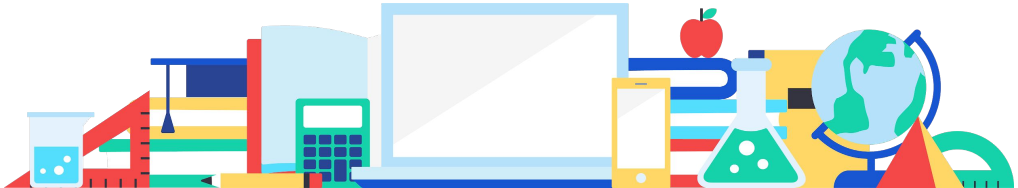
Trigonometry Review Guide

Updated January 2019

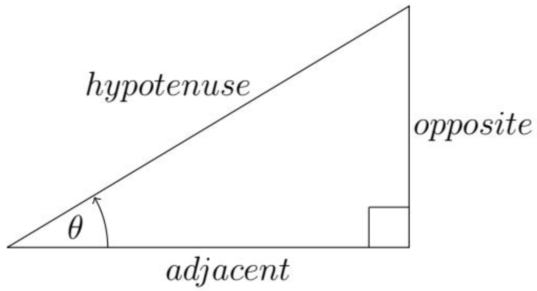


Key Definition and Properties of the Trigonometric Functions

Definition of a Right Triangle

Key Vocabulary

Assume that $0<\boldsymbol{\theta}<(\pi/2)$ or $0°<\boldsymbol{\theta}<90°$

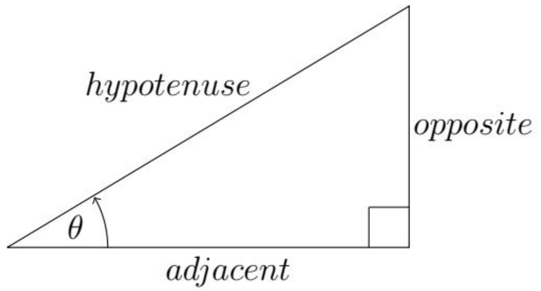


- ➔ **Hypotenuse:** side opposite the right angle and the longest side of the triangle
- ➔ **Opposite:** side opposite θ
- ➔ **Adjacent:** non-hypotenuse side next to θ

Definition of a Right Triangle

Key Vocabulary

Assume that $0<\boldsymbol{\theta}<(\pi/2)$ or $0°<\boldsymbol{\theta}<90°$



- ➔ **Sine:** sin θ = opp / hyp
- ➔ **Cosine:** cos θ = adj / hyp
- ➔ **Tangent:** tan θ = opp / adj
- ➔ **Cosecant:** csc θ = hyp / opp = 1 / sine
- ➔ **Secant:** sec θ = hyp / adj = 1 / cosine
- ➔ **Cotangent:** cot θ = adj / opp = 1 / tangent

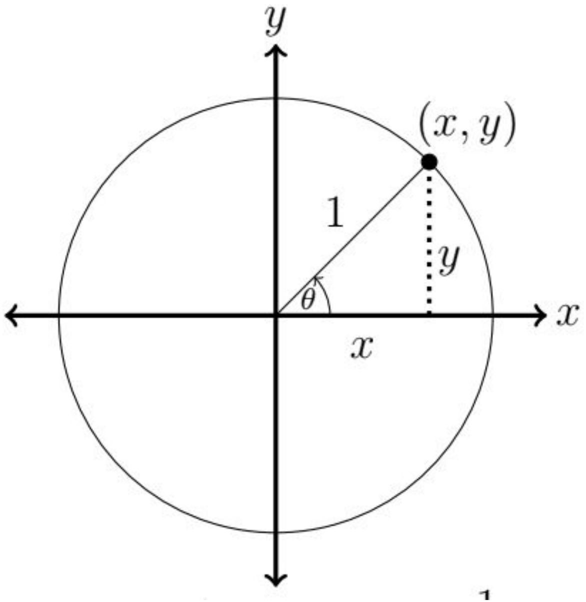
Remember: **SOHCAHTOA** (**s**in = **o**pp / **h**yp, **c**os = **a**dj / **h**yp, **t**an = **o**pp / **a**dj)

Definition of the Unit Circle

Key Vocabulary

Assume θ can be any angle

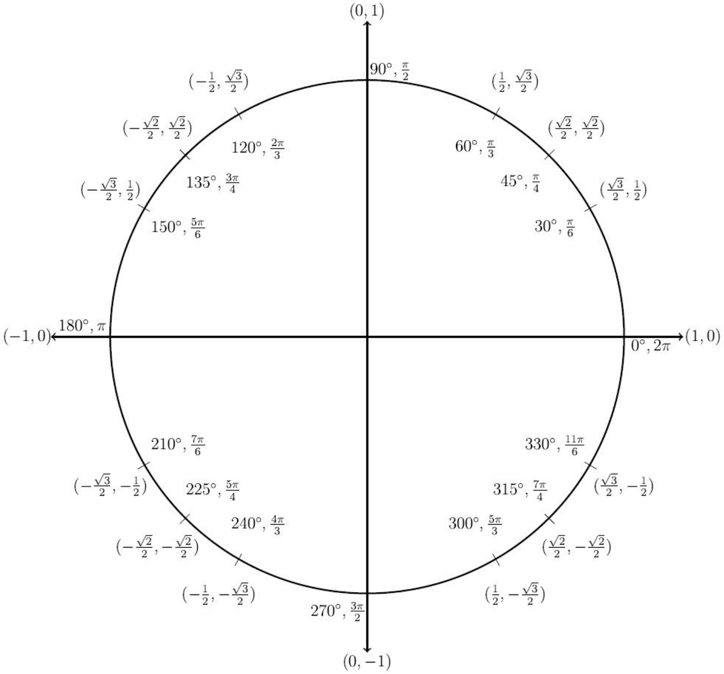
- ➔ sin θ = y / 1
- ➔ cos θ = x / 1
- ➔ tan θ = y / x
- ➔ csc θ = 1 / y
- ➔ sec θ = 1 / x
- ➔ cot θ = x / y



Definition of the Unit Circle

Key Vocabulary

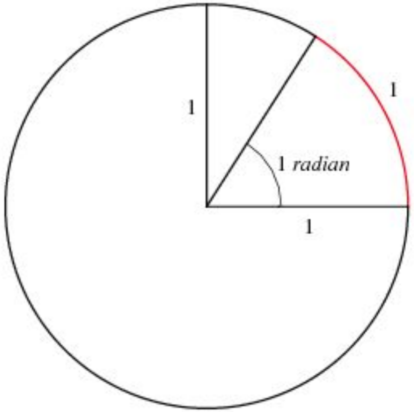
For any ordered pair on the unit circle (x,y): x = cos θ and y = sin θ



Degrees and Radians

Key Vocabulary

Angles can be measured using degrees or radians. A **radian** is the measure of the angle that cuts off an arc of length 1 on the unit circle



If x is an angle in degrees, and t is an angle in radians, then:

- ➔ $\pi / 180° = t / x$
- ➔ $t = (\pi x) / 180°$
- ➔ $x = (180° t) / \pi$

Commonly used angles in degrees and radians:

Deg:	0°	30°	45°	60°	90°	120°	135°	150°	180°
Rad:	0	π/6	π/4	π/3	π/2	2π/3	3π/4	5π/6	π

Deg:	180°	210°	225°	240°	270°	300°	315°	330°	360°
Rad:	π	7π/6	5π/4	4π/3	3π/2	5π/3	7π/4	11π/6	2π

Domains of the Trig Functions

Key Property

Domain: the set of all possible values of the **independent** variable (θ below)

Domains of the trig functions:

- ➔ sin θ: $\forall\ \theta \in (-\infty,\infty)$
- ➔ cos θ: $\forall\ \theta \in (-\infty,\infty)$
- ➔ tan θ: $\forall\ \theta \neq (k+½)\pi$, where $k \in \mathbb{Z}$
- ➔ csc θ: $\forall\ \theta \neq k\pi$, where $k \in \mathbb{Z}$
- ➔ sec θ: $\forall\ \theta \neq (k+½)\pi$, where $k \in \mathbb{Z}$
- ➔ cot θ: $\forall\ \theta \neq k\pi$, where $k \in \mathbb{Z}$

Symbol definitions:

- ➔ $\forall$: for all
- ➔ $\in$: in
- ➔ $\mathbb{Z}$: set of all integers

Ranges of the Trig Functions

Key Property

Range: the set of all possible values of the **dependent** variable after substituting the domain

Ranges of the trig functions:

- ➔ $-1 \leq \sin \theta \leq 1$
- ➔ $-1 \leq \cos \theta \leq 1$
- ➔ $-\infty \leq \tan \theta \leq \infty$
- ➔ $\csc \theta \geq 1$ *and* $\csc \theta \leq -1$
- ➔ $\sec \theta \geq 1$ *and* $\sec \theta \leq -1$
- ➔ $-\infty \leq \cot \theta \leq \infty$

Periods of the Trig Functions

Key Property

Period: The distance required for the function to complete one full cycle. It is the number, T, such that $f(\theta + T) = f(\theta)$

Let k be a fixed number and θ be any angle. Then the periods of the trig functions are:

- ➔ $\sin(k\theta)$: $T = 2\pi / k$
- ➔ $\cos(k\theta)$: $T = 2\pi / k$
- ➔ $\tan(k\theta)$: $T = \pi / k$
- ➔ $\csc(k\theta)$: $T = 2\pi / k$
- ➔ $\sec(k\theta)$: $T = 2\pi / k$
- ➔ $\cot(k\theta)$: $T = \pi / k$

Periodic Formulas

Let n be an integer. Then,

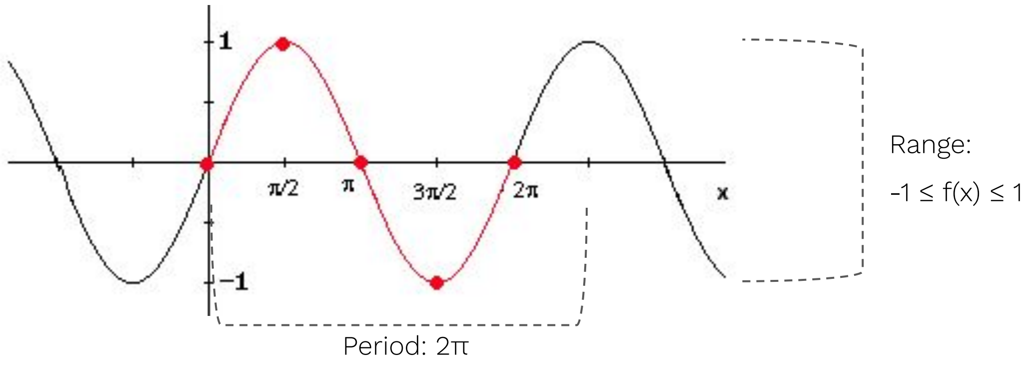
- ➔ $\sin(\theta + 2\pi n) = \sin\theta$
- ➔ $\cos(\theta + 2\pi n) = \cos\theta$
- ➔ $\tan(\theta + \pi n) = \tan\theta$
- ➔ $\csc(\theta + 2\pi n) = \csc\theta$
- ➔ $\sec(\theta + 2\pi n) = \sec\theta$
- ➔ $\cot(\theta + \pi n) = \cot\theta$

Graphs of the Trigonometric Functions

Graphs of the Trig Functions

Sine Graph

The below graph shows the function **f(x) = sin(x)**



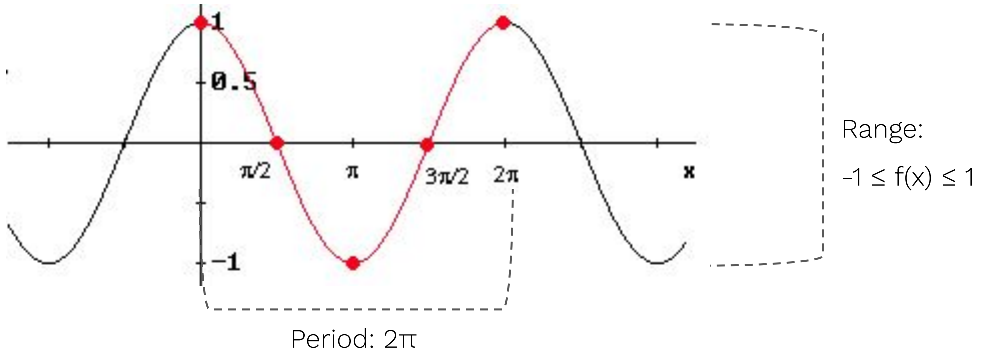
Key Properties

- ➔ Domain: $-\infty \le x \le \infty$
- ➔ Range: $-1 \le f(x) \le 1$
- ➔ Period: 2π
- ➔ x-intercept: $x = k\pi$, where k is an integer
- ➔ y-intercept: $y = 0$
- ➔ Maximum: $(\pi/2 + 2k\pi, 1)$, where k is an integer
- ➔ Minimum: $(3\pi / 2 + 2k\pi, -1)$, where k is an integer

Graphs of the Trig Functions

Cosine Graph

The below graph shows the function **f(x) = cos(x)**



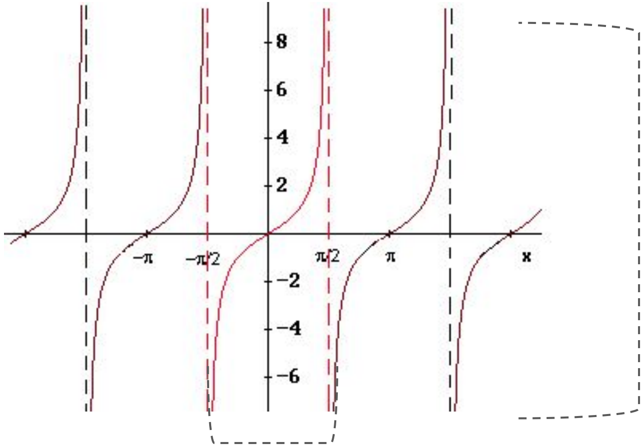
Key Properties

- ➔ Domain: $-\infty \leq x \leq \infty$
- ➔ Range: $-1 \leq f(x) \leq 1$
- ➔ Period: 2π
- ➔ x-intercept: $x = \pi/2 + k\pi$, where k is an integer
- ➔ y-intercept: $y = 1$
- ➔ Maximum: $(2k\pi, 1)$, where k is an integer
- ➔ Minimum: $(\pi + 2k\pi, -1)$, where k is an integer

Graphs of the Trig Functions

Tangent Graph

The below graph shows the function **f(x) = tan(x)**



Range:

$-\infty \leq f(x) \leq \infty$

Period: π

Key Properties

- Domain: $\forall\ x \neq (k+\frac{1}{2})\pi$, where k is an integer
- Range: $-\infty \leq f(x) \leq \infty$
- Period: π
- x-intercept: $x = k\pi$, where k is an integer
- y-intercept: $y = 0$

Key Identities and Formula of the Trigonometric Functions

Tangent and Reciprocal Identities

Key Identities

Tangent and Cotangent Identities

- ➔ $\tan\theta = \sin\theta / \cos\theta$
- ➔ $\cot\theta = \cos\theta / \sin\theta$

Reciprocal Identities

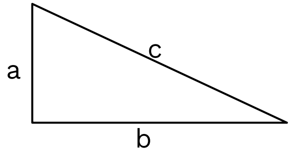
- ➔ $\sin\theta = 1 / \csc\theta$
- ➔ $\cos\theta = 1 / \sec\theta$
- ➔ $\tan\theta = 1 / \cot\theta$
- ➔ $\csc\theta = 1 / \sin\theta$
- ➔ $\sec\theta = 1 / \cos\theta$
- ➔ $\cot\theta = 1 / \tan\theta$

Pythagorean Theorem and Identities

Key Formulas

Pythagorean Theorem

- ➔ Let a, b and c be the lengths of the sides of a right triangle (where c is the length of the hypotenuse). Then $a^2 + b^2 = c^2$



Pythagorean Identities

- ➔ $\sin^2 \theta + \cos^2 \theta = 1$
- ➔ $\tan^2 \theta + 1 = \sec^2 \theta$
- ➔ $1 + \cot^2 \theta = \csc^2 \theta$

Pythagorean Triples

- ➔ A **pythagorean triple** is any set of three positive integers (a,b,c) such that $a^2 + b^2 = c^2$
- ➔ Common pythagorean triples include: (3,4,5), (6,8,10), (5,12,13), (8,15,17)
- ➔ If (a,b,c) is a Pythagorean triple, then so is (ka, kb, kc) for any positive integer k

Even and Odd

Key Formulas

Even and Odd Formulas

- ➔ $\sin(-\theta) = -\sin\theta$
- ➔ $\cos(-\theta) = \cos\theta$
- ➔ $\tan(-\theta) = -\tan\theta$
- ➔ $\csc(-\theta) = -\csc\theta$
- ➔ $\sec(-\theta) = \sec\theta$
- ➔ $\cot(-\theta) = -\cot\theta$

Double Angle and Half Angle

Key Formulas

Double Angle Formulas

- ➔ $\sin(2\theta) = 2 \sin\theta \cos\theta$
- ➔ $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- ➔ $\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Half Angle Formulas

- ➔ $\sin(\theta/2) = \pm\sqrt{\dfrac{1 - \cos(2\theta)}{2}}$
- ➔ $\cos(\theta/2) = \pm\sqrt{\dfrac{1 + \cos(2\theta)}{2}}$
- ➔ $\tan(\theta/2) = \pm\sqrt{\dfrac{1 - \cos(2\theta)}{1 + \cos(2\theta)}}$

Sum, Difference, and Product

Key Formulas

Sum and Difference

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Product to Sum Formulas

- $\sin\alpha\sin\beta = ½\,[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$
- $\cos\alpha\cos\beta = ½\,[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$
- $\sin\alpha\cos\beta = ½\,[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$
- $\cos\alpha\sin\beta = ½\,[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

Sum to Product Formulas

- $\sin\alpha + \sin\beta = 2\sin(\alpha/2 + \beta/2)\cos(\alpha/2 - \beta/2)$
- $\sin\alpha - \sin\beta = 2\cos(\alpha/2 + \beta/2)\sin(\alpha/2 - \beta/2)$
- $\cos\alpha + \cos\beta = 2\cos(\alpha/2 + \beta/2)\cos(\alpha/2 - \beta/2)$
- $\cos\alpha - \cos\beta = 2\sin(\alpha/2 + \beta/2)\sin(\alpha/2 - \beta/2)$

Cofunctions

Key Formulas

Cofunction Formulas

- ➔ $\sin(\pi/2 - \theta) = \cos\theta$
- ➔ $\cos(\pi/2 - \theta) = \sin\theta$
- ➔ $\csc(\pi/2 - \theta) = \sec\theta$
- ➔ $\sec(\pi/2 - \theta) = \csc\theta$
- ➔ $\tan(\pi/2 - \theta) = \cot\theta$
- ➔ $\cot(\pi/2 - \theta) = \tan\theta$

Inverse Trig Functions

Definition of the inverse trig functions

Key Vocabulary

Definition of the inverse trig functions

- ➔ $\theta = \sin^{-1}(x)$ is equivalent to $x = \sin\theta$
- ➔ $\theta = \cos^{-1}(x)$ is equivalent to $x = \cos\theta$
- ➔ $\theta = \tan^{-1}(x)$ is equivalent to $x = \tan\theta$

The inverse trig functions can also be notated as

- ➔ $\sin^{-1}(x) = \arcsin(x)$
- ➔ $\cos^{-1}(x) = \arccos(x)$
- ➔ $\tan^{-1}(x) = \arctan(x)$

Key properties of the inverse trig functions

Key Properties

Domain and range of the inverse trig functions

Function	Domain	Range
$\theta = \sin^{-1}(x)$	$-1 \leq x \leq 1$	$-\pi/2 \leq \theta \leq \pi/2$
$\theta = \cos^{-1}(x)$	$-1 \leq x \leq 1$	$0 \leq \theta \leq \pi$
$\theta = \tan^{-1}(x)$	$-\infty \leq x \leq \infty$	$-\pi/2 \leq \theta \leq \pi/2$

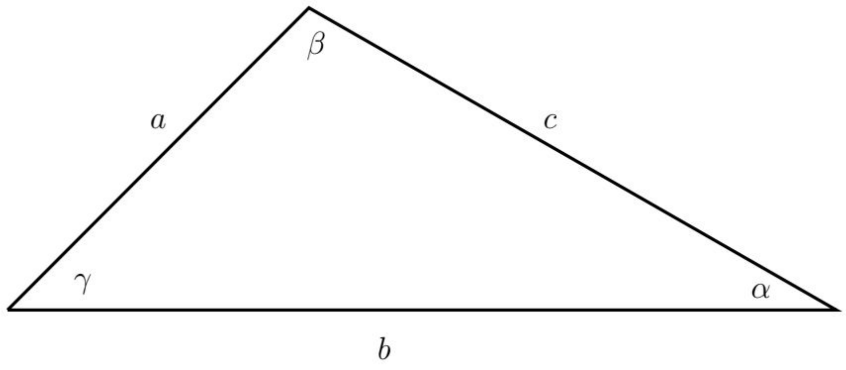
The following properties hold for x in the domain and θ in the range

- ➔ $\sin(\sin^{-1}(x)) = x$
- ➔ $\cos(\cos^{-1}(x)) = x$
- ➔ $\tan(\tan^{-1}(x)) = x$
- ➔ $\sin^{-1}(\sin(\theta)) = \theta$
- ➔ $\cos^{-1}(\cos(\theta)) = \theta$
- ➔ $\cos^{-1}(\cos\theta)) = \theta$

Laws of Sines, Cosines, and Tangents

Law of Sines

Key Vocabulary

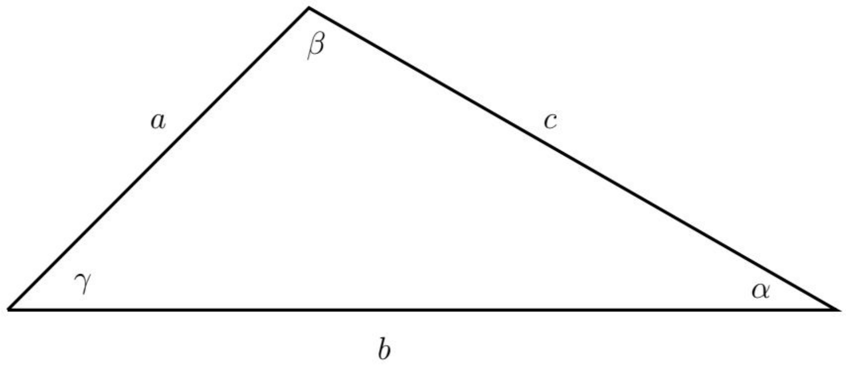


Law of Sines

➔ (sin α) / a = (sin β) / b = (sin γ) / c

Law of Cosines

Key Vocabulary

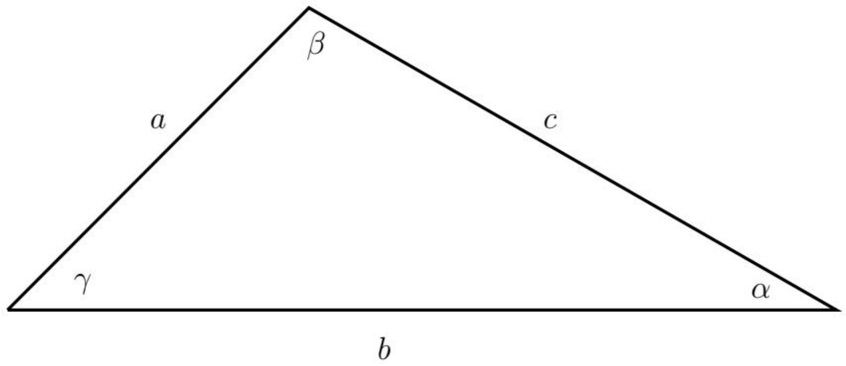


Law of Cosines

- ➔ $a^2 = b^2 + c^2 - 2bc(\cos \alpha)$
- ➔ $b^2 = a^2 + c^2 - 2ac(\cos \beta)$
- ➔ $c^2 = a^2 + b^2 - 2ab(\cos \gamma)$

Law of Tangents

Key Vocabulary



Law of Tangents

- ➔ $$\frac{a-b}{a+b} = \frac{\tan \tfrac{1}{2}(\alpha-\beta)}{\tan \tfrac{1}{2}(\alpha+\beta)}$$
- ➔ $$\frac{b-c}{b+c} = \frac{\tan \tfrac{1}{2}(\beta-\gamma)}{\tan \tfrac{1}{2}(\beta+\gamma)}$$
- ➔ $$\frac{a-b}{a+c} = \frac{\tan \tfrac{1}{2}(\alpha-\gamma)}{\tan \tfrac{1}{2}(\alpha+\gamma)}$$

Complex Numbers and DeMoivre’s Theorem

Definition of a complex number

Key Vocabulary

Definition of the **imaginary number *i***

- ➔ $i = \sqrt{(-1)}$
- ➔ $i^2 = -1$
- ➔ $i^3 = -i$
- ➔ $i^4 = 1$

A **complex number** is a combination of a real number and an imaginary number and can be written in the form a + b*i*, where a and b are real numbers

The **conjugate** of a complex number a + b*i*, denoted $\overline{a + bi}$ is given by

- ➔ $\overline{a + bi} = a - bi$

The absolute value of a complex number, or **complex modulus** is given by

- ➔ $|a + bi| = \sqrt{a^2 + b^2}$

DeMoivre's Theorem

DeMoivre's Theorem

Let $z = r\,(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then:

$$z^n = r^n\,(\cos n\theta + i\sin n\theta)$$

Example: Let $z = 1 - i$, find z^6

First write z in polar form

- ➔ The polar form of a complex number $x + yi$ is given by: $r\,(\cos\theta + i\sin\theta)$
- ➔ $r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$
- ➔ $\theta = \arg(z) = \tan^{-1}(-1/1) = -\pi/4$
- ➔ Polar form: $z = \sqrt{2}\,[\cos(-\pi/4) + i\sin(-\pi/4)]$

Applying DeMoivre's Theorem gives:

- ➔ $z^6 = (\sqrt{2})^6\,[\cos(6\times(-\pi/4)) + i\sin(6\times(-\pi/4))]$

$= 2^3\,[\cos(-3\pi/2) + i\sin(-3\pi/2)]$

$= 8\,[0 + i(1)]$

$= 8i$

DeMoivre's Theorem

Finding the *nth* roots of a number using DeMoivre's Theorem

Example: Find all the complex fourth roots of 4. That is, find all the complex solutions of $x^4 = 4$

For any positive integer n, a nonzero complex number z has exactly n distinct nth roots. If z is written in the trigonometric form r (cos θ + *i* sin θ), the nth roots of z are given by the following formula:

(*) $r^{1/n}$ [cos(θ/n + 360°k/n) + *i* sin(θ/n + 360°k/n)], for k = 0, 1, 2, ..., n−1

Writing the number 4 in trigonometric form using $r = \sqrt{a^2 + b^2}$ and θ = arg(z) = $\tan^{-1}$(b/a), we have the following:

➔ 4 = 4 + *i* (0) → $r = \sqrt{4^2 + 0^2}$ and θ = arg(z) = $\tan^{-1}$(0/4) = 0 → 4 = 4 (cos 0° + *i* sin 0°)

Using the formula **(*)** above with n = 4, we can find the fourth roots of 4 (cos 0° + *i* sin 0°):

➔ For k = 0, $4^{1/4}$ [cos(0°/4 + 360°*0/4) + *i* sin(0°/4 + 360°*0/4)] = $\sqrt{2}$ [cos(0°) + *i* sin(0°)] = $\sqrt{2}$

➔ For k = 1, $4^{1/4}$ [cos(0°/4 + 360°*1/4) + *i* sin(0°/4 + 360°*1/4)] = $\sqrt{2}$ [cos(90°) + *i* sin(90°)] = $\sqrt{2}\,i$

➔ For k = 2, $4^{1/4}$ [cos(0°/4 + 360°*2/4) + *i* sin(0°/4 + 360°*2/4)] = $\sqrt{2}$ [cos(180°) + *i* sin(180°)] = $-\sqrt{2}$

➔ For k = 3, $4^{1/4}$ [cos(0°/4 + 360°*3/4) + *i* sin(0°/4 + 360°*3/4)] = $\sqrt{2}$ [cos(270°) + *i* sin(270°)] = $-\sqrt{2}\,i$

Thus all of the complex roots of $x^4 = 4$ are: $\sqrt{2}$, $\sqrt{2}\,i$, $-\sqrt{2}$, $-\sqrt{2}\,i$

Conics

Circle

Key Formulas

The **standard form** of a circle is denoted by the formula

$$(x - h)^2 + (y - k)^2 = r^2$$

Where

- ➔ (h,k) = center of the circle
- ➔ r = radius of the circle

Ellipse

Key Formulas

The **standard form for horizontal major axis** is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The **standard form for vertical major axis** is given by

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Where

- ➔ (h,k) = center of the ellipse
- ➔ $2a$ = length of the major axis
- ➔ $2b$ = length of the minor axis
- ➔ $0 < b < a$

The **foci** of the ellipse can be found using the formula $c^2 = a^2 - b^2$

where c = foci length

Hyperbola

Key Formulas

The standard form for horizontal transverse is given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The standard form for vertical transverse axis is given by

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Where

- ➔ (h,k) = center of the hyperbola
- ➔ a = distance between center and either vertex

The **foci** can be found using $b^2 = c^2 - a^2$ where

- ➔ c is the distance between center and either focus
- ➔ $b > 0$

Parabola

Key Formulas

A parabola that is symmetric about a **vertical axis** is given by

$$y = a\,(x - h)^2 + k$$

A parabola that is symmetric about a **horizontal axis** is given by

$$x = a\,(y - k)^2 + h$$

Where

- ➔ (h,k) = **vertex**
- ➔ a = **scaling factor**

Additional Resources

Trigonometry

Additional Resources

- ➔ https://www.khanacademy.org/math/trigonometry
- ➔ http://web.mit.edu/jorloff/www/18.01a-esg/OCWTrig.pdf
- ➔ https://www.mathsisfun.com/algebra/trigonometry.html
- ➔ http://jwilson.coe.uga.edu/EMAT6680/Adcock/Adcock6690/RLAInstructUnit1/RLATrigMenu.htm