Problem 1

a. Addititive Identity

An addititive satisfies $a \oplus z = z \oplus a$ for all a

from definition: $a \oplus z = a + z - 1$

If $a + z - 1 = a \rightarrow z - 1 = 0 \rightarrow z = 1$

Symmetry check: $z \oplus a = z + a - 1 = 1 + a - 1 = a$

Uniqueness: suppose z' is another addititive identity

$z' = z' \oplus z = z$ by identity law

Therefore, $z' = 1 \rightarrow z = 1$

Therefore, the additive identity in the ring is integer 1.

b. Additive Inverse

An additive inverse b of a must satisfy $a \oplus b = b \oplus a = z = 1$

By operation, $a \oplus b = a + b - 1$

equal to identity, $a + b - 1 = 1 \rightarrow b = 2 - a$

Verification: $a \oplus (2-a) = a + (2-a) - 1 = 1$

$(2-a) \oplus a = (2-a) + a - 1 =$

Uniqueness: If b_1, b_2 are both inverse of a: $b_1 = b_1 \oplus a \oplus b_2 = 1 \oplus b_2 = b_2$

Therefore, the integar a has additive inverse $b = 2 - a$

c. Commutativity

Need to show $a \otimes b = b \otimes a$ for all $a, b \in \mathbb{Z}$:

$$a \otimes b = a + b - ab$$

$$b \otimes a = b + a - a$$

Since, $a \oplus b = a + b - 1 = b + a - 1 = b \oplus a$

Since $+$, $\cdot$ are commutative in $\mathbb{Z}$

Thus, both binary operations are commutative,

proving $a \otimes b = b \otimes a$.

d Multiplicative Indentity.

Find the (unique) integer u with $a \otimes u = a = u \otimes a$ $\forall a \in \mathbb{Z}$

Compute $a \otimes u$: $a \otimes u = a + u - au$

Impose requirement: $a + u - au = a \rightarrow u - au = 0$

$$\rightarrow u(1-a) = 0 \quad \forall a \in \mathbb{Z}$$

Since $1-a$ is not always zero, the only integer u

that makes $u(1-a) = 0$ for very a is $u = 0$.

Verification: $a \otimes 0 = a + 0 - a \cdot 0 = a$

$0 \otimes a = 0 + a - 0 \cdot a = a$

Uniqueness: If w is any other two-sided identity, plug $a = 0$: $0 = 0 \otimes w = w$

So, $w = 0$

Therefore, $w = 0$, and $u = 0 \neq 1 = z$ as required by ring axioms.

Problem 2

a. Deriving the closed form

If take numbers: 2, 3, 6, 14, 40, 152, 784

compare well-know sequence: $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$

$5! = 120$, $6! = 720$

Compute difference:

n	T_n	$n!$	$D_n = T_n - n!$
0	2	1	1
1	3	1	2
2	6	2	4
3	14	6	8
4	40	24	16
5	152	120	32

6 784 720 64

→ pattern $D_n = 2^n$

Hypothesis : $T_n = n! + 2^n$ $(n \geq 0)$

Therefore : $A_n = n!$ and $B_n = 2^n$

b. use introduction to prove

Prove : $T_n = n! + 2^n$ for all $n \geq 0$

Base cases $n = 0, 1, 2$:

$$\begin{cases} T_0 = 2 = 0! + 2^0 \\ T_1 = 3 = 1! + 2^1 \\ T_2 = 6 = 2! + 2^2 \end{cases}$$ → formula holds for initial data

Inductive hypothesis :

Assume the statement true for all integers up to $n-1$.

→ $T_{n-1} = (n-1)! + 2^{n-1}$,

$T_{n-2} = (n-2)! + 2^{n-2}$,

$T_{n-3} = (n-3)! + 2^{n-3}$

Inductive step

Insert the hypothess into the recurrence =

$$\rightarrow T_n = (n+4)\,T_{n-1} - 4n\,T_{n-2} + (4n-8)\,T_{n-3}$$

$$= (n+4)\left[(n-1)! + 2^{n-1}\right]$$

$$- 4n\left[(n-2)! + 2^{n-2}\right]$$

$$+ (4n-8)\left[(n-3)! + 2^{n-3}\right]$$

$$= \overbrace{\left[(n+4)(n-1)! - 4n(n-2)! + (4n-8)(n-3)!\right]}^{\text{faltorial part}}$$

$$+ \underbrace{\left[(n+4)\cdot 2^{n-1} - 4n\cdot 2^{n-2} + (4n-8)\cdot 2^{n-3}\right]}_{\text{power of two part}}$$

1- Faltorial part

$$(n+4)(n-1)! = (n+4)(n-1)(n-2)(n-3)!$$

$$4n(n-2)! = 4n(n-2)(n-3)!$$

$$(4n-8)(n-3)! = (4n-8)(n-3)!$$

$$\rightarrow (n-3)!\,\underbrace{\left[(n+4)(n-1)(n-2)(n-3) - 4n(n-2) + 4n-8\right]}$$

$$= (n-2)\left[(n+4)(n-1) - 4n\right] + (4n-8)$$

$$= (n-2)\left[n^2+4n-n-4-4n\right] + (4n-8)$$

$$= (n-2)\left[n^2-n-4\right] + 4n-8$$

$$= n^3 - 2n^2 - n^2 + 2n - 4n + 8 + 4n - 8$$

$$= n^3 - 3n^2 + 2n$$

$$= n(n^2 - 3n + 2n)$$

$$= n(n-1)(n-2)$$

$$\rightarrow = n(n-1)(n-2)(n-3)! = n!$$

2. Power of two part

$$\rightarrow 2^{n-3}[(n+4)2^2 - 4n \cdot 2 + (4n-8)]$$

$$= 2^{n-3}[4n + 16 - 8n + 4n - 8]$$

$$= 2^{n-3} \cdot 8 = 2^{n-3} \cdot 2^3 = 2^n$$

Combine back: $T_n = n! + 2^n$ ✓ Is exactly claimed formula

Therefore, by introduction, the closed form: $T_n = n! + 2^n$ for all $n \geq 0$ satisfy both the reccurrence and initial conditions.

Problem 3

Problem 4

a. space $P_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in \mathbb{R}\}$

all real polynomials of degree ≤ 2

$\dim P_2 = 3$

Set $S = \{v_1, v_2, v_3\}$ with $v_1 = 1$, $v_2 = 1-x$, $v_3 = (1-x)^2 = 1-2x+x^2$

1. Linear Independence

take an arbitrary linear combination + set zero polynomial

$$av_1 + bv_2 + cv_3 = a(1) + b(1-x) + c(1-2x+x^2) = 0$$

$$\rightarrow (a+b+c) + (-b-2c)x + (c)x^2 = 0$$

$$\rightarrow \begin{cases} a+b+c = 0 & ① \\ -b-2c = 0 & ② \\ c = 0 & ③ \end{cases}$$

$\rightarrow$ $c = 0$ plug in ② : $-b = 0 \rightarrow b = 0$

$b = 0$ plug in ① : $a = 0$

Since only the trivial solution exists, hence

S is linearly independent.

2. Spanning P_2

Take an arbitrary quadratic polynomial:

$$p(x) = Ax^2 + Bx + C \quad , \quad A, B, C \in \mathbb{R}$$

→ express as a combination of vectors in S:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = c_1 (1) + c_2 (1-x) + c_3 (1-2x+x^2)$$

$$\rightarrow \quad (c_1 + c_2 + c_3) + (-c_2 - 2c_3) x + (c_3) x^2$$

$$\rightarrow \quad \begin{cases} c_1 + c_2 + c_3 = C & ① \\ -c_2 - 2c_3 = B & ② \\ c_3 = A & ③ \end{cases}$$

$\rightarrow \quad c_3 = A$ plug in ② : $-c_2 - 2A = B \Rightarrow c_2 = -(B+2A)$

$c_2 = -(B+2A)$ plug in ① : $c_1 - (B+2A) + A = C \Rightarrow c_1 = A + B + C$

Thus, every polynomial in P_2 is representable:

$$p(x) = (A+B+C) v_1 - (B+2A) v_2 + A v_3$$

because a representation exists for arbitrary A, B, C, S spans P_2

Conclusion: S is linearly independent and spans a 3-dimension space, $S = \{1, 1-x, (1-x)^2\}$ is a basis of P_2.

3. Check:

$$p(x) = 3x^2 - 4x + 2$$

$\rightarrow$ $A = 3 \quad B = -4 \quad C = 2$

$\rightarrow$ $$\begin{cases} C_1 = A + B + C = 3 + (-4) + 2) = 1 \\ C_2 = -(B + 2A) = -(-4 + 2\times 3) = -2 \\ C_3 = A = 3 \end{cases}$$

$\rightarrow$ verify: $C_1(1) + C_2(1-x) + C_3(1-2x+x^2)$

$$= 1\times 1 + (-2)(1-x) + 3(1-2x+x^2)$$

$$= 1 - 2 + 2x + 3 - 6x + 3x^2$$

$$= (1-2+3) + (2x - 6x) + 3x^2$$

$$= 3x^2 - 4x + 2$$

$$= P(x)$$

b.

starting vectors:

$x_1(t) = t^2 \qquad x_2(t) = t \qquad x_3(t) = 1 \qquad t \in [-1, 1]$

Inner product:

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$$

1. Normalize x_1:

$$\|x_1\|^2 = \int_{-1}^{1} t^2 \cdot t^2\, dt = \int_{-1}^{1} t^4\, dt = \left[\frac{1}{5} t^5\right]_{-1}^{1} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$N_1 = \sqrt{\|x_1\|^2} = \sqrt{\frac{2}{5}}$ $\quad y_1(t) = \frac{x_1(t)}{N_1} = t^2\sqrt{\frac{5}{2}}$

check: $\langle y_1, y_1\rangle = \frac{1}{N_1^2}\langle x_1, x_1\rangle = 1$

2. Orthogonalise & normalise x_2

$$\langle x_2, y_1\rangle = \int_{-1}^{1} t\,(t^2\sqrt{\tfrac{5}{2}})\,dt = \sqrt{\tfrac{5}{2}}\int_{-1}^{1} t^3\,dt$$

$$= \sqrt{5/2}\,\left[\tfrac{1}{4}t^4\right]_{-1}^{1} = 0$$

Hence, no projection term: $w_2 = x_2 - (x_2, y_1)\,y_1 = x_2 = t$

Normalise:

$$\|w_2\|^2 = \int_{-1}^{1} t\cdot t\,dt = \int_{-1}^{1} t^2\,dt = \left[\tfrac{1}{3}t^3\right]_{-1}^{1} = \tfrac{2}{3}$$

$N_2 = \sqrt{\frac{2}{3}}$ $\quad y_2(t) = \frac{w_2(t)}{N_2} = \sqrt{\frac{3}{2}}\,t$

check: $\langle y_1, y_2\rangle = \sqrt{\frac{5}{2}}\sqrt{\frac{3}{2}}\int_{-1}^{1} t^3\,dt = 0 \rightarrow$ orthogonal

3. Orthogonalise & normalise x_3

$$\langle x_3, y_2\rangle = \int_{-1}^{1} 1\cdot y_2(t)\,dt = \sqrt{\tfrac{3}{2}}\int_{-1}^{1} t\,dt = 0$$

$$\langle x_3, y_1\rangle = \int_{-1}^{1} 1\cdot y_1(t)\,dt = \sqrt{\tfrac{5}{2}}\int_{-1}^{1} t^2\,dt = \tfrac{2}{3}\sqrt{\tfrac{5}{2}}$$

Orthogonal component: $w_3(t) = x_3(t) - \langle x_3, y_1\rangle\,y_1(t) - \langle x_3, y_2\rangle\,y_2(t)$

$$= 1 - \left(\tfrac{2}{3}\sqrt{\tfrac{5}{2}}\right)\left(t^2\sqrt{\tfrac{5}{2}}\right) - 0$$

$$= 1 - \tfrac{5}{3}t^2$$

normalise: $\|w_3\|^2 = \int_{-1}^{1} (1 - \frac{5}{3}t^2)^2\,dt$

$$= \int_{-1}^{1} \left(1 + \tfrac{25}{9}t^4 - \tfrac{10}{3}t^2\right)dt$$

$$= \left[t + \frac{5}{9}t^5 - \frac{10}{9}t^3 \right]_{-1}^{1}$$

$$= 1 + \frac{5}{9} - \frac{10}{9} - \left(-1 - \frac{5}{9} + \frac{10}{9}\right)$$

$$= \frac{8}{9}$$

$$N_3 = \sqrt{\frac{8}{9}} = \frac{2}{3}\sqrt{2} \quad , \quad y_3(t) = \frac{W_3(t)}{N_3} = \frac{3}{2\sqrt{2}}\left(1 - \frac{5}{3}t^2\right)$$

$$= \frac{3}{4}\sqrt{2} - \frac{5}{4}\sqrt{2}\,t^2$$

4. final orthonormal set & cross checks

$$\begin{cases} y_1(t) = t^2\sqrt{\frac{5}{2}} \\ y_2(t) = t\sqrt{\frac{3}{2}} \\ y_3(t) = \frac{3}{4}\sqrt{2} - \frac{5}{4}\sqrt{2}\,t^2 \end{cases}$$

✓ Orthogonality: ① $\langle y_1, y_2 \rangle = \sqrt{\frac{5}{2}}\sqrt{\frac{3}{2}} \int_{-1}^{1} t^3\,dt = 0$

② $\langle y_1, y_3 \rangle = \sqrt{\frac{5}{2}} \int_{-1}^{1} t^2\left(\frac{3}{4}\sqrt{2} - \frac{5}{4}\sqrt{2}\,t^2\right)dt$

$$= \sqrt{\frac{5}{2}} \int_{-1}^{1} \left(\frac{3}{4}\sqrt{2}\,t^2 - \frac{5}{4}\sqrt{2}\,t^4\right)dt$$

$$= \sqrt{\frac{5}{2}} \left[\frac{1}{4}\sqrt{2}\,t^3 - \frac{1}{4}\sqrt{2}\,t^5 \right]_{-1}^{1}$$

$$= \sqrt{\frac{5}{2}} \left[\frac{1}{4}\sqrt{2} - \frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{2} - \frac{1}{4}\sqrt{2} \right]$$

$$= 0$$

③ $\langle y_2, y_3 \rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} t\left(\frac{3}{4}\sqrt{2} - \frac{5}{4}\sqrt{2}\,t^2\right)dt$

$$= \sqrt{\frac{3}{2}} \int_{-1}^{1} \left(\frac{3}{4}\sqrt{2}\,t - \frac{5}{4}\sqrt{2}\,t^3\right)dt$$

$$= \sqrt{\frac{3}{2}} \left[\frac{3}{8}\sqrt{2}\,t^2 - \frac{5}{16}\sqrt{2}\,t^4 \right]_{-1}^{1}$$

$$= \sqrt{\frac{5}{2}} \left[\frac{3}{8}\sqrt{2} - \frac{5}{16}\sqrt{2} - \frac{3}{8}\sqrt{2} + \frac{5}{16}\sqrt{2} \right]$$

$$= 0$$

✓ normalisation: when constructing each y_i, already verified $\langle y_i, y_i \rangle = 1$.

Therefore, the set $\{y_1, y_2, y_3\}$ is orthonomal in $L^2[-1,1]$.