Fast Iterative Expectation Maximization

immediate

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1 Introduction

1.1 Qq notes sur l'état de l'art

Algo de gradient "full gradient" serait

$$\theta_{t+1} = \theta_t - \eta \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\theta_t).$$

La version stochastique pour éviter le calcul de la somme

$$\theta_{t+1} = \theta_t - \eta \nabla f_{I_t}(\theta_t).$$

En non convexe (voir Ghadimi, Lan, 2013), sous une condition du type

$$\sup_{\theta} \mathbb{E}\left[\|\nabla f_I(\theta) - \nabla f(\theta)\|^2\right] < \infty,$$

ils cherchent un résultat de complexité i.e. pour un échantillonnage uniforme K, pour un $\epsilon > 0$, combien d'appels à ∇f_i MAX (quelle valeur de $K_{\max}$) pour garantir

$$\mathbb{E}\left[\|\nabla f(\theta_K)\|^2\right] \leq \epsilon. \qquad (1)$$

eq:complexite

Ils prouvent que pour l'algo GdtSto, où le gradient exact est remplacé par $b_t^{-1} \sum_{i \in \mathcal{B}_t} \nabla f_i(\theta_t)$ pour une suite b_t déterministe, croissante ou constante, alors la complexité est $O(1/\epsilon^2)$.

SAG (Bach, Leroux, NIPS-NIPS) L'update est

$$\theta_{t+1} = \theta_t - \eta \widehat{\nabla f}^{t+1}$$

et l'estimé du gradient est construit comme dans i-EM:

$$\widehat{\nabla f}^{t+1} = \widehat{\nabla f}^{t} + \frac{1}{n}\left(\nabla f_{I_{t+1}}(\theta_t) - \widehat{\nabla f}^{<t+1, I_{t+1}}\right)$$

La difficulté des preuves c'est que le terme ajouté est biaisé; dans le cas convexe, il est montré que pour avoir (1), il faut $K_{\max} = O(n/\epsilon)$; et dans le cas fortement convexe, il faut $K_{\max} = O(-n \log(\epsilon))$. De plus, cet algo nécessite de garder en mémoire tous les gradients (l'analogue de notre S).

SVRG (Johnson, Zhang, 2013) L'objectif est de faire de la réduction de variance mais en prenant un estimateur du gradient qui ne soit pas biaisé. C'est un algorithme en deux boucles : pour $m = 1 : M$, puis pour $t = 0 : T$,

$$\widehat{\nabla f}^{t+1,m} = \nabla f_{I_t}(\theta_{t,m}) - \nabla f_{I_t}(\tilde{\omega}_m) + \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\tilde{\omega}_m)$$

$$\theta_{t+1,m} = \theta_{t,m} - \eta \widehat{\nabla f}^{t+1,m}$$

puis $\omega_{m+1} = \theta_{T+1,m}$.

Cet algorithme est coûteux car à chaque boucle en m, il faut calculer une somme de n termes pour définir la variable de contrôle. Un autre d"savantage c'est qu'il y a un peu plus de paramètres à régler (T, M). Mais l'avantage c'est que l'on n'a pas besoin de garder tout le vecteur des gradients. En complexité, on trouve des résultats en fortement convexe et on obtient $O(-n \log \epsilon)$ (ici, c'est M qui joue le rôle du $K_{\max}$, on regarde la trajectoire le long des ω_m)

Pierre: Après vérification, ce qui sert de $K_{\max}$ serait plutôt le produit MT. Dans l'algorithm original, la sortie est bien le dernier ω calculé. Par contre, dans l'article de Reddi, ils renvoient un sortie un θ tiré au hasard parmi tous les $\theta_{t,m}$ calculés. En pratique on se donne plutot $K_{\max}$ et T et on choisit M en conséquence. Dans leur analyse, ils prennent T de l'ordre de n, qui est un choix classique en pratique

. En convexe (voir Reddi, Sra et al.) et non convexe, pour une version un peu modifiée de l'algorithme ils ont $O(\sqrt{n}/\epsilon)$ et $O(n^{2/3}/\epsilon)$. Ils ont fait aussi en proximal non convexe (i.e. terme de pénalité non smooth en plus) et ont un $O(n^{2/3}/\epsilon)$ mais pour une version mini-batch de l'algorithme i.e. on ne tire pas qu'un seul indice I_t mais on fait une moyenne sur b tirages, avec $b = b_n$ (mais évidemment moins lourd que $b = n$!). *a vérifier : ils prendraient* $b = n^{2/3}$.

SAGA (Defazio, Bach, et al; NIPS, 2014) L'idée c'est de débiaiser SAG, via l'update

$$\theta_{t+1} = \theta_t - \eta \left(\nabla f_{I_t}(\theta_t) - Gmem_{t,I_t} + \frac{1}{n} \sum_{i=1}^{n} Gmem_{t,i} \right)$$

où $Gmem$ est l'analogue de S pour le gradient. Là encore, il y a un pbme de coût en mémoire. En convexe, ils ont $O(n/\epsilon)$; en fortement convexe ils ont $O(-n \log \epsilon)$; en non convexe (Reddi, Sra, etal 2016), ils ont $O(n^{2/3}/\epsilon)$; et dans le cas mini-batch (constant b - ce qui d'un point de vue ne change pas la complexité puisqu'on peut parlléliser) on obtient la même chose. Comme dans FIEM, on a deux indices indépendants, I_k, J_k. SAGA-Proximal existe aussi pour le cas non-smooth, et on retrouve en non convexe à la condition de prendre des mini batchs $b = n^{2/3}$, une complexité en $O(n^{2/3}/\epsilon)$. [en fait, il a $O(n/(\sqrt{b}\epsilon))$ sous la condition que $b < n^{2/3}$ et donc on prend le plus grand b].

MISO (Mairal) - à rapprocher de **FINITO** de Defazio. **c'est i-EM** On est dans le cas où on veut minimiser une moyenne $n^{-1} \sum_{i=1}^{n} f_i(\theta)$ ET on a l'existence de fonctions $g_i(\theta, \theta')$ telles que $f_i \leq g_i(, \theta')$ pour tout θ', et $f_i(\theta) = g_i(\theta, \theta)$. L'algo MM de base est de définir

$$\tau^{k+1} = \text{Argmin} \sum_{i=1}^{n} g(\theta, \tau^k)$$

ON se définit aussi un vecteur de fonctions memory $\hat{g}_t$: si $I_t = i$, alors $\hat{g}_{t,i} = \hat{g}_{t-1,i}$, et si $I_t = i$, $\hat{g}_{t,i} = g_i(\cdot, \theta^t)$. Puis ensuite, on fait l'analogue de SAG

$$\theta_{t+1} = \text{Argmin}_\theta \sum_{i=1}^{n} \hat{g}_{t,i}(\theta)$$

1.2 V0 d'une introduction

Pierre: Source principale : introduction des articles Karimi et al. (2019b), Chen et al. (2018) et Karimi et al. (2019a)

Pierre: Des travaux récents à inclure seraient des articles de Julien Mairal notamment, comme Kulunchakov and Mairal (2019), où ils regardent la convergence de SAGA/SVRG (et des variantes accélérées), sur un problème où

$$f_i(x) = \mathbb{E}[\widetilde{f}_i(x, \rho_i)]$$

Chez eux, ρ_i est une pertubation aléatoire qui dans certains cas sont introduites volontairement pour des soucis de généralisations, ou autre. Ils étudient les bornes non asymptotiques des algorithmes (un peu modifiés) où ils remplacent $\nabla f_{i_k}(x_k)$ par $g_k \stackrel{\text{def}}{=} \widetilde{f}_{i_k}(x, \rho_k)$ où ρ_k est une perturbation "tirée" aléatoirement. Leur hypothèse sur les perturbations sont que :

$$\mathbb{E}[g_k|\mathcal{F}_k] = \nabla f_{i_k}(x_k) \quad \mathbb{E}[\|g_k - \nabla f_{i_k}(x_k)\|] \leq \sigma$$

Chez eux, il ne s'agit pas de réduire l'erreur en faisant du MC ou autre, mais juste de regarder comment se répercutent les perturbations. Ils regardent les cas convexes et fortement convexes et obtiennent deux types de résultat: pour un pas constant, ils ont une partie convergente, et une partie liée aux perturbations qui ne tend pas vers 0 (avec Kmax), mais avec une suite de pas décroissante, les deux parties tendent vers 0. Il faudrait que je lise plus en détail pour comprendre dans les applications, d'où viennent ces perturbations., et voir quels autres articles avant eux ont traité du problème.

Many problems in machine learning boil down to solving an opitmization problem of the form:

$$\min_\theta \frac{1}{n}\sum_{i=1}^{n} l_i(\theta) + R(\theta),$$

where l_i is a loss function associated to the i-th observation and R is a regularization term. When the number of observations n becomes too big, typical algorithms like gradient descent become infeasible, and one has to fall back to stochastic gradient descent and its variance reduced counterparts: MISO (Mairal (2015)), SAG (Schmidt et al. (2017)), SAGA (Defazio et al. (2014)) or SVRG (Johnson and Zhang (2013)). The convergence of SAGA and SVRG has first been studied in the convex or strongly convex setting before being extended to the non-convex setting in Reddi et al. (2016b) and Reddi et al. (2016a).

In latent variable problems, l_i stands for an incomplete data likelihood. A classical tool to solve the optimization problem is the Expectation Maximization algorithm (Dempster et al. (1977)) which is divided in two steps: an E step that computes a surrogate function using the expectation of the sufficient statistics under the a posteriori distribution, given the current parameter, and a M step that maximizes this surrogate function. Its global convergence to a stationary point has been studied in Wu (1983), and for specific models, recent works has focused on giving non-asymptotic guarantees for the limit point of the algorithm (Balakrishnan et al. (2017), Xu et al. (2016)).

Again, in a large scale scenario, either the E step or the M step can become infeasible. In a high dimensional setting, the M step can be replaced by a gradient ascent iteration (Wang et al. (2015)), and incremental methods like SVRG has been used to reduce computational complexity (Zhu et al. (2017)).

In a big data setting, the E step becomes infeasible and incremental methods has also been developped to avoid a pass along the whole data set at each step: first in the seminal paper Neal and Hinton (1998), only one sample is used at each iteration to build an approximation of the E step, which convergence was first studied in Gunawardana and Byrne (2005). An online version of the EM has been studied in Cappé and Moulines (2009), and variance reduction methods related to gradient algorithms have been adapted to the EM.

In Chen et al. (2018), the authors give an SVRG-like procedure for the E step and study both the local convergence, when the starting point of the algorithm is close to a stationary point, and the global convergence. In Karimi et al. (2019b), the authors give non-asymptotic rates for the precedent algorithm, as well as for a new EM algorithm where the E step is approximated by a procedure similar to SAGA. They recover the nonconvex rates stated in Reddi et al. (2016b) and Reddi et al. (2016a), and give non-asymptotic rates for iEM, reinterpreting the algorithm in the MISO framework.

In the case where the E step is infeasible because the expectation under the a posteriori distribution is intractable, the Monte Carlo EM has been developped in Wei and Tanner (1990), where the expectation is replaced by a monte carlo sum, using either direct sampling or a MCMC procedure. Its convergence has been studied in Chan and Ledolter (1995), Sherman et al. (1999) and Fort and Moulines (2003) when the complete data distribution belongs to the curved exponential family. Alternatively, in the SAEM algorithm, (Delyon et al. (1999)) the authors replace the E step with a stochastic approximation procedure, where all the simulations are used at each step, leading to a more cost efficient algorithm. The authors show the convergence of the algorithm to a maxima of the likelihood when a direct sampling is used, Kuhn and Lavielle (2004) extend this result using a MCMC sampling.

1.3 Biblio

Versions EM: EM original Dempster et al. (1977); EM en ligne Cappé and Moulines (2009), Cappé (2011); EM incrémental original (iem) Neal and Hinton (1998); MCEM Wei and Tanner (1990); SAEM Delyon et al. (1999), avec reprojections Kuhn and Lavielle (2004) et Allassonnière et al. (2010).

Versions incrémentales: En gradient : SAG Schmidt et al. (2017), SAGA Defazio et al. (2014), SVRG Johnson and Zhang (2013), MISO Mairal (2015); en MCMC : Dubey et al. (2016), Chatterji et al. (2018); en EM pour l'étape E, fiEM (adapté de SAGA) Karimi et al. (2019b) et sEM-VR (adapté de SVRG) Chen et al. (2018); en EM pour l'étape M (gradient EM) Zhu et al. (2017)

Preuves des algorithmes Gradient en non-convexe : SAGA Reddi et al. (2016b), SVRG Reddi et al. (2016a), version prox J. Reddi et al. (2016); EM Wu (1983); MCEM Chan and Ledolter (1995), Sherman et al. (1999), Fort and Moulines (2003); iEM Gunawardana and Byrne (2005); points limites de EM dans des modèles précis: Balakrishnan et al. (2017), Xu et al. (2016).

2 Motivation

2.1 The problem

sec:motivation

The problem to be solved is

$$\operatorname{Argmin}_{\theta \in \Theta} F(\theta), \qquad \text{where} \quad F(\theta) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_i(\theta) + \mathsf{R}(\theta), \tag{2}$$

eq:problem

and

$$\mathcal{L}_i(\theta) \stackrel{\text{def}}{=} -\log \int_{\mathsf{Z}} h_i(z) \exp\left(\langle s_i(z), \phi(\theta)\rangle - \psi_i(\theta)\right) \ \mu(\mathrm{d}z). \tag{3}$$

eq:def:loss

hyp:Param

H1. *1. $\Theta \subseteq \mathbb{R}^d$ is an open set.*

hyp:curvedexpo

2. $(\mathsf{Z}, \mathcal{Z})$ is a measurable space and μ is a σ-finite positive measure on $\mathcal{Z}$. The functions $\mathsf{R} : \Theta \to \mathbb{R}$, $\phi : \Theta \to \mathbb{R}^q$, $\psi_i : \Theta \to \mathbb{R}$, $s_i : \mathsf{Z} \to \mathbb{R}^q$, $h_i : \mathsf{Z} \to \mathbb{R}_+$ for all $i \in \{1, \ldots, n\}$ are measurable functions. Finally, for any $\theta \in \Theta$, $-\infty < \mathcal{L}_i(\theta) < \infty$.

Under H1-item 2, for any $\theta \in \Theta$ and $i \in \{1, \ldots, n\}$, the quantity $p_i(z;\theta)\,\mu(\mathrm{d}z)$ where

$$p_i(z;\theta) \stackrel{\text{def}}{=} h_i(z) \exp\left(\langle s_i(z), \phi(\theta)\rangle - \psi_i(\theta) + \mathcal{L}_i(\theta)\right).$$

defines a probability distribution on $\mathcal{Z}$.

hyp:bars

H2. *For all $\theta \in \Theta$ and $i \in \{1, \ldots, n\}$, the expectation*

$$\bar{s}_i(\theta) \stackrel{\text{def}}{=} \int_{\mathsf{Z}} s_i(z) \ p_i(z;\theta)\mu(\mathrm{d}z)$$

exists.

Let us define $\mathsf{Q}_i : \Theta \times \Theta \to \mathbb{R}$ by

$$\mathsf{Q}_i(\theta, \theta') \stackrel{\text{def}}{=} \psi_i(\theta) - \langle \bar{s}_i(\theta'), \phi(\theta)\rangle .$$

Then, the Jensen's inequality implies that for any $\theta, \theta' \in \Theta$,

$$\mathcal{L}_i(\theta) \leq \mathsf{Q}_i(\theta, \theta') + \mathcal{L}_i(\theta') - \psi_i(\theta') + \langle \bar{s}_i(\theta'), \phi(\theta')\rangle .$$

Therefore, for any $\theta, \theta' \in \Theta$,

$$F(\theta) \leq \overline{\psi}(\theta) - \langle \bar{s}(\theta'), \phi(\theta)\rangle + \mathsf{R}(\theta) + \left\{ \frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_i(\theta') - \overline{\psi}(\theta') + \langle \bar{s}(\theta'), \phi(\theta')\rangle \right\} \tag{4}$$

eq:MMequation

where

$$\bar{s} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} \bar{s}_i, \qquad \overline{\psi} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} \psi_i. \tag{5}$$

eq:def:bars

The RHS of (4) defines a family of majorizing functions of F on the whole set Θ; this family is indexed by $\theta' \in \Theta$ and for any θ', the majorizing function is equal to F at the point $\theta = \theta'$.

hyp:Tmap

H3. *1. Let $\mathcal{S} \subseteq \mathbb{R}^q$ be a measurable open set such that*

$$\mathcal{S} \supset \left\{ \frac{1}{n} \sum_{i=1}^{n} u_i, \; u_i \in Conv(\bar{s}_i(\Theta)) \right\}.$$

For any $s \in \mathcal{S}$

$$\operatorname{Argmin}_{\theta \in \Theta} \; \left(\overline{\psi}(\theta) - \langle s, \phi(\theta) \rangle + \mathsf{R}(\theta)\right),$$

is a (non empty) singleton denoted by $\{\mathsf{T}(s)\}$.

2.2 An EM algorithm

sec:EM

From (4), a natural idea for solving (2) is the use of a MM algorithm defined as follows: define the sequence $\{\tau^k, \; k \in \mathbb{N}\}$ by $\tau^0 \in \Theta$, and for any $k \geq 0$,

$$\tau^{k+1} \stackrel{\text{def}}{=} \mathsf{T}(\bar{s}(\tau^k)). \tag{6}$$

eq:exact:update:tau

Starting from the current point τ^k, the algorithm first compute a point in $\bar{s}(\Theta)$ through the expectation $\bar{s}$, and then apply the map T to obtain the new point τ^{k+1}. It can therefore be equivalently defined in the $\bar{s}(\Theta)$-space, as follows: define $\{\bar{s}^k, \; k \in \mathbb{N}\}$ by $\bar{s}^0 \in \mathcal{S}$ and for any $k \geq 0$,

$$\bar{s}^{k+1} \stackrel{\text{def}}{=} \bar{s}\left(\mathsf{T}(\bar{s}^k)\right); \tag{7}$$

eq:exact:update:bars

upon noting that we have $\tau^{k+1} = \mathsf{T}(\bar{s}^{k+1})$ (for the initialization, choose $\tau^0 \stackrel{\text{def}}{=} \mathsf{T}(\bar{s}^0)$). The following lemma shows that there exists a natural Lyapunov function for these algorithms; it also establishes that this MM algorithm is an EM algorithm.

Data: $K_{\max} \in \mathbb{N}$, $\bar{s}^0 \in \mathcal{S}$
Result: The EM sequence: $\bar{s}^k, k = 0, \ldots, K_{\max}$
for $k = 0, \ldots, K_{\max} - 1$ **do**
  $\bar{s}^{k+1} = \bar{s} \circ \mathsf{T}(\bar{s}^k)$

Algorithm 1: EM algorithm

lem:lyapunovEM

Lemma 1. *Assume H1-item 1,item 2, H2 and H3-item 1. Let $\{\bar{s}^k, k \geq 0\}$ be given by (7) and set $\tau^k \stackrel{\text{def}}{=} \mathsf{T}(\bar{s}^k)$. Then for any $k \geq 0$,*

$$F(\tau^{k+1}) \leq F(\tau^k), \qquad F \circ \mathsf{T}(\bar{s}^{k+1}) \leq F \circ \mathsf{T}(\bar{s}^k),$$

and the sequence $\{\tau^k, k \geq 0\}$ is an EM sequence.

Proof. By definition of the map T,

$$\overline{\psi}(\tau^{k+1}) - \left\langle \bar{s}^{k+1}, \phi(\tau^{k+1}) \right\rangle + \mathsf{R}(\tau^{k+1}) \leq \overline{\psi}(\tau^k) - \left\langle \bar{s}^{k+1}, \phi(\tau^k) \right\rangle + \mathsf{R}(\tau^k).$$

In addition, by (4), we have

$$F(\tau^{k+1}) \leq \overline{\psi}(\tau^{k+1}) - \left\langle \bar{s}^{k+1}, \phi(\tau^{k+1}) \right\rangle + \mathsf{R}(\tau^{k+1}) + \frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_i(\tau^k) - \overline{\psi}(\tau^k) + \left\langle \bar{s}^{k+1}, \phi(\tau^k) \right\rangle .$$

Combining these inequalities yields

$$F(\tau^{k+1}) \leq \frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_i(\tau^k) + \mathsf{R}(\tau^k) = F(\tau^k).$$

This concludes the proof of the first statement. We now show this MM algorithm is equivalent to the EM algorithm. Given the current value τ^k, the E-step would compute at iteration k the quantity:

$$\begin{aligned}
\widetilde{Q}_i(\tau, \tau^k) &\stackrel{\mathrm{def}}{=} \int_{\mathsf{Z}} \log(h_i(z) \exp\left(\langle s_i(z), \phi(\tau)\rangle - \psi_i(\tau)\right) p_i(z;\tau^k)\mu(\mathrm{d}z) \\
&= \int_{\mathsf{Z}} \log(h_i(z)) \; p_i(z;\tau^k)\mu(\mathrm{d}z) + \mathsf{Q}_i(\tau, \tau^k)
\end{aligned}$$

and the M step defined the next value $\tau_{\mathrm{EM}}^{k+1} \stackrel{\mathrm{def}}{=} \mathrm{Argmax}_{\tau\in\Theta} \frac{1}{n}\sum_{i=1}^{n} \widetilde{Q}_i(\tau, \tau^k) - \mathsf{R}(\tau)$, that is

$$\begin{aligned}
\tau_{\mathrm{EM}}^{k+1} &= \mathrm{Argmax}_{\tau\in\Theta} \frac{1}{n}\sum_{i=1}^{n} \mathsf{Q}_i(\tau, \tau^k) - \mathsf{R}(\tau) \\
&= \mathrm{Argmax}_{\tau\in\Theta} \left\langle \bar{s}(\tau^k), \phi(\tau) \right\rangle - \overline{\psi}(\tau) - \mathsf{R}(\tau) = \mathsf{T}(\bar{s}^{k+1});
\end{aligned}$$

thus showing that $\tau^{k+1} = \tau_{\mathrm{EM}}^{k+1}$. □

The updating rule (7) shows that if the algorithm converges to $s^\star$, then $s^\star$ is a root of

$$s \mapsto h(s) \stackrel{\mathrm{def}}{=} \bar{s} \circ \mathsf{T}(s) - s. \tag{8}$$

eq:meanfield

The computational cost of the algorithm is proportional to n per iteration (since it requires the computation of $\bar{s}$, a sum over n terms); it is therefore untractable in the large scale learning framework. To overcome this drawback, and upon noting that

$$h(s) = \mathbb{E}\left[\bar{s}_I \circ \mathsf{T}(s) - s\right],$$

where I is a uniform random variable on $\{1, \ldots, n\}$, a natural idea is to replace $\bar{s} \circ \mathsf{T}(s^k)$ by a stochastic approximation involving the computation of one (or let us say, a fixed small number) expectation $\bar{s}_i \circ \mathsf{T}(s^k)$ at each iteration. Among possible strategies, we introduce in Section 2.3 and Section 2.4 two different algorithms where the deterministic algorithm producing $\bar{s}^{k+1} = \bar{s} \circ \mathsf{T}(\bar{s}^k)$ is replaced with a stochastic algorithm producing a sequence $\{\widehat{S}^k, k \geq 0\}$ satisfying

$$\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1} H(\widehat{S}^k, U_{k+1}) \tag{9}$$

eq:SAscheme

for two different choices of the field H and of the random sequence $\{U^k,\ k \in \mathbb{N}\}$. $\{\gamma^k,\ k \in \mathbb{N}\}$ is a deterministic positive stepsize sequence chosen by the user, and the random variable U_{k+1} is usually chosen such that

$$\mathbb{E}\left[H(\widehat{S}^k, U_{k+1}) | \widehat{S}^0, U_1, \ldots, U_k\right] = h(\widehat{S}^k),$$

(see Section 2.4) but not necessarily (see Section 2.3).

2.3 The incremental EM algorithm

sec:i-EM

Incremental EM (iEM) defines a sequence $\{\widehat{S}^k,\ k \in \mathbb{N}\}$ as described in algorithm 2.

Data: $K_{\max} \in \mathbb{N}$, $\widehat{S}^0 \in \mathcal{S}$, $\gamma_k \in (0, \infty)$ for $k = 1, \ldots, K_{\max}$
Result: The iEM sequence: $\widehat{S}^k, k = 0, \ldots, K_{\max}$

$\mathsf{S}_{0,i} = \bar{s}_i \circ \mathsf{T}(\widehat{S}^0)$ for all $i = 1, \ldots, n$;
$\widetilde{S}^0 = n^{-1} \sum_{i=1}^n \mathsf{S}_{0,i}$;
for $k = 0, \ldots, K_{\max} - 1$ **do**
  $I_{k+1} \sim \mathcal{U}(\{1, \ldots, n\})$;
  $\mathsf{S}_{k+1,i} = \mathsf{S}_{k,i}$ for $i \neq I_{k+1}$;
  $\mathsf{S}_{k+1,I_{k+1}} = \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$;
  $\widetilde{S}^{k+1} = \widetilde{S}^k + n^{-1}\left(\mathsf{S}_{k+1,I_{k+1}} - \mathsf{S}_{k,I_{k+1}}\right)$;
  $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1}(\widetilde{S}^{k+1} - \widehat{S}^k)$

Algorithm 2: The incremental EM (iEM) algorithm algo:iEM

Upon noting that for any $k \geq 0$,

$$\frac{1}{n}\left(\mathsf{S}_{k+1,I_{k+1}} - \mathsf{S}_{k,I_{k+1}}\right) = \frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k,i},$$

a trivial induction shows that $\widetilde{S}^k = \frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k,i}$ for any $k \geq 0$. Note however that the above algorithmic description allows the computation of this sum of n terms (at each iteration k) through a call to a single $\bar{s}_i$ at each iteration. $\widetilde{S}^{k+1}$ is an approximation of $\bar{s} \circ \mathsf{T}(\widehat{S}^k)$; more precisely, we have for any $k \geq 0$,

$$\widetilde{S}^k = \frac{1}{n}\sum_{i=1}^n \bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k,i})$$

where $\widehat{S}^{<0,i} \overset{\text{def}}{=} \widehat{S}^0$ for all $i \in \{1, \ldots, n\}$ and for $k \geq 0$,

$$\widehat{S}^{<k+1,i} = \widehat{S}^\ell, \begin{cases} \ell = k & \text{if } I_{k+1} = i \\ 1 \leq \ell \leq k-1 & \text{if } I_{k+1} \neq i, I_k \neq i, \ldots, I_{\ell+1} = i \\ \ell = 0 & \text{otherwise} \end{cases} \tag{10}$$

eq:memory:lastupdate

The above algorithm slightly extends the original incremental EM (see Neal and Hinton (1998)) by introducing a stepsize sequence. In Neal and Hinton (1998), we have $\gamma_{k+1} = 1$ for any $k \geq 0$ so that $\widehat{S}^k = \widetilde{S}^k = n^{-1} \sum_{i=1}^n \mathsf{S}_{k,i}$ for any $k \geq 0$.

This algorithm is defined as soon as $\mathsf{T}(\widehat{S}^k)$ exists that is $\widehat{S}^k \in \mathcal{S}$ at each iteration. Based on H3item 1, a sufficient condition is $\mathcal{S} = \mathbb{R}^q$ or $\gamma_k \in (0,1]$ for any k (note that in the original incremental EM by Neal and Hinton (1998), $\widehat{S}^k \in \mathcal{S}$ for any k under the assumption H3item 1).

In the literature, several variants of the EM algorithm share a similar procedure, meaning that at each step, a stochastic approximation of an expectation is defined through

$$\widehat{S}^k = (1-\gamma_k)\widehat{S}^{k-1} + \gamma_k H_{k+1}$$

where H_{k+1} is in the convex hull of $\cup_i s_i(\mathsf{Z})$; and then the updated parameter is obtained by $\mathsf{T}(\widehat{S}^k)$. There exist different ways to ensure $\widehat{S}^k \in \mathcal{S}$. In (Delyon et al., 1999, Section 4), $\mathcal{S}$ is assumed to contain the convex hull of $\cup_i s_i(\mathsf{Z})$ and the step sizes γ_k are in $(0,1)$. In (Kuhn and Lavielle, 2004, Theorem 1), (Allassonnière et al., 2010, Theorem 1) and (Donnet and Samson, 2007, Theorem 6), it is assumed $\gamma_k \in (0,1)$ and that the sequence $\widehat{S}^k$ remains in a compact subset of $\mathcal{S}$; the first two papers verify these assumptions in their applications. Furthermore, in (Cappé and Moulines, 2009, Assumption 1), $\gamma_k \in (0,1)$, and $\mathcal{S}$ is assumed to be convex and to contain the whole sequence $\widehat{S}^k$; they show that these conditions hold in their application as soon as the algorithm is suitably initialized. Finally, in the algorithm proposed in (Le Corff and Fort, 2013, Section 4.1) (which corresponds to the case $\gamma_k = 1$), it is again assumed that $\mathcal{S}$ contains the convex hull of $\cup_i s_i(\mathsf{Z})$.

`lem:iEM:equivalent`

Lemma 2. *Assume H1item 1-item 2, H2 and H3item 1. Let $\{\gamma^k,\ k \in \mathbb{N}\}$ be a positive stepsize sequence such that for any k, $\widehat{S}^k \in \mathcal{S}$. The incremental EM algorithm is equivalent to the following algorithm : initialize*

$$\widehat{S}^0 \in \mathcal{S}, \qquad \mathsf{S}_0 \stackrel{\text{def}}{=} (\bar{s}_1 \circ \mathsf{T}(\widehat{S}^0), \ldots, \bar{s}_n \circ \mathsf{T}(\widehat{S}^0)),$$

and then repeat for $k \geq 0$:

Draw: $I_{k+1} \sim \mathcal{U}(\{1,\ldots,n\})$,
Update: $\mathsf{S}_{k+1,i} = \mathsf{S}_{k,i}, i \neq I_{k+1}, \qquad \mathsf{S}_{k+1,I_{k+1}} = \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$,
Update: $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1}\left(\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \widehat{S}^k\right)$.

In addition, if $\mathcal{F}_0 \stackrel{\text{def}}{=} \sigma(\widehat{S}^0)$, and for any $k \geq 1$, $\mathcal{F}_k \stackrel{\text{def}}{=} \sigma\left(\widehat{S}^0, I_1, \ldots, I_k\right)$, then

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \widehat{S}^k \Big| \mathcal{F}_k\right] = h(\widehat{S}^k) + \left(1 - \frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\right).$$

Note that the sequence $\{\widehat{S}^k,\ k \in \mathbb{N}\}$ is not a Stochastic Approximation algorithm, but the bivariate sequence $\{(\widehat{S}^k, \mathsf{S}_{k,\cdot}), k \in \mathbb{N}\}$ is. While being an equivalent algorithmic description, the implementation is not equivalent: in Lemma 2, a sum over n terms is required for each update of $\widehat{S}^{k+1}$ (i.e. the computational cost is equivalent to the cost of the EM algorithm) while the first description of the algorithm is based on a recursive computation of this sum through the quantity $\widetilde{S}^{k+1}$.

2.4 The Fast Increment EM algorithm

sec:Fi-EM

Fast Incremental EM (FIEM) defines a sequence $\{\widehat{S}^k,\ k \in \mathbb{N}\}$ based on the scheme (9) where

$$H(\widehat{S}, U) \stackrel{\mathrm{def}}{=} \left(\bar{s}_J \circ \mathsf{T}(\widehat{S}) - \widehat{S}\right) + \left(\frac{1}{n}\sum_{i=1}^n \mathsf{S}_i - \mathsf{S}_J\right), \quad U \stackrel{\mathrm{def}}{=} (J, \mathsf{S}) \in \{1, \ldots, n\} \times \mathcal{S}^n.$$

This field can be seen as the sum of two terms: the natural field associated to the mean field h (8) when conditionally to $(\widehat{S}, \mathsf{S})$, J is sampled from the uniform distribution on the integers $\{1, \ldots, n\}$; and a random variable whose conditional expectation is zero. The fundamental property is that, conditionnally to $(\widehat{S}, \mathsf{S})$ these two terms are correlated through the use of the same random index J (see the variance reduction technique based on control variates, e.g. in (Glasserman, 2004, Section 4.1.).) We introduce a slight modification of the original algorithm, by using a sequence of coefficients $\{\lambda_k,\ k \in \mathbb{N}\}$ of real numbers. In the original algorithm (see Karimi et al. (2019b)), $\lambda_k = 1$.

FIEM is defined by algorithm 3. As in Section 2.3, it is easily seen that the

Data: $K_{\max} \in \mathbb{N}$, $\widehat{S}^0 \in \mathcal{S}$, $\gamma_k \in (0, \infty)$ for $k = 1, \ldots, K_{\max}$
Result: The iEM sequence: $\widehat{S}^k, k = 0, \ldots, K_{\max}$

$\mathsf{S}_{0,i} = \bar{s}_i \circ \mathsf{T}(\widehat{S}^0)$ for all $i = 1, \ldots, n$;
$\widetilde{S}^0 = n^{-1} \sum_{i=1}^n \mathsf{S}_{0,i}$;
for $k = 0, \ldots, K_{\max} - 1$ **do**
    $I_{k+1} \sim \mathcal{U}(\{1, \ldots, n\})$;
    $\mathsf{S}_{k+1,i} = \mathsf{S}_{k,i}$ for $i \neq I_{k+1}$;
    $\mathsf{S}_{k+1,I_{k+1}} = \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$;
    $\widetilde{S}^{k+1} = \widetilde{S}^k + n^{-1}\left(\mathsf{S}_{k+1,I_{k+1}} - \mathsf{S}_{k,I_{k+1}}\right)$;
    $J_{k+1} \sim \mathcal{U}(\{1, \ldots, n\})$;
    $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1}(\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \lambda_{k+1}(\widetilde{S}^{k+1} - \mathsf{S}_{k+1,J_{k+1}}))$

Algorithm 3: The Fast Incremental EM (FIEM) algorithm algo:FIEM

second and third instructions of the algorithm are a recursive computation of a sum of n terms: it holds for any $k \geq 0$ (see (10) for the definition of $\widehat{S}^{<k,i}$)

$$\widetilde{S}^k = \frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k,i} = \frac{1}{n} \sum_{i=1}^n \bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k,i}).$$

Here again, this algorithm is well defined as soon as $\widehat{S}^k \in \mathcal{S}$ for any $k \geq 0$, which ensures that $\mathsf{T}(\widehat{S}^k)$ exists. This is trivial when $\mathcal{S} = \mathbb{R}^q$; it holds true when $\lambda_k = 0$ and $\gamma_k \in (0, 1]$ under H3item 1. In the literature, there exist results which, similarly to FIEM, combine an update scheme of the form $\widehat{S}^{k+1} = (1 - \gamma_{k+1})\widehat{S}^k + \gamma_{k+1} H_{k+1}$ when H_{k+1} is **not** in the convex hull of $\bigcup_{i=1}^n s_i(\mathsf{Z})$, and a condition that $\widehat{S}^k$ remains in a definition set $\mathcal{S} \subseteq \mathbb{R}^q$ of a transformation: nevertheless, they assume $\mathcal{S} = \mathbb{R}^q$ (see e.g. Johnson and Zhang (2013), Defazio et al. (2014) and Chen et al. (2018)). Therefore, to our best knowledge, the case when $\mathcal{S} \neq \mathbb{R}^q$ is an open question.

Algorithm 3 is equivalent to the following one, but as for iEM, the computational cost of the implementation is not equivalent.

lem:FIEM:MeanField

Lemma 3. *Assume H1item 1-item 2, H2 and H3item 1. Let $\{\gamma_k,\ k \in \mathbb{N}\}$ be a positive stepsize sequence and $\{\lambda_k,\ k \in \mathbb{N}\}$ be a real valued sequence, such that $\widehat{S}^k \in \mathcal{S}$ for any k.*

1. *Fast Incremental EM is equivalent to the following algorithm : initialize*

$$\widehat{S}^0 \in \mathcal{S}, \qquad \mathsf{S}_{0,i} \stackrel{\text{def}}{=} \bar{s}_i \circ \mathsf{T}(\widehat{S}^0),\ 1 \leq i \leq n$$

and repeat for $k \geq 0$: draw independently $I_{k+1}, J_{k+1} \sim \mathcal{U}(\{1, \ldots, n\})$ and set

$$\mathsf{S}_{k+1,i} = \mathsf{S}_{k,i}, i \neq I_{k+1}, \qquad \mathsf{S}_{k+1,I_{k+1}} = \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k),$$
$$\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1} \left(\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \lambda_{k+1} \left\{ \frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,J_{k+1}} \right\} \right).$$

2. *Define the filtrations $\mathcal{F}_0 \stackrel{\text{def}}{=} \sigma(\widehat{S}^0)$, $\mathcal{F}_{1/2} \stackrel{\text{def}}{=} \sigma(\widehat{S}^0, I_1)$ and for $k \geq 1$,*

$$\mathcal{F}_k \stackrel{\text{def}}{=} \sigma\left(\widehat{S}^0, I_1, J_1, I_2, \ldots, I_k, J_k\right),$$
$$\mathcal{F}_{k+1/2} \stackrel{\text{def}}{=} \sigma\left(\widehat{S}^0, I_1, J_1, I_2, \ldots, I_k, J_k, I_{k+1}\right);$$

then $\widehat{S}^k \in \mathcal{F}_k$ and

$$\mathbb{E}\left[\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \lambda_{k+1} \left\{ \frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,J_{k+1}} \right\} \Big| \mathcal{F}_{k+1/2} \right] = h(\widehat{S}^k),$$

where h is given by (8).

Lemma 3 outlines that the sequence $\{(\widehat{S}^k, \mathsf{S}_{k,\cdot}), k \in \mathbb{N}\}$ is a Stochastic Approximation scheme. It necessitates the storage of a quantity $\mathsf{S}_{k,\cdot}$ whose size is proportional to n; in some cases (see e.g. (Schmidt et al., 2017, Section 4.1)), it is exactly a vector of length n.

sec:toymodel:exact

2.5 A toy example

2.5.1 Description

Pierre: On peut remettre la dépendence en i des X et des A

We observe n vectors $(Y_1, \ldots, Y_n) \in \mathbb{R}^y$, modeled as follows: conditionally to $(Z_1, \ldots, Z_n)$, the r.v. are independent with distribution $Y_i \sim \mathcal{N}_y(AZ_i, \mathrm{I}_y)$ where $A \in \mathbb{R}^{y \times p}$ is a deterministic matrix; $(Z_1, \ldots, Z_n)$ are i.i.d. under the distribution $\mathcal{N}_p(X\theta, \mathrm{I}_p)$, where $\theta \in \Theta \stackrel{\text{def}}{=} \mathbb{R}^q$ and $X \in \mathbb{R}^{p \times q}$ is a deterministic matrix. Here, X and A are known, and the θ is unknown; the objective is to compute an estimator defined as a solution of a (possibly) penalized maximum likelihood estimator, with penalty term $\mathsf{R}(\theta) \stackrel{\text{def}}{=} \lambda \|\theta\|^2/2$ for some $\lambda \geq 0$. If $\lambda = 0$, it is assumed that the rank of both X and AX is q. In this model, the r.v. $(Y_1, \ldots, Y_n)$ are i.i.d. with distribution $\mathcal{N}_y(AX\theta; \mathrm{I}_y + AA^T)$. The minimum of the function $\theta \mapsto F(\theta) \stackrel{\text{def}}{=} -n^{-1} \log g(Y_{1:n}; \theta) + \mathsf{R}(\theta)$ is unique and given by

$$\theta_\star \stackrel{\text{def}}{=} \left(\lambda \mathrm{I}_q + X^T A^T \left(\mathrm{I}_y + AA^T\right)^{-1} AX\right)^{-1} X^T A^T \left(\mathrm{I}_y + AA^T\right)^{-1} \overline{Y}_n,$$
$$\overline{Y}_n \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n Y_i.$$

Nevertheless, using the above description of the distribution of Y_i, this optimization problem can be casted into the general framework described in Section 2.1. The loss function (see (3)) is the normalized negative log-likelihood of the distribution of Y_i and is of the form (3) with

$$\phi(\theta) \stackrel{\text{def}}{=} \theta, \quad \psi_i(\theta) \stackrel{\text{def}}{=} \frac{1}{2} \theta^T X^T X \theta, \quad s_i(z) \stackrel{\text{def}}{=} X^T z.$$

Under the stated assumptions on X, the function $s \mapsto +\overline{\psi}(\theta) - \langle s, \phi(\theta) \rangle + R(\theta)$ is defined on $\mathcal{S} \stackrel{\text{def}}{=} \mathbb{R}^q$ and possesses an unique minimum given by

$$\mathsf{T}(s) \stackrel{\text{def}}{=} (\lambda \mathrm{I}_q + X^T X)^{-1} s.$$

Define

$$\Pi_1 \stackrel{\text{def}}{=} X^T (\mathrm{I}_p + A^T A)^{-1} A^T \in \mathbb{R}^{q \times y}$$
$$\Pi_2 \stackrel{\text{def}}{=} X^T (\mathrm{I}_p + A^T A)^{-1} X (\lambda \mathrm{I}_q + X^T X)^{-1} \in \mathbb{R}^{q \times q},$$

The a posteriori distribution $p_i(\cdot, \theta) \mathrm{d}\mu$ of the latent variable Z_i given the observation Y_i is a Gaussian distribution

$$\mathcal{N}_p\left((\mathrm{I}_p + A^T A)^{-1} (A^T Y_i + X\theta), (\mathrm{I}_p + A^T A)^{-1}\right),$$

so that for all $i \in \{1, \ldots, n\}$,

$$\bar{s}_i(\theta) \stackrel{\text{def}}{=} X^T (\mathrm{I}_p + A^T A)^{-1} (A^T Y_i + X\theta) = \Pi_1 Y_i + X^T (\mathrm{I}_p + A^T A)^{-1} X\theta \in \mathbb{R}^q,$$
$$\bar{s}_i \circ \mathsf{T}(s) = \Pi_1 Y_i + \Pi_2 s.$$

Therefore, the assumptions H1, H2, H3 and H4-item 1 are satisfied. Since $\phi \circ \mathsf{T}(s) = \mathsf{T}(s)$ then $B(s) = (\lambda \mathrm{I}_q + X^T X)^{-1}$ for any $s \in \mathcal{S}$. The assumption H4-item 2 holds with

$$v_{\min} \stackrel{\text{def}}{=} \frac{1}{\lambda + \text{max\_eig}(X^T X)}, \qquad v_{\max} \stackrel{\text{def}}{=} \frac{1}{\lambda + \text{min\_eig}(X^T X)};$$

here, max_eig and min_eig denotes resp. the maximum and the minimum of the eigenvalues. $\bar{s} \circ \mathsf{T}(s) = \Pi_1 \overline{Y}_n + \Pi_2 s$ thus showing that H4-item 3 holds with $L_i = L \stackrel{\text{def}}{=} \text{max\_eig}(\Pi_2)$. Finally, $s \mapsto B^T(s)\left(\bar{s} \circ \mathsf{T}(s) - s\right)$ is globally Lipschitz with constant

$$L_{\dot{V}} \stackrel{\text{def}}{=} \max \left| \text{eig}\left((\lambda \mathrm{I}_q + X^T X)^{-1} (\Pi_2 - \mathrm{I}_q) \right) \right|;$$

here eig denotes the eigenvalues. This concludes the proof of H4-item 4

2.5.2 EM, SA, i-EM and FIEM

Given the current value $\widehat{S}^k$, one iteration of EM update is given by the algorithm 4 For i-EM and FIEM, additional input variables are required such as the

Data: $\widehat{S}^k \in \mathcal{S}$, Π_1, Π_2 and $\overline{Y}_n$
Result: $\widehat{S}^{k+1}_{\text{EM}}$
1 $\widehat{S}^{k+1}_{\text{EM}} = \Pi_1 \overline{Y}_n + \Pi_2 \widehat{S}^k$

Algorithm 4: Toy example: one iteration of the EM algorithm. `algo:toy:EM`

storage of the quantities $\mathsf{S}_{k,i}$ for $i = 1, \cdots, n$; and the recursive computation of its mean $\widetilde{S}^k$. One iteration of these algorithms are given resp. in Algorithms 5 and 6. We will also run the *Stochastic Approximation (SA)* algorithm, which

Data: $\widehat{S}^k \in \mathcal{S}$, $\mathsf{S} \in \mathcal{S}^n$, $\widetilde{S} \in \mathcal{S}$; a step size $\gamma_{k+1} \in (0, 1]$; the matrices Π_1, Π_2; the examples $Y_1, \cdots, Y_n$
Result: $\widehat{S}^{k+1}_{\text{iEM}}$
1 Sample $I_{k+1} \sim \mathcal{U}(\{1, \cdots, n\})$;
2 Store $s = \mathsf{S}_{I_{k+1}}$;
3 Update $\mathsf{S}_{I_{k+1}} = \Pi_1 Y_{I_{k+1}} + \Pi_2 \widehat{S}^k$;
4 Update $\widetilde{S} = \widetilde{S} + n^{-1}(\mathsf{S}_{I_{k+1}} - s)$;
5 Update $\widehat{S}^{k+1}_{\text{iEM}} = \widehat{S}^k + \gamma_{k+1} \left(\widetilde{S} - \widehat{S}^k \right)$

Algorithm 5: Toy example: one iteration of the i-EM algorithm. `algo:toy:iEM`

corresponds to the FIEM one applied with $\lambda_{k+1} = 0$.

Data: $\widehat{S}^k \in \mathcal{S}$, $\mathsf{S} \in \mathcal{S}^n$, $\widetilde{S} \in \mathcal{S}$; a step size $\gamma_{k+1} \in (0,1]$ and a coefficient λ_{k+1}; the matrices Π_1, Π_2; the examples $Y_1, \cdots, Y_n$

Result: $\widehat{S}^{k+1}_{\text{FIEM}}$

1 Sample independently $I_{k+1}, J_{k+1} \sim \mathcal{U}(\{1, \cdots, n\})$;

2 Store $s = \mathsf{S}_{I_{k+1}}$;

3 Update $\mathsf{S}_{I_{k+1}} = \Pi_1 Y_{I_{k+1}} + \Pi_2 \widehat{S}^k$;

4 Update $\widetilde{S} = \widetilde{S} + n^{-1}(\mathsf{S}_{I_{k+1}} - s)$;

5 Update $\widehat{S}^{k+1}_{\text{FIEM}} = \widehat{S}^k + \gamma_{k+1}\left(\Pi_1 Y_{J_{k+1}} + \Pi_2 \widehat{S}^k - \widehat{S}^k + \lambda_{k+1}\left\{\widetilde{S} - \mathsf{S}_{J_{k+1}}\right\}\right)$

Algorithm 6: Toy example: one iteration of the FIEM algorithm. `algo:toy:FIEM`

2.5.3 Numerical analysis

The examples. In the numerical illustrations, we choose $Y_i \in \mathbb{R}^{15}$, $Z_i \in \mathbb{R}^{10}$ and $\theta \in \mathbb{R}^{20}$. The entries of the matrix A (resp. X) are obtained as i.i.d. samples from a centered distribution with variance 2 (resp. 1).

The problem. The data set is of length $n = 1e3$. The regularisation parameter λ is set to 0.1.

The i-EM algorithm. The i-EM algorithm is run with a constant stepsize sequence $\gamma_k = \gamma$; different values are compared. The maximal number of iterations is $K_{\max} = 12e3$.

The FIEM algorithm. The FIEM algorithm is run with a constant stepsize sequence $\gamma_k = \gamma$; different values are compared. Different values for λ_k are also considered. The maximal number of iterations is $K_{\max} = 12e3$.

Fair comparison. For a fair comparison, all the algorithms and all the runs for each of them, are started from the same initial value $\widehat{S}^0$ (which in turn fix the initial values of $\mathsf{S}_{0,\cdot}$ and $\widetilde{S}^0$ for i-EM and FIEM, see algorithms 2 and 3).
In addition, a path of length $K_{\max}$ for i-EM (resp. FIEM) require $K_{\max}$ (resp. $2\,K_{\max}$) samples from a uniform distribution on $\{1, \cdots, n\}$. For i-EM (resp. FIEM), when comparing D different strategies for the choice of some design parameters through a criterion involving Q paths of the algorithm, *(i)* we first sample Q (resp. $2Q$) such integer-valued sequences, *(ii)* and for each of the D values, we run Q paths of the algorithm by usung these Q sequences. This allows to compare the strategies by "freezing" the randomness due to the random choice of the examples, and to really explain the different behaviors by the values of the design parameters only.

Role of the stepsize in i-EM For different values of γ, a single path of i-EM is run. Figure 1 shows $k \mapsto \|\theta_k - \theta_\star\|$, where $\theta_k = (\lambda \mathrm{I}_q + X'X)^{-1}\widehat{S}^k_{\text{iEM}}$.

On the left plot, values larger than one are considered $\gamma \in \{2^{0.6}, 2^{0.4}, 2^{0.2}, 1\}$, and the first 20 iterations are displayed; on the center and the right plot, values smaller than one are considered $\gamma \in \{1, 2^{-1}, \cdots, 2^{-4}\}$ and the path from $k = 0$ to $k = K_{\max}$ is displayed (right) with a zoom on the first iterations (center). The conclusion is that, for the burn-in phase, the best value is $\gamma = 1$; but at convergence, many strategies yield the same convergence rate. As a comparison, we also display on Figure 1[right], the error $\|\theta_{k,\mathrm{EM}} - \theta_\star\|$ obtained along the EM sequence (started at the same value $\widehat{S}^0$) - here again, $\theta_{k,\mathrm{EM}} = (\lambda \mathrm{I}_q + X'X)^{-1} \widehat{S}^k_{\mathrm{EM}}$; for a fair comparison, iteration ℓ of EM is compared to iteration ℓn of i-EM (the comparison is based on the use of the same number of examples i.e. one per iteration of i-EM and n per iteration for EM). Not surprisingly, in the convergence phase at least, EM is more efficient than i-EM.

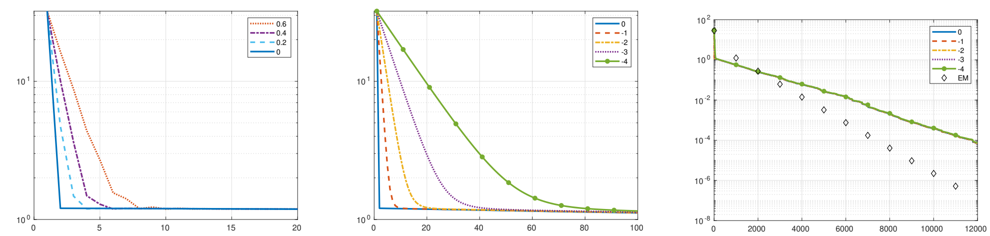

Figure 1: i-EM with a constant stepsize sequence $\gamma_k = \gamma$. A path of the error $k \mapsto \|\theta_k - \theta_\star\|$ along different runs of i-EM obtained with ifferent values of $\gamma = 2^\ell$ are considered. [left] $\ell \in \{0, \ldots, 0.6\}$; [center,right] $\ell \in \{0, \cdots, -4\}$; the plot in the center is a zoom, on the first 100 iterations, of the right one.

fig:toy:iEM:gamma-1

In the convergence phase, we compare the role of the constant γ on the variability of the path $k \mapsto \|\theta_k - \theta_\star\|$. To that goal, 100 independent paths are obtained, for different values of $\gamma \in \{1, 1/2, 1/4\}$. The results are displayed on Figure 2, for $k = \{25, 50, 75\}$ on the left, and $k \in \{8e3, 10e3, 12e3\}$ on the right. In the convergence phase, the results obtained with different values of γ are equivalent; in the burn-in phase, the bias is lower with $\gamma = 1$ but the variability is weakened with smaller values of γ.

Role of γ in FIEM. FIEM is run with a constant sequence of coefficient $\lambda_k = 1$; different strategies for the constant stepsize sequence are compared. Figure 3 displays the error $k \mapsto \|\theta_k - \theta_\star\|$ along FIEM paths obtained with $\gamma = \{1, 1\mathrm{e}{-1}, 1\mathrm{e}{-2}, \ldots, 1\mathrm{e}{-5}, \gamma_{\mathrm{GFM}}, \gamma_{\mathrm{KM}}\}$ where γ_{GFM} and γ_{KM} are given resp. in Corollary 5 and Eq.(94). We can observe the burn-in phase (left) and the convergence phase (right). Large values of γ, while being appealing in terms of convergence rate, may cause numerical instability as shown on the left for $\gamma = 1$ and $\gamma = 0.1$. The value γ_{GFM} is far better than the value γ_{KM} since it is larger and looks adequately tuned (it prevents from numerical instability and ensures a better convergence rate than the one obtained with γ_{KM}). In this example,

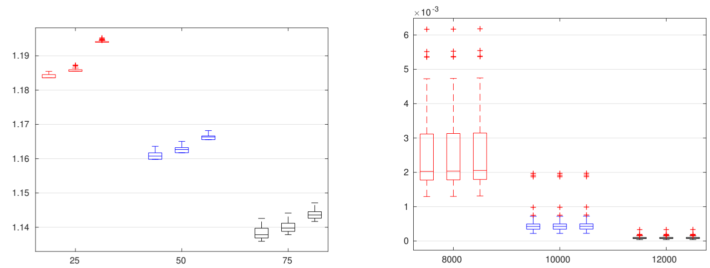

Figure 2: i-EM. Boxplot, over 100 independent realizations, of the error $\|\theta_k - \theta_\star\|$ when $k = \{25, 50, 75\}$ (left) and $k \in \{8e3, 10e3, 12e3\}$ (right). Each group compares three cases: $\gamma = 1, 0.5, 0.25$ from left to right.

fig:toy:iEM:gamma-2

$\gamma_{\mathrm{GFM}} \approx 1.32\mathrm{e}{-3}$ and $\gamma_{\mathrm{KM}} \approx 3.44\mathrm{e}{-6}$.

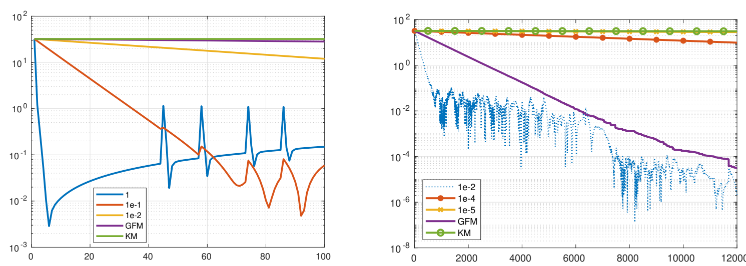

Figure 3: FIEM, with a constant coefficient $\lambda_k = 1$ and different strategies for a constant stepsize sequence: $\gamma_k = \gamma \in \{1, 1\mathrm{e}{-1}, \ldots, 1\mathrm{e}{-5}, \gamma_{\mathrm{GFM}}, \gamma_{\mathrm{KM}}\}$. [right] the path $k \mapsto \|\theta_k - \theta_\star\|$ is displayed, with [left] a zoom on the burn-in phase.

fig:toy:FIEM:gamma

Role of λ in FIEM. For a constant stepsize sequence $\gamma_k = \gamma$ fixed to $\gamma = \gamma_{\mathrm{GFM}}$ (see Corollary 5), we compare different strategies for the coefficients λ_k: the case $\lambda_k = 0$ for any k (in that case, FIEM is SA), the case $\lambda_k = 1$ for any k, and the case $\lambda_k = \lambda_k^\star$ (see Section 2.6). The criterion for the comparison, is the behavior of the error $\|\theta_k - \theta_\star\|$: the boxplots, obtained with 200 independent realizations, for $k = 3e3, 6e3, 8e3, 10e3, 12e3$ are displayed on Figure 4. On Figure 4[left], each group displays the strategies $\lambda = 0$ and $\lambda = 1$ (from left to right); it is clear that SA has a poor behavior when compared to FIEM, thus illustrating that the introduction of an unbiased additional field in the FIEM update rule, improves the convergence, both in terms of rate and of variability at convergence. On Figure 4[right], each group displays the strategies $\lambda = 1$ and $\lambda_k = \lambda_k^\star$ (from left to right); they look equivalent.

Nevertheless, the value $\lambda_k^\star$ is defined as optimizing another criterion: in

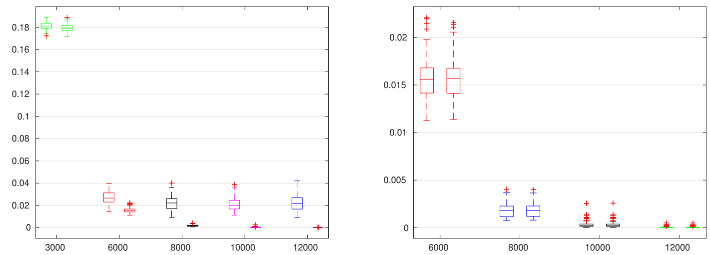

Figure 4: FIEM. Boxplot, over 100 independent realizations, of the error $\|\theta_k - \theta_\star\|$ when $k = \{3e3, 6e3, 8e3, 10e3, 12e3\}$. On the left, the strategies $\lambda = 0$ and $\lambda = 1$ are compared; on the right, the strategies $\lambda = 1$ and $\lambda_k = \lambda_k^\star$ are compared.

fig:toy:FIEM:lambda-1

the update rule $\widehat{S}^{k+1}_{\text{FIEM}} = \widehat{S}^{k}_{\text{FIEM}} + \gamma_{k+1} H_{k+1}$, $\lambda^\star_{k+1}$ provides the minimum of $\mathbb{E}\left[\|H_{k+1}\|^2 | \mathcal{F}_k\right]$. Therefore, we now compare the strategies $\lambda = 1$ and $\lambda_k = \lambda_k^\star$ by displaying a Monte Carlo approximation of the expectation $\mathbb{E}\left[\|H_{k+1}\|^2\right]$ for $k = 2e3, \ldots, 6e3$ (see Figure 5); the Monte Carlo approximation is computed with $1e3$ independent runs. A strategy is all the more better that the value of this criterion is small (see the role played by this quantity in the computation of the error rate, sketch of proof of Theorem 4, page 23).

Not surprisingly, there is a gain in using $\lambda^\star_{k+1}$. Nevertheless, this optimal value - while being computationally available in this toy example - is in general intractable as it is defined as a correlation coefficient; numerical approximation strategies can be derived, but it is not clear if, with this approximation, the benefit will be significant with respect to the simple rule $\lambda = 1$.

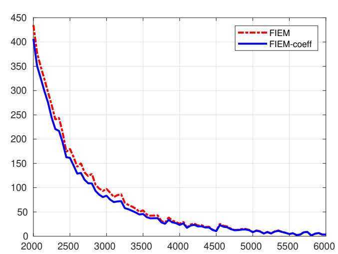

Figure 5: FIEM. For $k = 2e3, \ldots, 6e3$, a Monte Carlo approximation of the conditional expectation $\gamma_{k+1}^{-2}\mathbb{E}\left[\|\widehat{S}^{k+1}_{\text{FIEM}} - \widehat{S}^{k}_{\text{FIEM}}\|^2\right]$ along two different FIEM implementations: one with $\lambda = 1$ (red dash-dot line) and one with $\lambda = \lambda_k^\star$ (blue solid line)

fig:toy:FIEM:lambda-2

2.6 Beyond FIEM

sec:beyondFIEM

On the choice of the coefficient λ_{k+1}. Following the idea of the control variate technique, λ_{k+1} could be chosen as minimizing the conditional variance given $\mathcal{F}_{k+1/2}$, of the quantity

$$H_{k+1} \stackrel{\text{def}}{=} \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \lambda_{k+1} \left\{ \frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,J_{k+1}} \right\} \in \mathbb{R}^q.$$

• First assume that $q = 1$ i.e. $H_{k+1} \in \mathbb{R}$. In that case, the conditional variance is minimized at

$$\begin{aligned} \lambda_{k+1}^\star &\stackrel{\text{def}}{=} -\frac{\operatorname{Cov}\left(\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k, \frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,J_{k+1}} | \mathcal{F}_{k+1/2}\right)}{\operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,J_{k+1}} | \mathcal{F}_{k+1/2}\right)} \\ &= -\frac{n^{-1}\sum_{j=1}^n \bar{s}_j \circ \mathsf{T}(\widehat{S}^k)\left(\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,j}\right)}{n^{-1}\sum_{i=1}^n \mathsf{S}_{k+1,i}^2 - \left(n^{-1}\sum_{i=1}^n \mathsf{S}_{k+1,i}\right)^2} \end{aligned}$$

With this choice of the coefficient, we have

$$\operatorname{Var}(H_{k+1}|\mathcal{F}_{k+1/2}) \leq \operatorname{Var}(\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k | \mathcal{F}_{k+1/2}). \tag{11}$$

eq:variance:reduction

However, this value $\lambda_{k+1}^\star$ is untractable since it requires the computation of $\bar{s}_i \circ \mathsf{T}(\widehat{S}^k)$ for any i - which is unrealistic in the large scale learning framework. An idea, which can not guarantee anymore the property (11), is to replace $\bar{s}_i \circ \mathsf{T}(\widehat{S}^k)$ by $\bar{s}_i \circ \mathsf{T}(\widehat{S}^{\leq k+1,i}) = \mathsf{S}_{k+1,i}$; in that case we obtain $\lambda_{k+1} = 1$ - which is the choice in the original algorithm. Another idea is to mimic the optimal choice $\lambda_{k+1}^\star$ by defining

$$\begin{aligned} \lambda_{k+1} &\stackrel{\text{def}}{=} -\frac{\bar{s}_{L_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\left(\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,L_{k+1}}\right)}{n^{-1}\sum_{i=1}^n \mathsf{S}_{k+1,i}^2 - \left(n^{-1}\sum_{i=1}^n \mathsf{S}_{k+1,i}\right)^2} \\ &= -\frac{\bar{s}_{L_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\left(\widetilde{S}^{k+1} - \mathsf{S}_{k+1,L_{k+1}}\right)}{\check{S}^{k+1} - \left(\widetilde{S}^{k+1}\right)^2} \end{aligned}$$

where L_{k+1} is independent from $\mathcal{F}_{k+1/2}$ and J_{k+1}, and sampled from a uniform distribution on $\{1, \ldots, n\}$; and $\check{S}^{k+1}$ is defined iteratively by $\check{S}^0 \stackrel{\text{def}}{=} n^{-1}\sum_{i=1}^n (\mathsf{S}_{0,i})^2$ and

$$\check{S}^{k+1} = \check{S}^k + \frac{1}{n}\left(\mathsf{S}_{k+1,I_{k+1}}\right)^2 - \frac{1}{n}\left(\mathsf{S}_{k,I_{k+1}}\right)^2.$$

• In the case $q > 1$, the criterion could be to minimize the trace of the conditional variance-covariance matrix $\operatorname{Var}(H_{k+1}|\mathcal{F}_{k+1/2})$. In that case, the optimal coefficient is

$$\lambda_{k+1}^\star \stackrel{\text{def}}{=} -\frac{\operatorname{Tr}\left\{n^{-1}\sum_{j=1}^n \bar{s}_j \circ \mathsf{T}(\widehat{S}^k)\left(\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,j}\right)^T\right\}}{n^{-1}\sum_{i=1}^n \|\mathsf{S}_{k+1,i}\|^2 - \left\|\widetilde{S}^{k+1}\right\|^2}$$

and a numerical approximation is the choice $\lambda_{k+1} = 1$ - here again with no guarantees that it implies a variance reduction. As above in the scalar case, another strategy is to mimic the optimal choice $\lambda_{k+1}^\star$ by defining

$$\lambda_{k+1} - \frac{\mathrm{Tr}\left\{\bar{s}_{L_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\left(\widetilde{S}^{k+1} - \mathsf{S}_{k+1,L_{k+1}}\right)^T\right\}}{\check{S}^{k+1} - \left\|\widetilde{S}^{k+1}\right\|^2}$$

where L_{k+1} is independent from $\mathcal{F}_{k+1/2}$ and J_{k+1}, and sampled from a uniform distribution on $\{1, \ldots, n\}$; and $\check{S}^{k+1}$ is defined iteratively by $\check{S}^0 \stackrel{\mathrm{def}}{=} n^{-1}\sum_{i=1}^n \|\mathsf{S}_{0,i}\|^2$ and

$$\check{S}^{k+1} = \check{S}^k + \frac{1}{n}\left\|\mathsf{S}_{k+1,I_{k+1}}\right\|^2 - \frac{1}{n}\left\|\mathsf{S}_{k,I_{k+1}}\right\|^2 .$$

3 Error rates for a randomly stopped FIEM

sec:FIEM:complexity

3.1 Error rates: a general result

Given a maximal number of iterations $K_{\max}$, we address the choice of a random strategy to stop the algorithm at some (random) time $K \in \{1, \ldots, K_{\max}\}$. The quality criterion relies on an upper bound of the mean value of

$$\|\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2$$

which can be seen as an error when replacing a root of the function $s \mapsto h(s) = \bar{s} \circ \mathsf{T}(s) - s$ (see (8)) by a current estimate $\widehat{S}^k$. Under H4, this criterion is also related to the distance of the gradient of the Lyapunov function $V \stackrel{\text{def}}{=} F \circ \mathsf{T}$ (see Lemma 1) to zero:

$$\|\dot{V}(\widehat{S}^k)\|^2.$$

We derive explicit controls in Theorem 4 for a family of random stopping rules and then discuss the choice of this distribution.

hyp:model hyp:regV

H4. *1. The functions ϕ, ψ and R are continuously differentiable on Θ. T is continuously differentiable on $\mathcal{S}$.*

hyp:regV:C1

2. For any $s \in \mathcal{S}$, $B(s) \stackrel{\text{def}}{=} (\phi \,\dot{\circ}\, \mathsf{T})(s)$ is a symmetric $q \times q$ matrix and there exist $0 < v_{min} \leq v_{max} < \infty$ such that for all $s \in \mathcal{S}$, the spectrum of $B(s)$ is in $[v_{min}, v_{max}]$.

hyp:Tmap:smooth

3. For any $i \in \{1, \ldots, n\}$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on $\mathcal{S}$ with constant L_i.

hyp:regV:DerLip

4. The function $s \mapsto B^T(s)\,(\bar{s} \circ T(s) - s)$ is globally Lipschitz on $\mathcal{S}$ with constant $L_{\dot{V}}$.

Under the additional assumptions that for any $s \in \mathcal{S}$, $\tau \mapsto L(s,\tau) \stackrel{\text{def}}{=} \overline{\psi}(\tau) - \langle s, \phi(\tau)\rangle + \mathsf{R}(\tau)$ is twice continously differentiable on Θ, $q \leq d$ and $\mathrm{rank}(\dot{\mathsf{T}}(s)) = q$, then $B(s)$ is a symmetric matrix and its minimal eigenvalue is positive (see Lemma 8). Lemma 9 in Section 3.4.5 provides a sufficient condition for H4-item 3, condition which was essentially given in Karimi et al. (2019b).

Theorem 4 provides an explicit control for the sum of two terms: *(i)* some cumulated weighted errors $\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right]$ along a FIEM sequence $\{\widehat{S}^k,\ k \in \mathbb{N}\}$ of length $K_{\max}$; and *(ii)* some cumulated weighted errors when approximating $\bar{s} \circ \mathsf{T}(\widehat{S}^k)$ by $n^{-1}\sum_{i=1}^n \mathsf{S}_{k+1,i}$ along a FIEM sequence $\{\widehat{S}^k,\ k \in \mathbb{N}\}$ of length $K_{\max}$.

We then provide two applications of this result, both of them consisting in tuning some design parameters (the choice of the stepsize sequence $\{\gamma_k,\ k \in \mathbb{N}\}$ for example) in order to have non-negative weights with sum equal to one, in the cumulated errors $\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right]$. With these applications, Theorem 4 provides an explicit control on how far the algorithm is from the limiting set (caracterized by $\{s \in \mathcal{S}, h(s) = 0\}$) when stopped at a random time $K \in \{1, \ldots, K_{\max}\}$. The "distance" to this set is given by $\mathbb{E}\left[\|h(\widehat{S}^K)\|^2\right]$.

theo:FIEM:NonUnifStop

Theorem 4. *Assume H1item 1-item 2, H2item 1-item 1 and H3 and H4-item 1 to H4-item 4. Define* $L^2 \stackrel{\mathrm{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$.

Let $K_{\max}$ *be a positive integer. Let* $\{\gamma_k,\ k \in \mathbb{N}\}$ *be a sequence of positive stepsize and consider the FIEM sequence* $\{\widehat{S}^k,\ k \in \mathbb{N}\}$ *obtained with* $\lambda_{k+1} = 1$ *for any* k*; and assume that* $\widehat{S}^k \in \mathcal{S}$ *for any* $k \leq K_{\max}$*.*

For any positive numbers $\beta_1, \ldots, \beta_{K_{\max}-1}$*, we have*

$$\sum_{k=0}^{K_{\max}-1} \alpha_k \ \mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right] + \sum_{k=0}^{K_{\max}-1} \delta_k \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] \\ \leq \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right],$$

where for any $k = 0, \ldots, K_{\max}-1$,

$$\alpha_k \stackrel{\mathrm{def}}{=} \gamma_{k+1} v_{min} - \frac{L_{\dot{V}}}{2}\gamma_{k+1}^2\left(1 + L^2\Lambda_k\right), \quad \delta_k \stackrel{\mathrm{def}}{=} \frac{L_{\dot{V}}}{2}\gamma_{k+1}^2\left(1 + \frac{L^2\Lambda_k}{(1+\beta_{k+1}^{-1})}\right),$$

with $\Lambda_{K_{\max}-1} = 0$ *and for* $k = 0, \ldots, K_{\max}-2$,

$$\Lambda_k \stackrel{\mathrm{def}}{=} \left(1 + \frac{1}{\beta_{k+1}}\right) \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \prod_{\ell=k+2}^{j} \left(1 - \frac{1}{n} + \beta_\ell + \gamma_\ell^2 L^2\right).$$

pageref:sketch

Proof. The detailed proof is provided in Section 3.4; we just give here a sketch of proof. Set

$$H_{k+1} \stackrel{\mathrm{def}}{=} \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \lambda_{k+1}\left\{\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,J_{k+1}}\right\},$$

so that $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1} H_{k+1}$. In Lemma 11, it is proved that V is regular enough so that

$$V(\widehat{S}^{k+1}) - V(\widehat{S}^k) - \gamma_{k+1}\left\langle H_{k+1}, \dot{V}(\widehat{S}^k)\right\rangle \leq \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2}\|H_{k+1}\|^2.$$

Taking the expectation, using a contraction inequality satisfied by V (see Lemma 11), and applying Lemma 3, yield

$$\mathbb{E}\left[V(\widehat{S}^{k+1})\right] - \mathbb{E}\left[V(\widehat{S}^k)\right] + \gamma_{k+1}\left(v_{\min} - \gamma_{k+1}\frac{L_{\dot{V}}}{2}\right)\mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right] \\ \leq \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2}\mathbb{E}\left[\|H_{k+1} - \mathbb{E}\left[H_{k+1}|\mathcal{F}_{k+1/2}\right]\|^2\right].$$

By summation from $k=0$ to $k = K_{\max}-1$, it holds

$$\sum_{k=0}^{K_{\max}-1} \gamma_{k+1}\left(v_{\min} - \gamma_{k+1}\frac{L_{\dot{V}}}{2}\right)\mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right] \leq \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^K_{\max})\right] \\ + \frac{L_{\dot{V}}}{2}\sum_{k=0}^{K_{\max}-1}\gamma_{k+1}^2 \mathbb{E}\left[\|H_{k+1} - \mathbb{E}\left[H_{k+1}|\mathcal{F}_{k+1/2}\right]\|^2\right]. \tag{12}$$

eq:condition:lambdak

Finally, in Lemma 12 and Proposition 13, we prove that the last term on the RHS is upper bounded by

$$\frac{L_{\dot V} L^2}{2} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1}^2 \Lambda_k \ \Big\{ \mathbb{E}\left[\| \bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k \|^2 \right] \\ -(1+\beta_{k+1}^{-1})^{-1} \ \mathbb{E}\left[\| \frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k) \|^2 \right] \Big\} \ .$$

□

When $\alpha_k \geq 0$, we have by Lemma 11,

$$\sum_{k=0}^{K_{\max}-1} \alpha_k \mathbb{E}\left[\| \dot V(\widehat{S}^k) \|^2 \right] \leq v_{\max}^2 \sum_{k=0}^{K_{\max}-1} \alpha_k \mathbb{E}\left[\| \bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k \|^2 \right]$$

so that Theorem 4 also provides an upper bound for the gradient of the objective function along the FIEM path.

In Sections 3.2 and 3.3, we discuss how to choose some design parameters so that for any $k \in \{0, \ldots, K_{\max}-1\}$, the coefficient α_k is non-negative and such that $A_{K_{\max}} \stackrel{\text{def}}{=} \sum_{k=0}^{K_{\max}-1} \alpha_k$ is positive and maximal. We then deduce from Theorem 4 that

$$\sum_{k=0}^{K_{\max}-1} \frac{\alpha_k}{A_{K_{\max}}} \mathbb{E}\left[\| \bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k \|^2 \right] \leq \frac{1}{A_{K_{\max}}} \left\{ \mathbb{E}\left[V(\widehat{S}^0) \right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}}) \right] \right\}, \tag{13}$$

eq:FIEM:error

which gives an upper bound on a weighted cumulated distance to $\{s \in \mathcal{S} : h(s) = \bar{s} \circ \mathsf{T}(s) - s = 0\}$ expressed by $\mathbb{E}\left[\| \bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k \|^2 \right]$. Note that the LHS is equal to

$$\mathbb{E}\left[\| \bar{s} \circ \mathsf{T}(\widehat{S}^K) - \widehat{S}^K \|^2 \right] \tag{14}$$

eq:FIEM:randomlecture

where $K \in \{0, \ldots, K_{\max}-1\}$ is a r.v. with distribution $\{\alpha_k / A_{K_{\max}}, 0 \leq k \leq K_{\max}-1\}$, and sampled independently of $\{\widehat{S}^k, 0 \leq k \leq K_{\max}-1\}$. Therefore, the RHS in Theorem 4 (and therefore, in the following propositions), provides an explicit control for a random stopping rule of FIEM.

3.2 A uniform random stopping rule with constant step-sizes

sec:FIEM:errorrate:case1

In Proposition 5, we propose to choose constant step sizes γ_k, depending on n; this strategy yields to the uniform weights $\alpha_k / A_{K_{\max}} = 1/K_{\max}$ in (13) and to the uniform distribution for the stopping time K in (14).

Set

$$
\begin{aligned}
\mathsf{E}_0 &\stackrel{\text{def}}{=} \frac{1}{v_{\max}^2 K_{\max}} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\dot{V}(\widehat{S}^k)\|^2\right] = \frac{1}{v_{\max}^2}\mathbb{E}\left[\|\dot{V}(\widehat{S}^K)\|^2\right] \;, \\
\mathsf{E}_1 &\stackrel{\text{def}}{=} \frac{1}{K_{\max}} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\bar{s}\circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right] = \mathbb{E}\left[\|\bar{s}\circ \mathsf{T}(\widehat{S}^K) - \widehat{S}^K\|^2\right] \;, \\
\mathsf{E}_2 &\stackrel{\text{def}}{=} \frac{1}{K_{\max}} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s}\circ \mathsf{T}(\widehat{S}^k)\|^2\right] \\
&= \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{K+1,i} - \bar{s}\circ \mathsf{T}(\widehat{S}^K)\|^2\right] \;,
\end{aligned}
$$

where $K \sim \mathcal{U}(\{1,\ldots,n\})$ is independent of $\mathcal{F}_{K_{\max}}$.

coro:optimal:sampling

Proposition 5 (following of Theorem 4)**.** *Let $\mu \in (0,1)$. There exists $C \in (0,1)$ such that for any $n \geq 2$ and $K_{\max} \geq 1$, we have*

$$
\mathsf{E}_0 \leq \mathsf{E}_1 + \frac{L_{\dot{V}}}{2L}\frac{\sqrt{C}\mathsf{E}_2}{v_{\min} n^{2/3}} \leq \frac{n^{2/3}}{K_{\max}} \frac{L}{\sqrt{C}(1-\mu)v_{\min}} \left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right) \;,
$$

where the FIEM sequence $\{\widehat{S}^k, k\in\mathbb{N}\}$ is obtained with $\gamma_\ell = \sqrt{C} n^{-2/3} L^{-1}$. The constant C can be chosen such that

$$
\sqrt{C}\left(\frac{1}{n^{2/3}} + \frac{1}{1-n^{-1/3}}\left(\frac{1}{n} + \frac{1}{1-C}\right)\right) \leq \frac{2L}{L_{\dot{V}}}\mu v_{\min} \;. \tag{15}
$$

eq:C:uniform

Upon noting that in (15), the LHS is an increasing function of C which is lower bounded by the increasing function $x \mapsto \sqrt{x}/(1-x)$, any constant C satisfying (15) is upper bounded by $C^+ \in (0,1)$ solving

$$
\sqrt{x} L_{\dot{V}} - 2L\mu v_{\min}(1-x) = 0 \;.
$$

There exist similar results in the literature, in the case $p_k = 1/K_{\max}$ for any k. In Karimi et al. (2019b), FIEM is run with a constant step size sequence equal to

$$
\gamma_{\mathrm{KM}} = \frac{v_{min}}{\max(6, 1+4v_{\min})\, \max(L_{\dot{V}}, L_1, \ldots, L_n)\; n^{2/3}};
$$

and the upper bound is as in Proposition 5 where the constant

$$
C_{\mathrm{GFM}} \stackrel{\text{def}}{=} \frac{L}{\sqrt{C}(1-\mu)v_{\min}}
$$

is replaced with (see (Karimi et al., 2019b, Theorem 2))

$$
C_{\mathrm{KM}} \stackrel{\text{def}}{=} \left(\max(6, 1+4v_{\min})\right)^2\; \max(L_{\dot{V}}, L_1, \ldots, L_n). \tag{16}
$$

eq:C:KM

Definition of C from an asymptotic point of view. Proposition 5 indicates how to fix the constant C which plays a role in the definition of the stepsize sequence $\{\gamma_k,\ k \in \mathbb{N}\}$ and in the control of the errors E_i. Based on an asymptotic point of view, another strategy which is only available for n large enough can be derived (see Section 3.4.2): choose

$$C_\star \stackrel{\text{def}}{=} \frac{1}{4}\left(\frac{v_{\min} L}{L_{\dot V}}\right)^{2/3} \; ; \tag{17}$$

eq:C:optimal:asymptotique

for any $\mu_\star \in (0,1)$, there exists $N_\star$ (depending upon $v_{\min}$, L, $L_{\dot V}$) such that for $n \geq N_\star$,

$$\mathsf{E}_0 \leq \mathsf{E}_1 + \frac{L_{\dot V}}{2L}\frac{\sqrt{C}\mathsf{E}_2}{v_{\min} n^{2/3}} \leq \mu_\star \, \frac{8}{3}\frac{n^{2/3}}{K_{\max}} \frac{L^{2/3} L_{\dot V}^{1/3}}{v_{\min}^{4/3}} \left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right) \,.$$

The definitions of C in Proposition 5 and of $C_\star$ in (17) are most often of no numerical interest since in many applications $v_{\min}$, L or $L_{\dot V}$ is unknown. Nevertheless, they attest that given a tolerance $\varepsilon > 0$, there exists a constant M depending upon $v_{\min}, L, L_{\dot V}$ such that

$$K_{\max} = M n^{2/3} \varepsilon^{-1} \Longrightarrow \mathsf{E}_0 \leq \mathsf{E}_1 + \frac{L_{\dot V}}{2L}\frac{\sqrt{C}\mathsf{E}_2}{v_{\min} n^{2/3}} \leq \varepsilon \left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right) \,.$$

In Proposition 6, we propose to choose constant step sizes γ_k depending both on n and $K_{\max}$; we obtain another control of the errors E_i computed along a FIEM path obtained with $\gamma_\ell \propto n^{-1/3} K_{\max}^{-1/3}$.

coro:optimal:sampling:Ketn

Proposition 6 (following of Theorem 4)**.** *Let $\mu \in (0,1)$. Choose $\lambda \in (0,1)$ and $C > 0$ such that*

$$\sqrt{C}\left(1 + C\left(1 + \frac{1}{1-\lambda}\right)\right) \leq \frac{2L}{L_{\dot V}} \mu v_{\min} \,.$$

Then for any $n, K_{\max} \geq 1$ such that $n^{1/3} K_{\max}^{-2/3} \leq \lambda / C$, we have

$$\mathsf{E}_0 \leq \mathsf{E}_1 + \frac{L_{\dot V}}{2L}\frac{\sqrt{C}}{v_{\min}}\frac{\mathsf{E}_2}{(n K_{\max})^{1/3}} \leq \frac{n^{1/3}}{K_{\max}^{2/3}} \frac{L\left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right)}{\sqrt{C}(1-\mu) v_{\min}} \,,$$

where the FIEM sequence $\{\widehat{S}^k, k \in \mathbb{N}\}$ is obtained with $\gamma_\ell = \sqrt{C} n^{-1/3} K_{\max}^{-1/3} L^{-1}$. There exists a constant $M \in (1, +\infty)$ depending upon $v_{\min}, L, L_{\dot V}, \mu$ such that for any $\varepsilon \in (0,1)$, we have

$$\frac{n^{1/3}}{K_{\max}^{2/3}} \frac{L}{\sqrt{C}(1-\mu) v_{\min}} \leq \varepsilon \,,$$

by setting $K_{\max} = M\sqrt{n}\varepsilon^{-3/2}$.

Note that when $K_{\max} \propto \sqrt{n}$ then $\gamma_\ell \propto 1/\sqrt{n}$: this second upper bound is obtained with a slower step size than what was required in Proposition 5.

Definition of C from an asymptotic point of view. Proposition 6 indicates how to fix the constant C which plays a role in the definition of the stepsize sequence $\{\gamma_k,\ k \in \mathbb{N}\}$ and in the control of the errors E_i. Based on an asymptotic point of view, another strategy which is only available for n large enough can be derived (see Section 3.4.3): choose

$$C_\star \stackrel{\text{def}}{=} \left(\frac{v_{\min} L}{2L_{\dot{V}}}\right)^{2/3} (1-\lambda_\star)^{2/3} , \tag{18}$$

eq:C:optimal:asymptotique:

where $\lambda_\star$ is the unique solution in $(0,1)$ of

$$\kappa \left(\frac{v_{\min} L}{2L_{\dot{V}}}\right)^{2} (1-\lambda_\star)^{2} - \lambda_\star^3 = 0$$

for $\kappa > 0$. For any $\mu_\star \in (0,1)$ and $\kappa > 0$, there exists $N_\star$ (depending upon $v_{\min}$, L, $L_{\dot{V}}$) such that for $n \geq N_\star$ such that $n^{1/3} K_{\max}^{-2/3} \leq \kappa$,

$$\begin{aligned}\mathsf{E}_0 &\leq \mathsf{E}_1 + \frac{L_{\dot{V}}}{2L} \frac{\sqrt{C}\mathsf{E}_2}{v_{\min}(nK_{\max})^{1/3}} \\ &\leq \frac{\mu_\star}{(1-\lambda_\star)^{1/3}} \, \frac{n^{1/3}}{K_{\max}^{-2/3}} \frac{4}{3} \left(\frac{2L^2 L_{\dot{V}}}{v_{\min}^4}\right)^{1/3} \left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right) .\end{aligned}$$

The definitions of C in Proposition 6 and of $C_\star$ in (18) are most often of no numerical interest since in many applications $v_{\min}$, L or $L_{\dot{V}}$ is unknown. But here again, they attest that given a tolerance $\varepsilon > 0$, there exists a constant M depending upon $v_{\min}, L, L_{\dot{V}}$ such that

$$\begin{aligned}&K_{\max} = M n^{1/2} \varepsilon^{-3/2} \\ &\qquad \Longrightarrow \mathsf{E}_0 \leq \mathsf{E}_1 + \frac{L_{\dot{V}}}{2L} \frac{\sqrt{C}\mathsf{E}_2}{v_{\min}(nK_{\max})^{1/3}} \leq \varepsilon \left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right) .\end{aligned}$$

As a corollary of Proposition 5 and Proposition 6, we have two upper bounds of the errors $\mathsf{E}_1, \mathsf{E}_2$: the first one is $O(n^{2/3} K_{\max}^{-1})$ and the second one is $O(n^{1/3} K_{\max}^{-2/3})$. Given a tolerance $\varepsilon > 0$, the first or second strategy will be chosen depending on how $n^{1/2}\varepsilon^{-3/2}$ and $n^{2/3}\varepsilon^{-1}$ compare i.e. how $\sqrt{\varepsilon}$ and $n^{-1/6}$ compare.

sec:FIEM:errorrate:case2

3.3 A non-uniform random stopping rule

For a $\{0, \ldots, K_{\max}-1\}$-valued random variable K, define

$$\begin{aligned}\mathsf{E}_3 &\stackrel{\text{def}}{=} \frac{1}{v_{\max}^2} \mathbb{E}\left[\|\dot{V}(\widehat{S}^K)\|^2\right] , \\ \mathsf{E}_4 &\stackrel{\text{def}}{=} \mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^K) - \widehat{S}^K\|^2\right] .\end{aligned}$$

Given a distribution $p_0, \ldots, p_{K_{\max}-1}$ for the r.v. K, we show how to fix the step sizes $\gamma_1, \ldots, \gamma_{K_{\max}}$ in order to deduce from Theorem 4 a control of the errors E_3 and E_4. The proof of Proposition 7 is in Section 3.4.1.

For $C \in (0,1)$ and $n \geq 2$, define the function $F_{n,C}$

$$F_{n,C}(x) \mapsto \frac{1}{Ln^{2/3}} x \left(v_{\min} - x f_n(C)\right) \ ,$$
$$f_n(C) \stackrel{\text{def}}{=} \frac{L_{\dot{V}}}{2L} \left(\frac{1}{n^{2/3}} + \frac{1}{1-n^{-1/3}} \left(\frac{1}{n} + \frac{1}{1-C} \right) \right) \ ;$$

$F_{n,C}$ is positive, increasing and continuous on $(0, v_{\min}/(2f_n(C))]$.

coro:given:sampling

Proposition 7 (following of Theorem 4)**.** *Let K be a $\{0, \ldots, K_{\max}-1\}$-valued random variable with positive weights $p_0, \ldots, p_{K_{\max}-1}$. Let $C \in (0,1)$ solving*

$$2\sqrt{C} f_n(C) = v_{\min} \ . \tag{19}$$

eq:FIEM:NonUnifStep:C

For any $n \geq 2$ and $K_{\max} \geq 1$, we have

$$\mathsf{E}_3 \leq \mathsf{E}_4 \leq n^{2/3} \ \max_k p_k \, \frac{4Lf_n(C)}{v_{\min}^2} \left(\mathbb{E}\left[V(\widehat{S}^0) \right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}}) \right] \right) \ ,$$

where the FIEM sequence $\{\widehat{S}^k, k \in \mathbb{N}\}$ is obtained with

$$\gamma_{k+1} = \frac{1}{n^{2/3}L} \ F_{n,C}^{-1} \left(\frac{p_k}{\max_\ell p_\ell} \frac{\sqrt{C} v_{\min}}{4L} \frac{1}{n^{2/3}} \right).$$

The constant C satisfying (19) is upper bounded by the unique point C^+ solving

$$v_{\min} L(1-x) - \sqrt{x} L_{\dot{V}} = 0 \ ;$$

thus showing that $L_{\dot{V}}(1-C^+)^{-1}/(2L) \leq f_n(C) \leq \sup_n f_n(C^+) < \infty$

Note that since $\sum_k p_k = 1$, we have $\max_k p_k \geq 1/K_{\max}$ thus showing that among the dstributions, this term is minimal with the uniform distribution. In that case, the results in Proposition 7 can be compared to the results of Proposition 5: the control evolves as $n^{2/3}/K_{\max}$; the constant C solving the equality in (15) in the case $\mu = 1/2$ is the same as the constant C solving (19), and as a consequence, it is easily seen by using the equation (19), that

$$\frac{4Lf_n(C)}{v_{\min}^2} = \frac{2L}{\sqrt{C} v_{\min}} = \frac{L}{\sqrt{C}(1-\mu) v_{\min}} \ , \qquad \mu = 1/2.$$

Finally, when p_k is constant, the step sizes given by Proposition 7 are constant as in Proposition 5; and they are equal since $F_{n,C}^{-1}(v_{\min}^2 n^{-2/3}/(4Lf_n(C))) = \sqrt{C} = v_{\min}/(2f_n(C))$.

Analogue de la proposition 6 à faire.

3.4 Proofs

sec:proofs

Define the filtrations, for $k \geq 0$,

$$\mathcal{F}_k \stackrel{\text{def}}{=} \sigma(\theta^0, I_1, J_1, I_2, \ldots, I_k, J_k), \qquad \mathcal{F}_{k+1/2} \stackrel{\text{def}}{=} \sigma(\theta^0, I_1, J_1, I_2, \ldots, I_k, J_k, I_{k+1});$$

note that $\widehat{S}^k \in \mathcal{F}_k$ and $\mathsf{S}_{k+1,\cdot} \in \mathcal{F}_{k+1/2}$. Throughout this section, we consider the FIEM algorithm (see Section 2.4) for a fixed positive stepsize sequence $\{\gamma_k,\ k \in \mathbb{N}\}$ and a real valued sequence of coefficients $\{\lambda_k,\ k \in \mathbb{N}\}$. Set

$$H_{k+1} \stackrel{\text{def}}{=} \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \lambda_{k+1} \left(\frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,J_{k+1}} \right);$$

and recall that for any $s \in \mathcal{S}$,

$$h(s) \stackrel{\text{def}}{=} \bar{s} \circ \mathsf{T}(s) - s. \tag{20}$$

eq:meanfield:rappeldef

3.4.1 Proof of Theorem 4

sec:proof:theo

By assumption, $\dot{V}$ is $L_{\dot{V}}$-Lipschitz, and we have

$$\begin{aligned} V(\widehat{S}^{k+1}) &\leq V(\widehat{S}^k) + \left\langle \widehat{S}^{k+1} - \widehat{S}^k, \dot{V}(\widehat{S}^k) \right\rangle + \frac{L_{\dot{V}}}{2} \|\widehat{S}^{k+1} - \widehat{S}^k\|^2 \\ &\leq V(\widehat{S}^k) + \gamma_{k+1} \left\langle H_{k+1}, \dot{V}(\widehat{S}^k) \right\rangle + \frac{\gamma_{k+1}^2 L_{\dot{V}}}{2} \|H_{k+1}\|^2. \end{aligned}$$

Taking the expectation yields, upon noting that $\widehat{S}^k \in \mathcal{F}_k$

$$\begin{aligned} &\mathbb{E}\left[V(\widehat{S}^{k+1})\right] - \mathbb{E}\left[V(\widehat{S}^k)\right] \\ &\leq \gamma_{k+1} \mathbb{E}\left[\left\langle \mathbb{E}\left[H_{k+1} | \mathcal{F}_k\right], \dot{V}(\widehat{S}^k) \right\rangle\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|H_{k+1}\|^2\right] \\ &\leq \gamma_{k+1} \mathbb{E}\left[\left\langle h(\widehat{S}^k), \dot{V}(\widehat{S}^k) \right\rangle\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|H_{k+1}\|^2\right] \\ &\leq -\gamma_{k+1} v_{min} \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|H_{k+1}\|^2\right] \\ &\leq -\gamma_{k+1} \left(v_{min} - \gamma_{k+1} \frac{L_{\dot{V}}}{2} \right) \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] \end{aligned}$$

where we used that $\mathbb{E}\left[H_{k+1} | \mathcal{F}_k\right] = h(\widehat{S}^k)$ (see Lemma 3) and Lemma 11. Furthermore, using Lemma 12, we have:

$$\begin{aligned} &\mathbb{E}\left[V(\widehat{S}^{k+1})\right] - \mathbb{E}\left[V(\widehat{S}^k)\right] \leq -\gamma_{k+1} \left(v_{min} - \gamma_{k+1} \frac{L_{\dot{V}}}{2} \right) \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] \\ &-\gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|\frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}}\|^2\right] \end{aligned}$$

By summing from $k=0$ to $k=K_{\max}-1$, we have by Proposition 13

$$\sum_{k=0}^{K_{\max}-1} \gamma_{k+1}\left(v_{min} - \gamma_{k+1}\frac{L_{\dot{V}}}{2}\right)\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] +$$
$$\leq \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right] + \frac{L_{\dot{V}}\, L^2}{2}\sum_{k=0}^{K_{\max}-2}\gamma_{k+1}^2 \Lambda_k \; \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right]$$
$$-\frac{L_{\dot{V}}}{2}\sum_{k=0}^{K_{\max}-1}\gamma_{k+1}^2\left(L^2\Xi_k + 1\right)\mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right],$$

where for $0 \leq k \leq K_{\max}-2$ and with the convention $\Lambda_{K_{\max}-1} = \Xi_{K_{\max}-1} = 0$,

$$\Lambda_k \stackrel{\text{def}}{=} \left(1 + \frac{1}{\beta_{k+1}}\right)\sum_{j=k+1}^{K_{\max}-1}\gamma_{j+1}^2 \; \left(\frac{n-1}{n}\right)^{j-k} \prod_{\ell=k+2}^{j}\left(1+\beta_\ell + \gamma_\ell^2 L^2\right)$$
$$\leq \left(1 + \frac{1}{\beta_{k+1}}\right)\sum_{j=k+1}^{K_{\max}-1}\gamma_{j+1}^2 \prod_{\ell=k+2}^{j}\left(1 - \frac{1}{n} + \beta_\ell + \gamma_\ell^2 L^2\right),$$
$$\Xi_k \stackrel{\text{def}}{=} \left(1 + \frac{1}{\beta_{k+1}}\right)^{-1}\Lambda_k.$$

Hence,

$$\sum_{k=0}^{K_{\max}-1}\{\gamma_{k+1}\left(v_{min} - \gamma_{k+1}\frac{L_{\dot{V}}}{2}\right) - \gamma_{k+1}^2\frac{L_{\dot{V}}L^2}{2}\Lambda_k\}\;\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right]$$
$$+\sum_{k=0}^{K_{\max}-1}\frac{\gamma_{k+1}^2 L_{\dot{V}}}{2}\{1+L^2\Xi_k\}\mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right]$$
$$\leq \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]. \quad (21)$$ eq:nonunifavg

3.4.2 Proof of Proposition 5

roof:coro:optimal:sampling

We consider the case when

$$\beta_\ell \stackrel{\text{def}}{=} \frac{1-\lambda}{n^{\mathsf{b}}}, \qquad \gamma_\ell^2 \stackrel{\text{def}}{=} \frac{C}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}}$$

for some $\lambda \in (0,1)$, $C>0$ and $\mathsf{b},\mathsf{c},\mathsf{d}$ to be defined in the proof in such a way that *(i)* $\alpha_k \geq 0$, *(ii)* $\sum_{k=0}^{K_{\max}-1}\alpha_k$ is positive and as large as possible.

With theses definitions, we have

$$1 - \frac{\rho_n}{n} \stackrel{\text{def}}{=} 1 - \frac{1}{n} + \beta_\ell + L^2\gamma_\ell^2 = 1 - \frac{1}{n}\left(1 - \frac{1-\lambda}{n^{\mathsf{b}-1}} - \frac{C}{n^{2\mathsf{c}-1}K_{\max}^{2\mathsf{d}}}\right)$$

and choose $(\mathsf{b}, \mathsf{c}, \mathsf{d}, \lambda, C)$ such that

$$\frac{1-\lambda}{n^{\mathsf{b}-1}} + \frac{C}{n^{2\mathsf{c}-1} K_{\max}^{2\mathsf{d}}} < 1 \;, \tag{22}$$

eq:proof:coro:optsample:a

which ensures that $\rho_n \in (0,1)$. Hence, for any $0 \le k \le K_{\max} - 2$,

$$\begin{aligned}
\Lambda_k &\le n^{\mathsf{b}} \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda} \right) \frac{C}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}} \sum_{j=k+1}^{K_{\max}-1} \left(1 - \frac{\rho_n}{n}\right)^{j-k-1} \\
&\le \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda} \right) \frac{C}{L^2 \rho_n} \frac{1}{n^{2\mathsf{c}-\mathsf{b}-1} K_{\max}^{2\mathsf{d}}} \;.
\end{aligned}$$

From this upper bound, we deduce for any $0 \le k \le K_{\max} - 1$: $\alpha_k \ge \underline{\alpha}$ where

$$\underline{\alpha} \overset{\text{def}}{=} \frac{\sqrt{C}}{L n^{\mathsf{c}} K_{\max}^{\mathsf{d}}} \left(v_{\min} - \frac{L_{\dot V}}{2L} \frac{\sqrt{C}}{n^{\mathsf{c}} K_{\max}^{\mathsf{d}}} - \frac{L_{\dot V}}{2L} \frac{C^{3/2}}{\rho_n \, n^{3\mathsf{c}-\mathsf{b}-1} K_{\max}^{3\mathsf{d}}} \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda} \right) \right) \;. \tag{23}$$

eq:proof:coro:optsample:d

Based on (23) and on (22), we choose $\mathsf{b} = 1$, $\mathsf{c} = 2/3$, $\mathsf{d} = 0$; which yields for $n \ge 1$, since $\rho_n = \lambda - C n^{-1/3}$

$$\underline{\alpha} \ge \frac{\sqrt{C}}{L n^{2/3}} \left(v_{\min} - \frac{L_{\dot V}}{2L} \frac{\sqrt{C}}{n^{2/3}} - \frac{L_{\dot V}}{2L} \frac{C^{3/2}}{\lambda - C n^{-1/3}} \left(\frac{1}{n} + \frac{1}{1-\lambda} \right) \right) \;.$$

Let $\mu \in (0,1)$. Fix $\lambda \in (0,1)$ and $C > 0$ such that (see (22) for the second condition)

$$\frac{L_{\dot V}}{2L} \sqrt{C} \left(\frac{1}{n^{2/3}} + \frac{C}{\lambda - C n^{-1/3}} \left(\frac{1}{n} + \frac{1}{1-\lambda} \right) \right) \le \mu v_{\min} \;, \qquad \frac{1}{n^{1/3}} < \frac{\lambda}{C} \;. \tag{24}$$

eq:proof:coro:optsample:b

This implies that $\underline{\alpha} \ge \alpha_\star \overset{\text{def}}{=} \sqrt{C}(1-\mu) v_{\min} / (L n^{2/3})$, and

$$\frac{1}{K_{\max} \, \alpha_\star} = \frac{n^{2/3}}{K_{\max}} \frac{L}{\sqrt{C}(1-\mu) v_{\min}} \;. \tag{25}$$

eq:proof:coro:optsample:c

We obtain the upper bound on E_1 and E_2 by using

$$\mathsf{E}_1 \le \frac{1}{K_{\max} \, \alpha_\star} \sum_{k=0}^{K_{\max}-1} \alpha_k \mathbb{E}\left[\| \bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k \|^2 \right] \;;$$

and, since $\delta_k \ge L_{\dot V} \gamma_{k+1}^2 / 2$ and $\alpha_\star \le \sqrt{C} v_{\min} (L n^{2/3})$,

$$\begin{aligned}
\frac{L_{\dot V} \sqrt{C}}{2 L n^{2/3}} \frac{1}{v_{\min}} \mathsf{E}_2 &\le \frac{L_{\dot V} \, C}{2 L^2 n^{4/3}} \frac{1}{K_{\max} \, \alpha_\star} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\| \frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k) \|^2 \right] \\
&\le \frac{1}{K_{\max} \, \alpha_\star} \sum_{k=0}^{K_{\max}-1} \delta_k \mathbb{E}\left[\| \frac{1}{n} \sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k) \|^2 \right] \;.
\end{aligned}$$

Let $\varepsilon \in (0,1]$. Since $n \geq 2$, the second condition in (24) is satisfied with $\lambda = C$; then, the RHS in (25) is upper bounded by ε as soon as

$$K_{\max} = \frac{n^{2/3}}{\varepsilon} \frac{L}{\sqrt{C}(1-\mu)v_{\min}}$$

where $C \in (0,1)$ satisfies (see the first condition in (24))

$$\sqrt{C}\left(1 + \frac{2^{1/3}}{2^{1/3}-1}\left(\frac{1}{2} + \frac{1}{1-C}\right)\right) \leq \frac{2L}{L_{\dot{V}}} \mu v_{\min} \,.$$

Asymptotic point of view for the choice of (λ, C). From (23) applied with $\mathsf{b} = 1$, $\mathsf{c} = 2/3$ and $\mathsf{d} = 0$, we have

$$n^{2/3}\underline{\alpha} = \mathcal{L}(n, C, \lambda) \stackrel{\text{def}}{=} \frac{\sqrt{C}}{L}\left(v_{\min} - \frac{L_{\dot{V}}}{2L}\frac{\sqrt{C}}{n^{2/3}} - \frac{L_{\dot{V}}}{2L}\frac{C^{3/2}}{\rho_n}\left(\frac{1}{n} + \frac{1}{1-\lambda}\right)\right) \,,$$

for any $\lambda \in (0,1)$, $n \geq 2$ and $C > 0$ such that $Cn^{-1/3} < \lambda$. The lower bound $\mathcal{L}(n, C, \lambda)$ is a signed quantity, and upon noting that $\rho_n = \lambda - C/n^{1/3}$, we have $\mathcal{L}(n, C, \lambda) \uparrow \mathcal{L}(\infty, C, \lambda)$ when $n \to \infty$ where

$$\mathcal{L}(\infty, C, \lambda) \stackrel{\text{def}}{=} \frac{\sqrt{C}}{L}\left(v_{\min} - \frac{L_{\dot{V}}}{2L}\frac{C^{3/2}}{\lambda}\frac{1}{1-\lambda}\right) \,.$$

It is easily seen that $\mathcal{L}(\infty, C, \lambda) \leq \mathcal{L}(\infty, C_\star, \lambda_\star)$ where [1]

$$\lambda_\star = \frac{1}{2}, \qquad C_\star \stackrel{\text{def}}{=} \frac{1}{4}\left(\frac{v_{\min} L}{L_{\dot{V}}}\right)^{2/3} \,.$$

Therefore, we have for any $\lambda \in (0,1)$, $C > 0$, $n \geq 2$ such that $Cn^{-1/3} < \lambda$,

$$\lim_n n^{2/3}\underline{\alpha} = \mathcal{L}(\infty, C_\star, \lambda_\star) \geq \mathcal{L}(n, C, \lambda) \,.$$

As a conclusion,

$$\lim_n n^{2/3}\underline{\alpha} = \mathcal{L}(\infty, C_\star, \lambda_\star) = \frac{3}{8}\frac{v_{\min}}{L}\left(\frac{v_{\min}L}{L_{\dot{V}}}\right)^{1/3} > 0 \,,$$

thus showing that there exists $N_\star$ such that for any $n \geq N_\star$, $\mathcal{L}(n, C_\star, \lambda_\star) > 0$; and the lower bound $\mathcal{L}(\infty, C_\star, \lambda_\star)$ is the optimal one among the bounds $\mathcal{L}(n, C, \lambda)$, for any $\lambda \in (0,1)$, $C > 0$, $n \geq 2$ such that $Cn^{-1/3} < \lambda$.

[1] $x \mapsto Ax - Bx^3$ is increasing on $[0, x_\star]$ and then decreasing; and $x_\star = A^{1/3}B^{-1/3}4^{-1/3}$. The function evaluated at $x_\star$ is equal to $3A^{4/3}B^{-1/3}4^{-1/3}/4$. Here, the result is applied with $C_\star = x_\star^2$, $A = v_{\min}/L$ and $B = 2L_{\dot{V}}/L^2$.

3.4.3 Proof of Proposition 6

coro:optimal:sampling:Ketn

We consider the case when

$$\beta_\ell \stackrel{\text{def}}{=} \frac{1-\lambda}{n^{\mathsf{b}}}, \qquad \gamma_\ell^2 \stackrel{\text{def}}{=} \frac{C}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}}$$

for some $\lambda \in (0,1)$, $C > 0$ and $\mathsf{b}, \mathsf{c}, \mathsf{d}$ to be defined in the proof in such a way that *(i)* $\alpha_k \geq 0$, *(ii)* $\sum_{k=0}^{K_{\max}-1} \alpha_k$ is positive and as large as possible.

With theses definitions, we have

$$\rho \stackrel{\text{def}}{=} 1 - \frac{1}{n} + \beta_\ell + L^2\gamma_\ell^2 = 1 - \frac{1}{n}\left(1 - \frac{1-\lambda}{n^{\mathsf{b}-1}} - \frac{C}{n^{2\mathsf{c}-1}K_{\max}^{2\mathsf{d}}}\right)$$

and choose $(\mathsf{b}, \mathsf{c}, \mathsf{d}, \lambda, C)$ such that

$$\frac{1-\lambda}{n^{\mathsf{b}-1}} + \frac{C}{n^{2\mathsf{c}-1}K_{\max}^{2\mathsf{d}}} \leq 1 \ , \tag{26}$$

eq:proof:coro:optsample:Ke

which ensures that $\rho \in (0,1]$. Hence, for any $0 \leq k \leq K_{\max} - 2$,

$$\Lambda_k \leq n^{\mathsf{b}}\left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\frac{C}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}} \sum_{j=k+1}^{K_{\max}-1} \rho^{j-k-1} \leq \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\frac{C}{L^2 n^{2\mathsf{c}-\mathsf{b}} K_{\max}^{2\mathsf{d}-1}} \ .$$

From this upper bound, we obtain the following lower bound on α_k, for any $0 \leq k \leq K_{\max} - 1$,

$$\alpha_k \geq \underline{\alpha} \stackrel{\text{def}}{=} \frac{\sqrt{C}}{L n^{\mathsf{c}} K_{\max}^{\mathsf{d}}}\left(v_{\min} - \frac{L_{\dot{V}}}{2L}\frac{\sqrt{C}}{n^{\mathsf{c}}K_{\max}^{\mathsf{d}}} - \frac{L_{\dot{V}}}{2L}\frac{C^{3/2}}{n^{3\mathsf{c}-\mathsf{b}}K_{\max}^{3\mathsf{d}-1}}\left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\right) \ .$$

Based on this inequality and on the condition (26), we choose $\mathsf{b} = 1$ and $\mathsf{c} = \mathsf{d} = 1/3$; which yields for $n \geq 1$,

$$\underline{\alpha} \geq \frac{\sqrt{C}}{L(nK_{\max})^{1/3}}\left(v_{\min} - \frac{L_{\dot{V}}}{2L}\sqrt{C} - \frac{L_{\dot{V}}}{2L}C^{3/2}\left(\frac{1}{n} + \frac{1}{1-\lambda}\right)\right) \ .$$

Let $\mu \in (0,1)$. Fix $\lambda \in (0,1)$ and $C > 0$ such that (see (26) for the second condition)

$$\frac{L_{\dot{V}}}{2L}\sqrt{C}\left(1 + C\left(\frac{1}{n} + \frac{1}{1-\lambda}\right)\right) \leq \mu v_{\min} \ , \qquad \frac{n^{1/3}}{K_{\max}^{2/3}} \leq \frac{\lambda}{C} \ . \tag{27}$$

eq:proof:coro:optsample:Ke

This implies that $\underline{\alpha} \geq \alpha_\star \stackrel{\text{def}}{=} \sqrt{C}(1-\mu)v_{\min}/(Ln^{1/3}K_{\max}^{1/3})$ and

$$\frac{1}{K_{\max}\alpha_\star} = \frac{n^{1/3}}{K_{\max}^{2/3}} \frac{L}{\sqrt{C}(1-\mu)v_{\min}} \ . \tag{28}$$

eq:proof:coro:optsample:Ke

We obtain the upper bound on E_1 and E_2 by using

$$\mathsf{E}_1 \leq \frac{1}{K_{\max}\ \alpha_\star} \sum_{k=0}^{K_{\max}-1} \alpha_k \mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right] \ ;$$

and, since $\delta_k \geq L_{\dot{V}} \gamma_{k+1}^2/2$ and $\alpha_\star \leq \sqrt{C} v_{\min}/(Ln^{1/3} K_{\max}^{1/3})$

$$\begin{aligned}
&\frac{L_{\dot{V}}\sqrt{C}}{2Ln^{1/3}} \frac{1}{K_{\max}^{1/3} v_{\min}} \mathsf{E}_2 \\
&\quad \leq \frac{L_{\dot{V}}\, C}{2L^2 n^{2/3} K_{\max}^{2/3}} \frac{1}{K_{\max}\ \alpha_\star} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] \\
&\qquad \leq \frac{1}{K_{\max}\ \alpha_\star} \sum_{k=0}^{K_{\max}-1} \delta_k \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] \ .
\end{aligned}$$

Let $\varepsilon \in (0,1]$. By using the second condition in (27), the RHS in (28) is upper bounded by ε as soon as

$$\frac{\lambda}{C} \frac{L}{\sqrt{C}(1-\mu) v_{\min}} = \varepsilon \ .$$

We now fix

$$\lambda \stackrel{\text{def}}{=} \frac{\varepsilon}{M}, \qquad C^{3/2} \stackrel{\text{def}}{=} \frac{L}{M(1-\mu)v_{\min}}$$

where $M > 1$ is chosen large enough so that the first condition in (27) is satisfied, which holds as soon as

$$\frac{L^{1/3}}{M^{1/3}(1-\mu)^{1/3} v_{\min}^{1/3}} + \frac{L}{M(1-\mu)v_{\min}} \left(1 + \frac{1}{1-(1/M)}\right) \leq \frac{2L}{L_{\dot{V}}} \mu v_{\min} \ .$$

Such a M always exists since the LHS is a decreasing function, which tends to $+\infty$ when $M \downarrow 1$ and to 0 when $M \to +\infty$.

Asymptotic point of view for the choice of (λ, C)**.** From (23) applied with $\mathsf{b} = 1$, $\mathsf{c} = \mathsf{d} = 1/3$, we have

$$n^{1/3} K_{\max}^{1/3} \underline{\alpha} = \mathcal{L}(n, C, \lambda) \stackrel{\text{def}}{=} \frac{\sqrt{C}}{L}\left(v_{\min} - \frac{L_{\dot{V}}}{2L}\frac{\sqrt{C}}{n^{1/3}K_{\max}^{1/3}} - \frac{L_{\dot{V}}}{2L} C^{3/2}\left(\frac{1}{n} + \frac{1}{1-\lambda}\right)\right) \ ,$$

for any $\lambda \in (0,1)$ and $C > 0$ such that $n^{1/3} K_{\max}^{-2/3} \leq \lambda / C$. The lower bound $\mathcal{L}(n, C, \lambda)$ is a signed quantity and we have $\mathcal{L}(n, C, \lambda) \uparrow \mathcal{L}(\infty, C, \lambda)$ when $n \to \infty$ and $nK_{\max} \to \infty$ where

$$\mathcal{L}(\infty, C, \lambda) \stackrel{\text{def}}{=} \frac{\sqrt{C}}{L}\left(v_{\min} - \frac{L_{\dot{V}}}{2L}\frac{C^{3/2}}{1-\lambda}\right) \ .$$

It is easily seen that, under the additional condition: there exists $\kappa > 0$ such that $\kappa \geq \lim_{n, K_{\max}} n^{1/3} K_{\max}^{-2/3}$, it holds $\mathcal{L}(\infty, C, \lambda) \leq \mathcal{L}(\infty, C_\star, \lambda_\star)$ where [2]

$$C_\star \stackrel{\text{def}}{=} \left(\frac{v_{\min} L}{2 L_{\dot V}}\right)^{2/3} (1-\lambda_\star)^{2/3} \,,$$
$$\lambda_\star \text{ s.t. } \kappa \left(\frac{v_{\min} L}{2 L_{\dot V}}\right)^{2} (1-\lambda)^{2} - \lambda^3 = 0 \,.$$

Therefore, we have for any $\lambda \in (0,1)$, $C > 0$, $n, K_{\max} \geq 1$ such that $\lim_n n^{1/3} K_{\max}^{-2/3} = \kappa > 0$ and $n^{1/3} K_{\max}^{-2/3} \leq \lambda / C$,

$$\lim_n n^{1/3} K_{\max}^{1/3} \underline{\alpha} = \mathcal{L}(\infty, C_\star, \lambda_\star) \geq \mathcal{L}(n, C, \lambda) \,.$$

As a conclusion (the limit is under the condition $\lim_n n^{1/3} K_{\max}^{-2/3} \leq \kappa$),

$$\lim_n n^{1/3} K_{\max}^{1/3} \underline{\alpha} = \mathcal{L}(\infty, C_\star, \lambda_\star) = \frac{3}{4} \left(\frac{v_{\min}^4}{2 L^2 L_{\dot V}}\right)^{1/3} (1-\lambda_\star)^{1/3} \,,$$

thus showing that there exists $N_\star$ such that for any $n \geq N_\star$, $\mathcal{L}(n, C_\star, \lambda_\star) > 0$; and the lower bound $\mathcal{L}(\infty, C_\star, \lambda_\star)$ is the optimal one among the bounds $\mathcal{L}(n, C, \lambda)$, for any $\lambda \in (0,1)$, $C > 0$, $n, K_{\max} \geq 1$ such that $\lim_n n^{1/3} K_{\max}^{-2/3} \leq \kappa$ and $n^{1/3} K_{\max}^{-2/3} \leq \lambda / C$.

3.4.4 Proof of Proposition 7

sec:proof:coro:givensample

Let $p_0, \ldots, p_{K_{\max}-1}$ be positive real numbers such that $\sum_{k=0}^{K_{\max}-1} p_k = 1$. We consider the case when

$$\beta_\ell \stackrel{\text{def}}{=} \frac{1-\lambda}{n^{\mathsf{b}}}, \qquad \gamma_\ell^2 \stackrel{\text{def}}{=} \frac{C_\ell}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}}, \qquad 1 \leq \ell \leq K_{\max},$$

for $\lambda \in (0,1)$, $C_\ell > 0$, and $\mathsf{b}, \mathsf{c}, \mathsf{d}$ to be defined in the proof.

The first step consists in the definition of a function F and of a family $\mathcal{C}$ of vectors $\underline{C} = (C_1, \ldots, C_{K_{\max}}) \in \mathbb{R}_+^{K_{\max}}$ such that $\alpha_k \geq F(C_{k+1}) \geq 0$, and $\sum_{\ell=0}^{K_{\max}-1} F(C_{\ell+1}) > 0$. Then, it is proved that we can find $\underline{C} \in \mathcal{C}$ such that $p_k = F(C_{k+1}) / \sum_{\ell=0}^{K_{\max}-1} F(C_{\ell+1})$ for any $k = 0, \ldots, K_{\max}-1$. Such a pais $(F, \underline{C})$ is not unique, and among the possible ones, we indicate three strategies, all motivated by making the sum $\sum_{\ell=0}^{K_{\max}-1} F(C_{\ell+1})$ as large as possible.

[2] $x \mapsto Ax - Bx^3$ is increasing on $[0, x_\star]$ and then decreasing; and $x_\star = A^{1/3} B^{-1/3} 4^{-1/3}$. The function evaluated at $x_\star$ is equal to $3 A^{4/3} B^{-1/3} 4^{-4/3}$. This result is applied by setting $C_\star \stackrel{\text{def}}{=} x_\star^2$, $A = v_{\min}/L$ and $B = (1-\lambda)^{-1} L_{\dot V} / (2L^2)$. The function evaluated at $\sqrt{C_\star(\lambda)}$ depends on λ as $(1-\lambda)^{1/3}$: we are therefore interested by the smallest value of λ. On the other hand, $C_\star(\lambda)$ is of the form $\tilde\kappa (1-\lambda)^{2/3}$ for some $\tilde\kappa > 0$, and we have the condition $C_\star(\lambda)\, \kappa \leq \lambda$. Since the function $\lambda \mapsto (1-\lambda)^2 \lambda^{-3}$ is decreasing on $(0,1)$, the condition is satisfied for any $\lambda \geq \lambda_{\min}$ where $\lambda_{\min}$ is the unique solution of the equation $\kappa^3 \tilde\kappa^3 (1-\lambda)^2 - \lambda^3 = 0$. We obtain $\lambda_\star = \lambda_{\min}$ and $C_\star = C_\star(\lambda_{\min})$.

Step 1- Definition of the function F. With the definition of the sequences γ_ℓ and β_ℓ, we have

$$1 - \frac{\rho_{n,\ell}}{n} \stackrel{\text{def}}{=} 1 - \frac{1}{n} + \beta_\ell + L^2\gamma_\ell^2 = 1 - \frac{1}{n}\left(1 - \frac{1-\lambda}{n^{\mathsf{b}-1}} - \frac{C_\ell}{n^{2\mathsf{c}-1}K_{\max}^{2\mathsf{d}}}\right)$$

and choose $(\mathsf{b}, \mathsf{c}, \mathsf{d}, \lambda, C_\ell)$ such that

$$\frac{1-\lambda}{n^{\mathsf{b}-1}} + \frac{C_{\max}}{n^{2\mathsf{c}-1}K_{\max}^{2\mathsf{d}}} < 1\ , \text{ where } C_{\max} \stackrel{\text{def}}{=} \max_\ell C_\ell\ , \tag{29}$$

`eq:proof:coro:givensample:`

which ensures that $\rho_{n,\ell} \in (0,1)$. Define

$$\rho_n \stackrel{\text{def}}{=} \max_\ell \rho_{n,\ell} = 1 - \frac{1-\lambda}{n^{\mathsf{b}-1}} - \frac{C_{\max}}{n^{2\mathsf{c}-1}K_{\max}^{2\mathsf{d}}}\ .$$

Hence, for any $0 \le k \le K_{\max} - 2$,

$$\begin{aligned}\Lambda_k &\le n^{\mathsf{b}}\left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\frac{1}{L^2 n^{2\mathsf{c}}K_{\max}^{2\mathsf{d}}}\sum_{j=k+1}^{K_{\max}-1} C_{j+1}\left(1 - \frac{\rho_n}{n}\right)^{j-k-1} \\ &\le \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\frac{C_{\max}}{L^2\rho_n}\frac{1}{n^{2\mathsf{c}-\mathsf{b}-1}K_{\max}^{2\mathsf{d}}}\ .\end{aligned}$$

From this upper bound, we obtain the following lower bound on α_k, for any $0 \le k \le K_{\max} - 1$,

$$\alpha_k \ge \frac{\sqrt{C_{k+1}}}{Ln^{\mathsf{c}}K_{\max}^{\mathsf{d}}}\left(v_{\min} - \frac{L_{\dot V}}{2L}\frac{\sqrt{C_{k+1}}}{n^{\mathsf{c}}K_{\max}^{\mathsf{d}}} - \frac{L_{\dot V}}{2L}\frac{C_{\max}\sqrt{C_{k+1}}}{\rho_n\, n^{3\mathsf{c}-\mathsf{b}-1}K_{\max}^{3\mathsf{d}}}\left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\right)\ .$$

Based on this inequality and on condition (29), we choose $\mathsf{b} = 1$, $\mathsf{c} = 2/3$, $\mathsf{d} = 0$: this yields $\alpha_k \ge \underline{\alpha}_k$ with

$$\underline{\alpha}_k \stackrel{\text{def}}{=} \frac{\sqrt{C_{k+1}}}{Ln^{2/3}}\left[v_{\min} - \sqrt{C_{k+1}}\frac{L_{\dot V}}{2L}\left(\frac{1}{n^{2/3}} + \frac{C_{\max}}{\lambda - C_{\max}n^{-1/3}}\left(\frac{1}{n} + \frac{1}{1-\lambda}\right)\right)\right]\ ; \tag{30}$$

`eq:proof:coro:givensample:`

and the condition (29) gets into $n^{-1/3} < \lambda/C_{\max}$.

Define the quadratic function $x \mapsto F(x) \stackrel{\text{def}}{=} Ax(v_{\min} - Bx)$ where

$$A \stackrel{\text{def}}{=} \frac{1}{Ln^{2/3}}, \quad B \stackrel{\text{def}}{=} \frac{L_{\dot V}}{2L}\left(\frac{1}{n^{2/3}} + \frac{C_{\max}}{\lambda - n^{-1/3}C_{\max}}\left(\frac{1}{n} + \frac{1}{1-\lambda}\right)\right). \tag{31}$$

`eq:proof:coro:givensample:`

By Lemma 14, F is increasing on $(0, v_{\min}/(2B)]$, reaches its maximum at $x_\star \stackrel{\text{def}}{=} v_{\min}/(2B)$ and its maximal value is $F_\star \stackrel{\text{def}}{=} Av_{\min}^2/(4B)$. In addition, its inverse F^{-1} exists on $(0, F_\star]$.

We established that for any $\lambda \in (0,1)$, $C_{\max} > 0$ and $n \ge 1$ such that $n^{-1/3} < \lambda/C_{\max}$, there exists a quadratic function F such that $\underline{\alpha}_k = F(\sqrt{C_{k+1}})$.

Step 2- Choice of $C_1, \dots, C_{K_{\max}}$. We are now looking for $C_1, \dots, C_{K_{\max}}$ such that $p_k = F(\sqrt{C_{k+1}})/\sum_{\ell=0}^{K_{\max}-1} F(\sqrt{C_{\ell+1}})$ or equivalently

$$\frac{p_k}{p_I} = \frac{F(\sqrt{C_{k+1}})}{F(\sqrt{C_I})}, \qquad I \in \mathrm{argmax}_k p_k \; . \tag{32}$$

eq:fromptoC

It remains to fix $F(\sqrt{C_I})$ in such a way that F is invertible on $\big(0, \sqrt{C_I}\big]$. Since we also want $\sum_\ell F(\sqrt{C_{\ell+1}}) = F(\sqrt{C_I})/p_I$ as large as possible, we choose

$$\sqrt{C_I} = \sqrt{C_{\max}} \leq x_\star = \frac{v_{\min}}{2B} \tag{33}$$

eq:cond:Cmax:F

the best value being $\sqrt{C_{\max}} = v_{\min}/(2B)$. Therefore, $C_{\max}$ solves the equation $\sqrt{C_{\max}} \leq v_{\min}/(2B)$ or equivalently

$$\frac{v_{\min} L}{L_{\dot{V}}} \geq \sqrt{C_{\max}} \left(\frac{1}{n^{2/3}} + \frac{C_{\max}}{\lambda - n^{-1/3} C_{\max}} \left(\frac{1}{n} + \frac{1}{1-\lambda} \right) \right) \tag{34}$$

eq:Cmax:lambda

under the constraint that $\lambda \in (0,1)$ and $n^{-1/3} < \lambda / C_{\max}$. When $C_{\max}$ is fixed, we set

$$\sqrt{C_{k+1}} \overset{\text{def}}{=} F^{-1} \left(\frac{p_k}{\max_\ell p_l} F(\sqrt{C_{\max}}) \right).$$

With these definitions, we have (see (32))

$$\frac{1}{\sum_{k=0}^{K_{\max}-1} F(\sqrt{C_{k+1}})} = \frac{\max_\ell p_\ell}{F(\sqrt{C_{\max}})}.$$

Step 3 - Strategy 1 for the choice of $(C_{\max}, \lambda)$ A simple strategy is to choose $n \geq 2$ and $C_{\max} = \lambda$ solution of

$$\frac{v_{\min} L}{L_{\dot{V}}} = \sqrt{x} \left(\frac{1}{n^{2/3}} + \frac{1}{n - n^{2/3}} + \frac{1}{1 - n^{-1/3}} \frac{1}{1-x} \right) \; .$$

The RHS is increasing on $(0,1)$, tends to zero when $x \to 0$ and to $+\infty$ when $x \to 1$. There exists an unique solution $x_\star$; and upon noting that the RHS is lower bounded by $\sqrt{x}/(1-x)$, $x_\star$ is upper bounded by the root of $v_{\min} L(1-x) - \sqrt{x} L_{\dot{V}} = 0$ which only depends upon $v_{\min}, L, L_{\dot{V}}$.

Step 4 - Strategy 2 for the choice of $(C_{\max}, \lambda)$ A strategy is to choose $\lambda, C_{\max}$ in order to maximize $F(\sqrt{C_{\max}})$ which means that $\sqrt{C_{\max}} = v_{\min}/(2B)$ and $A v_{\min}^2/(4B)$ is maximal. Equivalently, we want $2\sqrt{C_{\max}} B = v_{\min}$ and B minimal. Upon noting that when $n \to \infty$, B depends on λ through $(\lambda(1-\lambda))^{-1}$ which is minimal at $\lambda = 1/2$, we propose to fix $\lambda = 1/2$. Set

$$\Psi(x) \overset{\text{def}}{=} \sqrt{x} \left(\frac{1}{n^{2/3}} + \frac{x}{1/2 - n^{-1/3} x} \left(\frac{1}{n} + 2 \right) \right).$$

When $x \to 0$, then $\Psi(x) \to 0$. When $x \uparrow \lambda n^{1/3}$, then $\Psi(x) \to +\infty$. On $\big(0, \lambda n^{1/3}\big)$, $x \mapsto \Psi(x)$ is increasing and continuous. Therefore, there exists an

unique solution to the equation $\Psi(x) = v$ whatever $v > 0$: there exists an unique solution to $2\sqrt{C_{\max}}B = v_{\min}$.
Upon noting that the $\Psi(x)$ is lower bounded by $x \mapsto 4x^{3/2}$ (which is also increasing and continuous), we have

$$C_{\max} \leq \left(\frac{v_{\min} L}{4 L_{\dot V}} \right)^{2/3} .$$

thus showing that the constraint $n^{-1/3} < \lambda / C$ is satisfied for any n such that $8n > (v_{\min} L / L_{\dot V})^2$.

Step 5 - Strategy 3 for the choice of $(C_{\max}, \lambda)$ The last strategy consists in minimizing the function

$$(\lambda, x) \mapsto \Psi_1(\lambda, x) \stackrel{\text{def}}{=} \frac{L_{\dot V}}{2L} \left(\frac{1}{n^{2/3}} + \frac{x}{\lambda - n^{-1/3} x} \left(\frac{1}{n} + \frac{1}{1-\lambda} \right) \right),$$

under the constraints

$$x > 0, \lambda \in (0,1) , \qquad 2\sqrt{x}\Psi_1(\lambda, x) = v_{\min} ,$$

and then conclude that this strategy is available for all n large enough.

3.4.5 Auxiliary results

subsec:auxiliary

lem:vmin:positive

Lemma 8. *Assume H1-item 1-item 2, H3-item 1 and $q \leq d$. Assume also that for any $s \in \mathcal{S}$, $\mathrm{rank}(\dot{\mathsf{T}}(s)) = q$, $\tau \mapsto L(s, \tau) \stackrel{\text{def}}{=} \overline{\psi}(\tau) - \langle s, \phi(\tau) \rangle + \mathsf{R}(\tau)$ is twice continously differentiable on Θ. Then for any $s \in \mathcal{S}$, $\dot\phi \circ \mathsf{T}(s)$ is a symmetric $q \times q$ matrix and its minimal eigenvalue is positive.*

Proof. The proof is adapted from the one of (Delyon et al., 1999, Lemma 2). The condition H3-item 1 implies that for any $s \in \mathcal{S}$: $\partial_\tau L(s, \mathsf{T}(s)) = 0$ and for any non null vector $\lambda \in \mathbb{R}^d$, $\lambda^T \partial_\tau^2 L(s, \mathsf{T}(s)) \lambda > 0$. Upon noting that for any $s \in \mathcal{S}$,

$$\partial_\tau L(s, \mathsf{T}(s)) = 0 \Longleftrightarrow \dot{\overline{\psi}}(\mathsf{T}(s)) - \left(\dot\phi(\mathsf{T}(s)) \right)^T s + \dot{\mathsf{R}}(\mathsf{T}(s)) = 0,$$

we obtain by differentiating w.r.t. s,

$$\partial_\tau^2 L(s, \mathsf{T}(s))\, \dot{\mathsf{T}}(s) - \left(\dot\phi(\mathsf{T}(s)) \right)^T = 0, \qquad s \in \mathcal{S}.$$

This yields for any $\mu \in \mathbb{R}^q$,

$$\left(\dot{\mathsf{T}}(s)\mu \right)^T \, \partial_\tau^2 L(s, \mathsf{T}(s)) \, \left(\dot{\mathsf{T}}(s)\mu \right) = \left(\dot{\mathsf{T}}(s)\mu \right)^T \, \left(\dot\phi(\mathsf{T}(s)) \right)^T \, \mu = \mu^T \, B(s) \, \mu;$$

the proof is concluded by noting that by assumptions, the LHS is positive (for any non null vector μ). $\square$

lem:sbarcircT **Lemma 9.** *Assume H1item 1-item 2, H2 and H3item 1. Assume in addition that there exists $L_{i,p}$ such that for any $\theta, \theta' \in \Theta$*

$$\sup_{z \in \mathsf{Z}} |p_i(z;\theta) - p_i(z;\theta')| \leq L_{i,p} \, \|\theta - \theta'\|;$$

T is globally Lipschitz on $\mathcal{S}$, and $\int_{\mathsf{Z}} \|s_i\| \mathrm{d}\mu < \infty$. Then there exists a constant $0 < L_i < \infty$ such that for all $s, s' \in \mathcal{S}$,

$$\|\bar{s}_i \circ \mathsf{T}(s) - \bar{s}_i \circ \mathsf{T}(s')\| \leq L_i \, \|s - s'\|.$$

Proof. Let $s, s' \in \mathcal{S}$. It holds

$$\bar{s}_i \circ \mathsf{T}(s) - \bar{s}_i \circ \mathsf{T}(s') = \int_{\mathsf{Z}} s_i(z) \left[p_i(z; \mathsf{T}(s)) - p_i(z; \mathsf{T}(s')) \right] \mu(\mathrm{d}z)$$

so that

$$\begin{aligned} \|\bar{s}_i \circ \mathsf{T}(s) - \bar{s}_i \circ \mathsf{T}(s')\| &\leq \int_{\mathsf{Z}} \|s_i(z)\| \, |p_i(z; \mathsf{T}(s)) - p_i(z; \mathsf{T}(s'))| \, \mu(\mathrm{d}z) \\ &\leq L_p \int_{\mathsf{Z}} \|s_i(z)\| \mu(\mathrm{d}z) \, \|\mathsf{T}(s) - \mathsf{T}(s')\|. \end{aligned}$$

□

lem:expfam:reg **Lemma 10.** *Assume H1-item 1-item 2 and H4-item 1. For all $i \in \{1, ..., n\}$, $\mathcal{L}_i$ is continuously differentiable.*

Proof. H1-item 1-item 2 and (**?**, Proposition 3.8) (see also (**?**, Theorem 2.2.)) imply that $C : \theta \mapsto \int_Z e^{\langle s_i(z), \theta \rangle} h_i(z) \mu(\mathrm{d}z)$ is continuously differentiable on Θ. H4-item 1 and the equality $\mathcal{L}_i = -\log(C \circ \phi) + \psi$ yield the result. □

lem:nablaV **Lemma 11.** *Assume H1item 1-item 2, H3-item 1 and H4item 1-item 2. Then $V \stackrel{\text{def}}{=} F \circ \mathsf{T}$ is continuously differentiable on $\mathcal{S}$ and for all $s \in \mathcal{S}$,*

$$\begin{aligned} &- v_{\max} \, \|h(s)\|^2 \leq \left\langle \bar{s} \circ \mathsf{T}(s) - s, \dot{V}(s) \right\rangle \leq -v_{\min} \, \|h(s)\|^2, \\ &\|\dot{V}(s)\|^2 \leq v_{max}^2 \|h(s)\|^2. \end{aligned}$$

Proof. The assumption H4-item 1 implies that T is continuously differentiable on $\mathcal{S}$. In addition, a classical result on exponential families and H1-item 1 (see Lemma 10) implies that $\theta \mapsto \mathcal{L}_i(\theta)$ is continuously differentiable on Θ for any $i \in \{1, \ldots, n\}$ and we have for any $\theta \in \Theta$, $\dot{\mathcal{L}}_i(\theta) = \dot{\psi}_i(\theta) - \dot{\phi}(\theta)^T \, \bar{s}_i(\theta)$. Therefore V is continuously differentiable on $\mathcal{S}$. Using a chain rule, we have :

$$\dot{V}(s) = \left(\dot{\mathsf{T}}(s) \right)^T \, \left\{ \dot{\mathsf{R}}(\mathsf{T}(s)) + \frac{1}{n} \sum_{i=1}^n \left(\dot{\psi}_i(\theta) - \dot{\phi}(\theta)^T \, \bar{s}_i(\theta) \right) |_{\theta = \mathsf{T}(s)} \right\}$$

Moreover, using H3 and H1-item 1, we have for any $s \in \mathcal{S}$,

$$\dot{\mathsf{R}}(\mathsf{T}(s)) + \frac{1}{n} \sum_{i=1}^n \dot{\psi}_i(\mathsf{T}(s)) - \dot{\phi}(\mathsf{T}(s))^T s = 0$$

so that

$$\dot{V}(s) = -\left(\dot{\mathsf{T}}(s)\right)^T \ \left(\dot{\phi}(\mathsf{T}(s))\right)^T \ \{\bar{s} \circ \mathsf{T}(s) - s\} = -\left(B(s)\right)^T \ (\bar{s} \circ \mathsf{T}(s) - s).$$

With H4-item 2, this yields the two inequalities on $\|\bar{s} \circ \mathsf{T}(s) - s\|$. □

lem:control:field

Lemma 12. *Assume H1item 1-item 2, H2 and H3item 1. For any $k \geq 0$,*

$$\mathbb{E}\left[\|H_{k+1}\|^2\right] = \mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] + \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right],$$

and

$$\mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] + \mathbb{E}\left[\|\frac{\lambda_{k+1}}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] \\ = \mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \lambda_{k+1}\mathsf{S}_{k+1,J_{k+1}}\|^2\right].$$

Proof. By Lemma 3, $\mathbb{E}\left[H_{k+1}\middle|\mathcal{F}_{k+1/2}\right] = h(\widehat{S}^k)$, which yields

$$\mathbb{E}\left[\|H_{k+1}\|^2\right] = \mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] + \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right].$$

In addition, upon noting that $\mathsf{S}_{k+1,i} \in \mathcal{F}_{k+1/2}$ for any i,

$$H_{k+1} - h(\widehat{S}^k) = \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \lambda_{k+1}\mathsf{S}_{k+1,J_{k+1}} - \bar{s} \circ \mathsf{T}(\widehat{S}^k) + \frac{\lambda_{k+1}}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i}$$
$$= \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \lambda_{k+1}\mathsf{S}_{k+1,J_{k+1}} - \mathbb{E}\left[\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \lambda_{k+1}\mathsf{S}_{k+1,J_{k+1}}\middle|\mathcal{F}_{k+1/2}\right],$$

we have

$$\mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] + \mathbb{E}\left[\|\frac{\lambda_{k+1}}{n}\sum_{i=1}^n \mathsf{S}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] \\ = \mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \lambda_{k+1}\mathsf{S}_{k+1,J_{k+1}}\|^2\right].$$

□

prop:variance:field

Proposition 13. *Assume H1item 1-item 2, H2, H3item 1 and H4-item 3. Set $L^2 \overset{\text{def}}{=} n^{-1}\sum_{i=1}^n L_i^2$. Then for any $k \geq 1$ and $\beta_1, \ldots, \beta_k > 0$,*

Gers 2all: cette preuve ne marche que si $\lambda_{k+1} = 1$ sinon c'est faux

$$\mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}}\|^2\right]$$
$$\leq \sum_{j=1}^k \widetilde{\Lambda}_{j,k}\left\{\mathbb{E}\left[\|h(\widehat{S}^{j-1})\|^2\right] - \left(1 + \frac{1}{\beta_j}\right)^{-1}\mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{j,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^{j-1})\|^2\right]\right\}$$

where

$$\widetilde{\Lambda}_{j,k} \stackrel{\text{def}}{=} L^2 \left(\frac{n-1}{n}\right)^{k-j+1} \gamma_j^2 \left(1+\frac{1}{\beta_j}\right) \prod_{\ell=j+1}^{k} \left(1+\beta_\ell+\gamma_\ell^2 L^2\right).$$

By convention, $\prod_{\ell=k+1}^{k} a_\ell = 1$.

Proof. Let us upper bound the RHS in the second statement of Lemma 12. We write

$$\mathsf{S}_{k+1,i} = \mathsf{S}_{k,i} \mathbb{1}_{I_{k+1}\neq i} + \bar{s}_i \circ \mathsf{T}(\widehat{S}^k) \mathbb{1}_{I_{k+1}=i} = \bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k,i}) \mathbb{1}_{I_{k+1}\neq i} + \bar{s}_i \circ \mathsf{T}(\widehat{S}^k) \mathbb{1}_{I_{k+1}=i},$$

where $\widehat{S}^{<\ell,i}$ is defined by (10). This yields, by H3-item 3

$$\frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^k) - \lambda_{k+1}\mathsf{S}_{k+1,i}\|^2\right] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^k) - \lambda_{k+1}\bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k,i})\|^2 \mathbb{1}_{I_{k+1}\neq i}\right]$$

$$\leq \Delta_k \stackrel{\text{def}}{=} \frac{n-1}{n^2}\sum_{i=1}^n L_i^2\, \mathbb{E}\left[\|\widehat{S}^k - \lambda_{k+1}\widehat{S}^{<k,i}\|^2\right]. \qquad (35)$$

eq:definition:Delta

Gers 2all: c'est là que tout déraille: ce que j'ai écrit avant (l'argument Lipschitz) n'est vrai que si $\lambda_{k+1} = 1$. Donc dans la suite, je ne mets plus de λ.

We have

$$\Delta_k = \frac{n-1}{n^2}\sum_{i=1}^n L_i^2\, \mathbb{E}\left[\|\widehat{S}^k - \widehat{S}^{k-1} + \left(\widehat{S}^{k-1} - \widehat{S}^{<k,i}\right)\mathbb{1}_{I_k\neq i}\|^2\right]$$

where we used in the last inequality that

$$\widehat{S}^{<k,i} = \widehat{S}^{k-1}\mathbb{1}_{I_k=i} + \widehat{S}^{<k-1,i}\mathbb{1}_{I_k\neq i}.$$

Upon noting that $2\left\langle \tilde{U}, V\right\rangle \leq \beta^{-1}\|\tilde{U}\|^2 + \beta\|V\|^2$ for any $\beta > 0$, we have for any $\mathcal{G}$-measurable r.v. V

$$\mathbb{E}\left[\|U+V\|^2\right] \leq \mathbb{E}\left[\|U\|^2\right] + \beta^{-1}\mathbb{E}\left[\|\mathbb{E}\left[U|\mathcal{G}\right]\|^2\right] + (1+\beta)\mathbb{E}\left[\|V\|^2\right].$$

Applying this inequality with $\beta \leftarrow \beta_k$, $U \leftarrow \widehat{S}^k - \widehat{S}^{k-1} = \gamma_k H_k$ and $\mathcal{G} \leftarrow \mathcal{F}_{k-1/2}$ yields

$$\Delta_k \leq \gamma_k^2 \frac{n-1}{n} L^2\, \mathbb{E}\left[\|H_k\|^2\right] + \frac{\gamma_k^2}{\beta_k}\frac{n-1}{n} L^2\, \mathbb{E}\left[\|\mathbb{E}\left[H_k|\mathcal{F}_{k-1/2}\right]\|^2\right]$$
$$+ \frac{n-1}{n^2}(1+\beta_k)\sum_{i=1}^n L_i^2\, \mathbb{E}\left[\|\widehat{S}^{k-1} - \widehat{S}^{<k-1,i}\|^2 \mathbb{1}_{I_k\neq i}\right].$$

By Lemma 12 and (35), we have

$$\mathbb{E}\left[\|H_k\|^2\right] \leq \mathbb{E}\left[\|h(\widehat{S}^{k-1})\|^2\right] + \Delta_{k-1} - \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k,i} - \bar{s}\circ\mathsf{T}(\widehat{S}^{k-1})\|^2\right];$$

by Lemma 3
$$\mathbb{E}\left[\|\mathbb{E}\left[H_k|\mathcal{F}_{k-1/2}\right]\|^2\right] = \mathbb{E}\left[\|h(\widehat{S}^{k-1})\|^2\right];$$
and since $I_k \in \mathcal{F}_{k-1/2}$, $\widehat{S}^{k-1} \in \mathcal{F}_{k-1}$, $\widehat{S}^{<k-1,i} \in \mathcal{F}_{k-1}$,
$$\sum_{i=1}^n L_i^2\, \mathbb{E}\left[\|\widehat{S}^{k-1} - \widehat{S}^{<k-1,i}\|^2 \mathbb{1}_{I_k \neq i}\right] = n\Delta_{k-1}.$$
Therefore, we established
$$\Delta_k \leq \frac{n-1}{n}\left(1+\beta_k+\gamma_k^2 L^2\right)\Delta_{k-1} + \gamma_k^2 L^2 (1+\frac{1}{\beta_k})\ \frac{n-1}{n}\,\mathbb{E}\left[\|h(\widehat{S}^{k-1})\|^2\right] \\ - \gamma_k^2 L^2 \frac{n-1}{n}\mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \mathsf{S}_{k,i} - \bar{s}\circ\mathsf{T}(\widehat{S}^{k-1})\|^2\right].$$
The proof is then concluded by standard algebra upon noting that $\Delta_0 = 0$. □

3.4.6 Technical lemmas

lem:polynomial:function

Lemma 14. *Let $A, B, v > 0$ and define $F(g) \stackrel{\text{def}}{=} Ag(v - gB)$ on $\mathbb{R}$. Then the roots of F are $\{0, v/B\}$; F is positive on $(0, v/B)$; the maximal value of F is $Av^2/(4B)$ and it is reached at $g_\star \stackrel{\text{def}}{=} v/2B$.*

lem:choix:C:lambda

Lemma 15. *For any $\lambda, C \in (0,1)$, it holds*
$$\frac{C(1+(1-C)\lambda)}{\lambda(1-\lambda)(1-C)^2} = \frac{C}{(1-C)^2}\left(\frac{1}{\lambda} + \frac{2-C}{1-\lambda}\right).$$
For any $C \in (0,1)$, this quantity is minimal at
$$\lambda(C) \stackrel{\text{def}}{=} \frac{\sqrt{2-C}-1}{1-C} \in (0,1),$$
and is equal to
$$\frac{C}{1-C}\frac{\sqrt{2-C}+1}{\sqrt{2-C}-1} = \frac{C}{(\sqrt{2-C}-1)^2}.$$
In addition, for all $p \geq 0$,
$$C \mapsto \sqrt{C}\left(\frac{1}{n^p} + \frac{C}{(\sqrt{2-C}-1)^2}\right)$$
is increasing on $(0,1)$, tends to zero when $C \downarrow 0$ and to $+\infty$ when $C \uparrow 1$. This implies that for any $u \in (0, n^{-p})$, we have $C_u \leq u^2 n^{2p}$ where C_u is the unique root of
$$C \mapsto u - \sqrt{C}\left(\frac{1}{n^p} + \frac{C}{(\sqrt{2-C}-1)^2}\right).$$

Proof. The derivative of the function F_C defined on $(0,1)$ by $F_C : \lambda \mapsto \lambda^{-1} + (2-C)(1-\lambda)^{-1}$, is

$$\frac{\lambda^2(1-C) + 2\lambda - 1}{\lambda^2(1-\lambda)^2}$$

It is negative on $(0, \lambda(C))$ and positive on $(\lambda(C), 1)$. We have

$$1 - \lambda(C) = \frac{1 - C - \sqrt{2-C} + 1}{1-C} = \frac{\sqrt{2-C}}{1-C}\left(\sqrt{2-C} - 1\right)$$

which yields

$$\frac{C}{(1-C)^2}\left(\frac{1}{\lambda(C)} + \frac{2-C}{1-\lambda(C)}\right) = \frac{C}{1-C}\frac{\sqrt{2-C}+1}{\sqrt{2-C}-1}.$$

The monotonicity property is trivial (sum and product of positive increasing functions). It implies that for $p \geq 0$, given $u \in (0, n^{-p})$, C_u is unique and we have $C \in [C_u, 1)$ iff $u \leq \sqrt{C}\left(\frac{1}{n^p} + \frac{C}{(\sqrt{2-C}-1)^2}\right)$. Since $\tilde{C} \stackrel{\text{def}}{=} u^2 n^{2p}$ satisfies

$$u = \sqrt{\tilde{C}}\,\frac{1}{n^p} \leq \sqrt{\tilde{C}}\left(\frac{1}{n^p} + \frac{\tilde{C}}{(\sqrt{2-\tilde{C}}-1)^2}\right),$$

then $C_u \leq \tilde{C}$. $\square$

4 Perturbed FIEM

We here consider the case where the explicit computation of $\bar{s}_i$ is not available and has to be replaced at each step by an approximation.

4.1 Description of the algorithm

sec:montecarlo

Data: $K_{\max} \in \mathbb{N}$, $\widehat{S}^0 \in \mathcal{S}$ and $\widetilde{\mathsf{S}}_{0,i} \in \mathcal{S}$ for any $i \in \{1, \dots, n\}$.
Result: The P-FIEM sequence: $\widehat{S}^k, k = 0, \dots, K_{\max}$
for $k = 0, \dots, K_{\max} - 1$ **do**
  Sample $I_{k+1} \sim \mathcal{U}(\{1, \dots, n\})$ independently from the past ;
  Compute $\widetilde{\mathsf{S}}_{k+1,I_{k+1}}$, an approximation of $\bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$ and set $\widetilde{\mathsf{S}}_{k+1,i} = \widetilde{\mathsf{S}}_{k,i}$ for $i \neq I_{k+1}$;
  Sample $J_{k+1} \sim \mathcal{U}(\{1, \dots, n\})$ independently from the past ;
  Compute $\tilde{s}_{k+1}$ an approximation of $\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$;
  Set $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1} \left(\tilde{s}_{k+1} - \widehat{S}^k + \frac{1}{n} \sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \widetilde{\mathsf{S}}_{k+1,J_{k+1}} \right)$.

Algorithm 7: Perturbed FIEM algorithm

Define the error when approximating expectations of the form $\bar{s}_i \circ \mathsf{T}(\widehat{S}^k)$: for $k \geq 0$,

$$\varepsilon^{(0)} \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n \|\widetilde{\mathsf{S}}_{0,i} - \bar{s}_i \circ \mathsf{T}(\widehat{S}^0)\|^2,$$

$$\eta_{k+1}^{(1)} \overset{\text{def}}{=} \widetilde{\mathsf{S}}_{k+1,I_{k+1}} - \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k), \quad \eta_{k+1}^{(2)} \overset{\text{def}}{=} \tilde{s}_{k+1} - \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k).$$

Note that the case addressed in Section 2.4 corresponds to the results in this section, applied with $\eta_{k+1}^{(2)} = \eta_{k+1}^{(1)} = 0$, and $\varepsilon^{(0)} = 0$.

4.2 Case of stochastic approximations

sec:MC

When the approximations are random, introduce the filtrations $\mathcal{F}_0 \overset{\text{def}}{=} \sigma(\widehat{S}^0, \widetilde{\mathsf{S}}_{0,\cdot})$ and for $k \geq 0$,

$$\mathcal{F}_{k+1/4} \overset{\text{def}}{=} \mathcal{F}_k \vee \sigma\left(I_{k+1}\right), \qquad \mathcal{F}_{k+1/2} \overset{\text{def}}{=} \mathcal{F}_{k+1/4} \vee \sigma\left(\widetilde{\mathsf{S}}_{k+1,\cdot}\right)$$

$$\mathcal{F}_{k+3/4} \overset{\text{def}}{=} \mathcal{F}_{k+1/2} \vee \sigma(J_{k+1}), \qquad \mathcal{F}_{k+1} \overset{\text{def}}{=} \mathcal{F}_{k+3/4} \vee \sigma(\tilde{s}_{k+1});$$

Note also that, for all $k \geq 0$, $\eta_{k+1}^{(1)}$ is $\mathcal{F}_{k+1/2}$-measurable and $\eta_{k+1}^{(2)}$ is $\mathcal{F}_{k+1}$-measurable. The approximations will be said *unbiased* if, with probability one, for any $k \geq 0$

$$\mathbb{E}\left[\eta_{k+1}^{(1)} | \mathcal{F}_{k+1/4}\right] = 0, \quad \text{and} \quad \mathbb{E}\left[\eta_{k+1}^{(2)} | \mathcal{F}_{k+3/4}\right] = 0.$$

As an example of stochastic approximation, consider the Monte Carlo case: the expectation

$$\bar{s}_i \circ \mathsf{T}(s) = \int_{\mathsf{Z}} s_i(z)\ p_i(z; \mathsf{T}(s))\mu(\mathrm{d}z)$$

can be approximated by a Monte Carlo sum. It holds

$$\bar{s}_i \circ \mathsf{T}(s) \approx \frac{1}{m} \sum_{j=1}^{m} s_i(Z_j^{\mathsf{T}(s),i})$$

where $\{Z_j^{\mathsf{T}(s),i}, j \geq 0\}$ are i.i.d. samples with distribution $p_i(\cdot, \mathsf{T}(s))\,\mathrm{d}\mu$; or, when exact sampling is intractable, the points are from a Markov chain designed to be ergodic with unique invariant distribution $p_i(\cdot, \mathsf{T}(s))\,\mathrm{d}\mu$.

The Monte Carlo approximation is unbiased, for example, when for any $k \geq 0$, conditionally to $\mathcal{F}_k$, the samples $\{Z_j^{\mathsf{T}(\widehat{S}^k),i}, j \geq 0\}$ are i.i.d. under the distribution $p_i(z; \mathsf{T}(\widehat{S}^k))\,\mathrm{d}\mu(z)$.

4.3 A general result on the error rate

The following theorem is available whatever the approximations $\widetilde{\mathsf{S}}$ and $\tilde{s}$: they can be deterministic or random, and if such, possibly based on a Monte Carlo approximation. Th results are derived under a control on the errors $\eta_{k+1}^{(1)}$ and $\eta_{k+1}^{(2)}$ as described by H5. Typically, the controls exhibited below are of interest when m_k and $\overline{m}_k$ increase with k.

hyp:approx:MC

H5. *There exist positive sequences $\{m_k, k \geq 0\}$ and $\{\overline{m}_k, k \geq 0\}$, positive numbers $M^{(1)}$ and $M^{(2)}$ and $M_\nu^{(2)} \geq 0$ such that for all $k \geq 0$, the approximations $\tilde{s}_{k+1}$ and $\widetilde{\mathsf{S}}_{k+1,I_{k+1}}$ satisfy*

$$\mathbb{E}[\|\eta_{k+1}^{(1)}\|^2] \leq \frac{M^{(1)}}{\overline{m}_{k+1}}, \quad \mathbb{E}[\|\mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_{k+3/4}\right]\|^2] \leq \frac{M_\nu^{(2)}}{m_{k+1}^2}, \quad \mathbb{E}[\|\eta_{k+1}^{(2)}\|^2] \leq \frac{M^{(2)}}{m_{k+1}}\ .$$

Note that $M_\nu^{(2)} = 0$ iff the approximation is unbiased.

theo:PFIEM:NonUnifStop

Theorem 16. *Assume H1item 1-item 2, H2, H3 and H4-item 1 to H4-item 4. Define $L^2 \stackrel{\mathrm{def}}{=} n^{-1} \sum_{i=1}^n L_i^2$.*

Let $K_{\max}$ be a positive integer. Let $\{\gamma_k,\ k \in \mathbb{N}\}$ be a sequence of positive step sizes and consider the FIEM sequence $\{\widehat{S}^k,\ k \in \mathbb{N}\}$ obtained with $\lambda_{k+1} = 1$ for any k. Assume that $\widehat{S}^k \in \mathcal{S}$ for any $k \leq K_{\max}$.

Let $\nu, \bar{\nu} \in \{0, 1\}$ with the convention $\nu = 0$ iff the approximations are unbiased, and $\bar{\nu} = 0$ iff for any $k \geq 0$, $\|\eta_{k+1}^{(1)}\| = \|\eta_{k+1}^{(2)}\| = \varepsilon^{(0)} = 0$.

For any positive numbers $\beta_1, \cdots, \beta_{K_{\max}-1}$ and $\beta_0 \in \big(0, v_{\min}/v_{\max}^2\big)$, it holds

$$\sum_{k=0}^{K_{\max}-1} \alpha_k \mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right] + \sum_{k=0}^{K_{\max}-1} \delta_k \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right]$$
$$\leq \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]$$
$$+ \xi_0(K_{\max}, n)\ \mathbb{E}\left[\varepsilon^{(0)}\right] + \bar{\nu}\,\Xi_1(\eta_{\cdot}^{(1)}, K_{\max}, n) + \bar{\nu}\,\Xi_2(\eta_{\cdot}^{(2)}, K_{\max}, n); \tag{36}$$

eq:theo:conclusion:perturb

for any $k = 0, \ldots, K_{\max} - 1$,

$$\alpha_k \stackrel{\text{def}}{=} \gamma_{k+1}\left(v_{\min} - \nu v_{\max}^2 \beta_0 - (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_{k+1}\left\{1 + (1+\bar{\nu})(1+\nu)L^2\Lambda_k\right\}\right)$$
$$\delta_k \stackrel{\text{def}}{=} (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_{k+1}^2\left(1 + (1+\bar{\nu})(1+\nu)\frac{\Lambda_k}{(1+\beta_{k+1}^{-1})}\right)$$

with $\Lambda_{K_{\max}-1} = 0$ and for $k = 0, \ldots, K_{\max} - 2$,

$$\Lambda_k \stackrel{\text{def}}{=} \left(1 + \frac{1}{\beta_{k+1}}\right) \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \prod_{\ell=k+2}^{j}\left(1 - \frac{1}{n} + \beta_\ell + (1+\bar{\nu})(1+\nu)L^2\gamma_\ell^2\right);$$

ξ_0, Ξ_1 and Ξ_2 are non negative real numbers; their explicit expressions can be found in Section 4.6, Eqs (54), (55) *and* (56)*. By convention, $\prod_{\ell\in\emptyset} a_\ell = 1$.*

The sketch of the proof of this theorem is on the same lines as the proof of Theorem 4: the main part of the proof consists in the computation of an upper bound for the moment $\mathbb{E}[\|\widehat{S}^{k+1} - \widehat{S}^k\|^2]$. The proof is given in Section 4.6.

The LHS in (36) is the sum of two terms: in some sense, the first one is a distance to the set $\{s \in \mathcal{S} : h(s) \stackrel{\text{def}}{=} \bar{s} \circ \mathsf{T}(s) - s = 0\}$; and the second one is a measure of the approximation of the sum $\bar{s} \circ \mathsf{T}(\widehat{S}^k)$ by $n^{-1}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i}$. Therefore this LHS can be seen as a convergence analysis of the algorithm as soon as $\alpha_k \geq 0$ and $\delta_k \geq 0$. In the next section, we propose a choice of the stepsize sequence $\{\gamma_k, k \geq 1\}$ and of the positive numbers $\beta_0, \ldots, \beta_{K_{\max}-1}$ implying that $\alpha_k \geq 0$, $A_{K_{\max}} \stackrel{\text{def}}{=} \sum_{k=0}^{K_{\max}-1}\alpha_k > 0$ and $\delta_k \geq (1+\nu)L_{\dot{V}}\gamma_{k+1}^2/2$. As a consequence, we obtain an upper bound for

$$\mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^K) - \widehat{S}^K\|^2\right] + \mathcal{G}_{K_{\max}},$$

where K is a $\{0, \ldots, K_{\max} - 1\}$-valued random variable, independent of $\mathcal{F}_{K_{\max}}$, and with distribution $\alpha_k/A_{K_{\max}}$; and

$$\mathcal{G}_{K_{\max}} \stackrel{\text{def}}{=} (1+\nu)\frac{L_{\dot{V}}}{2}\sum_{k=0}^{K_{\max}-1}\gamma_{k+1}^2 \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right]. \tag{37}$$

eq:Gronde

Finally, note that when $\alpha_k \geq 0$, we have by Lemma 11

$$\frac{1}{v_{\max}^2}\sum_{k=0}^{K_{\max}-1}\alpha_k \mathbb{E}\left[\|\dot{V}(\widehat{S}^k)\|^2\right] \leq \sum_{k=0}^{K_{\max}-1}\alpha_k \mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right];$$

hence, Theorem 16 (and therefore, all the corollaries in Section 4.4) also provides an explicit control of the gradient $\dot{V}$ of the objective function along the path of the algorithm.

4.4 Error rates for specific stopping rules

sec:coro:perturbed

In Proposition 17, we propose a definition of the step sizes γ_k yielding to $A_{K_{\max}}$ positive and maximal among the considered family of weights α_k (see the proof, section 4.7): the step sizes have to be constant, and yield to the uniform weights $\alpha_k/A_{K_{\max}} = 1/K_{\max}$ for any k.

coro:pFIEM:optimal

Proposition 17 (following Theorem 16)**.** *Let $C \in (0,1)$ satisfying*

$$v_{\min} \leq (1+\nu)\sqrt{\frac{1+\nu}{1+\bar{\nu}}}\frac{L_{\dot{V}}}{L}\sqrt{C}\left(\frac{1}{n^{2/3}} + \frac{C}{(\sqrt{2-C}-1)^2}\right); \qquad (38)$$

eq:def:Cstar

the optimal choice $C_\star$ being the unique C satisfying the equality. By choosing the constant stepsizes

$$\gamma_k \stackrel{\text{def}}{=} \frac{2v_{min}}{(1+\nu)^2 C_{\text{GFM}} n^{2/3}}, \qquad C_{\text{GFM}} \stackrel{\text{def}}{=} 2L_{\dot{V}}\left(\frac{1}{n^{2/3}} + \frac{C}{(\sqrt{2-C}-1)^2}\right),$$

we obtain

$$\begin{aligned}
&\frac{1}{K_{\max}}\sum_{k=0}^{K_{\max}-1}\mathbb{E}\left[\|\dot{V}(\widehat{S}^k)\|^2\right] + v_{\max}^2\mathcal{G}_{K_{\max}} \\
&\leq \frac{v_{\max}^2}{K_{\max}}\sum_{k=0}^{K_{\max}-1}\mathbb{E}\left[\|\bar{s}\circ\mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right] + v_{\max}^2\mathcal{G}_{K_{\max}} \\
&\leq (1+\nu)^3 C_{\text{GFM}}\frac{n^{2/3}}{K_{\max}}\frac{v_{\max}^2}{v_{\min}^2}\left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right) && (39a) \\
&+ \frac{C_0}{n^{2/3}}\{1\wedge\frac{n}{K_{\max}}\}\mathbb{E}\left[\varepsilon^{(0)}\right] && (39b) \\
&+ \frac{C_0}{n^{5/3}}\{1\wedge\frac{n}{K_{\max}}\}\sum_{k=0}^{K_{\max}-1}\mathbb{E}\left[\|\eta_{k+1}^{(1)}\|^2\right] && (39c) \\
&+ C_1\frac{1}{K_{\max}}\sum_{k=0}^{K_{\max}-1}\mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_k\right]\|^2\right] && (39d) \\
&+ \frac{C_0}{2(1+\nu)}\frac{1}{n^{2/3}K_{\max}}\sum_{k=0}^{K_{\max}-1}\left(\mathbb{E}\left[\|\eta_{k+1}^{(2)}\|^2\right] + \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_{k+3/4}\right]\|^2\right]\right); && (39e)
\end{aligned}$$

eq:coro:pfiem:opt:a

eq:coro:pfiem:opt:b

eq:coro:pfiem:opt:c

eq:coro:pfiem:opt:d

eq:coro:pfiem:opt:e

$\mathcal{G}_{K_{\max}}$ *is given by* (37) *and the constants* C_0 *and* C_1 *are given by*

$$C_0 \overset{\text{def}}{=} (1+\nu)\sqrt{\frac{1+\nu}{1+\bar{\nu}}}\frac{L_{\dot{V}}}{L}\frac{2v_{\max}^2}{v_{\min}}\left\{1 \wedge \frac{1}{\sqrt{2-C}(\sqrt{2-C}-1)}\frac{1}{n^{1/3}}\right\},$$
$$C_1 \overset{\text{def}}{=} (1+\nu)\frac{v_{\max}^4}{v_{\min}^2} + \sqrt{\frac{1+\nu}{1+\bar{\nu}}}2\frac{L_{\dot{V}}}{L}\frac{v_{\max}^2}{v_{\min}}\frac{1}{\sqrt{2-C}(\sqrt{2-C}-1)^2} + \frac{C_0}{2(1+\nu)n^{2/3}}.$$

It is easily seen that the RHS of (38) is an increasing function of C on $(0,1)$, which tends to zero when $C \to 0$ and to infinity when $C \to 1$, thus showing that $C_\star$ is unique (see Lemma 15).

Gers: Pierre: rajouter ici un commentaire sur la constante C qui n'explose pas quand $n \to \infty$. De même pour C_1

We now derive the upper bounds when the approximations $\widetilde{\mathsf{S}}_k$ and $\tilde{s}_k$ satisfy H5 and, for $u \geq 0$ and $\varepsilon > 0$, we discuss how to choose $K_{\max}, \overline{m}_k$ and m_k as a function of u, ε so that the RHS in Proposition 17 is upper bounded by $O(\varepsilon n^{-u})$. Then we have $\bar{\nu} = 1$, and if the Monte Carlo approximation is unbiased $\nu = M_\nu^{(2)} = 0$. We will use the inequality

$$\mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_k\right]\|^2\right] \leq \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_{k+3/4}\right]\|^2\right].$$

The term in (39a) says that $K_{\max} \propto n^{u+2/3}/\varepsilon$. With this choice of $K_{\max}$, the term in (39b) is $O(n^{-2/3} \wedge \{\varepsilon/n^{u+1/3}\})$. If we choose $\overline{m}_k = \overline{m}$, then the term in (39c) is $\overline{m}^{-1}\, O(n^{-2/3} \wedge \{n^{u-1}\epsilon^{-1}\})$; and it is upper bounded by $O(\varepsilon n^{-u})$ by choosing $\overline{m} \gtrsim \{\varepsilon^{-1}n^{u-2/3}\} \wedge \{n^{2u-1}\varepsilon^{-2}\}$. Finally, let us consider $m_k = m$: the first term (39e) exists for both biased and unbiased approximations. It is controlled by $O(n^{-2/3}m^{-1})$ and is upper bounded by $O(\varepsilon n^{-u})$ by choosing $m \gtrsim n^{u-2/3}\varepsilon^{-1}$. When $M_\nu^{(2)} \neq 0$, the two conditional expectations in (39d) and (39e) are a term which is $O(1/m^2)$ and this term can be bounded by $O(\varepsilon n^{-u})$ by setting $m \gtrsim n^{u/2}\varepsilon^{-1/2}$.

To summarize, the RHS (39a) to (39e) is upper bounded by $O(\varepsilon n^{-u})$ by choosing

Gers: c'est le max ou le min pour $\bar{m}$? dans les deux discussions

$$K_{\max} \gtrsim \frac{n^{u+2/3}}{\varepsilon}, \quad \overline{m} \gtrsim \frac{1}{n^{2/3-u}\varepsilon} \wedge \frac{1}{n^{1-2u}\varepsilon^2}, \quad m \gtrsim \frac{1}{n^{2/3-u}\varepsilon}$$

in the unbiased case and

$$K_{\max} \gtrsim \frac{n^{u+2/3}}{\varepsilon}, \quad \overline{m} \gtrsim \frac{1}{n^{2/3-u}\varepsilon} \wedge \frac{1}{n^{1-2u}\varepsilon^2}, \quad m \gtrsim \frac{1}{n^{2/3-u}\varepsilon} \vee \frac{n^{u/2}}{\sqrt{\varepsilon}}$$

in the biased one. Not surprisingly, the biased Monte Carlo case requires stronger conditions on the Monte Carlo batch size than the unbiased one.

In the following statement, we propose a different application of Theorem 16. In Proposition 17, we gave an upper bound on the quantity $\mathbb{E}\left[\|h(\widehat{S}^K)\|^2\right]$, where

K acts as a stopping rule for the algorithm, sampled uniformly in $\{0, \ldots, K_{\max}-1\}$. In Proposition 18, we address the case when the distribution of K is chosen among any probability distribution on $\{0, \ldots, K_{\max}-1\}$.

Gers: Proposition qui suit NON RELUE.

coro:pFIEM:given:sampling

Proposition 18 (following Theorem 16)**.** *Let $p_0, \ldots, p_{K_{\max}-1}$ be non negative real numbers such that $\sum_{k=0}^{K_{\max}-1} p_k = 1$. Let $C \in (0,1)$ satisfying*

$$v_{\min} \leq \frac{(1+\nu)}{1+2\nu}\sqrt{\frac{1+\nu}{1+\bar{\nu}}}\frac{L_{\dot{V}}}{L}\sqrt{C}\left(\frac{1}{n^{2/3}} + \frac{C}{(\sqrt{2-C}-1)^2}\right);$$

the optimal choice $C_\star$ being the unique C satisfying the equality. Define

$$F_\star(g) \stackrel{\text{def}}{=} \frac{\sqrt{C}}{Ln^{2/3}} g \left(\sqrt{\frac{1+\nu}{1+\bar{\nu}}}\frac{v_{\min}}{1+2\nu} - g\frac{1}{1+\bar{\nu}}\frac{L_{\dot{V}}}{2L}\sqrt{C}\left(\frac{1}{n^{2/3}} + \frac{C}{\left(\sqrt{2-C}-1\right)^2}\right)\right),$$

$$g_\star \stackrel{\text{def}}{=} \frac{\sqrt{(1+\bar{\nu})(1+\nu)}}{1+2\nu}\frac{v_{\min}L}{L_{\dot{V}}\sqrt{C}}\left(\frac{1}{n^{2/3}} + \frac{C}{\left(\sqrt{2-C}-1\right)^2}\right)^{-1}.$$

$F_\star$ is positive, continuous and increasing on $(0, g_\star)$, and by choosing the step sizes

$$\begin{aligned}\gamma_k &\stackrel{\text{def}}{=} \frac{\sqrt{C}}{\sqrt{(1+\bar{\nu})(1+\nu)}Ln^{2/3}}\; F_\star^{-1}\left(\frac{p_k}{\max_\ell p_\ell}\; F_\star(g_\star)\right) \\ &= \frac{\sqrt{C}}{\sqrt{(1+\bar{\nu})(1+\nu)}Ln^{2/3}}\; F_\star^{-1}\left(\frac{p_k}{\max_\ell p_\ell}\;\frac{v_{min}\sqrt{C}}{2Ln^{2/3}}\sqrt{\frac{1+\nu}{1+\bar{\nu}}}\frac{g_\star}{1+2\nu}\right)\end{aligned}$$

we obtain

$$
\begin{aligned}
&\sum_{k=0}^{K_{\max}-1} p_k \mathbb{E}\left[\|\dot{V}(\widehat{S}^k)\|^2\right] + v_{\max}^2 \mathcal{G}_{K_{\max}} \\
&\leq v_{\max}^2 \sum_{k=0}^{K_{\max}-1} p_k \mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\|^2\right] + v_{\max}^2 \mathcal{G}_{K_{\max}} \\
&\leq \frac{(1+2\nu)^2}{1+\nu} C_{\mathrm{GFM}} n^{2/3} \max_k p_k \ \frac{v_{\max}^2}{v_{\min}^2} \ \left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right) \\
&+ \bar{C}_0 \frac{K_{\max} \max_k p_k}{n^{2/3}} \{1 \wedge \frac{n}{K_{\max}}\} \mathbb{E}\left[\varepsilon^{(0)}\right] \\
&+ \bar{C}_0 \frac{K_{\max} \max_k p_k}{n^{5/3}} \{1 \wedge \frac{n}{K_{\max}}\} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\eta_{k+1}^{(1)}\|^2\right] \\
&+ \bar{C}_1 \max_k p_k \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)} | \mathcal{F}_k\right]\|^2\right] \\
&+ \frac{\bar{C}_0}{2(1+\nu)} \frac{\max_k p_k}{n^{2/3}} \sum_{k=0}^{K_{\max}-1} \left(\mathbb{E}\left[\|\eta_{k+1}^{(2)}\|^2\right] + \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)} | \mathcal{F}_{k+3/4}\right]\|^2\right]\right);
\end{aligned}
$$

$\mathcal{G}_{K_{\max}}$ *is given by* (37), C_{GFM} *is given in Proposition 17 and the constants* $\bar{C}_0$ *and* $\bar{C}_1$ *are given by*

$$
\begin{aligned}
\bar{C}_0 &\stackrel{\text{def}}{=} \frac{(1+2\nu)^2}{(1+\bar{\nu})(1+\nu)} \frac{L_{\dot{V}} C_{\mathrm{GFM}}}{L^2} \frac{v_{\max}^2}{v_{\min}^2} \left\{1 \wedge \frac{1}{\sqrt{2-C}(\sqrt{2-C}-1)} \frac{1}{n^{1/3}}\right\}, \\
\bar{C}_1 &\stackrel{\text{def}}{=} \frac{(1+2\nu)^2}{\sqrt{(1+\bar{\nu})}(1+\nu)^{3/2}} \frac{3C_{\mathrm{GFM}}}{4L} \frac{v_{\max}^4}{v_{\min}^3} \\
&\quad + \frac{(1+2\nu)^2}{(1+\bar{\nu})(1+\nu)^2} \frac{L_{\dot{V}} C_{\mathrm{GFM}}}{L^2} \frac{v_{\max}^2}{v_{\min}^2} \frac{1}{\sqrt{2-C}(\sqrt{2-C}-1)^2} + \frac{\bar{C}_0}{2(1+\nu) n^{2/3}}.
\end{aligned}
$$

Notice that in the case of a uniform stopping time, meaning $\max_k p_k = 1/K_{\max}$, we recover a similar upper bound to the one in Proposition 17 regarding the dependency in n and $K_{\max}$, the constant being slightly different. More precisely, assuming we took the same constant C for both corollaries, we have:

$$
\bar{C}_0 \geq \frac{(1+2\nu)^3}{(1+\nu)^4} C_0 \quad \text{and} \quad \bar{C}_1 \geq \frac{3}{2} \frac{(1+2\nu)^3}{(1+\nu)^4} C_1
$$

4.5 Proofs

sec:proof:pfiem

We write $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1} H_{k+1}$, where

$$
\begin{aligned}
H_{k+1} &\stackrel{\text{def}}{=} \tilde{s}_{k+1} - \widehat{S}^k + \frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \widetilde{\mathsf{S}}_{k+1,J_{k+1}} && (40) \\
&= \eta_{k+1}^{(2)} + \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \widetilde{\mathsf{S}}_{k+1,J_{k+1}}, \\
&= h(\widehat{S}^k) + \eta_{k+1}^{(2)} + \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \bar{s} \circ \mathsf{T}(\widehat{S}^k) + \frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \widetilde{\mathsf{S}}_{k+1,J_{k+1}}, \\
&= h(\widehat{S}^k) + \eta_{k+1}^{(2)} + \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widetilde{\mathsf{S}}_{k+1,J_{k+1}} \\
&\quad - \mathbb{E}\left[\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widetilde{\mathsf{S}}_{k+1,J_{k+1}} | \mathcal{F}_{k+1/2}\right]. && (41)
\end{aligned}
$$

eq:perturbed:defH

eq:perturbed:defH:espcond

Throughout the following proofs, we will largely use the inequality,

$$
\forall \beta > 0: \; 2\langle A, B\rangle \leq \frac{\nu}{\beta}\|A\|^2 + \beta\|B\|^2 \qquad (42)
$$

eq:use:nu

where $\nu = 0$ iff $B = 0$, and $\nu = 1$ otherwise.

4.5.1 Auxiliary results

em:control:field:perturbed

Lemma 19. *Assume H1-item 1-item 2, H2, H3-item 1. Let $\nu \in \{0,1\}$ defined by $\nu = 0$ iff the approximations are unbiased. Then for all $k \geq 0$,*

$$
\begin{aligned}
&\mathbb{E}[\|H_{k+1}\|^2] + (1+\nu)\,\mathbb{E}\left[\left\|\frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \bar{s}\circ\mathsf{T}(\widehat{S}^k)\right\|^2\right] \\
&\leq (1+\nu)\left\{\mathbb{E}[\|h(\widehat{S}^k)\|^2] + \mathbb{E}\left[\left\|\widetilde{\mathsf{S}}_{k+1,J_{k+1}} - \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\right\|^2\right]\right\} + \mathcal{E}_{k+1}^{(2)},
\end{aligned}
$$

where

$$
\mathcal{E}_{k+1}^{(2)} \stackrel{\text{def}}{=} \mathbb{E}\left[\|\eta_{k+1}^{(2)}\|^2\right] + \mathbb{E}\left[\|\mathbb{E}[\eta_{k+1}^{(2)}|\mathcal{F}_{k+3/4}]\|^2\right] + \mathbb{E}\left[\|\mathbb{E}[\eta_{k+1}^{(2)}|\mathcal{F}_k]\|^2\right].
$$

Proof. Upon noting that $\widehat{S}^k \in \mathcal{F}_k$ and $\mathbb{E}[H_{k+1}|\mathcal{F}_k] = \mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_k\right] + h(\widehat{S}^k)$, (see (41)), we have

$$
\mathbb{E}[\|H_{k+1}\|^2] = \mathbb{E}[\|h(\widehat{S}^k)\|^2] + \mathbb{E}[\|H_{k+1} - h(\widehat{S}^k)\|^2] + 2\mathbb{E}\left[\left\langle h(\widehat{S}^k), \mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_k\right]\right\rangle\right].
$$

First, we write (see (42) applied with $\beta \leftarrow 1$)

$$
2\mathbb{E}\left[\left\langle h(\widehat{S}^k), \mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_k\right]\right\rangle\right] \leq \nu\,\mathbb{E}[\|h(\widehat{S}^k)\|^2] + \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_k\right]\|^2\right].
$$

Set $Z_{k+1} \stackrel{\text{def}}{=} \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widetilde{\mathsf{S}}_{k+1,J_{k+1}}$. Since Z_{k+1} is $\mathcal{F}_{k+3/4}$-measurable, we have by using again (41) and (42) with $\beta \leftarrow 1$,

$$\mathbb{E}[\|H_{k+1} - h(\widehat{S}^k)\|^2] = \mathbb{E}\left[\|\eta^{(2)}_{k+1} + Z_{k+1} - \mathbb{E}\left[Z_{k+1}|\mathcal{F}_{k+1/2}\right]\|^2\right]$$
$$\leq \mathbb{E}[\|\eta^{(2)}_{k+1}\|^2] + \mathbb{E}\left[\|\mathbb{E}\left[\eta^{(2)}_{k+1}|\mathcal{F}_{k+3/4}\right]\|^2\right] + (1+\nu)\,\mathbb{E}\left[\left\|Z_{k+1} - \mathbb{E}\left[Z_{k+1}|\mathcal{F}_{k+1/2}\right]\right\|^2\right].$$

The proof is concluded upon noting $\mathbb{E}[\|U - \mathbb{E}[U|\mathcal{G}]\|^2] = \mathbb{E}\left[\|U\|^2\right] - \mathbb{E}[\|\mathbb{E}\left[U|\mathcal{G}\right]\|^2]$. □

lem:recursion:aux

Lemma 20. *Assume H1-item 1-item 2, H2, H3-item 1 and H4-item 3. Set* $L^2 \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^n L_i^2$. *Define* $\nu, \bar{\nu} \in \{0,1\}$ *with the convention that* $\nu = 0$ *iff the approximations are unbiased; and* $\bar{\nu} = 0$ *iff for all* $k \geq 0$, $\|\eta^{(1)}_{k+1}\| = \|\eta^{(2)}_{k+1}\| = \varepsilon^{(0)} = 0$ *with probability one.Let* $\{\beta_k, k \geq 1\}$ *be positive numbers. We have* $\mathbb{E}\left[\left\|\widetilde{\mathsf{S}}_{1,J_1} - \bar{s}_{J_1} \circ \mathsf{T}(\widehat{S}^0)\right\|^2\right] \leq n^{-1}\mathcal{E}^{(1)}_1$ *and for any* $k \geq 1$,

$$\mathbb{E}\left[\left\|\widetilde{\mathsf{S}}_{k+1,J_{k+1}} - \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\right\|^2\right] \leq \frac{1+\bar{\nu}}{n}\Delta_k + \frac{2}{n}\mathcal{E}^{(1)}_{k+1}$$

where $\Delta_0 = 0$ *and* Δ_k *satisfies the inequality for any* $k \geq 1$,

$$\Delta_k \leq (1+\beta_k)\frac{n-1}{n}\Delta_{k-1} + (n-1)L^2\gamma_k^2\mathbb{E}\left[\|H_k\|^2\right]$$
$$+ (n-1)L^2\frac{\gamma_k^2}{\beta_k}\left\{(1+\nu)\mathbb{E}\left[\|h(\widehat{S}^{k-1})\|^2\right] + 2\mathbb{E}\left[\left\|\mathbb{E}\left[\eta^{(2)}_k|\mathcal{F}_{k-1+1/2}\right]\right\|^2\right]\right\};$$

and for any $k \geq 0$,

$$\mathcal{E}^{(1)}_{k+1} \stackrel{\text{def}}{=} n\left(\frac{n-1}{n}\right)^{k+1}\mathbb{E}\left[\varepsilon^{(0)}\right] + \sum_{j=0}^{k}\left(\frac{n-1}{n}\right)^{k-j}\mathbb{E}\left[\left\|\eta^{(1)}_{j+1}\right\|^2\right].$$

Proof. Since J_{k+1} and $(\widehat{S}^k, \widetilde{\mathsf{S}}_{k+1,\cdot})$ are independent, we have:

$$\mathbb{E}\left[\left\|\widetilde{\mathsf{S}}_{k+1,J_{k+1}} - \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\right\|^2\right] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\left\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^k) - \widetilde{\mathsf{S}}_{k+1,i}\right\|^2\right].$$

Let $i \in \{1,\ldots,n\}$ and $\widehat{S}^{<k+1,i}$ be given by (10); note that by definition, $\widehat{S}^{<0,i} = \widehat{S}^{<1,i} = \widehat{S}^0$. Upon noting that if $\bar{\nu} = 0$, then $\bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k+1,i}) = \widetilde{\mathsf{S}}_{k+1,i}$ for all $i \in \{1,\ldots,n\}$, it holds by (42),

$$\mathbb{E}\left[\left\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^k) - \widetilde{\mathsf{S}}_{k+1,i}\right\|^2\right]$$
$$\leq (1+\bar{\nu})\mathbb{E}\left[\left\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^k) - \bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k+1,i})\right\|^2\right] + 2\mathbb{E}\left[\left\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k+1,i}) - \widetilde{\mathsf{S}}_{k+1,i}\right\|^2\right]$$
$$\leq (1+\bar{\nu})L_i^2\mathbb{E}\left[\left\|\widehat{S}^k - \widehat{S}^{<k+1,i}\right\|^2\right] + 2\mathbb{E}\left[\left\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k+1,i}) - \widetilde{\mathsf{S}}_{k+1,i}\right\|^2\right],$$

where we used H4-item 3 in the last inequality. This yields

$$\mathbb{E}\left[\left\|\widetilde{\mathsf{S}}_{k+1,J_{k+1}} - \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\right\|^2\right] \leq \frac{(1+\bar{\nu})}{n} \Delta_k + \frac{2}{n}\mathcal{E}^{(1)}_{k+1}, \tag{43}$$

eq:error:smemMC:deltak

where

$$\Delta_k \overset{\text{def}}{=} \sum_{i=1}^n L_i^2 \mathbb{E}\left[\left\|\widehat{S}^k - \widehat{S}^{<k+1,i}\right\|^2\right] = \frac{n-1}{n}\sum_{i=1}^n L_i^2 \mathbb{E}\left[\left\|\widehat{S}^k - \widehat{S}^{<k,i}\right\|^2\right], \tag{44}$$

eq:def:perturbed:delta

$$\mathcal{E}^{(1)}_{k+1} \overset{\text{def}}{=} \sum_{i=1}^n \mathbb{E}\left[\left\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k+1,i}) - \widetilde{\mathsf{S}}_{k+1,i}\right\|^2\right].$$

In the RHS of (44), we used $(\widehat{S}^k, \widehat{S}^{<k,i}) \in \mathcal{F}_k$, $I_{k+1} \in \mathcal{F}_{k+1/4}$ and:

$$\widehat{S}^{<k+1,i} = \widehat{S}^{<k,i}\mathbb{1}_{I_{k+1}\neq i} + \widehat{S}^k \mathbb{1}_{I_{k+1}=i}. \tag{45}$$

eq:hats<rec

• Since $(\widehat{S}^{<k,i}, \widetilde{\mathsf{S}}_{k,i}) \in \mathcal{F}_k$, $I_{k+1} \in \mathcal{F}_{k+1/4}$ and

$$\bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k+1,i}) - \widetilde{\mathsf{S}}_{k+1,i} = \eta^{(1)}_{k+1}\mathbb{1}_{I_{k+1}=i} + \left(\bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k,i}) - \widetilde{\mathsf{S}}_{k,i}\right)\mathbb{1}_{I_{k+1}\neq i},$$

it holds

$$\mathcal{E}^{(1)}_{k+1} = \frac{n-1}{n}\mathcal{E}^{(1)}_k + \mathbb{E}\left[\|\eta^{(1)}_{k+1}\|^2\right].$$

The expression of $\mathcal{E}^{(1)}_{k+1}$ follows by a trivial induction, using that $\mathcal{E}^{(1)}_0 = n\mathbb{E}\left[\varepsilon^{(0)}\right]$.
• Let us consider Δ_k. By (45),

$$\frac{n}{n-1}\Delta_k = \sum_{i=1}^n L_i^2 \mathbb{E}\left[\|\widehat{S}^k - \widehat{S}^{k-1} + \left(\widehat{S}^{k-1} - \widehat{S}^{<k-1,i}\right)\mathbb{1}_{I_k\neq i}\|^2\right],$$

By the inequality $2\left\langle \tilde{U}, V\right\rangle \leq \beta^{-1}\|\tilde{U}\|^2 + \beta\|V\|^2$ which holds true for any $\beta > 0$, we have for any $\mathcal{G}$-measurable r.v. V

$$\mathbb{E}\left[\|U+V\|^2\right] \leq \mathbb{E}\left[\|U\|^2\right] + \beta^{-1}\mathbb{E}\left[\|\mathbb{E}\left[U|\mathcal{G}\right]\|^2\right] + (1+\beta)\mathbb{E}\left[\|V\|^2\right].$$

Applying this inequality with $U \leftarrow \widehat{S}^k - \widehat{S}^{k-1} = \gamma_k H_k$, $V \leftarrow \left(\widehat{S}^{k-1} - \widehat{S}^{<k-1,i}\right)\mathbb{1}_{I_k\neq i}$, $\mathcal{G} \leftarrow \mathcal{F}_{k-1+1/4}$ and $\beta \leftarrow \beta_k$, and since $I_k \in \mathcal{F}_{k-1+1/4}$ and $(\widehat{S}^{k-1}, \widehat{S}^{<k,i}) \in \mathcal{F}_{k-1}$, it holds

$$\frac{\Delta_k}{n-1} \leq \frac{(1+\beta_k)}{n}\Delta_{k-1} + L^2\gamma_k^2\left\{\mathbb{E}\left[\|H_k\|^2\right] + \beta_k^{-1}\mathbb{E}\left[\left\|\mathbb{E}\left[H_k|\mathcal{F}_{k-1+1/4}\right]\right\|^2\right]\right\}. \tag{46}$$

eq:recurrence:deltatilde

From (41) and by definition of $\eta^{(2)}_k$,

$$\mathbb{E}\left[H_k|\mathcal{F}_{k-1+1/4}\right] = h(\widehat{S}^{k-1}) + \mathbb{E}\left[\eta^{(2)}_k|\mathcal{F}_{k-1+1/4}\right] = h(\widehat{S}^{k-1}) + \mathbb{E}\left[\eta^{(2)}_k|\mathcal{F}_{k-1+1/2}\right],$$

which yields, using again (42) applied with $\beta \leftarrow 1$,

$$\mathbb{E}\left[\left\|\mathbb{E}\left[H_k|\mathcal{F}_{k-1+1/4}\right]\right\|^2\right] \leq (1+\nu)\mathbb{E}\left[\left\|h(\widehat{S}^{k-1})\right\|^2\right] + 2\mathbb{E}\left[\left\|\mathbb{E}\left[\eta_k^{(2)}|\mathcal{F}_{k-1+1/2}\right]\right\|^2\right].$$

This concludes the proof. □

Proposition 21 combines Lemma 19 and Lemma 20 in order to provide an upper bound of the errors $\mathbb{E}\left[\left\|\widetilde{\mathsf{S}}_{k+1,J_{k+1}} - \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\right\|^2\right]$, $k \geq 0$, in terms of $h(\widehat{S}^k)$, $\eta_{k+1}^{(1)}$ and $\eta_{k+1}^{(2)}$ for $k \geq 0$.

n:variance:field:perturbed **Proposition 21.** *Assume H1-item 1-item 2, H2, H3-item 1 and H4-item 3. Set* $L^2 \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^n L_i^2$. *Define* $\nu, \bar{\nu} \in \{0,1\}$ *with the convention that* $\nu = 0$ *iff the approximations are unbiased; and* $\bar{\nu} = 0$ *iff for all* $k \geq 0$, $\|\eta_{k+1}^{(1)}\| = \|\eta_{k+1}^{(2)}\| = \varepsilon^{(0)} = 0$ *with probability one. We have*

$$\mathbb{E}\left[\left\|\widetilde{\mathsf{S}}_{1,J_1} - \bar{s}_{J_1} \circ \mathsf{T}(\widehat{S}^0)\right\|^2\right] \leq \frac{1}{n}\mathcal{E}_1^{(1)},$$

and for any $k \geq 1$,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\widetilde{\mathsf{S}}_{k+1,J_{k+1}} - \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k)\right\|^2\right] \\
&\qquad + (1+\bar{\nu})(1+\nu)\frac{n-1}{n}L^2\sum_{j=1}^k \omega_{j,k}\gamma_j^2\,\mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{j,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^{j-1})\|^2\right] \\
&\leq (1+\bar{\nu})(1+\nu)\frac{n-1}{n}L^2\sum_{j=1}^k \omega_{j,k}\gamma_j^2\left(1+\frac{1}{\beta_j}\right)\mathbb{E}\left[\|h(\widehat{S}^{j-1})\|^2\right] \\
&+ \frac{2}{n}\mathcal{E}_{k+1}^{(1)} + 2(1+\bar{\nu})(1+\nu)\frac{n-1}{n^2}L^2\sum_{j=1}^k \omega_{j,k}\gamma_j^2\mathcal{E}_j^{(1)} \\
&+ (1+\bar{\nu})\frac{n-1}{n}L^2\sum_{j=1}^k \omega_{j,k}\gamma_j^2\mathcal{E}_j^{(2)} \\
&+ 2(1+\bar{\nu})\frac{n-1}{n}L^2\sum_{j=1}^k \omega_{j,k}\frac{\gamma_j^2}{\beta_j}\mathbb{E}\left[\left\|\mathbb{E}\left[\eta_j^{(2)}|\mathcal{F}_{j-1+1/2}\right]\right\|^2\right].
\end{aligned}$$

The quantities $\mathcal{E}_k^{(i)}, i = 1,2$ *are defined in Lemma 19 and 20, for* $1 \leq j \leq k-1$

$$\omega_{j,k} \stackrel{\text{def}}{=} \left(\frac{n-1}{n}\right)^{k-j}\prod_{\ell=j+1}^k \left(1+\beta_\ell+(1+\bar{\nu})(1+\nu)L^2\gamma_\ell^2\right);\ \omega_{jj} \stackrel{\text{def}}{=} 1. \tag{47}$$ eq:omega

Proof. Let Δ_k be given by Lemma 20. By plugging the upper bound on $\mathbb{E}\left[\|H_k\|^2\right]$ given by Lemma 19 and the upper bound on Δ_k given by Lemma 20, we have for any $k \geq 1$,

$$\Delta_k \leq \frac{n-1}{n} \alpha_k \Delta_{k-1} + R_k$$

where $\alpha_k \stackrel{\text{def}}{=} 1 + \beta_k + (1+\bar{\nu})(1+\nu) L^2 \gamma_k^2$ and

$$\begin{aligned} R_k &\stackrel{\text{def}}{=} (1+\nu)(n-1) L^2 \gamma_k^2 \left(1 + \beta_k^{-1}\right) \mathbb{E}\left[\|h(\widehat{S}^{k-1})\|^2\right] \\ &\quad - (1+\nu)(n-1) L^2 \gamma_k^2 \mathbb{E}\left[\|\frac{1}{n} \sum_{i=1}^n \widetilde{\mathsf{S}}_{k,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^{k-1})\|^2\right] \\ &+ 2(1+\nu) \frac{n-1}{n} L^2 \gamma_k^2 \mathcal{E}_k^{(1)} + (n-1) L^2 \gamma_k^2 \left\{ \mathcal{E}_k^{(2)} + \frac{2}{\beta_k} \mathbb{E}\left[\left\| \mathbb{E}\left[\eta_k^{(2)} | \mathcal{F}_{k-1+1/2}\right] \right\|^2\right] \right\}; \end{aligned}$$

$\mathcal{E}_j^{(1)}$ and $\mathcal{E}_j^{(2)}$ are defined in Lemma 19 and 20. This yields, using that $\Delta_0 = 0$, that for any $k \geq 1$,

$$\Delta_k \leq \sum_{j=1}^k \omega_{j,k} R_j .$$

This inequality and Lemma 20 conclude the proof. □

4.6 Proof of Theorem 16

sec:proof:theo:perturbed

Gers: Pierre-1: il faudrait soigner un peu la mise en page, il y a des équations qui dépassent les lignes ...

Step 1. Let $k \geq 0$. By Lemma 11, it holds

$$\begin{aligned} V(\widehat{S}^{k+1}) &\leq V(\widehat{S}^k) + \left\langle \widehat{S}^{k+1} - \widehat{S}^k, \dot{V}(\widehat{S}^k) \right\rangle + \frac{L_{\dot{V}}}{2} \|\widehat{S}^{k+1} - \widehat{S}^k\|^2 \\ &\leq V(\widehat{S}^k) + \gamma_{k+1} \left\langle H_{k+1}, \dot{V}(\widehat{S}^k) \right\rangle + \frac{L_{\dot{V}}}{2} \gamma_{k+1}^2 \|H_{k+1}\|^2. \end{aligned} \tag{48}$$

eq:gradV:lipsch

Let $\beta_0 \in \left(0, v_{\min}/v_{\max}^2\right)$ - where $v_{\min}, v_{\max}$ are given by H4-item 2. By (41), (42) applied with $\beta \leftarrow \beta_0$ and using $\widehat{S}^k \in \mathcal{F}_k$,

$$\begin{aligned} \mathbb{E}\left[\left\langle H_{k+1}, \dot{V}(\widehat{S}^k) \right\rangle\right] &= \mathbb{E}\left[\left\langle h(\widehat{S}^k), \dot{V}(\widehat{S}^k) \right\rangle\right] + \mathbb{E}\left[\left\langle \mathbb{E}[\eta_{k+1}^{(2)} | \mathcal{F}_k], \dot{V}(\widehat{S}^k) \right\rangle\right] \\ &\leq -v_{\min} \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] + \nu \beta_0 \mathbb{E}\left[\|\dot{V}(\widehat{S}^k)\|^2\right] + \frac{1}{4\beta_0} \mathbb{E}\left[\left\| \mathbb{E}[\eta_{k+1}^{(2)} | \mathcal{F}_k] \right\|^2\right] \\ &\leq -(v_{\min} - \nu v_{max}^2 \beta_0) \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] + \frac{1}{4\beta_0} \mathbb{E}\left[\left\| \mathbb{E}[\eta_{k+1}^{(2)} | \mathcal{F}_k] \right\|^2\right], \end{aligned} \tag{49}$$

eq:pscal:lyap

where we used again Lemma 11 in the last inequality. Hereafter, set $\widetilde{v}_{\min} \stackrel{\text{def}}{=} v_{\min} - \nu v_{max}^2 \beta_0$, which is positive by definition of β_0. (48) and (49) yield

$$\gamma_{k+1}\widetilde{v}_{\min}\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] \leq \mathbb{E}\left[V(\widehat{S}^k)\right] - \mathbb{E}\left[V(\widehat{S}^{k+1})\right] + \frac{\gamma_{k+1}}{4\beta_0}\mathbb{E}\left[\left\|\mathbb{E}[\eta_{k+1}^{(2)}|\mathcal{F}_k]\right\|^2\right] + \frac{L_{\dot{V}}}{2}\gamma_{k+1}^2\,\mathbb{E}\left[\|H_{k+1}\|^2\right].$$

By Lemma 19 and Proposition 21, it holds for $k \geq 1$,

$$\begin{aligned}
&\gamma_{k+1}\left(\widetilde{v}_{\min} - (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_{k+1}\right)\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] \\
&+ (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_{k+1}^2\mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right] \\
&+ (1+\bar{\nu})(1+\nu)^2\frac{n-1}{n}\frac{L^2 L_{\dot{V}}}{2}\gamma_{k+1}^2\sum_{j=1}^k \omega_{j,k}\gamma_j^2\mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{j,i} - \bar{s}\circ\mathsf{T}(\widehat{S}^{j-1})\|^2\right] \\
&\leq \mathbb{E}\left[V(\widehat{S}^k)\right] - \mathbb{E}\left[V(\widehat{S}^{k+1})\right] \\
&+ (1+\bar{\nu})(1+\nu)^2\frac{n-1}{n}\frac{L^2\,L_{\dot{V}}}{2}\gamma_{k+1}^2\sum_{j=1}^k \omega_{j,k}\gamma_j^2\left(1+\frac{1}{\beta_j}\right)\mathbb{E}\left[\|h(\widehat{S}^{j-1})\|^2\right] \\
&+ (1+\nu)\frac{1}{n}L_{\dot{V}}\gamma_{k+1}^2\mathcal{E}_{k+1}^{(1)} + (1+\bar{\nu})(1+\nu)^2\frac{n-1}{n^2}L^2\,L_{\dot{V}}\gamma_{k+1}^2\sum_{j=1}^k \omega_{j,k}\gamma_j^2\mathcal{E}_j^{(1)} \\
&+ \frac{L_{\dot{V}}}{2}\gamma_{k+1}^2\mathcal{E}_{k+1}^{(2)} + (1+\bar{\nu})(1+\nu)\frac{n-1}{n}\frac{L^2\,L_{\dot{V}}}{2}\gamma_{k+1}^2\sum_{j=1}^k \omega_{j,k}\gamma_j^2\mathcal{E}_j^{(2)} \\
&+ (1+\bar{\nu})(1+\nu)\frac{n-1}{n}L^2\,L_{\dot{V}}\gamma_{k+1}^2\sum_{j=1}^k \omega_{j,k}\frac{\gamma_j^2}{\beta_j}\mathbb{E}\left[\left\|\mathbb{E}\left[\eta_j^{(2)}|\mathcal{F}_{j-1+1/2}\right]\right\|^2\right] \\
&+ \frac{\gamma_{k+1}}{4\beta_0}\mathbb{E}\left[\left\|\mathbb{E}[\eta_{k+1}^{(2)}|\mathcal{F}_k]\right\|^2\right];
\end{aligned}$$

and for $k = 0$,

$$\begin{aligned}
&\gamma_1\left(\widetilde{v}_{\min} - (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_1\right)\mathbb{E}\left[\|h(\widehat{S}^0)\|^2\right] + (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_1^2\mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{1,i} - \bar{s}\circ\mathsf{T}(\widehat{S}^0)\|^2\right] \\
&\leq \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^1)\right] + (1+\nu)\frac{1}{n}\frac{L_{\dot{V}}}{2}\gamma_1^2\mathcal{E}_1^{(1)} + \frac{L_{\dot{V}}}{2}\gamma_1^2\mathcal{E}_1^{(2)} + \frac{\gamma_1}{4\beta_0}\mathbb{E}\left[\left\|\mathbb{E}[\eta_1^{(2)}|\mathcal{F}_0]\right\|^2\right].
\end{aligned}$$

These two cases ($k \geq 1$ and $k = 0$) can be casted into the expression given for $k \geq 1$, setting by convention that $\sum_{j=q}^{q-1} = 0$. We now sum from $k = 0$ to

$k = K_{\max} - 1$ and we obtain

$$\sum_{k=0}^{K_{\max}-1} \widetilde{\alpha}_k \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] + \sum_{k=0}^{K_{\max}-1} \widetilde{\delta}_{k+1} \mathbb{E}\left[\|\frac{1}{n}\sum_{i=1}^n \widetilde{\mathsf{S}}_{k+1,i} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right]$$
$$\leq \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right] + \sum_{k=0}^{K_{\max}-1} \widetilde{\beta}_{k+1}^{(1)} \mathcal{E}_{k+1}^{(1)} + \sum_{k=0}^{K_{\max}-1} \widetilde{\beta}_{k+1}^{(2)} \mathcal{E}_{k+1}^{(2)}$$
$$+ \sum_{k=0}^{K_{\max}-2} \widetilde{\beta}_{k+1}^{(3)} \mathbb{E}\left[\left\|\mathbb{E}\left[\eta_{k+1}^{(2)} | \mathcal{F}_{k+1/2}\right]\right\|^2\right] + \frac{1}{4\beta_0} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1} \mathbb{E}\left[\left\|\mathbb{E}[\eta_{k+1}^{(2)} | \mathcal{F}_k]\right\|^2\right],$$

where

$$\widetilde{\alpha}_k \stackrel{\text{def}}{=} \gamma_{k+1} \widetilde{v}_{\min} - (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_{k+1}^2 \left\{1 + (1+\bar{\nu})(1+\nu)\frac{n-1}{n} L^2 \left(1 + \frac{1}{\beta_{k+1}}\right) \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \omega_{k+1,j}\right\}, \tag{50}$$

$$\widetilde{\delta}_{k+1} \stackrel{\text{def}}{=} (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_{k+1}^2 \left\{1 + (1+\bar{\nu})(1+\nu)\frac{n-1}{n} L^2 \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \omega_{k+1,j}\right\},$$

$$\widetilde{\beta}_{k+1}^{(1)} \stackrel{\text{def}}{=} (1+\nu)\frac{L_{\dot{V}}}{n}\gamma_{k+1}^2 \left\{1 + (1+\bar{\nu})(1+\nu)\frac{n-1}{n} L^2 \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \omega_{k+1,j}\right\}, \tag{51}$$

eq:def:betatilde:1

$$\widetilde{\beta}_{k+1}^{(2)} \stackrel{\text{def}}{=} \frac{L_{\dot{V}}}{2}\gamma_{k+1}^2 \left\{1 + (1+\bar{\nu})(1+\nu)\frac{n-1}{n} L^2 \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \omega_{k+1,j}\right\}, \tag{52}$$

eq:def:betatilde:2

$$\widetilde{\beta}_{k+1}^{(3)} \stackrel{\text{def}}{=} (1+\bar{\nu})(1+\nu)\frac{n-1}{n} L^2 \, L_{\dot{V}} \frac{\gamma_{k+1}^2}{\beta_{k+1}} \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \omega_{k+1,j}. \tag{53}$$

eq:def:betatilde:3

This concludes the proof, upon noting that for any $k \geq 0$, with probability one, $\mathbb{E}[\eta_{k+1}^{(2)} | \mathcal{F}_k] = \mathbb{E}[\eta_{k+1}^{(2)} | \mathcal{F}_{k+1/2}]$. We have

$$\xi_0(K_{\max}, n) \stackrel{\text{def}}{=} n \sum_{k=0}^{K_{\max}-1} \widetilde{\beta}_{k+1}^{(1)} \left(\frac{n-1}{n}\right)^{k+1}, \tag{54}$$

eq:def:Xi0

$$\Xi_1(\eta_\cdot^{(1)}, K_{\max}, n) \stackrel{\text{def}}{=} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\eta_{k+1}^{(1)}\|^2\right] \sum_{j=k}^{K_{\max}-1} \widetilde{\beta}_{j+1}^{(1)} \left(\frac{n-1}{n}\right)^{j-k}, \tag{55}$$

eq:def:Xi1

$$\Xi_2(\eta_\cdot^{(2)}, K_{\max}, n) \stackrel{\text{def}}{=} \sum_{k=0}^{K_{\max}-1} \widetilde{\beta}_{k+1}^{(2)} \left(\mathbb{E}\left[\|\eta_{k+1}^{(2)}\|^2\right] + \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)} | \mathcal{F}_{k+3/4}\right]\|^2\right]\right)$$
$$+ \sum_{k=0}^{K_{\max}-1} \left(\frac{\gamma_{k+1}}{4\beta_0} + \widetilde{\beta}_{k+1}^{(2)} + \widetilde{\beta}_{k+1}^{(3)}\right) \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)} | \mathcal{F}_k\right]\|^2\right]. \tag{56}$$

eq:def:Xi2

4.7 Proof of Proposition 17

Gers: Pierre: si on garde l'évolution des preuves des propositions du cas déterministe, il faudra répercuter ici les mêmes changements. Et réfléchir à la meilleure façon de choisir les paramétrages λ, C

c:proof:coro:pFIEM:optimal

We now consider sequences of the form

$$\beta_\ell \stackrel{\text{def}}{=} \frac{\lambda(1-C)}{n^{\mathsf{b}}}, \qquad \gamma_\ell \stackrel{\text{def}}{=} \frac{1}{\sqrt{(1+\bar{\nu})(1+\nu)}} \frac{\sqrt{C}}{Ln^{\mathsf{c}}} g_\ell,$$

where $\lambda, C \in (0,1)$, $g_1, \ldots, g_{K_{\max}}$ in $(0,1]$, $\mathsf{b} \in [1,+\infty)$ and $\mathsf{c} \in [1/2, \infty)$. Note that this choice of λ, C implies that

$$0 < \lambda(1-C) + C < 1\ . \tag{57}$$

eq:condition:lambda-c

We discuss how to choose $\lambda, C, \mathsf{b}, \mathsf{c}$ so that the term

$$\alpha_k \stackrel{\text{def}}{=} \gamma_{k+1}\left(v_{\min} - \nu v_{\max}^2 \beta_0 - (1+\nu)\frac{L_{\dot{V}}}{2}\gamma_{k+1}\left\{1 + (1+\bar{\nu})(1+\nu)L^2\Lambda_k\right\}\right)$$

is non-negative, $\sum_{k=0}^{K_{\max}-1} \alpha_k > 0$ and is as large as possible.

Step 2a- Choice of the sequences γ_l and β_l. We first derive a choice of the design parameters such that $\sum_{k=0}^{K_{\max}-1} \alpha_k$ is as large as possible. We write for any $n \geq 1$,

$$\begin{aligned}\gamma_{k+1}^2(1+\beta_{k+1}^{-1}) &= \frac{1}{(1+\bar{\nu})(1+\nu)}\left(\frac{\sqrt{C}}{Ln^{\mathsf{c}}}\right)^2 g_{k+1}^2 \left(1 + \frac{n^{\mathsf{b}}}{\lambda(1-C)}\right) \\ &\leq \frac{1}{(1+\bar{\nu})(1+\nu)} \frac{C}{L^2 n^{2\mathsf{c}-\mathsf{b}}} g_{k+1}^2 \left(1 + \frac{1}{\lambda(1-C)}\right)\ ;\end{aligned}$$

and by setting

$$\mathsf{a}_n \stackrel{\text{def}}{=} 1 - \left\{\lambda(1-C)n^{1-\mathsf{b}} + Cn^{1-2\mathsf{c}}\right\}, \tag{58}$$

eq:def:an

we write

$$\left(1 - \frac{1}{n} + \beta_\ell + (1+\bar{\nu})(1+\nu)\gamma_\ell^2 L^2\right) = 1 - \frac{\mathsf{a}_n}{n}, \tag{59}$$

eq:majoration:pa

$$\mathsf{a}_n \geq 1 - \{\lambda(1-C) + C\} = (1-\lambda)(1-C). \tag{60}$$

eq:minoration:pa

The condition (57) implies that $\mathsf{a}_n \in (0,1)$. Finally,

$$\begin{aligned}&\sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \prod_{\ell=k+2}^{j}\left(1 - \frac{1}{n} + \beta_\ell + (1+\bar{\nu})(1+\nu)L^2\gamma_\ell^2\right) \\ &\quad\leq \frac{1}{(1+\bar{\nu})(1+\nu)} \frac{C}{L^2 n^{2\mathsf{c}}} \sum_{j=k+1}^{K_{\max}-1}\left(1 - \frac{\mathsf{a}_n}{n}\right)^{j-k-1} \\ &\quad\leq \frac{1}{(1+\bar{\nu})(1+\nu)} \frac{C}{L^2 n^{2\mathsf{c}}}\left(\frac{n}{\mathsf{a}_n} \wedge K_{\max}\right).\end{aligned}$$

As a conclusion

$$\gamma_{k+1}^2 \Lambda_k \leq \frac{1}{(1+\bar{\nu})^2(1+\nu)^2} \frac{C^2}{L^4 n^{4\mathsf{c}-\mathsf{b}}} g_{k+1}^2 \left(1 + \frac{1}{\lambda(1-C)}\right) \left(\frac{n}{\mathsf{a}_n} \wedge K_{\max}\right) .$$

With these upper bounds, we obtain

$$\alpha_k \geq \frac{1}{\sqrt{(1+\bar{\nu})(1+\nu)}} \frac{\sqrt{C}\widetilde{v}_{\min}}{L n^{\mathsf{c}}} g_{k+1} - \frac{1}{(1+\bar{\nu})} \frac{C L_{\dot{V}}}{2L^2 n^{2\mathsf{c}}} g_{k+1}^2 - \frac{1}{(1+\bar{\nu})} \frac{C^2 L_{\dot{V}}}{2L^2 n^{4\mathsf{c}-\mathsf{b}}} g_{k+1}^2 \left(1 + \frac{1}{\lambda(1-C)}\right) \left(\frac{n}{\mathsf{a}_n} \wedge K_{\max}\right) . \quad (61)$$ eq:min1:alphak

We want c and b to be minimal and (i) if $K_{\max}\mathsf{a}_n < n$, $4\mathsf{c} - \mathsf{b} \geq \mathsf{c}$ i.e $(\mathsf{b}, \mathsf{c}) = (1, 1/2)$; and if $K_{\max}\mathsf{a}_n \geq n$, $4\mathsf{c} - \mathsf{b} - 1 \geq \mathsf{c}$ i.e. $(\mathsf{b}, \mathsf{c}) = (1, 2/3)$.

Set

$$A \stackrel{\text{def}}{=} \frac{1}{L n^{\mathsf{c}}} \sqrt{\frac{C}{(1+\bar{\nu})(1+\nu)}} \quad (62)$$ eq:def:cstA

$$B \stackrel{\text{def}}{=} \sqrt{\frac{1+\nu}{1+\bar{\nu}}} \frac{L_{\dot{V}}}{2L} \sqrt{C} \left(\frac{1}{n^{\mathsf{c}}} + \frac{C(1+(1-C)\lambda)}{\lambda(1-\lambda)(1-C)^2} \left\{\frac{1}{n^{3\mathsf{c}-2}} \wedge \frac{K_{\max}\mathsf{a}_n}{n^{3\mathsf{c}-1}}\right\}\right) \quad (63)$$

$$= \sqrt{\frac{1+\nu}{1+\bar{\nu}}} \frac{L_{\dot{V}}}{2L} \sqrt{C} \left(\frac{1}{n^{\mathsf{c}}} + \frac{C(1+(1-C)\lambda)}{\lambda(1-\lambda)(1-C)^2} \left\{\mathbb{1}_{K_{\max}\mathsf{a}_n \geq n} + \frac{K_{\max}\mathsf{a}_n}{\sqrt{n}} \mathbb{1}_{K_{\max}\mathsf{a}_n < n}\right\}\right) . \quad (64)$$ eq:def:cstB

This yields

$$\alpha_k \geq A g_{k+1} \left(\widetilde{v}_{\min} - g_{k+1} B\right) .$$

The function $g \mapsto Ag\left(\widetilde{v}_{\min} - gB\right)$ is maximal at $g_\star \stackrel{\text{def}}{=} \widetilde{v}_{\min}/(2B)$ and its maximal value is $A\widetilde{v}_{\min}^2/(4B)$ (see Lemma 14). Therefore by choosing (λ, C) such that $g_\star \in (0, 1]$ that is satisfying the condition

$$\widetilde{v}_{\min} \leq 2B , \quad (65)$$ eq:condition:root:MC-v2

we obtain

$$\alpha_k \geq \frac{A\widetilde{v}_{\min}^2}{4B} = \frac{\widetilde{v}_{\min}^2}{2(1+\nu)L_{\dot{V}}} \begin{cases} n^{-2/3} \left(\frac{1}{n^{2/3}} + \frac{C(1+(1-C)\lambda)}{\lambda(1-\lambda)(1-C)^2}\right)^{-1} & K_{\max}\mathsf{a}_n \geq n, \\ \left(1 + \frac{C(1+(1-C)\lambda)}{\lambda(1-\lambda)(1-C)^2} K_{\max}\mathsf{a}_n\right)^{-1} & K_{\max}\mathsf{a}_n < n \end{cases} \quad (66)$$ eq:lowerbound:pk:MC-v2

For any $C \in (0, 1)$, $\lambda \mapsto (1 + (1-C)\lambda)/(\lambda(1-\lambda))$ is minimal at (see Lemma 15)

Gers: Pierre-1: reprendre une partie du lemme pour virer la constante $C/(1-C)^2$

$$\lambda_\star(C) \stackrel{\text{def}}{=} \frac{\sqrt{2-C} - 1}{1-C} \in (0, 1) . \quad (67)$$ eq:lambda:star:C

With this choice of λ, the condition (65) is satisfied for any $C \in (0,1)$ such that

$$\widetilde{v}_{\min} \frac{L}{L_{\dot{V}}} \sqrt{\frac{1+\bar{\nu}}{1+\nu}} \leq \sqrt{C} \left(\frac{1}{n^{2/3}} + \frac{C}{(\sqrt{2-C}-1)^2} \right) \; ; \tag{68}$$

eq:condition:sur:C

note that we can always choose C given by

$$\widetilde{C} \text{ s.t. } \widetilde{v}_{\min} \frac{L}{L_{\dot{V}}} \sqrt{\frac{1+\bar{\nu}}{1+\nu}} = \frac{\widetilde{C}^{3/2}}{(\sqrt{2-\widetilde{C}}-1)^2} \; .$$

For the pair $(\lambda_\star(C), C)$ where the constant $C \in (0,1)$ satisfies (68), we have for any $\ell \geq 1$

$$\gamma_\ell \overset{\text{def}}{=} \frac{2\widetilde{v}_{\min}}{(1+\nu) C_{\text{GFM}} n^{2/3}}, \qquad C_{\text{GFM}} \overset{\text{def}}{=} 2 L_{\dot{V}} \left(\frac{1}{n^{2/3}} + \frac{C}{(\sqrt{2-C}-1)^2} \right); \tag{69}$$

eq:gamma:def:perturbed

and

$$\alpha_k \geq \frac{\widetilde{v}^2_{\min}}{2(1+\nu) L_{\dot{V}}} n^{-2/3} \left(\frac{1}{n^{2/3}} + \frac{C}{(\sqrt{2-C}-1)^2} \right)^{-1} = \frac{\widetilde{v}^2_{\min}}{(1+\nu) C_{\text{GFM}}} \frac{1}{n^{2/3}} \; . \tag{70}$$

eq:minoration:alphak

As a conclusion of these discussions, we also have

$$\beta_\ell = \frac{\sqrt{2-C}-1}{n}, \qquad \ell = 1, \ldots, K_{\max}. \tag{71}$$

eq:beta:def:perturbed

Step 2b- Upper bounds for $\widetilde{\beta}^{(i)}_k$. In the sequel, γ_k and β_k for $k = 1, \ldots, K_{\max}$ are fixed to the values given by (69) and (71). We derive upper bounds for the quantities $\widetilde{\beta}^{(i)}_{k+1}$, $i = 1, 2, 3$ defined by (51), (52) and (53). We will largely use the following inequalities

$$4 \left(\frac{\widetilde{v}_{\min}}{(1+\nu) C_{\text{GFM}}} \right)^2 \leq \frac{C}{(1+\bar{\nu})(1+\nu) L^2} \leq \frac{1}{(1+\bar{\nu})(1+\nu) L^2}, \tag{72}$$

eq:ratio:vmintilde:CGFM

which are obtained from (65), (69) and $C \in (0,1)$. By definition of $\omega_{k+1,j}$ (see (47)), and by (59), (60), (67), (69) and (72),

$$\sum_{j=k+1}^{K_{\max}-1} \gamma^2_{j+1} \omega_{k+1,j} \leq \gamma_1^2 \left(\frac{n}{\mathsf{a}_n} \wedge K_{\max} \right) \leq \frac{1}{(1+\bar{\nu})(1+\nu) L^2 n^{1/3}} \left(\Omega(C) \wedge \frac{K_{\max}}{n} \right)$$

$$\Omega(C) \overset{\text{def}}{=} \frac{1}{\sqrt{2-C}\; (\sqrt{2-C}-1)}.$$

Consequently, the following upper bounds hold.

$$\widetilde{\beta}^{(1)}_{k+1} \leq (1+\nu) \frac{L_{\dot{V}}}{n^{7/3}} \frac{4\widetilde{v}^2_{\min}}{((1+\nu) C_{\text{GFM}})^2} \left\{ 1 + \frac{\Omega(C)}{n^{1/3}} \right\}, \tag{73}$$

eq:majoration:tildebeta1

$$\widetilde{\beta}^{(2)}_{k+1} \leq \frac{L_{\dot{V}}}{2} \frac{1}{n^{4/3}} \frac{4\widetilde{v}^2_{\min}}{((1+\nu) C_{\text{GFM}})^2} \left\{ 1 + \frac{\Omega(C)}{n^{1/3}} \right\}, \tag{74}$$

eq:majoration:tildebeta2

$$\widetilde{\beta}^{(3)}_{k+1} \leq L_{\dot{V}} \frac{4\widetilde{v}^2_{\min}}{((1+\nu) C_{\text{GFM}})^2} \frac{\Omega(C)}{\sqrt{2-C}-1} \frac{1}{n^{2/3}}. \tag{75}$$

eq:majoration:tildebeta3

Step 2c- Upper bounds for $\widetilde{\beta}_k^{(i)} / \sum_{k=1}^{K_{\max}-1} \alpha_k$. Now that C is chosen so that $\alpha_k > 0$, we can now derive from Theorem 16 an upper bound for $\mathbb{E}\left[\|\bar{s} \circ \mathsf{T}(\widehat{S}^K) - \widehat{S}^K\|^2\right]$ where $K \in \{0, \ldots, K_{\max}-1\}$ is a random variable independent of $\mathcal{F}_{K_{\max}}$ and with distribution $\{\alpha_k / \sum_{\ell=0}^{K_{\max}-1} \alpha_\ell, k = 0, \ldots, K_{\max}-1\}$. We derive below upper bounds for the quantities ξ_0, Ξ_1 and Ξ_2 divided by $\sum_{k=0}^{K_{\max}-1} \alpha_k$ (see (54), (55) and (56)).

Gers: Pierre-1: attention, des refs ont sauté; je te laisse les reprendre

From (70), we have

$$\left(\sum_{k=0}^{K_{\max}-1} \alpha_k\right)^{-1} \leq \frac{n^{2/3}}{K_{\max}} \frac{(1+\nu) C_{\mathrm{GFM}}}{\widetilde{v}_{\min}^2}.$$

From (54), (73), (**??**) and (69)

$$\left(\sum_{k=0}^{K_{\max}-1} \alpha_k\right)^{-1} \xi_0(K_{\max}) \leq \ \Lambda_0 \stackrel{\text{def}}{=} \left(\frac{1}{n^{2/3}} \wedge \frac{n^{1/3}}{K_{\max}}\right) \sqrt{\frac{1+\nu}{1+\bar{\nu}}} \frac{L_{\dot{V}}}{L} \frac{2}{\widetilde{v}_{\min}} \left\{1 + \frac{\Omega(C)}{n^{1/3}}\right\}. \tag{76}$$

eq:def:Lambda0

From (55), (73), (**??**) and (69), we write

$$\left(\sum_{k=0}^{K_{\max}-1} \alpha_k\right)^{-1} \Xi_1(\eta^{(1)}, K_{\max}, n) \leq \Lambda_1 \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\eta_{k+1}^{(1)}\|^2\right]$$

where

$$\Lambda_1 \stackrel{\text{def}}{=} \frac{1}{n^{2/3}} \left(\frac{1}{K_{\max}} \wedge \frac{1}{n}\right) \sqrt{\frac{1+\nu}{1+\bar{\nu}}} \frac{L_{\dot{V}}}{L} \frac{2}{\widetilde{v}_{\min}} \left\{1 + \frac{\Omega(C)}{n^{1/3}}\right\}. \tag{77}$$

eq:def:Lambda1

Finally, from (56), (74), (75) (**??**) and (69), we write

$$\begin{aligned}\left(\sum_{k=0}^{K_{\max}-1} \alpha_k\right)^{-1} \Xi_2(\eta^{(2)}, K_{\max}, n) &\leq \Lambda_2 \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)} | \mathcal{F}_k\right]\|^2\right] \\ &+ \Lambda_3 \sum_{k=0}^{K_{\max}-1} \left(\mathbb{E}\left[\|\eta_{k+1}^{(2)}\|^2\right] + \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)} | \mathcal{F}_{k+3/4}\right]\|^2\right]\right),\end{aligned}$$

where

$$\Lambda_2 \stackrel{\text{def}}{=} \frac{1}{2\beta_0 \widetilde{v}_{\min} K_{\max}} + \frac{1}{\sqrt{(1+\bar{\nu})(1+\nu)}} \frac{2L_{\dot{V}}}{\widetilde{v}_{\min} L K_{\max}} \frac{\Omega(C)}{\sqrt{2-C}-1} + \Lambda_3 \tag{78}$$

eq:def:Lambda2

$$\Lambda_3 \stackrel{\text{def}}{=} \frac{1}{\sqrt{(1+\bar{\nu})(1+\nu)}} \frac{L_{\dot{V}}}{L\widetilde{v}_{\min}} \frac{1}{n^{2/3} K_{\max}} \left\{1 + \frac{\Omega(C)}{n^{1/3}}\right\}. \tag{79}$$

eq:def:Lambda3

Step 2d- Choice of β_0. Finally we chose β_0 so that the first term in Λ_2 is minimal. Denoting by $\xi \stackrel{\text{def}}{=} v_{\max}^2 \beta_0 / v_{\min} \in (0,1)$, we have:

$$\beta_0 \widetilde{v}_{\min} = v_{\min}^2 \xi (1 - \nu\xi) / v_{\max}^2.$$

This quantity is maximal with $\xi_\star \stackrel{\text{def}}{=} \frac{1}{2}$ which yields:

$$\beta_0 = \frac{v_{\min}}{2v_{\max}^2} \quad \text{and} \quad \widetilde{v}_{\min} = v_{\min}(1 - \frac{\nu}{2}) = \frac{v_{\min}}{1+\nu}.$$

Step 2e- Lower bound on δ_k. By (60), $\Lambda_k \geq 0$ which implies that

$$\delta_k \geq (1+\nu) \frac{L_{\dot{V}}}{2} \gamma_{k+1}^2.$$

4.8 Proof of Proposition 18

Step a- Choice of the sequence (g_k) We are looking for $g_1, \ldots, g_{K_{\max}}$ in $(0,1)$ such that

$$p_k = \frac{F(g_{k+1})}{\sum_{\ell=1}^{K_{\max}} F(g_\ell)}$$

where $F(g) \stackrel{\text{def}}{=} Ag(\widetilde{v}_{\min} - Bg)$ and A, B are defined in (62) and (64). Fix $I \in \mathrm{Argmax}_k\, p_k$. By Lemma 14, this function is increasing on $(0, \widetilde{v}_{\min}/(2B))$, reaches its maximum at $g_\star \stackrel{\text{def}}{=} \widetilde{v}_{\min}/(2B)$ and its maximum is $A\widetilde{v}_{\min}^2/(4B)$. We have for any $k = 0, \ldots, K_{\max}-1$, $p_k/p_I = F(g_{k+1})/F(g_{I+1})$ which yields

$$1 = \sum_{k=0}^{K_{\max}-1} p_k = \frac{p_I}{F(g_{I+1})} \sum_{k=1}^{K_{\max}} F(g_k).$$

This leads to $F(g_{k+1}) = F(g_{I+1}) p_k / p_I$. Recall we want to maximize $\sum_{k=0}^{K_{\max}-1} \alpha_k \geq F(g_{I+1})/p_I$ (see (61)); we then choose $g_{I+1} \stackrel{\text{def}}{=} g_\star$ by choosing the design parameters (C, λ) so that $g_\star \in (0,1)$ i.e. satisfying the condition (**??**). With this definition, we have $F(g_I) = A\widetilde{v}_{\min}^2/(4B)$. The optimization of the choice of λ, C is along the same lines as in the proof of Corollary 17 and is omitted.

Step b- Upper bounds for $\widetilde{\beta}_k^{(i)}$. The choice of g_k made above and (**??**) leads to the following definition:

$$\begin{aligned}
\gamma_{k+1} &\stackrel{\text{def}}{=} \frac{G_2}{n^{2/3}} F^{-1}\left(\frac{p_k}{\max_k p_k} F(g_\star)\right) \\
&= \frac{\sqrt{C}}{\sqrt{(1+\bar{\nu})(1+\nu)} L n^{2/3}} F^{-1}\left(\frac{p_k}{\max_k p_k} \frac{\widetilde{v}_{\min}^2}{(1+\nu) C_{\mathrm{GFM}} n^{2/3}}\right) \\
&\leq \frac{1}{\sqrt{(1+\bar{\nu})(1+\nu)} L n^{2/3}} \qquad (80) \quad \boxed{\texttt{eq:maj:gammapk}}
\end{aligned}$$

Using this upper bound, the definition of $\omega_{k+1,j}$ in (47), and by (59) and (60),

$$\sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \omega_{k+1,j} \le \frac{1}{(1+\bar{\nu})(1+\nu)L^2 n^{4/3}} \frac{n}{\mathsf{a}_n} \le \frac{\Omega(C)}{(1+\bar{\nu})(1+\nu)L^2 n^{1/3}},$$
$$\Omega(C) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2-C}\,(\sqrt{2-C}-1)}.$$

Substituting this upper bound and (80) in (51), (52) and (53) yields:

$$\widetilde{\beta}^{(1)}_{k+1} \le \frac{L_{\dot{V}}}{(1+\bar{\nu})L^2 n^{7/3}} \left\{1 + \frac{\Omega(C)}{n^{1/3}}\right\}, \tag{81}$$

eq:majoration:tildebeta1:p

$$\widetilde{\beta}^{(2)}_{k+1} \le \frac{L_{\dot{V}}}{2(1+\nu)(1+\bar{\nu})L^2 n^{4/3}} \left\{1 + \frac{\Omega(C)}{n^{1/3}}\right\}, \tag{82}$$

eq:majoration:tildebeta2:p

$$\widetilde{\beta}^{(3)}_{k+1} \le \frac{L_{\dot{V}}}{(1+\nu)(1+\bar{\nu})L^2} \frac{1}{n^{2/3}} \frac{\Omega(C)}{\sqrt{2-C}-1}. \tag{83}$$

eq:majoration:tildebeta3:p

Step c- Upper bounds for $\widetilde{\beta}^{(i)}_k / \sum_{k=1}^{K_{\max}-1} \alpha_k$. With our choice of (g_{k+1}), we have:

$$\left(\sum_{k=0}^{K_{\max}-1} \alpha_k\right)^{-1} \le \left(\sum_{k=0}^{K_{\max}-1} F(g_{k+1})\right)^{-1} \le n^{2/3} \max_k p_k \frac{(1+\nu)C_{\mathrm{GFM}}}{\widetilde{v}^2_{\min}} \tag{84}$$

eq:maj:sumalphak:pk

In particular, we have $\alpha_k > 0$, we can then derive from Theorem 16 an upper bound on $\mathbb{E}\left[\|h(\widehat{S}^K)\|^2\right]$ where $K \in \{0, \dots, K_{\max}-1\}$ is a random variable sampled independently from $\mathcal{F}_{K_{\max}}$, with the following distribution:

$$\mathbb{P}(K = k) = \frac{\alpha_k}{\sum_{k=0}^{K_{\max}-1} \alpha_k} = p_k.$$

From (54), (81) and (84),

$$\left(\sum_{k=0}^{K_{\max}-1} \alpha_k\right)^{-1} \xi_0(K_{\max})$$
$$\le \bar{\Lambda}_0 \stackrel{\text{def}}{=} \max_k p_k \left(\frac{K_{\max}}{n^{2/3}} \wedge n^{1/3}\right) \frac{1+\nu}{1+\bar{\nu}} \frac{L_{\dot{V}} C_{\mathrm{GFM}}}{L^2 \widetilde{v}^2_{\min}} \left\{1 + \frac{\Omega(C)}{n^{1/3}}\right\}.$$

eq:def:Lambdabar0

From (55), (81) and (84),

$$\left(\sum_{k=0}^{K_{\max}-1} \alpha_k\right)^{-1} \Xi_1(\eta^{(1)}, K_{\max}, n) \le \bar{\Lambda}_1 \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\eta^{(1)}_{k+1}\|^2\right]$$

where

$$\bar{\Lambda}_1 = \frac{\max_k p_k}{n^{2/3}} \left(\frac{K_{\max}}{n} \wedge 1\right) \frac{1+\nu}{1+\bar{\nu}} \frac{L_{\dot{V}} C_{\mathrm{GFM}}}{L^2 \widetilde{v}^2_{\min}} \left\{1 + \frac{\Omega(C)}{n^{1/3}}\right\}. \tag{85}$$

eq:def:Lambdabar1

From (56), (80), (82), (83) and (84),

$$\left(\sum_{k=0}^{K_{\max}-1} \alpha_k\right)^{-1} \Xi_2(\eta^{(2)}, K_{\max}, n) \leq \bar{\Lambda}_2 \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_k\right]\|^2\right]$$
$$+ \bar{\Lambda}_3 \sum_{k=0}^{K_{\max}-1} \left(\mathbb{E}\left[\|\eta_{k+1}^{(2)}\|^2\right] + \mathbb{E}\left[\|\mathbb{E}\left[\eta_{k+1}^{(2)}|\mathcal{F}_{k+3/4}\right]\|^2\right]\right),$$

where

$$\bar{\Lambda}_2 \stackrel{\text{def}}{=} \max_k p_k \sqrt{\frac{1+\nu}{1+\bar{\nu}}} \frac{C_{\mathrm{GFM}}}{4\beta_0 L \widetilde{v}_{\min}^2} + \max_k p_k \frac{L_{\dot{V}} C_{\mathrm{GFM}}}{(1+\bar{\nu}) L^2 \widetilde{v}_{\min}^2} \frac{\Omega(C)}{\sqrt{2-C}-1} + \bar{\Lambda}_3,$$
$$\bar{\Lambda}_3 \stackrel{\text{def}}{=} \frac{\max_k p_k}{n^{2/3}} \frac{L_{\dot{V}} C_{\mathrm{GFM}}}{2(1+\bar{\nu}) L^2 \widetilde{v}_{\min}^2}.$$

Step d- Choice of β_0. We choose β_0 following the same reasoning as in step 2d of the proof of Proposition 17, which leads to:

$$\beta_0 = \frac{v_{min}}{3v_{max}^2} \quad \text{and} \quad \widetilde{v}_{min} = v_{\min}(1 - \frac{\nu}{3}) = v_{\min}\frac{1+\nu}{1+2\nu}$$

5 Numerical experimentations

5.1 P-FIEM for the toy model

sec:toy:MC

We go back to the toy example introduced in Section 2.5. Recall that the conditional distribution of the latent variable Z_i given the observation Y_i, for the statistical model indexed by θ is

$$\mathcal{N}_p\left((\mathrm{I}_p + A^T A)^{-1}(A^T Y_i + X\theta), (\mathrm{I}_p + A^T A)^{-1}\right) . \tag{86}$$

eq:toyexample:posterior

P-FIEM is applied with a Monte Carlo approximation of the expectations $\bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$ and $\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$; each Monte Carlo sum is approximated with $m = \overline{m}$ samples at each iteration k (see Section 4.2).

ex:toy:pFIEM:H5

Proposition 22. *Let the approximations $\widetilde{\mathsf{S}}_{k+1}$ and $\tilde{s}_{k+1}$ be defined as a Monte Carlo sum involving resp. $\overline{m}_{k+1}$ and m_{k+1} i.i.d. samples from the distribution (86) with $\theta \leftarrow \mathsf{T}(\widehat{S}^k)$ and $i \leftarrow I_{k+1}$, $i \leftarrow J_{k+1}$ resp. Then H5 is verified with $\nu = 0$, $M_\nu^{(2)} = 0$, and $M^{(1)} = M^{(2)} = \mathbb{E}[\|W\|^2]$ where $W \sim \mathcal{N}_q(0, X^T(\mathrm{I}_p + A^T A)^{-1} X)$.*

Gers: attention, tu prenais k pour l'indice de Monte Carlo; k est plutôt l'indice d'itération et du coup c'est "confusing". Dans la démo de la proposition, j'ai utilisé ℓ, gardons cette convention.

Proof. Conditionally to $\mathcal{F}_{k+1/4}$ and $\{I_{k+1} = i\}$, let $\{Z_{\ell+1}^{(i)}, \ell \geq 0\}$ be i.i.d. samples from the distribution (86), where $\theta \leftarrow \mathsf{T}(\widehat{S}^k)$. We have for all $k \geq 0$, on the set $\{I_{k+1} = i\}$,

$$\mathbb{E}\left[X^T Z_{\ell+1}^{(i)} | \mathcal{F}_{k+1/4}\right] = X^T(\mathrm{I}_p + A^T A)^{-1}(A^T Y_i + X\mathsf{T}(\widehat{S}^k)) = \bar{s}_i \circ \mathsf{T}(\widehat{S}^k) ;$$

see (**??**) for the definition of $\bar{s}_i \circ \mathsf{T}$.

Gers: reference: rajouter le nom de l'équation sur la première ligne qui donne la définition de $\bar{s}_i(\theta)$

Therefore $\mathbb{E}\left[\eta_{k+1}^{(1)} | \mathcal{F}_{k+1/4}\right] = 0$. Furthermore, since the r.v. $\{Z_{\ell+1}^{(i)}, \ell \geq 0\}$ are i.i.d. conditionally to the past, we have on the set $\{I_{k+1} = i\}$,

$$\begin{aligned} \mathbb{E}\left[\|\eta_{k+1}^{(1)} | \mathcal{F}_{k+1/4}\|^2\right] &= \frac{1}{m}\mathbb{E}\left[\left\|X^T\left(Z_1^{(i)} - \mathbb{E}\left[Z_1^{(i)} | \mathcal{F}_{k+1/4}\right]\right)\right\|^2 | \mathcal{F}_{k+1/4}\right] \\ &= \frac{\mathbb{E}\left[\|W\|^2\right]}{m} . \end{aligned}$$

□

P-FIEM is compared to perturbed versions of i-EM and SA, when the perturbation results from a Monte Carlo approximation of the intractable expectations $\bar{s}_i \circ \mathsf{T}(s)$; in this toy example, we will consider that the Monte Carlo approximation of the expectation of $\mathcal{N}_q(\mu, \Gamma)$ are of the form

$$\mu + \sqrt{\Gamma} \left(\frac{1}{m} \sum_{j=1}^{m} Z_j \right) ,$$

where the r.v. $\{Z_j, j \geq 1\}$ are i.i.d. $\mathcal{N}_q(0, \mathrm{I}_q)$. Hereafter, $\Gamma \overset{\text{def}}{=} X^T(\mathrm{I}_p + A^T A)^{-1} X$.

Gers: C'est vraiment débile comme approximation Monte Carlo, parce que pour calculer notre approximation, on se sert explicitement de la quantité inconnue. Parfois, on arrive quand même à faire un peu moins jouet ... Je ne suis pas sûre que l'on arrive à voir grand chose de plus que ce que l'on voit en déterministe ?

One iteration of Pi-EM and P-SA are resp. given by algorithm 8 and algorithm 9.

Data: $\widehat{S}^k \in \mathcal{S}$; a step size $\gamma_{k+1} \in (0, 1]$ and a positive batch size $m \in \mathbb{N}$, the matrices X, Π_2, $\sqrt{\Gamma}$ and the vectors $\Pi_1 Y_i$ for $i = 1, \ldots, n$.
Result: $\widehat{S}^{k+1}_{\text{p-SA}}$

Sample independently from the past $I_{k+1} \sim \mathcal{U}(\{1, \cdots, n\})$;
Sample independent standard $\mathbb{R}^q$-valued Gaussian r.v. $\{Z_j, 1 \leq j \leq m\}$ and set $W \overset{\text{def}}{=} \Pi_1 Y_{I_{k+1}} + \Pi_2 \widehat{S}^k + \sqrt{\Gamma} \left\{ \frac{1}{m} \sum_{j=1}^m Z_j \right\}$;
Set $\widehat{S}^{k+1}_{\text{p-SA}} = \widehat{S}^k + \gamma_{k+1} \left(W - \widehat{S}^k \right)$

Algorithm 8: Toy example: one iteration of the p-SA algorithm. `algo:toy:pSA`

Data: $\widehat{S}^k \in \mathcal{S}$, $\mathsf{S} \in \mathcal{S}^n$, $\widetilde{S} \in \mathcal{S}$; a step size $\gamma_{k+1} \in (0, 1]$ and a positive batch size $m \in \mathbb{N}$; the matrices X, Π_2, $\sqrt{\Gamma}$ and the vectors $\Pi_1 Y_i$ for $i = 1, \ldots, n$.
Result: $\widehat{S}^{k+1}_{\text{p-iEM}}$

Independently from the past, sample $I_{k+1} \sim \mathcal{U}(\{1, \cdots, n\})$;
Sample independent standard $\mathbb{R}^q$-valued Gaussian r.v. $\{Z_j, 1 \leq j \leq m\}$ and set $W \overset{\text{def}}{=} \Pi_1 Y_{I_{k+1}} + \Pi_2 \widehat{S}^k + \sqrt{\Gamma} \left\{ \frac{1}{m} \sum_{j=1}^m Z_j \right\}$;
Store $s = \mathsf{S}_{I_{k+1}}$;
Update $\mathsf{S}_{I_{k+1}} = W$;
Update $\widetilde{S} = \widetilde{S} + n^{-1}(\mathsf{S}_{I_{k+1}} - s)$;
Update $\widehat{S}^{k+1}_{\text{p-iEM}} = \widehat{S}^k + \gamma_{k+1} \left(\widetilde{S} - \widehat{S}^k \right)$

Algorithm 9: Toy example: one iteration of the p-iEM algorithm. `algo:toy:pIEM`

Gers: Pierre-Bien relire tous ces algos car j'ai fait pas mal de changements

Data: $\widehat{S}^k \in \mathcal{S}$, $\mathsf{S} \in \mathcal{S}^n$, $\widetilde{S} \in \mathcal{S}$; a step size $\gamma_{k+1} \in (0,1]$ and a positive batch size $m \in \mathbb{N}$; the matrices X, Π_2, $\sqrt{\Gamma}$ and the vectors $\Pi_1 Y_i$ for $i = 1, \ldots, n$.

Result: $\widehat{S}^{k+1}_{\mathrm{p-FIEM}}$

1 Sample independently $I_{k+1}, J_{k+1} \sim \mathcal{U}(\{1, \cdots, n\})$;

2 Sample independent standard $\mathbb{R}^q$-valued Gaussian r.v. $\{Z_j, 1 \leq j \leq \overline{m} + m\}$ and set $W_1 \overset{\mathrm{def}}{=} \Pi_1 Y_{I_{k+1}} + \Pi_2 \widehat{S}^k + \sqrt{\Gamma} \left\{ \frac{1}{\overline{m}} \sum_{j=1}^{\overline{m}} Z_j \right\}$ and $W_2 \overset{\mathrm{def}}{=} \Pi_1 Y_{J_{k+1}} + \Pi_2 \widehat{S}^k + \sqrt{\Gamma} \left\{ \frac{1}{m} \sum_{j=\overline{m}+1}^{\overline{m}+m} Z_j \right\}$.;

3 Store $s = \mathsf{S}_{I_{k+1}}$;

4 Update $\mathsf{S}_{I_{k+1}} = W_1$;

5 Update $\widetilde{S} = \widetilde{S} + n^{-1}(\mathsf{S}_{I_{k+1}} - s)$;

6 Update $\widehat{S}^{k+1}_{\mathrm{p-FIEM}} = \widehat{S}^k + \gamma_{k+1} \left(W_2 - \widehat{S}^k + \widetilde{S} - \mathsf{S}_{J_{k+1}} \right)$

Algorithm 10: Toy example: one iteration of the P-FIEM algorithm. `algo:toy:pFIEM`

5.1.1 Numerical analysis

Pierre: J'ai repris le cadre des simulations du cas exact. Quand je compare fiEM et iEM en version exacte sur mes codes, j'ai quelque chose de bizarre: pour ces dimensions là, ils sont quasiment équivalent. En particulier iEM converge très (trop?) vite. Tandis que pour de petite dimension (par exemple 2 pour Y_i), on retrouve une différence significative entre les deux. Seule la dimension des observations semble avoir une influence. Je ne sais pas si c'est juste quelque chose d'étrange ou si ça peut vraiment avoir un sens par rapport à l'algorithme.

Gers: Je pense surtout que regarder de telles dimensions n'a aucun intérêt, avec un n grand. Si on cherche des algorithmes qui savent traiter beaucoup de données (n grand) c'est plutôt pour explorer des problèmes de grande dimension. Donc ne pas se torturer sur cet exemple jouet en regardant des dimensions qui ont vraiment trop peu d'intérêt. Regarde les courbes que j'avais faites, et tu verras que EM bat tout le monde.

Pierre: Pour la version MC, prendre une petite dimension rendait p-iEM moins bon, et donnait une convergence à long terme vers la même valeur que P-FIEM, là où il donne une valeur plus faible ici (en terme d'erreur par rapport à $S_\star$). Mais ce n'est potentiellement pas un problème (voir commentaire en dessous)

Gers: Il ne faut pas prendre des valeurs trop petites des paramètres. On veut quand même comaprer des algorithmes sur la grande dimension. Il faudrait faire tourner tout cela sur des trucs vraiment d'envergure et cesser de tirer des conclusions sur des trucs de dimension petite quand on dit qu'on vise les données massives ... A relancer sur du un peu plus dur avant le CIRM, qu'on ait les résultats au retour ?

Setting We run all algorithms for $K_{\max} = 25n$ iterations.

Gers: Pierre, il faut reprendre les mêmes données que pour le déterministe (la même matrice X, A, les mêmes Y_i, le même n et idéalement les mêmes I_k, J_k. Est-ce ce que tu as fait ? (je ne sais plus si je les avais mis sur le git ou pas). Du coup la taille de l'échantillon ($n = 1e3$) doit être la même. Il faut aussi reprendre le même point intial $\widehat{S}^0$.

P-FIEM: Role of the step size and the batch size Here we run the algorithm P-FIEM in different settings to study the role of the step size and of the Monte Carlo batch sizes $\overline{m}, m$.

Three stepsize sequences are considered. The first one is the constant one equal to γ_{GFM}, where γ_{GFM} is defined in Proposition 17. The second one is again a constant one equal to $\gamma_{best} \stackrel{\mathrm{def}}{=} 3/(2n^{2/3})$; it is chosen as the step size which yields the best convergence rate for fiEM in the exact setting.

Gers: Je ne comprends pas cette notion de "best"; est-ce que tu prends le γ donné par une des propositions ?

The last stepsize sequence is equal to γ_{best} until iteration n and then equal to γ_{GFM}; this third case in denoted by γ_{burnin} hereafter.

We also consider two different values for the Monte Carlo batch sizes $\overline{m} = m \in \{1e2, 1e3\}$.

This yields to six different pairs (γ, m); for each pair, 100 independent paths of P-FIEM, of length $K_{\max}$ are run. For each path, the values of $\|\widehat{S}^k - \bar{s}(\theta_\star)\|^2$, where $\bar{s}(\theta_\star)$ is the unique root of the field h, (see XXX)

Gers: Pierre, insérer une ref à la valeur $\theta_\star$ et à l'expression de h dans cet exemple

are stored for different values of the iteation index k. The boxplots of these squared errors are displayed on Figure 6, for a focus on the first iterations, and in Figure 7, for a long term behavior analysis. For a fair comparison, the $100 \times K_{\max} \times 2$ random indices I_k, J_k and the $100 \times K_{\max} \times (\overline{m} + m)$ Gaussian random variables are the same for the six values of the pairs (stepsize, batch size).

Gers: Pierre, c'est bien l'erreur quadratique que tu as tracé ? parce que dans la légende de la figure tu n'as pas mis le carré. Je l'ai rajouté

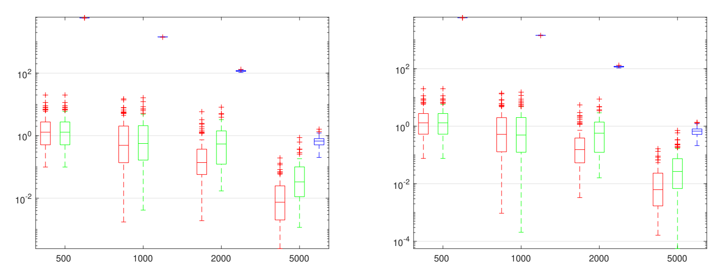

fig:toy:Comp:pfiEM:burnin

Figure 6: P-FIEM. Boxplot over 100 independent realizations, of the error $\|\widehat{S}^k - \bar{s}(\theta_\star)\|^2$ when $k = \{500, 1e3, 2e3, 5e3\}$, $m = 1e2$ (left), and $m = 1e3$ (right). For each value of k, three boxplots are displayed corresponding to the step sizes (from left to right) $\gamma_{burnin}, \gamma_{best}$ and γ_{GFM}.

In the transient phase, the algorithms with γ_{burnin} and γ_{best} converge much more rapidly than the algorithm using γ_{GFM}, which was expected since, numerically, γ_{GFM} is the smallest step size.

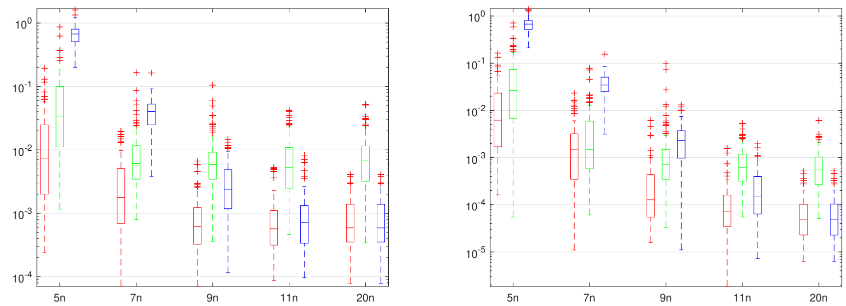

fig:toy:Comp:pfiEM:long

Figure 7: P-FIEM. Boxplot over 100 independent realizations, of the error $\|\widehat{S}^k - \bar{s}(\theta_\star)\|^2$ when $k = \{5n, 7n, 9n, 11n, 20n\}$, $m = 1e2$ (left), and $m = 1e3$ (right). For each value of k, three boxplots are displayed corresponding to the step sizes (from left to right) $\gamma_{burnin}, \gamma_{best}$ and γ_{GFM}.

Gers: Pierre, le nouveau γ proposé (ou les deux, selon que l'on se place dans le cas de la proposition 5 ou 6) sont-ils aussi petits ou plus petits que γ_{GFM} ?

After n iterations, even though γ_{burnin} becomes much smaller than γ_{best}, P-FIEM with γ_{burnin} converges faster that the one with γ_{GFM}, showing that the burnin period was long enough to compensate the poor initialization. Taking a larger value of m seems to yield no benefit in these first iterations.

Gers: Pierre, pour tirer qq chose sur le rôle de m, je pense que ça va être dur. Car déjà 100 points pour approcher zero avec des gaussiennes centrées de variance 1, c'est un estimateur d'écart type 1/10 dont 95% des estimations de zero sont dans l'intervalle $[-2/10, 2/10]$ grosso modo. Et probable que cette fluctuation impacte très peu notre $\widetilde{\mathsf{S}}$ (et l'autre aussi) qui sont explicitement égaux à S + bruit. Ne pas garder les deux graphes : garder uniquement $m = 1e2$; et explorer plus de γ et aussi le γ_{KM}; et enfin, pour comparaison, rajouter ce que donne la trajectoire de l'algo déterministe avec les mêmes tirages I, J. Je crains que notre Monte Carlo ici soit vraiment trop simple et ne permette de tirer aucune conclusion sur ce qu'il se passe sur les algos perturbés.

For the long time behavior, we observe that all algorithms eventually enter a stationary regime since they do not diminish significantly anymore, which is in line with the result of Proposition 17 for a fixed batch of samples.

Gers: Je ne comprends pas quel commentaire de cette proposition tu évoques. Mettre une référence ?

Furthermore, P-FIEM with γ_{best} reaches this regime earlier than the other two, but has a larger variability. P-FIEM with γ_{burnin} and P-FIEM with γ_{GFM} reach the same precision, although γ_{burnin} converges faster.

Gers: Faire des explorations sur plus de γ, mais uniquement pour $m = 1e2$.

Comparison between p-SA, p-iEM and P-FIEM We now compare the algorithms through boxplots computed from 100 independent paths of each algorithm. P-SA and P-FIEM are run with $\gamma_k = \gamma_{burnin}$ and P-iEM is run with $\gamma_k = 1$.

Gers: J'ai sorti les graphes du cas $m = 1e3$; ne mettre que le cas $m = 1e2$. Rajouter le boxplot du déterministe, comme comparaison - pour que l'on puisse visualiser la fluctuation due aux tirages des indices, et celle due aux tirages MC

Again, to make the comparison fair, the uniform and Gaussian random draws required for a single path are shared between the three algorithms; in addition, a budget of m Gaussian draws per iteration is allocated to each algorithm (which means that for P-FIEM, $m/2$ draws are used for $\widetilde{\mathsf{S}}$ and $m/2$ draws are used for $\tilde{s}$. The algorithms are compared through the boxplot of $\|\widehat{S}^k - \bar{s}(\theta_\star)\|^2$ along their paths, during the transient phase - see Figure 8 and the converging phase - see Figure 9.

In the transient phase and in the converging phase as well, P-SA and P-FIEM are mostly equivalent while P-iEM suffer from the poor initialization. Then, P-SA does not diminish anymore while the errors in P-FIEM are still decreasing.

Gers: Reprendre l'anglais du paragraphe qui suit, en s'inspirant de ce que j'ai écrit avant

Again, this is to be expected using a fixed batch size m and a fixed step size. p-SA reaches its stationary regime before P-FIEM, however P-FIEM is far superior in term of precision. p-iEM takes two times more iterations to converge than P-FIEM, thus necessiting overall two times more simulations; however, its precision appears to be better than the one of P-FIEM.

Pierre: Une remarque à rajouter, mais qui serait peut etre un peu trop spéculative, serait de remarquer que quand on multiplie m par 10, on divise l'erreur moyenne finale par 10, en regardant les boxplot. Ca correspondrait à notre borne, mais on ne regarde pas la même quantité que le LHS.

Gers: Je pense que notre Monte Carlo est trop simple pour faire ce genre de conclusions. A faire sur des exemples moins jouets si cela s'avère encore vrai

Pierre: A partir de là, on pourrait dire que p-iEM "converge" en deux fois plus itérations, et donc en deux fois plus de MC. On pourrait donc dire que son erreur moyenne finale n'est vraiment une amélioration que si elle est inférieure à la moitié de l'erreur finale de P-FIEM; et ici elle est à peu près égale à la moitié. Donc même si on dirait que p-iEM a une meilleure convergence à très long terme, on aurait la même avec P-FIEM avec un MC équivalent, tout en gardant un comportement globalement meilleur sur tout l'algorithme (par rapport à la phase transiente).

Gers: Franchement: je ne suis pas du tout fan de ce genre de commentaires, parce que c'est très très dépendant des tailles des objets que l'on regarde, de la pénalité, des modèles, du point initial .Etc. Je ne me permettrai jamais de conclure sur de telles recos d'un algo sur l'autre. On ne peut utiliser les simulations que pour attirer l'attention sur des paramètres d'implémentation importants pour l'efficacité des algos.

5.2 Application to Logistic Regression

sec:logreg

Gers: Je n'ai pas le temps de lire cette section avant le CIRM. Mais dans l'optique d'une publication, il va quand même falloir faire tourner sur des choses de plus grande dimension. Prendre des jeux de données plus gros et surtout estimer des paramètres plus gros. Karimi et al. font des exemples dans ces dimensions ?

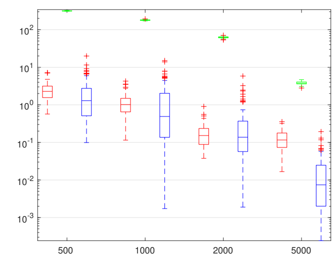

Figure 8: P-SA, P-iEM, P-FIEM. Boxplot, over 100 independent paths, of the error $\|\widehat{S}^k - \bar{s}(\theta_\star)\|^2$ when $k = \{500, 1e3, 2e3, 5e3\}$ and $m = 1e2$. For each value of k, the group of three boxplots correspond to (from left to right) P-SA, P-iEM and P-FIEM.

fig:toy:Comp:IncEm:burnin

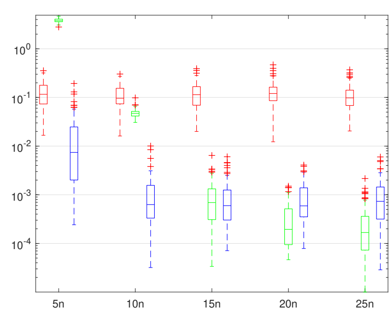

Figure 9: P-SA, P-iEM, P-FIEM. Boxplot, over 100 independent paths, of the error $\|\widehat{S}^k - \bar{s}(\theta_\star)\|^2$ when $k = \{5n, 10n, 15n, 20n, 25n\}$ and $m = 1e2$. For each value of k, the group of three boxplots correspond to (from left to right) P-SA, P-iEM and P-FIEM

fig:toy:Comp:IncEm:long

5.2.1 Model

We consider a binary regression model, where we observe n variables $(Y_1, \dots, Y_n)$ in $\{0, 1\}$, which conditionally to latent variables $(Z_1, \dots, Z_n) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d$, are independent; and such that for all $i \leq n$, conditionally to Z_i, we have:

$$\mathbb{P}(Y_i = 1 | Z_i) = \frac{1}{1 + e^{-\delta_i^T Z_i}},$$

where $\delta_i \in \mathbb{R}^d$ are known explanatory variables. Moreover, we assume the latent variables to be independent and identically distributed under $\mathcal{N}(\theta, \mathrm{I}_d)$. We want to estimate the mean parameter $\theta \in \Theta \overset{\text{def}}{=} \mathbb{R}^d$ by minimizing the negative log likelihood of the sample $(Y_1, \dots, Y_n)$ without penalty, meaning $R(\theta) \overset{\text{def}}{=} 0$. The negative log-likelihood of the i-th observation is given by:

$$g(Y_i; \theta) \overset{\text{def}}{=} -\log \int_{\mathbb{R}^d} \frac{\left(e^{-\delta_i^T z}\right)^{1 - Y_i}}{1 + e^{-\delta_i^T z}} \frac{e^{-\frac{1}{2}\|z - \theta\|^2}}{(2\pi)^{d/2}} \mathrm{d}z. \tag{87}$$

eq:def:logreg:nll

This quantity can be rewritten in the form of (3) with:

$$s_i(z) \overset{\text{def}}{=} z, \quad \phi(\theta) \overset{\text{def}}{=} \theta \quad \text{and} \quad \psi(\theta) \overset{\text{def}}{=} \frac{1}{2}\theta^T \theta.$$

With these definitions, we verify assumption H1.Setting $\mathcal{S} \overset{\text{def}}{=} \mathbb{R}^d$, we have for all $s \in \mathcal{S}$:

$$\mathsf{T}(s) \overset{\text{def}}{=} \mathrm{Argmin}_{\theta \in \Theta}\, \psi(\theta) - \langle s, \phi(\theta) \rangle = s,$$

so that H3 and H4 - item 1 are also satisfied. Moreover, in this example, we have $B(s) \overset{\text{def}}{=} \mathrm{I}_d$; we then verify H4 - item 2 with $v_{\min} = v_{\max} = 1$. Here the posterior distribution is only known up to a multiplicative constant:

$$p_i(\cdot; \theta) \overset{\text{def}}{\propto} \frac{\left(e^{-\delta_i^T z}\right)^{1 - Y_i}}{1 + e^{-\delta_i^T z}} e^{-\frac{1}{2}\|z - \theta\|^2}. \tag{88}$$

eq:def:logreg:p_i

Because of this, the functions $\bar{s}_i$, here defined as:

$$\bar{s}_i(\theta) \overset{\text{def}}{=} \int_{\mathbb{R}^d} z p_i(z; \theta) \mathrm{d}z,$$

are not available in close form. However, we can verify assumptions H4 - item 3 and H4 - item 4 (see Proposition 24), with

$$L_i \overset{\text{def}}{=} 4\sqrt{d(2 + d)}\|\delta_i\|_\infty + 1 \quad \text{and} \quad L_{\dot{V}} \overset{\text{def}}{=} \frac{4\sqrt{d(2 + d)}}{n} \sum_{i=1}^n \|\delta_i\|_\infty + 2.$$

5.2.2 MCMC sampling

Since the functions $\bar{s}_i$ are intractable, we resort to use p-fiEM to solve the optimization problem. We build a Monte Carlo approximation of the quantity $\bar{s}_i \circ \mathsf{T}(\widehat{S}^k)$ according to Section 4.2 at each iteration. We sample the posterior distribution $p_i(z;\theta)$ using a Gibbs sampler based on data augmentation first established in (Polson et al., 2013, Section 3). More specifically, we perform a Gibbs sampler on the density defined for $z \in \mathbb{R}^d$ and $\omega > 0$ by:

$$\widetilde{\pi}_{\theta,i}(z,\omega) \stackrel{\text{def}}{=} \pi_{PG}(\omega, \delta_i^T z) p_i(z;\theta) \tag{89}$$ eq:def:logreg:augmented

where $\pi_{PG}(\cdot, c)$ is the density of the Polya-Gamma distribution of parameter $(1, c)$, whose expression is given by:

$$\pi_{PG}(\omega, c) \stackrel{\text{def}}{=} \cosh(c/2) e^{-\frac{c^2}{2}\omega} \rho(\omega) \mathbb{1}_{\omega > 0} \tag{90}$$ eq:def:logreg:PG

where $\rho(\omega) \propto \sum_{n=0}^{\infty} (-1)^n (2n+1) \exp(-(2n+1)^2/(8\omega)) \omega^{-3/2}$. With this density, the conditional distribution of z given ω is $\mathcal{N}(m_\omega, V_\omega)$, where:

$$V_\omega \stackrel{\text{def}}{=} \left(\mathrm{I}_d + \omega \delta_i \delta_i^T\right)^{-1} \quad \text{and} \quad m_\omega \stackrel{\text{def}}{=} V_\omega((Y_i - \frac{1}{2})\delta_i + \theta)).$$

The details of this computation can be found in Lemma 25. Note that there is a close form for V_ω thanks to the Sherman–Morrison formula:

$$V_\omega = \mathrm{I}_d - \frac{\omega}{1 + \omega \|\delta_i\|^2} \delta_i \delta_i^T$$

We can now state the MCMC sampler described in algorithm 11. To sample from a distribution $PG(1, c)$, we refer to (Polson et al., 2013, Algorithm 1). Note that here, the assumption H5 is not verified in general.

Pierre: Dans Atchadé et al. (2017), il y marqué que l'algorithme MCMC est uniformément ergodique, et donc le H5 du papier est vérifié, ce qui suffit à entrainer notre H5 sans supposer Θ compact?

Data: $Z_{init} \in \mathbb{R}^d$; a batch size $m \geq 1$; the vector δ_i; the example Y_i; the current parameter θ
Result: $\left(Z^{(j)}\right)_{1 \leq j \leq m}$
1 Set $Z^{(0)} = Z_{init}$;
2 **for** $k \leftarrow 1$ **to** m **do**
3   Sample ω from $PG(1, \delta_i^T Z^{(k-1)})$;
4   Sample $Z^{(k)}$ from $\mathcal{N}(m_\omega, V_\omega)$

algo:logreg:gibbs **Algorithm 11:** Logistic Regression: MCMC sampler of the posterior distribution $p_i(\cdot;\theta)$.

Finally the iteration of p-fiEM is displayed in algorithm 12.

Data: $\widehat{S}^k \in \mathcal{S}$, $\mathsf{S} \in \mathcal{S}^n$, $\widetilde{S} \in \mathcal{S}$; a step size $\gamma_{k+1} \in (0,1]$ and a batch size $m \geq 1$; the vectors $\delta_1, \ldots, \delta_n$; the examples $Y_1, \cdots, Y_n$; a starting point Z_{init}

Result: $\widehat{S}^{k+1}_{\mathrm{FIEM}}$

Sample independently $I_{k+1}, J_{k+1} \sim \mathcal{U}(\{1, \cdots, n\})$;

Sample $(Z^{(j)}_{k+1,1})_{1 \leq j \leq m}$ from $p_{I_{k+1}}(\cdot; \widehat{S}^k)$ using algorithm 11 ;

Set $Z_{init} = Z^{(m)}_{k+1,1}$;

Sample $(Z^{(j)}_{k+1,1})_{1 \leq j \leq m}$ from $p_{J_{k+1}}(\cdot; \widehat{S}^k)$ using algorithm 11 ;

Store $s = \mathsf{S}_{I_{k+1}}$;

Update $\mathsf{S}_{I_{k+1}} = \frac{1}{m} \sum_{j=1}^m Z^{(j)}_{k+1,1}$;

Update $\widetilde{S} = \widetilde{S} + n^{-1}(\mathsf{S}_{I_{k+1}} - s)$;

Update $\widehat{S}^{k+1}_{\mathrm{FIEM}} = \widehat{S}^k + \gamma_{k+1} \left(\frac{1}{m} \sum_{j=1}^m Z^{(j)}_{k+1,2} - \widehat{S}^k + \widetilde{S} - \mathsf{S}_{J_{k+1}} \right)$

Algorithm 12: Logistic Regression: one iteration of the p-fiEM algorithm.

algo:logreg:p-FIEM

5.2.3 Numerical exploration

Setting We simulate $n = 10^3$ samples from a parameter $\theta \in \mathbb{R}^3$, obtain from a uniform law on $[-5, 5]$, and a matrix δ, whose entries are sampled from a uniform law on $[-1, 1]$. We run the p-fiEM and the p-iEM algorithm using $m = 20$ Monte Carlo samples at each iterations for each algorithm during $K_{\max} = 200n$ iterations. We run the p-fiEM algorithm for three different step sizes, following the conclusion of last section: $\gamma_{best} = 0.3n^{-2/3}$, γ_{GFM} as stated in Proposition 17 and γ_{burnin} defined as:

$$\gamma_{burnin,k} = \begin{cases} \gamma_{best} & \text{if} \quad k \leq 100n \\ \gamma_{\mathrm{GFM}} & \text{if} \quad k > 100n \end{cases}$$

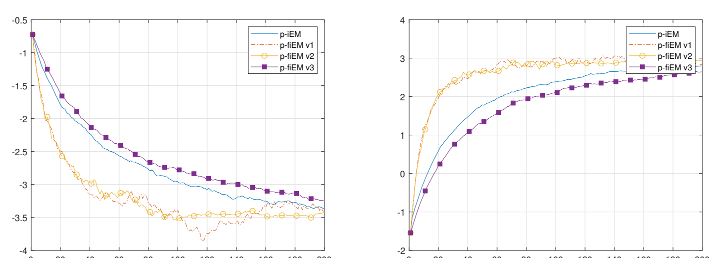

Figure 10: p-iEM and p-fiEM. Evolution over 200 epochs, of the second (left) and third (right) coordinate of the sequence $\widehat{S}^k$ for p-iEM and three versions of p-fiEM. v1 : γ_{best}; v2 : γ_{burnin}; v3 : γ_{GFM}.

fig:logreg:evocoord

Comparison of the perturbed incremental algorithms We compare the results looking at the convergence of the second and third coordinates of the sequence $(\widehat{S}^k)$ in Figure 10. We can see the conclusions of Section 5.1 still apply. The step size γ_{GFM} offer little variability but can suffer from a poor initialization, where the step size γ_{best} is more efficient during the convergence phase, but if far less precise in the long run. The step size γ_{burnin} offer a compromise between speed during the transient phase and variability in the large iterations, and a better convergence overall than the p-iEM algorithm.

A Proofs of Section 5.2

lem:logreg:sup **Lemma 23.** *For all $i \in \{1, \dots, n\}$, we have:*

$$\sup_{\theta \in \Theta} \int_{\mathbb{R}^d} \|z\|^2 p_i(z + \theta; \theta) \mathrm{d}z \leq \sqrt{d(2 + d)}$$

Proof. The proof can be found in (Atchadé et al., 2017, Appendix A). We adapt it here to our notations for completeness.

Denote:

$$h_i(z) \stackrel{\text{def}}{=} -(1 - Y_i)(\delta_i^T z) - \ln(1 + e^{-\delta_i^T z}) \tag{91}$$ eq:def:logreg:h_i

so that, using (88), $p_i(z+\theta, \theta) = e^{h_i(z+\theta)+g(Y_i;\theta)}\pi(z)$, where π is the density of a standard Gaussian vector. Using these notations, with the definition of $g(T_i;\theta)$ in (87),

$$\int_{\mathbb{R}^d} \|z\|^2 p_i(z + \theta; \theta) \mathrm{d}z = \left(\int_{\mathbb{R}^d} \|z\|^2 e^{h_i(z+\theta)} \pi(z) \mathrm{d}z \right) \left(\int_{\mathbb{R}^d} e^{h_i(z+\theta)} \pi(z) \mathrm{d}z \right)^{-1}.$$

The Cauchy-Schwartz inequality yields:

$$\begin{aligned}\int_{\mathbb{R}^d} \|z\|^2 e^{h_i(z+\theta)} \pi(z) \mathrm{d}z &\leq \left(\int_{\mathbb{R}^d} \|z\|^4 \pi(z) \mathrm{d}z \right)^{1/2} \left(\int_{\mathbb{R}^d} e^{2h_i(z+\theta)} \pi(z) \mathrm{d}z \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^d} \|z\|^4 \pi(z) \mathrm{d}z \right)^{1/2} \left(\int_{\mathbb{R}^d} e^{h_i(z+\theta)} \pi(z) \mathrm{d}z \right)\end{aligned}$$

Pierre: Il y a une inégalité que je comprends pas dans le papier d'origine (première inégalité de l'appendice A), on dit que par Cauchy Schwartz,

$$\left(\int_{\mathbb{R}^d} e^{\frac{1}{2} h_i(z+\theta)} \pi(z) \mathrm{d}z \right)^{1/2} \leq \int_{\mathbb{R}^d} e^{h_i(z+\theta)} \pi(z) \mathrm{d}z$$

Substituting this inequality, for all $\theta \in \Theta$

$$\int_{\mathbb{R}^d} \|z\|^2 p_i(z + \theta; \theta) \mathrm{d}z \leq \left(\int_{\mathbb{R}^d} \|z\|^4 \pi(z) \mathrm{d}z \right)^{1/2}.$$

Using that $\int_{\mathbb{R}^d} \|z\|^4 \pi(z) \mathrm{d}z = d(2 + d)$ yields the result. □

prop:logreg:barsi:lipsch

Proposition 24. *For all $i \in \{1, \dots, n\}$, the function $\bar{s}_i \circ \mathsf{T}$ is globally lipschitz on Θ with constant $L_i = 4\sqrt{d(2+d)}\|\delta_i\|_\infty + 1$.*

Proof. Recall that $\mathsf{T}(s) = s$ for all $s \in \mathcal{S}$, we then prove the function $\bar{s}_i$ is globally lipschitz on Θ. Using a change of variable,

$$\bar{s}_i(\theta) = \int_{\mathbb{R}^d} z p_i(z;\theta)\mathrm{d}z = \int_{\mathbb{R}^d} z p_i(z+\theta;\theta)\mathrm{d}z + \theta = \int_{\mathbb{R}^d} z e^{h_i(z+\theta)+g(Y_i;\theta)}\pi(z)\mathrm{d}z + \theta.$$

where $h_i(z)$ is defined in (91) and π is the density of a standard Gaussian vector. Taking the derivative, and using the definition of $g(Y_i;\theta)$ in (87), we have:

$$\begin{aligned}
&\dot{\bar{s}}_i(\theta) = \int_{\mathbb{R}^d} \left(Y_i - \frac{1}{1+e^{-\delta_i^T(z+\theta)}}\right) z\delta_i^T p_i(z+\theta;\theta)\mathrm{d}z \\
&- \left(\int_{\mathbb{R}^d} \left(Y_i - \frac{1}{1+e^{-\delta_i^T(z+\theta)}}\right) p_i(z+\theta;\theta)\mathrm{d}z\right)\left(\int_{\mathbb{R}^d} z p_i(z+\theta;\theta)\mathrm{d}z\right)\delta_i^T + \mathrm{I}_d
\end{aligned}$$

Using that $|Y_i - (1+e^{-\delta_i^T(z+\theta)})^{-1}| \leq 2$ and the result of Lemma 23, we have:

$$\|\dot{\bar{s}}_i(\theta)\|_\infty \leq 4\sqrt{d(2+d)}\|\delta_i\|_\infty + 1$$

Taking $L_i = 4\sqrt{d(2+d)}\|\delta_i\|_\infty + 1$ yields the result. □

lem:logreg:gibbs

Lemma 25. *Denote by $\widetilde{\pi}_{\theta,i}(z|\omega)$ the conditional density of z given ω. We have $\widetilde{\pi}_{\theta,i}(z|\omega) = \mathcal{N}(m_\omega, V_\omega)[z]$, where*

$$V_\omega \stackrel{\text{def}}{=} \left(\mathrm{I}_d + \omega\delta_i\delta_i^T\right)^{-1} \quad \textit{and} \quad m_\omega \stackrel{\text{def}}{=} V_\omega\left(\left(Y_i - \frac{1}{2}\right)\delta_i + \theta\right).$$

Proof. Starting from (89), considering ω is a fixed parameter and substituting (88) and (90), we have for all $z \in \mathbb{R}^d$:

$$\begin{aligned}
\widetilde{\pi}_{\theta,i}(z|\omega) &\propto \left(e^{\frac{\delta_i^T z}{2}} + e^{-\frac{\delta_i^T z}{2}}\right)\frac{\left(e^{-\delta_i^T z}\right)^{1-Y_i}}{1+e^{-\delta_i^T z}} e^{-\frac{1}{2}\left(\|z-\theta\|^2 + \omega(\delta_i^T z)^2\right)} \\
&\propto \exp\left(\delta_i^T z\left(Y_i - \frac{1}{2}\right) - \frac{1}{2}\left(\|z-\theta\|^2 + \omega(\delta_i^T z)^2\right)\right).
\end{aligned}$$

Upon noticing that $(\delta_i^T z)^2 = z^T\delta_i\delta_i^T z$, we have:

$$\begin{aligned}
\widetilde{\pi}_{\theta,i}(z|\omega) &\propto \exp\left(\delta_i^T z\left(Y_i - \frac{1}{2}\right) - \frac{1}{2}\left(\|z-\theta\|^2 + \omega z^T\delta_i\delta_i^T z\right)\right) \\
&\propto \exp\left(\left(\delta_i\left(Y_i - \frac{1}{2}\right) + \theta\right)^T z - \frac{1}{2}z^T\left(\mathrm{I}_d + \omega\delta_i\delta_i^T\right)z\right) \\
&\propto \exp\left((z-m_\omega)^T\left(\mathrm{I}_d + \omega\delta_i\delta_i^T\right)(z-m_\omega)\right) \\
&= \mathcal{N}(m_\omega, V_\omega)[z].
\end{aligned}$$

□

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B Toutes les choses pour la suite

B.1 On the definition of $\mathcal{S}$

B.1.1 On a toy example

Gers: Pierre-1: mettre ici l'exemple sur les mélanges de loi et l'estimation des poids. Détailler ce que donnent $\phi, \psi, \bar{s}, \mathsf{T}, \mathcal{S}$ pour les deux paramétrages des poids. Conclure sur la nécessité (ou pas) d'envisager un ensemble $\mathcal{S}$ qui ne soit pas $\mathbb{R}^q$.

Gers: Réduire à l'estimation des poids et de la moyenne - sortir la variance; pour les poids, que donne le paramétrage en $\exp(\alpha_i)/\sum_{j=1}^M \exp(\alpha_j)$ avec la convention que $\alpha_1 = 0$ poru l'identifiabilité du mélange; attention au sur-paramétrage de la fonction $\bar{s}$

B.1.2 Discussion

Gers: Pierre-1: Travailler cette section.

If the sequence γ_k is defined such as $\inf_k \gamma_k > 0$ and $\sup_k \gamma_k < 1$, we can show that the iterates $\widehat{S}^k$ remain in the set :

$$E \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{n} (\alpha_i a_i - \beta_i b_i),\ \alpha_i, \beta_i \in [0, M],\ a_i, b_i \in Conv(s_i(\mathsf{Z})),\ i \in \{1, ..., n\} \right\}$$

where $Conv(A)$ is the convex hull of the set A and $M > 0$ depends only on $\inf_k \gamma_k$ and $\sup_k \gamma_k$. For $\epsilon > 0$, we define:

$$\mathcal{S} \stackrel{\text{def}}{=} \left\{ s \in \mathbb{R}^d, d(s, E) < \epsilon \right\}$$

which, under item 1, defines an open bounded set. For the study of the algorithm, we set $V \stackrel{\text{def}}{=} F \circ T$. The following assumption allows to show V behaves as a lypaunov function.

Gers: Reflechir encore sur (a) la pertinence de raisonner sur $s_i(\mathsf{Z})$ ou sur $\bar{s}_i(\Theta)$; (ii) ne peut-on pas éviter de supposer que $\sup_k \gamma_k < 1$? notamment en exploitant le fait que $\sup_{k \leq K_{\max}} \gamma_k < \infty$? (iii) à voir si c'est intéressant, mais on peut noter que s_i est défini à une constante multiplicative près qui, néanmoins, doit être choisie indep de i (car on l'absorbe dans ϕ); (iv) Ainsi défini, E dépend de n et en particulier tout supremum pris sur E peut dépendre de n ? Attention donc dans les conclusions sur les constantes.

B.2 Gaussian Mixture Model

Gers: Pierre-1: là encore, décrire en 2 à 5 lignes max, ce que l'on veut illustrer avec cet exemple; présenter en 2 lignes les valeurs numériques prises; montrer les graphes qui répondent au plan de simulation et conclure

On regarde un mélange de M gaussiennes en dimension 1, où on veut estimer les poids du mélange, les moyennes et les variances. La vraisemblance d'une observation y est :

$$g(y; \theta) = \sum_{m=1}^{M} \frac{\omega_m}{\sqrt{2\pi\sigma_m^2}} e^{-\frac{(y-\mu_m)^2}{2\sigma_m^2}}$$

où $\theta = ((\omega_m)_{1\leq m\leq M-1}, (\mu_m)_{1\leq m\leq M}, (\sigma_m)_{1\leq m\leq M})$. On a $\theta \in \Theta$ où

$$\Theta = \Delta_M \times \mathbb{R}^M \times (0,+\infty)^M$$

avec

$$\Delta_m = \left\{(\omega_1,\ldots,\omega_{M-1}) \in [0,+\infty)^M,\ \sum_{m=1}^{M-1}\omega_m \leq 1\right\}$$

On note à chaque fois par commodité, $\omega_M = 1 - \sum_{m=1}^{M}\omega_m$. En notant μ la mesure de comptage sur $\{1,\ldots,M\}$, on le réinterprète en notant :

$$g(y;\theta) = \int_{\{1,\ldots,M\}} f(y,z;\theta)\mu(\mathrm{d}z)$$

où :

$$f(y,z;\theta) = \prod_{m=1}^{M}\left(\frac{\omega_m}{\sqrt{2\pi\sigma_m^2}}e^{-\frac{(y-\mu_m)^2}{2\sigma_m^2}}\right)^{\mathbb{1}_{z=m}}$$

On peut écrire :

$$\log(f(y,z;\theta)) = \langle s(y,z), \phi(\theta)\rangle - \psi(\theta) + h(y,z)$$

avec :

$$s(y,z) = \left((\mathbb{1}_{z=m})_{1\leq m\leq M}, (y\mathbb{1}_{z=m})_{1\leq m\leq M}, (y^2\mathbb{1}_{z=m})_{1\leq m\leq M}\right)$$
$$\phi(\theta) = \left(\left(\log(\omega_m) - \frac{1}{2}\log(\sigma_m^2) - \frac{\mu_m^2}{2\sigma_m^2}\right)_{1\leq m\leq M}, \left(\frac{\mu_m}{\sigma_m^2}\right)_{1\leq m\leq M}, \left(-\frac{1}{2\sigma_m^2}\right)_{1\leq m\leq M}\right)$$
$$\psi(\theta) = 0.$$

A partir d'un échantillon iid $\{y_1,\ldots,y_n\}$, on cherche à minimiser sur Θ :

$$F(\theta) = \frac{1}{n}\sum_{i=1}^{n} -\log(g(y_i;\theta)) + R(\theta)$$

avec :

$$R(\theta) = -\epsilon\sum_{m=1}^{M}\log(\omega_m)$$

Ici, les fonctions $\bar{s}_i$ et T sont définies pour $\theta \in \Theta$ et $s \in \mathbb{R}^{3M}$ par :

$$\bar{s}_i(\theta) = \left((\widetilde{\omega}_m(y_i;\theta))_{1\leq m\leq M}, (y_i\widetilde{\omega}_m(y_i;\theta))_{1\leq m\leq M}, (y_i^2\widetilde{\omega}_m(y_i;\theta))_{1\leq m\leq M}\right)$$

$$\text{où } \widetilde{\omega}_m(y_i;\theta) = \frac{\omega_m/\sqrt{\sigma_m^2}e^{-\frac{(y-\mu_m)^2}{2\sigma_m^2}}}{\sum_{m=1}^{M}\omega_m/\sqrt{\sigma_m^2}e^{-\frac{(y-\mu_m)^2}{2\sigma_m^2}}}$$

$$T(s) = \left(\left(\frac{s_m+\epsilon}{\sum_{m=1}^{M}s_m+\epsilon M}\right)_{1\leq m\leq M-1}, \left(\frac{s_{m+M}}{s_m}\right)_{1\leq m\leq M}, \left(\frac{s_{m+2M}}{s_m} - \left(\frac{s_{m+M}}{s_m}\right)^2\right)_{1\leq m\leq M}\right)$$

B.2.1 Méthodes avec indices aléatoires

On simule n observations à partir de ω, μ et σ tirés uniformément dans $[0,1]$, $[-\frac{M}{2}, \frac{M}{2}]$ et $[0, 0.1]$. On calcule une partition des observations avec un k-means et on choisit les valeurs initiales de *omega*, μ et σ qui correspondent à cette partition. On fait tourner l'algorithme pour :

- $M = 5$ et $n = 10^3$, avec une version avec $\epsilon = 0$ (non pénalisé) et une version avec $\epsilon = 0.05$. (Figure 1 et 2)
- $M = 10$ et $n = 10^4$ avec $\epsilon = 0$. (Figure 3)

A chaque fois, les algorithmes iEM et EM convergent, et on ne voit pas de différence de rapidité. L'algorithme fiEM converge pour un γ suffisamment petit, et plus γ est grand, plus il converge rapidement. Pour $\gamma \leq \frac{1}{n}$ environ, il devient moins rapide que EM ou iEM.

Ici on a pris $\gamma = \frac{0.15}{n^{2/3}}$ pour les versions non pénalisées, et $\gamma = \frac{0.75}{n^{2/3}}$ pour la version pénalisée, qui correspondent aux valeurs maximales pour lesquelles fiEM converge. On compare la convergence des algorithmes en nombre "d'époques", où une époque correspond à n appels à une foncion $\bar{s}_i$, donc n itérations de fiEM ou iEM et 1 itération de EM.

L'ajout d'une pénalité permet de faire converger les algorithmes plus facilement, et en particulier pour fiEM de prendre une valeur de γ plus grande.

Les algorithmes convergent vers la même valeur de F mais les (ω, μ, σ^2) limites ne sont pas forcément les mêmes.

B.2.2 Méthodes avec indice cyclique

Ici, on regarde une méthode ou les indices I et J de iEM et fiEM ne sont pas choisis aléatoirement mais de manière cyclique. On prend $I = J$ dans fiEM. L'indice cyclique permet d'actualiser chaque composante de S le plus fréquemment possible (toutes les n itérations). Avec les méthodes uniformes, le temps maximum d'actualisations pour chaque composantes était de $20n$ environ.

On lance les algorithmes avec les mêmes paramètres que précédemment sauf pour γ. Ici il faut en général prendre un γ plus petit que dans le cas ou les indices sont uniformes. Le γ maximal est de $\frac{0.15}{n^{2/3}}$, que ce soit avec ou sans pénalité, dans le cas $n = 10^3$. On obtient les figures 4 et 5.

La convergence de fiEM a l'air inchangé par le changement de méthode, mais la convergence de iEM est bien améliorée par rapport à celle de EM. Remarque : Ici en prenant $I = J$ à chaque étape dans l'algorithme fiEM, pour $\gamma = \frac{1}{n}$, on retrouve exactement iEM.

B.3 On the choice of λ_{k+1}

Gers 2all: DEBUT de mes reflexions

Soit $\lambda \in \mathbb{R}_\star$. Vue la définition du map T, on voit que si $\mathsf{T}(s)$ existe alors $\mathsf{T}(\lambda s)$ existe aussi ? il suffit de changer ϕ en ϕ/λ. Ca ne change aucune des

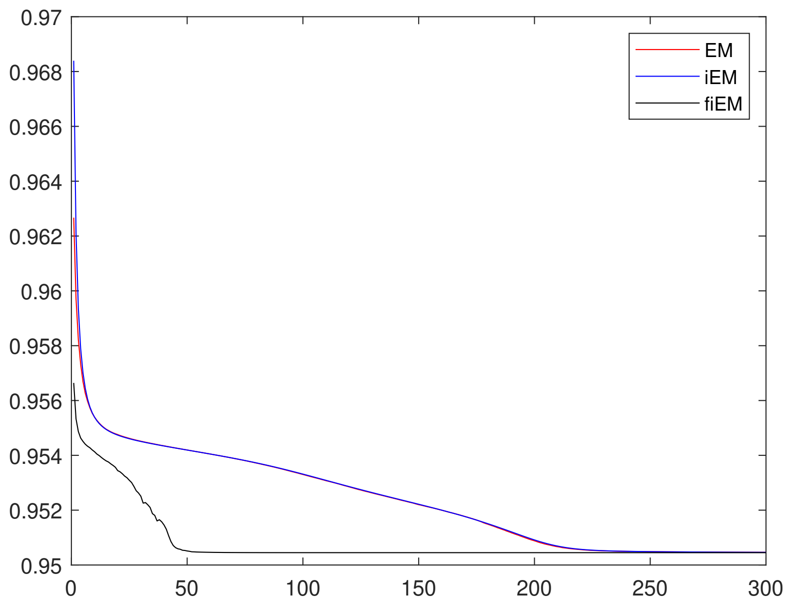

Figure 11: Evolution de F sans pénalité en fonction du nombre "d'époques" avec indices uniformes et $n = 10^3$

hypothèses faites sur ϕ. En fait, il y a une indétermination de la statistique S à un coefficient multiplicatif près. Si on multiplis S par λ, alors: il faut multiplier ϕ par $1/\lambda$; $\mathsf{T}(s)$ reste inchangé; $v_{\min}$ devient $v_{\min}/\lambda$, L_i devient λL_i, $L_{\dot{V}}$ est inchangé; $C_\star$ (et tous les C) sont inchangés; donc C_{GFM} est inchangée; γ_yGFM est divisé par λ. En revanche, C_{KM} a un drôle de comportement.

Gers 2all: FIN de mes reflexions

Gers 2all: un truc que je ne comprends pas; l'inégalité (12) montre que l'on a intérêt à prendre λ_{k+1} qui minimise la variance conditionnelle de H_{k+1} (voir section 2.4 pour le choix optimal). Avec ce choix optimal, la variance conditionnelle est de la forme (résultats standard sur les variables de contrôle)

$$(1-\rho^2)\ \mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right]$$

et c'est donc le meilleur majorant que l'on peut faire du terme de variance conditionnelle. OR, pour des arguments calculatoires, on va prendre $\lambda_{k+1} = 1$. J'aime pas ... dire que je fixe un paramètre à une valeur qui n'est pas la meilleure uniquement parce que avec ce choix, je sais construire un contrôle du terme ...

B.4 On the assumption H5

Commenter comment on vérifie cette hypothèse. En particulier, aller voir le proximal-SAGA et si ce n'est pas sur la statistique, essayer de l'adapter.

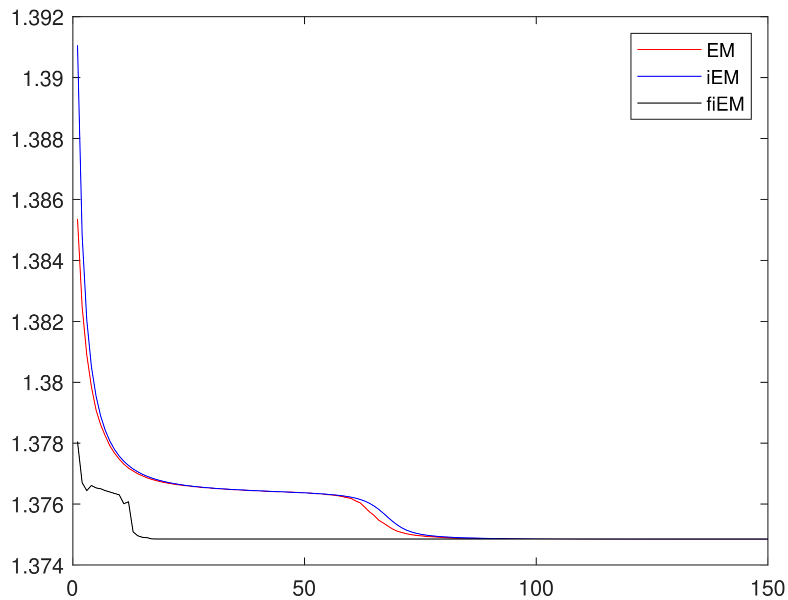

Figure 12: Evolution de F avec pénalité en fonction du nombre "d'époques" avec indices uniformes et $n = 10^3$, $\epsilon = 5 * 10^{-2}$

B.5 On the caracterization of an optimum, which is not a root of the gradient

Gers: (A) voir MISO pour une caractérisation de l'optimum sous contrainte, autre (forcément) que zero du gradient; (B) surveiller ce que l'on demande sur $\mathcal{S}$ et faire évoluer la définition en conséquence. (C) Vue la définition de T, il me semble que si $s \in \mathcal{S}$ alors $-s \in \mathcal{S}$? donc $\mathcal{S}$ est symétrique par rapport à l'origine ?

B.6 Sur le stockage

Gers 2all: franchement, la façon de relire leur algorithme, présentée ci-dessus (voir section 2.4, lemme 3), montre que le stockage de S est nécessaire uniquement pour construire la variable de contrôle. On peut tout à fait la construire avec un petit nombre d'exemples (choix ?) et prenant une convention sur quoi faire quand J_{k+1} pointe un indice qui n'est pas dans ce sous-ensemble; mais il faut quand même maintenir une corrélation entre les deux termes du champ H sinon on perd l'intérêt de la variable de contrôle.

B.7 Autres

Gers: Regarder sur SAGA, ce que l'on appelle "Sample complexity"; voir pour l'étendre afin d'avoir un outil qui analyse l'intérêt de prendre un aléa K plutôt qu'un autre.

Gers: "Stochastic approximation EM for logistic regression with missing values" (Julie Josse and co). // voir ce papier pour un exemple de régression logistique

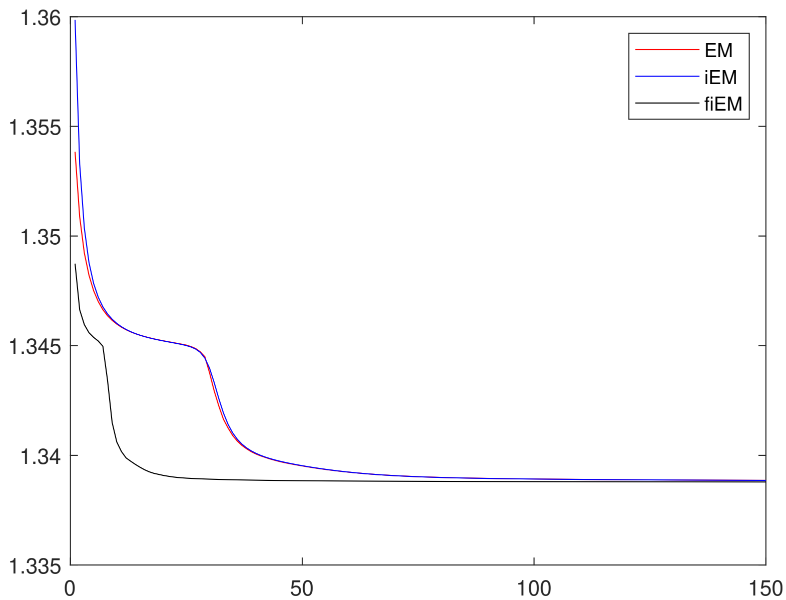

Figure 13: Evolution de F sans pénalité en fonction du nombre "d'époques" avec indices uniformes et $n = 10^4$

Gers: En stochastique : la vitesse $n^{2/3}$ sort-elle toujours ? (a) on peut déjà revenir au SAGA de base (sans l'EM) quand on fait une approx sto du gradient avec de l'iid tout simple; (b) puis exploiter que fiEM et SAGA sont proches pour passer le résultat à SAEM

Gers: Le critère pour comparer les méthodes est vraiment basé sur la dépendance en n (c'est ce qui intéresse les gens). SAGA apporte une dépendance en $n^{2/3}$ qui bat tout le monde. En particulier SAG.
Pour ezssayer de comprendre pourquoi SAGA utilise un levier $\lambda_k = 1$, discuter avec Francis Bach; ou Julien Mayral; ou regarder la preuve initiale de cvge de SAGA par Bach-Leroux (assez difficile à suivre) ou celle sur laquelle Eric s'est basé (Sra ?)

B.8 Commentaires en attente

Gers: Pierre-1: quelle est la logique de toutes les discussions qui suivent ? il faut les ordonner pour que ce soit un discours progressif. Là ça part un peu dans tous les sens : parfois on est en uniforme, parfois pas; parfois $n \to \infty$, parfois $C \to 0$, parfois $v_{\min} \to 0$. Bref : réfléchir à un discours progressif vers un objectif précis, et réordonner la suite en conséquence.

On the choice of C In order to make the constant C independant of n,

Gers: Pierre-1: quel est le gain à cette stratégie ? si il y en a un, pourquoi n'est-ce pas le choix à mettre dans la proposition ? et s'il n'y en a pas, doit-on garder la remarque ?

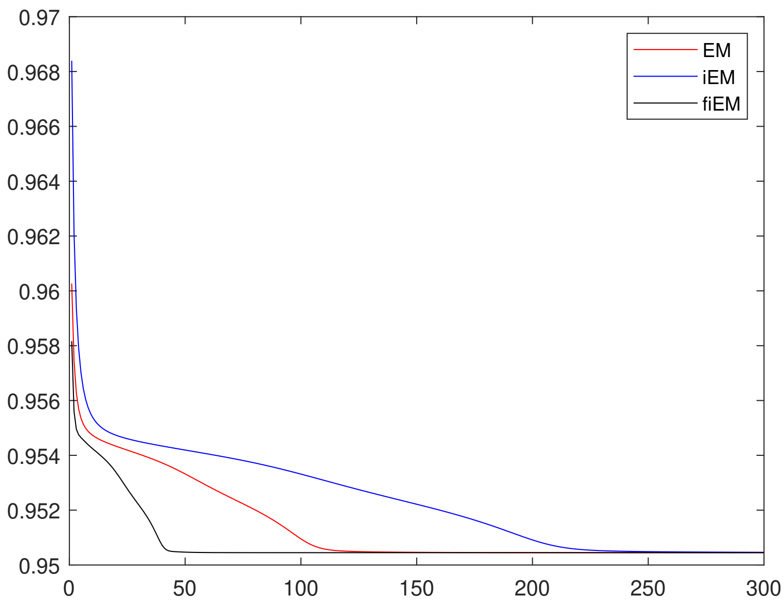

Figure 14: Evolution de F sans pénalité en fonction du nombre "d'époques" avec indices cycliques $n = 10^3$

a possibility is to choose C_{alt} such that

$$\frac{C_{\text{alt}}^{3/2}}{(\sqrt{2 - C_{\text{alt}}} - 1)^2} = \frac{v_{\min} L}{L_{\dot{V}}} \tag{92}$$

eq:def:Calt

Notice that in this case, we have:

$$C_{\text{alt}} \leq (\sqrt{2} - 1)^{4/3} \left(\frac{v_{\min} L}{L_{\dot{V}}} \right)^{2/3} \tag{93}$$

eq:control:Calt

Choice of C when n is large Here we set $p_k = 1/K_{\max}$.

Gers: Pierre-1: donc le titre du paragraphe n'est pas suffisant : tu veux dire aussi “dans le cas de poids uniforme” ?

We propose a different choice of C in the case when we neglect the term that goes to 0 when $n \to \infty$ in the proof of Proposition 7(see Section 3.4.4).

Gers: Pierre-1: on ne peut pas rédiger ainsi, tu ne peux pas demander au lecteur d'avoir digéré toutes les preuves pour comprendre tes remarques. Reprendre - voir ce que j'ai écrit pour la proposition 5.

More pecisely, we set

$$\sqrt{C_\star} \stackrel{\text{def}}{=} \left(\frac{(\sqrt{2} - 1)^2}{2} \right)^{1/3} \left(\frac{v_{\min} L}{L_{\dot{V}}} \right)^{1/3}$$

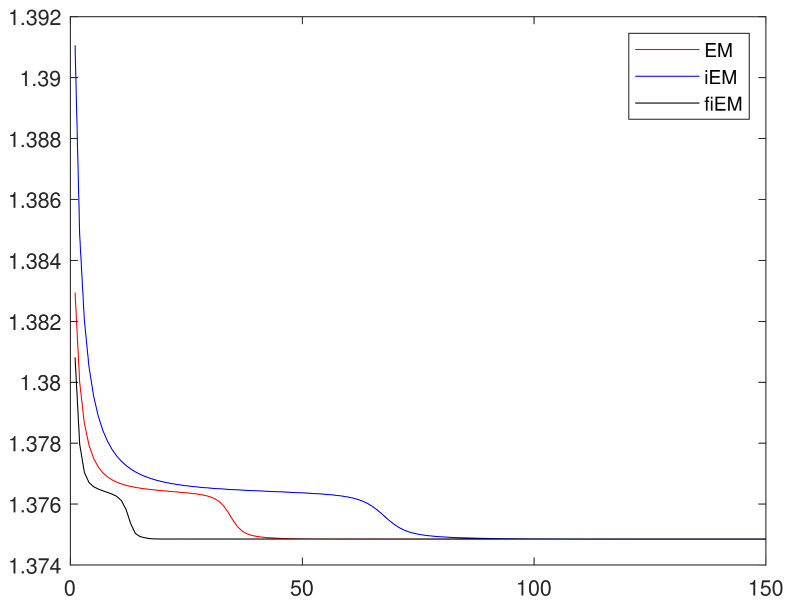

Figure 15: Evolution de F avec pénalité en fonction du nombre "d'époques" avec indices cycliques $n = 10^3$ et $\epsilon = 5 * 10^{-2}$
.

which would yield to an error controlled by

$$\frac{4}{3}\left(\frac{2}{(\sqrt{2}-1)^2}\right)^{1/3} \frac{n^{2/3}}{K_{\max}} \frac{L}{v_{\min}} \left(\frac{L_{\dot{V}}}{v_{\min} L}\right)^{1/3} \left(\mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]\right) .$$

Notice in this case that the multiplicative constant is slightly larger than the one given by Proposition 5 in a similar setting (see Section 3.2)

Gers: Pierre-1: que veux-tu dire par "similar setting", pas clair pour le lecteur - reprendre.

Pierre: En dessous, j'essaie de comparer la prop 5 et la prop 8 dans le cas uniforme, mais ça fait peut etre beaucoup de texte pour pas grand chose. La conclusion est que pour $v_{\min}$ petit, il faut mieux le γ de la prop 8, et quand $v_{\min}$ est grad, il vaut mieux celui de la prop 5. Je le fais avec la version corrigée

Comments on the case $p_k = 1/K_{\max}$ Since $\sum_{k=0}^{K_{\max}-1} p_k = 1$ and $p_k \geq 0$ for all k,

Gers: Pierre-1: $p_k \geq 0$ ou $p_k > 0$? attention, beaucoup de conventions en dépendent (positive / non negative; intervalles ouverts ou fermés. A fixer une bonne fois et s'y tenir

we have $\max_k p_k \geq 1/K_{\max}$ and the equality occurs when $p_k = 1/K_{\max}$ for any k. Therefore, this corollary

Gers: Pierre-1: pbme de ref

shows that among the sampling strategies $(p_0, \ldots, p_{K_{\max}-1})$, the RHS is optimal by choosing the uniform sampling strategy ($p_k = \alpha_k / A_{K_{\max}} = 1/K_{\max}$).

Gers: pierre-1: tout ce qui suit est difficilement digeste et le détail des calculs importe peu. Soit ça tient en cinq lignes, soit on fait un lemme ou un corollaire; et on met des calculs dans une section preuve

We now compare the LHS of Proposition 5 and Proposition 7 in a uniform setting. We take the optimal constant $\bar{C}$ for $\mu = 1/2$ in Proposition 5,

Gers: Pierre-1 : pourquoi ce choix de μ ?

meaning $\bar{C}$ is the root of

$$\sqrt{\bar{C}}\left(3 + \frac{2}{1-\bar{C}}\right) = \frac{v_{\min} L}{L_{\dot{V}}}$$

We then have in the RHS of Proposition 5

$$\frac{L}{\sqrt{\bar{C}}(1-\mu)v_{\min}} = \frac{2L}{\sqrt{\bar{C}}v_{\min}} \frac{2L_{\dot{V}}}{v_{\min}^2}\left(3 + \frac{2}{1-\bar{C}}\right),$$

and, taking C_{alt} (see (92)) in the RHS of Proposition 7,

Gers: Pierre-1: expliquer au lecteur aussi pourquoi c'est ce choix de la constante que tu prends.

$$\frac{C_{\text{GFM}}}{v_{\min}^2} = \frac{2L_{\dot{V}}}{v_{\min}^2}\left(\frac{1}{n^{2/3}} + \frac{C_{\text{alt}}}{(\sqrt{2-C_{\text{alt}}}-1)^2}\right)$$

Using (93) and the definition of $\bar{C}$, we have the following control on the constants:

$$C_{\text{alt}} \le (\sqrt{2}-1)^{4/3}\left(\frac{v_{\min}L}{L_{\dot{V}}}\right)^{2/3} \quad \text{and} \quad \bar{C} \le \frac{1}{25}\left(\frac{v_{\min}L}{L_{\dot{V}}}\right)^2$$

Gers: pierre: je ne suis pas convaincue par ce qui suit, plus généralement par la discussion du cas "$v_{\min} \to 0$" (d'ailleurs il me semble qu'on avait convenu de ne plus raisonner ainsi. Y a-t-il une raison pour revenir sur cela ?): je ne vois pas dans quelles situations cela est pertinent. De plus, cela est difficilement compatible avec les bornes que nous faisons où on dit qu'on prend des trucs indep de n, ou qu'on veut contrôler par ϵ: et jamais on ne se préoccupe d'une dépendance en n que pourrait avoir $v_{\min}$. N'y a-t-il pas inconsistance de nos propos ?

When $v_{\min} \to 0$, both C_{alt} and $\bar{C}$ goes to 0, so that the RHS of Proposition 7 becomes better than the one of Proposition 5 by a factor of $5n^{-2/3}$, meaning that the stepsize of Proposition 7 is preferable for small values of $v_{\min}$. Alternatively, when $v_{\min}$ is large, and so C_{alt} and $\bar{C}$ are close to 1, we have, using the implicit definition of C_{alt},

$$\frac{C_{\text{GFM}}}{v_{\min}^2} = \frac{2L}{v_{\min}}\left(\frac{1}{\sqrt{C_{\text{alt}}}} + \frac{L_{\dot{V}}}{v_{\min}Ln^{2/3}}\right)$$

In this case, the stepsize of Proposition 5 becomes preferable.

Comparison with other results There exist similar results in the literature, in the case the random stopping rule is the uniform one (i.e. $\alpha_k / A_{K_{\max}} = 1/K_{\max}$).

Gers: Maintenant, on ne parle plus de α_k, A_k, mais de p_k. Reprendre ici et ailleurs (if any)

In Karimi et al. (2019b), FIEM is run with a constant step size sequence equal to

$$\gamma_{\mathrm{KM}} = \frac{v_{min}}{\max(6, 1+4v_{\min}) \ \max(L_{\dot{V}}, L_1, \ldots, L_n) \ n^{2/3}};$$

and the upper bound is as in Proposition 7 with $\max_k p_k = 1/K_{\max}$, where the constant C_{GFM} is replaced with

$$C_{\mathrm{KM}} \overset{\mathrm{def}}{=} \left(\max(6, 1+4v_{\min})\right)^2 \ \max(L_{\dot{V}}, L_1, \ldots, L_n). \tag{94}$$

eq:C:KM

The first comment is that in Proposition 7 and in (Karimi et al., 2019b, Theorem 2) as well, the RHS is inversely proportional to $K_{\max}$ and proportional to $n^{2/3}$.

Gers: Pierre-1 : oui et non; vu que l'on n'est pas encore convaincu que là aussi, on peut avoir un autre contrôle en $n, K_{\max}$. L'as-tu exploré, donne-t-il qq chose ?

In that sense, the results are equivalent. Let us now compare the constants: we have

$$C_{\mathrm{KM}} \geq C_{\mathrm{GFM}} \frac{18}{\frac{1}{n^{2/3}} + \frac{C}{\left(\sqrt{2-C}-1\right)^2}} \overset{C \to 0}{\sim} 18 \, n^{2/3},$$

where the limiting case $C \to 0$ occurs for example when $C = C_{\mathrm{alt}}$ and $v_{\min} \to 0$ (see (93)).

Gers: Pierre-1: comment insères-tu cette discussion, après avoir parlé d'une constante indep de n etc ?