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# A Doubly Stochastic Surrogate Optimization Scheme for Non-convex Finite-sum Problems

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## Abstract

1 Many constrained, non-convex optimization problems can be tackled using the  
2 Majorization-Minimization (MM) method which alternates between constructing  
3 a surrogate function which upper bounds the objective function, and then mini-  
4 mizing this surrogate. For problems which minimize a finite sum of functions,  
5 a stochastic version of the MM method selects a batch of functions at random  
6 at each iteration and optimizes the accumulated surrogate. However, in many  
7 cases of interest such as variational inference for latent variable models, the sur-  
8rogate functions are expressed as an expectation. In this contribution, we propose  
9 a doubly stochastic MM method based on Monte Carlo approximation of these  
10 stochastic surrogates. We establish asymptotic and non-asymptotic convergence  
11 of our scheme in a constrained, non-convex, non-smooth optimization setting. We  
12 apply our new framework for inference of logistic regression model with missing  
13 covariates and for variational inference of autoencoder on the MNIST dataset.

## 14 1 Introduction

15 We consider the *constrained* minimization problem of a finite sum of functions:

$$\min_{\theta \in \Theta} \mathcal{L}(\theta) := \frac{1}{n} \sum_{i=1}^n \mathcal{L}_i(\theta), \quad (1)$$

16 where  $\Theta$  is a convex, compact, and closed subset of  $\mathbb{R}^p$ , and for any  $i \in \llbracket 1, n \rrbracket$ , the function  $\mathcal{L}_i : \mathbb{R}^p \rightarrow \mathbb{R}$  is bounded from below and is (possibly) non-convex and non-smooth.

17  
18 To tackle the optimization problem (1), a popular approach is to apply the majorization-minimization  
19 (MM) method which iteratively minimizes a majorizing surrogate function. A large number of ex-  
20 isting procedures fall into this general framework, for instance gradient-based or proximal methods  
21 or the Expectation-Maximization (EM) algorithm [McLachlan and Krishnan, 2008] and some vari-  
22 ational Bayes inference techniques [Jordan et al., 1999]; see for example [Razaviyayn et al., 2013]  
23 and [Lange, 2016] and the references therein. When the number of terms  $n$  in (1) is large, the  
24 vanilla MM method may be intractable because it requires to construct a surrogate function for all  
25 the  $n$  terms  $\mathcal{L}_i$  at each iteration. Here, a remedy is to apply the Minimization by Incremental Sur-  
26 surrogate Optimization (MISO) method proposed by Mairal [2015], where the surrogate functions are  
27 updated incrementally. The MISO method can be interpreted as a combination of MM and ideas  
28 which have emerged for variance reduction in stochastic gradient methods [Schmidt et al., 2017].

29 The success of the MISO method rests upon the efficient minimization of surrogates such as convex  
30 functions, see [Mairal, 2015, Section 2.3]. In many applications of interest, the natural surrogate  
31 functions are intractable, yet they are defined as expectation of tractable functions. This for exam-  
32 ple the case for inference in latent variable models. Another application is variational inference,

[Ghahramani, 2015], in which the goal is to approximate the posterior distribution of parameters given the observations; see for example [Neal, 2012, Blundell et al., 2015, Polson et al., 2017, Rezende et al., 2014, Li and Gal, 2017].

This paper fills the gap in the literature by proposing a new method called *Minimization by Incremental Stochastic Surrogate Optimization (MISSO)* which is designed for the finite sum optimization with a finite-time convergence guarantee. Our contributions can be summarized as follows.

- We propose a unifying framework of analysis for incremental stochastic surrogate optimization when the surrogates are defined by expectations of tractable functions. The proposed MISSO method is built on the Monte Carlo integration of the intractable surrogate function, *i.e.*, a doubly stochastic surrogate optimization scheme. In addition, we present an incremental variational inference and Monte-Carlo EM methods as two special cases of this framework.
- We establish both asymptotic and non-asymptotic convergence for the MISSO method. In particular, the MISSO method converges almost surely to a stationary point and in  $\mathcal{O}(n/\epsilon)$  iterations to an  $\epsilon$ -stationary point.

In Section 2, we review the techniques for incremental minimization of finite sum functions based on the MM principle; specifically, we review the MISO method as introduced in [Mairal, 2015], and present a class of surrogate functions expressed as an expectation over a latent space. The MISSO method is then introduced for the latter class of surrogate functions. In Section 3, we provide the asymptotic and non-asymptotic convergence analysis for the MISSO method. Finally, Section 4 presents numerical applications to illustrate our findings including parameter inference for logistic regression with missing covariates and variational inference for Bayesian neural network.

**Notations** We denote  $\llbracket 1, n \rrbracket = \{1, \dots, n\}$ . Unless otherwise specified,  $\|\cdot\|$  denotes the standard Euclidean norm and  $\langle \cdot | \cdot \rangle$  is the inner product in Euclidean space. For any function  $f : \Theta \rightarrow \mathbb{R}$ ,  $f'(\theta, d)$  is the directional derivative of  $f$  at  $\theta$  along the direction  $d$ , *i.e.*,

$$f'(\theta, d) := \lim_{t \rightarrow 0^+} \frac{f(\theta + td) - f(\theta)}{t}. \quad (2)$$

The directional derivative is assumed to exist for the functions introduced throughout this paper.

## 2 Incremental Minimization of Finite Sum Non-convex Functions

The objective function in (1) is composed of a finite sum of possibly non-smooth and non-convex functions. A popular approach here is to apply the MM method. The MM method tackles (1) through alternating between two steps — (i) minimizing a *surrogate* function which upper bounds the original objective function; and (ii) updating the surrogate function to tighten the upper bound.

As mentioned in the Introduction, the MISO method proposed by Mairal [2015] is developed as an iterative scheme that only updates the surrogate functions *partially* at each iteration. Formally, for any  $i \in \llbracket 1, n \rrbracket$ , we consider a surrogate function  $\hat{\mathcal{L}}_i(\theta; \bar{\theta})$  which satisfies

**S1.** For all  $i \in \llbracket 1, n \rrbracket$  and  $\bar{\theta} \in \Theta$ , the function  $\hat{\mathcal{L}}_i(\theta; \bar{\theta})$  is convex w.r.t.  $\theta$ , and it holds

$$\hat{\mathcal{L}}_i(\theta; \bar{\theta}) \geq \mathcal{L}_i(\theta), \quad \forall \theta \in \Theta, \quad (3)$$

where the equality holds when  $\theta = \bar{\theta}$ .

**S2.** For any  $\bar{\theta}_i \in \Theta$ ,  $i \in \llbracket 1, n \rrbracket$  and some  $\epsilon > 0$ , the difference function  $\hat{e}(\theta; \{\bar{\theta}_i\}_{i=1}^n) := \frac{1}{n} \sum_{i=1}^n \hat{\mathcal{L}}_i(\theta; \bar{\theta}_i) - \mathcal{L}(\theta)$  is defined for all  $\theta \in \Theta_\epsilon$  and differentiable for all  $\theta \in \Theta$ , where  $\Theta_\epsilon = \{\theta \in \mathbb{R}^d, \inf_{\theta' \in \Theta} \|\theta - \theta'\| < \epsilon\}$  is an  $\epsilon$ -neighborhood set of  $\Theta$ . Moreover, for some constant  $L$ , the gradient satisfies

$$\|\nabla \hat{e}(\theta; \{\bar{\theta}_i\}_{i=1}^n)\|^2 \leq 2L \hat{e}(\theta; \{\bar{\theta}_i\}_{i=1}^n), \quad \forall \theta \in \Theta. \quad (4)$$

**S1** is a common condition used for surrogate optimization, see [Mairal, 2015, Section 2.3]. Meanwhile, **S2** can be satisfied when the difference function  $\hat{e}(\theta; \{\bar{\theta}_i\}_{i=1}^n)$  is  $L$ -smooth for all  $\theta \in \mathbb{R}^d$ , where the condition can be implied through applying [Razaviyayn et al., 2013, Proposition 1].

The inequality (3) implies  $\hat{\mathcal{L}}_i(\theta; \bar{\theta}) \geq \mathcal{L}_i(\theta) > -\infty$  for any  $\theta \in \Theta$ . The MISO method is an incremental version of the MM method, as summarized by Algorithm 1. As seen in the pseudo code, the MISO method maintains an iteratively updated set of surrogate upper-bound functions  $\{\mathcal{A}_i^k(\theta)\}_{i=1}^n$  and updates the iterate through minimizing the average of the surrogate functions.

Particularly, only one out of the  $n$  surrogate functions is updated at each iteration [cf. Line 5] and the sum function  $\frac{1}{n} \sum_{i=1}^n \mathcal{A}_i^{k+1}(\theta)$  is designed to be ‘easy to optimize’, for example, it can be a sum of quadratic functions. As such, the MISO method is suitable for large-scale optimization as the computation cost per iteration is independent of  $n$ . Moreover, under S1, S2, it was shown that the MISO method converges almost surely to a stationary point of (1) [Mairal, 2015, Proposition 3.1].

We now consider the case when the surrogate functions  $\hat{\mathcal{L}}_i(\theta; \bar{\theta})$  are intractable. Let  $Z$  be a measurable set,  $p_i : Z \times \Theta \rightarrow \mathbb{R}_+$  be a pdf,  $r_i : \Theta \times \Theta \times Z \rightarrow \mathbb{R}$  be a measurable function and  $\mu_i$  be a  $\sigma$ -finite measure, we consider surrogate functions which satisfy S1, S2 that can be expressed as an expectation:

$$\hat{\mathcal{L}}_i(\theta; \bar{\theta}) := \int_Z r_i(\theta; \bar{\theta}, z_i) p_i(z_i; \bar{\theta}) \mu_i(dz_i) \quad \forall (\theta, \bar{\theta}) \in \Theta \times \Theta. \quad (5)$$

Plugging (5) into the MISO method is not feasible since the update step in Step 6 involves a minimization of an expectation. Several motivating examples of (1) are given in Section 2.

We propose the *Minimization by Incremental Stochastic Surrogate Optimization* (MISSO) method which replaces the expectation in (5) by *Monte Carlo* integration and then optimizes (1) incrementally. Denote by  $M \in \mathbb{N}$  the Monte Carlo batch size and let  $z_m \in Z$ ,  $m = 1, \dots, M$  be a set of samples. These samples can be drawn (Case 1) i.i.d. from the distribution  $p_i(\cdot; \bar{\theta})$  or (Case 2) from a Markov chain with the stationary distribution  $p_i(\cdot; \bar{\theta})$ ; see Section 3 for illustrations. To this end, we define

$$\tilde{\mathcal{L}}_i(\theta; \bar{\theta}, \{z_m\}_{m=1}^M) := \frac{1}{M} \sum_{m=1}^M r_i(\theta; \bar{\theta}, z_m) \quad (6)$$

and we summarize the proposed MISSO method in Algorithm 2. As seen, the procedure is similar to the MISO method but it involves two types of randomness. The first randomness comes from the selection of  $i_k$  in Line 5. The second randomness is that a set of Monte-Carlo approximated functions  $\tilde{\mathcal{A}}_i^k(\theta)$  is used in lieu of  $\mathcal{A}_i^k(\theta)$  when optimizing for the next iterate  $\theta^{(k)}$ . We now discuss two applications of the MISSO method.

**Example 1: Maximum Likelihood Estimation for Latent Variable Model** Latent variable models [Bishop, 2006] are constructed by introducing unobserved (latent) variables which help explain the observed data. We consider  $n$  independent observations  $((y_i, z_i), i \in \llbracket n \rrbracket)$  where  $y_i$  is observed and  $z_i$  is latent. In this incomplete data framework, define  $\{f_i(z_i, \theta), \theta \in \Theta\}$  to be the complete data likelihood models, i.e., joint likelihood of the observations and latent variables. Let

$$g_i(\theta) := \int_Z f_i(z_i, \theta) \mu_i(dz_i), \quad i \in \llbracket 1, n \rrbracket \quad (9)$$

denote the incomplete data likelihood, i.e., the marginal likelihood of the observations. For ease of notations, the dependence on the observations is made implicit. The maximum likelihood (ML) estimation problem takes  $\mathcal{L}_i(\theta)$  to be the  $i$ th negated incomplete data log-likelihood  $\mathcal{L}_i(\theta) := -\log g_i(\theta)$ .

Assume without loss of generality that  $g_i(\theta) \neq 0$  for all  $\theta \in \Theta$ , we define by  $p_i(z_i, \theta) := f_i(z_i, \theta)/g_i(\theta)$  the conditional distribution of the latent variable  $z_i$  given the observation  $y_i$ . A surrogate function  $\hat{\mathcal{L}}_i(\theta; \bar{\theta})$  satisfying S1 can be obtained through writing  $f_i(z_i, \theta) = \frac{f_i(z_i, \theta)}{p_i(z_i, \bar{\theta})} p_i(z_i, \bar{\theta})$

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**Algorithm 1** MISO method [Mairal, 2015]

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- 1: **Input:** initialization  $\theta^{(0)}$ .
- 2: Initialize the surrogate function as  $\mathcal{A}_i^0(\theta) := \hat{\mathcal{L}}_i(\theta; \theta^{(0)})$ ,  $i \in \llbracket 1, n \rrbracket$ .
- 3: **for**  $k = 0, 1, \dots$  **do**
- 4: Pick  $i_k$  uniformly from  $\llbracket 1, n \rrbracket$ .
- 5: Update  $\mathcal{A}_{i_k}^{k+1}(\theta)$  as:

$$\mathcal{A}_{i_k}^{k+1}(\theta) = \begin{cases} \hat{\mathcal{L}}_{i_k}(\theta; \theta^{(k)}), & \text{if } i = i_k \\ \mathcal{A}_{i_k}^k(\theta), & \text{otherwise.} \end{cases}$$

- 6: Set  $\theta^{(k+1)} \in \arg \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \mathcal{A}_i^{k+1}(\theta)$ .
  - 7: **end for**
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**Algorithm 2** MISSO method

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- 1: **Input:** initialization  $\theta^{(0)}$ ; a sequence of non-negative numbers  $\{M_{(k)}\}_{k=0}^{\infty}$ .
- 2: For all  $i \in \llbracket 1, n \rrbracket$ , draw  $M_{(0)}$  Monte-Carlo samples with the stationary distribution  $p_i(\cdot; \theta^{(0)})$ .
- 3: Initialize the surrogate function as

$$\tilde{\mathcal{A}}_i^0(\theta) := \tilde{\mathcal{L}}_i(\theta; \theta^{(0)}, \{z_{i,m}^{(0)}\}_{m=1}^{M_{(0)}}), \quad i \in \llbracket 1, n \rrbracket. \quad (7)$$

- 4: **for**  $k = 0, 1, \dots$  **do**
- 5:   Pick a function index  $i_k$  uniformly on  $\llbracket 1, n \rrbracket$ .
- 6:   Draw  $M_{(k)}$  Monte-Carlo samples with the stationary distribution  $p_{i_k}(\cdot; \theta^{(k)})$ .
- 7:   Update the individual surrogate functions recursively as:

$$\tilde{\mathcal{A}}_i^{k+1}(\theta) = \begin{cases} \tilde{\mathcal{L}}_i(\theta; \theta^{(k)}, \{z_{i,m}^{(k)}\}_{m=1}^{M_{(k)}}), & \text{if } i = i_k \\ \tilde{\mathcal{A}}_i^k(\theta), & \text{otherwise.} \end{cases} \quad (8)$$

- 8:   Set  $\theta^{(k+1)} \in \arg \min_{\theta \in \Theta} \tilde{\mathcal{L}}^{(k+1)}(\theta) := \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{A}}_i^{k+1}(\theta)$ .
  - 9: **end for**
- 

123 and applying the Jensen inequality:

$$\hat{\mathcal{L}}_i(\theta; \bar{\theta}) = \int_Z \underbrace{\log(p_i(z_i, \bar{\theta})/f_i(z_i, \theta))}_{=r_i(\theta; \bar{\theta}, z_i)} p_i(z_i, \bar{\theta}) \mu_i(dz_i), \quad (10)$$

124 We note that S2 can also be verified for common distribution models. We can apply the MISSO  
125 method following the above specification of  $r_i(\theta; \bar{\theta}, z_i), p_i(z_i, \bar{\theta})$ .

126 **Example 2: Variational Inference** Let  $((x_i, y_i), i \in \llbracket 1, n \rrbracket)$  be i.i.d. input-output pairs and  $w \in$   
127  $W \subseteq \mathbb{R}^d$  be a latent variable. When conditioned on the input  $x = (x_i, i \in \llbracket 1, n \rrbracket)$ , the joint  
128 distribution of  $y = (y_i, i \in \llbracket 1, n \rrbracket)$  and  $w$  is given by:

$$p(y, w|x) = \pi(w) \prod_{i=1}^n p(y_i|x_i, w). \quad (11)$$

129 Our goal is to compute the posterior distribution  $p(w|y, x)$ . In most cases, the posterior distribution  
130  $p(w|y, x)$  is intractable and is approximated using a family of parametric distributions,  $\{q(w, \theta), \theta \in$   
131  $\Theta\}$ . The variational inference (VI) problem [Blei et al., 2017] boils down to minimizing the KL  
132 divergence between  $q(w, \theta)$  and the posterior distribution  $p(w|y, x)$ , as follows:

$$\min_{\theta \in \Theta} \mathcal{L}(\theta) := \text{KL}(q(w; \theta) || p(w|y, x)) := \mathbb{E}_{q(w; \theta)} [\log(q(w; \theta)/p(w|y, x))] . \quad (12)$$

133 Using (11), we decompose  $\mathcal{L}(\theta) = n^{-1} \sum_{i=1}^n \mathcal{L}_i(\theta) + \text{const.}$  where:

$$\mathcal{L}_i(\theta) := -\mathbb{E}_{q(w; \theta)} [\log p(y_i|x_i, w)] + \frac{1}{n} \mathbb{E}_{q(w; \theta)} [\log q(w; \theta)/\pi(w)] = r_i(\theta) + d(\theta). \quad (13)$$

134 Directly optimizing the finite sum objective function in (12) can be difficult. First, with  $n \gg 1$ ,  
135 evaluating the objective function  $\mathcal{L}(\theta)$  requires a full pass over the entire dataset. Second, for some  
136 complex models, the expectations in (13) can be intractable even if we assume a simple parametric  
137 model for  $q(w; \theta)$ . Assume that  $\mathcal{L}_i$  is L-smooth, i.e.,  $\mathcal{L}_i$  is differentiable on  $\Theta$  and its gradient  $\nabla \mathcal{L}_i$   
138 is L-Lipschitz. We apply the MISSO method with a quadratic surrogate function defined as:

$$\hat{\mathcal{L}}_i(\theta; \bar{\theta}) := \mathcal{L}_i(\bar{\theta}) + \langle \nabla_{\theta} \mathcal{L}_i(\bar{\theta}) | \theta - \bar{\theta} \rangle + \frac{L}{2} \|\bar{\theta} - \theta\|^2. \quad (14)$$

139 It is easily checked that  $\hat{\mathcal{L}}_i(\theta; \bar{\theta})$  satisfies S1, S2. To compute the gradient  $\nabla \mathcal{L}_i(\bar{\theta})$ , we apply the  
140 re-parametrization technique suggested in [Paisley et al., 2012, Kingma and Welling, 2014, Blundell  
141 et al., 2015]. Let  $t: \mathbb{R}^d \times \Theta \mapsto \mathbb{R}^d$  be a differentiable function w.r.t.  $\theta \in \Theta$  which is designed such  
142 that the law of  $w = t(z, \bar{\theta})$ , where  $z \sim \mathcal{N}_d(0, \mathbf{I})$ , is  $q(\cdot, \bar{\theta})$ . By [Blundell et al., 2015, Proposition 1],  
143 the gradient of  $-r_i(\cdot)$  in (13) is:

$$\nabla_{\theta} \mathbb{E}_{q(w; \theta)} [\log p(y_i|x_i, w)] = \mathbb{E}_{z \sim \mathcal{N}_d(0, \mathbf{I})} [\mathbf{J}_{\theta}^t(z, \bar{\theta}) \nabla_w \log p(y_i|x_i, w)|_{w=t(z, \bar{\theta})}] , \quad (15)$$

where for each  $z \in \mathbb{R}^d$ ,  $J_{\theta}^t(z, \bar{\theta})$  is the Jacobian of the function  $t(z, \cdot)$  with respect to  $\theta$  evaluated at  $\bar{\theta}$ . In addition, for most cases, the term  $\nabla d(\bar{\theta})$  can be evaluated in closed form.

$$r_i(\theta; \bar{\theta}, z) := \left\langle \nabla_{\theta} d(\bar{\theta}) - J_{\theta}^t(z, \bar{\theta}) \nabla_w \log p(y_i | x_i, w) \Big|_{w=t(z, \bar{\theta})} \mid \theta - \bar{\theta} \right\rangle + \frac{L}{2} \|\theta - \bar{\theta}\|^2. \quad (16)$$

Finally, using (14) and (16), the surrogate function (6) is given by  $\tilde{\mathcal{L}}_i(\theta; \bar{\theta}, \{z_m\}_{m=1}^M) := M^{-1} \sum_{m=1}^M r_i(\theta; \bar{\theta}, z_m)$  where  $\{z_m\}_{m=1}^M$  is an i.i.d sample from  $\mathcal{N}(0, \mathbf{I})$ .

### 3 Convergence Analysis

We provide non-asymptotic convergence bound for the MISSO method and show that it converges asymptotically to a stationary point. Consider the following assumptions.

**H1.** For all  $i \in \llbracket 1, n \rrbracket$ ,  $\bar{\theta} \in \Theta$ ,  $z_i \in \mathbf{Z}$ , the measurable function  $r_i(\theta; \bar{\theta}, z_i)$  is convex in  $\theta$  and is lower bounded.

**H2.** For the samples  $\{z_{i,m}\}_{m=1}^M$ , there exists finite constants  $C_r$  and  $C_{gr}$  such that

$$C_r := \sup_{\bar{\theta} \in \Theta} \sup_{M > 0} \frac{1}{\sqrt{M}} \mathbb{E}_{\bar{\theta}} \left[ \sup_{\theta \in \Theta} \left| \sum_{m=1}^M \left\{ r_i(\theta; \bar{\theta}, z_{i,m}) - \hat{\mathcal{L}}_i(\theta; \bar{\theta}) \right\} \right| \right] \quad (17)$$

$$C_{gr} := \sup_{\bar{\theta} \in \Theta} \sup_{M > 0} \sqrt{M} \mathbb{E}_{\bar{\theta}} \left[ \sup_{\theta \in \Theta} \left| \frac{1}{M} \sum_{m=1}^M \frac{\hat{\mathcal{L}}'_i(\theta, \theta - \bar{\theta}; \bar{\theta}) - r'_i(\theta, \theta - \bar{\theta}; \bar{\theta}, z_{i,m})}{\|\bar{\theta} - \theta\|} \right|^2 \right] \quad (18)$$

for all  $i \in \llbracket 1, n \rrbracket$ , and we denoted by  $\mathbb{E}_{\bar{\theta}}[\cdot]$  the expectation w.r.t. a Markov chain  $\{z_{i,m}\}_{m=1}^M$  with initial distribution  $\xi_i(\cdot; \bar{\theta})$ , transition kernel  $P_{i,\bar{\theta}}$ , and stationary distribution  $p_i(\cdot; \bar{\theta})$ .

**H2** essentially requires to control the expectation of the supremum of an empirical process [Shapiro et al., 2009, Boucheron et al., 2013]. In particular, if  $M \rightarrow \infty$ , the surrogate function's value and its directional derivative approximate that of  $\hat{\mathcal{L}}_i(\theta; \bar{\theta})$  uniformly for all  $\theta \in \Theta$ . As discussed before, there are two relevant cases here:

**Case 1:** When the samples  $\{z_m\}_{m=1}^M$  used to construct the approximation  $\tilde{\mathcal{L}}_i(\cdot; \cdot)$  are drawn i.i.d. directly from  $p_i(\cdot; \bar{\theta})$  and  $\Theta$  is bounded, then **H2** can be implied by the concentration of measure under certain additional regularity conditions.

**Case 2:** When the samples are generated by an MCMC procedure, **H2** can be achieved through an maximal inequality for beta-mixing sequences obtained in [Doukhan et al., 1995]. The condition may also be implied by a number of drift and minorization conditions [Meyn and Tweedie, 2012].

**Stationarity measure** As problem (1) is a constrained optimization, we consider the following stationarity measure:

$$g(\bar{\theta}) := \inf_{\theta \in \Theta} \frac{\mathcal{L}'(\bar{\theta}, \theta - \bar{\theta})}{\|\bar{\theta} - \theta\|} \quad \text{and} \quad g(\bar{\theta}) = g_+(\bar{\theta}) - g_-(\bar{\theta}), \quad (19)$$

where  $g_+(\bar{\theta}) := \max\{0, g(\bar{\theta})\}$ ,  $g_-(\bar{\theta}) := -\min\{0, g(\bar{\theta})\}$  denote the positive and negative part of  $g(\bar{\theta})$ , respectively. Note that  $\bar{\theta}$  is a stationary point if and only if  $g_-(\bar{\theta}) = 0$  [Fletcher et al., 2002]. Furthermore, suppose that the sequence  $\{\theta^{(k)}\}_{k \geq 0}$  has a limit point  $\bar{\theta}$  that is a stationary point, then one has  $\lim_{k \rightarrow \infty} g_-(\theta^{(k)}) = 0$ . In this sense, the sequence  $\{\theta^{(k)}\}_{k \geq 0}$  is said to satisfy an *asymptotic stationary point condition*. This is equivalent to [Mairal, 2015, Definition 2.4].

To explain the condition (19), observe that if  $\bar{\theta} \in \text{int}(\Theta)$ , the directional derivative can be replaced by the inner product between the gradient  $\nabla \mathcal{L}(\bar{\theta})$  and  $\theta - \bar{\theta}$ , i.e.,  $\mathcal{L}'(\bar{\theta}, \theta - \bar{\theta}) = \langle \nabla \mathcal{L}(\bar{\theta}) \mid \theta - \bar{\theta} \rangle$ . Therefore, from the definition we have  $g(\bar{\theta}) = -\|\nabla \mathcal{L}(\bar{\theta})\| = -g_-(\bar{\theta})$ . If in addition  $g_-(\bar{\theta}) = 0$ , then  $\bar{\theta}$  is a stationary point to (1) in the same sense as in unconstrained optimization.

To facilitate our analysis, we define  $\tau_i^k$  as the iteration index where the  $i$ th function is last accessed in the MISSO method prior to iteration  $k$ . For example, we have  $\tau_{i_k}^{k+1} = k$ . We define:

$$\hat{\mathcal{L}}^{(k)}(\theta) := \frac{1}{n} \sum_{i=1}^n \hat{\mathcal{L}}_i(\theta; \theta^{(\tau_i^k)}), \quad \hat{e}^{(k)}(\theta) := \hat{\mathcal{L}}^{(k)}(\theta) - \mathcal{L}(\theta). \quad (20)$$

180 We first establish a non-asymptotic convergence rate for the MISSO method:

181 **Theorem 1.** *Under S1, S2, H1, H2. For any  $K_{\max} \in \mathbb{N}$ , let  $K$  be an independent discrete r.v. drawn*  
 182 *uniformly from  $\{0, \dots, K_{\max} - 1\}$  and define the following quantity:*

$$\Delta_{(K_{\max})} := 2nL\mathbb{E}[\tilde{\mathcal{L}}^{(0)}(\boldsymbol{\theta}^{(0)}) - \tilde{\mathcal{L}}^{(K_{\max})}(\boldsymbol{\theta}^{(K_{\max})})] + \sum_{k=0}^{K_{\max}-1} \frac{4LC_r}{\sqrt{M_{(k)}}}, \quad (21)$$

183 Then we have following non-asymptotic bounds:

$$\mathbb{E}[\|\nabla \hat{e}^{(K)}(\boldsymbol{\theta}^{(K)})\|^2] \leq \frac{\Delta_{(K_{\max})}}{K_{\max}}, \quad \mathbb{E}[g_{-}(\boldsymbol{\theta}^{(K)})] \leq \sqrt{\frac{\Delta_{(K_{\max})}}{K_{\max}}} + \frac{C_{\text{gr}}}{K_{\max}} \sum_{k=0}^{K_{\max}-1} M_{(k)}^{-1/2}. \quad (22)$$

184 Note that  $\Delta_{(K_{\max})}$  is finite for any  $K_{\max} \in \mathbb{N}$ . As expected, the MISSO method converges to a  
 185 stationary point of (1) asymptotically and at a sublinear rate  $\mathbb{E}[g_{-}^{(K)}] \leq \mathcal{O}(\sqrt{1/K_{\max}})$ . Furthermore,  
 186 we remark that the MISO method can be analyzed in Theorem 1 as a special case of the MISSO  
 187 method satisfying  $C_r = C_{\text{gr}} = 0$ . In this case, while the asymptotic convergence is well known  
 188 from [Mairal, 2015] [cf. H2], Eq. (22) gives a non-asymptotic rate of  $\mathbb{E}[g_{-}^{(K)}] \leq \mathcal{O}(\sqrt{nL/K_{\max}})$   
 189 which is new to our best knowledge.

190 Next, we show that under an additional assumption on the sequence of batch size  $M_{(k)}$ , the MISSO  
 191 method converges almost surely to a stationary point:

192 **Theorem 2.** *Under S1, S2, H1, H2. In addition, assume that  $\{M_{(k)}\}_{k \geq 0}$  is a non-decreasing*  
 193 *sequence of integers which satisfies  $\sum_{k=0}^{\infty} M_{(k)}^{-1/2} < \infty$ . Then:*

- 194 1. the negative part of the stationarity measure converges almost surely to zero,  
 195 i.e.,  $\lim_{k \rightarrow \infty} g_{-}(\boldsymbol{\theta}^{(k)}) = 0$  a.s..
- 196 2. the objective value  $\mathcal{L}(\boldsymbol{\theta}^{(k)})$  converges almost surely to a finite number  $\underline{\mathcal{L}}$ ,  
 197 i.e.,  $\lim_{k \rightarrow \infty} \mathcal{L}(\boldsymbol{\theta}^{(k)}) = \underline{\mathcal{L}}$  a.s..

198 In particular, the first result above shows that the sequence  $\{\boldsymbol{\theta}^{(k)}\}_{k \geq 0}$  produced by the MISSO  
 199 method satisfies an *asymptotic stationary point condition*.

## 200 4 Numerical Experiments

### 201 4.1 Binary logistic regression with missing values

202 This application follows **Example 1** described in Section 2. We consider a binary regression setup,  
 203  $((y_i, z_i), i \in \llbracket n \rrbracket)$  where  $y_i \in \{0, 1\}$  is a binary response and  $z_i = (z_{i,j}, j \in \llbracket p \rrbracket)$  is a covariate  
 204 vector. The vector of covariates  $z_i = [z_{i,\text{mis}}, z_{i,\text{obs}}]$  is not fully observed where we denote by  $z_{i,\text{mis}}$   
 205 the missing values and  $z_{i,\text{obs}}$  the observed covariate. It is assumed that  $(z_i, i \in \llbracket n \rrbracket)$  are i.i.d. and  
 206 marginally distributed according to  $\mathcal{N}(\beta, \Omega)$  where  $\beta \in \mathbb{R}^p$  and  $\Omega$  is a positive definite  $p \times p$  matrix.

207 We define the conditional distribution of the observations  $y_i$  given  $z_i = (z_{i,\text{mis}}, z_{i,\text{obs}})$  as:

$$p_i(y_i|z_i) = S(\boldsymbol{\delta}^\top \bar{z}_i)^{y_i} (1 - S(\boldsymbol{\delta}^\top \bar{z}_i))^{1-y_i} \quad (23)$$

208 where for  $u \in \mathbb{R}$ ,  $S(u) = 1/(1+e^{-u})$ ,  $\boldsymbol{\delta} = (\delta_0, \dots, \delta_p)$  are the logistic parameters and  $\bar{z}_i = (1, z_i)$ .  
 209 We are interested in estimating  $\boldsymbol{\delta}$  and finding the latent structure of the covariates  $z_i$ . Here,  $\boldsymbol{\theta} =$   
 210  $(\boldsymbol{\delta}, \beta, \Omega)$  is the parameter to estimate. For  $i \in \llbracket n \rrbracket$ , the complete data log-likelihood is expressed  
 211 as:

$$\log f_i(z_{i,\text{mis}}, \boldsymbol{\theta}) \propto y_i \boldsymbol{\delta}^\top \bar{z}_i - \log(1 + \exp(\boldsymbol{\delta}^\top \bar{z}_i)) - \frac{1}{2} \log(|\Omega|) + \frac{1}{2} \text{Tr}(\Omega^{-1}(z_i - \beta)(z_i - \beta)^\top).$$

212 **Choice of surrogate function for MISO:** We recall the MISO deterministic surrogate defined in  
 213 (10):

$$\hat{\mathcal{L}}_i(\boldsymbol{\theta}; \bar{\boldsymbol{\theta}}) = \int_{\mathcal{Z}} \log(p_i(z_{i,\text{mis}}, \bar{\boldsymbol{\theta}})/f_i(z_{i,\text{mis}}, \boldsymbol{\theta})) p_i(z_{i,\text{mis}}, \bar{\boldsymbol{\theta}}) \mu_i(dz_i). \quad (24)$$

214 where  $\boldsymbol{\theta} = (\boldsymbol{\delta}, \beta, \Omega)$  and  $\bar{\boldsymbol{\theta}} = (\bar{\boldsymbol{\delta}}, \bar{\beta}, \bar{\Omega})$ . We adapt it to our missing covariates problem and decom-  
 215 pose the surrogate function defined above into an observed and a missing part (see Appendix  
 216 D.1.3).



217 **MISSE update:** At the  $k$ -th iteration, and after the initialization, for all  $i \in \llbracket n \rrbracket$ , of the latent  
 218 variables  $(z_i^{(0)})$ , the MISSE algorithm consists in picking an index  $i_k$  uniformly on  $\llbracket n \rrbracket$ , complet-  
 219 ing the observations by sampling a Monte Carlo batch  $\{z_{i_k, \text{mis}, m}^{(k)}\}_{m=1}^{M_{(k)}}$  of missing values from the  
 220 conditional distribution  $p(z_{i_k, \text{mis}} | z_{i_k, \text{obs}}, y_{i_k}; \theta^{(k-1)})$  using an MCMC sampler and computing the  
 221 estimated parameters as follows:

$$\begin{aligned} \beta^{(k)} &= \arg \min_{\beta \in \Theta} \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{L}}_i^{(2)}(\beta, \Omega^{(k)}, \theta^{(\tau_i^k)}, \{z_{i,m}\}_{m=1}^{M_{(\tau_i^k)}}) = \frac{1}{n} \sum_{i=1}^n \frac{1}{M_{(\tau_i^k)}} \sum_{m=1}^{M_{(\tau_i^k)}} z_{i,m}^{(k)} \\ \Omega^{(k)} &= \arg \min_{\Omega \in \Theta} \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{L}}_i^{(2)}(\beta^{(k)}, \Omega, \theta^{(\tau_i^k)}, \{z_{i,m}\}_{m=1}^{M_{(\tau_i^k)}}) = \frac{1}{n} \sum_{i=1}^n \frac{1}{M_{(\tau_i^k)}} \sum_{m=1}^{M_{(\tau_i^k)}} z_{i,m}^{(k)} (z_{i,m}^{(k)})^\top - \beta^{(k)} (\beta^{(k)})^\top \\ \delta^{(k)} &= \frac{1}{n} \sum_{i=1}^n \delta^{(\tau_i^k)} - (\tilde{H}^{(k)})^{-1} \tilde{D}^{(k)}. \end{aligned} \quad (25)$$

222 where  $z_{i,m}^{(k)} = (z_{i, \text{mis}, m}^{(k)}, z_{i, \text{obs}})$  is composed of a simulated and an observed part and  
 223  $\tilde{D}^{(k)} = \frac{1}{n} \sum_{i=1}^n \tilde{D}_i^{(\tau_i^k)}$  and  $\tilde{H}^{(k)} = \frac{1}{n} \sum_{i=1}^n \tilde{H}_i^{(\tau_i^k)}$ . Besides,  $\tilde{\mathcal{L}}_i^{(1)}(\beta, \Omega, \bar{\theta}, \{z_m\}_{m=1}^M)$  and  
 224  $\tilde{\mathcal{L}}_i^{(2)}(\beta, \Omega, \bar{\theta}, \{z_m\}_{m=1}^M)$  are defined as MC approximation of  $\hat{\mathcal{L}}_i^{(1)}(\beta, \Omega, \bar{\theta})$  and  $\hat{\mathcal{L}}_i^{(2)}(\beta, \Omega, \bar{\theta})$ , for  
 225 all  $i \in \llbracket n \rrbracket$ .

226 See Appendix D.1.3 for more explanation.

227 **Fitting a logistic regression model on the TraumaBase dataset** We apply the MISSE method  
 228 to fit a logistic regression model on the TraumaBase (<http://traumabase.eu>) dataset, which  
 229 consists of data collected from 15 trauma centers in France, covering measurements on patients  
 230 from the initial to last stage of trauma.

231 Similar to [Jiang et al., 2018], we select  $p = 16$  influential quantitative measurements, described  
 232 in Appendix D.1.1, on  $n = 6384$  patients, and we adopt the logistic regression model with missing  
 233 covariates in (23) to predict the risk of a severe hemorrhage which is one of the main cause of  
 234 death after a major trauma. Note as the dataset considered is heterogeneous – coming from multiple  
 235 sources with frequently missed entries – we apply the latent data model described in the above.  
 236 For the Monte-Carlo sampling of  $z_{i, \text{mis}}$ , we run a Metropolis Hastings algorithm with the target  
 237 distribution  $p(\cdot | z_{i, \text{obs}}, y_i; \theta^{(k)})$  whose procedure is detailed in Appendix D.1.2.

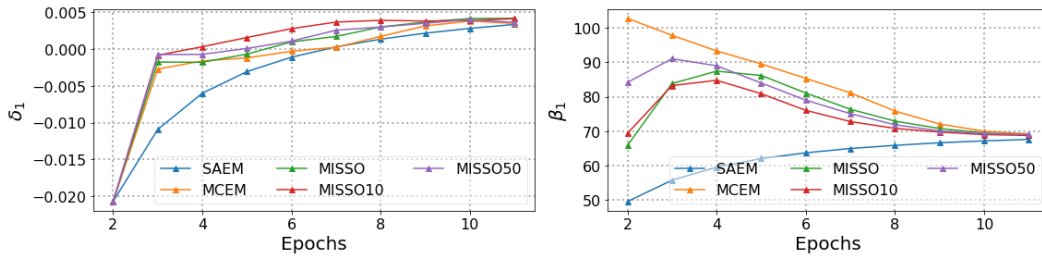


Figure 1: Convergence of first component of the vector of parameters  $\delta$  and  $\beta$  for the SAEM, the MCEM and the MISSE methods. The convergence is plotted against the number of passes over the data.

238 We compare in Figure 1 the convergence behavior of the estimated parameters  $\beta$  using SAEM  
 239 [Delyon et al., 1999] (with stepsize  $\gamma_k = 1/k$ ), MCEM [Wei and Tanner, 1990] and the proposed  
 240 MISSE method. For the MISSE method, we set the batch size to  $M_{(k)} = 10 + k^2$  and we examine  
 241 with selecting different number of functions in Line 5 in the method – the default settings with  
 242 1 function (MISSE), 10% (MISSE10) and 50% (MISSE50) of the functions per iteration. From  
 243 Figure 1, the MISSE method converges to a static value with less number of epochs than the MCEM,  
 244 SAEM methods. It is worth noting that the difference among the MISSE runs for different number  
 245 of selected functions demonstrates a variance-cost tradeoff.

## 4.2 Fitting Bayesian LeNet-5 on MNIST

This application follows **Example 2** described in Section 2. We apply the MISSO method to fit a Bayesian variant of LeNet-5 [LeCun et al., 1998] (see Appendix D.2.1). We train this network on the MNIST dataset [LeCun, 1998]. The training set is composed of  $N = 55\,000$  handwritten digits,  $28 \times 28$  images. Each image is labelled with its corresponding number (from zero to nine). Under the prior distribution  $\pi$ , see (11), the weights are assumed independent and identically distributed according to  $\mathcal{N}(0, 1)$ . We also assume that  $q(\cdot; \theta) \equiv \mathcal{N}(\mu, \sigma^2 \mathbf{I})$ . The variational posterior parameters are thus  $\theta = (\mu, \sigma)$  where  $\mu = (\mu_\ell, \ell \in \llbracket d \rrbracket)$  where  $d$  is the number of weights in the neural network. We use the re-parametrization as  $w = t(\theta, z) = \mu + \sigma z$  with  $z \sim \mathcal{N}(0, \mathbf{I})$ .

At iteration  $k$ , minimizing the sum of stochastic surrogates defined as in (6) and (16) yields the following MISSO update — **step (i)** pick a function index  $i_k$  uniformly on  $\llbracket n \rrbracket$ ; **step (ii)** sample a Monte Carlo batch  $\{z_m^{(k)}\}_{m=1}^{M(k)}$  from  $\mathcal{N}(0, \mathbf{I})$ ; and **step (iii)** update the parameters as

$$\mu_\ell^{(k)} = \frac{1}{n} \sum_{i=1}^n \mu_\ell^{(\tau_i^k)} - \frac{\gamma}{n} \sum_{i=1}^n \hat{\delta}_{\mu_\ell, i}^{(k)} \quad \text{and} \quad \sigma^{(k)} = \frac{1}{n} \sum_{i=1}^n \sigma^{(\tau_i^k)} - \frac{\gamma}{n} \sum_{i=1}^n \hat{\delta}_{\sigma, i}^{(k)}, \quad (26)$$

where  $\hat{\delta}_{\mu_\ell, i}^{(k)} = \hat{\delta}_{\mu_\ell, i}^{(k-1)}$  and  $\hat{\delta}_{\sigma, i}^{(k)} = \hat{\delta}_{\sigma, i}^{(k-1)}$  for  $i \neq i_k$  and:

$$\begin{aligned} \hat{\delta}_{\mu_\ell, i_k}^{(k)} &= -\frac{1}{M(k)} \sum_{m=1}^{M(k)} \nabla_w \log p(y_{i_k} | x_{i_k}, w) \Big|_{w=t(\theta^{(k-1)}, z_m^{(k)})} + \nabla_{\mu_\ell} d(\theta^{(k-1)}), \\ \hat{\delta}_{\sigma, i_k}^{(k)} &= -\frac{1}{M(k)} \sum_{m=1}^{M(k)} z_m^{(k)} \nabla_w \log p(y_{i_k} | x_{i_k}, w) \Big|_{w=t(\theta^{(k-1)}, z_m^{(k)})} + \nabla_\sigma d(\theta^{(k-1)}) \end{aligned}$$

with  $d(\theta) = n^{-1} \sum_{\ell=1}^d (-\log(\sigma) + (\sigma^2 + \mu_\ell^2)/2 - 1/2)$ .

We compare the convergence of the *Monte Carlo variants* of the following state of the art optimization algorithms — the ADAM [Kingma and Ba, 2015], the Momentum [Sutskever et al., 2013] and the SAG [Schmidt et al., 2017] methods versus the *Bayes by Backprop* (BBB) [Blundell et al., 2015] and our proposed MISSO method. For all these methods, the loss function (13) and its gradients were computed by Monte Carlo integration using Tensorflow Probability library [Dillon et al., 2017], based on the re-parametrization described above. Update rules for each algorithm are performed using their vanilla implementations on TensorFlow [Abadi et al., 2015] as detailed in Appendix D.2.2. We use the following hyperparameters for all runs — the learning rate is  $10^{-3}$ , we run 100 epochs with a mini-batch size of 128 and use the batchsize of  $M(k) = k$ .

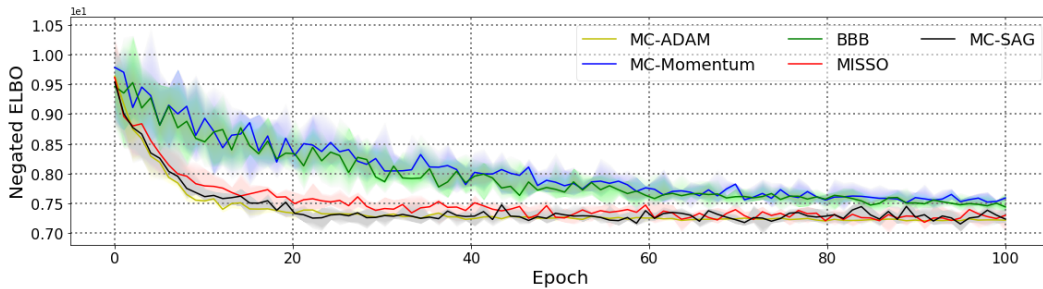


Figure 2: (Incremental Variational Inference) Negated ELBO versus epochs elapsed for fitting the Bayesian LeNet-5 on MNIST using different algorithms. The solid curve is obtained from averaging over 5 independent runs of the methods, and the shaded area represents the standard deviation.

Figure 2 shows the convergence of the negated evidence lower bound against the number of passes over data (one pass represents an epoch). As observed, the proposed MISSO method outperforms *Bayes by Backprop* and Momentum, while similar convergence rates are observed with the MISSO, ADAM and SAG methods.



## 273 5 Conclusions

274 We present a unifying framework for minimizing a non-convex finite-sum objective function using  
275 incremental surrogates when the latter functions are expressed as an expectation and are intractable.  
276 Our approach covers a large class of non-convex applications in machine learning such as logistic  
277 regression with missing values and variational inference. We provide both finite-time and asymptotic  
278 guarantees of our incremental stochastic surrogate optimization technique and illustrate our findings  
279 training a binary logistic regression with missing covariates to predict hemorrhagic shock and a  
280 Bayesian variant of LeNet-5 on MNIST.

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## 361 A Proof of Theorem 1

362 **Theorem.** Under S1, S2, H1, H2. For any  $K_{\max} \in \mathbb{N}$ , let  $K$  be an independent discrete r.v. drawn  
 363 uniformly from  $\{0, \dots, K_{\max} - 1\}$  and define the following quantity:

$$\Delta_{(K_{\max})} := 2nL\mathbb{E}[\tilde{\mathcal{L}}^{(0)}(\boldsymbol{\theta}^{(0)}) - \tilde{\mathcal{L}}^{(K_{\max})}(\boldsymbol{\theta}^{(K_{\max})})] + \sum_{k=0}^{K_{\max}-1} \frac{4LC_r}{\sqrt{M_{(k)}}},$$

364 Then we have following non-asymptotic bounds:

$$\mathbb{E}[\|\nabla \hat{e}^{(K)}(\boldsymbol{\theta}^{(K)})\|^2] \leq \frac{\Delta_{(K_{\max})}}{K_{\max}}, \quad \mathbb{E}[g_{-}(\boldsymbol{\theta}^{(K)})] \leq \sqrt{\frac{\Delta_{(K_{\max})}}{K_{\max}}} + \frac{C_{\text{gr}}}{K_{\max}} \sum_{k=0}^{K_{\max}-1} M_{(k)}^{-1/2}.$$

365 **Proof** We begin by recalling the definition

$$\tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}) := \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{A}}_i^k(\boldsymbol{\theta}). \quad (27)$$

366 Notice that

$$\begin{aligned} \tilde{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}) &= \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{L}}_i(\boldsymbol{\theta}; \boldsymbol{\theta}^{(\tau_i^{k+1})}, \{z_{i,m}^{(\tau_i^{k+1})}\}_{m=1}^{M_{(\tau_i^{k+1})}}) \\ &= \tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}) + \frac{1}{n} (\tilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}; \boldsymbol{\theta}^{(k)}, \{z_{i_k,m}^{(k)}\}_{m=1}^{M_{(k)}}) - \tilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}, \{z_{i_k,m}^{(\tau_{i_k}^k)}\}_{m=1}^{M_{(\tau_{i_k}^k)}})). \end{aligned} \quad (28)$$

367 Furthermore, we recall that

$$\hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}) := \frac{1}{n} \sum_{i=1}^n \hat{\mathcal{L}}_i(\boldsymbol{\theta}; \boldsymbol{\theta}^{(\tau_i^k)}), \quad \hat{e}^{(k)}(\boldsymbol{\theta}) := \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}) - \mathcal{L}(\boldsymbol{\theta}). \quad (29)$$

368 Due to S2, we have

$$\|\nabla \hat{e}^{(k)}(\boldsymbol{\theta}^{(k)})\|^2 \leq 2L\hat{e}^{(k)}(\boldsymbol{\theta}^{(k)}). \quad (30)$$

369 To prove the first bound in (22), using the optimality of  $\boldsymbol{\theta}^{(k+1)}$ , one has

$$\begin{aligned} \tilde{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)}) &\leq \tilde{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k)}) \\ &= \tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) + \frac{1}{n} (\tilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)}, \{z_{i_k,m}^{(k)}\}_{m=1}^{M_{(k)}}) - \tilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}, \{z_{i_k,m}^{(\tau_{i_k}^k)}\}_{m=1}^{M_{(\tau_{i_k}^k)}})) \end{aligned} \quad (31)$$

370 Let  $\mathcal{F}_k$  be the filtration of random variables up to iteration  $k$ , i.e.,  $\{i_{\ell-1}, \{z_{i_{\ell-1},m}^{(\ell-1)}\}_{m=1}^{M_{(\ell-1)}}, \boldsymbol{\theta}^{(\ell)}\}_{\ell=1}^k$ .

371 We observe that the conditional expectation evaluates to

$$\begin{aligned} \mathbb{E}_{i_k} [\mathbb{E}[\tilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)}, \{z_{i_k,m}^{(k)}\}_{m=1}^{M_{(k)}}) | \mathcal{F}_k, i_k] | \mathcal{F}_k] \\ = \mathcal{L}(\boldsymbol{\theta}^{(k)}) + \mathbb{E}_{i_k} [\mathbb{E}[\frac{1}{M_{(k)}} \sum_{m=1}^{M_{(k)}} r_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)}, z_{i_k,m}^{(k)}) - \hat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)}) | \mathcal{F}_k, i_k] | \mathcal{F}_k] \\ \leq \mathcal{L}(\boldsymbol{\theta}^{(k)}) + \frac{C_r}{\sqrt{M_{(k)}}}, \end{aligned} \quad (32)$$

372 where the last inequality is due to H2. Moreover,

$$\mathbb{E}[\tilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}, \{z_{i_k,m}^{(\tau_{i_k}^k)}\}_{m=1}^{M_{(\tau_{i_k}^k)}}) | \mathcal{F}_k] = \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{L}}_i(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}, \{z_{i,m}^{(\tau_i^k)}\}_{m=1}^{M_{(\tau_i^k)}}) = \tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}). \quad (33)$$

373 Taking the conditional expectations on both sides of (31) and re-arranging terms give:

$$\tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \mathcal{L}(\boldsymbol{\theta}^{(k)}) \leq n\mathbb{E}[\tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \tilde{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)}) | \mathcal{F}_k] + \frac{C_r}{\sqrt{M_{(k)}}} \quad (34)$$

374 Proceeding from (34), we observe the following lower bound for the left hand side

$$\begin{aligned}
& \tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \mathcal{L}(\boldsymbol{\theta}^{(k)}) \stackrel{(a)}{=} \tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) + \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) \\
& \stackrel{(b)}{\geq} \tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) + \frac{1}{2L} \|\nabla \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})\|^2 \\
& = \frac{1}{n} \sum_{i=1}^n \left\{ \frac{1}{M_{(\tau_i^k)}} \sum_{m=1}^{M_{(\tau_i^k)}} r_i(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}, z_{i,m}^{(\tau_i^k)}) - \hat{\mathcal{L}}_i(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}) \right\} + \frac{1}{2L} \|\nabla \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})\|^2 \\
& \quad \underbrace{\hspace{10em}}_{:= -\delta^{(k)}(\boldsymbol{\theta}^{(k)})}
\end{aligned} \tag{35}$$

375 where (a) is due to  $\hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) = 0$  [cf. S1], (b) is due to (30) and we have defined the summation in  
376 the last equality as  $-\delta^{(k)}(\boldsymbol{\theta}^{(k)})$ . Substituting the above into (34) yields

$$\frac{\|\nabla \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})\|^2}{2L} \leq n \mathbb{E}[\tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \tilde{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)}) | \mathcal{F}_k] + \frac{C_r}{\sqrt{M_{(k)}}} + \delta^{(k)}(\boldsymbol{\theta}^{(k)}) \tag{36}$$

377 Observe the following upper bound on the total expectations:

$$\mathbb{E}[\delta^{(k)}(\boldsymbol{\theta}^{(k)})] \leq \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \frac{C_r}{\sqrt{M_{(\tau_i^k)}}}\right], \tag{37}$$

378 which is due to H2. It yields

$$\mathbb{E}[\|\nabla \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})\|^2] \leq 2nL \mathbb{E}[\tilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \tilde{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)})] + \frac{2LC_r}{\sqrt{M_{(k)}}} + \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\frac{2LC_r}{\sqrt{M_{(\tau_i^k)}}}\right]$$

379 Finally, for any  $K_{\max} \in \mathbb{N}$ , we let  $K$  be a discrete r.v. that is uniformly drawn from  $\{0, 1, \dots, K_{\max} -$   
380  $1\}$ . Using H2 and taking total expectations lead to

$$\begin{aligned}
\mathbb{E}[\|\nabla \hat{\mathcal{L}}^{(K)}(\boldsymbol{\theta}^{(K)})\|^2] &= \frac{1}{K_{\max}} \sum_{k=0}^{K_{\max}-1} \mathbb{E}[\|\nabla \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})\|^2] \\
&\leq \frac{2nL \mathbb{E}[\tilde{\mathcal{L}}^{(0)}(\boldsymbol{\theta}^{(0)}) - \tilde{\mathcal{L}}^{(K_{\max})}(\boldsymbol{\theta}^{(K_{\max})})]}{K_{\max}} + \frac{2LC_r}{K_{\max}} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\frac{1}{\sqrt{M_{(k)}}} + \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{M_{(\tau_i^k)}}}\right]
\end{aligned} \tag{38}$$

381 For all  $i \in [1, n]$ , the index  $i$  is selected with a probability equal to  $\frac{1}{n}$  when conditioned indepen-  
382 dently on the past. We observe:

$$\mathbb{E}[M_{(\tau_i^k)}^{-1/2}] = \sum_{j=1}^k \frac{1}{n} \left(1 - \frac{1}{n}\right)^{j-1} M_{(k-j)}^{-1/2} \tag{39}$$

383 Taking the sum yields:

$$\begin{aligned}
\sum_{k=0}^{K_{\max}-1} \mathbb{E}[M_{(\tau_i^k)}^{-1/2}] &= \sum_{k=0}^{K_{\max}-1} \sum_{j=1}^k \frac{1}{n} \left(1 - \frac{1}{n}\right)^{j-1} M_{(k-j)}^{-1/2} = \sum_{k=0}^{K_{\max}-1} \sum_{l=0}^{k-1} \frac{1}{n} \left(1 - \frac{1}{n}\right)^{k-(l+1)} M_{(l)}^{-1/2} \\
&= \sum_{l=0}^{K_{\max}-1} M_{(l)}^{-1/2} \sum_{k=l+1}^{K_{\max}-1} \frac{1}{n} \left(1 - \frac{1}{n}\right)^{k-(l+1)} \leq \sum_{l=0}^{K_{\max}-1} M_{(l)}^{-1/2}
\end{aligned} \tag{40}$$

384 where the last inequality is due to upper bounding the geometric series. Plugging this back into (38)  
385 yields

$$\begin{aligned}
\mathbb{E}[\|\nabla \hat{\mathcal{L}}^{(K)}(\boldsymbol{\theta}^{(K)})\|^2] &= \frac{1}{K_{\max}} \sum_{k=0}^{K_{\max}-1} \mathbb{E}[\|\nabla \hat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})\|^2] \\
&\leq \frac{2nL \mathbb{E}[\tilde{\mathcal{L}}^{(0)}(\boldsymbol{\theta}^{(0)}) - \tilde{\mathcal{L}}^{(K_{\max})}(\boldsymbol{\theta}^{(K_{\max})})]}{K_{\max}} + \frac{1}{K_{\max}} \sum_{k=0}^{K_{\max}-1} \frac{4LC_r}{\sqrt{M_{(k)}}} = \frac{\Delta_{(K_{\max})}}{K_{\max}}.
\end{aligned} \tag{41}$$

386 This concludes our proof for the first inequality in (22).

387 To prove the second inequality of (22), we define the shorthand notations  $g^{(k)} := g(\boldsymbol{\theta}^{(k)})$ ,  $g_-^{(k)} :=$   
 388  $-\min\{0, g^{(k)}\}$ ,  $g_+^{(k)} := \max\{0, g^{(k)}\}$ . We observe that

$$\begin{aligned} g^{(k)} &= \inf_{\boldsymbol{\theta} \in \Theta} \frac{\mathcal{L}'(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)})}{\|\boldsymbol{\theta}^{(k)} - \boldsymbol{\theta}\|} \\ &= \inf_{\boldsymbol{\theta} \in \Theta} \left\{ \frac{\frac{1}{n} \sum_{i=1}^n \widehat{\mathcal{L}}'_i(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)})}{\|\boldsymbol{\theta}^{(k)} - \boldsymbol{\theta}\|} - \frac{\langle \nabla \widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)}) | \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)} \rangle}{\|\boldsymbol{\theta}^{(k)} - \boldsymbol{\theta}\|} \right\} \\ &\geq -\|\nabla \widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)})\| + \inf_{\boldsymbol{\theta} \in \Theta} \frac{\frac{1}{n} \sum_{i=1}^n \widehat{\mathcal{L}}'_i(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)})}{\|\boldsymbol{\theta}^{(k)} - \boldsymbol{\theta}\|} \end{aligned} \quad (42)$$

389 where the last inequality is due to the Cauchy-Schwarz inequality and we have defined  
 390  $\widehat{\mathcal{L}}'_i(\boldsymbol{\theta}, \boldsymbol{d}; \boldsymbol{\theta}^{(\tau_i^k)})$  as the directional derivative of  $\widehat{\mathcal{L}}_i(\cdot; \boldsymbol{\theta}^{(\tau_i^k)})$  at  $\boldsymbol{\theta}$  along the direction  $\boldsymbol{d}$ . Moreover,  
 391 for any  $\boldsymbol{\theta} \in \Theta$ ,

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n \widehat{\mathcal{L}}'_i(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}) \\ &= \underbrace{\widetilde{\mathcal{L}}^{(k)'}(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}) - \widetilde{\mathcal{L}}^{(k)'}(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)})}_{\geq 0} + \frac{1}{n} \sum_{i=1}^n \widehat{\mathcal{L}}'_i(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}) \\ &\geq \frac{1}{n} \sum_{i=1}^n \left\{ \widehat{\mathcal{L}}'_i(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}) - \frac{1}{M_{(\tau_i^k)}} \sum_{m=1}^{M_{(\tau_i^k)}} r'_i(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}, z_{i,m}^{(\tau_i^k)}) \right\} \end{aligned} \quad (43)$$

392 where the inequality is due to the optimality of  $\boldsymbol{\theta}^{(k)}$  and the convexity of  $\widetilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta})$  [cf. H1]. Denoting  
 393 a scaled version of the above term as:

$$\epsilon^{(k)}(\boldsymbol{\theta}) := \frac{\frac{1}{n} \sum_{i=1}^n \left\{ \frac{1}{M_{(\tau_i^k)}} \sum_{m=1}^{M_{(\tau_i^k)}} r'_i(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}, z_{i,m}^{(\tau_i^k)}) - \widehat{\mathcal{L}}'_i(\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta} - \boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_i^k)}) \right\}}{\|\boldsymbol{\theta}^{(k)} - \boldsymbol{\theta}\|}.$$

394 We have

$$g^{(k)} \geq -\|\nabla \widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)})\| + \inf_{\boldsymbol{\theta} \in \Theta} (-\epsilon^{(k)}(\boldsymbol{\theta})) \geq -\|\nabla \widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)})\| - \sup_{\boldsymbol{\theta} \in \Theta} |\epsilon^{(k)}(\boldsymbol{\theta})|. \quad (44)$$

395 Since  $g^{(k)} = g_+^{(k)} - g_-^{(k)}$  and  $g_+^{(k)} g_-^{(k)} = 0$ , this implies

$$g_-^{(k)} \leq \|\nabla \widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)})\| + \sup_{\boldsymbol{\theta} \in \Theta} |\epsilon^{(k)}(\boldsymbol{\theta})|. \quad (45)$$

396 Consider the above inequality when  $k = K$ , i.e., the random index, and taking total expectations on  
 397 both sides gives

$$\mathbb{E}[g_-^{(K)}] \leq \mathbb{E}[\|\nabla \widehat{e}^{(K)}(\boldsymbol{\theta}^{(K)})\|] + \mathbb{E}[\sup_{\boldsymbol{\theta} \in \Theta} \epsilon^{(K)}(\boldsymbol{\theta})] \quad (46)$$

398 We note that

$$\left( \mathbb{E}[\|\nabla \widehat{e}^{(K)}(\boldsymbol{\theta}^{(K)})\|] \right)^2 \leq \mathbb{E}[\|\nabla \widehat{e}^{(K)}(\boldsymbol{\theta}^{(K)})\|^2] \leq \frac{\Delta(K_{\max})}{K_{\max}}, \quad (47)$$

399 where the first inequality is due to the convexity of  $(\cdot)^2$  and the Jensen's inequality, and

$$\begin{aligned} \mathbb{E}[\sup_{\boldsymbol{\theta} \in \Theta} \epsilon^{(K)}(\boldsymbol{\theta})] &= \frac{1}{K_{\max}} \sum_{k=0}^{K_{\max}} \mathbb{E}[\sup_{\boldsymbol{\theta} \in \Theta} \epsilon^{(k)}(\boldsymbol{\theta})] \stackrel{(a)}{\leq} \frac{C_{\text{gr}}}{K_{\max}} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n M_{(\tau_i^k)}^{-1/2}\right] \\ &\stackrel{(b)}{\leq} \frac{C_{\text{gr}}}{K_{\max}} \sum_{k=0}^{K_{\max}-1} M_{(k)}^{-1/2} \end{aligned} \quad (48)$$

400 where (a) is due to H2 and (b) is due to (40). This implies

$$\mathbb{E}[g_-^{(K)}] \leq \sqrt{\frac{\Delta(K_{\max})}{K_{\max}}} + \frac{C_{\text{gr}}}{K_{\max}} \sum_{k=0}^{K_{\max}-1} M_{(k)}^{-1/2}, \quad (49)$$

401 and concludes the proof of the theorem.  $\square$



## B Proof of Theorem 2

**Theorem.** Under S1, S2, H1, H2. In addition, assume that  $\{M_{(k)}\}_{k \geq 0}$  is a non-decreasing sequence of integers which satisfies  $\sum_{k=0}^{\infty} M_{(k)}^{-1/2} < \infty$ . Then:

1. the negative part of the stationarity measure converges almost surely to zero, i.e.,  $\lim_{k \rightarrow \infty} g_{-}(\boldsymbol{\theta}^{(k)}) = 0$  a.s..
2. the objective value  $\mathcal{L}(\boldsymbol{\theta}^{(k)})$  converges almost surely to a finite number  $\underline{\mathcal{L}}$ , i.e.,  $\lim_{k \rightarrow \infty} \mathcal{L}(\boldsymbol{\theta}^{(k)}) = \underline{\mathcal{L}}$  a.s..

**Proof** We apply the following auxiliary lemma which proof can be found in Appendix C for the readability of the current proof:

**Lemma 1.** Let  $(V_k)_{k \geq 0}$  be a non negative sequence of random variables such that  $\mathbb{E}[V_0] < \infty$ . Let  $(X_k)_{k \geq 0}$  a non negative sequence of random variables and  $(E_k)_{k \geq 0}$  be a sequence of random variables such that  $\sum_{k=0}^{\infty} \mathbb{E}[|E_k|] < \infty$ . If for any  $k \geq 1$ :

$$V_k \leq V_{k-1} - X_{k-1} + E_{k-1} \quad (50)$$

then:

(i) for all  $k \geq 0$ ,  $\mathbb{E}[V_k] < \infty$  and the sequence  $(V_k)_{k \geq 0}$  converges a.s. to a finite limit  $V_{\infty}$ .

(ii) the sequence  $(\mathbb{E}[V_k])_{k \geq 0}$  converges and  $\lim_{k \rightarrow \infty} \mathbb{E}[V_k] = \mathbb{E}[V_{\infty}]$ .

(iii) the series  $\sum_{k=0}^{\infty} X_k$  converges almost surely and  $\sum_{k=0}^{\infty} \mathbb{E}[X_k] < \infty$ .

We proceed from (31) by re-arranging terms and observing that

$$\begin{aligned} \widehat{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)}) &\leq \widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \frac{1}{n} (\widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}) - \widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)})) \\ &\quad - (\widetilde{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)}) - \widehat{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)})) + (\widetilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})) \\ &\quad + \frac{1}{n} (\widetilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)}, \{z_{i_k, m}^{(k)}\}_{m=1}^{M_{(k)}}) - \widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)})) \\ &\quad + \frac{1}{n} (\widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}) - \widetilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}, \{z_{i_k, m}^{(\tau_{i_k}^k)}\}_{m=1}^{M_{(\tau_{i_k}^k)}})) \end{aligned} \quad (51)$$

Our idea is to apply Lemma 1. Under S1, the finite sum of surrogate functions  $\widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta})$ , defined in (20), is lower bounded by a constant  $c_k > -\infty$  for any  $\boldsymbol{\theta}$ . To this end, we observe that

$$V_k := \widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \inf_{k \geq 0} c_k \geq 0 \quad (52)$$

is a non-negative random variable.

Secondly, under H1, the following random variable is non-negative

$$X_k := \frac{1}{n} (\widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(\tau_{i_k}^k)}; \boldsymbol{\theta}^{(k)}) - \widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)})) \geq 0. \quad (53)$$

Thirdly, we define

$$\begin{aligned} E_k &= -(\widetilde{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)}) - \widehat{\mathcal{L}}^{(k+1)}(\boldsymbol{\theta}^{(k+1)})) + (\widetilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})) \\ &\quad + \frac{1}{n} (\widetilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)}, \{z_{i_k, m}^{(k)}\}_{m=1}^{M_{(k)}}) - \widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)})) \\ &\quad + \frac{1}{n} (\widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}) - \widetilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}, \{z_{i_k, m}^{(\tau_{i_k}^k)}\}_{m=1}^{M_{(\tau_{i_k}^k)}})). \end{aligned} \quad (54)$$

Note that from the definitions (52), (53), (54), we have  $V_{k+1} \leq V_k - X_k + E_k$  for any  $k \geq 1$ .

Under H2, we observe that

$$\mathbb{E}[|\widetilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)}, \{z_{i_k, m}^{(k)}\}_{m=1}^{M_{(k)}}) - \widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)})|] \leq C_r M_{(k)}^{-1/2} \quad (55)$$

$$\mathbb{E} \left[ \left| \widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}) - \widetilde{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}, \{z_{i_k, m}^{(\tau_{i_k}^k)}\}_{m=1}^{M_{(\tau_{i_k}^k)}}) \right| \right] \leq C_r \mathbb{E} \left[ M_{(\tau_{i_k}^k)}^{-1/2} \right] \quad (56)$$

$$\mathbb{E} \left[ |\widetilde{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})| \right] \leq \frac{1}{n} \sum_{i=1}^n C_r \mathbb{E} \left[ M_{(\tau_i^k)}^{-1/2} \right] \quad (57)$$

Therefore,

$$\mathbb{E} [|E_k|] \leq \frac{C_r}{n} \left( M_{(k)}^{-1/2} + \mathbb{E} \left[ M_{(\tau_{i_k}^k)}^{-1/2} + \sum_{i=1}^n \{M_{(\tau_i^k)}^{-1/2} + M_{(\tau_{i+1}^k)}^{-1/2}\} \right] \right) \quad (58)$$

Using (40) and the assumption on the sequence  $\{M_{(k)}\}_{k \geq 0}$ , we obtain that

$$\sum_{k=0}^{\infty} \mathbb{E} [|E_k|] < \frac{C_r}{n} (2 + 2n) \sum_{k=0}^{\infty} M_{(k)}^{-1/2} < \infty. \quad (59)$$

Therefore, the conclusions in Lemma 1 hold. Precisely, we have  $\sum_{k=0}^{\infty} X_k < \infty$  and  $\sum_{k=0}^{\infty} \mathbb{E} [X_k] < \infty$  almost surely. Note that this implies

$$\begin{aligned} \infty &> \sum_{k=0}^{\infty} \mathbb{E} [X_k] = \frac{1}{n} \sum_{k=0}^{\infty} \mathbb{E} [\widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(\tau_{i_k}^k)}) - \widehat{\mathcal{L}}_{i_k}(\boldsymbol{\theta}^{(k)}; \boldsymbol{\theta}^{(k)})] \\ &= \frac{1}{n} \sum_{k=0}^{\infty} \mathbb{E} [\widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) - \mathcal{L}(\boldsymbol{\theta}^{(k)})] = \frac{1}{n} \sum_{k=0}^{\infty} \mathbb{E} [\widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)})] \end{aligned} \quad (60)$$

Since  $\widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)}) \geq 0$ , the above implies

$$\lim_{k \rightarrow \infty} \widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)}) = 0 \quad \text{a.s.} \quad (61)$$

and subsequently applying (30), we have  $\lim_{k \rightarrow \infty} \|\widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)})\| = 0$  almost surely. Finally, it follows from (30) and (45) that

$$\lim_{k \rightarrow \infty} g_-^{(k)} \leq \lim_{k \rightarrow \infty} \sqrt{2L} \sqrt{\widehat{e}^{(k)}(\boldsymbol{\theta}^{(k)})} + \lim_{k \rightarrow \infty} \sup_{\boldsymbol{\theta} \in \Theta} |\epsilon^{(k)}(\boldsymbol{\theta})| = 0, \quad (62)$$

where the last equality holds almost surely due to the fact that  $\sum_{k=0}^{\infty} \mathbb{E} [\sup_{\boldsymbol{\theta} \in \Theta} |\epsilon^{(k)}(\boldsymbol{\theta})|] < \infty$ . This concludes the asymptotic convergence of the MISSO method.

Finally, we prove that  $\mathcal{L}(\boldsymbol{\theta}^{(k)})$  converges almost surely. As a consequence of Lemma 1, it is clear that  $\{V_k\}_{k \geq 0}$  converges almost surely and so is  $\{\widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)})\}_{k \geq 0}$ , i.e., we have  $\lim_{k \rightarrow \infty} \widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) = \underline{\mathcal{L}}$ . Applying (61) implies that

$$\underline{\mathcal{L}} = \lim_{k \rightarrow \infty} \widehat{\mathcal{L}}^{(k)}(\boldsymbol{\theta}^{(k)}) = \lim_{k \rightarrow \infty} \mathcal{L}(\boldsymbol{\theta}^{(k)}) \quad \text{a.s.} \quad (63)$$

This shows that  $\mathcal{L}(\boldsymbol{\theta}^{(k)})$  converges almost surely to  $\underline{\mathcal{L}}$ .  $\square$

## C Proof of Lemma 1

**Lemma.** Let  $(V_k)_{k \geq 0}$  be a non negative sequence of random variables such that  $\mathbb{E}[V_0] < \infty$ . Let  $(X_k)_{k \geq 0}$  a non negative sequence of random variables and  $(E_k)_{k \geq 0}$  be a sequence of random variables such that  $\sum_{k=0}^{\infty} \mathbb{E}[|E_k|] < \infty$ . If for any  $k \geq 1$ :

$$V_k \leq V_{k-1} - X_{k-1} + E_{k-1}$$

then:

(i) for all  $k \geq 0$ ,  $\mathbb{E}[V_k] < \infty$  and the sequence  $(V_k)_{k \geq 0}$  converges a.s. to a finite limit  $V_{\infty}$ .

(ii) the sequence  $(\mathbb{E}[V_k])_{k \geq 0}$  converges and  $\lim_{k \rightarrow \infty} \mathbb{E}[V_k] = \mathbb{E}[V_{\infty}]$ .

(iii) the series  $\sum_{k=0}^{\infty} X_k$  converges almost surely and  $\sum_{k=0}^{\infty} \mathbb{E}[X_k] < \infty$ .

449 **Proof** We first show that for all  $k \geq 0$ ,  $\mathbb{E}[V_k] < \infty$ . Note indeed that:

$$0 \leq V_k \leq V_0 - \sum_{j=1}^k X_j + \sum_{j=1}^k E_j \leq V_0 + \sum_{j=1}^k E_j \quad (64)$$

450 showing that  $\mathbb{E}[V_k] \leq \mathbb{E}[V_0] + \mathbb{E}\left[\sum_{j=1}^k E_j\right] < \infty$ .

451 Since  $0 \leq X_k \leq V_{k-1} - V_k + E_k$  we also obtain for all  $k \geq 0$ ,  $\mathbb{E}[X_k] < \infty$ . Moreover, since  
 452  $\mathbb{E}\left[\sum_{j=1}^{\infty} |E_j|\right] < \infty$ , the series  $\sum_{j=1}^{\infty} E_j$  converges a.s. We may therefore define:

$$W_k = V_k + \sum_{j=k+1}^{\infty} E_j \quad (65)$$

453 Note that  $\mathbb{E}[|W_k|] \leq \mathbb{E}[V_k] + \mathbb{E}\left[\sum_{j=k+1}^{\infty} |E_j|\right] < \infty$ . For all  $k \geq 1$ , we get:

$$\begin{aligned} W_k &\leq V_{k-1} - X_k + \sum_{j=k}^{\infty} E_j \leq W_{k-1} - X_k \leq W_{k-1} \\ \mathbb{E}[W_k] &\leq \mathbb{E}[W_{k-1}] - \mathbb{E}[X_k] \end{aligned} \quad (66)$$

454 Hence the sequences  $(W_k)_{k \geq 0}$  and  $(\mathbb{E}[W_k])_{k \geq 0}$  are non increasing. Since for all  $k \geq 0$ ,  $W_k \geq$   
 455  $-\sum_{j=1}^{\infty} |E_j| > -\infty$  and  $\mathbb{E}[W_k] \geq -\sum_{j=1}^{\infty} \mathbb{E}[|E_j|] > -\infty$ , the (random) sequence  $(W_k)_{k \geq 0}$   
 456 converges a.s. to a limit  $W_{\infty}$  and the (deterministic) sequence  $(\mathbb{E}[W_k])_{k \geq 0}$  converges to a limit  $w_{\infty}$ .  
 457 Since  $|W_k| \leq V_0 + \sum_{j=1}^{\infty} |E_j|$ , the Fatou lemma implies that:

$$\mathbb{E}[\liminf_{k \rightarrow \infty} |W_k|] = \mathbb{E}[|W_{\infty}|] \leq \liminf_{k \rightarrow \infty} \mathbb{E}[|W_k|] \leq \mathbb{E}[V_0] + \sum_{j=1}^{\infty} \mathbb{E}[|E_j|] < \infty \quad (67)$$

458 showing that the random variable  $W_{\infty}$  is integrable.

459 In the sequel, set  $U_k \triangleq W_0 - W_k$ . By construction we have for all  $k \geq 0$ ,  $U_k \geq 0$ ,  $U_k \leq U_{k+1}$  and  
 460  $\mathbb{E}[U_k] \leq \mathbb{E}[|W_0|] + \mathbb{E}[|W_k|] < \infty$  and by the monotone convergence theorem, we get:

$$\lim_{k \rightarrow \infty} \mathbb{E}[U_k] = \mathbb{E}[\lim_{k \rightarrow \infty} U_k] \quad (68)$$

461 Finally, we have:

$$\lim_{k \rightarrow \infty} \mathbb{E}[U_k] = \mathbb{E}[W_0] - w_{\infty} \quad \text{and} \quad \mathbb{E}[\lim_{k \rightarrow \infty} U_k] = \mathbb{E}[W_0] - \mathbb{E}[W_{\infty}] \quad (69)$$

462 showing that  $\mathbb{E}[W_{\infty}] = w_{\infty}$  and concluding the proof of (ii). Moreover, using (66) we have that  
 463  $W_k \leq W_{k-1} - X_k$  which yields:

$$\begin{aligned} \sum_{j=1}^{\infty} X_j &\leq W_0 - W_{\infty} < \infty \\ \sum_{j=1}^{\infty} \mathbb{E}[X_j] &\leq \mathbb{E}[W_0] - w_{\infty} < \infty \end{aligned} \quad (70)$$

464 which concludes the proof of the lemma.  $\square$

## 465 **D Details about the Numerical Experiments**

### 466 **D.1 Binary Logistic Regression on the Traumabase**

#### 467 **D.1.1 Traumabase quantitative variables**

468 The list of the 16 quantitative variables we use in our experiments are as follows — *age*, *weight*,  
 469 *height*, *BMI (Body Mass Index)*, *the Glasgow Coma Scale*, *the Glasgow Coma Scale motor com-*  
 470 *ponent*, *the minimum systolic blood pressure*, *the minimum diastolic blood pressure*, *the maximum*

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**Algorithm 3** MH algorithm

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1: Input: initialization  $z_{i,\text{mis},0} \sim q(z_{i,\text{mis}}; \delta)$ 
2: for  $m = 1, \dots, M$  do
3:   Sample  $z_{i,\text{mis},m} \sim q(z_{i,\text{mis}}; \delta)$ 
4:   Sample  $u \sim \mathcal{U}([0, 1])$ 
5:   Calculate the ratio  $r = \frac{\pi(z_{i,\text{mis},m}; \theta) / q(z_{i,\text{mis},m}; \delta)}{\pi(z_{i,\text{mis},m-1}; \theta) / q(z_{i,\text{mis},m-1}; \delta)}$ 
6:   if  $u < r$  then
7:     Accept  $z_{i,\text{mis},m}$ 
8:   else
9:      $z_{i,\text{mis},m} \leftarrow z_{i,\text{mis},m-1}$ 
10:  end if
11: end for
12: Output:  $z_{i,\text{mis},M}$ 

```

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471 number of heart rate (or pulse) per unit time (usually a minute), the systolic blood pressure at ar-  
472 rival of ambulance, the diastolic blood pressure at arrival of ambulance, the heart rate at arrival  
473 of ambulance, the capillary Hemoglobin concentration, the oxygen saturation, the fluid expansion  
474 colloids, the fluid expansion cristalloids, the pulse pressure for the minimum value of diastolic and  
475 systolic blood pressure, the pulse pressure at arrival of ambulance.

### 476 D.1.2 Metropolis Hastings algorithm

477 During the simulation step of the MISSO method, the sampling from the target distribution  
478  $\pi(z_{i,\text{mis}}; \theta) := p(z_{i,\text{mis}} | z_{i,\text{obs}}, y_i; \theta)$  is performed using a Metropolis Hastings (MH) algorithm  
479 [Meyn and Tweedie, 2012] with proposal distribution  $q(z_{i,\text{mis}}; \delta) := p(z_{i,\text{mis}} | z_{i,\text{obs}}; \delta)$  where  
480  $\theta = (\beta, \Omega)$  and  $\delta = (\xi, \Sigma)$ . The parameters of the Gaussian conditional distribution of  $z_{i,\text{mis}} | z_{i,\text{obs}}$   
481 read:

$$\begin{aligned} \xi &= \beta_{\text{mis}} + \Omega_{\text{mis},\text{obs}} \Omega_{\text{obs},\text{obs}}^{-1} (z_{i,\text{obs}} - \beta_{\text{obs}}), \\ \Sigma &= \Omega_{\text{mis},\text{mis}} + \Omega_{\text{mis},\text{obs}} \Omega_{\text{obs},\text{obs}}^{-1} \Omega_{\text{obs},\text{mis}} \end{aligned} \quad (71)$$

482 where we have used the Schur Complement of  $\Omega_{\text{obs},\text{obs}}$  in  $\Omega$  and noted  $\beta_{\text{mis}}$  (resp.  $\beta_{\text{obs}}$ ) the missing  
483 (resp. observed) elements of  $\beta$ . The MH algorithm is summarized in Algorithm 3.

### 484 D.1.3 MISSO Update

485 **Surrogate function decomposition** We adapt it to our missing covariates problem and decompose  
486 the term depending on  $\theta$ , while  $\bar{\theta}$  is fixed, in two following parts leading to

$$\begin{aligned} \hat{\mathcal{L}}_i(\theta; \bar{\theta}) &= - \int_{\mathcal{Z}} \log f_i(z_{i,\text{mis}}, z_{i,\text{obs}}, \theta) p_i(z_{i,\text{mis}}, \bar{\theta}) \mu_i(dz_{i,\text{mis}}) \\ &= - \int_{\mathcal{Z}} \log [p_i(y_i | z_{i,\text{mis}}, z_{i,\text{obs}}, \delta) p_i(z_{i,\text{mis}}, \beta, \Omega)] p_i(z_i, \bar{\theta}) \mu_i(dz_{i,\text{mis}}) \\ &= \underbrace{- \int_{\mathcal{Z}} \log p_i(y_i | z_{i,\text{mis}}, z_{i,\text{obs}}, \delta) p_i(z_i, \bar{\theta}) \mu_i(dz_{i,\text{mis}})}_{=\hat{\mathcal{L}}_i^{(1)}(\delta, \bar{\theta})} - \underbrace{\int_{\mathcal{Z}} \log p_i(z_{i,\text{mis}}, \beta, \Omega) p_i(z_i, \bar{\theta}) \mu_i(dz_{i,\text{mis}})}_{=\hat{\mathcal{L}}_i^{(2)}(\beta, \Omega, \bar{\theta})} \end{aligned} \quad (72)$$

487 The mean  $\beta$  and the covariance  $\Omega$  of the latent structure can be estimated minimizing the sum of  
488 MISSO surrogates  $\tilde{\mathcal{L}}_i^{(2)}(\beta, \Omega, \bar{\theta}, \{z_m\}_{m=1}^M)$ , defined as MC approximation of  $\hat{\mathcal{L}}_i^{(2)}(\beta, \Omega, \bar{\theta})$ , for all  
489  $i \in \llbracket n \rrbracket$ , in closed-form expression.

490 We thus keep the surrogate  $\hat{\mathcal{L}}_i^{(2)}(\beta, \Omega, \bar{\theta})$  and consider the following quadratic approximation of  
 491  $\hat{\mathcal{L}}_i^{(1)}(\delta, \bar{\theta})$  to estimate the vector of logistic parameters  $\delta$ :

$$\begin{aligned} \hat{\mathcal{L}}_i^{(1)}(\bar{\delta}, \bar{\theta}) - \int_{\mathbf{Z}} \nabla \log p_i(y_i | z_{i,\text{mis}}, z_{i,\text{obs}}, \delta) \Big|_{\delta=\bar{\delta}} p_i(z_{i,\text{mis}}, \bar{\theta}) \mu_i(dz_{i,\text{mis}}) (\delta - \bar{\delta}) \\ - (\delta - \bar{\delta})/2 \int_{\mathbf{Z}} \nabla^2 \log p_i(y_i | z_{i,\text{mis}}, z_{i,\text{obs}}, \delta) p_i(z_{i,\text{mis}}, \bar{\theta}) p_i(z_{i,\text{mis}}, \bar{\theta}) \mu_i(dz_{i,\text{mis}}) (\delta - \bar{\delta})^\top \end{aligned} \quad (73)$$

492 Recall that:

$$\begin{aligned} \nabla \log p_i(y_i | z_{i,\text{mis}}, z_{i,\text{obs}}, \delta) &= z_i (y_i - S(\delta^\top z_i)) \\ \nabla^2 \log p_i(y_i | z_{i,\text{mis}}, z_{i,\text{obs}}, \delta) &= -z_i z_i^\top \dot{S}(\delta^\top z_i) \end{aligned} \quad (74)$$

493 where  $\dot{S}(u)$  is the derivative of  $S(u)$ . Note that  $\dot{S}(u) \leq 1/4$  and since, for all  $i \in \llbracket n \rrbracket$ , the  $p \times p$   
 494 matrix  $z_i z_i^\top$  is semi-definite positive we can assume:

495 **L1.** For all  $i \in \llbracket n \rrbracket$  and  $\epsilon > 0$ , there exist, for all  $z_i \in \mathbf{Z}$ , a positive definite matrix  $H_i(z_i) :=$   
 496  $\frac{1}{4}(z_i z_i^\top + \epsilon I_d)$  such that for all  $\delta \in \mathbb{R}^p$ ,  $-z_i z_i^\top \dot{S}(\delta^\top z_i) \leq H_i(z_i)$ .

497 Then, we use, for all  $i \in \llbracket n \rrbracket$ , the following surrogate function to estimate  $\delta$ :

$$\bar{\mathcal{L}}_i^{(1)}(\delta, \bar{\theta}) = \hat{\mathcal{L}}_i^{(1)}(\bar{\delta}, \bar{\theta}) - D_i^\top (\delta - \bar{\delta}) + \frac{1}{2} (\delta - \bar{\delta}) H_i (\delta - \bar{\delta})^\top \quad (75)$$

498 where:

$$\begin{aligned} D_i &= \int_{\mathbf{Z}} \nabla \log p_i(y_i | z_{i,\text{mis}}, z_{i,\text{obs}}, \delta) \Big|_{\delta=\bar{\delta}} p_i(z_{i,\text{mis}}, \bar{\theta}) \mu_i(dz_{i,\text{mis}}) \\ H_i &= \int_{\mathbf{Z}} H_i(z_{i,\text{mis}}) p_i(z_{i,\text{mis}}, \bar{\theta}) \mu_i(dz_{i,\text{mis}}) \end{aligned} \quad (76)$$

499 Finally, at iteration  $k$ , the total surrogate is:

$$\begin{aligned} \tilde{\mathcal{L}}^{(k)}(\theta) &= \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{L}}_i(\theta, \theta^{(\tau_i^k)}, \{z_{i,m}\}_{m=1}^{M(\tau_i^k)}) \\ &= \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{L}}_i^{(2)}(\beta, \Omega, \theta^{(\tau_i^k)}, \{z_{i,m}\}_{m=1}^{M(\tau_i^k)}) - \frac{1}{n} \sum_{i=1}^n \tilde{D}_i^{(\tau_i^k)} (\delta - \delta^{(\tau_i^k)}) + \frac{1}{2n} \sum_{i=1}^n (\delta - \delta^{(\tau_i^k)}) \left\{ \tilde{H}_i^{(\tau_i^k)} \right\} (\delta - \delta^{(\tau_i^k)})^\top \end{aligned} \quad (77)$$

500 where for all  $i \in \llbracket n \rrbracket$ :

$$\begin{aligned} \tilde{D}_i^{(\tau_i^k)} &= \frac{1}{M(\tau_i^k)} \sum_{m=1}^{M(\tau_i^k)} z_{i,m}^{(\tau_i^k)} \left( y_i - S\left( \left( \delta^{(\tau_i^k)} \right)^\top z_{i,m}^{(\tau_i^k)} \right) \right) \\ \tilde{H}_i^{(\tau_i^k)} &= \frac{1}{4M(\tau_i^k)} \sum_{m=1}^{M(\tau_i^k)} z_{i,m}^{(\tau_i^k)} (z_{i,m}^{(\tau_i^k)})^\top \end{aligned} \quad (78)$$

501 Minimizing the total surrogate (77) boils down to performing a quasi-Newton step. It is perhaps sen-  
 502 sible to apply some diagonal loading which is perfectly compatible with the surrogate interpretation  
 503 we just gave.

504 The logistic parameters are estimated as follows:

$$\delta^{(k)} = \arg \min_{\delta \in \Theta} \frac{1}{n} \sum_{i=1}^n \tilde{\mathcal{L}}_i^{(1)}(\delta, \theta^{(\tau_i^k)}, \{z_{i,m}\}_{m=1}^{M(\tau_i^k)}) \quad (79)$$

505 where  $\tilde{\mathcal{L}}_i^{(1)}(\delta, \theta^{(\tau_i^k)}, \{z_{i,m}\}_{m=1}^{M(\tau_i^k)})$  is the MC approximation of the MISO surrogate defined in  
 506 (75) and which leads to the following quasi-Newton step:

$$\delta^{(k)} = \frac{1}{n} \sum_{i=1}^n \delta^{(\tau_i^k)} - (\tilde{H}^{(k)})^{-1} \tilde{D}^{(k)} \quad (80)$$

507 with  $\tilde{D}^{(k)} = \frac{1}{n} \sum_{i=1}^n \tilde{D}_i^{(\tau_i^k)}$  and  $\tilde{H}^{(k)} = \frac{1}{n} \sum_{i=1}^n \tilde{H}_i^{(\tau_i^k)}$ .

## 508 D.2 Incremental Variational Inference for MNIST

### 509 D.2.1 Bayesian LeNet-5 Architecture

510 We describe in Table D.2.1 the architecture of the Convolutional Neural Network introduced in  
 511 [LeCun et al., 1998] and trained on MNIST:

layer type	width	stride	padding	input shape	nonlinearity
convolution ( $5 \times 5$ )	6	1	0	$1 \times 32 \times 32$	ReLU
max-pooling ( $2 \times 2$ )		2	0	$6 \times 28 \times 28$	
convolution ( $5 \times 5$ )	6	1	0	$1 \times 14 \times 14$	ReLU
max-pooling ( $2 \times 2$ )		2	0	$16 \times 10 \times 10$	
fully-connected	120			400	ReLU
fully-connected	84			120	ReLU
fully-connected	10			84	

Table 1: LeNet-5 architecture

### 512 D.2.2 Algorithms updates

513 First, we initialize the means  $\mu_\ell^{(0)}$  for  $\ell \in \llbracket d \rrbracket$  and variance estimates  $\sigma^{(0)}$ . In the sequel, at iteration  
 514  $k$  and for all  $i \in \llbracket n \rrbracket$  we define the following terms:

$$\begin{aligned}
 \hat{\delta}_{\mu_\ell, i}^{(k)} &= -\frac{1}{M_{(k)}} \sum_{m=1}^{M_{(k)}} \nabla_w \log p(y_i | x_i, w) \Big|_{w=t(\theta^{(k-1)}, z_m^{(k)})} + \nabla_{\mu_\ell} d(\theta^{(k-1)}), \\
 \hat{\delta}_{\sigma, i}^{(k)} &= -\frac{1}{M_{(k)}} \sum_{m=1}^{M_{(k)}} z_m^{(k)} \nabla_w \log p(y_i | x_i, w) \Big|_{w=t(\theta^{(k-1)}, z_m^{(k)})} + \nabla_{\sigma} d(\theta^{(k-1)}).
 \end{aligned} \tag{81}$$

515 For all benchmark algorithms, we pick, at iteration  $k$ , a function index  $i_k$  uniformly on  $\llbracket n \rrbracket$  and  
 516 sample a Monte Carlo batch  $\{z_m^{(k)}\}_{m=1}^{M_{(k)}}$  from the standard Gaussian distribution. The updates of the  
 517 parameters  $\mu_\ell$  for all  $\ell \in \llbracket d \rrbracket$  and  $\sigma$  break down as follows:

518 **Monte Carlo SAG update:** Set

$$\mu_\ell^{(k)} = \mu_\ell^{(k-1)} - \frac{\gamma}{n} \sum_{i=1}^n \hat{\delta}_{\mu_\ell, i}^{(k)} \quad \text{and} \quad \sigma^{(k)} = \sigma^{(k-1)} - \frac{\gamma}{n} \sum_{i=1}^n \hat{\delta}_{\sigma, i}^{(k)}, \tag{82}$$

519 where  $\hat{\delta}_{\mu_\ell, i}^{(k)} = \hat{\delta}_{\mu_\ell, i}^{(k-1)}$  and  $\hat{\delta}_{\sigma, i}^{(k)} = \hat{\delta}_{\sigma, i}^{(k-1)}$  for  $i \neq i_k$  and are defined by (81) for  $i = i_k$ . The learning  
 520 rate is set to  $\gamma = 10^{-3}$ .

521 **Bayes By Backprop update:** Set

$$\mu_\ell^{(k)} = \mu_\ell^{(k-1)} - \frac{\gamma}{n} \hat{\delta}_{\mu_\ell, i_k}^{(k)} \quad \text{and} \quad \sigma^{(k)} = \sigma^{(k-1)} - \frac{\gamma}{n} \hat{\delta}_{\sigma, i_k}^{(k)}, \tag{83}$$

522 where the learning rate  $\gamma = 10^{-3}$ .

523 **Monte Carlo Momentum update:** Set

$$\mu_\ell^{(k)} = \mu_\ell^{(k-1)} + \hat{v}_{\mu_\ell}^{(k)} \quad \text{and} \quad \sigma^{(k)} = \sigma^{(k-1)} + \hat{v}_{\sigma}^{(k)}, \tag{84}$$

524 where

$$\hat{v}_{\mu_\ell, i}^{(k)} = \alpha \hat{v}_{\mu_\ell, i}^{(k-1)} - \frac{\gamma}{n} \hat{\delta}_{\mu_\ell, i_k}^{(k)} \quad \text{and} \quad \hat{v}_{\sigma}^{(k)} = \alpha \hat{v}_{\sigma}^{(k-1)} - \frac{\gamma}{n} \hat{\delta}_{\sigma, i_k}^{(k)}, \tag{85}$$

525 where  $\alpha$  and  $\gamma$ , respectively the momentum and the learning rates, are set to  $10^{-3}$ .

526 **Monte Carlo ADAM update:** Set

$$\mu_\ell^{(k)} = \mu_\ell^{(k-1)} - \frac{\gamma}{n} \hat{m}_{\mu_\ell}^{(k)} / (\sqrt{\hat{m}_{\mu_\ell}^{(k)}} + \epsilon) \quad \text{and} \quad \sigma^{(k)} = \sigma^{(k-1)} - \frac{\gamma}{n} \hat{m}_{\sigma}^{(k)} / (\sqrt{\hat{m}_{\sigma}^{(k)}} + \epsilon), \tag{86}$$



527 where

$$\begin{aligned}\hat{\mathbf{m}}_{\mu_\ell}^{(k)} &= \mathbf{m}_{\mu_\ell}^{(k-1)} / (1 - \rho_1^k) \quad \text{with} \quad \mathbf{m}_{\mu_\ell}^{(k)} = \rho_1 \mathbf{m}_{\mu_\ell}^{(k-1)} + (1 - \rho_1) \hat{\boldsymbol{\delta}}_{\mu_\ell, i_k}^{(k)}, \\ \hat{\mathbf{v}}_{\mu_\ell}^{(k)} &= \mathbf{v}_{\mu_\ell}^{(k-1)} / (1 - \rho_2^k) \quad \text{with} \quad \mathbf{v}_{\mu_\ell}^{(k)} = \rho_2 \mathbf{v}_{\mu_\ell}^{(k-1)} + (1 - \rho_2) (\hat{\boldsymbol{\delta}}_{\mu_\ell, i_k}^{(k)})^2\end{aligned}\tag{87}$$

528 and

$$\begin{aligned}\hat{\mathbf{m}}_\sigma^{(k)} &= \mathbf{m}_\sigma^{(k-1)} / (1 - \rho_1^k) \quad \text{with} \quad \mathbf{m}_\sigma^{(k)} = \rho_1 \mathbf{m}_\sigma^{(k-1)} + (1 - \rho_1) \hat{\boldsymbol{\delta}}_{\sigma, i_k}^{(k)}, \\ \hat{\mathbf{v}}_\sigma^{(k)} &= \mathbf{v}_\sigma^{(k-1)} / (1 - \rho_2^k) \quad \text{with} \quad \mathbf{v}_\sigma^{(k)} = \rho_2 \mathbf{v}_\sigma^{(k-1)} + (1 - \rho_2) (\hat{\boldsymbol{\delta}}_{\sigma, i_k}^{(k)})^2.\end{aligned}\tag{88}$$

529 The hyperparameters are set as follows:  $\gamma = 10^{-3}$ ,  $\rho_1 = 0.9$ ,  $\rho_2 = 0.999$ ,  $\epsilon = 10^{-8}$ .