HW8 - Lzy219 - Ziyu Liao

Problem 1

a. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

plug in ↓

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} e^{-ikx} dx$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2} - ikx} dx$$

let $I = \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2} - ikx} dx$

for $-\frac{(x-\mu)^2}{2\sigma^2} - ikx = -\frac{1}{2\sigma^2}(x^2 + \mu^2 - 2\mu x) - ikx$

$$= -\frac{x^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} + \frac{2\mu x}{2\sigma^2} - ikx$$

$$= -\frac{x^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} + \left(\frac{\mu}{\sigma^2} - ik\right)x$$

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2} + \left(\frac{\mu}{\sigma^2} - ik\right)x - \frac{\mu^2}{2\sigma^2}\right) dx$$

$$= e^{-\frac{\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2} + \left(\frac{\mu}{\sigma^2} - ik\right)x\right) dx$$

let $a = \frac{1}{2\sigma^2}$, $b = \frac{\mu}{\sigma^2} - ik$

$$= e^{-\frac{\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \exp(-ax^2 + bx)\, dx$$

By Guassian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad \& \quad \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \quad a>0$$

$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx$ where $-ax^2+bx = -a(x^2 - \frac{b}{a}x)$

$$= -a(x-\frac{b}{2a})^2 + \frac{b^2}{4a}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-ax^2+bx} dx = e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{-a(x-\frac{b}{2a})^2} dx$$

let $u = x - \frac{b}{2a}$ $\quad = e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{-au^2} du$

$$= e^{\frac{b^2}{4a}} \cdot \sqrt{\frac{\pi}{a}}$$

Therefore $\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}} \quad a>0$

$\frac{1}{2\sigma^2} > 0$ Then, use the formula

$$\sqrt{\frac{\pi}{a}} = \sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} = \sqrt{2\sigma^2\pi} = \sqrt{2\pi}\,\sigma$$

$$b^2 = (\frac{M}{\sigma^2} - ik)^2 = \frac{M^2}{\sigma^4} - 2i\frac{Mk}{\sigma^2} - k^2 \quad (i^2=-1)$$

$$4a = 4 \cdot \frac{1}{2\sigma^2} = \frac{2}{\sigma^2}$$

$$\frac{b^2}{4a} = \frac{\frac{M^2}{\sigma^4} - 2i\frac{Mk}{\sigma^2} - k^2}{2/\sigma^2}$$

$$= \frac{\sigma^2}{2}\left(\frac{M^2}{\sigma^4} - 2i\frac{Mk}{\sigma^2} - k^2\right)$$

$$= \frac{M^2}{2\sigma^2} - iMk - \frac{\sigma^2 k^2}{2}$$

$$e^{\frac{b^2}{4a}} = \exp\left(\frac{\mu^2}{2\sigma^2} - i\mu k - \frac{\sigma^2 k^2}{2}\right)$$

$$\rightarrow \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2} + \left(\frac{\mu}{\sigma^2} - ik\right)x\right) dx = \sqrt{2\pi}\,\sigma \exp\left(\frac{\mu^2}{2\sigma^2} - i\mu k - \frac{\sigma^2 k^2}{2}\right)$$

plug in I :

$$I = e^{-\frac{\mu^2}{2\sigma^2}} \cdot \sqrt{2\pi}\,\sigma \exp\left(\frac{\mu^2}{2\sigma^2} - i\mu k - \frac{\sigma^2 k^2}{2}\right)$$

$$= \sqrt{2\pi}\,\sigma \exp\left(-i\mu k - \frac{\sigma^2 k^2}{2}\right)$$

back in $\hat{f}(k)$: $\hat{f}(k) = \frac{1}{2\pi\sigma} \cdot I$

$$= \frac{1}{2\pi\sigma} \cdot \sqrt{2\pi}\,\sigma \exp\left(-i\mu k - \frac{\sigma^2 k^2}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-i\mu k - \frac{\sigma^2 k^2}{2}\right)$$

$$\boxed{= \frac{1}{\sqrt{2\pi}} e^{-i\mu k - \frac{\sigma^2 k^2}{2}}}$$

b. $f(t) = \sin(\omega_0 t)$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ikt}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\omega_0 t)\, e^{-ikt}\, dt$$

* $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

since $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \Rightarrow \sin(\omega_0 t) = \dfrac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{i w_0 t} - e^{-i w_0 t}}{2i} \right) e^{-ikt} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2i} \left[\int_{-\infty}^{\infty} e^{i w_0 t} e^{-ikt} \, dt - \int_{-\infty}^{\infty} e^{-i w_0 t} e^{-ikt} \, dt \right.$$

$$= \frac{1}{2i\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{\infty} e^{i(w_0 - k)t} \, dt}_{1} - \underbrace{\int_{-\infty}^{\infty} e^{-i(w_0 + k)t} \, dt}_{2} \right.$$

given $\int_{-\infty}^{\infty} e^{iat} \, dt = 2\pi \delta(a)$

for 1: let $a = w_0 - k$

$$\int_{-\infty}^{\infty} e^{i(w_0 - k)t} \, dt = 2\pi \delta(w_0 - k)$$

since $\delta(x) = \delta(-x) \Rightarrow \delta(w_0 - k) = \delta(k - w_0)$

for 2: let $a = -w_0 - k$

$$\int_{-\infty}^{\infty} e^{-i(w_0 + k)t} \, dt = 2\pi \delta(-w_0 - k) = 2\pi \delta(k + w_0)$$

Therefore:

$$\hat{f}(k) = \frac{1}{2i\sqrt{2\pi}} \cdot \left[2\pi \delta(k - w_0) - 2\pi \delta(k + w_0) \right]$$

$$= \frac{1}{2i\sqrt{2\pi}} 2\pi \left[\delta(k - w_0) - \delta(k + w_0) \right]$$

$\frac{1}{i} = -i$

$$= -i \frac{\pi}{\sqrt{2\pi}} \left[\delta(k - w_0) - \delta(k + w_0) \right]$$

$$\boxed{= i\sqrt{\frac{\pi}{2}} \left[\delta(k + w_0) - \delta(k - w_0) \right]}$$

c. $f(x) = e^{-a|x|}, \ a > 0$

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, e^{-ikx}\, dx$$

Since absolute value $|x|$, seperate into 2 parts:

when $x < 0$, $|x| = -x \Rightarrow e^{-a|x|} = e^{-a(-x)}$

when $x > 0$, $|x| = x \Rightarrow e^{-a|x|} = e^{-ax}$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{\infty}^{0} e^{-a(-x)}\, e^{-ikx}\, dx + \int_{0}^{\infty} e^{-ax}\, e^{-ikx}\, dx\right.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(a-ik)x}\, dx + \int_{0}^{\infty} e^{-(a+ik)x}\, dx\right.$$

Let $I_1 = \int_{-\infty}^{0} e^{(a-ik)x}$

(since $a > 0$, the integral converges)

$$\int e^{(a-ik)x}\, dx = \frac{e^{(a-ik)x}}{a-ik} + C$$

$$\Rightarrow \quad I_1 = \left[\frac{e^{(a-ik)x}}{a-ik}\right]_{-\infty}^{0} = \frac{e^{0}}{a-ik} - \lim_{x\to-\infty}\frac{e^{(a-ik)x}}{a-ik}$$

when $x \to -\infty$, since $a > 0$, $Re(a-ik) = a > 0 \Rightarrow e^{(a-ik)x} \to 0$

$$\Rightarrow \quad I_1 = \frac{1}{a-ik} - 0 = \frac{1}{a-ik}$$

let $I_2 = \int_{0}^{\infty} e^{-(a+ik)x}$

$$\int_0^\infty e^{-(a+ik)x}\,dx = \frac{e^{-(a+ik)x}}{-(a+ik)} + C$$

$$I_2 = \left[\frac{e^{-(a+ik)x}}{-(a+ik)}\right]_0^\infty = \lim_{x\to\infty} \frac{e^{-(a+ik)x}}{-(a+ik)} - \frac{e^0}{-(a+ik)}$$

when $x \to \infty$, since $a > 0$, $Re(-(a+ik)) = -a < 0 \Rightarrow e^{-(a+ik)x} \to 0$

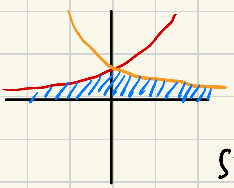
$$I_2 = 0 - \frac{e^0}{-(a+ik)} = \frac{1}{a+ik}$$

Therefore, $\hat{f}(k) = \frac{1}{\sqrt{2\pi}}(I_1 + I_2) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a-ik} + \frac{1}{a+ik}\right)$

where, $\frac{1}{a-ik} + \frac{1}{a+ik} = \frac{(a+ik)+(a-ik)}{(a-ik)(a+ik)} = \frac{2a}{a^2-(ik)^2} = \frac{2a}{a^2+k^2}$

Then, $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+k^2} = \frac{2a}{\sqrt{2\pi}(a^2+k^2)} = \boxed{\sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}}$

Check: if $k=0$: $\hat{f}(0) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2} = \sqrt{\frac{2}{\pi}}\frac{1}{a}$



$$\int_{-\infty}^{\infty} e^{-a|x|}\,dx = 2\int_0^\infty e^{-ax}\,dx = 2\left[-\frac{e^{-ax}}{a}\right]_0^\infty = \frac{2}{a}$$

Sub in: $\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{a} = \sqrt{\frac{2}{\pi}}\frac{1}{a}$

Same answer ✓

d. $f(t) = \delta(t)$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-ikt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t)\, e^{-ikt}\, dt$$

given def of dirac delta: $\int_{-\infty}^{\infty} \delta(t)\, \underset{\sim}{g(t)}\, dt = g(0)$

any continuous fun

here $g(t) = e^{-ikt} \Rightarrow \int_{-\infty}^{\infty} \delta(t)\, e^{-ikt}\, dt = e^{-ik0} = e^{0} = 1$

plug back: $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \cdot 1 = \boxed{\frac{1}{\sqrt{2\pi}}}$

Problem 2

given:

$$p(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases} \qquad q(t) = \begin{cases} 0 & t < 0 \\ 1-t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

correlation def: $(p \odot q)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau)\, q(t+\tau)\, d\tau$

since $p(t)$ & $q(t)$ are real-valued func: $(p \odot q)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau)\, q(t+\tau)\, d\tau$

Integral Interval Analysis:

$p(\tau) \neq 0$ when $\tau \in [0, 1)$

$q(\tau + t) \neq 0$ when $t + \tau \in [0, 1) \rightarrow \tau \in (-t, 1-t)$

$\Rightarrow p(\tau)\, q(t+\tau) \neq 0$ when τ in $[\max(0, -t), \min(1, 1-t))$

In this interval $p(\tau) = 1$ (since constant)

$$q(t+\tau) = 1 - (t+\tau)$$

Integration interval is non empty when $\max(0,-t) < \min(1,1-t)$

→ Analy drift t:

① when $t \le -1$ or $t \ge 1$, the integration interval is empty or single point (which are measure zero) ⇒ Integral $= 0$

② when $-1 < t < 0$, Integration interval $[-t, 1]$

③ when $0 \le t < 1$, Integration interval $[0, 1-t]$

Case 1: $-1 < t < 0$

Integration Interval: $\tau \in [-t, 1]$

In Interval: $p(\tau) = 1$, $q(t+\tau) = 1-(t+\tau)$

since $t+\tau \in [0, t+1] \subseteq [0, 1)$

$$\int_{-t}^{1} [1-(t+\tau)]\, d\tau$$

$$= \int_{-t}^{1} (1-t-\tau)\, d\tau$$

$$= \left[(1-t)\tau - \frac{1}{2}\tau^2\right]_{-t}^{1}$$

$$= (1-t)(1) - \frac{1}{2}(1)^2 - \left[(1-t)(-t) - \frac{1}{2}(-t)^2\right]$$

$$= 1-t - \frac{1}{2} - \left(-t + t^2 - \frac{1}{2}t^2\right)$$

$$= -\frac{1}{2}t^2 + \frac{1}{2} = \frac{1}{2}(1-t^2) = \int p(\tau)\, q(t+\tau)\, d\tau$$

Case 2: $0 \leq t \leq 1$

Integration Interval: $\tau \in [0, 1-t]$

In interval: $p(\tau) = 1$, $q(t+\tau) = 1-(t+\tau)$

since $t+\tau \in [t, 1] \subseteq [0, 1)$

$$\int_0^{1-t} [1-(t+\tau)]\, d\tau$$

$$= \int_0^{1-t} (1-t-\tau)\, d\tau$$

$$= \left[(1-t)\tau - \tfrac{1}{2}\tau^2 \right]_0^{1-t}$$

$$= (1-t)(1-t) - \tfrac{1}{2}(1-t)^2 - \left[(1-t)\times 0 - \tfrac{1}{2}0^2\right]$$

$$= (1-t)^2 - \tfrac{1}{2}(1-t)^2$$

$$= \frac{1}{2}(1-t)^2$$

Case 3: $t \leq -1$ or $t \geq 1$

Single point or empty interval

$\longrightarrow$ Integration $= 0$

Plug into correlation:

$$(p \circ q)(t) = \frac{1}{\sqrt{2\pi}} \times \int p(\tau)\, q(t+\tau)\, d\tau$$

- $-1 \le t < 0$: $\int = \frac{1}{2}(1-t^2)$
- $0 \le t < 1$: $\int = \frac{1}{2}(1-t)^2$
- $|t| > 1$: $\int = 0$

At $t=0$: $\frac{1}{2}(1-0^2) = \frac{1}{2} = \frac{1}{2}(1-0)^2$

At end points: $t=-1$, $t=1$

- $t=-1$: $\frac{1}{2}(1-(-1)^2) = \frac{1}{2}(1-1) = 0$
- $t=1$: $\frac{1}{2}(1-1)^2 = 0$

Therefore,

$$(p \odot q)(t) = \frac{1}{\sqrt{2\pi}} \times \begin{cases} \frac{1}{2}(1-t^2) & \text{if } -1 \le t \le 0 \\ \frac{1}{2}(1-t)^2 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{= \begin{cases} \frac{1}{2\sqrt{2\pi}}(1-t^2) & \text{if } -1 \le t \le 0 \\ \frac{1}{2\sqrt{2\pi}}(1-t)^2 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

Verify :

① at $t = -0.5$

$$RHS = (p \odot q)(-0.5) = \frac{1}{2\sqrt{2\pi}}(1-(-0.5)^2) = \frac{1}{2\sqrt{2\pi}}(1-0.25)$$

$$= \frac{0.375}{\sqrt{2\pi}}$$

LHS: Integration Interval $\tau \in [0.5, 1]$

$$\int_{0.5}^{1} [1-(-0.5+\tau)]\, d\tau$$

$$= \int_{0.5}^{1} (1.5-\tau)\, d\tau$$

$$= \left[1.5\tau - \frac{\tau^2}{2}\right]_{0.5}^{1}$$

$$= 1.5 - 0.5 - 0.75 + 0.125 = 0.375$$

$$(p \odot q)(-0.5) = \frac{1}{\sqrt{2\pi}} \times 0.375 = RHS \checkmark$$

② Uncorrelation condition

When 2 functions are uncorrelated $p \odot q = \langle p \rangle \langle q \rangle$

$$\langle p \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p(\tau)\, d\tau = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{1} 1\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{2T} = 0$$

$$\langle q \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q(\tau)\, d\tau = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{1} (1-\tau)\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{2T} \cdot \frac{1}{2} = 0$$

$\langle p\rangle\langle q\rangle = 0$

However, $(p \odot q)(0) = \frac{1}{2\sqrt{2\pi}}(1-0^2) = \frac{1}{2\sqrt{2\pi}}(1-0)^2$

$= \frac{1}{2\sqrt{2\pi}} \neq 0$ depend on t

Therefore, two functions are corelated.

It satisfies expectation.

Problem 3

given $p(t) = \begin{cases} 0 & t<0 \\ 1 & 0 \le t \le 1 \\ 0 & t>1 \end{cases}$

$(p \odot p)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau)\, p(t+\tau)\, d\tau$

since $p(t)$ is real-valued func: $p^*(\tau) = p(\tau)$

$(p \odot p)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau)\, p(t+\tau)\, d\tau$

Integral Interval Analysis:

$p(\tau) \neq 0$: $\tau \in [0,1]$

$p(\tau+t) \neq 0$: $t+\tau \in [0,1] \to \tau \in [-t, 1-t]$

$p(\tau)\,p(t+\tau) \neq 0$ when: $0 \le \tau \le 1$ & $-t \le \tau \le 1-t$

$\rightarrow \tau \in [\max(0,-t),\ \min(1,\ 1-t)]$

① $t \le -1$, $t \ge 1$, empty interval or single point

② $-1 < t < 0$, $[-t, 1]$

③ $0 \le t \le 1$, $[0, 1-t]$

In the integral interval, $p(\tau) = 1$ & $p(\tau+t) = 1$

Case 1: $-1 < t < 0$

Integral interval: $\tau \in [-t, 1]$

$$\int p(\tau)\,p(t+\tau)\,d\tau = \int_{-t}^{1} 1\,d\tau = [\tau]_{-t}^{1} = 1+t$$

Case 2: $0 \le t \le 1$

Integral interval: $\tau \in [0, 1-t]$

$$\int p(\tau)\,p(t+\tau)\,d\tau = \int_{0}^{1-t} 1\,d\tau = [\tau]_{0}^{1-t} = 1-t$$

case 3: $t \le -1$, $t \ge 1$

$$\int p(\tau)\,p(t+\tau)\,d\tau = 0$$

plug in: $-1 < t < 0$: $\frac{1}{\sqrt{2\pi}}(1+t)$

$0 \le t \le 1 : \frac{1}{\sqrt{2\pi}}(1-t)$

$|t| > 1 = 0$

at end points = $t = -1$, $t = 1$

$t = -1 = 1 + (-1) = 0$

$t = 1 = 1 - 1 = 0$

Therefore:

$$\boxed{(p \odot p)(t) = \begin{cases} \frac{1+t}{\sqrt{2\pi}} & \text{if } -1 \le t < 0 \\ \frac{1-t}{\sqrt{2\pi}} & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

Verify:

① at $t = 0$

LHS = $(p \odot p)(0) = \frac{1-0}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$

RHS = $\int_{-\infty}^{\infty} p(\tau)\, p(\tau)\, d\tau = \int_0^1 1\, d\tau = [\tau]_0^1 = 1$

$\rightarrow (p \odot p)(0) = \frac{1}{\sqrt{2\pi}} \times 1 = \frac{1}{\sqrt{2\pi}} = \text{LHS}$

② At $t = 0.5$:

$$LHS = (p \otimes p)(0.5) = \frac{1-0.5}{\sqrt{2\pi}} = \frac{0.5}{\sqrt{2\pi}}$$

$$RHS = \int_{-\infty}^{\infty} p(\tau)\, p(\tau+0.5)\, d\tau = \int_0^{0.5} 1\, d\tau = [\tau]_0^{0.5} = 0.5$$

$$\rightarrow (p \otimes p)(0.5) = \frac{1}{\sqrt{2\pi}} \times 0.5 = \frac{0.5}{\sqrt{2\pi}} = LHS$$

Problem 4

a. $$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad \text{given } T(x, 0)$$

$$\underbrace{\tau(k,t)}_{} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T(x,t)\, e^{-ikx}\, dx$$

$\rightarrow$ Fourier transform of $T(x,t)$

$$\mathcal{F}\left\{\frac{\partial T}{\partial t}\right\} = k\, \mathcal{F}\left\{\frac{\partial^2 T}{\partial x^2}\right\}$$

LHS: assume smooth

$$\mathcal{F}\left\{\frac{\partial T}{\partial t}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial T}{\partial t}\, e^{-ikx}\, dx$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} T e^{-ikx}\, dx\right)$$

$$= \frac{\partial \tau}{\partial t}$$

RHS: use property $\mathcal{F}\{\partial_x^2 T\} = (-ik)^2 \tau(k,t)$

$$\mathcal{F}\left\{\frac{\partial^2 T}{\partial x^2}\right\} = (-ik)^2 \tau(k,t) = -k^2 \tau(k,t)$$

$$\rightarrow \kappa_0 \mathcal{F}\left\{\frac{\partial^2 T}{\partial x^2}\right\} = -\kappa_0 k^2 \tau(k,t)$$

Therefore: $\frac{\partial \tau}{\partial t} = -\kappa_0 k^2 \tau(k,t) \Rightarrow$ k spacing ode

$$\frac{1}{\tau(k,t)} \frac{\partial \tau(k,t)}{\partial t} = -\kappa_0 k^2$$

$$\frac{\partial}{\partial t}[\ln \tau(k,t)] = -\kappa_0 k^2$$

$$\Rightarrow \int_0^t \frac{\partial}{\partial s}[\ln \tau(k,t)]\, ds = \int_0^t -\kappa k^2\, ds$$

$$\Rightarrow \ln[\tau(k,t)] - \ln[\tau(k,0)] = -\kappa k^2 t$$

$$\Rightarrow \tau(k,t) = \tau(k,0)\, e^{-\kappa_0 k^2 t}$$

Inverse to original spacing:

$$T(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau(k,t)\, e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau(k,0)\, e^{-\kappa k^2 t}\, e^{ikx}\, dk$$

expand $\tau(k,0)$ in def:

$$\tau(k,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T(x',0)\, e^{-ikx'} dx'$$

$$\Rightarrow T(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T(x',0)\, e^{-ikx'} dx'\right] e^{-\kappa k^2 t} e^{ikx} dk$$

change order:

$$T(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T'(x',0) \left[\int_{-\infty}^{\infty} e^{-\kappa k^2 t} e^{ik(x-x')} dk\right] dx'$$

let $I = \int_{-\infty}^{\infty} e^{-\kappa k^2 t} e^{ik(x-x')} dk$

for power: $-\kappa k^2 t + i(x-x')k = -\kappa t\left(k^2 - \frac{i(x-x')}{\kappa t} k\right)$

where: $k^2 - \frac{i(x-x')}{\kappa t} k = \left(k - \frac{i(x-x')}{2\kappa t}\right)^2 - \left(\frac{i(x-x')}{2\kappa t}\right)^2$

$$\rightarrow -\kappa t\left(k^2 - \frac{i(x-x')}{\kappa t} k\right) = -\kappa t\left(k - \frac{i(x-x')}{2\kappa t}\right)^2 - \frac{(x-x')^2}{4\kappa t}$$

$$I = e^{-\frac{(x-x')^2}{4\kappa t}} \int_{-\infty}^{\infty} \exp\left(-\kappa t\left(k - \frac{i(x-x')}{2\kappa t}\right)^2\right) dk$$

let $u = k - \frac{i(x-x')}{2\kappa t}$ $\quad du = dk$

the integration contour is shifted in the complex plane, but this does not affect convergence (since the integrand is analytic and decays rapidly)

Standard Guassian Intergral:

$$\int_{-\infty}^{\infty} e^{-au^2} du = \sqrt{\frac{\pi}{a}} \quad a>0$$

here $a = \kappa t$: $\int_{-\infty}^{\infty} \exp(-\kappa t u^2)\, du = \sqrt{\frac{\pi}{\kappa t}}$

Therefore, $I = e^{-\frac{(x-x')^2}{4\kappa t}} \cdot \sqrt{\frac{\pi}{\kappa t}}$

Plug back into $T(x,t)$:

$$T(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(x',0) \cdot e^{-\frac{(x-x')^2}{4\kappa t}} \cdot \sqrt{\frac{\pi}{\kappa t}}\, dx'$$

$$\boxed{= \frac{1}{\sqrt{4\pi \kappa t}} \int_{-\infty}^{\infty} T(x',0) \exp\left(-\frac{(x-x')^2}{4\kappa t}\right) dx'}$$

This solution is convolution of the initial temperature distribution $T(x',0)$ with Gaussian kernel.

Where kernel: $\frac{1}{\sqrt{4\pi \kappa t}} \exp\left(-\frac{(x-x')^2}{4\kappa t}\right)$ describes the diffusion of heat over time. (assuming $T(x,0)$ is sufficiently smooth).

b.

given $\kappa = 10^3\ m^2/s$

$$T(x,0) = \begin{cases} 0 & |x| > 1m \\ 100°C & |x| = 1m \end{cases}$$

from a: $T(x,t) = \frac{1}{\sqrt{4\pi \kappa t}} \int_{-\infty}^{\infty} T(x',0) \exp\left(-\frac{(x-x')^2}{4\kappa t}\right) dx'$

plug in initial condition: $T(x',0) \neq 0$ only $|x'| = 1$

$$T(x,t) = \frac{1}{\sqrt{4\pi \kappa t}} \int_{-1}^{1} 100 \exp\left(-\frac{(x-x')^2}{4\kappa t}\right) dx'$$

$$= \frac{100}{\sqrt{4\pi \kappa t}} \int_{-1}^{1} \exp\left(-\frac{(x-x')^2}{4\kappa t}\right) dx'$$

$$= \frac{50}{\sqrt{\pi \kappa t}} \int_{-1}^{1} \exp\left(-\frac{(x-x')^2}{4\kappa t}\right) dx'$$

let $u = \frac{x'-x}{\sqrt{4\kappa t}} = \frac{x'-x}{2\sqrt{\kappa t}}$

$du = \frac{dx'}{2\sqrt{\kappa t}} \quad dx' = 2\sqrt{\kappa t}\, du$

when $x' = -1$: $u = \frac{-1-x}{2\sqrt{\kappa t}} = -\frac{1+x}{2\sqrt{\kappa t}}$

when $x' = 1$: $u = \frac{1-x}{2\sqrt{\kappa t}}$

$$\Rightarrow \int_{-1}^{1} \exp\left(-\frac{(x-x')^2}{4\kappa t}\right) dx' = \int_{-\frac{1+x}{2\sqrt{\kappa t}}}^{\frac{1-x}{2\sqrt{\kappa t}}} e^{-u^2} \cdot 2\sqrt{\kappa t}\, du$$

since error function = $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$

$$\rightarrow \int_a^b e^{-u^2} du = \frac{\sqrt{\pi}}{2} (\operatorname{erf}(b) - \operatorname{erf}(a))$$

$$\rightarrow \int_{-\frac{1+x}{2\sqrt{kt}}}^{\frac{1-x}{2\sqrt{kt}}} e^{-u^2} du = \frac{\sqrt{\pi}}{2} [\operatorname{erf}(\frac{1-x}{2\sqrt{kt}}) - \operatorname{erf}(-\frac{1+x}{2\sqrt{kt}})$$

since error function is odd:

$$\operatorname{erf}(-z) = -\operatorname{erf}(z):$$

$$-\operatorname{erf}(-\frac{1+x}{2\sqrt{kt}}) = \operatorname{erf}(\frac{1+x}{2\sqrt{kt}})$$

$$\Rightarrow \int = \frac{\sqrt{\pi}}{2} [\operatorname{erf}(\frac{1-x}{2\sqrt{kt}}) + \operatorname{erf}(\frac{1+x}{2\sqrt{kt}})]$$

plug into original:

$$\int_{-1}^{1} \exp(-\frac{(x-x')^2}{4kt}) dx' = 2\sqrt{kt} \cdot \frac{\sqrt{\pi}}{2} [\operatorname{erf}(\frac{1-x}{2\sqrt{kt}}) + \operatorname{erf}(\frac{1+x}{2\sqrt{kt}})]$$

$$T(x,t) = \frac{50}{\sqrt{\pi k t}} \cdot \sqrt{\pi}\sqrt{kt} [\operatorname{erf}(\frac{1-x}{2\sqrt{kt}}) + \operatorname{erf}(\frac{1+x}{2\sqrt{kt}})]$$

$$\boxed{= 50 \cdot [\operatorname{erf}(\frac{1-x}{2\sqrt{kt}}) + \operatorname{erf}(\frac{1+x}{2\sqrt{kt}})]}$$

where $k = 10^3\ m^2/s$, $x \rightarrow$ meter, $t \rightarrow$ second

Verify:

1. $t \to 0^+$

If $|x| < 1$: $\lim\limits_{t \to 0^+} \frac{1 \pm x}{2\sqrt{kt}} \to \infty \Rightarrow erf(\infty) = 1$

$$T(x, 0^+) = 50[1+1] = 100°C$$

If $|x| > 1$: for example $x > 1$

$$\lim_{t \to 0^+} \frac{1-x}{2\sqrt{kt}} \to -\infty \Rightarrow erf(-\infty) = -1$$

$$\lim_{t \to 0^+} \frac{1+x}{2\sqrt{kt}} \to \infty \Rightarrow erf(\infty) = 1$$

$$T(x, 0^+) = 50[-1+1] = 0$$

This satisfy initial condition.

2. $|x| \to \infty$:

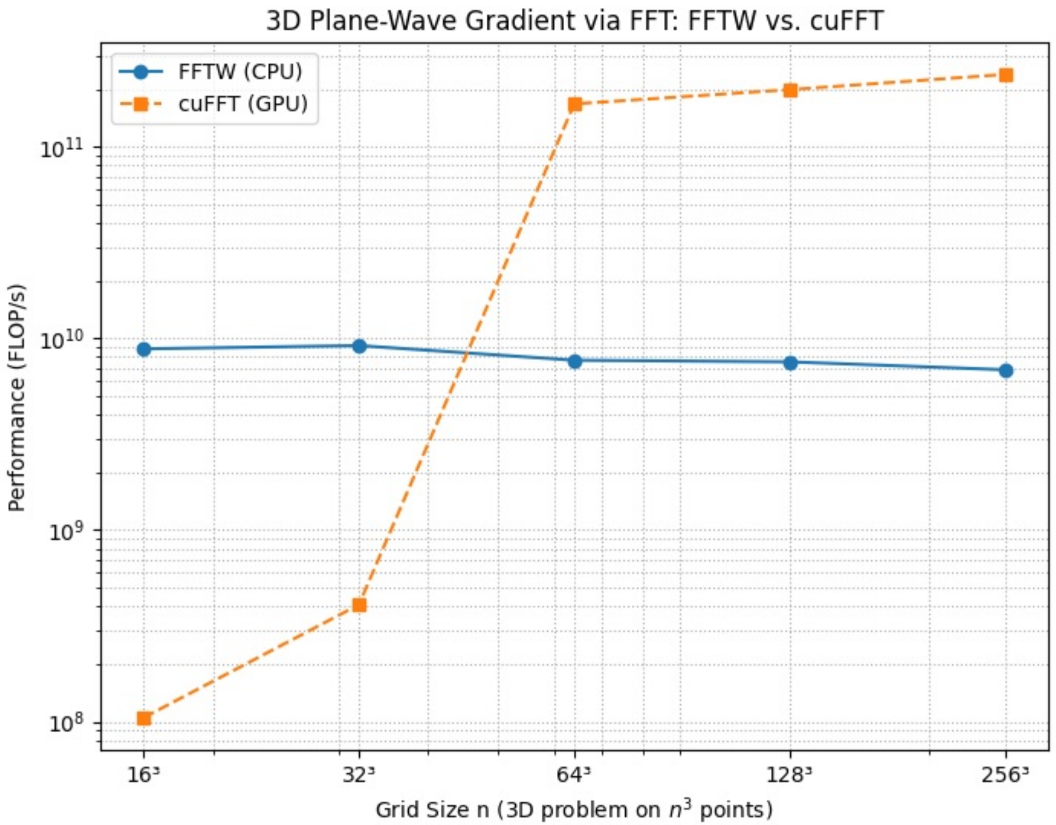
$$\lim_{|x| \to \infty} \frac{1 \pm x}{2\sqrt{kt}} \to \pm\infty \Rightarrow erf(\infty) + erf(-\infty)$$

$$\Rightarrow 1 + (-1) = 0 \Rightarrow T \to 0$$

3. $t \to \infty$:

$$\lim_{t \to \infty} \frac{1 \pm x}{2\sqrt{kt}} \to 0 \Rightarrow erf(0) = 0 \Rightarrow T \to 0$$

Problem 5



3D Plane-Wave Gradient via FFT: FFTW vs. cuFFT
FFTW (CPU)
cuFFT (GPU)
Performance (FLOP/s)
10^11
10^10
10^9
10^8
16³
32³
64³
128³
256³
Grid Size n (3D problem on n³ points)