Code Problems (include problem1, problem3, check for 2&4)

```cpp
// OUTPUT
// [Running] cd "/Users/liaoziyu/Desktop/课程文件/大三/Amath 483/assignment/assignment 2/" && g++
// Problem 1:
// Single precision machine epsilon (float, 32-bit) = 1.192092895507812e-07
// Double precision machine epsilon (double, 64-bit) = 2.220446049250313e-16
// Problem 3:
// Mutiply Result: -884901888
// Check Value for Problem 2:
// ---- FP 32 ----
// Largest:3.402823466385289e+38
// Normalized Smallest:1.175494350822288e-38
// Denormalized Smallest:1.401298464324817e-45
// ---- FP 64 ----
// Largest:1.797693134862316e+308
// Normalized Smallest:2.225073858507201e-308
// Denormalized Smallest:4.940656458412465e-324
// Check Value for Problem 4:
// Final Value of counter:4294967293

// [Done] exited with code=0 in 0.387 seconds

#include <iostream>
#include <iomanip>
#include <limits>

int main()
{
    // Problem 1
    std::cout << "Problem 1:" << std::endl;
    // float, 32-bit
    int j_float = 0;
    float epsilon_float;
    while (true)
    {
        float test_val = 1.0f + 1.0f / (1u << j_float);
        if (test_val == 1.0f)
        {
            break;
        }
        j_float++;
    }
    epsilon_float = 1.0f / (1u << (j_float - 1));
    std::cout << "Single precision machine epsilon (float, 32-bit) = "
        << std::setprecision(16) << epsilon_float << std::endl;

    // double, 64-bit
    int j_double = 0;
    double epsilon_double;
    while (true)
    {
        double test_val = 1.0 + 1.0 / (1ULL << j_double);
        if (test_val == 1.0)
        {
            break;
        }
        j_double++;
    }
```

```cpp
    epsilon_double = 1.0 / (1ULL << (j_double - 1));
    std::cout << "Double precision machine epsilon (double, 64-bit) = "
        << std::setprecision(16) << epsilon_double << std::endl;

    // Problem 3
    std::cout << "Problem 3:" << std::endl;
    int result = 200 * 300 * 400;
    std::cout << "Mutiply Result: " << result * 500 << std::endl; // avoid "overflow" warning

    // Check Value for Problem 2
    std::cout << "Check Value for Problem 2:" << std::endl;
    std::cout << "---- FP 32 ----" << std::endl;
    std::cout << "Largest:" << std::numeric_limits<float>::max() << std::endl;
    std::cout << "Normalized Smallest:" << std::numeric_limits<float>::min() << std::endl;
    std::cout << "Denormalized Smallest:" << std::numeric_limits<float>::denorm_min() << std::endl;

    std::cout << "---- FP 64 ----" << std::endl;
    std::cout << "Largest:" << std::numeric_limits<double>::max() << std::endl;
    std::cout << "Normalized Smallest:" << std::numeric_limits<double>::min() << std::endl;
    std::cout << "Denormalized Smallest:" << std::numeric_limits<double>::denorm_min() << std::endl;

    // Check Value for Problem 4
    std::cout << "Check Value for Problem 4:" << std::endl;
    unsigned int counter = 0;
    for (int i = 0; i < 3; ++i)
        --counter;
    std::cout << "Final Value of counter:" << counter << std::endl;

    return 0;
}
```

Handwrite Problems

Problem 2

Single Precision 32-bit

I choose all sign be 0, meaning the positive here, because if we only consider the sign, the smallest number actually is the same as the biggest number in magnitude. It's symmteric. To discusse the smallest number represented by IEEE, we use a same sign (0)

1. Format

sign bit (S) = 1 bit

exponent bits (E) = 8 bits

fraction bits (F) = 23 bits

Value of a normalized number =

$$\text{Value} = (-1)^S \times (1.F) \times 2^{(E_{decimal} - 127)}$$

$(-1)^0 = 1$; $1 - 254$

2. Largest normalized FP32

1) sign = $S = 0$ → positive number

2) exponent =

max = 254 = 11111110

$E_{actual} = 254 - 127 = 127$

3) fraction =

max → 23 bits all set 1

$F = 11111111111111111111111$

$0.F = 0.11111111111111111111111$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{23}}$

$= 1 - 2^{-23}$

$1.F = 1 + (1 - 2^{-23}) = 2 - 2^{-23}$

$S_n = \sum_{i=1}^{n} \frac{1}{2^i}$

first term $= \frac{1}{2}$

common ration $= \frac{1}{2}$

$S_n = \frac{1}{2} \cdot \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}}$

$= 1 - (\frac{1}{2})^n$

$= 1 - 2^{-n}$

4) value $= (-1)^0 \cdot (2 - 2^{-23}) \times 2^{127}$

$= 3.4028234664 \times 10^{38}$

Binary $= 0 \mid 11111110 \mid 11111111111111111111111$

3. Smallest normalized FP 32

1) Sign : $S = 0 \rightarrow$ positive

2) Exponent :

min $= 1 = 00000001$

$E_{actual} = 1 - 127 = -126$

3) Fraction :

min $\rightarrow$ all 0

$F = 0 \rightarrow 0.F = 0 \rightarrow 1.F = 1.0$

4) Value $= (-1)^0 \cdot 1.0 \times 2^{-126} = 1.17549435108 \times 10^{-38}$

Binary $= 0 \mid 00000001 \mid 00000000000000000000000$

4. Smallest Denormalized FP 32

$E = 0 \rightarrow$ fixed value $= 2^{-126}$

no implicit 1

1) sign $=$ $S = 0$

2) exponent: $E = 0$, binary $= 00000000$

3) fraction $=$

$F = 00000000000000000000001$

$0.F = 2^{-23}$

4) value $= 2^{-23} \times 2^{-126} = 2^{-149}$

$= 1.4012984643 \times 10^{-45}$

Binary $=$ 0 | 00000000 | 00000000000000000000001

5. Summary for FP 32 if take positive sign

① Largest $= 3.4028234664 \times 10^{38}$

② normal smallest $= 1.1754943508 \times 10^{-38}$

③ denormal smallest: $1.4012984643 \times 10^{-45}$

Double Precision 64-bit:

1. Format

sign bit = 1 bit → I take positive $s=0$

exponent bit : 11 bits

normalized E from 1 to 2046

bias is 1023

fraction bit = 52 bits with an implicit 1 for normalized number.

$$\text{value} = \underset{\substack{\downarrow \\ (-1)^0 = 1}}{(-1)^s} \times (1.F) \times 2^{(E-1023)}$$

2. Largest normalized FP 64

1) sign = $S = 0$ → positive

2) exponent:

max → $E = 2046$

Binary : 11111111110

actual $e = 2046 - 1023 = 1023$

3) fraction =

max → set all 1

$F = 1111111111111111111111111111111111111111111111111111$

$0.F = 0.1111111111111111111111111111111111111111111111111111$

$= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{52}}$

$= 1 - 2^{-52}$

$1.F = 2 - 2^{-52}$

4) value $= (2 - 2^{-52}) \times 2^{1023}$

$= 1.7976931349 \times 10^{308}$

Binary : 0 | 11111111110 | 1111111111111111111111111111111111111111111111111111

3. smallest normalized FP 64

1) sign : $S = 0$ → positive

2) exponent : min → $E = 1$ → 00000000001

actual $E = 1 - 1023 = -1022$

3) fraction =

min → all 0 → $F = 0$

$1.F$ → 1.0

4)

$$\text{value} = 1.0 \times 2^{-1022}$$

$$= 2.2250738585 \times 10^{-308}$$

Binary: 0 | 00000000001 | 0000000000000000000000000000000000000000000000000000

4. Smallest denormalized FP64

1) sign: $s = 0 \rightarrow$ positive

2) exponent: $E = 0 \rightarrow 00000000000$

fixed: $1 - 1023 = -1022$

no implicit leading 1

3) fraction: least significant in F to 1

$F = 0000000000000000000000000000000000000000000000000001$

$$0.F = 2^{-52}$$

4) $$\text{value} = 2^{-52} \times 2^{-1022} = 2^{-1074}$$

$$= 4.9406564584 \times 10^{-324}$$

Binary = 0 | 00000000000 | 0000000000000000000000000000000000000000000000000001

5. Summary for FP64 if take positive value

① Largest $= 1.7976931349 \times 10^{308}$

② normal smallest = $2.2250738585 \times 10^{-308}$

③ denormal smallest : $4.9406564584 \times 10^{-324}$

Problem 3

The result is a wrong value : -884901888

I get a warning of "overflow".

It is an integer overflow effect.

Problem 4

given the counter is a unsigned int,

Its ranger from 0 to $2^{32}-1$.

$i=0$ counter $-1 = 0-1$

the wrap-around happens

$\rightarrow$ counter $= 2^{32}-1$

— re-begin from the max value

$i=1$ counter $-1 = 2^{32}-1-1 = 2^{32}-2$

$i=2$ counter $-2 = 2^{32}-2-1 = 2^{32}-3$

$= 4294967293$

Problem 5

IEEE 754 Single - Precision

| S (1 bit) | E(8 bits) | F(23 bits) |

↓ Sign Bit
0 : positive
1 = negative

↓ Exponent
value between
0 ~ 255

↓ Fraction

Total = 2^{32} = 4294967296

1. Normalized Numbers

Exponent range: 1 to 254 , no 0 or 255 ⇒ 254

Fraction = 23 bit → 2^{23} possible pattern

Sign = 2 (0 or 1)

Total = $2 \times 254 \times 2^{23} = 4261412864$

2. Denormalized Numbers

Exponent = must be 0

Fraction = range from 1 to $2^{23}-1$

Sign = 2

Total = $2 \times (2^{23}-1) = 16777214$

3. Zeros

E stored = 0, F = 0, sign = 2

Total : 2

4. Infinities

E stored = 255, F = 0, sign = 2

Total : 2

6. NaNs → Not a Number

E stored = 255

Fraction : not 0, from 1 to $2^{23}-1$

Sign = 2

Total : $1 \times 2 \times (2^{23}-1) = 16777214$

Check: $4261412864 + 16777214 + 2 + 2 + 16777214$

$= 4294967296 = 2^{32} = \text{total}$

Problem 6

Based on data structure provided in the lectures

6 bit representation

$s = 1,\ k = 3,\ n = 2$

$V = \pm M \cdot 2^E$

$E,\ k = 3$

$M, f,\ n = 2$

$\text{bias},\ 2^{k-1} - 1 = 3$

normalized $\quad E = e - \text{bias} \quad\quad M = 1 + f$

$= (e_2 e_1 e_0) - \text{bias} \quad\quad f \sim f_{n_1} f_{n_0}$

$.f_{n_1} f_{n_0}$

$\{0,1\} \cdot \frac{1}{2^1} \quad \{0,1\} \cdot \frac{1}{2^2}$

$e_2\ e_1\ e_0$	e	$e - \text{bias}$	$n_1\ n_0$	f
0 0 0	0	-2	0 0	0
0 0 1	1	-2	0 1	$1 \cdot \frac{1}{2^2} = \frac{1}{4}$
0 1 0	2	-1	1 0	$1 \cdot \frac{1}{2} = \frac{1}{2}$
0 1 1	3	0	1 1	$1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2^2} = \frac{3}{4}$
1 0 0	4	1		
1 0 1	5	2		
1 1 0	6	3		
1 1 1	7	$\pm\infty$		

$M_{00} = 0 \rightarrow 0 \cdot 2^{-2}$

000 $\rightarrow$ denormalized, $M = f$, bias = 3, $E_{000} = 1 - 3 = -2$

111, $\pm\infty$

$M_{00} = 1 + 0 = 1 \rightarrow$

$1 \cdot 2^{-2} = \frac{1}{4} \quad\quad 1 \cdot 2^1 = 2$

$1 \cdot 2^{-1} = \frac{1}{2} \quad\quad 1 \cdot 2^2 = 4$

$1 \cdot 2^0 = 1 \quad\quad 1 \cdot 2^3 = 8$

a. Normalized numbers

① Based on M_{00}

If $s=1$, $v<0$, $\{-8, -4, -2, -1, -\frac{1}{2}, -\frac{1}{4}\}$

Values based M_{00} = $\{-8, -4, -2, -1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8\}$

② Based on M_{01}

$M_{01} = 1 + \frac{1}{4} = \frac{5}{4}$

If $s=0$, $v>0$:

$$M_{01} \cdot 2^{E_{001}} = \frac{5}{4} \times 2^{-2} = \frac{5}{16}$$

$$M_{01} \cdot 2^{E_{010}} = \frac{5}{4} \times 2^{-1} = \frac{5}{8}$$

$$M_{01} \cdot 2^{E_{011}} = \frac{5}{4} \times 2^{0} = \frac{5}{4}$$

$$M_{01} \cdot 2^{E_{100}} = \frac{5}{4} \times 2^{1} = \frac{5}{2}$$

$$M_{01} \cdot 2^{E_{101}} = \frac{5}{4} \times 2^{2} = 5$$

$$M_{01} \cdot 2^{E_{110}} = \frac{5}{4} \times 2^{3} = 10$$

If $s=1$, $v<0$: $\{-10, -5, -\frac{5}{2}, -\frac{5}{4}, -\frac{5}{8}, -\frac{5}{16}\}$

Values based on M_{01}: $\{-10, -5, -\frac{5}{2}, -\frac{5}{4}, -\frac{5}{8}, -\frac{5}{16}, \frac{5}{16}, \frac{5}{8}, \frac{5}{4}, \frac{5}{2}, 5, 10\}$

③ Based on M_{10}: $M_{10} = 1 + \frac{1}{2} = \frac{3}{2}$

If $s=0$, $v>1$:

$$M_{10} \cdot 2^{E001} = \frac{3}{2} \times 2^{-2} = \frac{3}{8}$$

$$M_{10} \cdot 2^{E010} = \frac{3}{2} \times 2^{-1} = \frac{3}{4}$$

$$M_{10} \cdot 2^{E011} = \frac{3}{2} \times 2^{0} = \frac{3}{2}$$

$$M_{10} \cdot 2^{E100} = \frac{3}{2} \times 2^{1} = 3$$

$$M_{10} \cdot 2^{E101} = \frac{3}{2} \times 2^{2} = 6$$

$$M_{10} \cdot 2^{E110} = \frac{3}{2} \times 2^{3} = 12$$

If $s=1$, $v<0$: $\{-12, -6, -3, -\frac{3}{2}, -\frac{3}{4}, -\frac{3}{8}\}$

Values based on M_{10}: $\{-12, -6, -3, -\frac{3}{2}, -\frac{3}{4}, -\frac{3}{8}, \frac{3}{8}, \frac{3}{4}, \frac{3}{2}, 3, 6, 12\}$

④ Based on M_{11}: $M_{11} = 1 + \frac{3}{4} = \frac{7}{4}$

If $s=0$, $v>0$:

$$M_{11} \cdot 2^{E001} = 7/4 \times 2^{-2} = \frac{7}{16}$$

$$M_{11} \cdot 2^{E010} = 7/4 \times 2^{-1} = 7/8$$

$$M_{11} \cdot 2^{E011} = 7/4 \times 2^{0} = 7/4$$

$M_{11} \cdot 2^{E_{100}} = 7/4 \times 2^1 = \frac{7}{2}$

$M_{11} \cdot 2^{E_{101}} = 7/4 \times 2^2 = 7$

$M_{11} \cdot 2^{E_{110}} = 7/4 \times 2^3 = 14$

If $s=1$, $v<0$: $\{-14, -7, -\frac{7}{2}, -7/4, -7/8, -\frac{7}{16}\}$

values based on M_{11} = $\{-14, -7, -\frac{7}{2}, -7/4, -7/8, -7/16, 7/16, 7/8, 7/4, 7/2, 7, 14\}$

Total values $= 12+12+12+12 = 48 = 2 \cdot 6 \cdot 2^2$

b. Denormalized Numbers

$M = f \rightarrow$ $M_{00} = 0$ → this is used for the case $=0$ $\quad E_{000} = -2$

$M_{01} = \frac{1}{4}$

$M_{10} = \frac{1}{2}$

$M_{11} = \frac{3}{4}$

If $s=0$, $v>0$, take different M:

$M_{01} \cdot 2^{E_{000}} = \frac{1}{4} \cdot 2^{-2} = \frac{1}{16}$

$M_{10} \cdot 2^{E_{000}} = \frac{1}{2} \cdot 2^{-2} = \frac{1}{8}$

$M_{11} \cdot 2^{E_{000}} = 3/4 \cdot 2^{-2} = \frac{3}{16}$

If $s=1$, $v<0$, $\{-\frac{3}{16}, -\frac{1}{8}, -\frac{1}{16}\}$

all values : $\{-\frac{3}{16}, -\frac{1}{8}, -\frac{1}{16}, \frac{1}{16}, \frac{1}{8}, \frac{3}{16}\}$

c. plot set of numbers

considering the situation of zeros:

$M_{00} = 0$ = if $s=0$, $v>0$ $\quad M_{00} \times 2^{E_{000}} = 0 \cdot 2^{-2} = +0$

if $s=1$, $v<0$ $\quad -M_{00} \times 2^{E_{00}} = -0 \cdot 2^{-2} = -0$

values = $\{0, -0\}$

normalized:

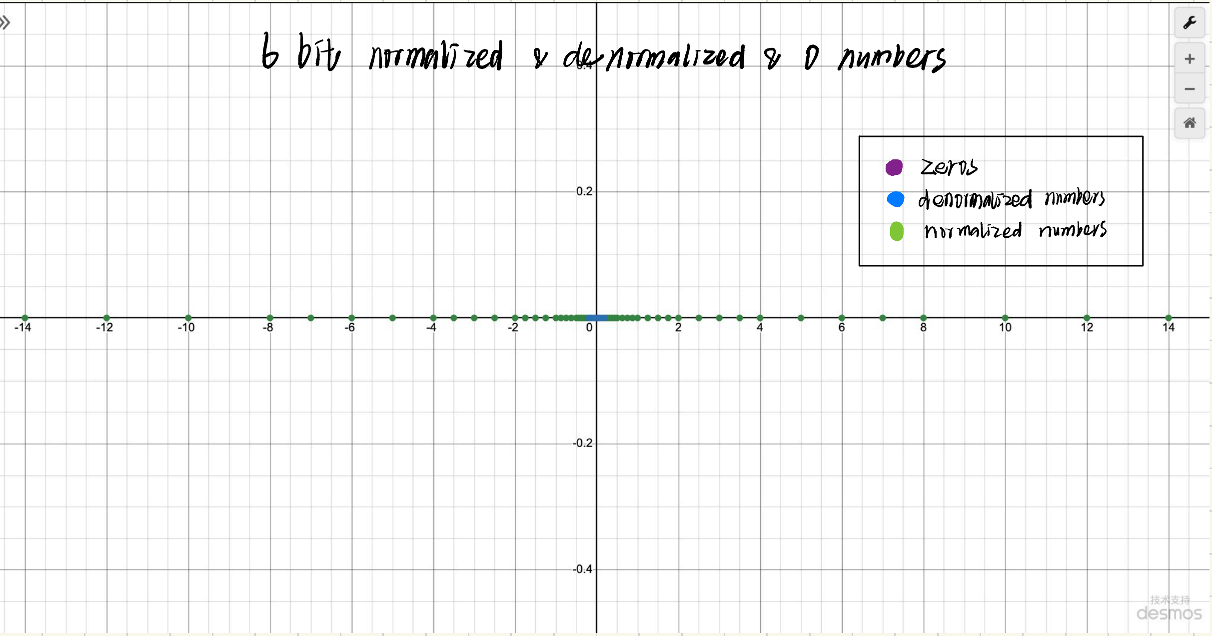
$M_{00} = \{-8, -4, -2, -1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8\}$

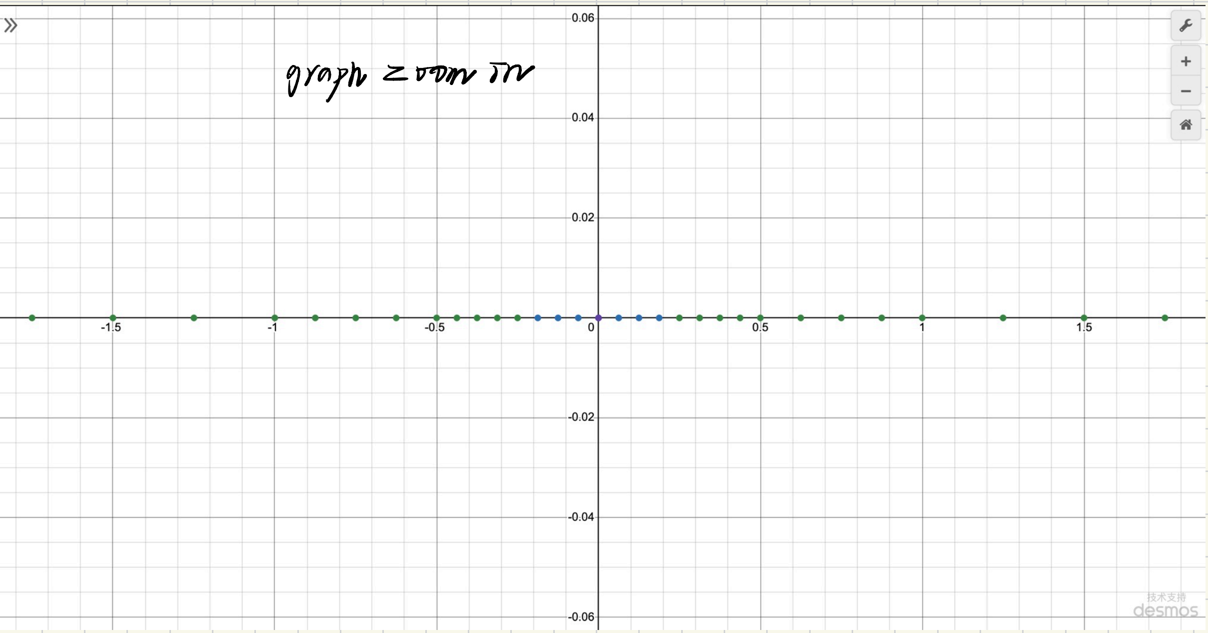
$M_{01}: \{-10, -5, -\frac{5}{2}, -\frac{5}{4}, -\frac{5}{8}, -\frac{5}{16}, \frac{5}{16}, \frac{5}{8}, \frac{5}{4}, \frac{5}{2}, 5, 10\}$

$M_{10}: \{-12, -6, -3, -\frac{3}{2}, -\frac{3}{4}, -\frac{3}{8}, \frac{3}{8}, \frac{3}{4}, \frac{3}{2}, 3, 6, 12\}$

$M_{11}: \{-14, -7, -\frac{7}{2}, -7/4, -7/8, -7/16, 7/16, 7/8, 7/4, 7/2, 7, 14\}$

denormalized: $\{-\frac{3}{16}, -\frac{1}{8}, -\frac{1}{16}, \frac{1}{16}, \frac{1}{8}, \frac{3}{16}\}$


6 bit normalized & denormalized & 0 numbers
zeros
denormalized numbers
normalized numbers
-14
-12
-10
-8
-6
-4
-2
0
2
4
6
8
10
12
14
0.2
-0.2
-0.4
desmos


graph zoom in
0.06
0.04
0.02
-0.02
-0.04
-0.06
-1.5
-1
-0.5
0
0.5
1
1.5
desmos

Problem 7

Hexademical Number

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

a. $(D3B701)_{16}$

$= D \times 16^5 + 3 \times 16^4 + B \times 16^3 + 7 \times 16^2 + 0 \times 16^1 + 1 \times 16^0$

$= 13 \times 16^5 + 3 \times 16^4 + 11 \times 16^3 + 7 \times 16^2 + 0 \times 16^1 + 1 \times 16^0$

$= 13631488 + 196608 + 45056 + 1792 + 0 + 1$

$= 13874945$

b.

$(1010000100111111)_2$

$\overset{3210}{1010} \mid \overset{3210}{0001} \mid \overset{3210}{0011} \mid \overset{3210}{1111}$

↓ ↓ ↓ ↓

$1 \cdot 2^3 + 1 \cdot 2^1$ $\quad 1 \cdot 2^0$ $\quad 1 \cdot 2^1 + 1 \cdot 2^0$ $\quad 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$

↓ ↓ ↓ ↓

10 1 3 15

↓ ↓ ↓ ↓

A 1 3 F

$= (A13F)_{16}$

Problem 8

$a, b, c \in \mathbb{Z}$ s.t $6a + 9b + 15c = 107$

$\text{GCD}(6,9)$: $9 = 1 \times 6 + 3$

$6 = 2 \times 3 + 0$

$\text{GCD}(6,9) = 3$

$\text{GCD}(3,15)$: $15 = 5 \times 3$

$\text{GCD}(3,15) = 3$

$\Rightarrow \text{GCD}(6,9,15) = 3$

$$6a + 9b + 15c = 3(\underbrace{2a + 3b + 5c}_{\text{Integer}})$$

$107 \div 3 = 35.666\ldots \overset{\text{conflict}}{\neq}$ Integer

Since 107 is not divisible by 3, 107 cannot be expressed as a combination of $6a + 9b + 15c$, which are all multiples of 3.

Problem 9

To check if $\mathbb{Z}_n = \{[0],[1],\cdots [n-1]\}$ is a ring $\{\mathbb{Z}_n, +, \cdot\}$

Check whether it is well defined:

① Addition

Let $a \in [i]$, $b \in [j]$

$a = k_1 n + i \qquad b = k_2 n + j$

$a+b = k_1 n + i + k_2 n + j$

$= (k_1 + k_2) n + (i+j)$

$a+b - (i+j) = (k_1 + k_2) n$

$\rightarrow n \mid (a+b) - (i+j)$

$\rightarrow a+b \equiv i+j \pmod n$

$\rightarrow [i] + [j] = [i+j]$ ✓

② Multiplication

$a \cdot b = (k_1 n + i) \cdot (k_2 n + j)$

$= (k_1 n + i) \cdot k_2 n + (k_1 n + i) \cdot j$

$= (k_1 k_2 n + i k_2 + j k_1) n + i \cdot j$

$a \cdot b - i \cdot j = (k_1 k_2 n + i k_2 + j k_1) n$

$\rightarrow n \mid (a\cdot b) - (i\cdot j)$

$\rightarrow a\cdot b \equiv i\cdot j \pmod{n}$

$\rightarrow [i]\cdot[j] = [i\cdot j]$ ✓

To show $(\mathbb{Z}_n, +, \cdot)$ is a ring, weed to confirm:

1. $(\mathbb{Z}_n, +)$ is an Abelian group
2. $(\mathbb{Z}_n, \cdot)$ is a semigroup
3. The distributive laws hold between addition & multiplication

1. $(\mathbb{Z}_n, +)$ is Abelian Group

① Closure ✓

if $[a],[b] \in \mathbb{Z}_n$, then $[a]+[b] = [a+b] \in \mathbb{Z}_n$ proved above

② Assoliativity ✓ Let $c \in [f]$, $c = k_3 n + f$

$[i+j] + [f]$

$= [i] + [j] + [f]$

$= [i] + [j+f]$

③ Existence of an additive identity ✓

$[i] + [j] = [j] + [i] = [i]$

$$a+b = (k_1 n + i) + (k_2 n + j)$$

$$= (k_1 + k_2) i + (i + j)$$

To satisfy $[i] + [j] = [i]$, $j = 0$ here

$$= (k_1 + k_2) i + i + 0$$

$$a + b - i = (k_1 + k_2) i$$

$\rightarrow n \mid (a+b) - i$

$\rightarrow a + b \equiv i \pmod{n}$

$\rightarrow$ $[0]$ serves as the additive Identity

$\rightarrow [i] = [i + 0] = [0 + i] = [i] + [0] = [0] + [i]$

④ Existence of additive Inverse ✓ $[i] + [j] = [0] \rightarrow [j]$

$$a+b = (k_1 n + i) + (k_2 n + j) = (k_1 + k_2) n + (i + j)$$

To make $[i] + [j] = [0]$, $i + j$ should be a multiple of n then it can combine together

$$i + j = n \quad \rightarrow \quad j = n - i$$

$$a+b = (k_1 + k_2) n + i + n - i = (k_1 + k_2 + 1) \cdot n$$

$\rightarrow a+b-0 = (k_1+k_2+1)\cdot n$

$\rightarrow n \mid (a+b)$

$\rightarrow a+b \equiv 0 \pmod n$

$\rightarrow$ exist $[i]+[j] = [i+j] = kn = [0]$

⑤ Commutativity of addition ✓

$$[i]+[j] = (k_1 n+i)+(k_2 n+j)$$
$$= (k_1+k_2)n + (i+j)$$
$$= (k_2 n+j)+(k_1 n+i)$$
$$= [j] + [i]$$

2. $(\mathbb{Z}_n, \cdot)$ is semigroup

① Closure ✓

already proved at begin $[a]\cdot[b] = [a\cdot b]$ still in $\mathbb{Z}$

② Associativity ✓

$$(a\cdot b)\cdot c = ((k_1 n+i)\cdot(k_2 n+j))\cdot(k_3 n+f)$$
$$= (k_1 k_2 nn + i k_2 n + j k_1 n + ij)(k_3 n+f)$$

$$= k_1k_2k_3nnn + ik_2k_3nn + jk_1k_3nn + ijk_3n$$
$$+ k_1k_2nnf + ik_2nf + jk_1fn + ijf$$

$$= (k_1k_2k_3nn + ik_2k_3n + jk_1k_3n + ijk_3$$
$$+ k_1k_2nf + ik_2f \; ik_1f)n + ijf$$

$$(a\cdot b)\cdot c - ijf = (n)\,n$$

$$\rightarrow n \mid (a\cdot b)\cdot c - ijf$$

$$\rightarrow (a\cdot b)\cdot c \equiv ijf \pmod n$$

$$(a\cdot(b\cdot c)) = (k_1n + i)((k_2n+j) + (k_3n+f))$$

$$= (k_1n+i)(k_2k_3nn + k_3jn + k_2fn + jf)$$

$$= k_1k_2k_3nnn + k_1k_3jnn + k_1k_2fnn + k_1jfn$$
$$+ k_2k_3inn + k_3ijn + k_2ifn + ijf$$

$$(a\cdot(b\cdot c)) - ijf = (k_1k_2k_3nn + k_1k_3jn + k_1k_2fn + k_1jf + k_2k_3in$$
$$+ k_3ij + k_2if)n$$

$$\rightarrow n \mid (a\cdot(b\cdot c)) - ijf$$

$$\rightarrow a\cdot(b\cdot c) \equiv ijf \pmod n$$

$$\rightarrow ([i]\cdot[j])\cdot[f] = [i]\cdot([j]\cdot[f])$$

3. Distributive Laws

① Left

$a \cdot (b+c)$

$= (k_1 n + i) \cdot ((k_2 n + j) + (k_3 n + f))$

$= k_1 k_2 nn + k_1 n j + k_1 n k_3 n + k_1 n f + i k_2 n + ij + i k_3 n + if$

$a \cdot (b+c) - i(j+f) = (k_1 k_2 n + k_1 j + k_1 k_3 n + k_1 f + i k_2 + i k_3) n$

$\to \quad n \mid a \cdot (b+c) - i(j+f)$

$\to \quad a \cdot (b+c) \equiv i(j+f) \pmod{n}$

$ab + a \cdot c$

$= (k_1 n + i) \cdot (k_2 n + j) + (k_1 n + i)(k_3 n + f)$

$= k_1 k_2 nn + k_2 i n + k_1 j n + ij + k_1 k_3 nn + k_3 i n + k_1 f n + if$

$a \cdot b + ac - (ij + if) = (k_1 k_2 n + k_2 i + k_1 j + k_1 k_3 n + k_3 i + k_1 f) \cdot n$

$\to \quad n \mid ab + ac - (ij + if)$

$\to \quad ab + ac = ij + if \pmod{n}$

$\to [i] \cdot ([j] + [f]) = [i] \cdot [j] + [i] \cdot [f]$

② Right

$([j] + [f]) \cdot [i] = [j+f] \cdot [i] = [(j+f) \cdot i] = [ji + fi]$

$= [ji] + [fi] = [j][i] + [f][i]$ (similar logic in left)

Since we prove all properties above, $(\mathbb{Z}_n, +, \cdot)$ is a ring.