HW 1

1.

a. $\int_0^{\frac{\pi}{2}} \ln \sin(x)\, dx$

let $x = \frac{\pi}{2} - x$

$= \int_0^{\frac{\pi}{2}} \ln(\sin(\frac{\pi}{2} - x))\, dx$

$= \int_0^{\frac{\pi}{2}} \ln(\cos x)\, dx \quad = I$

Therefore,

$2I = \int_0^{\frac{\pi}{2}} \ln(\sin x) + \int_0^{\frac{\pi}{2}} \ln(\cos x)\, dx$

$= \int_0^{\frac{\pi}{2}} [\ln(\sin x) + \ln(\cos x)]\, dx$

$= \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)\, dx$

$= \int_0^{\frac{\pi}{2}} \ln(\frac{2 \sin x \cos x}{2})\, dx$

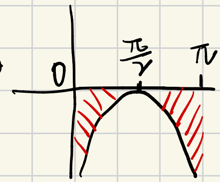
$= \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx - \int_0^{\frac{\pi}{2}} \ln(2)\, dx$

$$\int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx$$

$u = 2x \quad du = 2\, dx$

$$= \frac{1}{2}\int_0^{\pi} \ln(\sin u)\, du$$

$$= \frac{1}{2}\int_0^{\pi} \ln(\sin x)\, dx$$



$\rightarrow$ symmetric at $\frac{\pi}{2}$

$$= \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = I$$

$$\int_0^{\frac{\pi}{2}} \ln(2)\, dx$$

$$= \left[\ln(2)\, x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\ln 2$$

$$2I = I - \frac{\pi}{2}\ln 2$$

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = -\frac{\pi}{2}\ln(2)$$

b. $$I = \int_0^{\pi} \frac{x \sin(x)}{1 + \cos^2(x)}\, dx$$

let $x = \pi - x$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\, dx$$

$$\cos(\pi - x) = -\cos(x)$$

$$\cos^2(\pi - x) = (-\cos(x))^2 = \cos^2(x)$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(x)}\, dx$$

$$2I = \int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)} dx + \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(x)} dx$$

$$\sin(\pi - x) = \sin(x)$$

$$2I = \int_0^{\pi} \frac{x\sin(x) + \pi\sin(x) - x\sin x}{1+\cos^2 x} dx$$

$$= \int_0^{\pi} \frac{\pi \sin(x)}{1+\cos^2 x}\, dx$$

let $u = \cos x$ $\quad \cos 0 = 1 \quad \cos\pi = -1$

$$du = -\sin x\, dx \to -du = \sin x\, dx$$

$$= \pi \int_1^{-1} \frac{-1}{1+u^2}\, du$$

$$= \pi \int_{-1}^{1} \frac{1}{1+u^2}\, du$$

$$= \pi \left[\tan^{-1} u\right]_{-1}^{1}$$

$$= \pi \left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$= \pi\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{2} = 2I$$

$$I = \int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\,dx = \frac{\pi^2}{4}$$

c. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx = I$

let $x = \frac{\pi}{2} - x$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan(\frac{\pi}{2}-x))^{\sqrt{2}}}\,dx$$

$$(\tan(x))^{\sqrt{2}} = (\tan(\tfrac{\pi}{2}-x))^{-\sqrt{2}}$$

$$(\tan(x))^{-\sqrt{2}} = (\tan(\tfrac{\pi}{2}-x))^{\sqrt{2}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{(\tan(x))^{\sqrt{2}}}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\tan(x))^{\sqrt{2}}}{(\tan(x))^{\sqrt{2}}+1}\,dx = I$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx + \int_0^{\frac{\pi}{2}} \frac{(\tan(x))^{\sqrt{2}}}{(\tan(x))^{\sqrt{2}}+1}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+(\tan(x))^{\sqrt{2}}}{1+(\tan(x))^{\sqrt{2}}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$= \left[x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} = 2I$$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx = \frac{\pi}{4}$$

2.

a. find $a, b \rightarrow (1+i\sqrt{3})^{11} = a + ib$

let $z = 1 + i\sqrt{3}$

Convert to polar form:

$$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$z = r e^{i\theta} = 2 e^{\frac{\pi}{3} i}$$

$$(1+i\sqrt{3})^{11} = z^{11} = \left(2e^{\frac{\pi}{3}i}\right)^{11} = 2^{11} \cdot e^{\frac{11}{3}\pi i}$$

$$= 2^{11}\left(\cos\left(\frac{11}{3}\pi\right) + i\sin\left(\frac{11}{3}\pi\right)\right)$$

$$= 2^{11} \cdot \cos\left(\frac{11}{3}\pi\right) + i\,2^{11} \sin\left(\frac{11}{3}\pi\right)$$

$$\cos\left(\frac{11}{3}\pi\right) = \cos\left(\frac{11}{3}\pi - 4\pi\right) = \cos\left(-\frac{1}{3}\pi\right)$$

$$= \cos\left(\frac{1}{3}\pi\right) = \frac{1}{2}$$

$$\sin\left(\frac{11}{3}\pi\right) = \sin\left(-\frac{1}{3}\pi\right)$$

$$= -\sin\left(\frac{1}{3}\pi\right) = -\frac{\sqrt{3}}{2}$$

$$(1+i\sqrt{3})^{11} = 2^{11}\cdot\frac{1}{2} - 2^{11}\cdot\frac{\sqrt{3}}{2}$$

$$= 2^{10} - 2^{10}\sqrt{3}$$

$a = 2^{10}$ $\quad b = -2^{10}\sqrt{3}$

b. find $(1+i\sqrt{3})^{\frac{1}{5}}$

follow the same process in a,

$z = 1+i\sqrt{3} \rightarrow z = 2e^{\frac{\pi}{3}i}$

to find $z^{\frac{1}{5}}$, but the complex number has multiple roots. The fifth root has 5 distinct values, spaced equally around the unit circle.

$z = re^{i\theta} \Rightarrow z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)}$ $\quad k=0,\cdots n-1$

$$z^{\frac{1}{5}} = 2^{\frac{1}{5}}\cdot e^{i\left(\frac{\frac{1}{3}\pi + 2\pi k}{5}\right)}$$

$$= 2^{\frac{1}{5}}\cdot e^{(\frac{1}{15}\pi + \frac{2}{5}\pi k)i} \quad k=0,1\cdots,4$$

$k=0 \quad 2^{\frac{1}{5}} \cdot e^{\frac{1}{15}\pi i}$

$k=1 \quad 2^{\frac{1}{5}} \cdot e^{(\frac{1}{15}\pi + \frac{2}{5}\pi) i} = 2^{\frac{1}{5}} \cdot e^{\frac{7}{15}\pi i}$

$k=2 \quad 2^{\frac{1}{5}} \cdot e^{(\frac{1}{15}\pi + \frac{4}{5}\pi) i} = 2^{\frac{1}{5}} \cdot e^{\frac{13}{15}\pi i}$

$k=3 \quad 2^{\frac{1}{5}} \cdot e^{(\frac{1}{15}\pi + \frac{6}{5}\pi) i} = 2^{\frac{1}{5}} \cdot e^{\frac{19}{15}\pi i}$

$k=4 \quad 2^{\frac{1}{5}} \cdot e^{(\frac{1}{15}\pi + \frac{8}{5}\pi) i} = 2^{\frac{1}{5}} \cdot e^{\frac{5}{3}\pi i}$

c. $w \rightarrow \quad w = w^{4/3} + 2i = 0$

$w^{4/3} = -2i$

$w = (-2i)^{3/4}$

let $z = -2i$

$|z| = \sqrt{(-2)^2} = 2$

$\theta = -\frac{\pi}{2}$

same logic in b, we should have

four roots when take $\frac{1}{4}$.

$$z = 2e^{-\frac{\pi}{2}i}$$

If $z^{\frac{3}{4}} = ae^{bi}$ for some complex numbers

$$z = a^{\frac{4}{3}} e^{b\cdot\frac{4}{3}i}$$

$$\to a^{\frac{4}{3}} = 2 \to a = 2^{\frac{3}{4}}$$

$$\to b\cdot\frac{4}{3} = -\frac{\pi}{2}.$$

$$\frac{4}{3}b = -\frac{\pi}{2} + 2\pi k \quad k = 0, 1, 2, 3$$

$$b = -\frac{3}{8}\pi + \frac{6}{4}\pi k$$

$$W = z^{\frac{3}{4}} = 2^{\frac{3}{4}} \cdot e^{(-\frac{3}{8}\pi + \frac{12}{8}\pi k)i}$$

$k = 0$, $\quad w_1 = 2^{\frac{3}{4}} \cdot e^{-\frac{3}{8}\pi i}$

$k = 1$, $\quad w_2 = 2^{\frac{3}{4}} \cdot e^{\frac{9}{8}\pi i}$

$k = 2$, $\quad w_3 = 2^{\frac{3}{4}} \cdot e^{\frac{21}{8}\pi i}$

$k=3$, $\quad w_4 = 2^{\frac{3}{4}} \cdot e^{\frac{33}{8}\pi i}$

check result:

$$w_1 = 2 \cdot e^{-\frac{1}{2}\pi i} + 2i$$

$$= 2 \cdot (\cos(-\tfrac{\pi}{2}) + i\sin(-\tfrac{\pi}{2})) + 2i$$

$$= 2 \cdot (0 - i) + 2i$$

$$= 0$$

$$w_2 = 2 \cdot e^{\frac{3}{2}\pi i} + 2i$$

$$= 2 \cdot (\cos(\tfrac{3}{2}\pi) + i\sin(\tfrac{3}{2}\pi)) + 2i$$

$$= 2 \cdot (0 - i) + 2i$$

$$= 0$$

$$w_3 = 2 \cdot e^{\frac{7}{2}\pi i} + 2i$$

$$= 2 \cdot (\cos(\tfrac{7}{2}\pi) + i\sin(\tfrac{7}{2}\pi)) + 2i$$

$$= 2 \cdot (0 - i) + 2i$$

$$= 0$$

$$w_4 = 2 \cdot e^{\frac{11}{2}\pi i} + 2i$$

$$= 2 \cdot (\cos(\tfrac{11}{2}\pi) + i\sin(\tfrac{11}{2}\pi)) + 2i$$

$$= 2 \cdot (0 - i) + 2i$$

$$= 0$$

3.

a. $1 + 10^{-2} + 10^{-4} + 10^{-6} + \cdots$

$= 1 + \frac{1}{(10)^2} + \frac{1}{(10^2)^2} + \frac{1}{(10^2)^3} + \cdots$

$= 1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots$

$= \left(\frac{1}{100}\right)^0 + \left(\frac{1}{100}\right)^1 + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots$

$= \sum_{i=0}^{\infty} \left(\frac{1}{100}\right)^i$

It is a geometric series with

a_n - first term $= 1$

r - common ratio $= \frac{1}{100}$

since $|r| < 1$

$= \frac{a_n}{1-r}$

$= \frac{1}{1 - \frac{1}{100}}$

$= \frac{100}{99}$

b. $376.376376\ldots$

$$= 376 \times \underbrace{1.001001\ldots}_{1 + 0.001 + 0.000001 + \cdots}$$

$$= 1 + 10^{-3} + 10^{-6} + \cdots$$

$$= 1 + \left(\frac{1}{1000}\right)^{1} + \left(\frac{1}{1000}\right)^{2} + \cdots$$

$$= \sum_{i=0}^{\infty} \left(\frac{1}{1000}\right)^{i}$$

geometric series $a_n = 1$ $r = \frac{1}{1000} < 1$

$$= \frac{1}{1 - \frac{1}{1000}}$$

$$= \frac{1000}{999}$$

$$376.376\ldots = \frac{376000}{999}$$

c. $0.999\overline{9}$

$= 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

$= 9 \times 10^{-1} + 9 \times 10^{-2} + 9 \times 10^{-3} + 9 \times 10^{-4} + \cdots$

$= 9 \times \left(\frac{1}{10}\right) + 9 \times \left(\frac{1}{10}\right)^2 + 9 \times \left(\frac{1}{10}\right)^3 + 9 \times \left(\frac{1}{10}\right)^4 + \cdots$

$= 9 \sum_{i=1}^{\infty} \left(\frac{1}{10}\right)^i$

$= 9 \underbrace{\sum_{i=0}^{\infty} \left(\frac{1}{10}\right)^i}_{} - 9$

geometric series $a_0 = 1$, $r = \frac{1}{10} < 1$

$= 9 \times \frac{1}{1 - \frac{1}{10}} - 9$

$= 9 \times \frac{10}{9} - 9$

$= 10 - 9$

$= 1$

4. Use secant approximation

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = hf'(x) + f(x)$$

a. $(1.1)^{\frac{1}{3}}$

let $f(x) = x^{1/3}$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$$

let $x+h = 1.1$, $x = 1$, $h = 0.1$

$$f(1) = 1^{1/3} = 1$$

$$f'(1) = \frac{1}{3} \cdot 1^{-2/3} = \frac{1}{3}$$

$$f(1.1) = f(1+0.1)$$

$$= 0.1 \cdot \frac{1}{3} + 1$$

$$= \frac{1}{10} \times \frac{1}{3} + 1$$

$$= \frac{31}{30}$$

6. $\sqrt{8.5}$

let $f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

let $x + h = 8.5$, $x = 9$, $h = -0.5$

$f(x) = \sqrt{9} = 3$

$f'(x) = \frac{1}{2} \times \frac{1}{\sqrt{9}} = \frac{1}{6}$

$f(8.5) = f(9 + (-0.5))$

$= (-0.5) \times \frac{1}{6} + 3$

$= -\frac{1}{12} + 3$

$= \frac{35}{12}$

5. given $(x+y+z)^7$, find expansion coff based on multinomial theorem =

$$(x+y+z)^7 = \sum_{\substack{i+j+k=7 \\ i,j,k \geq 0}} \frac{7!}{i!\,j!\,k!} x^i y^j z^k$$

a. $x^2 y^2 z^3$

$i=2$, $j=2$, $k=3$

$$\text{coefficient} = \frac{7!}{2!\,2!\,3!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3!}{(2 \times 1) \times (2 \times 1) \times 3!}$$

$$= 210$$

b. $x^3 z^4$

$i = 3, \; j = 0, \; k = 4$

coefficient $= \dfrac{7!}{3! \cdot 0! \cdot 4!}$

$= \dfrac{7 \times 6 \times 5 \times 4!}{(3 \times 2 \times 1) \cdot 1 \cdot 4!}$

$= 35$

6. given $(x+2y-3z+2w+5)^{16}$, find coff

Multinomial theorem =

$$(a_1+a_2+\cdots a_k)^n = \sum \frac{n!}{k_1!k_2!\cdots k_m!} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}$$

$k_1+k_2+\cdots k_m = n$

a. $x^2 y^3 z^2 w^5$

let $a_1 = x$

$a_2 = 2y$

$a_3 = -3z$ $\Rightarrow (a_1+a_2+a_3+a_4+a_5)^{16}$

$a_4 = 2w$

$a_5 = 5$

here, $k_1 = 2$

$k_2 = 3$

$k_3 = 2$

$k_4 = 5$

$$k_5 = 16 - k_1 - k_2 - k_3 - k_4 = 4$$

$$\longrightarrow \left(\frac{16!}{2! \cdot 3! \cdot 2! \cdot 5! \cdot 4!} \right)$$

$$= \frac{16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5!}{(2\times1) \times (3\times2\times1) \times (2\times1) \times 5! \times (4\times3\times2\times1)}$$

$$= 8 \times 15 \times 7 \times 13 \times 11 \times 5 \times 3 \times 4 \times 7 \times 6$$

$$= 3027024000$$

also, $a_1{}^2 = x^2$

$a_2{}^3 = 8y^3$

$a_3{}^2 = 9z^2$

$a_4{}^5 = 32w^5$

$a_5{}^4 = 5^4 = 625$

$$\text{coeff} = 3027024000 \times 1 \times 8 \times 9 \times 32 \times 625$$

$$= 4358914560000000$$

7.

a. prove $8n+3$ and $5n+2$ are relatively prime

$\rightarrow$ prove $\gcd(8n+3, 5n+2) = 1$

Use Euclidean Algorithm =

$$8n+3 - (5n+2) = 3n+1 > 0$$

Therefore $8n+3 > 5n+2$

$$8n+3 = (5n+2) \cdot 1 + 3n+1 \quad 0 < 3n+1 < 5n+2$$

$$5n+2 = (3n+1) \cdot 1 + 2n+1 \quad 0 < 2n+1 < 3n+1$$

$$3n+1 = (2n+1) \cdot 1 + n \quad 0 < n < 2n+1$$

$$2n+1 = n \cdot 2 + 1 \quad 0 < 1 < n$$

$$n = 1 \cdot n + 0$$

The last non-zero remainder is 1

Therefore, $\gcd(8n+3, 5n+2) = 1$ for any $n \in \mathbb{Z}^+$

b. find gcd(250, 111)

Similar with a, given $250 > 111$

$$250 = 2 \times 111 + 28 \quad 0 < 28 < 111$$
$$111 = 3 \times 28 + 27 \quad 0 < 27 < 28$$
$$28 = 1 \times 27 + 1 \quad 0 < 1 < 27$$
$$27 = 27 \times 1 + 0$$

Therefore, $\gcd(250, 111) = 1$

Back - substitute:

$$1 = 28 - 27$$
$$= 28 - (111 - 3 \cdot 28)$$
$$= 4 \cdot 28 - 1 \cdot 111$$
$$= 4 \cdot (250 - 2 \times 111) - 1 \cdot 111$$
$$= 4 \cdot 250 - 9 \cdot 111$$

$\gcd(250, 111) = 1 = 4 \cdot 250 - 9 \cdot 111$

8. write the prime factorization of 980220

$980220 = 2 \times (490110)$

$= 2 \times 2 \times (245055)$ try from 2, 3

try from 2, 3, 5 $= 2 \times 2 \times 3 \times (81685)$

$= 2 \times 2 \times 3 \times 5 \times (16337)$ try from 2, 3, 5, 7, 11, 13, 17

try from 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 $= 2 \times 2 \times 3 \times 5 \times 17 \times (961)$

$= 2 \times 2 \times 3 \times 5 \times 17 \times 31 \times (31)$ → last prime

$= 2^2 \times 3 \times 5 \times 17 \times 31^2$