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# On the Convergence of Decentralized Adaptive Gradient Methods

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## Abstract

Adaptive gradient methods including Adam, AdaGrad, and their variants have been very successful for training deep learning models, such as neural networks. Meanwhile, given the need for distributed computing, distributed optimization algorithms are rapidly becoming a focal point. With the growth of computing power and the need for using machine learning models on mobile devices, the communication cost of distributed training algorithms needs careful consideration. In this paper, we introduce novel convergent decentralized adaptive gradient methods and rigorously incorporate adaptive gradient methods into decentralized training procedures. Specifically, we propose a general algorithmic framework that can convert existing adaptive gradient methods to their decentralized counterparts. In addition, we thoroughly analyze the convergence behavior of the proposed algorithmic framework and show that if a given adaptive gradient method converges, under some specific conditions, then its decentralized counterpart is also convergent. We illustrate the benefit of our generic decentralized framework on a prototype method, *i.e.* AMSGrad, both theoretically and numerically.

## 1 Introduction

Distributed training of machine learning models is drawing growing attention in the past few years due to its practical benefits and necessities. Given the evolution of computing capabilities of CPUs and GPUs, computation time in distributed settings is gradually dominated by the communication time in many circumstances [10; 25]. As a result, a large amount of recent works has been focussing on reducing communication cost for distributed learning [3; 22; 36; 32; 35; 34]. In the traditional parameter (central) server setting, where a parameter server is employed to manage communication in the whole network, many effective communication reductions have been proposed based on gradient compression [2] and quantization [9; 14; 16] techniques. Despite these communication reduction techniques, its cost still, usually, scales linearly with the number of workers. Due to this limitation and with the sheer size of decentralized devices, the *decentralized training paradigm* [13], where the parameter server is removed and each node only communicates with its neighbors, is drawing attention. It has been shown in [21] that decentralized training algorithms can outperform parameter server-based algorithms when the training bottleneck is the communication cost. The decentralized paradigm is also preferred when a central parameter server is not available.

In light of recent advances in nonconvex optimization, an effective way to accelerate training is by using adaptive gradient methods like AdaGrad [12], Adam [17] or AMSGrad [29]. Their popularity are due to their practical benefits in training neural networks, featured by faster convergence and ease of parameter tuning compared with Stochastic Gradient Descent (SGD) [30]. Despite a large amount of studies within the distributed optimization literature, few works have considered bringing adaptive gradient methods into distributed training, largely due to the lack of understanding of their convergence behaviors. Notably, Reddi et al. [28] develop the first decentralized ADAM method for distributed optimization problems with a direct application to federated learning. An inner loop is employed to compute mini-batch gradients on each node and a global adaptive step is applied

to update the global parameter at each outer iteration. Yet, in the settings of our paper, nodes can only communicate *to their neighbors* on a fixed communication graph while a server/worker communication is required in [28]. Designing adaptive methods in such settings is highly non-trivial due to the already complex update rules and to the interaction between the effect of using adaptive learning rates and the decentralized communication protocols. This paper is an attempt at bridging the gap between both realms in nonconvex optimization. Our **contributions** are summarized as follows:

- We investigate the use of adaptive gradient methods in the decentralized training paradigm, where nodes have only a local view of the whole communication graph. We develop a general technique that converts an adaptive gradient method from a centralized method to its decentralized variant and highlight the importance of adaptive learning rate consensus.
- By using our proposed technique, we present a new decentralized optimization algorithm, called decentralized AMSGrad, as the decentralized counterpart of AMSGrad.
- We provide a theoretical verification interface, in Theorem 2, for analyzing the behavior of decentralized adaptive gradient methods obtained as a result of our technique. Thus, we characterize the convergence rate of decentralized AMSGrad, which is the first convergent decentralized adaptive gradient method, to the best of our knowledge.

The paper is organized as follows. In Section 2, we show the importance of adaptive learning rate consensus by proving a divergent example for a recently proposed decentralized adaptive gradient method, DADAM [26]. In Section 3, we develop our general framework for converting adaptive gradient methods into their decentralized counterparts along with convergence analysis and converted algorithms. Illustrative experiments are presented in Section 4. Section 5 concludes our work.

**Notations:**  $x_{t,i}$  denotes variable  $x$  at node  $i$  and iteration  $t$ .  $\|\cdot\|_{abs}$  denotes the entry-wise  $L_1$  norm of a matrix, i.e.  $\|A\|_{abs} = \sum_{i,j} |A_{i,j}|$ . We introduce important notations used throughout the paper: for any  $t > 0$ ,  $G_t := [g_{t,N}]$  where  $[g_{t,N}]$  denotes the matrix  $[g_{t,1}, g_{t,2}, \dots, g_{t,N}]$  (where  $g_{t,i}$  is a column vector),  $M_t := [m_{t,N}]$ ,  $X_t := [x_{t,N}]$ ,  $\overline{\nabla}f(X_t) := \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i})$ ,  $U_t := [u_{t,N}]$ ,  $\tilde{U}_t := [\tilde{u}_{t,N}]$ ,  $V_t := [v_{t,N}]$ ,  $\hat{V}_t := [\hat{v}_{t,N}]$ ,  $\bar{X}_t := \frac{1}{N} \sum_{i=1}^N x_{t,i}$ ,  $\bar{U}_t := \frac{1}{N} \sum_{i=1}^N u_{t,i}$  and  $\bar{\tilde{U}}_t := \frac{1}{N} \sum_{i=1}^N \tilde{u}_{t,i}$ .

## 2 Decentralized Adaptive Training and Divergence of DADAM

### 2.1 Related Work

**Decentralized optimization:** Traditional decentralized optimization methods include well-known algorithms such as ADMM [7], Dual Averaging [13], Distributed Subgradient Descent [27]. More recent algorithms include Extra [31], Next [11], Prox-PDA [15], GNSD [23], and Choco-SGD [18]. While these algorithms are commonly used in applications other than deep learning, recent algorithmic advances in the machine learning community have shown that decentralized optimization can also be useful for training deep models such as neural networks. Lian et al. [21] demonstrate that a stochastic version of Decentralized Subgradient Descent can outperform parameter server-based algorithms when the communication cost is high. Tang et al. [33] propose the D<sup>2</sup> algorithm improving the convergence rate over Stochastic Subgradient Descent. Assran et al. [4] propose the Stochastic Gradient Push that is more robust to network failures for training neural networks. The study of decentralized training algorithms in the machine learning community is only at its initial stage. No existing work, to our knowledge, has seriously considered integrating *adaptive gradient methods* in the setting of decentralized learning. One noteworthy work [26] proposes a decentralized version of AMSGrad [29] and it is proven to satisfy some non-standard regret.

**Adaptive gradient methods:** Adaptive gradient methods have been popular in recent years due to their superior performance in training neural networks. Most commonly used adaptive methods include AdaGrad [12] or Adam [17] and their variants. Key features of such methods lie in the use of momentum and adaptive learning rates (which means that the learning rate is changing during the optimization and is anisotropic, i.e. depends on the dimension). The method of reference, called Adam, has been analyzed in [29] where the authors point out an error in previous convergence analyses. Since then, a variety of papers have been focusing on analyzing the convergence behavior of the numerous existing adaptive gradient methods. Ward et al. [37], Li and Orabona [20] derive convergence guarantees for a variant of AdaGrad without coordinate-wise learning rates. Chen et al. [8] analyze the convergence behavior of a broad class of algorithms including AMSGrad and

92 AdaGrad. Zhou et al. [41] give a more refined analysis of AMSGrad with better convergence rate.  
 93 Zou and Shen [42] provide a unified convergence analysis for AdaGrad with momentum. Noticeable  
 94 recent works on adaptive gradient methods can be found in [1; 24; 40].

## 95 2.2 Decentralized Optimization

96 In distributed optimization (with  $N$  nodes), we aim at solving the following problem

$$\min_{x \in \mathbb{R}^d} \frac{1}{N} \sum_{i=1}^N f_i(x), \quad (1)$$

97 where  $x$  is the vector of parameters and  $f_i$  is only accessible by the  $i$ th node. Through the prism of em-  
 98 pirical risk minimization procedures,  $f_i$  can be viewed as the average loss of the data samples located  
 99 at node  $i$ , for all  $i \in [N]$ . Throughout the paper, we make the following mild assumptions required  
 100 for analyzing the convergence behavior of the different decentralized optimization algorithms:

101 **A1.** For all  $i \in [N]$ ,  $f_i$  is differentiable and the gradients are  $L$ -Lipschitz, i.e., for all  $(x, y) \in \mathbb{R}^d$ ,  
 102  $\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|$ .

103 **A2.** We assume that, at iteration  $t$ , node  $i$  accesses a stochastic gradient  $g_{t,i}$ . The stochastic gradients  
 104 and the gradients of  $f_i$  have bounded  $L_\infty$  norms, i.e.  $\|g_{t,i}\| \leq G_\infty$ ,  $\|\nabla f_i(x)\|_\infty \leq G_\infty$ .

105 **A3.** The gradient estimators are unbiased and each coordinate has bounded variance, i.e.  $\mathbb{E}[g_{t,i}] =$   
 106  $\nabla f_i(x_{t,i})$  and  $\mathbb{E}[(g_{t,i} - \nabla f_i(x_{t,i}))_j^2] \leq \sigma^2, \forall t, i, j$ .

107 Assumptions A1 and A3 are standard in distributed optimization literature. A2 is slightly stronger  
 108 than the traditional assumption that the estimator has bounded variance, but is commonly used for the  
 109 analysis of adaptive gradient methods [8; 37]. Note that the bounded gradient estimator assumption  
 110 in A2 implies the bounded variance assumption in A3. In decentralized optimization, the nodes are  
 111 connected as a graph and each node only communicates to its neighbors. In such case, one usually  
 112 constructs a  $N \times N$  matrix  $W$  for information sharing when designing new algorithms. We denote  
 113  $\lambda_i$  to be its  $i$ th largest eigenvalue and define  $\lambda \triangleq \max(|\lambda_2|, |\lambda_N|)$ . The matrix  $W$  cannot be arbitrary,  
 114 its required key properties are listed in the following assumption:

115 **A4.** The matrix  $W$  satisfies: (I)  $\sum_{j=1}^N W_{i,j} = 1$ ,  $\sum_{i=1}^N W_{i,j} = 1$ ,  $W_{i,j} \geq 0$ , (II)  $\lambda_1 = 1$ ,  $|\lambda_2| < 1$ ,  
 116  $|\lambda_N| < 1$  and (III)  $W_{i,j} = 0$  if node  $i$  and node  $j$  are not neighbors.

117 We now present the failure to converge of current decentralized adaptive method before introducing  
 118 our general framework for decentralized adaptive gradient methods.

## 119 2.3 Divergence of DADAM

120 Recently, Nazari et al. [26] initiated an attempt  
 121 to bring adaptive gradient methods into decen-  
 122 tralized optimization with Decentralized ADAM  
 123 (DADAM), shown in Algorithm 1. DADAM is  
 124 essentially a decentralized version of ADAM  
 125 and the key modification is the use of a con-  
 126 sensus step on the optimization variable  $x$  to  
 127 transmit information across the network, encour-  
 128 aging its convergence. The matrix  $W$  is a dou-  
 129 bly stochastic matrix (which satisfies A4) for  
 130 achieving average consensus of  $x$ . Introducing  
 131 such mixing matrix is standard for decentraliz-  
 132 ing an algorithm, such as distributed gradient de-  
 133 scent [27; 39]. It is proven in [26] that DADAM  
 134 admits a non-standard regret bound in the online  
 135 setting. Nevertheless, whether the algorithm can  
 136 converge to stationary points in standard offline settings such training neural networks is still unknown.  
 137 The next theorem shows that DADAM may fail to converge in the offline settings.

138 **Theorem 1.** There exists a problem satisfying A1-A4 where DADAM fails to converge to a stationary  
 139 points with  $\nabla f(\bar{X}_t) = 0$ .

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### Algorithm 1 DADAM (with N nodes)

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1: Input:  $\alpha$ , current point  $X_t$ ,  $u_{\frac{1}{2},i} = \hat{v}_{0,i} = \epsilon 1$ ,  

    $m_0 = 0$  and mixing matrix  $W$ 
2: for  $t = 1, 2, \dots, T$  do
3:   for all  $i \in [N]$  do in parallel
4:      $g_{t,i} \leftarrow \nabla f_i(x_{t,i}) + \xi_{t,i}$ 
5:      $m_{t,i} = \beta_1 m_{t-1,i} + (1 - \beta_1) g_{t,i}$ 
6:      $v_{t,i} = \beta_2 v_{t-1,i} + (1 - \beta_2) g_{t,i}^2$ 
7:      $\hat{v}_{t,i} = \beta_3 \hat{v}_{t,i} + (1 - \beta_3) \max(\hat{v}_{t-1,i}, v_{t,i})$ 
8:    $x_{t+\frac{1}{2},i} = \sum_{j=1}^N W_{ij} x_{t,j}$ 
9:    $x_{t+1,i} = x_{t+\frac{1}{2},i} - \alpha \frac{m_{t,i}}{\sqrt{\hat{v}_{t,i}}}$ 
10: end for

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140 *Proof.* Consider a two-node setting with objective function  $f(x) = 1/2 \sum_{i=1}^2 f_i(x)$  and  $f_1(x) =$   
141  $\mathbb{1}[|x| \leq 1]2x^2 + \mathbb{1}[|x| > 1](4|x| - 2)$ ,  $f_2(x) = \mathbb{1}[|x-1| \leq 1](x-1)^2 + \mathbb{1}[|x-1| > 1](2|x-1| - 1)$ .  
142 We set the mixing matrix  $W = [0.5, 0.5; 0.5, 0.5]$ . The optimal solution is  $x^* = 1/3$ . Both  $f_1$  and  
143  $f_2$  are smooth and convex with bounded gradient norm 4 and 2, respectively. We also have  $L = 4$   
144 (defined in A1). If we initialize with  $x_{1,1} = x_{1,2} = -1$  and run DADAM with  $\beta_1 = \beta_2 = \beta_3 = 0$   
145 and  $\epsilon \leq 1$ , we will get  $\hat{v}_{1,1} = 16$  and  $\hat{v}_{1,2} = 4$ . Since  $|g_{t,1}| \leq 4$ ,  $|g_{t,2}| \leq 2$  due to bounded gradient,  
146 and  $(\hat{v}_{t,1}, \hat{v}_{t,2})$  are non-decreasing, we have  $\hat{v}_{t,1} = 16$ ,  $\hat{v}_{t,2} = 4$ ,  $\forall t \geq 1$ . Thus, after  $t = 1$ , DADAM  
147 is equivalent to running decentralized gradient descent (D-PSGD) [39] with a re-scaled  $f_1$  and  $f_2$ , i.e.  
148 running D-PSGD on  $f'(x) = \sum_{i=1}^2 f'_i(x)$  with  $f'_1(x) = 0.25f_1(x)$  and  $f'_2(x) = 0.5f_2(x)$ , which  
149 unique optimal  $x' = 0.5$ . Define  $\bar{x}_t = (x_{t,1} + x_{t,2})/2$ , then by Theorem 2 in [39], we have when  
150  $\alpha < 1/4$ ,  $f'(\bar{x}_t) - f(x') = O(1/(\alpha t))$ . Since  $f'$  has a unique optima  $x'$ , the above bound implies  
151  $\bar{x}_t$  is converging to  $x' = 0.5$  which has non-zero gradient on function  $\nabla f(0.5) = 0.5$ .  $\square$

152 Theorem 1 shows that, even though DADAM is proven to satisfy some regret bounds [26], it can  
153 fail to converge to stationary points in the nonconvex offline setting (common for training neural  
154 networks). We conjecture that this inconsistency in the convergence behavior of DADAM is due to  
155 the definition of the regret in [26]. The next section presents decentralized adaptive gradient methods  
156 that are guaranteed to converge to stationary points under assumptions and provide a characterization  
157 of that convergence in finite-time and independently of the initialization.

### 158 3 On the Convergence of Decentralized Adaptive Gradient Methods

159 In this section, we discuss the difficulties of designing adaptive gradient methods in decentralized  
160 optimization and introduce an algorithmic framework that can turn existing convergent adaptive gra-  
161 dient methods to their decentralized counterparts. We also develop the first convergent decentralized  
162 adaptive gradient method, converted from AMSGrad, as an instance of this framework.

#### 163 3.1 Importance and Difficulties of Consensus on Adaptive Learning Rates

164 The divergent example in the previous section implies that we should synchronize the adaptive  
165 learning rates on different nodes. This can be easily achieved in the parameter server setting where  
166 all the nodes are sending their gradients to a central server at each iteration. The parameter server  
167 can then exploit the received gradients to maintain a sequence of synchronized adaptive learning  
168 rates when updating the parameters, see [28]. However, in our decentralized setting, every node can  
169 only communicate with its neighbors and such central server does not exist. Under that setting, the  
170 information for updating the adaptive learning rates can only be shared locally instead of broadcasted  
171 over the whole network. This makes it impossible to obtain, in a single iteration, a synchronized  
172 adaptive learning rate update using all the information in the network.

173 *Systemic Approach:* On a systemic level, one way to alleviate this bottleneck is to design communi-  
174 cation protocols in order to give each node access to the same aggregated gradients over the whole  
175 network, at least periodically if not at every iteration. Therefore, the nodes can update their individual  
176 adaptive learning rates based on the same shared information. However, such solution may introduce  
177 an extra communication cost since it involves broadcasting the information over the whole network.

178 *Algorithmic Approach:* Our contributions being on an algorithmic level, another way to solve the  
179 aforementioned problem is by letting the sequences of adaptive learning rates, present on different  
180 nodes, to gradually *consent*, through the iterations. Intuitively, if the adaptive learning rates can  
181 consent fast enough, the difference among the adaptive learning rates on different nodes will not  
182 affect the convergence behavior of the algorithm. Consequently, no extra communication costs need  
183 to be introduced. We now develop this exact idea within the existing adaptive methods stressing on  
184 the need for a relatively low-cost and easy-to-implement consensus of adaptive learning rates.

185 Below is main archetype of the adaptive rates consensus mechanism within a decentralized framework.

#### 186 3.2 Unifying Decentralized Adaptive Gradient Framework

187 While each node can have different  $\hat{v}_{t,i}$  in DADAM (Algorithm 1), one can keep track of the  
188 min/max/average of these adaptive learning rates and use that quantity as the new adaptive learning  
189 rate. The predefinition of some convergent lower and upper bounds may also lead to a gradual  
190 synchronization of the adaptive learning rates on different nodes as developed for AdaBound in [24].

In this paper, we present an algorithm framework for decentralized adaptive gradient methods as Algorithm 2, which uses average consensus of  $\hat{v}_{t,i}$  (see consensus update in line 8 and 11) to help convergence. Algorithm 2 can become different adaptive gradient methods by specifying  $r_t$  as different functions. E.g., when we choose  $\hat{v}_{t,i} = \frac{1}{t} \sum_{k=1}^t g_{k,i}^2$ , Algorithm 2 becomes a decentralized version of AdaGrad. When one chooses  $\hat{v}_{t,i}$  to be the adaptive learning rate for AMSGrad, we get decentralized AMSGrad (Algorithm 3). The intuition of using average consensus is that for adaptive gradient methods such as AdaGrad or Adam,  $\hat{v}_{t,i}$  approximates the second moment of the gradient estimator, the average of the estimations of those second moments from different nodes is an estimation of second moment on the whole network. Also, this design will not introduce any extra hyperparameters that can potentially complicate the tuning process ( $\epsilon$  in line 9 is important for numerical stability as in vanilla Adam). The following result gives a finite-time convergence rate for our framework described in Algorithm 2.

**Theorem 2.** Assume A1-A4. When  $\alpha \leq \frac{\epsilon^{0.5}}{16L}$ , Algorithm 2 yields the following regret bound

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \left( \frac{1}{T\alpha} (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \alpha \frac{d\sigma^2}{N} \right) + C_2 \alpha^2 d \\ &\quad + C_3 \alpha^3 d + \frac{1}{T\sqrt{N}} (C_4 + C_5 \alpha) \mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \end{aligned} \quad (2)$$

where  $\|\cdot\|_{abs}$  denotes the entry-wise  $L_1$  norm of a matrix (i.e.  $\|A\|_{abs} = \sum_{i,j} |A_{i,j}|$ ). The constants  $C_1 = \max(4, 4L/\epsilon)$ ,  $C_2 = 6((\beta_1/(1-\beta_1))^2 + 1/(1-\lambda)^2) LG_\infty^2/\epsilon^{1.5}$ ,  $C_3 = 16L^2(1-\lambda)G_\infty^2/\epsilon^2$ ,  $C_4 = 2/(\epsilon^{1.5}(1-\lambda))(\lambda + \beta_1/(1-\beta_1))G_\infty^2$ ,  $C_5 = 2/(\epsilon^2(1-\lambda))L(\lambda + \beta_1/(1-\beta_1))G_\infty^2 + 4/(\epsilon^2(1-\lambda))LG_\infty^2$  are independent of  $d$ ,  $T$  and  $N$ . In addition,  $\frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \leq \alpha^2 \left( \frac{1}{1-\lambda} \right)^2 dG_\infty^2 \frac{1}{\epsilon}$  which quantifies the consensus error.

In addition, one can specify  $\alpha$  to show convergence in terms of  $T$ ,  $d$ , and  $N$ . An immediate result, shown in Corollary 2.1, is by setting  $\alpha = \sqrt{N}/\sqrt{Td}$ :

**Corollary 2.1.** Assume A1-A4. Set  $\alpha = \sqrt{N}/\sqrt{Td}$ . When  $\alpha \leq \frac{\epsilon^{0.5}}{16L}$ , Algorithm 2 yields:

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \frac{\sqrt{d}}{\sqrt{TN}} \left( (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \sigma^2 \right) + C_2 \frac{N}{T} \\ &\quad + C_3 \frac{N^{1.5}}{T^{1.5}d^{0.5}} + \left( C_4 \frac{1}{T\sqrt{N}} + C_5 \frac{1}{T^{1.5}d^{0.5}} \right) \mathbb{E}[\mathcal{V}_T] \end{aligned} \quad (3)$$

where  $\mathcal{V}_T := \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}$  and  $C_1, C_2, C_3, C_4, C_5$  are defined in Theorem 2.

Corollary 2.1 indicates that if  $\mathbb{E}[\mathcal{V}_T] = o(T)$  and  $\bar{U}_t$  is upper bounded, then Algorithm 2 is guaranteed to converge to stationary points of the loss function. Intuitively, this means that if the adaptive learning rates on different nodes do not change too fast, the algorithm can converge. In convergence analysis, the term  $\mathbb{E}[\mathcal{V}_T]$  upper bounds the total bias in update direction caused by the correlation between  $m_{t,i}$  and  $\hat{v}_{t,i}$ . It is shown in [8] that when  $N = 1$ ,  $\mathbb{E}[\mathcal{V}_T] = \tilde{O}(d)$  for AdaGrad and AMSGrad. Besides,  $\mathbb{E}[\mathcal{V}_T] = \tilde{O}(Td)$  for Adam which do not converge. Later, we will show convergence of decentralized versions of AMSGrad and AdaGrad by bounding this term as  $O(Nd)$  and  $O(Nd \log(T))$ , respectively. Corollary 2.1 also conveys the benefits of using more nodes in the graph employed. When  $T$  is large enough such that the term  $O(\sqrt{d}/\sqrt{TN})$  dominates the right hand side of (3), then linear speedup can be achieved by increasing the number of nodes  $N$ .



Another point worth discussion is the choice of  $W$  since the convergence rate depends on  $\lambda$  which is dependent on  $W$ . A common way to set  $W$  for undirected graph is the maximum-degree method (MDM) in [5]. Denote  $d_i$  as degree of vertex  $i$  and  $d_{\max} = \max_i d_i$ , MDM sets  $W_{i,i} = 1 - d_i/d_{\max}$ ,  $W_{i,j} = 1/d_{\max}$  if  $i \neq j$  and  $(i, j)$  is an edge, and  $W_{i,j} = 0$  otherwise. This  $W$  ensures Assumption A4 for many common connected graph types, so does the variant  $\gamma I + (1 - \gamma)W$  for any  $\gamma \in [0, 1]$ . A more refined choice of  $W$  coupled with a comprehensive discussion on  $\lambda$  in our Theorem 2 can be found in [6], e.g.,  $1 - \lambda = O(1/N^2)$  for cycle graphs,  $1 - \lambda = O(1/\log(N))$  for hypercube graphs,  $\lambda = 0$  for fully connected graph. Intuitively,  $\lambda$  can be close to 1 for sparse graphs and to 0 for dense graphs. This is consistent (2), whose RHS is large for  $\lambda$  close to 1 and small for  $\lambda$  close to 0 since average consensus on sparser graphs is expected to take longer time.

### 3.3 Application to AMSGrad algorithm

We now present, in Algorithm 3, a notable special case of our algorithmic framework, namely Decentralized AMSGrad, which is a decentralized variant of AMSGrad. Compared with DADAM, the above algorithm exhibits a dynamic average consensus mechanism to keep track of the average of  $\{\hat{v}_{t,i}\}_{i=1}^N$ , stored as  $\tilde{u}_{t,i}$  on  $i$ th node, and uses  $u_{t,i} := \max(\tilde{u}_{t,i}, \epsilon)$  for updating the adaptive learning rate for  $i$ th node. As the number of iteration grows, even though  $\hat{v}_{t,i}$  on different nodes can converge to different constants, the  $u_{t,i}$  will converge to the same number  $\lim_{t \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \hat{v}_{t,i}$  if the limit exists.

This average consensus mechanism enables the consensus of adaptive learning rates on different nodes, which accordingly guarantees the convergence of the method to stationary points. The consensus of adaptive learning rates is the key difference between decentralized AMSGrad and DADAM and is the reason why decentralized AMSGrad is convergent while DADAM is not.

One may notice that decentralized AMSGrad does not reduce to AMSGrad for  $N = 1$  since the quantity  $u_{t,i}$  in line 10 is calculated based on  $v_{t-1,i}$  instead of  $v_{t,i}$ . This design encourages the execution of gradient computation and communication in a parallel manner. Specifically, line 4-7 (line 4-6) in Algorithm 3 (Algorithm 2) can be executed in parallel with line 8-9 (line 7-8) to overlap communication and computation time. If  $u_{t,i}$  depends on  $v_{t,i}$  which in turn depends on  $g_{t,i}$ , the gradient computation must finish before the consensus step of the adaptive learning rate in line 9. This can slow down the running time per-iteration of the algorithm. To avoid such delayed adaptive learning, adding  $\tilde{u}_{t-\frac{1}{2},i} = \tilde{u}_{t,i} - \hat{v}_{t-1,i} + \hat{v}_{t,i}$  before line 9 and getting rid of line 12 in Algorithm 2 is an option. Similar convergence guarantees will hold since one can easily modify our proof of Theorem 2 for such update rule. As stated above, Algorithm 3 converges, with the following rate:

**Theorem 3.** Assume A1-A4. Set  $\alpha = 1/\sqrt{Td}$ . When  $\alpha \leq \frac{\epsilon^{0.5}}{16L}$ , then Algorithm 3 satisfies:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \leq C'_1 \frac{\sqrt{d}}{\sqrt{TN}} (D_f + \sigma^2) + C'_2 \frac{N}{T} + C'_3 \frac{N^{1.5}}{T^{1.5}d^{0.5}} + C'_4 \frac{\sqrt{Nd}}{T} + C'_5 \frac{Nd^{0.5}}{T^{1.5}},$$

where  $D_f := \mathbb{E}[f(Z_1)] - \min_x f(x)$ ,  $C'_1 = C_1$ ,  $C'_2 = C_2$ ,  $C'_3 = C_3$ ,  $C'_4 = C_4 G_\infty^2$  and  $C'_5 = C_5 G_\infty^2$ .  $C_1, C_2, C_3, C_4, C_5$  are independent of  $d, T$  and  $N$  defined in Theorem 2. In addition, the consensus of variables at different nodes is given by  $\frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \leq \frac{N}{T} \left( \frac{1}{1-\lambda} \right)^2 G_\infty^2 \frac{1}{\epsilon}$ .

Theorem 3 shows that Algorithm 3 converges with a rate of  $\mathcal{O}(\sqrt{d}/\sqrt{T})$  when  $T$  is large, which is the best known convergence rate under the given assumptions. Note that in some related works, SGD admits a convergence rate of  $\mathcal{O}(1/\sqrt{T})$  without any dependence on the dimension of the problem. Such improved convergence rate is derived under the assumption that the gradient estimator have a

bounded  $L_2$  norm, which can thus hide a dependency of  $\sqrt{d}$  in the final convergence rate. Another remark is the convergence measure can be converted to  $\frac{1}{T} \sum_{t=1}^T \mathbb{E} [\|\nabla f(\bar{X}_t)\|^2]$  using the fact that  $\|\bar{U}_t\|_\infty \leq G_\infty^2$  (by update rule of Algorithm 3), for the ease of comparison with existing literature.

**Proof Sketch of Theorem 2:** The detailed proofs are reported in the supplementary material.

*Step 1: Reparameterization.* Similarly to [38; 8] with SGD (with momentum) and centralized adaptive gradient methods, define the following auxiliary sequence:  $Z_t = \bar{X}_t + \frac{\beta_1}{1-\beta_1}(\bar{X}_t - \bar{X}_{t-1})$ , with  $\bar{X}_0 \triangleq \bar{X}_1$ . Such an auxiliary sequence can help us deal with the bias brought by the momentum and simplifies the convergence analysis.

*Step 2: Bounding gradient.* With the help of  $Z_t$ , we can remove the complicated update dependence on  $m_t$ , and perform convergence analysis to bound gradient of  $Z_t$ . Then bound gradient of  $\bar{X}_t$  by smoothness of gradient, which yields:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \leq \frac{2}{T\alpha} \mathbb{E}[\Delta_f] + \frac{2}{T} \frac{\beta_1 D_1}{1-\beta_1} + \frac{2D_2}{T} + \frac{3D_3}{T} + \frac{L}{T\alpha} \sum_{t=1}^T \mathbb{E} [\|Z_{t+1} - Z_t\|^2], \quad (4)$$

where  $\Delta_f := \mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]$ .  $D_1, D_2$  and  $D_3$  are three terms, defined in the supplementary material, and which can be tightly bounded from above. We first bound  $D_3$  using the following quantities of interest:

$$\sum_{t=1}^T \|Z_t - \bar{X}_t\|^2 \leq T \left( \frac{\beta_1}{1-\beta_1} \right)^2 \alpha^2 d \frac{G_\infty^2}{\epsilon} \quad \text{and} \quad \sum_{t=1}^T \frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \leq T \alpha^2 \left( \frac{1}{1-\lambda} \right)^2 d G_\infty^2 \frac{1}{\epsilon}.$$

where  $\lambda = \max(|\lambda_2|, |\lambda_N|)$  and recall that  $\lambda_i$  is  $i$ th largest eigenvalue of  $W$ .

Then, bounding  $D_1$  and  $D_2$  give rise to the terms related to  $\mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right]$ .

*Step 3: Bounding the drift term variance.* An important term that needs upper bounding in our proof is the variance of the gradients multiplied (element-wise) by the adaptive learning rate,

$$\mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \leq \mathbb{E}[\|\Gamma_u^f\|^2] + \frac{d}{N} \frac{\sigma^2}{\epsilon}, \quad \text{where } \Gamma_u^f := \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) / \sqrt{u_{t,i}}.$$

We can then transform  $\mathbb{E}[\|\Gamma_u^f\|^2]$  into  $\mathbb{E}[\|\Gamma_{\bar{U}}^f\|^2]$  by splitting out two error terms, then bounding the error terms as operated for  $D_2$  and  $D_3$ . Then, by plugging it into (4), we obtain the desired bound in Theorem 2.

**Proof of Theorem 3:** Recall the bound in (3) of Theorem 2. Since Algorithm 3 is a special case of Algorithm 2, the remaining of the proof consists of characterizing the growth rate of  $\mathbb{E}[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}]$ . By construction,  $\hat{V}_t$  is non decreasing, so that  $\mathbb{E}[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}] = \mathbb{E}[\sum_{i=1}^N \sum_{j=1}^d (-\hat{v}_{0,i,j} + \hat{v}_{T-1,i,j})]$ . We can also prove  $|v_{t,i,j}| \leq G_\infty^2$  using  $\|g_{t,i}\|_\infty \leq G_\infty$ . Then we have  $\mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \leq \sum_{i=1}^N \sum_{j=1}^d \mathbb{E}[G_\infty^2] = NdG_\infty^2$ . Substituting into (3) yields the desired convergence bound for Algorithm 3.

### 3.4 Application to AdaGrad algorithm

In this section, we provide a decentralized version of AdaGrad [12] (optionally with momentum) converted by Algorithm 2, further supporting the usefulness of our decentralization framework. The required modification for decentralized AdaGrad is to specify line 4 of Algorithm 2 as follows:  $\hat{v}_{t,i} = \frac{t-1}{t} \hat{v}_{t-1,i} + \frac{1}{t} g_{t,i}^2$ , which is equivalent to  $\hat{v}_{t,i} = \frac{1}{t} \sum_{k=1}^t g_{k,i}^2$ . In this section, we call this algorithm decentralized AdaGrad.

The pseudo code of the algorithm is shown in Algorithm 4. There are two details in Algorithm 4 worth mentioning. The first one is that the introduced framework leverages momentum  $m_{t,i}$  in updates, while original AdaGrad does not use momentum. The momentum can be turned off by setting  $\beta_1 = 0$  and the convergence results will still hold. The other one is that in Decentralized AdaGrad, we use the average instead of the sum in the term  $\hat{v}_{t,i}$ . In other words, we write  $\hat{v}_{t,i} = \frac{1}{t} \sum_{k=1}^t g_{k,i}^2$ . This latter point is different from the original AdaGrad which actually uses  $\hat{v}_{t,i} = \sum_{k=1}^t g_{k,i}^2$ .

The reason is that in the original AdaGrad, a constant stepsize ( $\alpha$  independent of  $t$  or  $T$ ) is used with  $\hat{v}_{t,i} = \sum_{k=1}^t g_{k,i}^2$ . This is equivalent to using a well-known decreasing stepsize sequence  $\alpha_t = \frac{1}{\sqrt{t}}$  with  $\hat{v}_{t,i} = \frac{1}{t} \sum_{k=1}^t g_{k,i}^2$ . In our convergence analysis, which can be found below, we use a constant stepsize  $\alpha = O(\frac{1}{\sqrt{T}})$  to replace the decreasing stepsize sequence  $\alpha_t = O(\frac{1}{\sqrt{t}})$ . Such a replacement is popularly used in Stochastic Gradient Descent analysis for the sake of simplicity and to achieve a better convergence rate. In addition, it is easy to modify our theoretical framework to include decreasing stepsize sequences such as  $\alpha_t = O(\frac{1}{\sqrt{t}})$ . The convergence analysis for decentralized AdaGrad is shown in Theorem 4.

**Theorem 4.** Assume A1-A4. Set  $\alpha = \sqrt{N}/\sqrt{Td}$ . When  $\alpha \leq \frac{\epsilon_{0.5}}{16L}$ , decentralized AdaGrad yields the following regret bound

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \leq \frac{C'_1 \sqrt{d}}{\sqrt{TN}} D'_f + \frac{C'_2}{T} + \frac{C'_3 N^{1.5}}{T^{1.5} d^{0.5}} + \frac{\sqrt{N}(1 + \log(T))}{T} (dC'_4 + \frac{\sqrt{d}}{T^{0.5}} C'_5),$$

where  $D'_f := \mathbb{E}[f(Z_1)] - \min_z f(z) + \sigma^2$ ,  $C'_1 = C_1$ ,  $C'_2 = C_2$ ,  $C'_3 = C_3$ ,  $C'_4 = C_4 G_\infty^2$  and  $C'_5 = C_5 G_\infty^2$ .  $C_1, C_2, C_3, C_4, C_5$  are defined in Theorem 2 independent of  $d, T$  and  $N$ . In addition, the consensus of variables at different nodes is given by  $\frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \leq \frac{N}{T} \left( \frac{1}{1-\lambda} \right)^2 G_\infty^2 \frac{1}{\epsilon}$ .

## 4 Numerical Experiments

In this section, we conduct some experiments to test the performance of Decentralized AMSGrad, developed in Algorithm 3, on both *homogeneous* data and *heterogeneous* data distribution (i.e. the data generating distribution on different nodes are assumed to be different). Comparison with DADAM and the decentralized parallel stochastic gradient descent (D-PSGD) developed in [21] are conducted. We train a Convolutional Neural Network (CNN) with 3 convolution layers followed by a fully connected layer on MNIST [19]. We set  $\epsilon = 10^{-6}$  for both Decentralized AMSGrad and DADAM. The learning rate is chosen from the grid  $[10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}]$  based on validation accuracy for all algorithms. In the following experiments, the graph contains 5 nodes and each node can only communicate with its two adjacent neighbors forming a cycle. Regarding the mixing matrix  $W$ , we set  $W_{ij} = 1/3$  if nodes  $i$  and  $j$  are neighbors and  $W_{ij} = 0$  otherwise. More details on experiments can be found in the supplementary material of our paper.

### 4.1 Effect of heterogeneity

**Homogeneous data:** The whole dataset is shuffled and evenly split into different nodes. Such a setting is possible when the nodes are in a computer cluster. We see, Figure 1(a), that decentralized AMSGrad and DADAM perform quite similarly while D-PSGD (labelled as DGD) is much slower

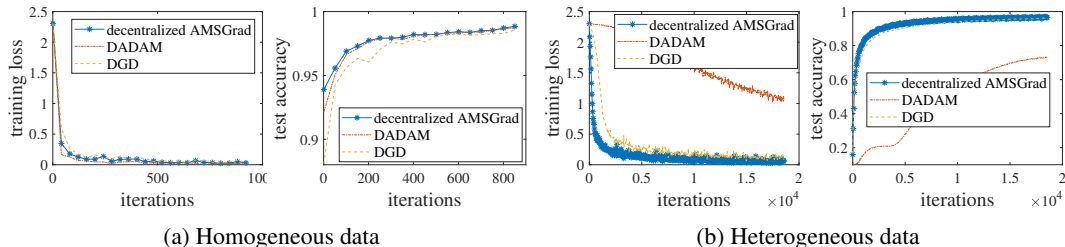


Figure 1: Training loss and Testing accuracy for homogeneous and heterogeneous data



both in terms of training loss and test accuracy. Though the (possible) non convergence of DADAM, mentioned in this paper, its performance are empirically good on homogeneous data. The reason is that the adaptive learning rates tend to be similar on different nodes in presence of homogeneous data distribution. We thus compare these algorithms under the heterogeneous regime.

*Heterogeneous data:* Here, each node only contains training data with two labels out of ten. Such a setting is common when data shuffling is prohibited, such as in federated learning. We can see that each algorithm converges significantly slower than with homogeneous data. Especially, the performance of DADAM deteriorates significantly. Decentralized AMSGrad achieves the best training and testing performance in that setting as observed in Figure 1(b).

## 4.2 Sensitivity to the Learning Rate

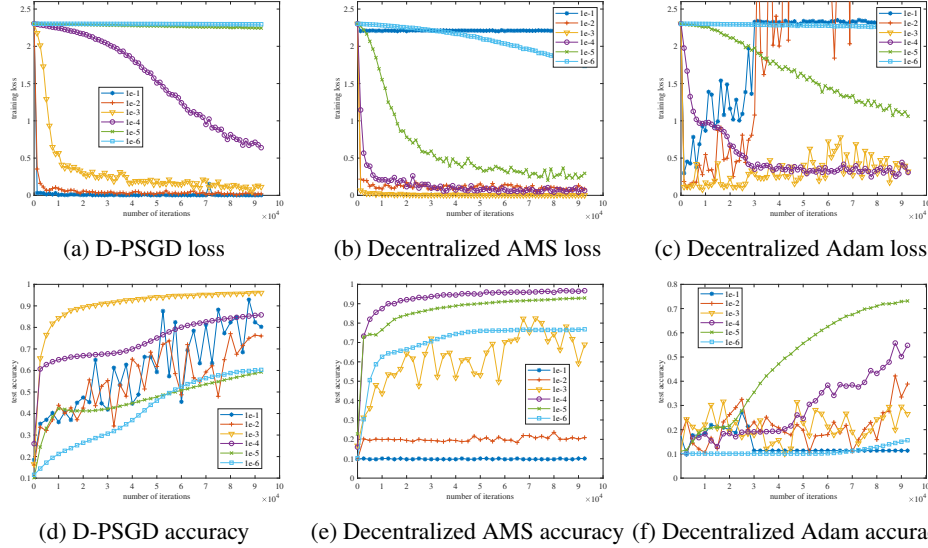


Figure 2: Training loss and testing accuracy comparison of different stepsizes for various methods

We compare the training loss and testing accuracies of different D-PSGD, DADAM, and our proposed Decentralized AMSGrad, with different stepsizes on *heterogeneous* data distribution. We use 5 nodes and the heterogeneous data distribution is created by assigning each node with data of only two labels. Note that there are no overlapping labels between different nodes. We observe Figure 2(a) and (d) that the stepsize  $10^{-3}$  works best for D-PSGD in terms of test accuracy and  $10^{-1}$  works best in terms of training loss. This difference is caused by the inconsistency among the model parameters on different nodes when the stepsize is large.

Figure 2(b) and (e) shows the performance of decentralized AMSGrad with different stepsizes. We see that its best performance is better than the one of D-PSGD and the performance is more stable (the test performance is less sensitive to stepsize tuning). As expected, the performance of DADAM is not as good as D-PSGD or decentralized AMSGrad, see Figure 2(c) and (f). Its divergence characteristic, highlighted Section 2.3, coupled with the heterogeneity in the data amplify its non-convergence issue in our experiments. From the experiments above, we can see the advantages of decentralized AMSGrad in terms of both performance and ease of parameter tuning, and the importance of ensuring the theoretical convergence of any newly proposed methods in the presented setting.

## 5 Conclusion

This paper studies the problem of designing adaptive gradient methods for decentralized training. We propose a unifying algorithmic framework that can convert existing adaptive gradient methods to decentralized settings. With rigorous convergence analysis, we show that if the original algorithm converges under some minor conditions, the converted algorithm obtained using our proposed framework is guaranteed to converge to stationary points of the regret function. By applying our framework to AMSGrad, we propose the first convergent adaptive gradient methods, namely Decentralized AMSGrad. We also give an extension to a decentralized variant of AdaGrad for completeness of our converting scheme. Experiments show that the proposed algorithm achieves better performance than the baselines.

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## Checklist

### 1. For all authors...

- (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
- (b) Did you describe the limitations of your work? [Yes]
- (c) Did you discuss any potential negative societal impacts of your work? [N/A]
- (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

### 2. If you are including theoretical results...

- (a) Did you state the full set of assumptions of all theoretical results? [Yes]
- (b) Did you include complete proofs of all theoretical results? [Yes]

### 3. If you ran experiments...

- (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No] Available upon demand
- (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes]
- (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [No]
- (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]

### 4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

- (a) If your work uses existing assets, did you cite the creators? [Yes]
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- (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A] Open source code and datasets.
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### 5. If you used crowdsourcing or conducted research with human subjects...

- (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
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- (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]



## 542 A Proof of Auxiliary Lemmas

543 **Lemma A.1.** *For the sequence defined in (38), we have*

$$Z_{t+1} - Z_t = \alpha \frac{\beta_1}{1 - \beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}}.$$

544 **Proof:** By update rule of Algorithm 2, we first have

$$\begin{aligned} \bar{X}_{t+1} &= \frac{1}{N} \sum_{i=1}^N x_{t+1,i} \\ &= \frac{1}{N} \sum_{i=1}^N \left( x_{t+0.5,i} - \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \right) \\ &= \frac{1}{N} \sum_{i=1}^N \left( \sum_{j=1}^N W_{ij} x_{t,j} - \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \right) \\ &\stackrel{(i)}{=} \left( \frac{1}{N} \sum_{j=1}^N x_{t,j} \right) - \frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \\ &= \bar{X}_t - \frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}}, \end{aligned}$$

545 where (i) is due to an interchange of summation and  $\sum_{i=1}^N W_{ij} = 1$ . Then, we have

$$\begin{aligned} Z_{t+1} - Z_t &= \bar{X}_{t+1} - \bar{X}_t + \frac{\beta_1}{1 - \beta_1} (\bar{X}_{t+1} - \bar{X}_t) - \frac{\beta_1}{1 - \beta_1} (\bar{X}_{t+1} - \bar{X}_t) \\ &= \frac{1}{1 - \beta_1} (\bar{X}_{t+1} - \bar{X}_t) - \frac{\beta_1}{1 - \beta_1} (\bar{X}_{t+1} - \bar{X}_t) \\ &= \frac{1}{1 - \beta_1} \left( -\frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t,i}}{\sqrt{u_{t,i}}} \right) - \frac{\beta_1}{1 - \beta_1} \left( -\frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t-1,i}}{\sqrt{u_{t-1,i}}} \right) \\ &= \frac{1}{1 - \beta_1} \left( -\frac{1}{N} \sum_{i=1}^N \alpha \frac{\beta_1 m_{t-1,i} + (1 - \beta_1) g_{t,i}}{\sqrt{u_{t,i}}} \right) - \frac{\beta_1}{1 - \beta_1} \left( -\frac{1}{N} \sum_{i=1}^N \alpha \frac{m_{t-1,i}}{\sqrt{u_{t-1,i}}} \right) \\ &= \alpha \frac{\beta_1}{1 - \beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}}, \end{aligned}$$

546 which is the desired result.  $\square$

547 **Lemma A.2.** *Given a set of numbers  $a_1, \dots, a_n$  and denote their mean to be  $\bar{a} = \frac{1}{n} \sum_{i=1}^n a_i$ . Define*  
 548  $b_i(r) \triangleq \max(a_i, r)$  *and  $\bar{b}(r) = \frac{1}{n} \sum_{i=1}^n b_i(r)$ . For any  $r$  and  $r'$  with  $r' \geq r$  we have*

$$\sum_{i=1}^n |b_i(r) - \bar{b}(r)| \geq \sum_{i=1}^n |b_i(r') - \bar{b}(r')| \quad (5)$$

549 and when  $r \leq \min_{i \in [n]} a_i$ , we have

$$\sum_{i=1}^n |b_i(r) - \bar{b}(r)| = \sum_{i=1}^n |a_i - \bar{a}|. \quad (6)$$

550 **Proof:** Without loss of generality, assume  $a_i \leq a_j$  when  $i < j$ , i.e.  $a_i$  is a non-decreasing sequence.  
 551 Define

$$h(r) = \sum_{i=1}^n |b_i(r) - \bar{b}(r)| = \sum_{i=1}^n \left| \max(a_i, r) - \frac{1}{n} \sum_{j=1}^n \max(a_j, r) \right|.$$

552 We need to prove that  $h$  is a non-increasing function of  $r$ . First, it is easy to see that  $h$  is a continuous  
 553 function of  $r$  with non-differentiable points  $r = a_i, i \in [n]$ , thus  $h$  is a piece-wise linear function.

554 Next, we will prove that  $h(r)$  is non-increasing in each piece. Define  $l(r)$  to be the largest index  
 555 with  $a(l(r)) < r$ , and  $s(r)$  to be the largest index with  $a_{s(r)} < \bar{b}(r)$ . Note that we have for  $i \leq l(r)$ ,  
 556  $b_i(r) = r$  and for  $i \leq s(r)$   $b_i(r) - \bar{b}(r) \leq 0$  since  $a_i$  is a non-decreasing sequence. Therefore, we  
 557 have

$$h(r) = \sum_{i=1}^{l(r)} (\bar{b}(r) - r) + \sum_{i=l(r)+1}^{s(r)} (\bar{b}(r) - a_i) + \sum_{i=s(r)+1}^n (a_i - \bar{b}(r))$$

558 and

$$\bar{b}(r) = \frac{1}{n} \left( l(r)r + \sum_{i=l(r)+1}^n a_i \right).$$

559 Taking derivative of the above form, we know the derivative of  $h(r)$  at differentiable points is

$$\begin{aligned} h'(r) &= l(r) \left( \frac{l(r)}{n} - 1 \right) + (s(r) - l(r)) \frac{l(r)}{n} - (n - s(r)) \frac{l(r)}{n} \\ &= \frac{l(r)}{n} ((l(r) - n) + (s(r) - l(r)) - (n - s(r))). \end{aligned}$$

560 Since we have  $s(r) \leq n$  we know  $(l(r) - n) + (s(r) - l(r)) - (n - s(r)) \leq 0$  and thus

$$h'(r) \leq 0,$$

561 which means  $h(r)$  is non-increasing in each piece. Combining with the fact that  $h(r)$  is continuous,  
 562 (5) is proven. When  $r \leq a(i)$ , we have  $b(i) = \max(a_i, r) = r$ , for all  $r \in [n]$  and  $\bar{b}(r) =$   
 563  $\frac{1}{n} \sum_{i=1}^n a_i = \bar{a}$  which proves (6).  $\square$

## 564 B Proof of Theorem 2

565 To prove convergence of the algorithm, we first define an auxiliary sequence

$$Z_t = \bar{X}_t + \frac{\beta_1}{1 - \beta_1} (\bar{X}_t - \bar{X}_{t-1}), \quad (7)$$

566 with  $\bar{X}_0 \triangleq \bar{X}_1$ . Since  $\mathbb{E}[g_{t,i}] = \nabla f(x_{t,i})$  and  $u_{t,i}$  is a function of  $G_{1:t-1}$  (which denotes  
567  $G_1, G_2, \dots, G_{t-1}$ ), we have

$$\mathbb{E}_{G_t|G_{1:t-1}} \left[ \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right] = \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}}.$$

568 Assuming smoothness (A1) we have

$$f(Z_{t+1}) \leq f(Z_t) + \langle \nabla f(Z_t), Z_{t+1} - Z_t \rangle + \frac{L}{2} \|Z_{t+1} - Z_t\|^2.$$

569 Using Lemma A.1 into the above inequality and take expectation over  $G_t$  given  $G_{1:t-1}$ , we have

$$\begin{aligned} & \mathbb{E}_{G_t|G_{1:t-1}} [f(Z_{t+1})] \\ & \leq f(Z_t) - \alpha \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\rangle + \frac{L}{2} \mathbb{E}_{G_t|G_{1:t-1}} [\|Z_{t+1} - Z_t\|^2] \\ & \quad + \alpha \frac{\beta_1}{1 - \beta_1} \mathbb{E}_{G_t|G_{1:t-1}} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right]. \end{aligned}$$

570 Then take expectation over  $G_{1:t-1}$  and rearrange, we have

$$\alpha \mathbb{E} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\rangle \right] \quad (8)$$

$$\begin{aligned} & \leq \mathbb{E}[f(Z_t)] - \mathbb{E}[f(Z_{t+1})] + \frac{L}{2} \mathbb{E} [\|Z_{t+1} - Z_t\|^2] \\ & \quad + \alpha \frac{\beta_1}{1 - \beta_1} \mathbb{E} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right]. \end{aligned} \quad (9)$$

571 In addition, we have

$$\begin{aligned} & \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\rangle \\ & = \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\rangle + \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left( \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right) \right\rangle \end{aligned} \quad (10)$$

572 and the first term on RHS of the equality can be lower bounded as

$$\begin{aligned} & \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\rangle \\ & = \frac{1}{2} \left\| \frac{\nabla f(Z_t)}{\bar{U}_t^{1/4}} \right\|^2 + \frac{1}{2} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i})}{\bar{U}_t^{1/4}} \right\|^2 - \frac{1}{2} \left\| \frac{\nabla f(Z_t) - \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i})}{\bar{U}_t^{1/4}} \right\|^2 \\ & \geq \frac{1}{4} \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 + \frac{1}{4} \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{1}{2} \left\| \frac{\nabla f(Z_t) - \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i})}{\bar{U}_t^{1/4}} \right\|^2 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{1}{2} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \\
& \geq \frac{1}{2} \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{3}{2} \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{3}{2} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2, \quad (11)
\end{aligned}$$

573 where the inequalities are all due to Cauchy-Schwartz. Substituting (11) and (10) into (8), we get

$$\begin{aligned}
\frac{1}{2} \alpha \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] & \leq \mathbb{E}[f(Z_t)] - \mathbb{E}[f(Z_{t+1})] + \frac{L}{2} \mathbb{E}[\|Z_{t+1} - Z_t\|^2] \\
& + \alpha \frac{\beta_1}{1 - \beta_1} \mathbb{E} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right] \\
& - \alpha \mathbb{E} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left( \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right) \right\rangle \right] \\
& + \frac{3}{2} \alpha \mathbb{E} \left[ \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 + \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right].
\end{aligned}$$

574 Then sum over the above inequality from  $t = 1$  to  $T$  and divide both sides by  $T\alpha/2$ , we have

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \\
& \leq \frac{2}{T\alpha} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) + \frac{L}{T\alpha} \sum_{t=1}^T \mathbb{E}[\|Z_{t+1} - Z_t\|^2] \\
& + \frac{2}{T} \frac{\beta_1}{1 - \beta_1} \sum_{t=1}^T \underbrace{\mathbb{E} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right]}_{D_1} \\
& + \frac{2}{T} \sum_{t=1}^T \underbrace{\mathbb{E} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left( \frac{1}{\sqrt{\bar{U}_t}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right]}_{D_2} \\
& + \frac{3}{T} \sum_{t=1}^T \underbrace{\mathbb{E} \left[ \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 + \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right]}_{D_3}. \quad (12)
\end{aligned}$$

575 Now we need to upper bound all the terms on RHS of the above inequality to get the convergence  
576 rate. For the terms composing  $D_3$  in (12), we can upper bound them by

$$\begin{aligned}
\left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 & \leq \frac{1}{\min_{j \in [d]} [\bar{U}_t^{1/2}]_j} \|\nabla f(Z_t) - \nabla f(\bar{X}_t)\|^2 \\
& \leq L \frac{1}{\min_{j \in [d]} [\bar{U}_t^{1/2}]_j} \underbrace{\|Z_t - \bar{X}_t\|^2}_{D_4} \quad (13)
\end{aligned}$$

577 and

$$\begin{aligned} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 &\leq \frac{1}{\min_{j \in [d]} [\bar{U}_t^{1/2}]_j} \frac{1}{N} \sum_{i=1}^N \|\nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)\|^2 \\ &\leq L \underbrace{\frac{1}{\min_{j \in [d]} [\bar{U}_t^{1/2}]_j} \frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2}_{D_5}, \end{aligned} \quad (14)$$

578 using Jensen's inequality, Lipschitz continuity of  $f_i$ , and the fact that  $f = \frac{1}{N} \sum_{i=1}^N f_i$ . Next we need  
579 to bound  $D_4$  and  $D_5$ . Recall the update rule of  $X_t$ , we have

$$X_t = X_{t-1}W - \alpha \frac{M_{t-1}}{\sqrt{U_{t-1}}} = X_1 W^{t-1} - \alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} W^k, \quad (15)$$

580 where we define  $W^0 = \mathbf{I}$ . Since  $W$  is a symmetric matrix, we can decompose it as  $W = Q\Lambda Q^T$   
581 where  $Q$  is a orthonormal matrix and  $\Lambda$  is a diagonal matrix whose diagonal elements correspond  
582 to eigenvalues of  $W$  in an descending order, i.e.  $\Lambda_{ii} = \lambda_i$  with  $\lambda_i$  being  $i$ th largest eigenvalue of  
583  $W$ . In addition, because  $W$  is a doubly stochastic matrix, we know  $\lambda_1 = 1$  and  $q_1 = \frac{1}{\sqrt{N}}$ . With  
584 eigen-decomposition of  $W$ , we can rewrite  $D_5$  as

$$\sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 = \|X_t - \bar{X}_t \mathbf{1}_N^T\|_F^2 = \|X_t Q Q^T - X_t \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T\|_F^2 = \sum_{l=2}^N \|X_t q_l\|^2. \quad (16)$$

585 In addition, we can rewrite (15) as

$$X_t = X_1 W^{t-1} - \alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} W^k = X_1 - \alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} Q \Lambda^k Q^T, \quad (17)$$

586 where the last equality is because  $x_{1,i} = x_{1,j}$ , for all  $i, j$  and thus  $X_1 W = X_1$ . Then we have when  
587  $l > 1$ ,

$$X_t q_l = (X_1 - \alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} Q \Lambda^k Q^T) q_l = -\alpha \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} q_l \lambda_l^k, \quad (18)$$

588 since  $Q$  is orthonormal and  $X_1 q_l = x_{1,1} \mathbf{1}_N^T q_l = x_{1,1} \sqrt{N} q_1^T q_l = 0$ , for all  $l \neq 1$ .

589 Combining (16) and (18), we have

$$\begin{aligned} D_5 &= \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 = \sum_{l=2}^N \|X_t q_l\|^2 \\ &= \sum_{l=2}^N \alpha^2 \left\| \sum_{k=0}^{t-2} \frac{M_{t-k-1}}{\sqrt{U_{t-k-1}}} \lambda_l^k q_l \right\|^2 \\ &\leq \alpha^2 \left( \frac{1}{1-\lambda} \right)^2 N d G_\infty^2 \frac{1}{\epsilon}, \end{aligned} \quad (19)$$

590 where the last inequality follows from the fact that  $g_{t,i} \leq G_\infty$ ,  $\|q_l\| = 1$ , and  $|\lambda_l| \leq \lambda < 1$ . Now let  
591 us turn to  $D_4$ , it can be rewritten as

$$\begin{aligned} \|Z_t - \bar{X}_t\|^2 &= \left\| \frac{\beta_1}{1-\beta_1} (\bar{X}_t - \bar{X}_{t-1}) \right\|^2 = \left( \frac{\beta_1}{1-\beta_1} \right)^2 \alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{m_{t-1,i}}{\sqrt{u_{t-1,i}}} \right\|^2 \\ &\leq \left( \frac{\beta_1}{1-\beta_1} \right)^2 \alpha^2 d \frac{G_\infty^2}{\epsilon}. \end{aligned} \quad (20)$$

592 Now we know both  $D_4$  and  $D_5$  are in the order of  $\mathcal{O}(\alpha^2)$  and thus  $D_3$  is in the order of  
593  $\mathcal{O}(\alpha^2)$ . Next we will bound  $D_2$  and  $D_1$ . Define  $G_1 \triangleq \max_{t \in [T]} \max_{i \in [N]} \|\nabla f_i(x_{t,i})\|_\infty$ ,



594  $G_2 \triangleq \max_{t \in [T]} \|\nabla f(Z_t)\|_\infty$ ,  $G_3 \triangleq \max_{t \in [T]} \max_{i \in [N]} \|g_{t,i}\|_\infty$  and  $G_\infty = \max(G_1, G_2, G_3)$ .  
 595 Then we have

$$\begin{aligned}
 D_2 &= \sum_{t=1}^T \mathbb{E} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left( \frac{1}{\sqrt{\bar{U}_t}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right] \\
 &\leq \sum_{t=1}^T \mathbb{E} \left[ G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{1}{\sqrt{[\bar{U}_t]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right| \right] \\
 &= \sum_{t=1}^T \mathbb{E} \left[ G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{1}{\sqrt{[\bar{U}_t]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right| \frac{\sqrt{[\bar{U}_t]_j} + \sqrt{[u_{t,i}]_j}}{\sqrt{[\bar{U}_t]_j} + \sqrt{[u_{t,i}]_j}} \right] \\
 &= \sum_{t=1}^T \mathbb{E} \left[ G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{[\bar{U}_t]_j - [u_{t,i}]_j}{[\bar{U}_t]_j \sqrt{[u_{t,i}]_j} + \sqrt{[\bar{U}_t]_j} [u_{t,i}]_j} \right| \right] \\
 &\leq \underbrace{\mathbb{E} \left[ \sum_{t=1}^T G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{[\bar{U}_t]_j - [u_{t,i}]_j}{2\epsilon^{1.5}} \right| \right]}_{D_6},
 \end{aligned} \tag{21}$$

596 where the last inequality is due to  $[u_{t,i}]_j \geq \epsilon$ , for all  $t, i, j$ . To simplify notations, define  $\|A\|_{abs} =$   
 597  $\sum_{i,j} |A_{ij}|$  to be the entry-wise  $L_1$  norm of a matrix  $A$ , then we obtain

$$\begin{aligned}
 D_6 &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \|\bar{U}_t \mathbf{1}^T - U_t\|_{abs} \leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \|\bar{U}_t \mathbf{1}^T - \tilde{U}_t\|_{abs} \\
 &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \|\tilde{U}_t \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T - \tilde{U}_t Q Q^T\|_{abs} \\
 &= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| - \sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs},
 \end{aligned}$$

598 where the second inequality is due to Lemma A.2, introduced Section A, and the fact that  $U_t =$   
 599  $\max(\tilde{U}_t, \epsilon)$  (element-wise max operator). Recall from update rule of  $U_t$ , by defining  $\hat{V}_{-1} \triangleq \hat{V}_0$  and  
 600  $U_0 \triangleq U_{1/2}$ , we have for all  $t \geq 0$ ,  $\tilde{U}_{t+1} = (\tilde{U}_t - \hat{V}_{t-1} + \hat{V}_t)W$ . Thus, we obtain

$$\tilde{U}_t = \tilde{U}_0 W^t + \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) W^k = \tilde{U}_0 + \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) Q \Lambda^k Q^T.$$

601 Then we further obtain when  $l \neq 1$ ,

$$\tilde{U}_t q_l = (\tilde{U}_0 + \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) Q \Lambda^k Q^T) q_l = \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) q_l \lambda_l^k,$$

602 where the last equality is due to the definition  $\tilde{U}_0 \triangleq U_{1/2} = \epsilon \mathbf{1}_d \mathbf{1}_N^T = \sqrt{N} \epsilon \mathbf{1}_d \mathbf{1}_N^T$  (recall that  
 603  $q_1 = \frac{1}{\sqrt{N}} \mathbf{1}_N^T$ ) and  $q_i^T q_j = 0$  when  $i \neq j$ . Note that by definition of  $\|\cdot\|_{abs}$ , we have for all

604  $A, B, \|A + B\|_{abs} \leq \|A\|_{abs} + \|B\|_{abs}$ , then

$$\begin{aligned}
D_6 &\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| - \sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs} \\
&= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| - \sum_{k=1}^t (-\hat{V}_{t-1-k} + \hat{V}_{t-k}) \sum_{l=2}^N q_l \lambda_l^k q_l^T \right\|_{abs} \\
&\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \sum_{j=1}^d \left\| \sum_{l=2}^N q_l \lambda_l^k q_l^T \right\|_1 \|(-\hat{V}_{t-1-k} + \hat{V}_{t-k})^T e_j\|_1 \\
&\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \sum_{j=1}^d \sqrt{N} \left\| \sum_{l=2}^N q_l \lambda_l^k q_l^T \right\|_2 \|(-\hat{V}_{t-1-k} + \hat{V}_{t-k})^T e_j\|_1 \\
&\leq \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \sum_{j=1}^d \|(-\hat{V}_{t-1-k} + \hat{V}_{t-k})^T e_j\|_1 \sqrt{N} \lambda^k \\
&= \frac{G_\infty^2}{N} \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \sum_{k=1}^t \|(-\hat{V}_{t-1-k} + \hat{V}_{t-k})\|_{abs} \sqrt{N} \lambda^k \\
&= \frac{G_\infty^2}{N} \frac{1}{2\epsilon^{1.5}} \sum_{o=0}^{T-1} \sum_{t=o+1}^T \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \sqrt{N} \lambda^{t-o} \\
&\leq \frac{G_\infty^2}{\sqrt{N}} \frac{1}{2\epsilon^{1.5}} \sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs},
\end{aligned} \tag{22}$$

605 where  $\lambda = \max(|\lambda_2|, |\lambda_N|)$ . Combining (21) and (22), we have

$$D_2 \leq \frac{G_\infty^2}{\sqrt{N}} \frac{1}{2\epsilon^{1.5}} \frac{\lambda}{1-\lambda} \mathbb{E} \left[ \sum_{o=0}^{T-1} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right].$$

606 Now we need to bound  $D_1$ , we have

$$\begin{aligned}
D_1 &= \sum_{t=1}^T \mathbb{E} \left[ \left\langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\rangle \right] \\
&\leq \sum_{t=1}^T \mathbb{E} \left[ G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{1}{\sqrt{[u_{t-1,i}]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right| \right] \\
&= \sum_{t=1}^T \mathbb{E} \left[ G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \left( \frac{1}{\sqrt{[u_{t-1,i}]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right) \frac{\sqrt{[u_{t,i}]_j} + \sqrt{[u_{t-1,i}]_j}}{\sqrt{[u_{t,i}]_j} + \sqrt{[u_{t-1,i}]_j}} \right| \right] \\
&\leq \sum_{t=1}^T \mathbb{E} \left[ G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \left| \frac{1}{2\epsilon^{1.5}} ([u_{t-1,i}]_j - [u_{t,i}]_j) \right| \right] \\
&\stackrel{(a)}{\leq} \sum_{t=1}^T \mathbb{E} \left[ G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \frac{1}{2\epsilon^{1.5}} |([\tilde{u}_{t-1,i}]_j - [\tilde{u}_{t,i}]_j)| \right] \\
&= G_\infty^2 \frac{1}{2\epsilon^{1.5}} \frac{1}{N} \mathbb{E} \left[ \sum_{t=1}^T \|\tilde{U}_{t-1} - \tilde{U}_t\|_{abs} \right],
\end{aligned} \tag{23}$$

607 where (a) is due to  $[\tilde{u}_{t-1,i}]_j = \max([u_{t-1,i}]_j, \epsilon)$  and the function  $\max(\cdot, \epsilon)$  is 1-Lipschitz. In  
 608 addition, by update rule of  $U_t$ , we have

$$\begin{aligned}
& \sum_{t=1}^T \|\tilde{U}_{t-1} - \tilde{U}_t\|_{abs} \\
&= \sum_{t=1}^T \|\tilde{U}_{t-1} - (\tilde{U}_{t-1} - \hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\
&= \sum_{t=1}^T \|\tilde{U}_{t-1}(QQ^T - Q\Lambda Q^T) + (-\hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\
&= \sum_{t=1}^T \|\tilde{U}_{t-1}(\sum_{l=2}^N q_l(1 - \lambda_l)q_l^T) + (-\hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\
&\leq \sum_{t=1}^T \left\| \sum_{k=1}^{t-1} (-\hat{V}_{t-2-k} + \hat{V}_{t-1-k}) \sum_{l=2}^N q_l \lambda_l^k (1 - \lambda_l) q_l^T \right\|_{abs} + \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})W\|_{abs} \\
&\leq \sum_{t=1}^T \left( \sum_{k=1}^{t-1} \|\hat{V}_{t-2-k} - \hat{V}_{t-1-k}\|_{abs} \sqrt{N} \lambda^k \right) + \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \\
&= \sum_{t=1}^T \left( \sum_{o=1}^{t-1} \|\hat{V}_{t-2-o} - \hat{V}_{t-1-o}\|_{abs} \sqrt{N} \lambda^{t-o} \right) + \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \\
&= \sum_{o=1}^{T-1} \sum_{t=o+1}^T \left( \|\hat{V}_{t-2-o} - \hat{V}_{t-1-o}\|_{abs} \sqrt{N} \lambda^{t-o} \right) + \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \\
&\leq \sum_{o=1}^{T-1} \frac{\lambda}{1-\lambda} \left( \|\hat{V}_{t-2-o} - \hat{V}_{t-1-o}\|_{abs} \sqrt{N} \right) + \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \\
&\leq \frac{1}{1-\lambda} \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \sqrt{N}.
\end{aligned} \tag{24}$$

609 Combining (23) and (24), we have

$$D_1 \leq G_\infty^2 \frac{1}{2\epsilon^{1.5}} \frac{1}{N} \mathbb{E} \left[ \frac{1}{1-\lambda} \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \sqrt{N} \right]. \tag{25}$$

610 What remains is to bound  $\sum_{t=1}^T \mathbb{E} [\|Z_{t+1} - Z_t\|^2]$ . By update rule of  $Z_t$ , we have

$$\begin{aligned}
& \|Z_{t+1} - Z_t\|^2 \\
&= \left\| \alpha \frac{\beta_1}{1-\beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&\leq 2\alpha^2 \left\| \frac{\beta_1}{1-\beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) \right\|^2 + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&\leq 2\alpha^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \frac{1}{\sqrt{\epsilon}} \left| \frac{1}{\sqrt{[u_{t-1,i}]_j}} - \frac{1}{\sqrt{[u_{t,i}]_j}} \right| + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&\leq 2\alpha^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \frac{1}{\sqrt{\epsilon}} \left| \frac{[u_{t,i}]_j - [u_{t-1,i}]_j}{2\epsilon^{1.5}} \right| + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \\
&\leq 2\alpha^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^d \frac{1}{2\epsilon^2} |\tilde{u}_{t,i,j} - \tilde{u}_{t-1,i,j}| + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2
\end{aligned}$$

$$= 2\alpha^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \frac{1}{2\epsilon^2} \|\tilde{U}_t - \tilde{U}_{t-1}\|_{abs} + 2\alpha^2 \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2, \quad (26)$$

611 where the last inequality is again due to the definition that  $[\tilde{u}_{t,i}]_j = \max([u_{t,i}]_j, \epsilon)$  and the fact that  
 612  $\max(\cdot, \epsilon)$  is 1-Lipschitz. Then, we have

$$\begin{aligned} & \sum_{t=1}^T \mathbb{E}[\|Z_{t+1} - Z_t\|^2] \\ & \leq 2\alpha^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 \frac{1}{N} \frac{1}{2\epsilon^2} \mathbb{E} \left[ \sum_{t=1}^T \|\tilde{U}_t - \tilde{U}_{t-1}\|_{abs} \right] + 2\alpha^2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \\ & \leq \alpha^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \frac{G_\infty^2}{\sqrt{N}} \frac{1}{\epsilon^2} \frac{1}{1-\lambda} \mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] + 2\alpha^2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right], \end{aligned}$$

613 where the last inequality is due to (24).

614 We now bound the last term on RHS of the above inequality. A trivial bound can be

$$\sum_{t=1}^T \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \leq \sum_{t=1}^T d G_\infty^2 \frac{1}{\epsilon},$$

615 due to  $\|g_{t,i}\| \leq G_\infty$  and  $[u_{t,i}]_j \geq \epsilon$ , for all  $j$  (verified from update rule of  $u_{t,i}$  and the assumption  
 616 that  $[v_{t,i}]_j \geq \epsilon$ , for all  $i$ ). However, the above bound is independent of  $N$ , to get a better bound, we  
 617 need a more involved analysis to show its dependency on  $N$ . To do this, we first notice that

$$\begin{aligned} & \mathbb{E}_{G_t|G_{1:t-1}} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \\ & = \mathbb{E}_{G_t|G_{1:t-1}} \left[ \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left\langle \frac{\nabla f_i(x_{t,i}) + \xi_{t,i}}{\sqrt{u_{t,i}}}, \frac{\nabla f_j(x_{t,j}) + \xi_{t,j}}{\sqrt{u_{t,j}}} \right\rangle \right] \\ & \stackrel{(a)}{=} \mathbb{E}_{G_t|G_{1:t-1}} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right] + \mathbb{E}_{G_t|G_{1:t-1}} \left[ \frac{1}{N^2} \sum_{i=1}^N \left\| \frac{\xi_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \\ & \stackrel{(b)}{=} \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 + \frac{1}{N^2} \sum_{i=1}^N \sum_{l=1}^d \frac{\mathbb{E}_{G_t|G_{1:t-1}}[\xi_{t,i}^2]}{[u_{t,i}]_l} \\ & \stackrel{(c)}{\leq} \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 + \frac{d}{N} \frac{\sigma^2}{\epsilon}, \end{aligned}$$

618 where (a) is due to  $\mathbb{E}_{G_t|G_{1:t-1}}[\xi_{t,i}] = 0$  and  $\xi_{t,i}$  is independent of  $x_{t,j}, u_{t,j}$  for all  $j$ , and  $\xi_j$ , for all  
 619  $j \neq i$ , (b) comes from the fact that  $x_{t,i}, u_{t,i}$  are fixed given  $G_{1:t}$ , (c) is due to  $\mathbb{E}_{G_t|G_{1:t-1}}[\xi_{t,i}^2] \leq \sigma^2$   
 620 and  $[u_{t,i}]_l \geq \epsilon$  by definition. Then we have

$$\begin{aligned} \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] & = \mathbb{E}_{G_{1:t-1}} \left[ \mathbb{E}_{G_t|G_{1:t-1}} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \right] \\ & \leq \mathbb{E}_{G_{1:t-1}} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 + \frac{d}{N} \frac{\sigma^2}{\epsilon} \right] \\ & = \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right] + \frac{d}{N} \frac{\sigma^2}{\epsilon}. \end{aligned} \quad (27)$$

621 In traditional analysis of SGD-like distributed algorithms, the term corresponding to  
 622  $\mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right]$  will be merged with the first order descent when the stepsize is cho-  
 623 sen to be small enough. However, in our case, the term cannot be merged because it is different from  
 624 the first order descent in our algorithm. A brute-force upper bound is possible but this will lead to a  
 625 worse convergence rate in terms of  $N$ . Thus, we need a more detailed analysis for the term in the  
 626 following.

$$\begin{aligned}
 & \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right] \\
 &= \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} + \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left( \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right) \right\|^2 \right] \\
 &\leq 2\mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2\mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) \odot \left( \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right) \right\|^2 \right] \\
 &\leq 2\mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2\mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N \left\| \nabla f_i(x_{t,i}) \odot \left( \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right) \right\|^2 \right] \\
 &\leq 2\mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2\mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \left\| \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right\|_1 \right].
 \end{aligned}$$

627 Summing over  $T$ , we have

$$\begin{aligned}
 & \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{u_{t,i}}} \right\|^2 \right] \\
 &\leq 2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2 \sum_{t=1}^T \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \left\| \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right\|_1 \right]. \quad (28)
 \end{aligned}$$

628 For the last term on RHS of (28), we can bound it similarly as what we did for  $D_2$  from (21) to (22),  
 629 which yields

$$\begin{aligned}
 & \sum_{t=1}^T \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \left\| \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{\bar{U}_t}} \right\|_1 \right] \leq \sum_{t=1}^T \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \frac{1}{2\epsilon^{1.5}} \|u_{t,i} - \bar{U}_t\|_1 \right] \\
 &= \sum_{t=1}^T \mathbb{E} \left[ \frac{1}{N} G_\infty^2 \frac{1}{2\epsilon^2} \|\bar{U}_t \mathbf{1}^T - U_t\|_{abs} \right] \\
 &\leq \sum_{t=1}^T \mathbb{E} \left[ \frac{1}{N} G_\infty^2 \frac{1}{2\epsilon^2} \left\| - \sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs} \right] \\
 &\leq \frac{1}{\sqrt{N}} G_\infty^2 \frac{1}{2\epsilon^2} \mathbb{E} \left[ \sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right]. \quad (29)
 \end{aligned}$$



630 Further, we have

$$\begin{aligned}
& \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] \\
& \leq 2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t)}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t) - \nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] \\
& = 2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t) - \nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right]
\end{aligned}$$

631 and the last term on RHS of the above inequality can be bounded following similar procedures from  
632 (14) to (19), as what we did for  $D_3$ . Completing the procedures yields

$$\begin{aligned}
\sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t) - \nabla f_i(x_{t,i})}{\sqrt{\bar{U}_t}} \right\|^2 \right] & \leq \sum_{t=1}^T \mathbb{E} \left[ L \frac{1}{\epsilon} \frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \right] \\
& \leq \sum_{t=1}^T \mathbb{E} \left[ L \frac{1}{\epsilon} \frac{1}{N} \alpha^2 \left( \frac{1}{1-\lambda} \right) N d G_\infty^2 \frac{1}{\epsilon} \right] \quad (30) \\
& = T L \frac{1}{\epsilon^2} \alpha^2 \left( \frac{1}{1-\lambda} \right) d G_\infty^2.
\end{aligned}$$

633 Finally, combining (27) to (30), we get

$$\begin{aligned}
\sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] & \leq 4 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\sqrt{\bar{U}_t}} \right\|^2 \right] + 4 T L \frac{1}{\epsilon^2} \alpha^2 \left( \frac{1}{1-\lambda} \right) d G_\infty^2 \\
& \quad + 2 \frac{1}{\sqrt{N}} G_\infty^2 \frac{1}{2\epsilon^2} \mathbb{E} \left[ \sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right] + T \frac{d}{N} \frac{\sigma^2}{\epsilon} \\
& \leq 4 \frac{1}{\sqrt{\epsilon}} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] + 4 T L \frac{1}{\epsilon^2} \alpha^2 \left( \frac{1}{1-\lambda} \right) d G_\infty^2 \\
& \quad + 2 \frac{1}{\sqrt{N}} G_\infty^2 \frac{1}{2\epsilon^2} \mathbb{E} \left[ \sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right] + T \frac{d}{N} \frac{\sigma^2}{\epsilon}.
\end{aligned}$$

634 where the last inequality is due to each element of  $\bar{U}_t$  is lower bounded by  $\epsilon$  by definition.

635 Combining all above, we obtain

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \\
& \leq \frac{2}{T\alpha} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) \\
& \quad + \frac{L}{T} \alpha \left( \frac{\beta_1}{1-\beta_1} \right)^2 \frac{G_\infty^2}{\sqrt{N}} \frac{1}{\epsilon^2} \frac{1}{1-\lambda} \mathbb{E}[\mathcal{V}_T] \\
& \quad + \frac{8L}{T} \alpha \frac{1}{\sqrt{\epsilon}} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] + 8 L^2 \alpha \frac{1}{\epsilon^2} \alpha^2 \left( \frac{1}{1-\lambda} \right) d G_\infty^2 \quad (31) \\
& \quad + \frac{4L}{T} \alpha \frac{1}{\sqrt{N}} G_\infty^2 \frac{1}{2\epsilon^2} \mathbb{E} \left[ \sum_{o=0}^{T-1} \frac{\lambda}{1-\lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs} \right] + 2 L \alpha \frac{d}{N} \frac{\sigma^2}{\epsilon}
\end{aligned}$$

$$\begin{aligned}
& + \frac{2}{T} \frac{\beta_1}{1-\beta_1} G_\infty^2 \frac{1}{2\epsilon^{1.5}} \frac{1}{\sqrt{N}} \mathbb{E} \left[ \frac{1}{1-\lambda} \mathcal{V}_T \right] \\
& + \frac{2}{T} \frac{G_\infty^2}{\sqrt{N}} \frac{1}{2\epsilon^{1.5}} \frac{\lambda}{1-\lambda} \mathbb{E} [\mathcal{V}_T] \\
& + \frac{3}{T} \left( \sum_{t=1}^T L \left( \frac{1}{1-\lambda} \right)^2 \alpha^2 d G_\infty^2 \frac{1}{\epsilon^{1.5}} + \sum_{t=1}^T L \left( \frac{\beta_1}{1-\beta_1} \right)^2 \alpha^2 d \frac{G_\infty^2}{\epsilon^{1.5}} \right) \\
& = \frac{2}{T\alpha} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) + 2L\alpha \frac{d}{N} \frac{\sigma^2}{\epsilon} + 8L\alpha \frac{1}{\sqrt{\epsilon}} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \\
& + 3\alpha^2 d \left( \left( \frac{\beta_1}{1-\beta_1} \right)^2 + \left( \frac{1}{1-\lambda} \right)^2 \right) L \frac{G_\infty^2}{\epsilon^{1.5}} + 8\alpha^3 L^2 \left( \frac{1}{1-\lambda} \right) d \frac{G_\infty^2}{\epsilon^2} \\
& + \frac{1}{T\epsilon^{1.5}} \frac{G_\infty^2}{\sqrt{N}} \frac{1}{1-\lambda} \left( L\alpha \left( \frac{\beta_1}{1-\beta_1} \right)^2 \frac{1}{\epsilon^{0.5}} + \lambda + \frac{\beta_1}{1-\beta_1} + 2L\alpha \frac{1}{\epsilon^{0.5}} \lambda \right) \mathbb{E} [\mathcal{V}_T] .
\end{aligned}$$

636 where  $\mathcal{V}_T := \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}$ . Set  $\alpha = \frac{1}{\sqrt{dT}}$  and when  $\alpha \leq \frac{\epsilon^{0.5}}{16L}$ , we further have

$$\begin{aligned}
& \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \\
& \leq \frac{4}{T\alpha} (\mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]) + 4L\alpha \frac{d}{N} \frac{\sigma^2}{\epsilon} \\
& \quad + 6\alpha^2 d \left( \left( \frac{\beta_1}{1-\beta_1} \right)^2 + \left( \frac{1}{1-\lambda} \right)^2 \right) L \frac{G_\infty^2}{\epsilon^{1.5}} + 16\alpha^3 L^2 \left( \frac{1}{1-\lambda} \right) d \frac{G_\infty^2}{\epsilon^2} \\
& \quad + \frac{2}{T\epsilon^{1.5}} \frac{G_\infty^2}{\sqrt{N}} \frac{1}{1-\lambda} \left( L\alpha \left( \frac{\beta_1}{1-\beta_1} \right)^2 \frac{1}{\epsilon^{0.5}} + \lambda + \frac{\beta_1}{1-\beta_1} + 2L\alpha \frac{1}{\epsilon^{0.5}} \lambda \right) \mathbb{E} [\mathcal{V}_T] \\
& \leq \frac{4}{T\alpha} (\mathbb{E}[f(Z_1)] - \min_x f(x)) + 4L\alpha \frac{d}{N} \frac{\sigma^2}{\epsilon} \\
& \quad + 6\alpha^2 d \left( \left( \frac{\beta_1}{1-\beta_1} \right)^2 + \left( \frac{1}{1-\lambda} \right)^2 \right) L \frac{G_\infty^2}{\epsilon^{1.5}} + 16\alpha^3 d L^2 \left( \frac{1}{1-\lambda} \right) \frac{G_\infty^2}{\epsilon^2} \\
& \quad + \frac{2}{T\epsilon^{1.5}} \frac{G_\infty^2}{\sqrt{N}} \frac{1}{1-\lambda} \left( L\alpha \left( \frac{\beta_1}{1-\beta_1} \right)^2 \frac{1}{\epsilon^{0.5}} + \lambda + \frac{\beta_1}{1-\beta_1} + 2L\alpha \frac{1}{\epsilon^{0.5}} \lambda \right) \mathbb{E} [\mathcal{V}_T] \\
& \leq C_1 \left( \frac{1}{T\alpha} (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \alpha \frac{d\sigma^2}{N} \right) + C_2 \alpha^2 d + C_3 \alpha^3 d + \frac{1}{T\sqrt{N}} (C_4 + C_5 \alpha) \mathbb{E} [\mathcal{V}_T]
\end{aligned} \tag{32}$$

637 where the first inequality is obtained by moving the term  $8L\alpha \frac{1}{\sqrt{\epsilon}} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right]$  on the

638 RHS of (31) to the LHS to cancel it using the assumption  $8L\alpha \frac{1}{\sqrt{\epsilon}} \leq \frac{1}{2}$  followed by multiplying both  
639 sides by 2. The constants introduced in the last step are defined as following

$$\begin{aligned}
C_1 &= \max(4, 4L/\epsilon), \\
C_2 &= 6 \left( \left( \frac{\beta_1}{1-\beta_1} \right)^2 + \left( \frac{1}{1-\lambda} \right)^2 \right) L \frac{G_\infty^2}{\epsilon^{1.5}}, \\
C_3 &= 16L^2 \left( \frac{1}{1-\lambda} \right) \frac{G_\infty^2}{\epsilon^2}, \\
C_4 &= \frac{2}{\epsilon^{1.5}} \frac{1}{1-\lambda} \left( \lambda + \frac{\beta_1}{1-\beta_1} \right) G_\infty^2,
\end{aligned}$$

$$C_5 = \frac{2}{\epsilon^2} \frac{1}{1-\lambda} L \left( \frac{\beta_1}{1-\beta_1} \right)^2 G_\infty^2 + \frac{4}{\epsilon^2} \frac{\lambda}{1-\lambda} L G_\infty^2.$$

640 Substituting into  $Z_1 = \overline{X}_1$  completes the proof.

□

### 641 C Proof of Theorem 3

642 Under some assumptions stated in Corollary 2.1, we have that

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \frac{\sqrt{d}}{\sqrt{TN}} \left( (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \sigma^2 \right) + C_2 \frac{N}{T} + C_3 \frac{N^{1.5}}{T^{1.5}d^{0.5}} \\ &\quad + \left( C_4 \frac{1}{T\sqrt{N}} + C_5 \frac{1}{T^{1.5}d^{0.5}} \right) \mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \end{aligned} \quad (33)$$

643 where  $\|\cdot\|_{abs}$  denotes the entry-wise  $L_1$  norm of a matrix (i.e.  $\|A\|_{abs} = \sum_{i,j} |A_{ij}|$ ) and  
644  $C_1, C_2, C_3, C_4, C_5$  are defined in Theorem 2.

645 Since Algorithm 3 is a special case of 2, building on result of Theorem 2, we just need to characterize  
646 the growth speed of  $\mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right]$  to prove convergence of Algorithm 3. By the  
647 update rule of Algorithm 3, we know  $\hat{V}_t$  is non decreasing and thus

$$\begin{aligned} \mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] &= \mathbb{E} \left[ \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^d |-\hat{v}_{t-2,i,j} + \hat{v}_{t-1,i,j}| \right] \\ &= \mathbb{E} \left[ \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^d (-\hat{v}_{t-2,i,j} + \hat{v}_{t-1,i,j}) \right] \\ &= \mathbb{E} \left[ \sum_{i=1}^N \sum_{j=1}^d (-\hat{v}_{-1,i,j} + \hat{v}_{T-1,i,j}) \right] \\ &= \mathbb{E} \left[ \sum_{i=1}^N \sum_{j=1}^d (-\hat{v}_{0,i,j} + \hat{v}_{T-1,i,j}) \right], \end{aligned}$$

648 where the last equality is because we defined  $\hat{V}_{-1} \triangleq \hat{V}_0$  previously.

649 Further, because  $\|g_{t,i}\|_\infty \leq G_\infty$  for all  $t, i$  and  $v_{t,i}$  is a exponential moving average of  $g_{k,i}^2, k =$   
650  $1, 2, \dots, t$ , we know  $|[v_{t,i}]_j| \leq G_\infty^2$ , for all  $t, i, j$ . In addition, by update rule of  $\hat{V}_t$ , we also know  
651 each element of  $\hat{V}_t$  also cannot be greater than  $G_\infty^2$ , i.e.  $|\hat{v}_{t,i,j}| \leq G_\infty^2$ , for all  $t, i, j$ . Given the fact  
652 that  $[\hat{v}_{0,i}]_j \geq 0$ , we have

$$\mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] = \mathbb{E} \left[ \sum_{i=1}^N \sum_{j=1}^d (-[\hat{v}_{0,i}]_j + [\hat{v}_{T-1,i}]_j) \right] \leq \mathbb{E} \left[ \sum_{i=1}^N \sum_{j=1}^d G_\infty^2 \right] = NdG_\infty^2.$$

653 Substituting the above into (33), we have

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \frac{\sqrt{d}}{\sqrt{TN}} \left( (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \sigma^2 \right) + C_2 \frac{N}{T} + C_3 \frac{N^{1.5}}{T^{1.5}d^{0.5}} \\ &\quad + \left( C_4 \frac{1}{T\sqrt{N}} + C_5 \frac{1}{T^{1.5}d^{0.5}} \right) NdG_\infty^2 \\ &= C'_1 \frac{\sqrt{d}}{\sqrt{TN}} \left( (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \sigma^2 \right) + C'_2 \frac{N}{T} + C'_3 \frac{N^{1.5}}{T^{1.5}d^{0.5}} \\ &\quad + C'_4 \frac{\sqrt{Nd}}{T} + C'_5 \frac{Nd^{0.5}}{T^{1.5}}, \end{aligned} \quad (34)$$

654 where we have

$$C'_1 = C_1 \quad C'_2 = C_2 \quad C'_3 = C_3 \quad C'_4 = C_4 G_\infty^2 \quad C'_5 = C_5 G_\infty^2. \quad (35)$$

655 and we conclude the proof.  $\square$

## 656 D Proof of Theorem 4

657 The proof follows the same flow as that of Theorem 3. Under assumptions stated in Corollary 2.1, set  
 658  $\alpha = \sqrt{N}/\sqrt{Td}$ , we have that

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \frac{\sqrt{d}}{\sqrt{TN}} \left( (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \sigma^2 \right) + C_2 \frac{N}{T} + C_3 \frac{N^{1.5}}{T^{1.5}d^{0.5}} \\ &\quad + \left( C_4 \frac{1}{T\sqrt{N}} + C_5 \frac{1}{T^{1.5}d^{0.5}} \right) \mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right], \end{aligned} \quad (36)$$

659 where  $\|\cdot\|_{abs}$  denotes the entry-wise  $L_1$  norm of a matrix (i.e.  $\|A\|_{abs} = \sum_{i,j} |A_{ij}|$ ) and  
 660  $C_1, C_2, C_3, C_4, C_5$  are defined in Theorem 2.

661 Again, Since decentralized AdaGrad is a special case of 2, we can apply Corollary 2.1 and what we  
 662 need is to upper bound  $\mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right]$  derive convergence rate. By the update rule  
 663 of decentralized AdaGrad, we have  $\hat{v}_{t,i} = \frac{1}{t} (\sum_{k=1}^t g_{k,i}^2)$  for  $t \geq 1$  and  $\hat{v}_{0,i} = \epsilon \mathbf{1}$ . Then we have for  
 664  $t \geq 3$ ,

$$\begin{aligned} &\mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \\ &= \mathbb{E} \left[ \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^d |-\hat{v}_{t-2,i} + \hat{v}_{t-1,i}| \right] \\ &\leq \mathbb{E} \left[ \sum_{t=3}^T \sum_{i=1}^N \sum_{j=1}^d \left| -\frac{1}{t-2} \left( \sum_{k=1}^{t-2} g_{k,i}^2 \right) + \frac{1}{t-1} \left( \sum_{k=1}^{t-1} g_{k,i}^2 \right) \right| \right] + Nd(G_\infty^2 - \epsilon) \\ &\leq \mathbb{E} \left[ \sum_{t=3}^T \sum_{i=1}^N \sum_{j=1}^d \left| \left( \frac{1}{t-1} - \frac{1}{t-2} \right) \left( \sum_{k=1}^{t-2} g_{k,i}^2 \right) + \frac{1}{t-1} g_{t-1,i}^2 \right| \right] + NdG_\infty^2 \\ &= \mathbb{E} \left[ \sum_{t=3}^T \sum_{i=1}^N \sum_{j=1}^d \left| \left( -\frac{1}{(t-1)(t-2)} \right) \left( \sum_{k=1}^{t-2} g_{k,i}^2 \right) + \frac{1}{t-1} g_{t-1,i}^2 \right| \right] + NdG_\infty^2 \\ &\leq \mathbb{E} \left[ \sum_{t=3}^T \sum_{i=1}^N \sum_{j=1}^d \max \left( \frac{1}{(t-1)(t-2)} \left( \sum_{k=1}^{t-2} g_{k,i}^2 \right), \frac{1}{t-1} g_{t-1,i}^2 \right) \right] + NdG_\infty^2 \\ &\leq \mathbb{E} \left[ Nd \sum_{t=3}^T \frac{G_\infty^2}{t-1} \right] + NdG_\infty^2 \\ &\leq NdG_\infty^2 \log(T) + NdG_\infty^2 \\ &= NdG_\infty^2 (\log(T) + 1) \end{aligned}$$

665 where the first equality is because we defined  $\hat{V}_{-1} \triangleq \hat{V}_0$  previously and  $\|g_{k,i}\|_\infty \leq G_\infty$  by assump-  
 666 tion.



667 Substituting the above into (36), we have

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] &\leq C_1 \frac{\sqrt{d}}{\sqrt{TN}} \left( (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \sigma^2 \right) + C_2 \frac{N}{T} + C_3 \frac{N^{1.5}}{T^{1.5} d^{0.5}} \\
&\quad + \left( C_4 \frac{1}{T\sqrt{N}} + C_5 \frac{1}{T^{1.5} d^{0.5}} \right) N d G_\infty^2 (\log(T) + 1) \\
&= C'_1 \frac{\sqrt{d}}{\sqrt{TN}} \left( (\mathbb{E}[f(Z_1)] - \min_x f(x)) + \sigma^2 \right) + C'_2 \frac{N}{T} + C'_3 \frac{N^{1.5}}{T^{1.5} d^{0.5}} \\
&\quad + C'_4 \frac{d\sqrt{N}(\log(T) + 1)}{T} + C'_5 \frac{(\log(T) + 1)N\sqrt{d}}{T^{1.5}},
\end{aligned}$$

668 where we have

$$C'_1 = C_1 \quad C'_2 = C_2 \quad C'_3 = C_3 \quad C'_4 = C_4 G_\infty^2 \quad C'_5 = C_5 G_\infty^2. \quad (37)$$

669 and we conclude the proof.  $\square$

## E Convergence Analysis: Proof Sketch

The detailed proofs of this section are reported in the supplementary material.

**Proof of Theorem 2:** We now present a proof sketch for our main convergence result of Algorithm 2. *Step 1: Reparameterization.* Similarly to [38; 8] with SGD (with momentum) and centralized adaptive gradient methods, define the following auxiliary sequence:

$$Z_t = \bar{X}_t + \frac{\beta_1}{1 - \beta_1} (\bar{X}_t - \bar{X}_{t-1}), \quad (38)$$

with  $\bar{X}_0 \triangleq \bar{X}_1$ . Such an auxiliary sequence can help us deal with the bias brought by the momentum and simplifies the convergence analysis. We recall an intermediary result needed to conduct our proof:

**Lemma A.1.** *For the sequence defined in (7), we have*

$$Z_{t+1} - Z_t = \alpha \frac{\beta_1}{1 - \beta_1} \frac{1}{N} \sum_{i=1}^N m_{t-1,i} \odot \left( \frac{1}{\sqrt{u_{t-1,i}}} - \frac{1}{\sqrt{u_{t,i}}} \right) - \alpha \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}}. \quad (39)$$

Lemma A.1 does not display any momentum term in  $\frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}}$ . This simplification is convenient since it is directly related to the current gradients instead of the exponential average of past gradients.

*Step 2: Smoothness.* Using smoothness assumption A1 involves the following scalar product term:  $\kappa_t := \langle \nabla f(Z_t), \frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) / \sqrt{U_t} \rangle$  which can be lower bounded by:

$$\kappa_t \geq \frac{1}{2} \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{3}{2} \left\| \frac{\nabla f(Z_t) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 - \frac{3}{2} \left\| \frac{\frac{1}{N} \sum_{i=1}^N \nabla f_i(x_{t,i}) - \nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2.$$

The above inequality substituted in the smoothness condition  $f(Z_{t+1}) \leq f(Z_t) + \langle \nabla f(Z_t), Z_{t+1} - Z_t \rangle + \frac{L}{2} \|Z_{t+1} - Z_t\|^2$  yields:

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\bar{U}_t^{1/4}} \right\|^2 \right] \leq \frac{2}{T\alpha} \mathbb{E}[\Delta_f] + \frac{2}{T} \frac{\beta_1 D_1}{1 - \beta_1} + \frac{2D_2}{T} + \frac{3D_3}{T} + \frac{L}{T\alpha} \sum_{t=1}^T \mathbb{E} [\|Z_{t+1} - Z_t\|^2], \quad (40)$$

where  $\Delta_f := \mathbb{E}[f(Z_1)] - \mathbb{E}[f(Z_{T+1})]$ .  $D_1, D_2$  and  $D_3$  are three terms, defined in the supplementary material, and which can be tightly bounded from above. We first bound  $D_3$  using the following quantities of interest:

$$\sum_{t=1}^T \|Z_t - \bar{X}_t\|^2 \leq T \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \alpha^2 d \frac{G_\infty^2}{\epsilon} \quad \text{and} \quad \sum_{t=1}^T \frac{1}{N} \sum_{i=1}^N \|x_{t,i} - \bar{X}_t\|^2 \leq T \alpha^2 \left( \frac{1}{1 - \lambda} \right)^2 d G_\infty^2 \frac{1}{\epsilon}.$$

where  $\lambda = \max(|\lambda_2|, |\lambda_N|)$  and recall that  $\lambda_i$  is  $i$ th largest eigenvalue of  $W$ .

Then, concerning the term  $D_2$ , few derivations, not detailed here for simplicity, yields:

$$D_2 \leq \frac{G_\infty^2}{N} \mathbb{E} \left[ \sum_{t=1}^T \frac{1}{2\epsilon^{1.5}} \left\| -\sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs} \right],$$

where  $q_l$  is the eigenvector corresponding to  $l$ th largest eigenvalue of  $W$  and  $\|\cdot\|_{abs}$  is the entry-wise  $L_1$  norm of matrices. We can also show that

$$\sum_{t=1}^T \left\| -\sum_{l=2}^N \tilde{U}_t q_l q_l^T \right\|_{abs} \leq \sqrt{N} \sum_{o=0}^{T-1} \frac{\lambda}{1 - \lambda} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs},$$

resulting in an upper bound for  $D_2$  proportional to  $\sum_{o=0}^{T-1} \|(-\hat{V}_{o-1} + \hat{V}_o)\|_{abs}$ . Similarly:

$$D_1 \leq G_\infty^2 \frac{1}{2\epsilon^{1.5}} \frac{1}{\sqrt{N}} \mathbb{E} \left[ \frac{1}{1 - \lambda} \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right].$$

693 *Step 3: Bounding the drift term variance.* An important term that needs upper bounding in our proof  
 694 is the variance of the gradients multiplied (element-wise) by the adaptive learning rate:

$$\mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{g_{t,i}}{\sqrt{u_{t,i}}} \right\|^2 \right] \leq \mathbb{E}[\|\Gamma_u^f\|^2] + \frac{d}{N} \frac{\sigma^2}{\epsilon},$$

695 where  $\Gamma_u^f := 1/N \sum_{i=1}^N \nabla f_i(x_{t,i})/\sqrt{u_{t,i}}$ . Two consecutive and simple bounding of the above yields:

$$\sum_{t=1}^T \mathbb{E}[\|\Gamma_u^f\|^2] \leq 2 \sum_{t=1}^T \mathbb{E}[\|\Gamma_U^f\|^2] + 2 \sum_{t=1}^T \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N G_\infty^2 \frac{1}{\sqrt{\epsilon}} \left\| \frac{1}{\sqrt{u_{t,i}}} - \frac{1}{\sqrt{U_t}} \right\| \right]$$

696 and

$$\sum_{t=1}^T \mathbb{E}[\|\Gamma_U^f\|^2] \leq 2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{\nabla f(\bar{X}_t)}{\sqrt{U_t}} \right\|^2 \right] + 2 \sum_{t=1}^T \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N \frac{\nabla f_i(\bar{X}_t) - \nabla f_i(x_{t,i})}{\sqrt{U_t}} \right\|^2 \right]. \quad (41)$$

697 Then, by plugging the LHS of (41) in (40), and further bounding as operated for  $D_2, D_3$  (see  
 698 supplement), we obtain the desired bound in Theorem 2.

699 **Proof of Theorem 3:** Recall the bound in (3) of Theorem 2. Since Algorithm 3 is a special  
 700 case of Algorithm 2, the remaining of the proof consists in characterizing the growth rate of  
 701  $\mathbb{E}[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}]$ . By construction,  $\hat{V}_t$  is non decreasing, then it can be shown that  
 702  $\mathbb{E}[\sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs}] = \mathbb{E}[\sum_{i=1}^N \sum_{j=1}^d (-[\hat{v}_{0,i}]_j + [\hat{v}_{T-1,i}]_j)]$ . Besides, since for all  
 703  $t, i$ ,  $\|g_{t,i}\|_\infty \leq G_\infty$  and  $v_{t,i}$  is an exponential moving average of  $g_{k,i}^2, k = 1, 2, \dots, t$ , we have  
 704  $|[v_{t,i}]_j| \leq G_\infty^2$  for all  $t, i, j$ . By construction of  $\hat{V}_t$ , we also observe that each element of  $\hat{V}_t$  cannot  
 705 be greater than  $G_\infty^2$ , i.e.  $[\hat{v}_{t,i}]_j \leq G_\infty^2$  for all  $t, i, j$ . Given that  $[\hat{v}_{0,i}]_j \geq 0$ , we have

$$\mathbb{E} \left[ \sum_{t=1}^T \|(-\hat{V}_{t-2} + \hat{V}_{t-1})\|_{abs} \right] \leq \sum_{i=1}^N \sum_{j=1}^d \mathbb{E}[G_\infty^2] = NdG_\infty^2.$$

706 Substituting into (3) yields the desired convergence bound for Algorithm 3.

## 707 F Additional Experiments and Details

708 In this section, we compare the training loss and testing accuracy of different algorithms, namely  
 709 Decentralized Stochastic Gradient Descent (D-PSGD), Decentralized Adam (DADAM) and our  
 710 proposed Decentralized AMSGrad, with different stepsizes on heterogeneous data distribution. We  
 711 use 5 nodes and the heterogeneous data distribution is created by assigning each node with data of  
 712 only two labels. Note that there are no overlapping labels between different nodes. For all algorithms,  
 713 we compare stepsizes in the grid  $[10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}]$ .

714 Figure 3 shows the training loss and test accuracy for D-PSGD algorithm. We observe that the  
 715 stepsize  $10^{-3}$  works best for D-PSGD in terms of test accuracy and  $10^{-1}$  works best in terms of  
 716 training loss. This difference is caused by the inconsistency among the value of parameters on  
 717 different nodes when the stepsize is large. The training loss is calculated as the average of the loss  
 718 value of different local models evaluated on their local training batch. Thus, while the training loss is  
 719 small at a particular node, the test accuracy will be low when evaluating data with labels not seen by  
 720 the node (recall that each node contains data with different labels since we are in the heterogeneous  
 721 setting).

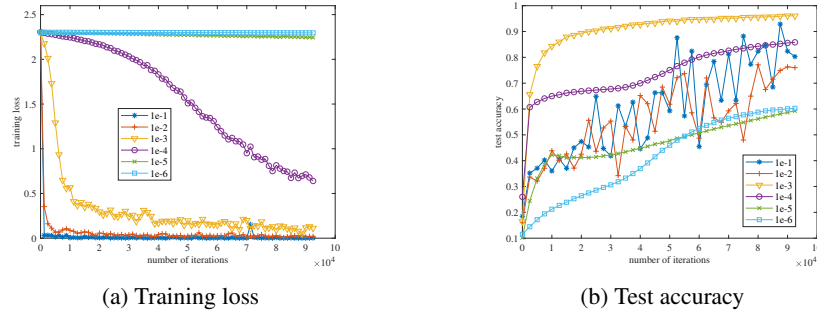


Figure 3: Performance comparison of different stepsizes for D-PSGD

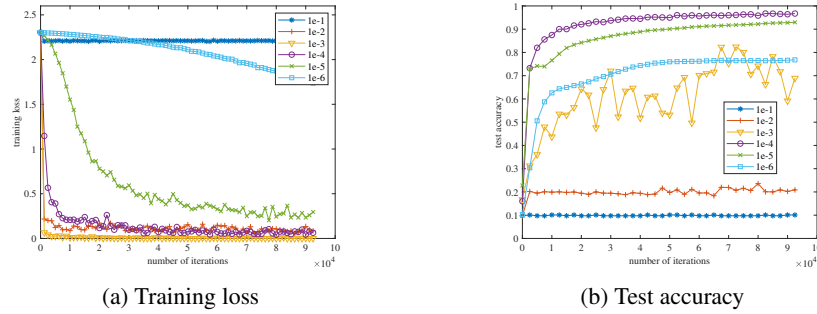


Figure 4: Performance comparison of different stepsizes for decentralized AMSGrad

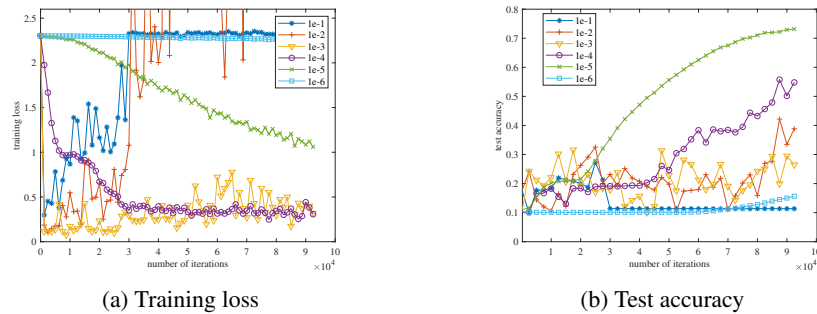


Figure 5: Performance comparison of different stepsizes for DADAM

722 Figure 4 shows the performance of decentralized AMSGrad with different stepsizes. We see that its  
723 best performance is better than the one of D-PSGD. Its performance is also more stable in the sense  
724 that the test performance is less sensitive to stepsize tuning according to our experiments.

725 Figure 5 displays the performance of Decentralized Adam algorithm. As expected, the performance  
726 of DADAM is not as good as D-PSGD or decentralized AMSGrad. Its divergence characteristic,  
727 highlighted Section 2.3, coupled with the heterogeneity in the data amplify its non-convergence issue  
728 in our experiments. From the experiments above, we can see the benefits of decentralized AMSGrad  
729 both in terms of performance and ease of parameter tuning, and the importance of ensuring the  
730 theoretical convergence of any newly proposed methods in the presented setting.