CVPR Supplementary Material for:
3D Point Cloud Registration for Localization using a Deep Neural Network Auto-Encoder

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Appendix A: Theoretical Analysis of the Number of Super-points Created by the Random Sphere Cover Set (RSCS) Method.

Theoretical Analysis

An upper bound for the number of super-points (SP) created by the RSCS algorithm given the intrinsic parameters of the local and global point clouds is shown in this section. This upper bound estimation is useful for understanding the algorithm's computational complexity prior to its application. The upper bound also serves as theoretical evidence that a manageable number of SPs are created, in addition to the empirical evidence given later in the appendix.

A super-point is generated on each iteration of the algorithm, covering a local region of the point cloud. An exponential decay relationship is shown between the number of iterations k and the number of non-covered points before that iteration, u_k.

As discussed in Section 2.2 of our paper, we select the volume V_{sphere} based on the density constant found from random sphere packing [1] and on the constraint of obtaining at least m matching super points. We use a $2m$ constraint as a safety factor, robust for edge-case geometries and varying point densities which are not included in our estimation model:

$$V_{sphere} \approx \frac{0.64 \cdot V_{local}}{2m}, \tag{1}$$

where, as defined in the paper, V_{local} is the volume of the minimal sphere encompassing all of the points in the local point cloud. In the problem definition, the global point cloud is substantially larger than the local point cloud: therefore, we analyze the number of super-points required to represent the global point cloud.

On average, if the total number of points uncovered before iteration k is u_k, then the number of points in a random sphere of volume V_{sphere} can be estimated to equal $\frac{u_k \cdot V_{sphere}}{V^{CH}_{global}}$, where V^{CH}_{global} is defined as the volume of the convex hull encompassing all the points in the global point cloud. This means that the relationship between the number of non-covered points on step k to the number of non-covered points on step k+1, can be expressed as:

$$u_{k+1} = u_k - \frac{u_k \cdot V_{sphere}}{V^{CH}_{global}}$$

$$\frac{u_{k+1} - u_k}{u_k} = \frac{\Delta u_k}{u_k} = -\frac{V_{sphere}}{V^{CH}_{global}}. \tag{2}$$

By moving from a discrete environment to a continuous one, we obtain:

$$\frac{\delta u_k}{u_k} = -\frac{V_{sphere}}{V^{CH}_{global}}.$$

Integration on both sides of the equation yields:

$$\log(u_k) = -\frac{V_{sphere}}{V^{CH}_{global}} \cdot k + C_1.$$

The constant C_1 is still undefined after the integration step, as seen here:

$$u_k = C_1 \cdot \exp\left(-\frac{V_{sphere}}{V^{CH}_{global}} \cdot k\right). \tag{3}$$

The start condition, where the initial number of non-covered points is equal to the total number of points, as applied to the equation:

$$u_1 = N_p$$

$$u_1 = C_1 \cdot \exp\left(-\frac{V_{sphere}}{V^{CH}_{global}} \cdot 0\right) = C_1$$

$$C_1 = N_p$$

In each step at least one point is selected, bringing the actual number of points per iteration to:

$$u_k = \max\left(N_p \cdot \exp\left(-\frac{V_{sphere}}{V^{CH}_{global}} \cdot k\right), 1\right). \quad (4)$$

For practical applications the method is stopped when a predefined percentage of the total points ($X_{covered}$) is covered. We set $X_{covered} = 95\% \left(\frac{u_k}{u_0} = 0.05\right)$, and when this condition is met the RSCS iterations are terminated. The number of super-points created can be estimated:

$$u_k = 0.05 \cdot u_1 = 0.05 \cdot N_p$$

$$N_p \cdot \exp\left(-\frac{V_{sphere}}{V^{CH}_{global}} \cdot k\right) = 0.05 \cdot N_p \quad (5)$$

$$k = -\frac{\log(0.05) \cdot V^{CH}_{global}}{V_{sphere}} \approx \frac{1.3 \cdot V^{CH}_{global}}{V_{sphere}} \leq 1.3 \cdot \frac{2m \cdot V^{CH}_{global}}{0.64 \cdot V_{local}} \quad (6)$$

We use $m = 6$, as the number of matching super-points in the final stage of the algorithm.

We define the density of points in the point cloud as the ratio between the volume of its convex hull, and the number of points within it:

$$d = \frac{V^{CH}}{N}$$

The volume of the encompassing sphere is larger than or equal to the volume of the convex hull:

$$V_{local} \geq V^{CH}_{local} = N_{local} \cdot d_{local},$$

therefore the following proceeds:

$$k \leq 24.275 \cdot \frac{V^{CH}_{global}}{V_{local}} \approx 25 \cdot \frac{V^{CH}_{global}}{V_{local}} \leq 25 \cdot \frac{N_{global}}{N_{local}} \cdot \frac{d_{global}}{d_{local}}.$$

It can be assumed that the density of the local point cloud d_{local} and of the global point cloud d_{global} are approximately the same on average ($d_{global} \approx d_{local}$). *Alternatively, the density ratios between the point clouds can be estimated by comparing the average radius of the sphere containing K nearest-neighbors.* Hence, the number of super-points created for the global cloud N^{global}_{sp} is bounded by:

$$N^{global}_{sp} \leq 25 \cdot \frac{N_{global}}{N_{local}}. \quad (7)$$

From an identical analysis on the local point cloud, its upper bound is:

$$N^{local}_{sp} \leq 25.$$

As mentioned in the paper, the RSCS algorithm is applied K times to the local point cloud (we used $K = 5$), so in practice $N^{multi-local}_{sp} \approx N^{local}_{sp} \cdot 5 \leq 125$ super-points represent the local point cloud. This limit guarantees that the maximal number of super-points remains manageable.

A quantitative example reflecting a real use case can help understand the scale of the theoretical bound: In this example LORAX is applied to a global point cloud of $N_{global} = 10,000,000$ points and a local point cloud of $N_{local} = 100,000$ points (1% of the global point cloud). The number of super-points generated is safely bounded using Eq. 7 to be $N_{sp} \leq 2500$. This number reflects a significant reduction in the problem dimension.

Theoretical Discussion

This approximated model is based on a statistical average number of non-covered points encompassed within V_{sphere}. In practice, the exact number of points covered in each iteration is dependent on the local geometry of the point cloud, the density of the point cloud in that area, and many other factors. Although a given iteration may be very different from that estimated by the model, it should return, on average over many iterations, a good approximation for the number of super-points created. This theoretical model is supported by empirical tests (described below), which show meaningful correlations.

This process can be seen as transforming three-dimensional point clouds of N_p points into N_{sp} super-points of a higher dimension (defined by all of the points within the SP). While the total amount of data increased after the application of this algorithm, using the SP information allows for its reduction later in the algorithm. By reducing the dimension of this data, powerful invariant features describing local surfaces of the scene in a compact and descriptive manner are created.

Empirical Backing

An empirical example of RSCS is applied to aerial data in Fig. 1, and its analysis is shown in Fig. .

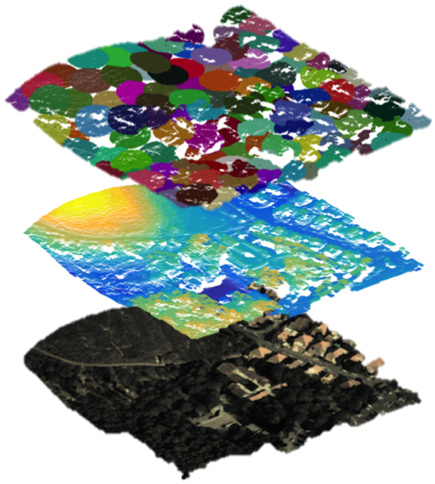

Figure 1. Example of RSCS application on aerial scanned data. Bottom: original 3D scene point cloud. Middle: height map point cloud of scene. Top: RSCS coverage of point cloud

The percentage of non-covered points out of all points, as a function of RSCS iteration, as well as the amount of covered/non-covered points covered in each given iteration are shown in Figure .

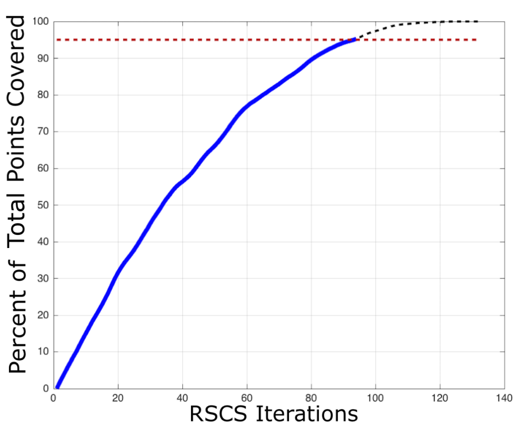

Figure 2. RSCS example analysis, total covered points as a function of super-points created

We indeed see an exponential decay of coverage per iteration. In this example the global point cloud consists of approximately 1.5 million points, and the local point cloud of approximately 150 thousand. Using the theoretical estimation $N_{sp} \leq 25 \cdot \frac{N_{global}}{N_{local}} = 250$, in this example approximately 100 super-points were created. A near-constant number of points were selected in each super-point, this corresponds to a uniform density of points and the fixed selection volume. In practice the number of total covered points in each iteration is greatly effected by geometry and is unique to each application of RSCS due to the random nature of the selection process. For this reason it is hard to create a exact prediction of the number of super-points created. In an extensive test, 100 global point clouds were analyzed by applying RSCS. On average, the number of super-points created were $39.23\% \pm 6.37\%$ of the theoretical upper bound.

RSCS Parallel Implementation

We would also like to note here that this algorithm can be adapted to run in parallel. Multiple random spheres can cover points in the point clouds simultaneously on each iteration, after which a quick filtration of spheres with overlapping centers can be applied. The large scale theoretical example shown above with the upper bound of 2500 super-points could be covered in approximately 25 iterations of 100 super-points each, where after each iteration a simple condition is checked on the 100 super-point centers. This method is fast and simple when implemented linearly and it would be even faster and more scalable implemented in parallel.

References

[1] GD Scott and DM Kilgour. The density of random close packing of spheres. *Journal of Physics D: Applied Physics*, 2(6):863, 1969.