Calculus Review Guide

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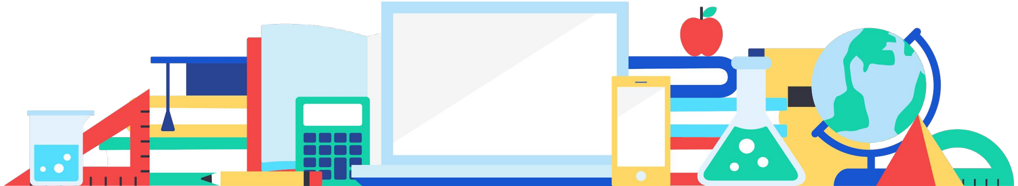

Limits

Definition of a Limit

Key Vocabulary

Formal Definition

- ➔ The **limit** $\lim_{x \to a} f(x) = L$ if for every $\boldsymbol{\varepsilon} > 0$ there is a $\boldsymbol{\delta} > 0$ such that whenever $0 < |x - a| < \boldsymbol{\delta}$ then $|f(x) - L| < \boldsymbol{\varepsilon}$

"Working" Definition

- ➔ The **limit** $\lim_{x \to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$

Right and Left Hand Limits

Key Vocabulary

Right hand limit: $\lim_{x\to a+} f(x) = L$ when $x > a$ (i.e. the limit when x approaches from the right)

Left hand limit: $\lim_{x\to a-} f(x) = L$ when $x < a$ (i.e. the limit when x approaches from the left)

Relationship between the limit and one-sided limits

- ➔ $\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = L$
- ➔ $\lim_{x\to a+} f(x) \neq \lim_{x\to a-} f(x) \rightarrow \lim_{x\to a} f(x)$ *does not exit*

Symbol Definitions

- ➔ $\Leftrightarrow$: if and only if

Limit at Infinity and Infinite Limit

Key Vocabulary (continued)

Limit at Infinity

- $\lim_{x\to\infty} f(x) = L$ if we can make f(x) as close to L as we want by taking x large enough and *positive*
- $\lim_{x\to-\infty} f(x) = L$ if we can make f(x) as close to L as we want by taking x large enough and *negative*

Infinite Limit

- $\lim_{x\to a} f(x) = \infty$ if we can make f(x) arbitrarily large and *positive* by taking x sufficiently close to a (on either side of a) without letting x = a
- $\lim_{x\to a} f(x) = -\infty$ if we can make f(x) arbitrarily large and *negative* by taking x sufficiently close to a (on either side of a) without letting x = a

Properties of Limits

Key Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number. Then:

1. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$
6. $\lim_{x\to a}[\sqrt[n]{f(x)}] = \sqrt[n]{\lim_{x\to a} f(x)}$
7. $\lim_{x\to a} c = c$

Basic Limit Evaluations

Problem Solving

1. $\lim_{x\to+\infty} e^x = \infty$
2. $\lim_{x\to-\infty} e^x = 0$
3. $\lim_{x\to\infty} \ln(x) = \infty$

   $\lim_{x\to 0+} \ln(x) = -\infty$
4. If $r > 0$ then $\lim_{x\to\infty} b / x^r = 0$
5. If $r > 0$ and x^r is real for negative x, then $\lim_{x\to-\infty} b / x^r = 0$
6. If n even, then $\lim_{x\to\pm\infty} x^n = \infty$
7. If n odd, then $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$

Limit Evaluation Techniques

Problem Solving

Continuous Functions

- ➔ If f(x) is continuous at a then $\lim_{x\to a} f(x) = f(a)$
- ➔ If f(x) is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$

Factor and Cancel

Ex: $\lim_{x\to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \lim_{x\to 2} \frac{(x-2)(x+6)}{x(x-2)} = \lim_{x\to 2} \frac{x+6}{x} = 8 / 2 = 4$

Rationalize Numerator / Denominator

Ex: $\lim_{x\to 9} \frac{3 - \sqrt{x}}{x^2 - 81} = \lim_{x\to 9} \frac{3 - \sqrt{x}}{x^2 - 81} \frac{(3 + \sqrt{x})}{(3 + \sqrt{x})} = \lim_{x\to 9} \frac{9 - x}{(x^2 - 81)(3 + \sqrt{x})}$

$= \lim_{x\to 9} \frac{-1}{(x+9)(3+\sqrt{x})} = \frac{-1}{(18)(6)} = -1 / 108$

Combine Rational Expressions

Ex: $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{x - (x+h)}{x(x+h)}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h\to 0} \frac{-1}{x(x+h)} = -1 / x^2$

Limit Evaluation Techniques

Problem Solving (Continued)

L'Hospital's Rule

If $\lim_{x\to a} f(x) / g(x) = 0$ or $\lim_{x\to a} f(x) / g(x) = \pm\infty / \pm\infty$, then

$$\lim_{x\to a} f(x)\, g(x) = \lim_{x\to a} f'(x) / g'(x)$$

where a is a number, $+\infty$ or $-\infty$

Polynomials at Infinity

Suppose p(x) and q(x) are polynomials. To compute $\lim_{\pm\infty} p(x) / q(x)$, factor the largest power of x in q(x) out of both p(x) and q(x), then compute the limit

➔ Ex: $\lim_{x\to-\infty} \dfrac{3x^2 - 4}{5x - 2x^2} = \dfrac{x^2(3 - 4/x^2)}{x^2(5/x - 2)} = \lim_{x\to-\infty} \dfrac{3 - 4/x^2}{5/x - 2} = -3/2$

Piecewise Function

Ex: $\lim_{x\to-2} g(x)$ where $g(x) = \begin{cases} x^2 + 5 \text{ if } x < -2 \\ 1 - 3x \text{ if } x > -2 \end{cases}$

Compute two one sided limits, $\lim_{x\to-2^-} x^2 + 5 = 9$ and $\lim_{x\to-2^+} 1 - 3x = 7$

The one sided limits are different, so $\lim_{x\to-2} g(x)$ doesn't exist. If the two one sided limits had been equal, then $\lim_{x\to-2} g(x)$ would have existed and had that same value

Derivatives

Definition of a Derivative

Key Vocabulary

If $y = f(x)$ then the **derivative** of $f(x)$ with respect to x is the function $f'(x)$ and is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

If $y = f(x)$ then the derivative may be denoted as

$$f'(x) = y' = df / dx = dy / dx = d / dx\,(f(x)) = Df(x)$$

A function $f(x)$ is **differentiable** at $x = a$ if $f'(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval

If $y = f(x)$ then the derivative of $f(x)$ at $x = a$ may be denoted as

$$f'(a) = y'|_{x=a} = df / dx\,|_{x=a} = dy / dx\,|_{x=a} = Df(a)$$

Interpretation of a Derivative

Key Vocabulary

If $y = f(x)$, then

- ➔ $m = f'(a)$ is the slope of the **tangent line** to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x - a)$
- ➔ $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$
- ➔ If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x = a$

Properties of Derivatives

Key Properties

If f(x) and g(x) are differentiable functions (the derivative exists), and c and n are any real numbers. Then:

1. $(cf)' = cf'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(fg)' = f'g + fg'$ (**product rule**)
4. $(f / g)' = (f'g - fg') / g^2$ (**quotient rule**)
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}(x^n) = nx^{n-1}$ (**power rule**)
7. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$ (**chain rule**)

Common Derivatives to Know

Problem Solving

General

- ➔ $\frac{d}{dx}(x) = 1$

Trig Derivatives

- ➔ $\frac{d}{dx}(\sin x) = \cos x$
- ➔ $\frac{d}{dx}(\cos x) = -\sin x$
- ➔ $\frac{d}{dx}(\tan x) = \sec^2 x$
- ➔ $\frac{d}{dx}(\sec x) = \sec x \tan x$
- ➔ $\frac{d}{dx}(\csc x) = -\csc x \cot x$
- ➔ $\frac{d}{dx}(\cot x) = -\csc^2 x$
- ➔ $\frac{d}{dx}(\sin^{-1} x) = 1 / (\sqrt{1 - x^2})$
- ➔ $\frac{d}{dx}(\cos^{-1} x) = -1 / (\sqrt{1 - x^2})$
- ➔ $\frac{d}{dx}(\tan^{-1} x) = -1 / (1 + x^2)$

Exponential and Log Derivatives

- ➔ $\frac{d}{dx}(a^x) = a^x \ln(a)$
- ➔ $\frac{d}{dx}(e^x) = e^x$
- ➔ $\frac{d}{dx}(\ln(x)) = 1 / x, x > 0$
- ➔ $\frac{d}{dx}(\ln|x|) = 1 / x, x \neq 0$
- ➔ $\frac{d}{dx}(\log_a(x)) = 1 / (x\ln(a)), x > 0$

Chain Rule and Variations

Key Properties

The **chain rule** for derivatives shows that

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

The chain rule applied to some specific functions follow:

- ➔ $\frac{d}{dx}([f(x)]^{n}) = n[f(x)]^{n-1}f'x)$
- ➔ $\frac{d}{dx}(e^{f(x)}) = f'(x)\ e^{f(x)}$
- ➔ $\frac{d}{dx}(\ln[f(x)]) = f'(x) / f(x)$
- ➔ $\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$
- ➔ $\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$
- ➔ $\frac{d}{dx}(\tan[f(x)]) = f'(x)\sec^{2}[f(x)]$
- ➔ $\frac{d}{dx}(\sec[f(x)]) = f'(x)\sec[f(x)]\tan[f(x)]$
- ➔ $\frac{d}{dx}(\tan^{-1}[f(x)]) = f'(x) / (1+[f(x)]^{2})$

Higher Order Derivatives

Key Vocabulary

The **second derivative** is defined as

$$f''(x) = (f'(x))'$$

I.e. is the derivatibe of the first derivative

The second derivative may be denoted as $f''(x) = f^{(2)}(x) = d^2f / dx^2$

The **nth derivative** is defined as

$$f^{(n)}(x) = (f^{(n-1)}(x))'$$

I.e. is the derivatibe of the (n – 1) derivative, $f^{(n-1)}(x)$

The second derivative may be denoted as $f''(x) = f^{(2)}(x) = d^2f / dx^2$

Implicit Differentiation

Problem Solving

Ex: Find y' if $e^{2x - 9y} + x^3y^2 = \sin(y) + 11x$

Remember, $y = y(x)$ here, so products / quotients of x and y will use the product / quotient rule and derivatives of y will use the chain rule

The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating, solve for y'.

- ➔ Step 1: Differentiate as normal, tacking on a y' each time

  $e^{2x - 9y}(2 - 9y') + 3x^2y^2 + 2x^3yy' = \cos(y)y' + 11$
- ➔ Solve for y'

  $2e^{2x - 9y} - 9y'2e^{2x - 9y} + 3x^2y^2 + 2x^3yy' = \cos(y)y' + 11$

  $(2x^3y - 9e^{2x - 9y} - \cos(y))y' = 11 - 2e^{2x - 9y} - 3x^2y^2$

  $y' = \dfrac{11 - 2e^{2x - 9y} - 3x^2y^2}{2x^3y - 9e^{2x - 9y} - \cos(y)}$

Increasing / Decreasing Functions

Key Vocabulary

Critical Points

- ➔ $x = c$ is a **critical point** of $f(x)$ provided either 1. $f'(c) = 0$ or 2. $f'(c)$ doesn't exist

Increasing / Decreasing

- ➔ If $f'(x) > 0$ for all x in an interval I then $f(x)$ is **increasing** on the interval I
- ➔ If $f'(x) < 0$ for all x in an interval I, then $f(x)$ is **decreasing** on the interval I
- ➔ If $f'(x) = 0$ for all x in an interval I then $f(x)$ is **constant** on the interval I

Concave Up / Concave Down

- ➔ If $f''(x) > 0$ for all x in an interval I then $f(x)$ is **concave up** on the interval I
- ➔ If $f''(x) < 0$ for all x in an interval I, then $f(x)$ is **concave down** on the interval I

Inflection Points

- ➔ $X = c$ is an **inflection point** of $f(x)$ if the concavity changes at $x = c$

Extrema

Key Vocabulary

Absolute Extrema

- ➔ $x = c$ is an **absolute maximum** of $f(x)$ if $f(c) \geq f(x)$ for all x in the domain
- ➔ $x = c$ is an **absolute minimum** of $f(x)$ if $f(c) \leq f(x)$ for all x in the domain

Relative (local) Extrema

- ➔ $x = c$ is a **relative (local) maximum** of $f(x)$ if $f(c) \geq f(x)$ for all x near c
- ➔ $x = c$ is a **relative (local) minimum** of $f(x)$ if $f(c) \leq f(x)$ for all x near c

Fermat's Theorem: If $f(x)$ has a relative (or local) extrema at $x = c$, then $x = c$ is a critical point of $f(x)$

Extreme Value Theorem: If $f(x)$ is continuous on the closed interval $[a,b]$, then there exists numbers c and d so that 1. $a \leq c, d \leq b$ 2. $f(x)$ is the absolute maximum in $[a,b]$ and 3. $f(d)$ is the absolute minimum in $[a,b]$

Extrema

Problem Solving

To find the absolute extrema of a continuous f(x) on the interval [a,b], use the following process:

- ➔ Step 1: Find all critical points of f(x) in [a,b]
- ➔ Step 2: Evaluate f(x) at all points found in Step 1
- ➔ Step 3: Evaluate f(a) and f(b)
- ➔ Step 4: Identify the absolute maximum (largest function value) and absolute minimum (smallest function value) from the evaluations in Steps 2 & 3

Extrema

Problem Solving (Continued)

1st Derivative Test: If $x = c$ is a critical point of $f(x)$ then $x = c$ is

1. A relative maximum of $f(x)$ if $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$
2. A relative minimum of $f(x)$ if $f'(x) < 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of c
3. Not a relative extrema of $f(x)$ if $f'(x)$ is the same sign on both sides of $x = c$

2nd Derivative Test: If $x = c$ is a critical point of $f(x)$ such that $f'(c) = 0$ then $x = c$

1. Is a relative maximum of $f(x)$ if $f''(c) < 0$
2. Is a relative minimum of $f(x)$ if $f''(c) > 0$
3. May be a relative maximum, relative minimum, or neither if $f''(c) = 0$

Relative Extrema and / or Classify Critical Points

- ➔ Step 1: Find all critical points of $f(x)$
- ➔ Step 2: Use the 1st derivative test or the 2nd derivative test on each critical point

Integrals

Definition of an Integral

Key Vocabulary

Suppose f(x) is continuous on [a,b]. Divide [a,b] into n subintervals of width Δx and choose x_i^* from each interval. Then the **integral** is defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{\infty} f(x_i^*)\Delta x$$

An **antiderivative** of f(x) is a function, F(x) such that F'(x) = f(x)

The **indefinite integral** is defined as

$$\int f(X)dx = F(x) + c$$

where F(x) is an anti-derivative of F(x) + c

Fundamental Theorem of Calculus

Key Theorem

The **first fundamental theorem of calculus** states that if $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)dt$ is also continuous on $[a,b]$ and

$$g'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

The **second fundamental theorem of calculus** states that if $f(x)$ is continuous on $[a,b]$ and $F(x)$ is an anti-derivative of $f(x)$ (i.e. $F(x) = \int f(x)dx$), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Variants of the first fundamental theorem of calculus

- ➔ $\frac{d}{dx} \int_a^{u(x)} f(t)dt = u'(x)f[u(x)]$
- ➔ $\frac{d}{dx} \int_{v(x)}^{b} f(t)dt = -v'(x)f[v(x)]$
- ➔ $\frac{d}{dx} \int_{v(x)}^{u(x)} f(t)dt = u'(x)f[u(x)]-v'(x)f[v(x)]$

Properties of Integrals

Key Properties

Key properties of integrals

1. $\int[f(x) \pm g(x)]dx = \int f(x)dx + \int g(x)dx$
2. $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
3. $\int_a^a f(x)dx = 0$
4. $\int_a^b f(x)dx = -\int_a^b f(x)dx$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any calue of c
6. If $f(x) \geq g(x)$ on a $a \leq x \leq b$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
7. If $f(x) \geq 0$ on $a \leq x \leq b$ then $\int_a^b f(x)dx \geq 0$
8. If $m \leq f(x) \leq M$ on $a \leq x \leq b$ then $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$
9. $\int cf(x)dx = c\int f(x)dx$, where c is a constant
10. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is a constant
11. $\int_a^b cf(x)dx = c(b - a)$
12. $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

Common Integrals to Know

Problem Solving

General

- $\int(k)dx = kx + c$
- $\int(x^n)dx = [x^{n+1} / (n+1)] + c,\ n \neq -1$

Trig Derivatives

- $\int(\cos u)du = \sin u + c$
- $\int(\sin u)du = -\cos u + c$
- $\int(\sec^2 u)du = \tan u + c$
- $\int[(\sec u)(\tan u)]du = \sec u + c$
- $\int[(\csc u)(\cot u)] = -\csc u + c$
- $\int(\tan u)du = \ln|\sec u| + c$
- $\int(\sec u)du = \ln|\sec u + \tan u| + c$
- $\int[1/(a^2 + u^2)]du = (1/a)\tan^{-1}(u/a) + c$
- $\int[1/\sqrt{(a^2 - u^2)}]du = \sin^{-1}(u/a) + c$

Exponential and Log Derivatives

- $\int(x^{-1})dx = \int(1/x)dx = \ln|x| + c$
- $\int[1 / (ax+b)]dx = (1/a)\ln|ax + b| + c$
- $\int(\ln u)dx = u\ln(u) - u + c$
- $\int(e^u)du = e^u + c$

Standard Integration Techniques

Problem Solving

u Substitution

The substitution $u = g(x)$ will convert $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$ using $du = g'(x)dx$. For indefinite integrals, drop the limits of integration.

Ex: Find $\int_1^2 (5x^2)\cos(x^3)dx$

- ➔ Step 1: Select a value for u

  Let $u = x^3 \rightarrow du = 3x^2dx$
- ➔ Step 2: Determine the value of dx

  $u = x^3 \rightarrow du = 3x^2dx \rightarrow dx = (⅓)(du/x^2)$ and $5x^2dx = (5/3)du$
- ➔ Step 3: Substitute in u and du where x and dx appear in the equation

  $\int_1^2 (5x^2)\cos(x^3)dx = \int_1^2 (5/3)\cos(u)du$
- ➔ Step 4: Evalute the integral

  $\int_1^2 (5/3)\cos(u)du = (5/3)\sin(u)|_1^8 = (5/3)[\sin(8) - \sin(1)] \approx .246$

Standard Integration Techniques

Problem Solving (Continued)

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Choose u and dv from the integral, and compute du by differentiating u and compute v using $v = \int dv$

Ex: Find $\int xe^{-x}dx$

- ➔ Step 1: Select values for u and dv

  Let $u = x$ and $dv = e^{-x}$
- ➔ Step 2: Compute du and v

  $u = x \rightarrow du = dx$

  $dv = e^{-x} \rightarrow v = \int dv = \int e^{-x} = -e^{-x}$
- ➔ Step 3: Substitute in the values for u, du, v, and dv into the equation

  $\int u\,dv = uv - \int v\,du$

  $\int xe^{-x}dx = \int u\,dv = uv - \int v\,du = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + c$

Additional Resources

Calculus

Additional Resources

- ➔ http://www.stat.wisc.edu/~ifischer/calculus.pdf
- ➔ http://tutorial.math.lamar.edu/Classes/CalcI/CalcI.aspx
- ➔ https://www.khanacademy.org/math/calculus-1
- ➔ https://notendur.hi.is/adl2/CalcI_Complete.pdf
- ➔ https://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/study-guide/
- ➔ http://www.math.nagoya-u.ac.jp/~richard/teaching/f2016/BasicCalculus.pdf