\ &\quad - (\eta_{t+1} + \sum_{k=t+1}^{\infty} \eta_k \beta_1^{k-t+2}) G^2 \mathbb{E}[\sum_{j=1}^d \left[(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \right]] \\ &\quad + \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1} \right) \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \end{aligned} \quad (49)$$

521 We now sum from $t = 1$ to $t = T$ the inequality in (49), and divide it by T :

$$\begin{aligned} &\eta \frac{(1-\beta_1)}{2} (\epsilon + \frac{(q^2+1)G^2}{1-\beta_2})^{-\frac{1}{2}} \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\leq \frac{\mathbb{E}[R_1] - \mathbb{E}[R_{T+1}]}{T} + \frac{q^2 \eta + \sum_{k=t+1}^{\infty} \eta \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}}{T} \\ &\quad + \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1} \right) \frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2] \end{aligned} \quad (50)$$

522 where we have used the fact that $(\hat{v}_t^j + \epsilon)^{-1/2} - (\hat{v}_{t+1}^j + \epsilon)^{-1/2} \geq 0$ for all dimension $j \in [d]$ by
523 construction of \hat{v}_{t+1}^j .

524 We now bound the two remaining terms:

525 **Bounding $-\mathbb{E}[R_{T+1}]$:**

526 By definition (42) of R_t we have, using Lemma 6:

$$\begin{aligned} -\mathbb{E}[R_{T+1}] &\leq \sum_{k=T+1}^{\infty} \eta_k \beta_1^{k-t+1} \mathbb{E}[\langle \nabla f(\theta_{t-1}) | (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \rangle] - f(\theta_{T+1}) \\ &\leq \left\| \sum_{k=T+1}^{\infty} \eta_k \beta_1^{k-t+1} \|\nabla f(\theta_{t-1})\| (\hat{V}_t + \epsilon \mathbf{I}_d)^{-1/2} m_t \right\| \\ &\leq \eta_{t+1} (1-\beta_1) \epsilon^{-\frac{1}{2}} \sqrt{(q^2+1)\sigma^2} - f(\theta_{T+1}) \end{aligned} \quad (51)$$

527 **Bounding $\sum_{t=1}^T \mathbb{E}[\|\theta_{t+1} - \theta_t\|^2]$:**

528 By definition in Algorithm 1:

$$\|\theta_{t+1} - \theta_t\|^2 = \eta_{t+1}^2 \left[(\hat{V}_{t+1} + \epsilon \mathbf{I}_d)^{-\frac{1}{2}} m_{t+1} \right]^2 = \eta_{t+1}^2 \sum_{j=1}^d \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j + \epsilon} \quad (52)$$

529 For any dimension $j \in [d]$,

$$\begin{aligned}
|m_{t+1}^j|^2 &= |\beta_1 m_t^j + (1 - \beta_1) \bar{g}_t^j|^2 \\
&\leq \beta_1 (\beta_1^2 |m_{t-1}^j|^2 + (1 - \beta_1)^2 |\bar{g}_{t-1}^j|^2) + |\bar{g}_t^j|^2 \\
&\leq \sum_{k=0}^t \beta_1^{2(t-k)} |\bar{g}_k^j|^2 \\
&\leq \sum_{k=0}^t \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}} \beta_2^{t-k} |\bar{g}_k^j|^2
\end{aligned} \tag{53}$$

530 Using Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
|m_{t+1}^j|^2 &\leq \sum_{k=0}^t \frac{\beta_1^{2(t-k)}}{\beta_2^{t-k}} \beta_2^{t-k} |\bar{g}_k^j|^2 \leq \sum_{k=0}^t \left(\frac{\beta_1^2}{\beta_2} \right)^{t-k} \sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2 \\
&\leq \frac{1}{1 - \frac{\beta_1^2}{\beta_2}} \sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2
\end{aligned} \tag{54}$$

531 On the other hand we have

$$\hat{v}_{t+1}^j \geq \beta_2 \hat{v}_t^j + (1 - \beta_2) (\bar{g}_t^j)^2 \tag{55}$$

532 and since it is also true for iteration $t = 1$, we have by induction replacing v_t^j in the above that

$$\hat{v}_{t+1}^j \geq (1 - \beta_2) \sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2 \iff \frac{\sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \leq (1 - \beta_2)^{-1} \tag{56}$$

533 Hence, we can derive from (52) that

$$\begin{aligned}
\|\theta_{t+1} - \theta_t\|^2 &= \eta_{t+1}^2 \sum_{j=1}^d \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j + \epsilon} \leq \eta_{t+1}^2 \sum_{j=1}^d \frac{|m_{t+1}^j|^2}{\hat{v}_{t+1}^j} \\
&\stackrel{(a)}{\leq} \eta_{t+1}^2 \sum_{j=1}^d \frac{1}{1 - \frac{\beta_1^2}{\beta_2}} \frac{\sum_{k=0}^t \beta_2^{t-k} |\bar{g}_k^j|^2}{\hat{v}_{t+1}^j} \\
&\stackrel{(b)}{\leq} \eta_{t+1}^2 d (1 - \beta_2)^{-1} \left(1 - \frac{\beta_1^2}{\beta_2}\right)^{-1}
\end{aligned} \tag{57}$$

534 where (a) uses (54) and (b) uses (56).

535 Plugging the two bounds in (50), we obtain the following bound:

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] &\leq \frac{\mathbb{E}[f(\theta_1) - f(\theta_{T+1})]}{\eta \Delta_1 T} + \frac{q^2 \eta + \sum_{k=t+1}^{\infty} \eta \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}}{\eta \Delta_1 T} \\
&\quad + \frac{1 - \beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1) \sigma^2} \\
&\quad + \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1 - \beta_1} \right) \frac{1}{\eta \Delta_1} \eta^2 d (1 - \beta_2)^{-1} \left(1 - \frac{\beta_1^2}{\beta_2}\right)^{-1}
\end{aligned} \tag{58}$$

536 where $\Delta_1 := \frac{(1 - \beta_1)}{2} \left(\epsilon + \frac{(q^2 + 1) \sigma^2}{1 - \beta_2} \right)^{-\frac{1}{2}}$

537 With a constant stepsize $\eta = \frac{L}{\sqrt{T}}$ we get the final convergence bound as follows:

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] &\leq \frac{\mathbb{E}[f(\theta_1) - f(\theta_{T+1})]}{L \Delta_1 \sqrt{T}} + d \frac{L \Delta_3}{\Delta_1 \sqrt{T}} \\
&\quad + \frac{\Delta_2}{\Delta_1 T} + \frac{1 - \beta_1}{\Delta_1} \epsilon^{-\frac{1}{2}} \sqrt{(q^2 + 1) \sigma^2}
\end{aligned} \tag{59}$$

538 where $\Delta_2 := q^2 + \sum_{k=t+1}^{\infty} \beta_1^{k-t+2} \frac{G^2}{\epsilon 2n^2}$ and $\Delta_3 := \left(\frac{L}{2} + 1 + \frac{\beta_1 L}{1-\beta_1} \right) (1-\beta_2)^{-1} (1 - \frac{\beta_1^2}{\beta_2})^{-1}$.

539

□

540 **C Distributed setting Xiaoyun**

541 **Assumption 5.** The true gradient deviation is bounded by $\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_t) - \nabla f(\theta_t)\|^2 \leq \sigma_g^2, \forall t$.

542 **Lemma 9.** For the distributed SPARS-AMS with n local workers, we have

$$\begin{aligned} \mathbb{E}[\|\bar{m}'_t\|^2] &\leq \frac{C\sigma^2}{n} + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2], \\ \mathbb{E}[\|\bar{m}_t\|^2] &\leq \frac{C\sigma^2}{n} + (3q^2 + \frac{4q^2(6q^2 + 3)}{(1 - q^2)^2})C\sigma^2 + (6q^2 + 3)C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2], \end{aligned}$$

543 where $C_1 = (1 - \beta_1^2)(1 + \frac{1}{4(1 - \beta_1^2)})$ and $C = \frac{C_1}{1 - \beta_1^2(2 - \beta_1^2)}$.

544 *Proof.* First we investigate the variance of average gradients. It holds that

$$\begin{aligned} \mathbb{E}[\|\bar{g}_t\|^2] &= \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n g_{t,i}\right\|^2\right] \\ &= \frac{1}{n^2} \mathbb{E}\left[\left\|\sum_{i=1}^n (g_{t,i} - \nabla f_i(\theta_t) + \nabla f_i(\theta_t))\right\|^2\right] \\ &\leq \frac{\sigma^2}{n} + \left\|\frac{1}{n} \sum_{i=1}^n \nabla f_i(\theta_t)\right\|^2 = \frac{\sigma^2}{n} + \|\nabla f(\theta_t)\|^2, \end{aligned}$$

545 as $g_{t,i} - \nabla f_i(\theta_t), i \in [n]$ are mean-zero and independent random variables. Analogous to Lemma 1,
546 we have

$$\mathbb{E}[\|m'_t\|^2] \leq \frac{C\sigma^2}{n} + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2], \quad (60)$$

547 with $C_1 = (1 - \beta_1^2)(1 + \frac{1}{4(1 - \beta_1^2)})$ and $C = \frac{C_1}{1 - \beta_1^2(2 - \beta_1^2)}$.

548 For \bar{m}_t , the first moment sequence based on averaged compressed stochastic gradients, the following
549 bound holds

$$\begin{aligned} \mathbb{E}[\|\bar{g}_t\|^2] &= \mathbb{E}\left[\left\|\frac{1}{n} \sum_{i=1}^n \mathcal{C}(g_{t,i} + e_{t,i})\right\|^2\right] \\ &= \mathbb{E}\left[\left\|\frac{1}{n} \sum_{t=1}^N (\mathcal{C}(g_{t,i} + e_{t,i}) - (g_{t,i} + e_{t,i}) + g_{t,i} + e_{t,i} - \nabla f_i(\theta_t) + \nabla f_i(\theta_t))\right\|^2\right] \\ &\leq \frac{\sigma^2}{n} + \frac{1}{n^2} \mathbb{E}\left[\left\|\sum_{t=1}^N (\mathcal{C}(g_{t,i} + e_{t,i}) - (g_{t,i} + e_{t,i})) + \sum_{t=1}^N e_{t,i} + \sum_{t=1}^N \nabla f_i(\theta_t)\right\|^2\right] \\ &\leq \frac{\sigma^2}{n} + \frac{3}{n} \sum_{i=1}^n \mathbb{E}[q^2 \|g_{t,i} + e_{t,i}\|^2 + \|e_{t,i}\|^2] + 3\left\|\frac{1}{n} \sum_{i=1}^n \nabla f_i(\theta_t)\right\|^2 \\ &\leq \frac{\sigma^2}{n} + (3q^2 + \frac{4q^2(6q^2 + 3)}{(1 - q^2)^2})\sigma^2 + (6q^2 + 3)\mathbb{E}[\|\nabla f(\theta_t)\|^2], \end{aligned}$$

BK:

$$3\left\|\frac{1}{n} \sum_{i=1}^n \nabla f_i(\theta_t)\right\|^2 = 3\frac{1}{n^2} \left\|\sum_{i=1}^n \nabla f(\theta_t) + \nabla f_i(\theta_t) - \nabla f(\theta_t)\right\|^2$$

So there should be

$$3\frac{1}{n} \left[\frac{1}{n} \left\|\sum_{i=1}^n \nabla f_i(\theta_t) - \nabla f(\theta_t)\right\|^2 \right] \leq \frac{3}{n} \sigma_g^2$$

550 that should appear according to the way you defined σ_g right?

551 where the first inequality is because of Assumption 1 and that the stochastic error $(g_t - \nabla f(\theta_t))$
 552 is mean-zero and independent of other terms. The bound on $\|e_t\|^2$ in the last inequality is due to
 553 Lemma 3 of [31]. Then by similar induction we can obtain

$$\mathbb{E}[\|m_t\|^2] \leq \frac{C\sigma^2}{n} + (3q^2 + \frac{4q^2(6q^2+3)}{(1-q^2)^2})C\sigma^2 + (6q^2+3)C_1 \sum_{\tau=1}^t (\beta_1^2(2-\beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2].$$

554

□

555 **Lemma 10.** For the averaged error sequence \bar{e}_t in distributed SPARS-AMS, under Assumption 4,
 556 for $\forall t$,

$$\mathbb{E}[\|\bar{e}_{t+1}\|^2] \leq \frac{4q^2}{(1-q^2)^2}\sigma^2 + \frac{2q^2}{1-q^2} \sum_{\tau=1}^t (\frac{1+q^2}{2})^{t-\tau} \mathbb{E}[\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_\tau)\|^2].$$

557 *Proof.* We have

$$\begin{aligned} \mathbb{E}[\|\bar{e}_{t+1}\|^2] &= \mathbb{E}[\|\frac{1}{n} \sum_{i=1}^n e_{t,i}\|^2] \\ &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|e_{t,i}\|^2] \\ &\leq \frac{4q^2}{(1-q^2)^2}\sigma^2 + \frac{2q^2}{1-q^2} \sum_{\tau=1}^t (\frac{1+q^2}{2})^{t-\tau} \mathbb{E}[\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_\tau)\|^2], \end{aligned}$$

558 where we use Lemma 4 for each local worker.

□

559 **Lemma 11.** For the moving average error sequence $\bar{\mathcal{E}}_t$ averaged over all local workers, we have

$$\sum_{t=1}^T \mathbb{E}[\|\bar{\mathcal{E}}_t\|^2] \leq \frac{4Tq^2}{(1-q^2)^2\epsilon}(\sigma^2 + \sigma_g^2) + \frac{4q^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}[\|\frac{1}{n} \sum_{i=1}^n \nabla f_i(\theta_t)\|^2],$$

560 *Proof.* The proof is similar to Lemma 5. Denote $K_t := \sum_{\tau=1}^t (\frac{1+q^2}{2})^{t-\tau} \mathbb{E}[\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_\tau)\|^2]$
 561 and $K_0 = 0$. We have

$$\begin{aligned} \mathbb{E}[\|\bar{\mathcal{E}}_t\|^2] &= \mathbb{E}[\|\frac{(1-\beta_1) \sum_{\tau=1}^t \beta_1^{t-\tau} \bar{e}_\tau}{\sqrt{\hat{v}_t} + \epsilon}\|^2] \\ &\leq \frac{(1-\beta_1)^2}{\epsilon} \sum_{i=1}^d \mathbb{E}[(\sum_{\tau=1}^t \beta_1^{t-\tau} \bar{e}_{\tau,i})^2] \\ &\stackrel{(a)}{\leq} \frac{(1-\beta_1)^2}{\epsilon} \sum_{i=1}^d \mathbb{E}[(\sum_{\tau=1}^t \beta_1^{t-\tau})(\sum_{\tau=1}^t \beta_1^{t-\tau} \bar{e}_{\tau,i}^2)] \\ &\leq \frac{1-\beta_1}{\epsilon} \sum_{\tau=1}^t \beta_1^{t-\tau} \mathbb{E}[\|\bar{e}_\tau\|^2] \\ &\stackrel{(b)}{\leq} \frac{4q^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{2q^2(1-\beta_1)}{(1-q^2)\epsilon} \sum_{\tau=1}^t \beta_1^{t-\tau} K_\tau, \end{aligned}$$

where (a) is due to Cauchy-Schwartz and (b) is a result of Lemma 10. Summing over $t = 1, \dots, T$ and using the similar technique as in Lemma 3 leads to

$$\begin{aligned}
\sum_{t=1}^T \mathbb{E}[\|\bar{\mathcal{E}}_t\|^2] &= \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{2q^2(1-\beta_1)}{(1-q^2)\epsilon} \sum_{t=1}^T \sum_{\tau=1}^t \beta_1^{t-\tau} K_\tau \\
&\leq \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{2q^2}{(1-q^2)\epsilon} \sum_{t=1}^T \sum_{\tau=1}^t \left(\frac{1+q^2}{2}\right)^{t-\tau} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_\tau)\|^2\right] \\
&\leq \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{4q^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_t)\|^2\right] \\
&= \frac{4Tq^2}{(1-q^2)^2\epsilon} \sigma^2 + \frac{4q^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_t)\|^2 + \frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_t) - \nabla f(\theta_t)\|^2\right] \\
&\leq \frac{4Tq^2}{(1-q^2)^2\epsilon} (\sigma^2 + \sigma_g^2) + \frac{4q^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta_t)\|^2\right],
\end{aligned}$$

where the last two lines hold because of variance decomposition and Assumption 5.

□

Denote the average gradient as $\bar{g}_t = \frac{1}{n} \sum_{i=1}^n \tilde{g}_{t,i}$, and $\bar{g}'_t = \frac{1}{n} \sum_{i=1}^n g_{t,i}$ be the average of true (uncompressed) local gradients. With a little change of notation, we denote $\bar{m}_0 = \bar{m}'_0 = 0$, and

$$\begin{aligned}
\bar{m}_t &= \beta_1 \bar{m}_{t-1} + (1-\beta_1) \bar{g}_t \quad \text{and} \quad \bar{m}'_t = \beta_1 \bar{m}'_{t-1} + (1-\beta_1) \bar{g}'_t \\
a_t &= \frac{\bar{m}_t}{\sqrt{\hat{v}_t} + \epsilon}, \quad \text{and} \quad a'_t = \frac{\bar{m}'_t}{\sqrt{\hat{v}_t} + \epsilon}.
\end{aligned}$$

By construction we have $m'_t = (1-\beta_1) \sum_{i=1}^k \beta_1^{t-i} \bar{g}_t$.

Let $\bar{e}_t = \frac{1}{n} \sum_{i=1}^n e_{t,i}$. Denote the following auxiliary sequences,

$$\begin{aligned}
\bar{\mathcal{E}}_{t+1} &:= \frac{(1-\beta_1) \sum_{i=1}^{t+1} \beta_1^{t+1-i} \bar{e}_i}{\sqrt{\hat{v}_t} + \epsilon} \\
\theta'_{t+1} &:= \theta_{t+1} - \eta \bar{\mathcal{E}}_{t+1}.
\end{aligned}$$

Then,

$$\begin{aligned}
\theta'_{t+1} &= \theta_{t+1} - \eta \bar{\mathcal{E}}_{t+1} \\
&= \theta_t - \eta \frac{(1-\beta_1) \sum_{i=1}^t \beta_1^{t-i} \bar{g}_i + (1-\beta_1) \sum_{i=1}^{t+1} \beta_1^{t+1-i} \bar{e}_i}{\sqrt{\hat{v}_t} + \epsilon} \\
&= \theta_t - \eta \frac{(1-\beta_1) \sum_{i=1}^t \beta_1^{t-i} (\bar{g}_i + \bar{e}_{i+1}) + (1-\beta) \beta_1^t \bar{e}_1}{\sqrt{\hat{v}_t} + \epsilon} \\
&= \theta_t - \eta \frac{(1-\beta_1) \sum_{i=1}^t \beta_1^{t-i} \bar{e}_i}{\sqrt{\hat{v}_t} + \epsilon} - \eta \frac{\bar{m}'_t}{\sqrt{\hat{v}_t} + \epsilon} \\
&\stackrel{(a)}{=} \theta'_t - \eta \frac{\bar{m}'_t}{\sqrt{\hat{v}_t} + \epsilon} := \theta'_t - \eta a'_t,
\end{aligned}$$

where (a) uses the fact that $\tilde{g}_{t,i} + e_{t+1,i} = g_{t,i} + e_{t,i}$ for $\forall i \in [N]$. By Assumption 2 we have

$$f(\theta'_{t+1}) \leq f(\theta'_t) - \eta \langle \nabla f(\theta'_t), a'_t \rangle + \frac{L}{2} \|\theta'_{t+1} - \theta'_t\|^2.$$

572 Thus,

$$\begin{aligned}
\mathbb{E}[f(\theta'_{t+1}) - f(\theta'_t)] &\leq -\eta \mathbb{E}[\langle \nabla f(\theta'_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] \\
&= -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] + \eta \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta'_t), a'_t \rangle] \\
&\leq -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \frac{\eta^2 L}{2} \mathbb{E}[\|a'_t\|^2] + \eta^2 L \mathbb{E}[\|\mathcal{E}_t\| \|a'_t\|] \\
&\leq -\eta \mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] + \eta^2 L \mathbb{E}[\|a'_t\|^2] + \frac{\eta^2 L}{2} \mathbb{E}[\|\mathcal{E}_t\|^2].
\end{aligned} \tag{61}$$

573 **Bounding the first term in (61).** We have

$$\begin{aligned}
M_t &:= -\mathbb{E}[\langle \nabla f(\theta_t), a'_t \rangle] = -\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}} \rangle] \\
&= \underbrace{-\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle]}_I + \underbrace{\mathbb{E}[\langle \nabla f(\theta_t), (\frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}}) m'_t \rangle]}_{II}.
\end{aligned}$$

574 To bound I, note that

$$\begin{aligned}
I &= -\mathbb{E}[\langle \nabla f(\theta_t), \frac{(1 - \beta_1)g_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&= -\mathbb{E}[\langle \nabla f(\theta_t), \frac{(1 - \beta_1)g_t}{\sqrt{\hat{v}_{t-1} + \epsilon}} | \mathcal{F}_{t-1} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&= -(1 - \beta_1) \mathbb{E}[\frac{\|\nabla f(\theta_t)\|^2}{\sqrt{\hat{v}_{t-1} + \epsilon}}] - \mathbb{E}[\langle \nabla f(\theta_t), \frac{\beta_1 m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&\leq -\frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] - \beta_1 \mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle].
\end{aligned}$$

575 Regarding the second term, we have

$$\begin{aligned}
&-\mathbb{E}[\langle \nabla f(\theta_t), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&= -\mathbb{E}[\langle \nabla f(\theta_{t-1}), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] - \mathbb{E}[\langle \nabla f(\theta_t) - \nabla f(\theta_{t-1}), \frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}} \rangle] \\
&= M_{t-1} + \eta L \mathbb{E}[\|\frac{m_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}}\| \|\frac{m'_{t-1}}{\sqrt{\hat{v}_{t-1} + \epsilon}}\|] \\
&\leq M_{t-1} + \frac{\eta L}{\epsilon} \mathbb{E}[\|m'_{t-1}\|^2] + \eta L \mathbb{E}[\|a_{t-1}\|^2] \\
&\leq M_{t-1} + \frac{\eta L}{\epsilon} (C\sigma^2 + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2]) + \frac{\eta L(1 - \beta_1)d}{(1 - \beta_2)(1 - \gamma)},
\end{aligned}$$

576 where Lemma 1 and Lemma 2 are used, with $C_1 = (1 - \beta_1^2)(1 + \frac{1}{4(1 - \beta_1^2)})$ and $C = \frac{C_1}{1 - \beta_1^2(2 - \beta_1^2)}$.

577 Putting parts together we obtain

$$\begin{aligned}
I &\leq \beta_1 M_{t-1} + \frac{\eta \beta_1 L C \sigma^2}{\epsilon} + \frac{\eta \beta_1 L C_1}{\epsilon} \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2] \\
&\quad + \frac{\eta L \beta_1(1 - \beta_1)d}{(1 - \beta_2)(1 - \gamma)} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2].
\end{aligned}$$

578 For II, it holds that

$$II \leq G^2 \mathbb{E}[\sum_{i=1}^d |\frac{1}{\sqrt{\hat{v}_{t-1} + \epsilon}} - \frac{1}{\sqrt{\hat{v}_t + \epsilon}}|].$$

579 Denoting $H_t := \mathbb{E}[\sum_{i=1}^d |\frac{1}{\sqrt{\hat{v}_{t-1}+\epsilon}} - \frac{1}{\sqrt{\hat{v}_t+\epsilon}}|]$, $S_t := \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2]$. We
 580 arrive at

$$\begin{aligned} M_t &\leq \beta_1 M_{t-1} + \frac{\eta\beta_1 LC\sigma^2}{\epsilon} + \frac{\eta\beta_1 LC_1}{\epsilon} S_t + G^2 H_t \\ &\quad + \frac{\eta L\beta_1(1 - \beta_1)d}{(1 - \beta_2)(1 - \gamma)} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\leq \beta_1 M_{t-1} + \frac{\eta\beta_1 LC\sigma^2}{\epsilon} + \frac{\eta\beta_1 LC_1}{\epsilon} S_t + G^2 H_t + \frac{\eta L\beta_1(1 - \beta_1)d}{(1 - \beta_2)(1 - \gamma)}. \end{aligned}$$

581 By induction, we have

$$\begin{aligned} M_t &\leq \beta_1^{t-1} M_1 + G^2 \sum_{\tau=0}^{t-2} \beta_1^\tau H_{t-\tau} + \frac{\eta\beta_1 LC_1}{\epsilon} \sum_{\tau=0}^{t-2} \beta_1^\tau S_{t-\tau} + \frac{\eta\beta_1 LC\sigma^2}{(1 - \beta_1)\epsilon} \\ &\quad + \frac{\eta L\beta_1 d}{(1 - \beta_2)(1 - \gamma)} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \mathbb{E}[\|\nabla f(\theta_t)\|^2], \end{aligned}$$

582 since $\beta_1 < 1$. For bounding the summations, we have the following result.

583 Summing over $t = 1, \dots, T$, we obtain

$$\begin{aligned} \sum_{t=1}^T M_t &\leq \sum_{t=1}^T \beta_1^{t-1} M_1 + G^2 \sum_{t=2}^T \sum_{\tau=0}^{t-2} \beta_1^\tau H_{t-\tau} + \frac{\eta\beta_1 LC_1}{\epsilon} \sum_{t=2}^T \sum_{\tau=0}^{t-2} \beta_1^\tau S_{t-\tau} \\ &\quad + \frac{T\eta\beta_1 LC\sigma^2}{(1 - \beta_1)\epsilon} + \frac{T\eta L\beta_1 d}{(1 - \beta_2)(1 - \gamma)} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\stackrel{(a)}{\leq} \frac{2dG^2}{(1 - \beta_1)\sqrt{\epsilon}} + \frac{T\eta\beta_1 LC\sigma^2}{(1 - \beta_1)\epsilon} + \frac{T\eta L\beta_1 d}{(1 - \beta_2)(1 - \gamma)} \\ &\quad + \left[\frac{\eta LC}{(1 - \beta_1)\epsilon} - \frac{1 - \beta_1}{\sqrt{(q^2 + 1)G^2 + \epsilon}} \right] \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\ &\leq \frac{2dG^2}{(1 - \beta_1)\sqrt{\epsilon}} + \frac{T\eta\beta_1 LC\sigma^2}{(1 - \beta_1)\epsilon} + \frac{T\eta L\beta_1 d}{(1 - \beta_2)(1 - \gamma)} - \frac{3(1 - \beta_1)}{4\sqrt{(q^2 + 1)G^2 + \epsilon}} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2], \end{aligned}$$

584 when η is chosen to be $\eta \leq \frac{(1 - \beta_1)^2 \epsilon}{4LC\sqrt{(q^2 + 1)G^2 + \epsilon}}$. Here, (a) is due to $M_1 = \mathbb{E}[\langle \nabla f(\theta_1), a'_0 \rangle] \leq$
 585 $\beta_1 dG^2 / \sqrt{\epsilon}$ and Lemma 3. It remains to bound the last two terms in (61).

586 **Bounding the last two terms in in (61).** We have

$$\mathbb{E}[\|a'_t\|^2] = \mathbb{E}[\|\frac{m'_t}{\sqrt{\hat{v}_t + \epsilon}}\|^2] \leq \frac{1}{\epsilon} \mathbb{E}[\|m'_t\|^2].$$

587 By Lemma 1, it follows that

$$\mathbb{E}[\|a'_t\|^2] \leq \frac{1}{\epsilon} (C\sigma^2 + C_1 \sum_{\tau=1}^t (\beta_1^2(2 - \beta_1^2))^{t-\tau} \mathbb{E}[\|\nabla f(\theta_\tau)\|^2]).$$

588 Summing over $t = 1, \dots, T$, we obtain

$$\sum_{t=1}^T \|a'_t\|^2 \leq \frac{TC\sigma^2}{\epsilon} + \frac{C}{\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2],$$

589 where the last inequality can be derived similar to Lemma 3.

590 For the last term in (61), we have by Lemma 5

$$\sum_{t=1}^T \mathbb{E}[\|\mathcal{E}_t\|^2] \leq \frac{4Tq^2}{(1 - q^2)^2 \epsilon} \sigma^2 + \frac{4q^2}{(1 - q^2)^2 \epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2].$$

591 **Completing the proof.** Summing (61) over $t = 1, \dots, T$ and integrating things together, we have

$$\begin{aligned}
& \mathbb{E}[f(\theta'_{T+1}) - f(\theta'_1)] \\
& \leq \eta \sum_{t=1}^T M_t + \frac{T\eta^2 CL\sigma^2}{\epsilon} + \frac{C\eta^2 L}{\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
& \quad + \frac{2T\eta^2 Lq^2\sigma^2}{(1-q^2)^2\epsilon} + \frac{2\eta^2 Lq^2}{(1-q^2)^2\epsilon} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
& \leq \frac{2\eta dG^2}{(1-\beta_1)\sqrt{\epsilon}} + \frac{T\eta^2\beta_1 LC\sigma^2}{(1-\beta_1)\epsilon} + \frac{T\eta^2 L\beta_1 d}{(1-\beta_2)(1-\gamma)} - \frac{3\eta(1-\beta_1)}{4\sqrt{(q^2+1)G^2+\epsilon}} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] \\
& \quad + \frac{T\eta^2 CL\sigma^2}{\epsilon} + \left[\frac{C\eta^2 L}{\epsilon} + \frac{2\eta^2 Lq^2}{(1-q^2)^2\epsilon} \right] \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{2T\eta^2 Lq^2\sigma^2}{(1-q^2)^2\epsilon} \\
& \leq -\frac{\eta(1-\beta_1)}{4\sqrt{(q^2+1)G^2+\epsilon}} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] + \frac{2\eta dG^2}{(1-\beta_1)\sqrt{\epsilon}} + \frac{T\eta^2 LC\sigma^2}{(1-\beta_1)\epsilon} \\
& \quad + \frac{T\eta^2 L\beta_1 d}{(1-\beta_2)(1-\gamma)} + \frac{2T\eta^2 Lq^2\sigma^2}{(1-q^2)^2\epsilon},
\end{aligned}$$

592 when $\eta \leq \frac{(1-q^2)^2(1-\beta_1)\epsilon}{8Lq^2\sqrt{(q^2+1)G^2+\epsilon}}$, where the last line is because $C\eta L \leq \frac{(1-\beta_1)\epsilon}{4\sqrt{(q^2+1)G^2+\epsilon}}$ also holds.

593 Re-arranging terms, we get that when $\eta \leq \min\left\{\frac{1-\beta_1}{C}, \frac{(1-q^2)^2}{2q^2}\right\} \frac{(1-\beta_1)\epsilon}{4L\sqrt{(q^2+1)G^2+\epsilon}}$,

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T \mathbb{E}[\|\nabla f(\theta_t)\|^2] & \leq C' \left(\frac{\mathbb{E}[f(\theta'_1) - f(\theta'_{T+1})]}{T\eta} + \frac{2dG^2}{T(1-\beta_1)\sqrt{\epsilon}} + \frac{\eta LC\sigma^2}{(1-\beta_1)\epsilon} \right. \\
& \quad \left. + \frac{\eta L\beta_1 d}{(1-\beta_2)(1-\gamma)} + \frac{2\eta Lq^2\sigma^2}{(1-q^2)^2\epsilon} \right) \\
& \leq C' \left(\frac{\mathbb{E}[f(\theta_1) - f(\theta^*)]}{T\eta} + \frac{2dG^2}{T(1-\beta_1)\sqrt{\epsilon}} + \frac{\eta LC\sigma^2}{(1-\beta_1)\epsilon} \right. \\
& \quad \left. + \frac{\eta L\beta_1 d}{(1-\beta_2)(1-\gamma)} + \frac{2\eta Lq^2\sigma^2}{(1-q^2)^2\epsilon} \right).
\end{aligned}$$

594 where $C' = \frac{4\sqrt{(q^2+1)G^2+\epsilon}}{1-\beta_1}$, and $C = \frac{(1-\beta_1)^2}{1-\beta_1^2(2-\beta_1)^2} \left(1 + \frac{1}{4(1-\beta_1^2)}\right)$. The last inequality is because
595 $\theta'_1 = \theta_1$, and $\theta^* = \arg \min_{\theta} f(\theta)$. The proof is complete.

596 □