COMP9024: Data Structures and Algorithms

Graphs (II)

Contents

- Depth-First Search
- Breadth-First Search
- Transitive Closure
- Topological Sorting

Properties

Property 1

$\sum_v \deg(\boldsymbol{v}) = 2\boldsymbol{m}$

Proof: each edge is counted twice

Property 2

In an undirected graph with no self-loops and no multiple edges

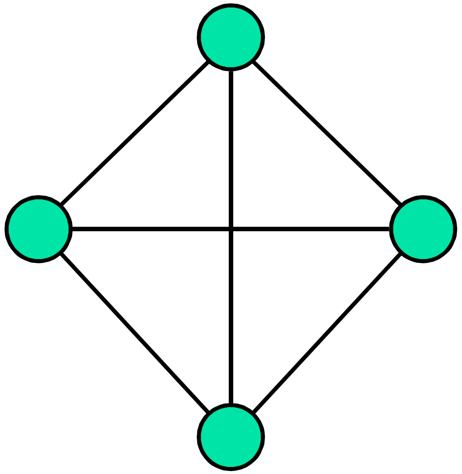
$\boldsymbol{m} \leq \boldsymbol{n}\,(\boldsymbol{n} - 1)/2$

Proof: each vertex has degree at most $(\boldsymbol{n} - 1)$

What is the bound for a directed graph?

Notation

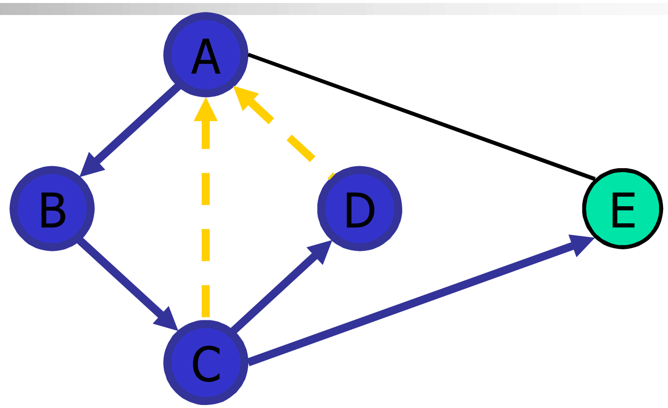
$\boldsymbol{n}$	number of vertices
$\boldsymbol{m}$	number of edges
$\deg(\boldsymbol{v})$	degree of vertex $\boldsymbol{v}$



Example

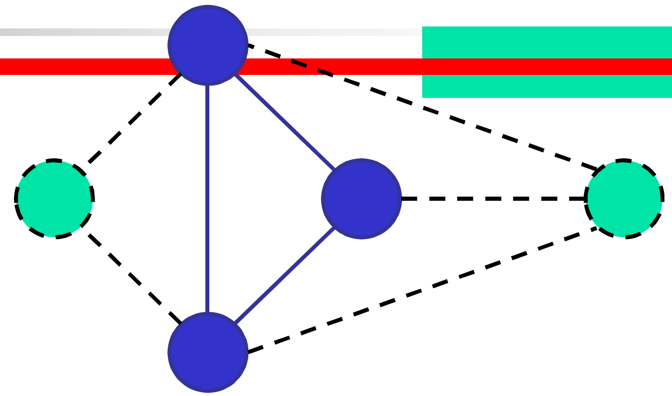
- $\boldsymbol{n} = 4$
- $\boldsymbol{m} = 6$
- $\deg(\boldsymbol{v}) = 3$

Depth-First Search

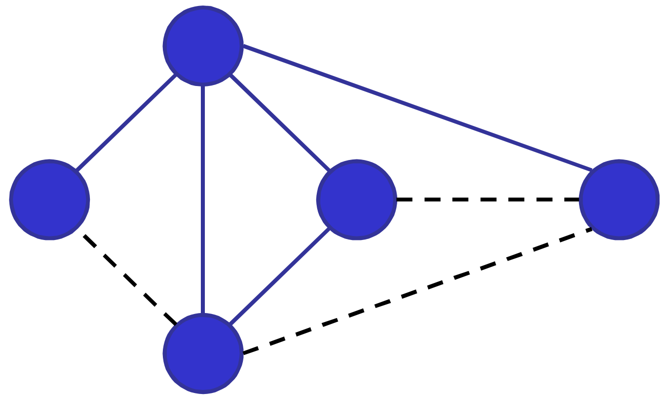


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



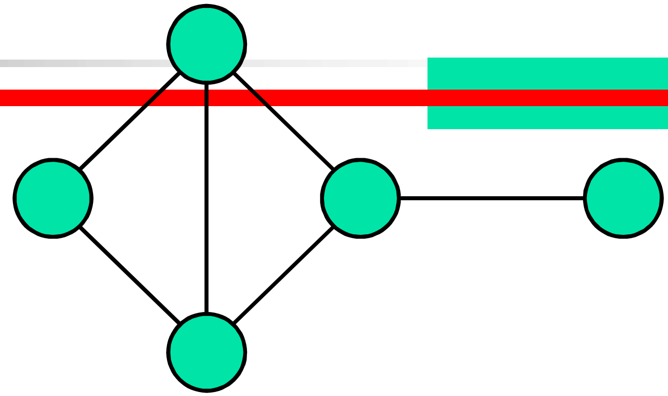
Subgraph



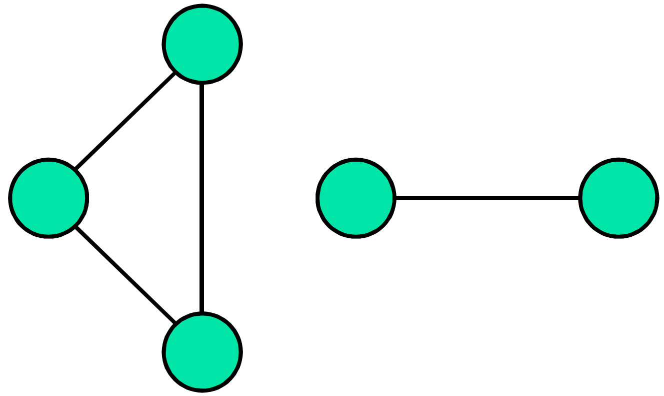
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



Connected graph

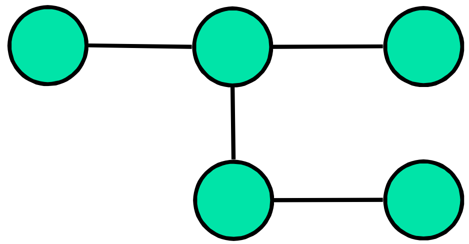


Non connected graph with two connected components

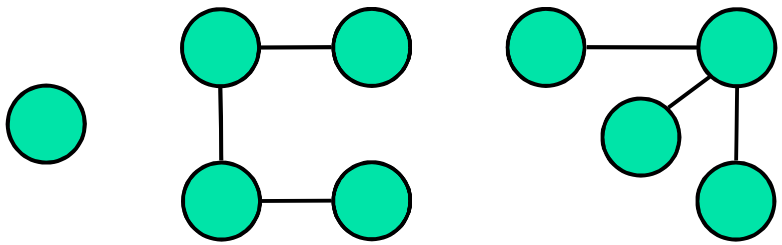
Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles

  This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



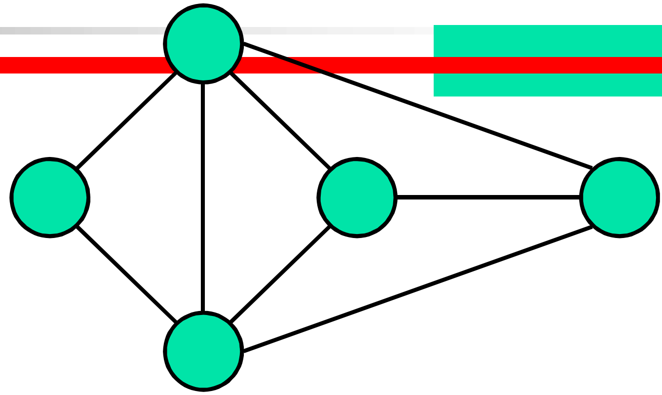
Tree



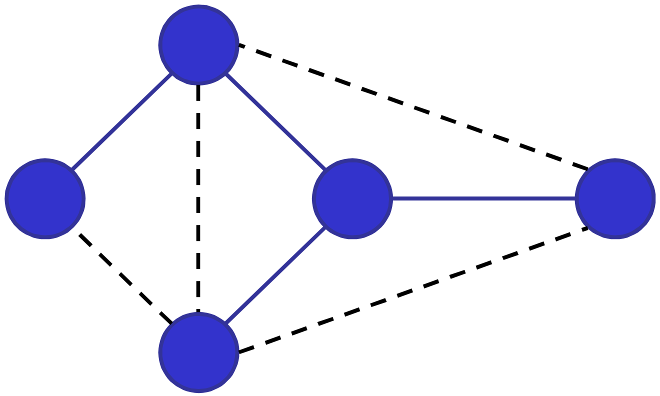
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs what Euler tour is to binary trees

DFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm DFS(G)
    Input graph G
    Output labeling of the edges of G
        as discovery edges and
        back edges
{ for all u ∈ G.vertices()
      setLabel(u, UNEXPLORED);
  for all e ∈ G.edges()
      setLabel(e, UNEXPLORED);
  for all v ∈ G.vertices()
      if ( getLabel(v) = UNEXPLORED )
          DFS(G, v);
}
```

```
Algorithm DFS(G, v)
  Input graph G and a start vertex v of G
  Output labeling of the edges of G
      in the connected component of v
      as discovery edges and back edges
{ setLabel(v, VISITED);
  for all e ∈ G.incidentEdges(v)
    if ( getLabel(e) = UNEXPLORED )
      { w = opposite(v, e);
        if ( getLabel(w) = UNEXPLORED )
          { setLabel(e, DISCOVERY);
            DFS(G, w);
          }
        else
            setLabel(e, BACK);
      }
}
```

Example (1/2)

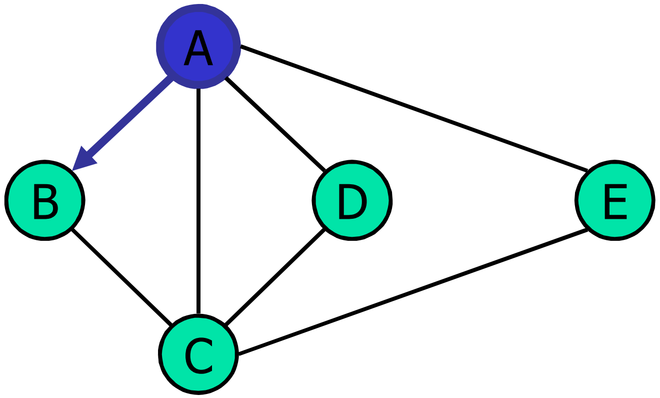
A unexplored vertex

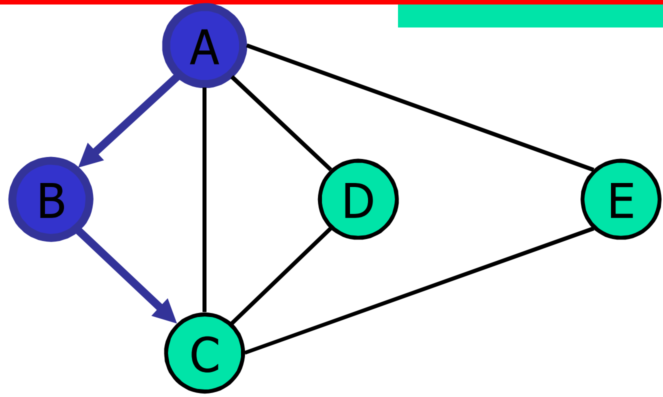
A visited vertex

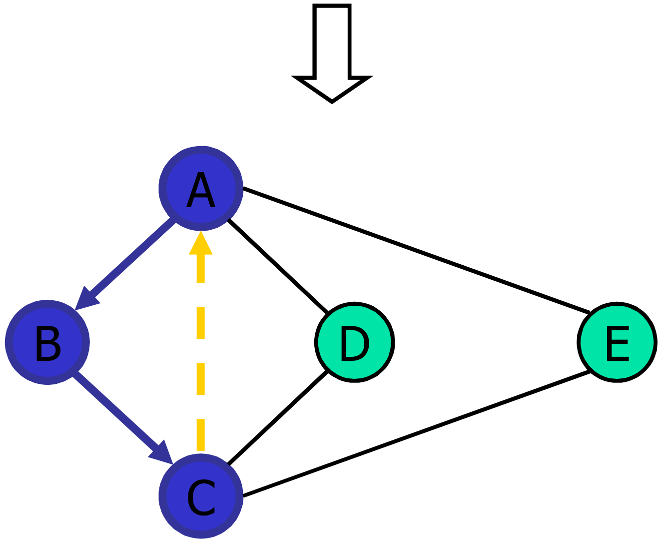
unexplored edge

discovery edge

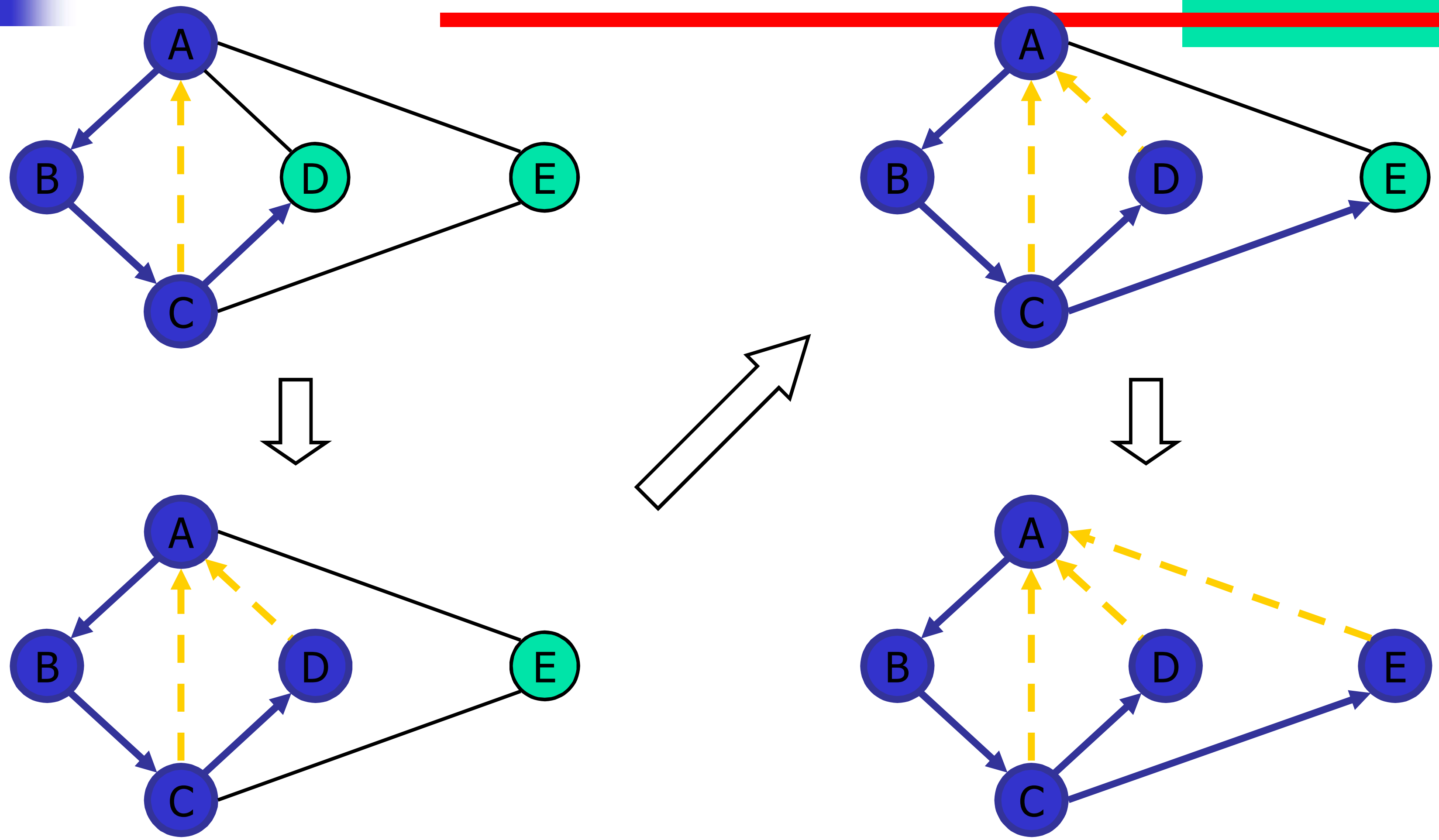
back edge



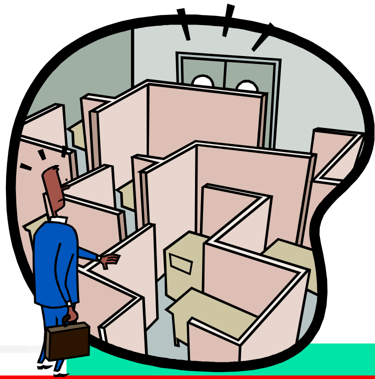




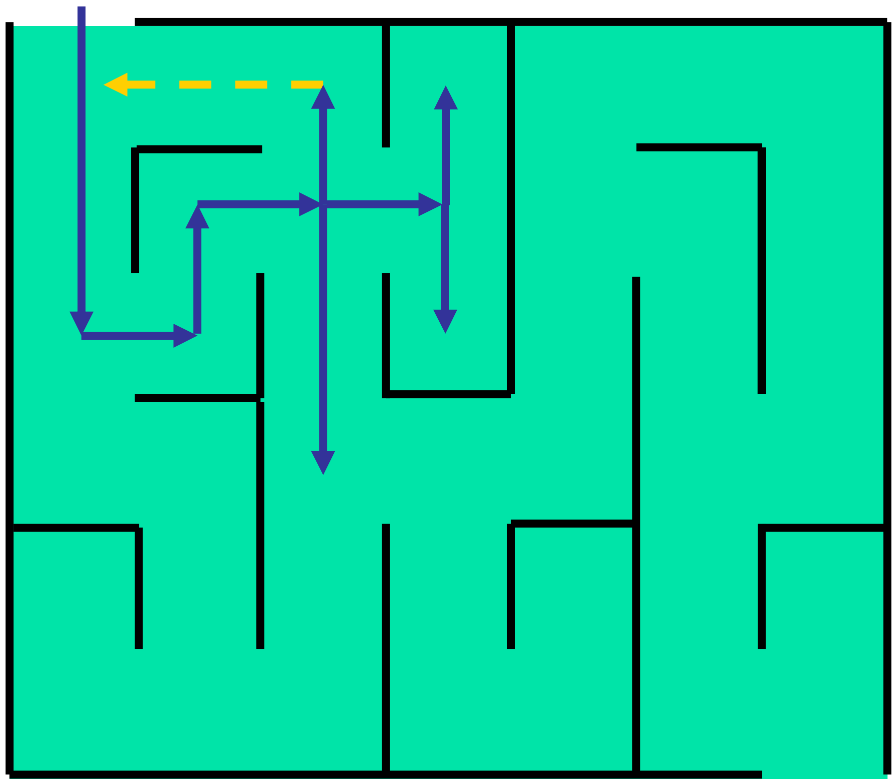
Example (2/2)



DFS and Maze Traversal



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



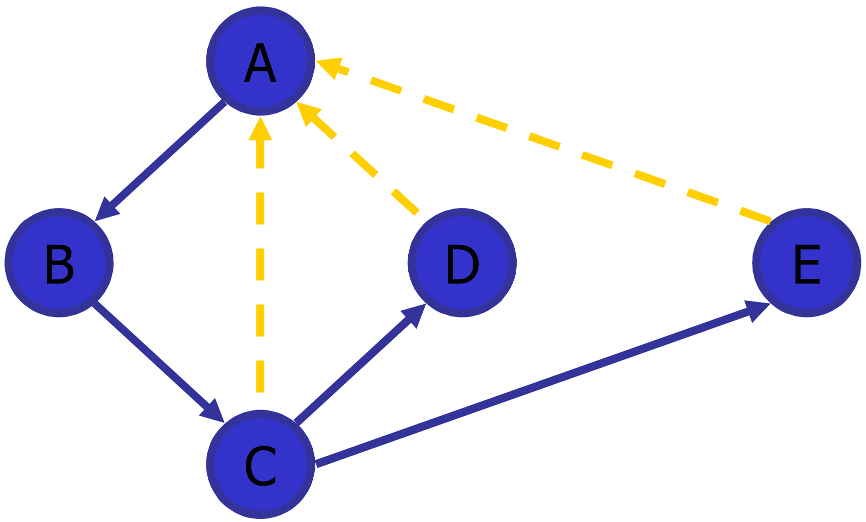
Properties of DFS

Property 1

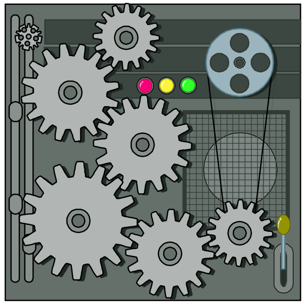
$DFS(G, v)$ visits all the vertices and edges in the connected component of v

Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of v



Analysis of DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices *v* and *z* using the template method pattern
- We call ***DFS***(***G, v***) with *v* as the start vertex
- We use a stack ***S*** to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex *z* is encountered, we return the path as the contents of the stack

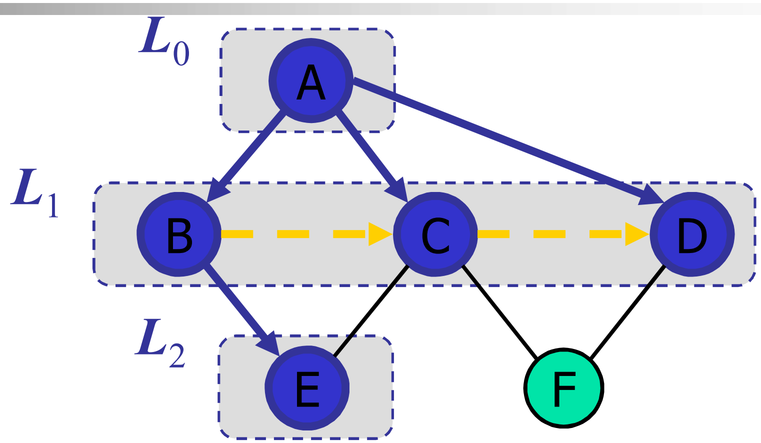
```
Algorithm pathDFS(G, v, z)
  { setLabel(v, VISITED);
    S.push(v);
    if ( v = z )
      return S.elements();
    for all  e ∈ G.incidentEdges(v)
      if  ( getLabel(e) = UNEXPLORED )
        {  w = opposite(v,e);
          if  ( getLabel(w) = UNEXPLORED )
            {  setLabel(e, DISCOVERY);
              S.push(e);
              pathDFS(G, w, z);
              S.pop(e);
            }
          else
            setLabel(e, BACK);
        }
    S.pop(v);
  }
```

Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack *S* to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (*v*, *w*) is encountered, we return the cycle as the portion of the stack from the top to vertex *w*

```
Algorithm cycleDFS(G, v)
 { setLabel(v, VISITED);
   S.push(v);
   for all  e ∈ G.incidentEdges(v)
     if  ( getLabel(e) = UNEXPLORED )
       {  w = opposite(v,e);
          S.push(e);
          if  ( getLabel(w) = UNEXPLORED )
            {  setLabel(e, DISCOVERY);
               cycleDFS(G, w);
               S.pop(e);  }
          else
            {  T = new empty stack
               repeat
                 {  o = S.pop();
                    T.push(o);  }
               until ( o = w );
               return T.elements();   }
   S.pop(v);
 }
```

Breadth-First Search



Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

BFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm BFS(G)
    Input graph G
    Output labeling of the edges
        and partition of the
        vertices of G
{
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED);
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED);
  for all v ∈ G.vertices()
    if ( getLabel(v) = UNEXPLORED )
      BFS(G, v);
}
```

```
Algorithm BFS(G, s)
 { L0 = new empty sequence;
   L0.insertLast(s);
   setLabel(s, VISITED);
   i = 0;
   while ( ¬Li.isEmpty() )
    { Li+1 = new empty sequence;
      for all v ∈ Li.elements()
        for all e ∈ G.incidentEdges(v)
          if ( getLabel(e) = UNEXPLORED )
           { w = opposite(v,e);
             if ( getLabel(w) = UNEXPLORED )
              { setLabel(e, DISCOVERY);
                setLabel(w, VISITED);
                Li+1.insertLast(w); }
             else
                setLabel(e, CROSS);
           }
      i = i +1;
    }
 }
```

Example (1/3)

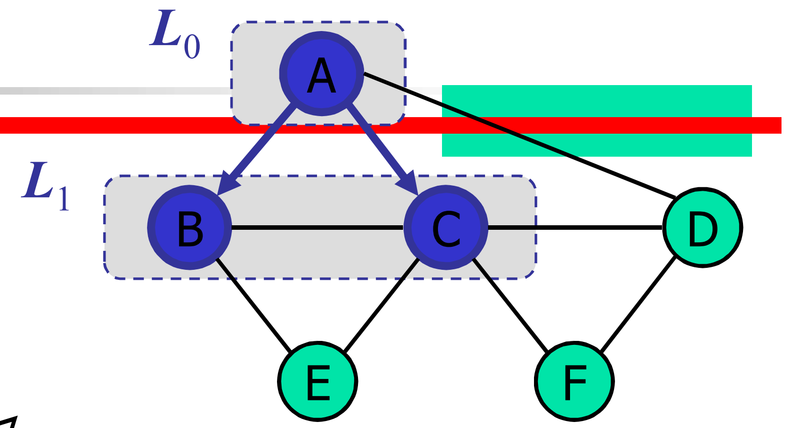
A unexplored vertex

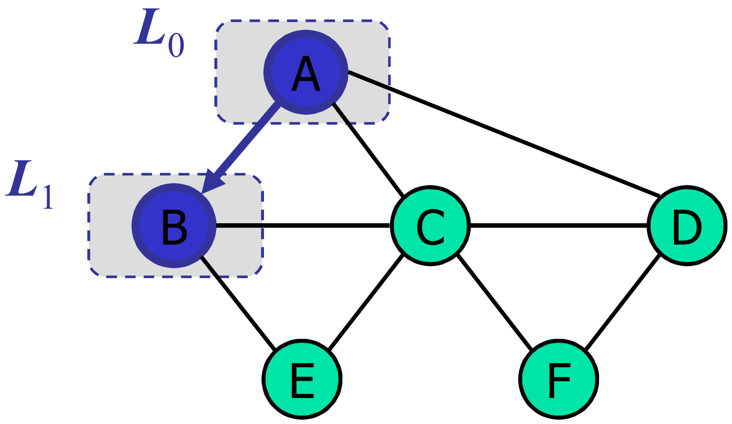
A visited vertex

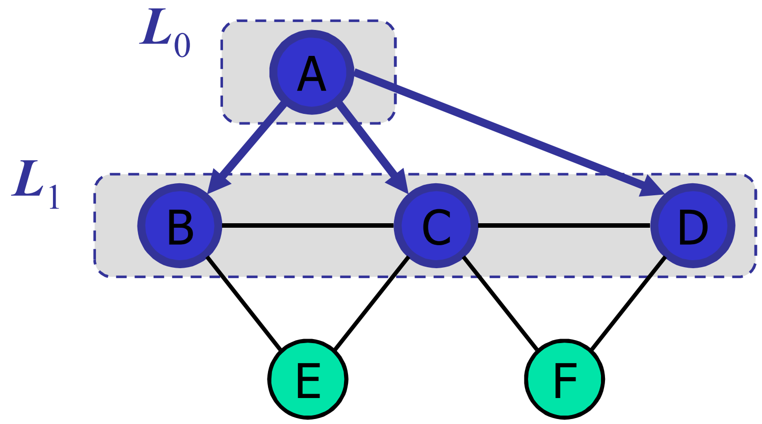
unexplored edge

discovery edge

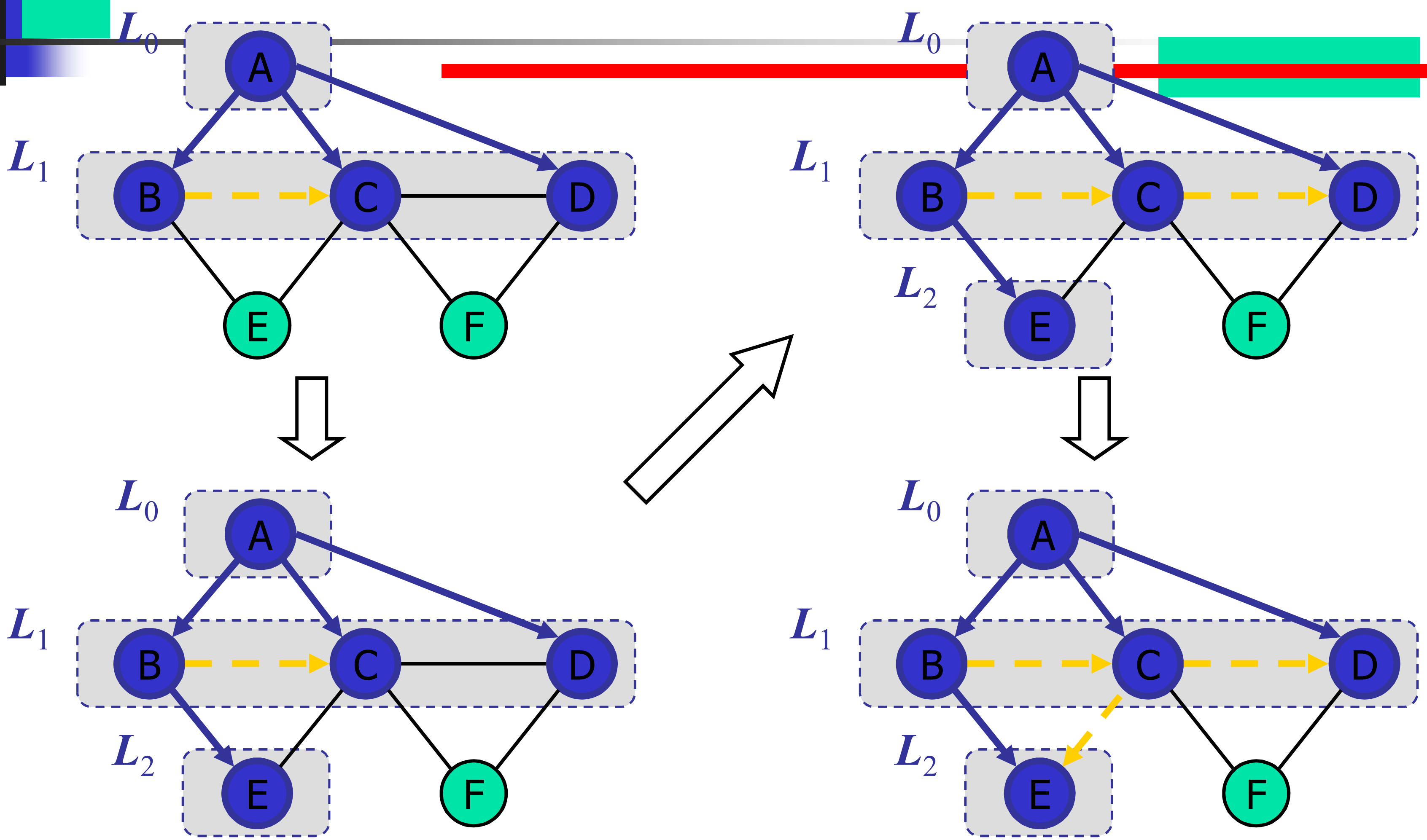
cross edge



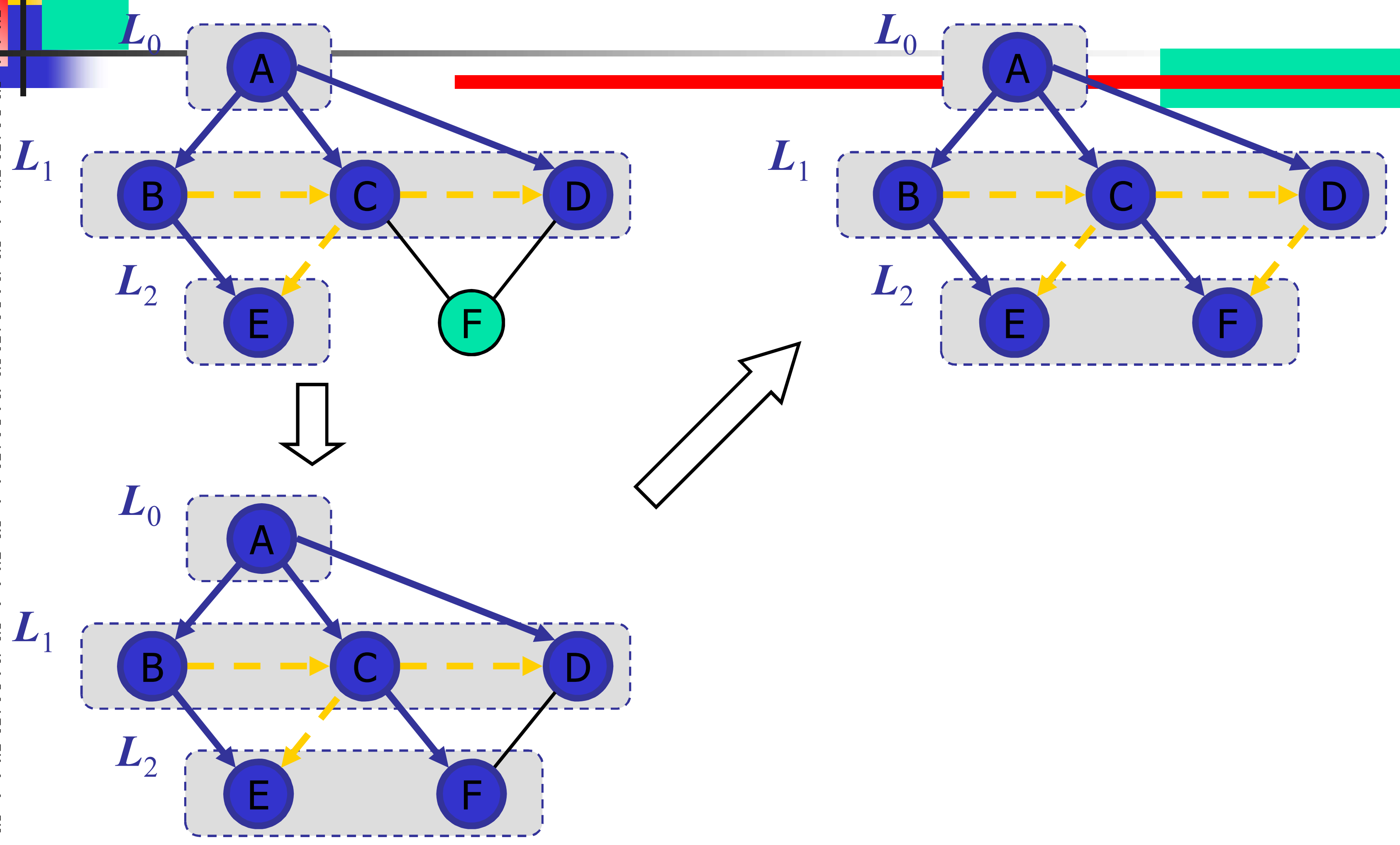




Example (2/3)



Example (3/3)



Properties

Notation

G_s: connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

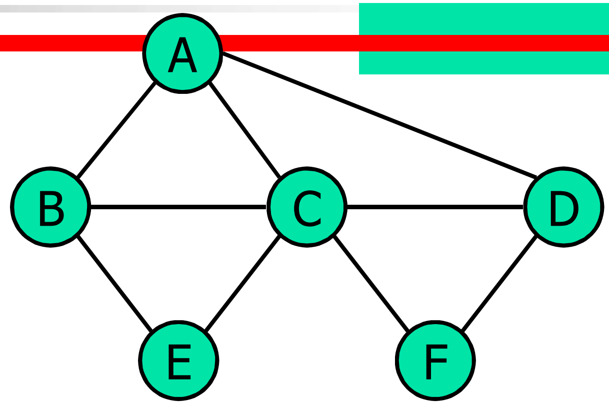
Property 2

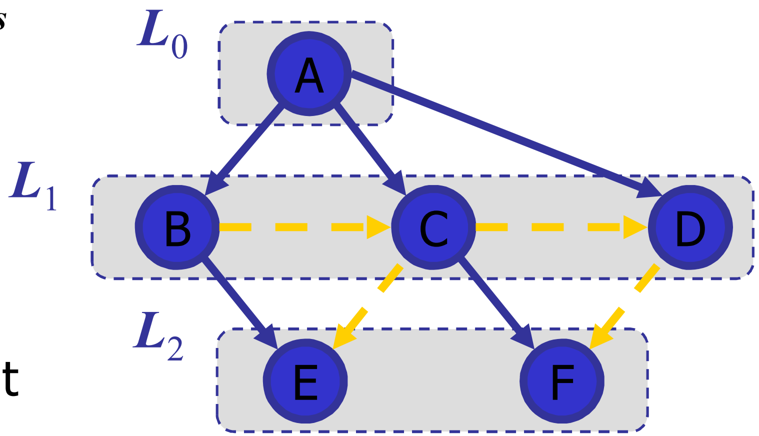
The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- The path of T_s from s to v has i edges
- Every path from s to v in G_s has at least i edges





Analysis

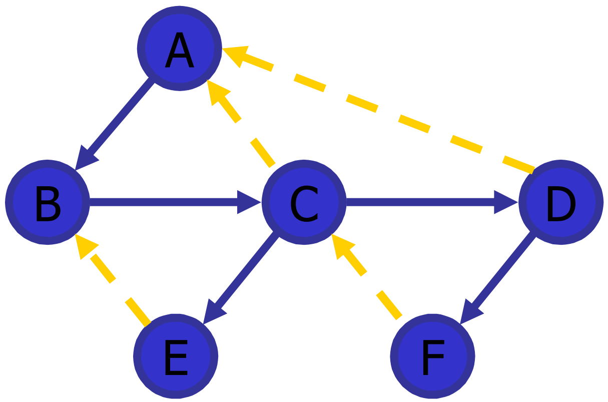
- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Applications

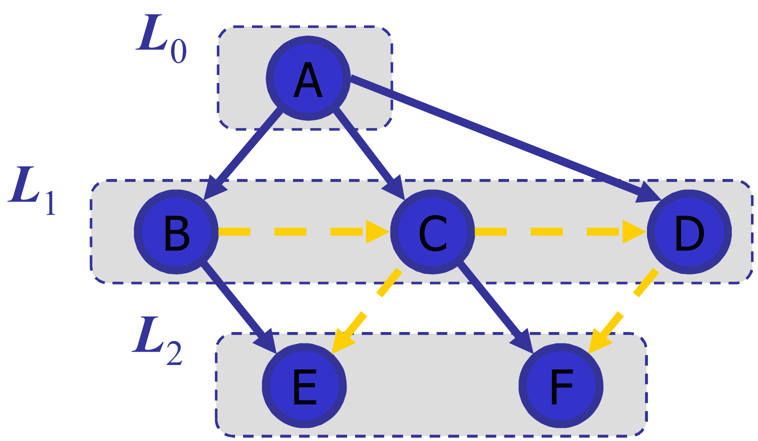
- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS vs. BFS (1/2)

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



BFS

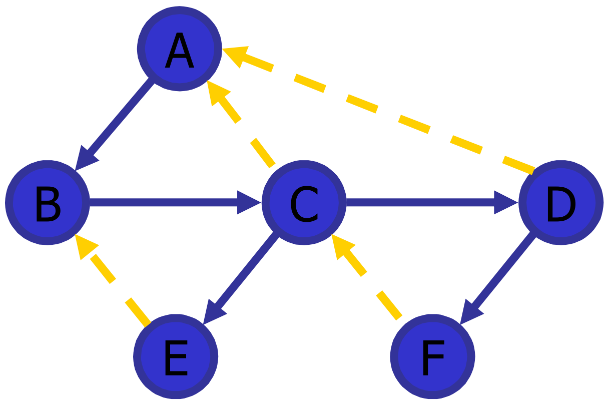
DFS vs. BFS (2/2)

Back edge (v,w)

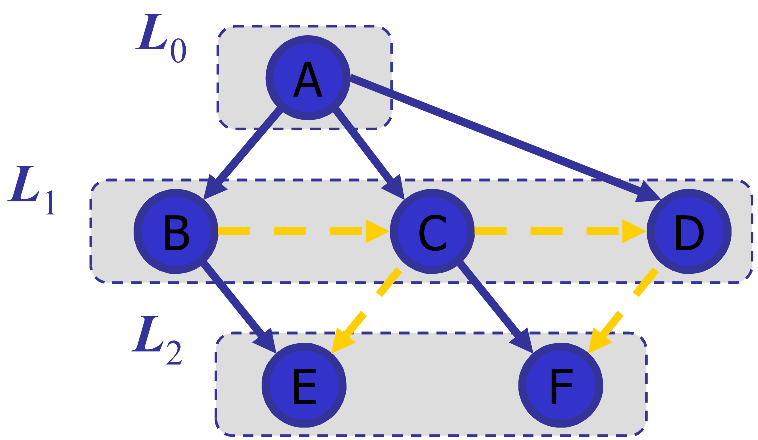
- w is an ancestor of v in the tree of discovery edges

Cross edge (v,w)

- w is in the same level as v or in the next level in the tree of discovery edges

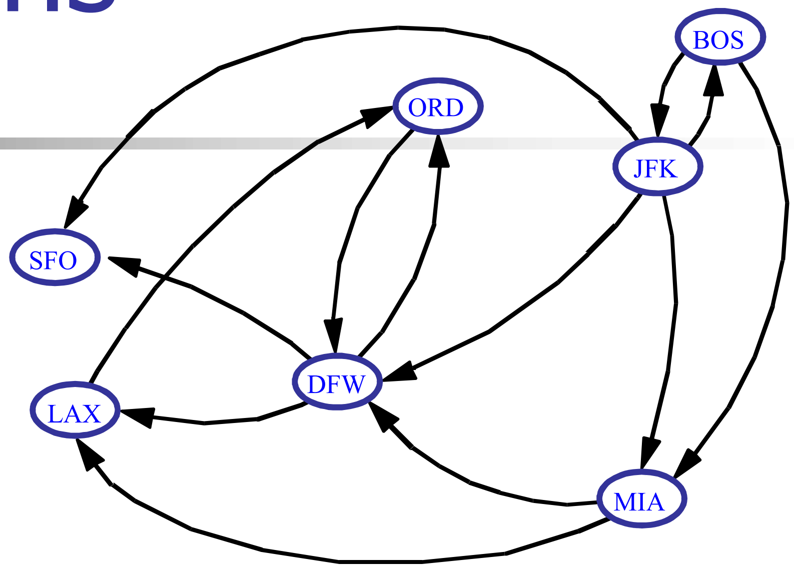


DFS



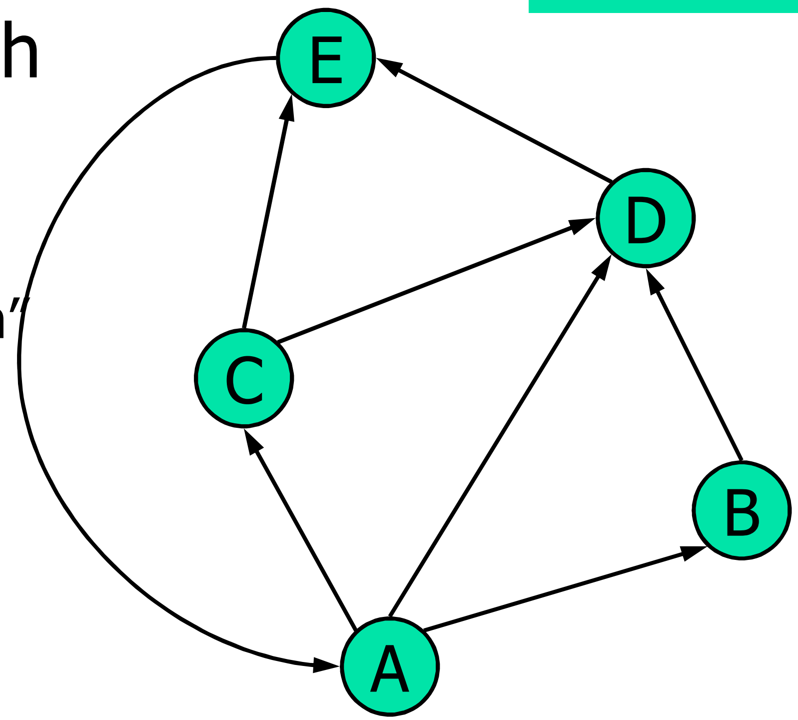
BFS

Directed Graphs

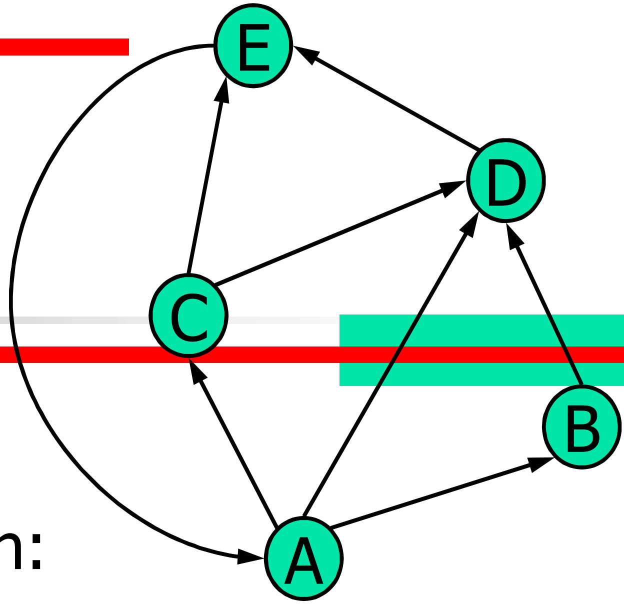


Digraphs

- A **digraph** is a graph whose edges are all directed
  - Short for "directed graph"
- Applications
  - one-way streets
  - flights
  - task scheduling



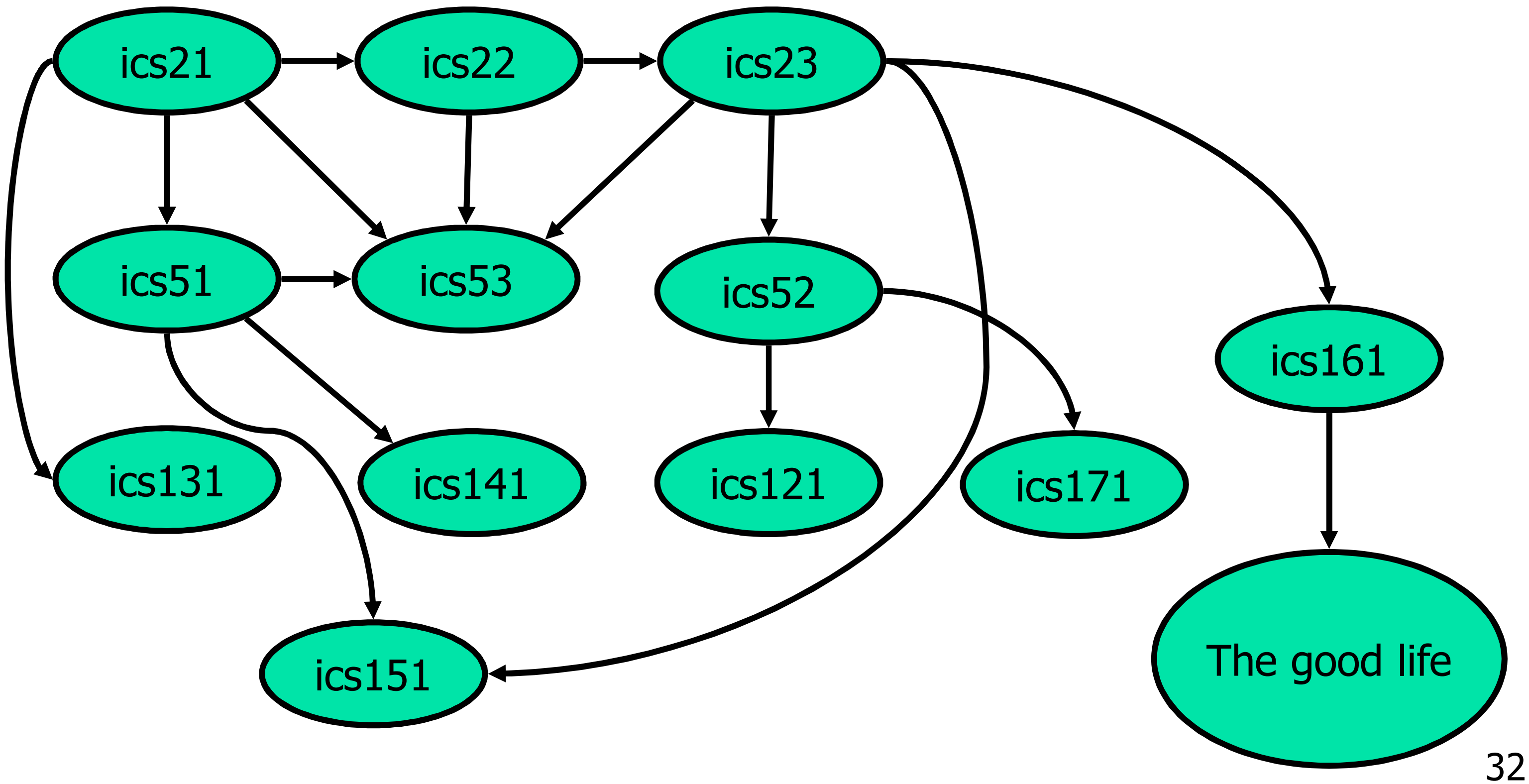
Digraph Properties



- A graph G=(V,E) such that
  - Each edge goes in one direction:
    - Edge (a,b) goes from a to b, but not b to a.
- If G is simple, $m \leq n*(n-1)$.
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of in-edges and out-edges in time proportional to their size.

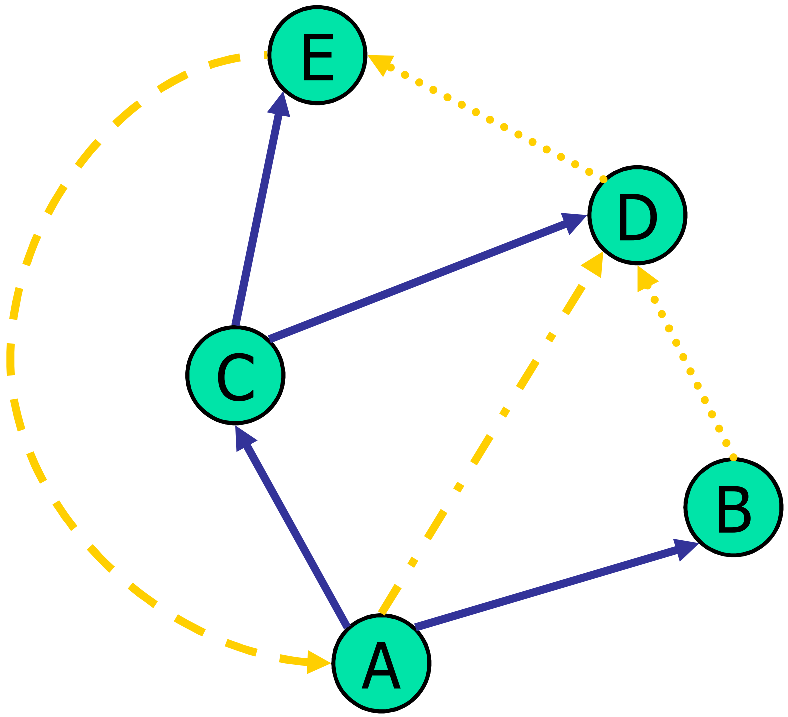
Digraph Application

- Scheduling: edge (a,b) means task a must be completed before b can be started



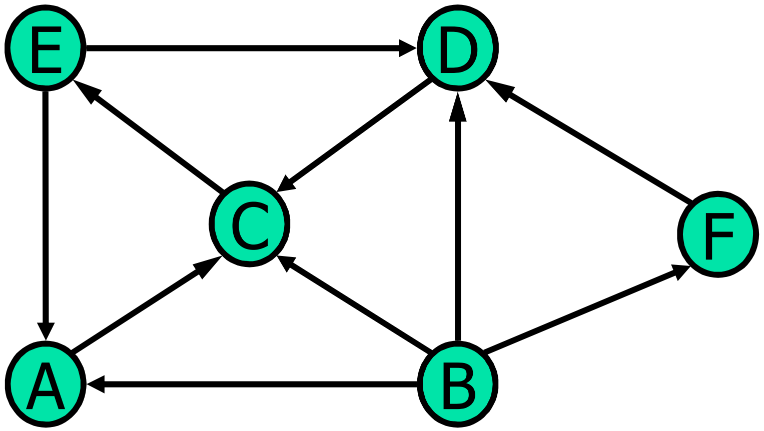
Directed DFS

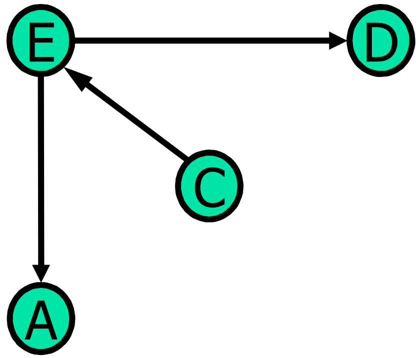
- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting a a vertex s determines the vertices reachable from s

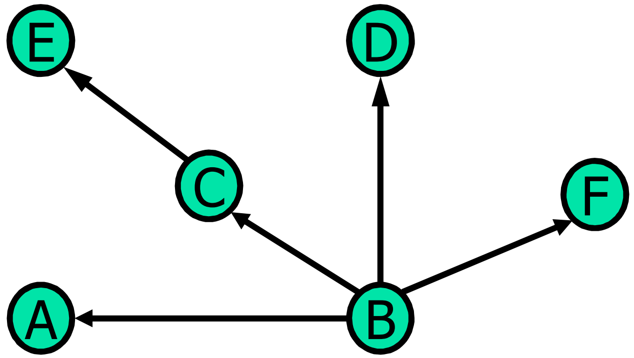


Reachability

- DFS tree rooted at v: vertices reachable from v via directed paths

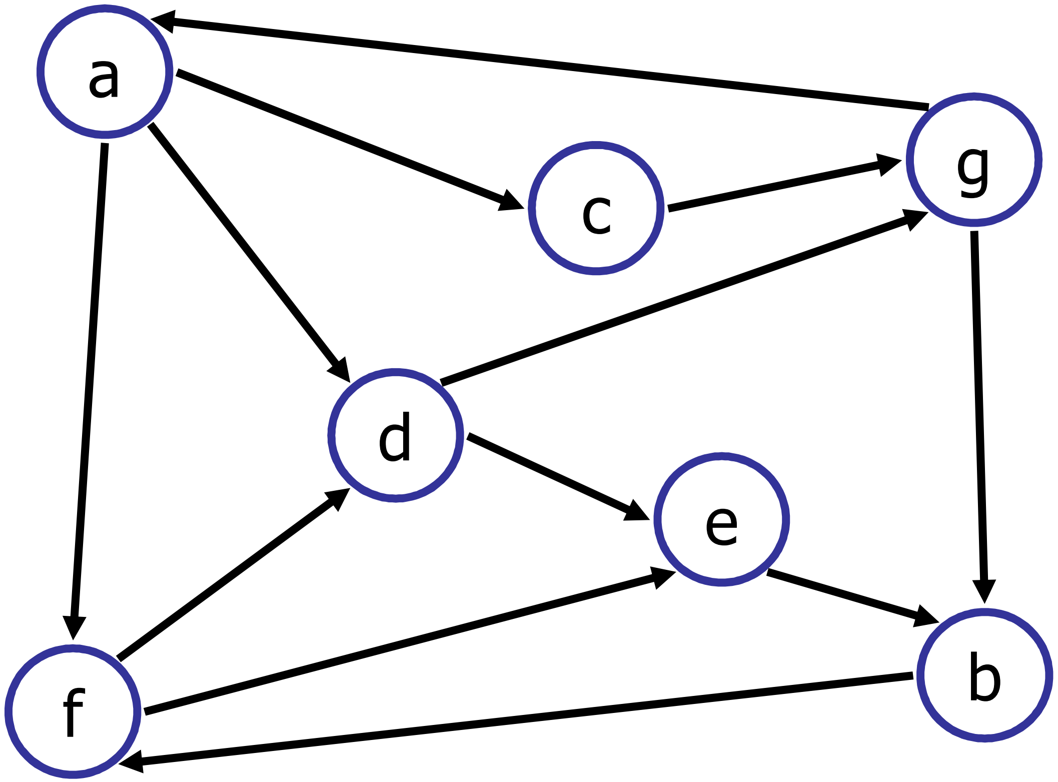




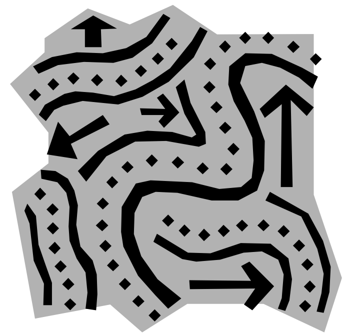


Strong Connectivity

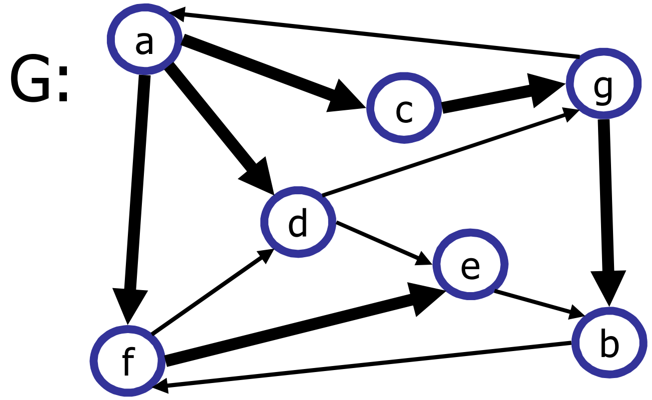
- Each vertex can reach all other vertices

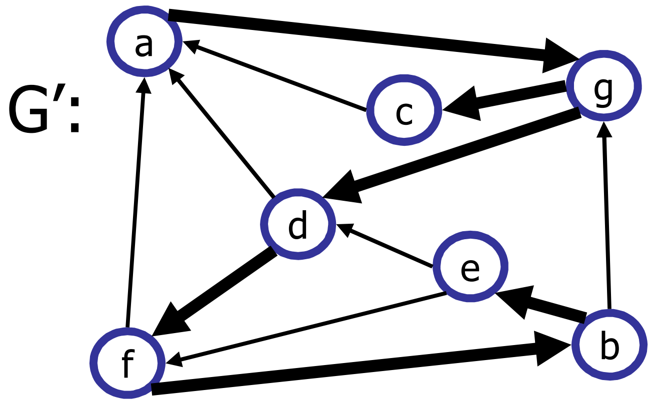


Strong Connectivity Algorithm



- Pick a vertex v in G.
- Perform a DFS from v in G.
  - If there's a w not visited, print "no".
- Let G' be G with edges reversed.
- Perform a DFS from v in G'.
  - If there's a w not visited, print "no".
  - Else, print "yes".

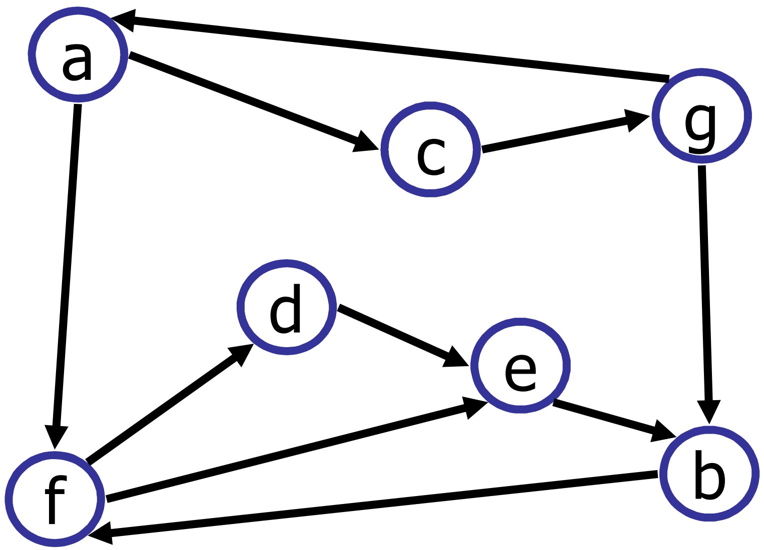




- Running time: O(n+m).

Strongly Connected Components

- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in O(n+m) time using DFS, but is more complicated (similar to biconnectivity).

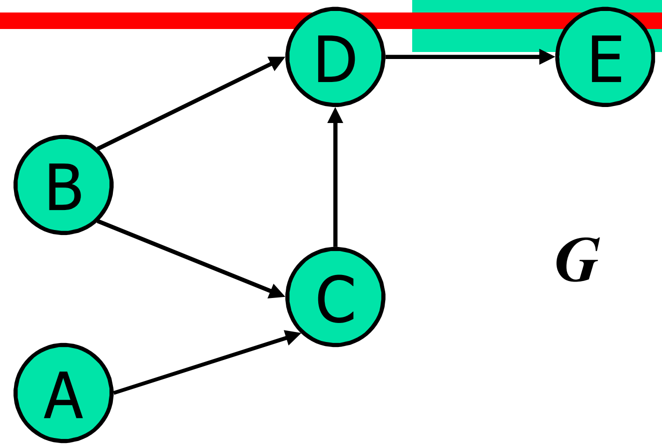


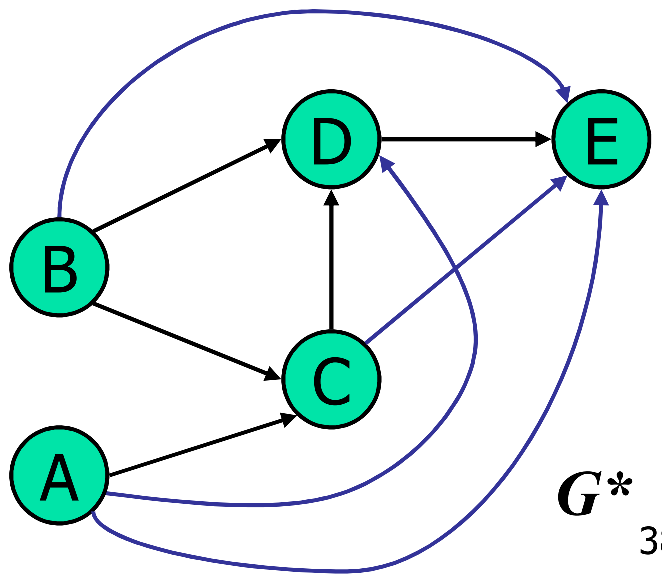
{ a , c , g }

{ f , d , e , b }

Transitive Closure

- Given a digraph G, the transitive closure of G is the digraph G^* such that
  - G^* has the same vertices as G
  - if G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v
- The transitive closure provides reachability information about a digraph

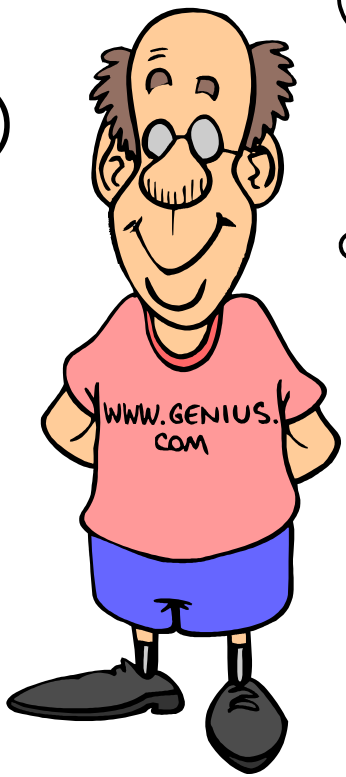




Computing the Transitive Closure

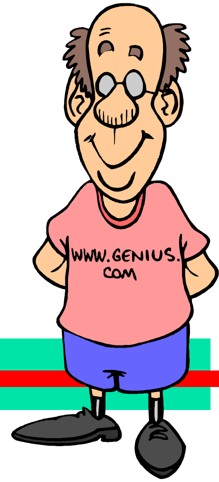
- We can perform DFS starting at each vertex
  - O(n(n+m))

If there's a way to get from A to B and from B to C, then there's a way to get from A to C.

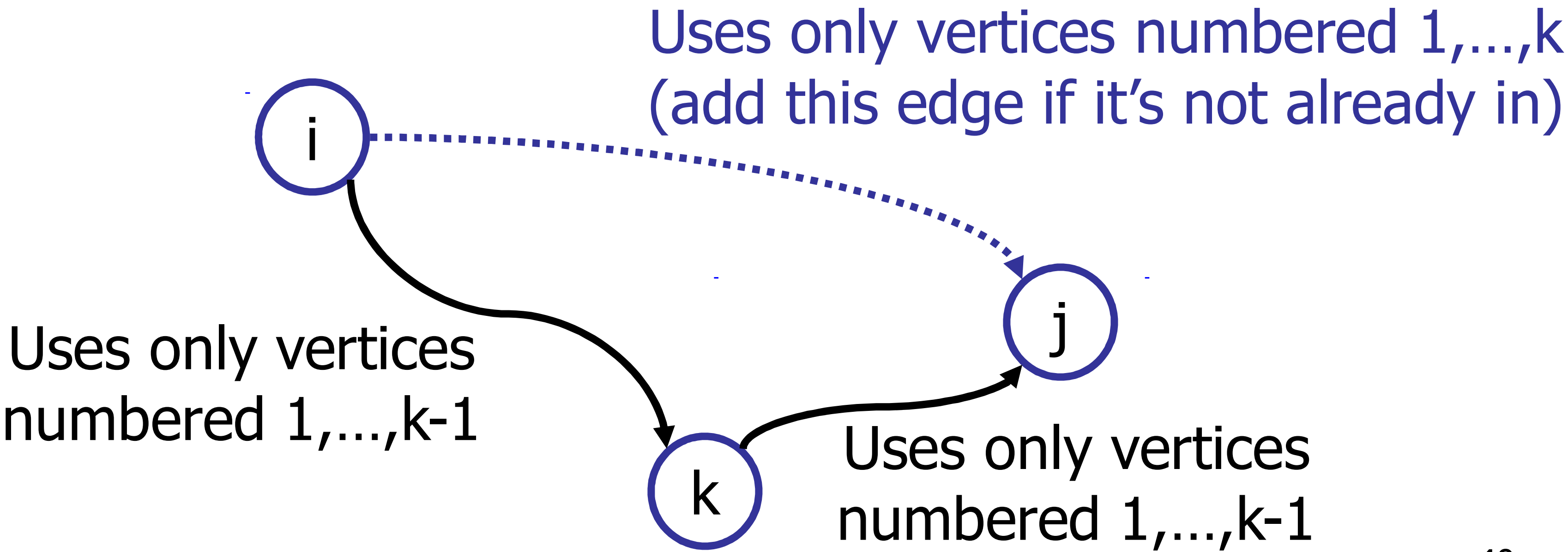


◆ Alternatively ... Use dynamic programming: The Floyd-Warshall Algorithm

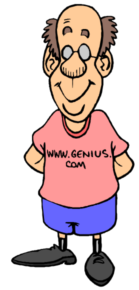
Floyd-Warshall Transitive Closure



- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:



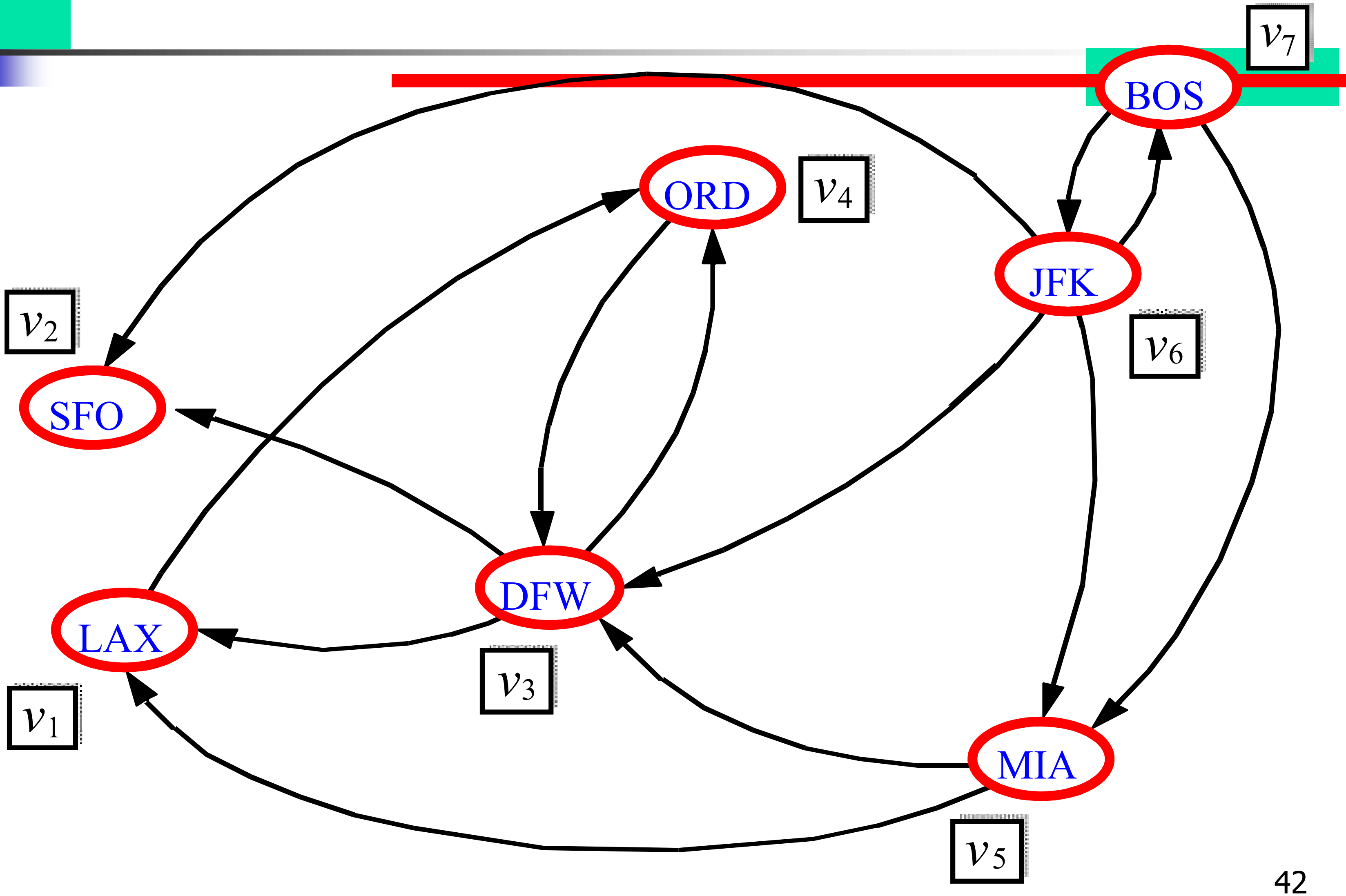
Floyd-Warshall's Algorithm



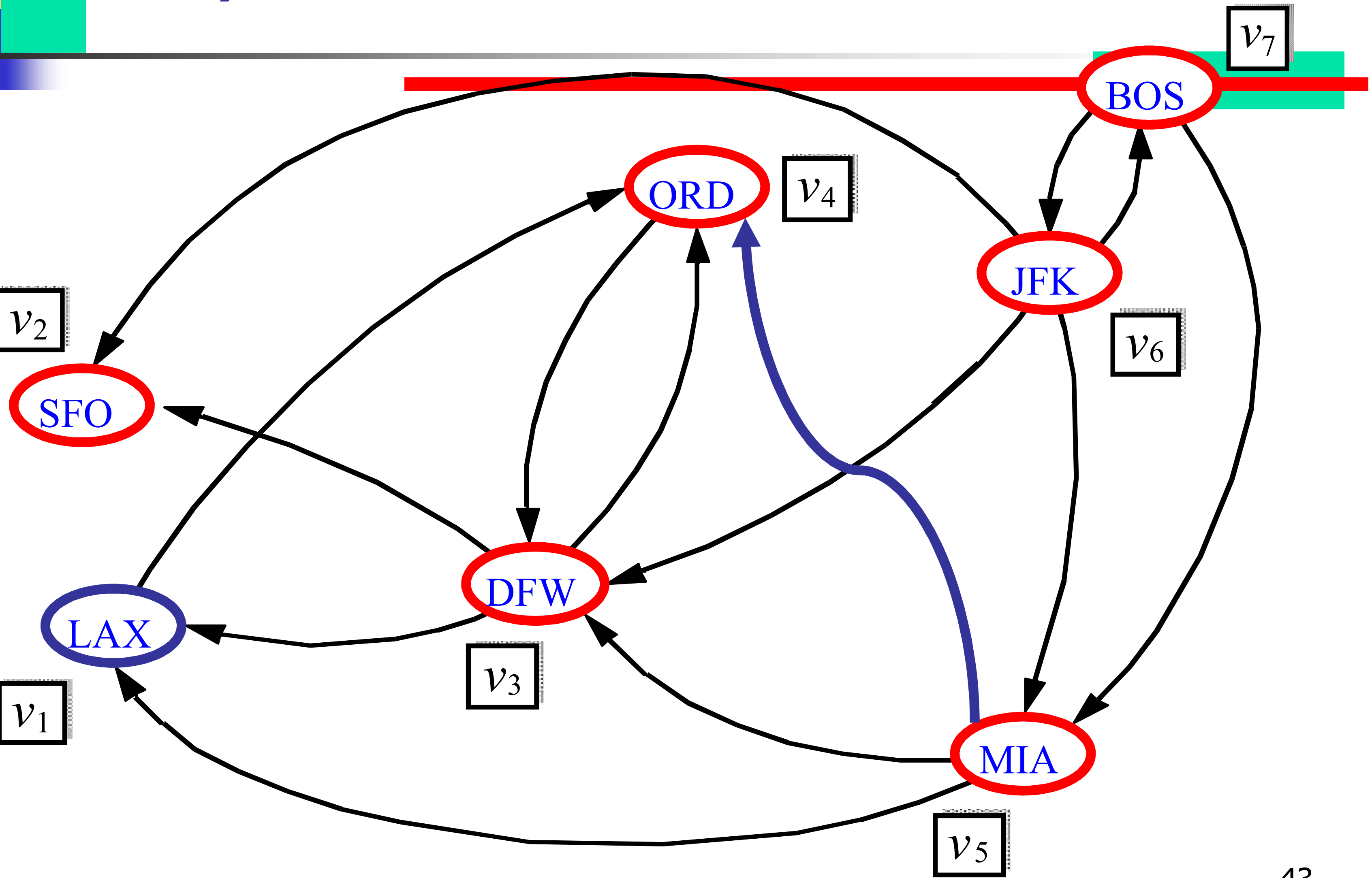
- Floyd-Warshall's algorithm numbers the vertices of G as $v_1, \ldots, v_n$ and computes a series of digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - G_k has a directed edge (v_i, v_j) if G has a directed path from v_i to v_j with intermediate vertices in the set $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase k, digraph G_k is computed from G_{k-1}
- Running time: O(n^3), assuming areAdjacent is O(1) (e.g., adjacency matrix)

```
Algorithm FloydWarshall(G)
  Input digraph G
  Output transitive closure G* of G
{ i = 1;
  for all v ∈ G.vertices()
    { denote v as v_i ;
      i = i + 1; }
  G_0 = G;
  for k = 1 to n do
    { G_k = G_{k-1};
      for i = 1 to n (i ≠ k) do
        for j = 1 to n (j ≠ i, k) do
          if G_{k-1}.areAdjacent(v_i, v_k) ∧
               G_{k-1}.areAdjacent(v_k, v_j)
            if ¬G_k.areAdjacent(v_i, v_j)
               G_k.insertDirectedEdge(v_i, v_j , k);
    }
  return G_n
}
```

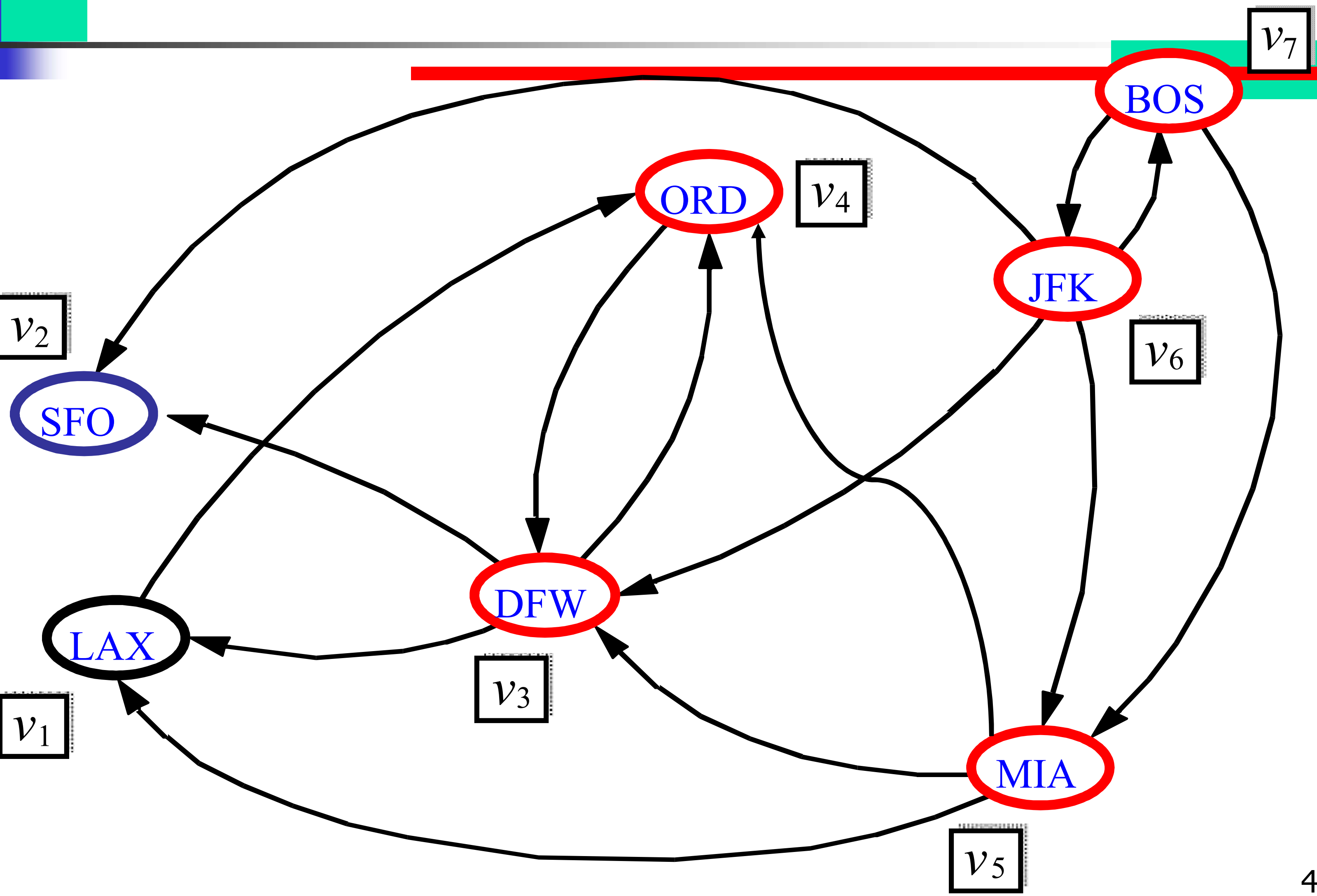
Floyd-Warshall Example



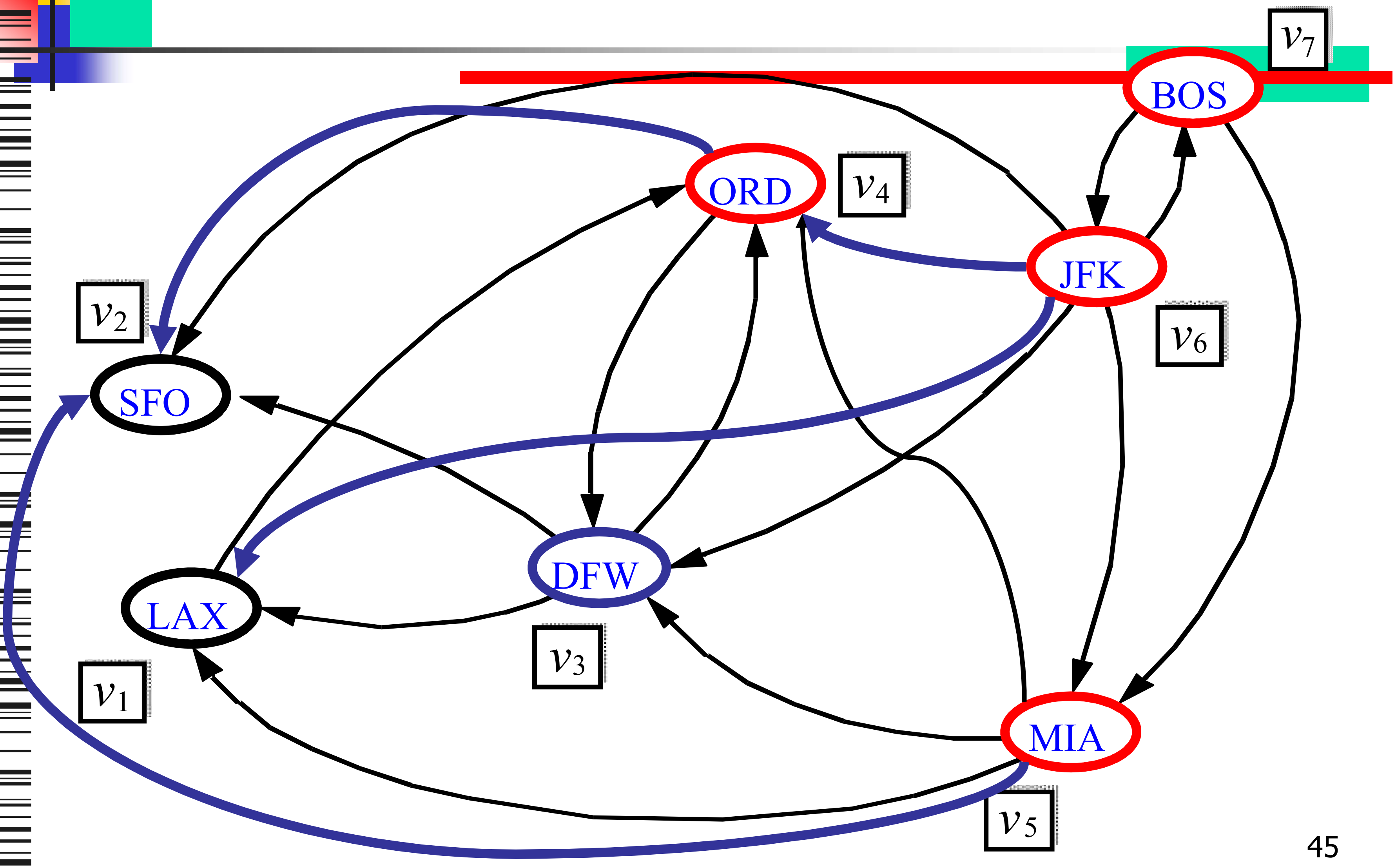
Floyd-Warshall, Iteration 1



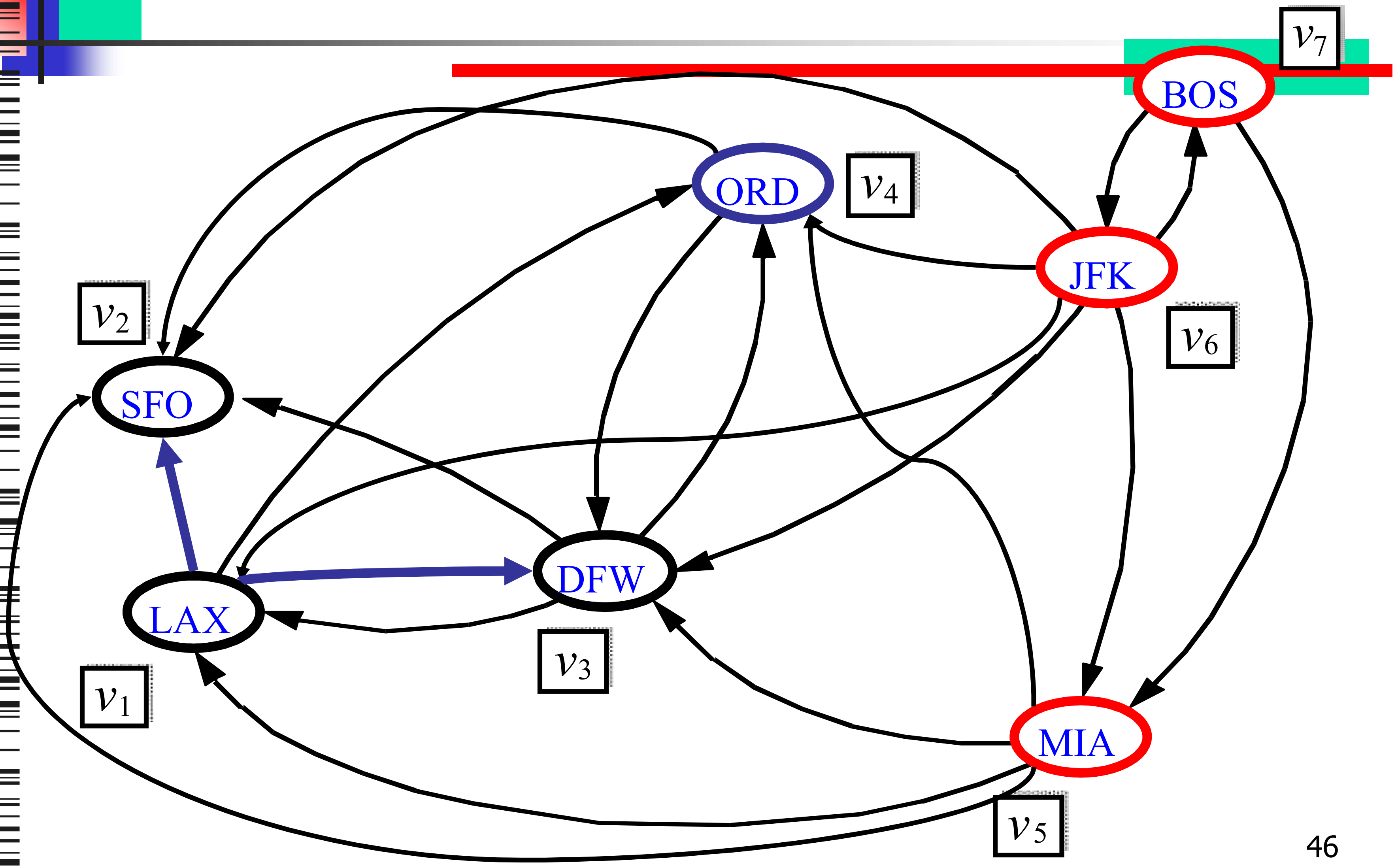
Floyd-Warshall, Iteration 2



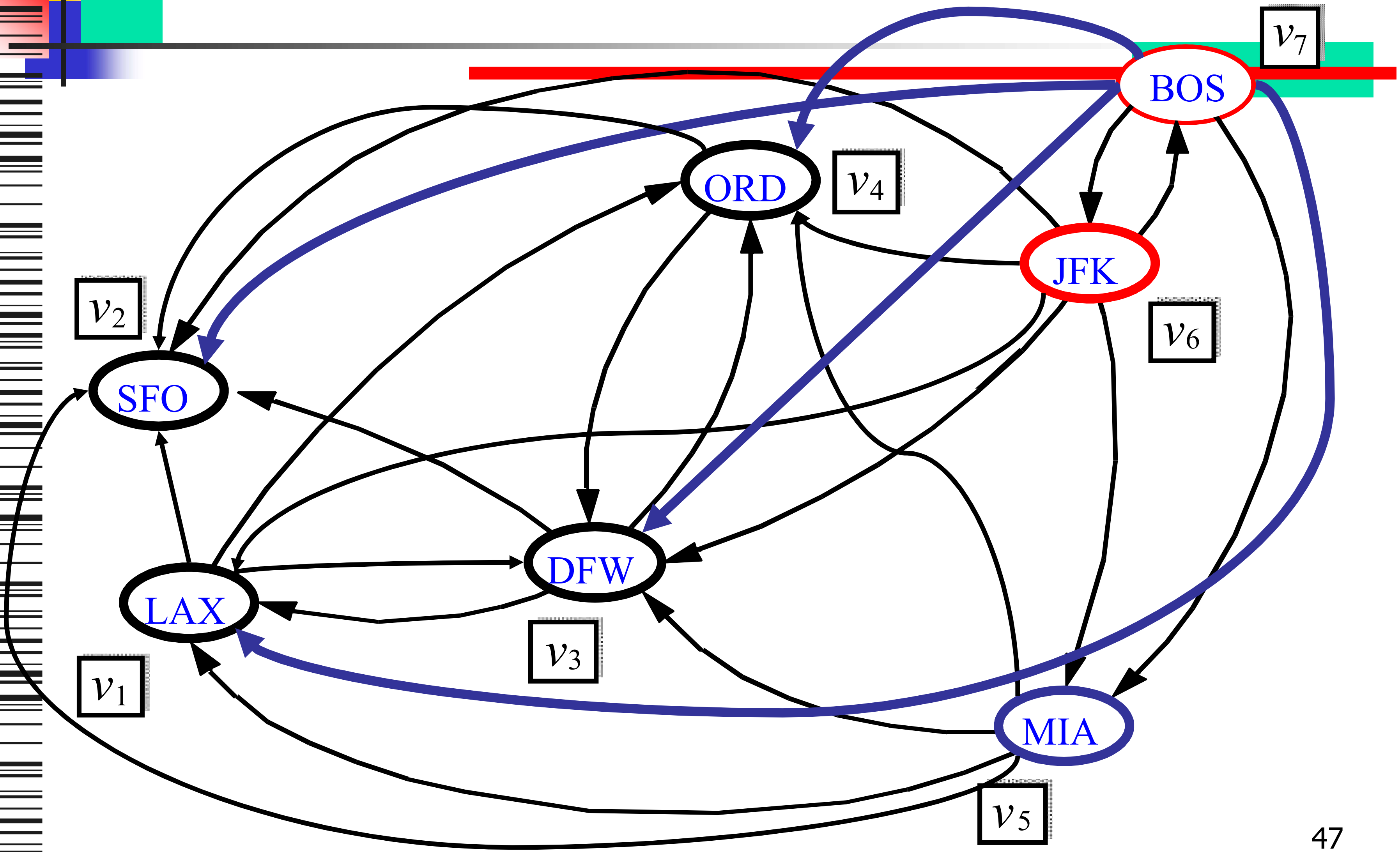
Floyd-Warshall, Iteration 3



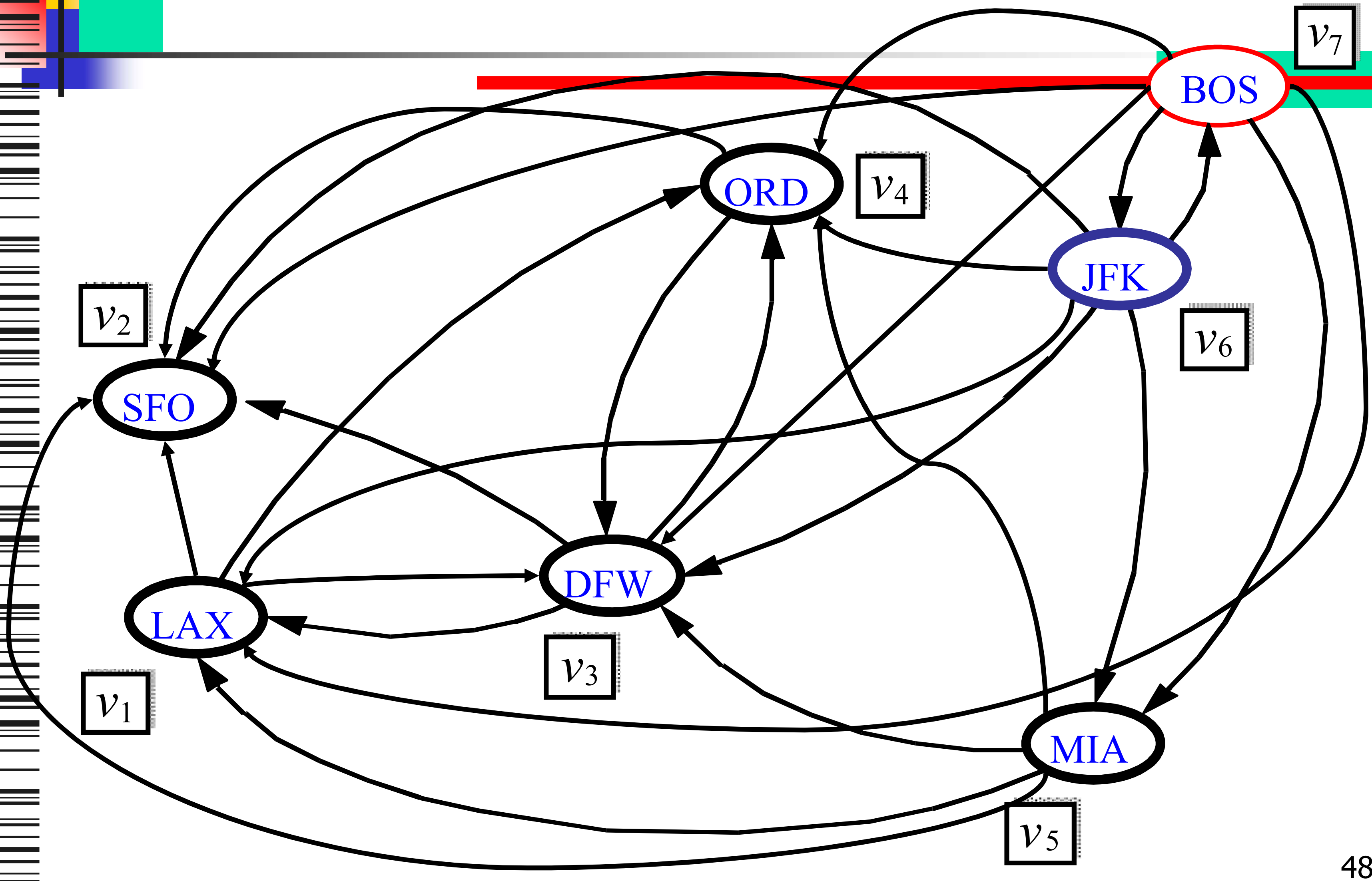
Floyd-Warshall, Iteration 4



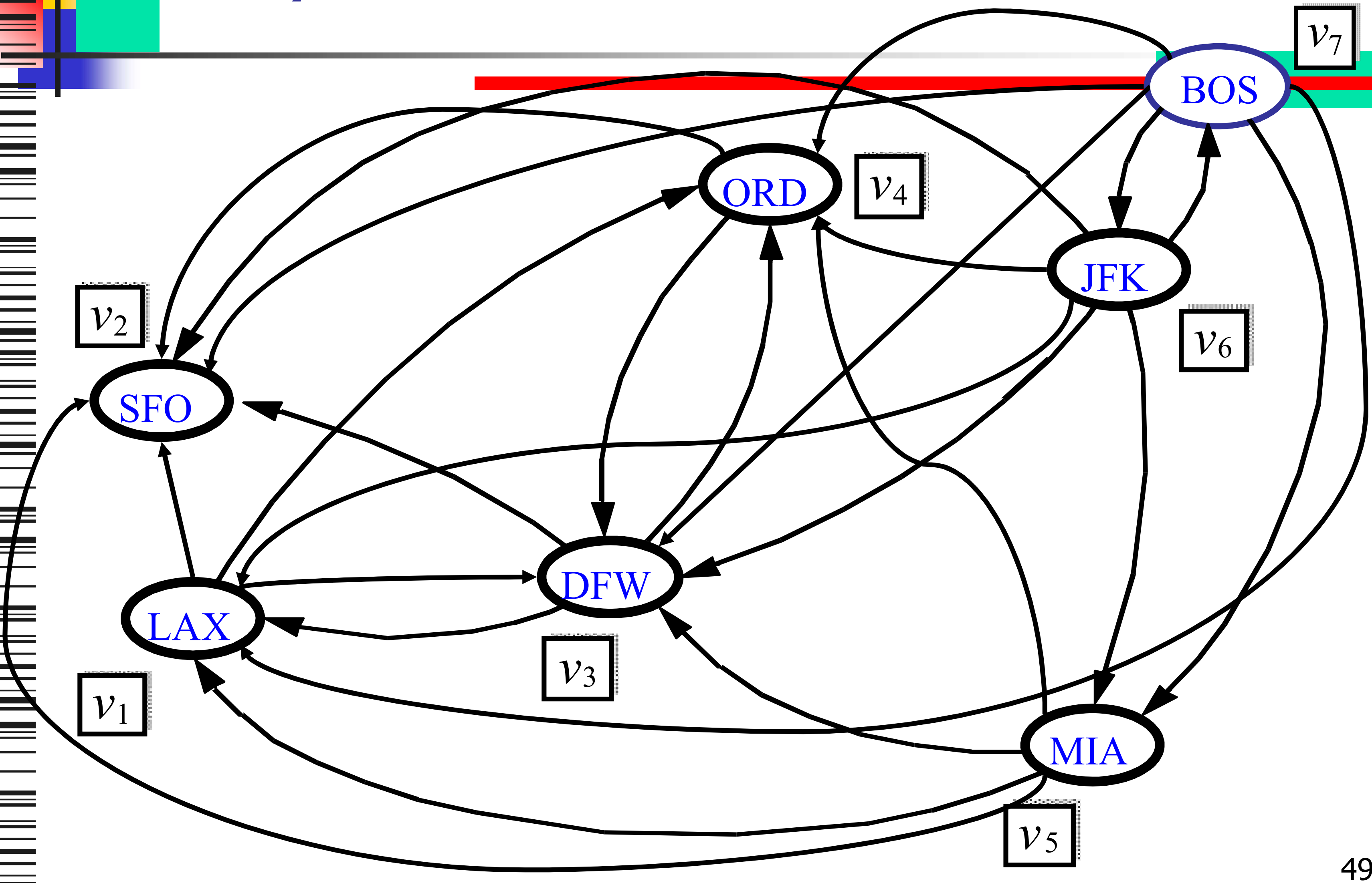
Floyd-Warshall, Iteration 5



Floyd-Warshall, Iteration 6



Floyd-Warshall, Conclusion



DAGs and Topological Ordering

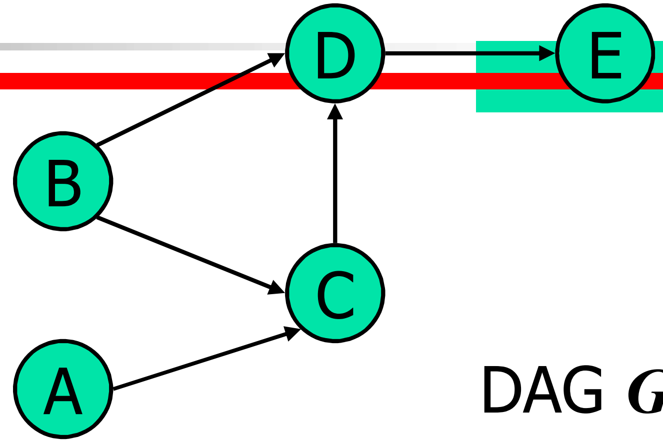
- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering

  $$v_1, \ldots, v_n$$

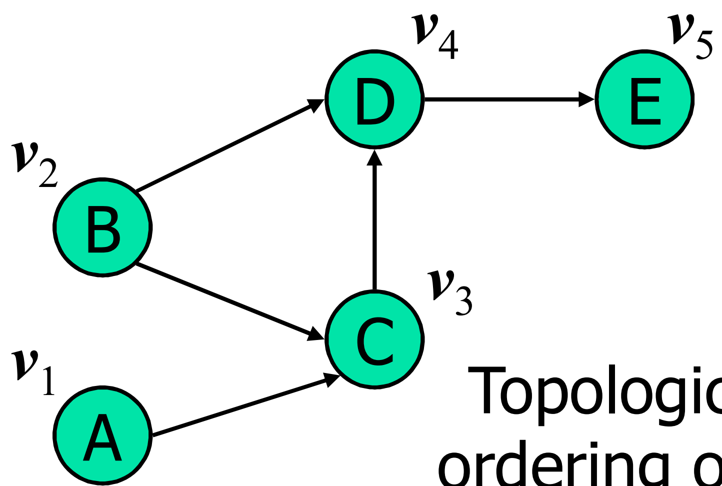
  of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints

Theorem

A digraph admits a topological ordering if and only if it is a DAG

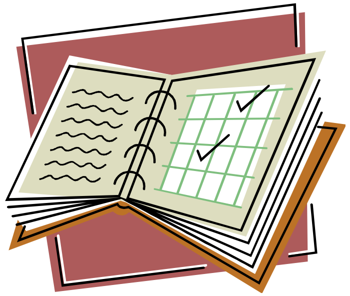


DAG G

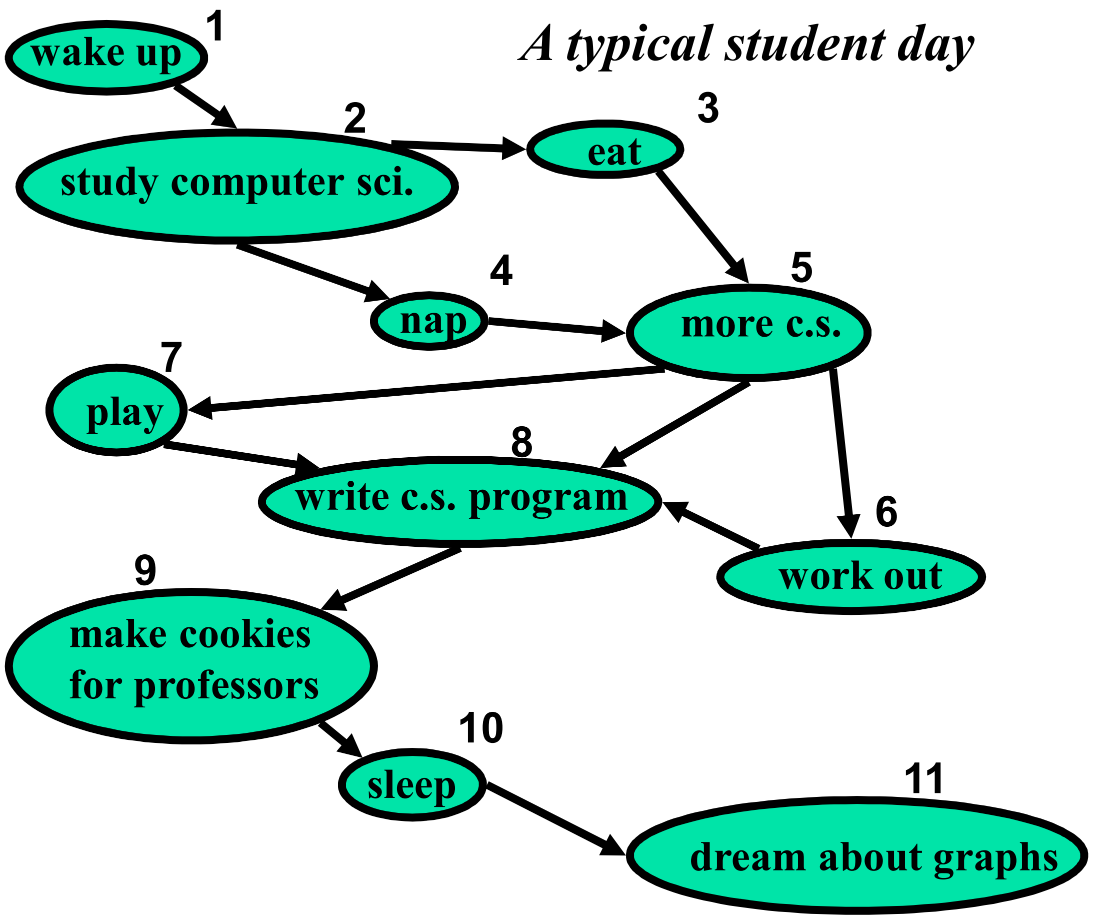


Topological ordering of G

Topological Sorting



- Number vertices, so that (u,v) in E implies u < v



Algorithm for Topological Sorting

```
Method TopologicalSort(G)
  { H = G;          // Temporary copy of G
    n = G.numVertices();
    while H is not empty do
       { Let v be a vertex with no outgoing edges;
         Label v = n;
         n = n – 1;
         Remove v from H;
       }
  }
```

- Running time: O(n + m). How…?

Topological Sorting Algorithm using DFS

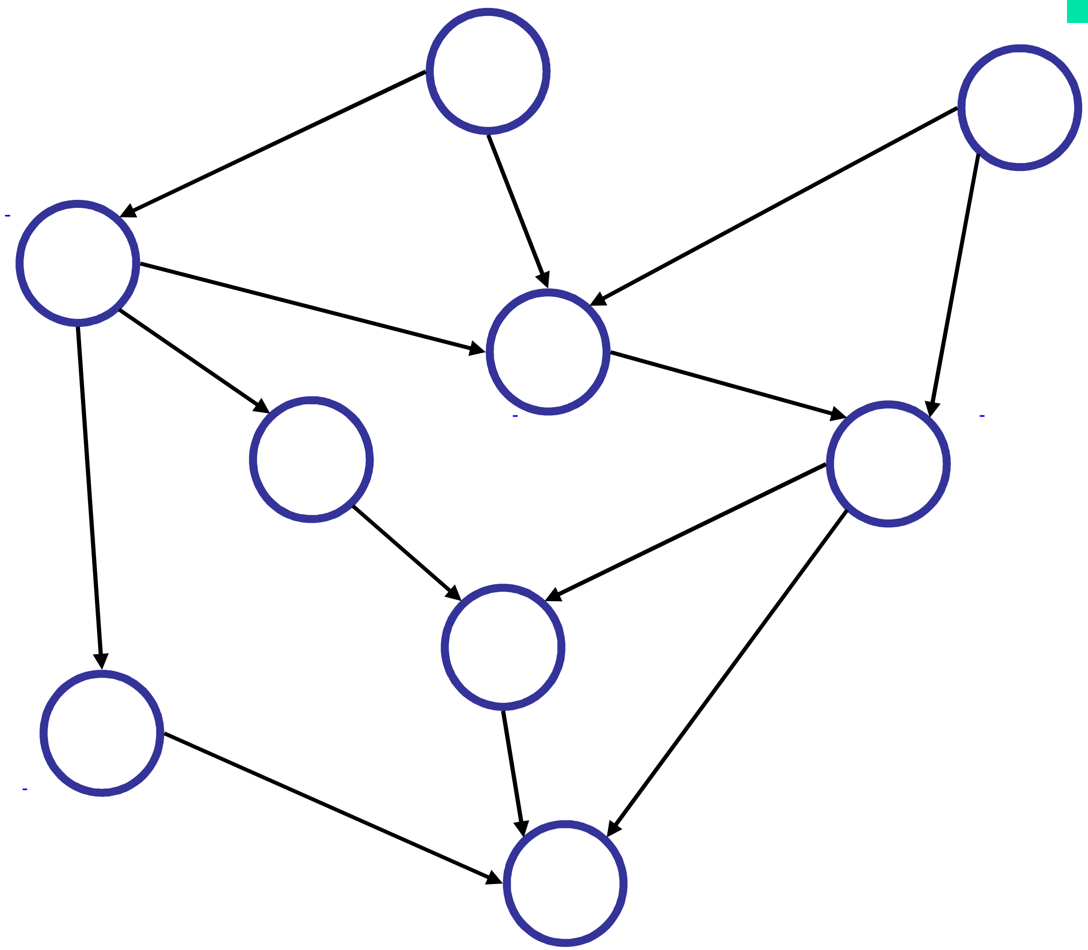
- Simulate the algorithm by using depth-first search

```
Algorithm topologicalDFS(G)
    Input dag G
    Output topological ordering of G
{ n = G.numVertices();
   for all u ∈ G.vertices()
     setLabel(u, UNEXPLORED);
   for all e ∈ G.edges()
     setLabel(e, UNEXPLORED);
   for all v ∈ G.vertices()
     if ( getLabel(v) = UNEXPLORED )
        topologicalDFS(G, v);
}
```

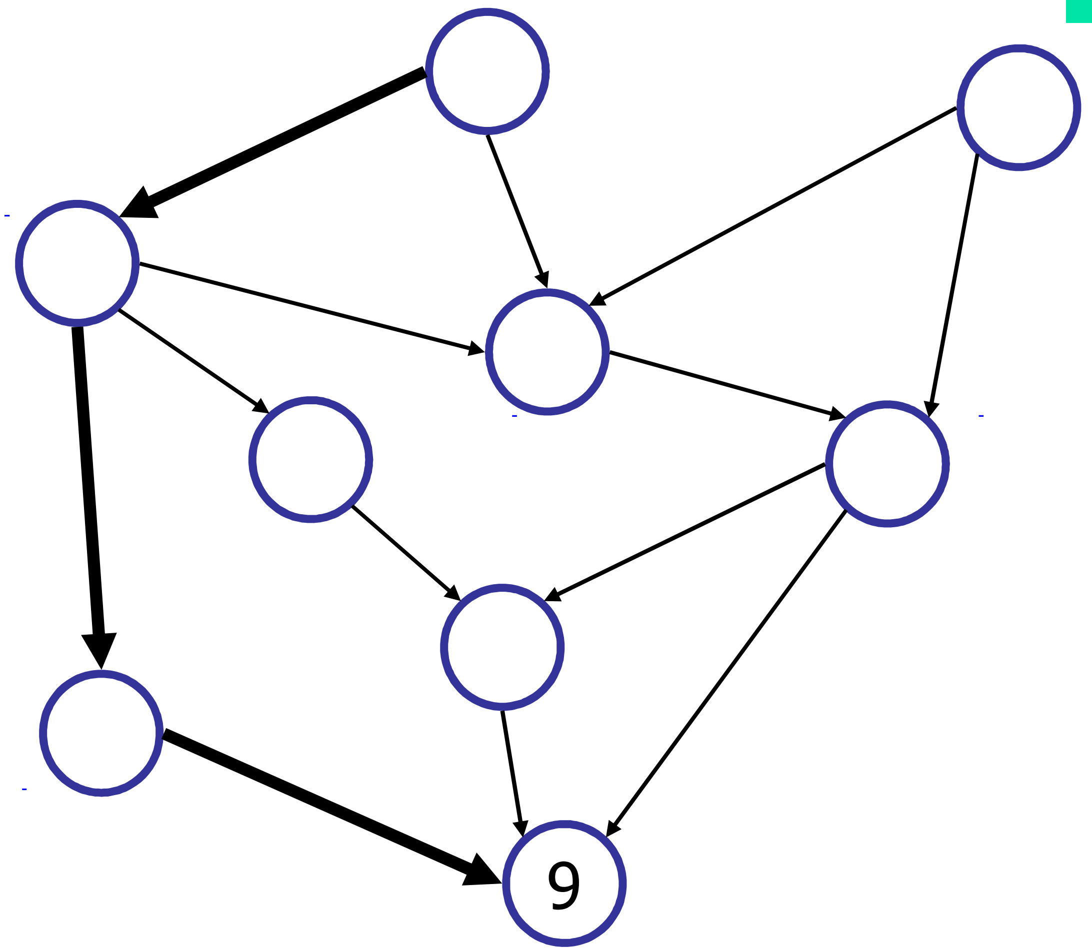
```
Algorithm topologicalDFS(G, v)
   Input graph G and a start vertex v of G
   Output labeling of the vertices of G
      in the connected component of v
{ setLabel(v, VISITED);
   for all e ∈ G.incidentEdges(v)
      if ( getLabel(e) = UNEXPLORED )
        { w = opposite(v,e);
           if ( getLabel(w) = UNEXPLORED )
             { setLabel(e, DISCOVERY);
               topologicalDFS(G, w); }
        }
   Label v with topological number n;
   n = n – 1;
   return;
}
```

- O(n+m) time.

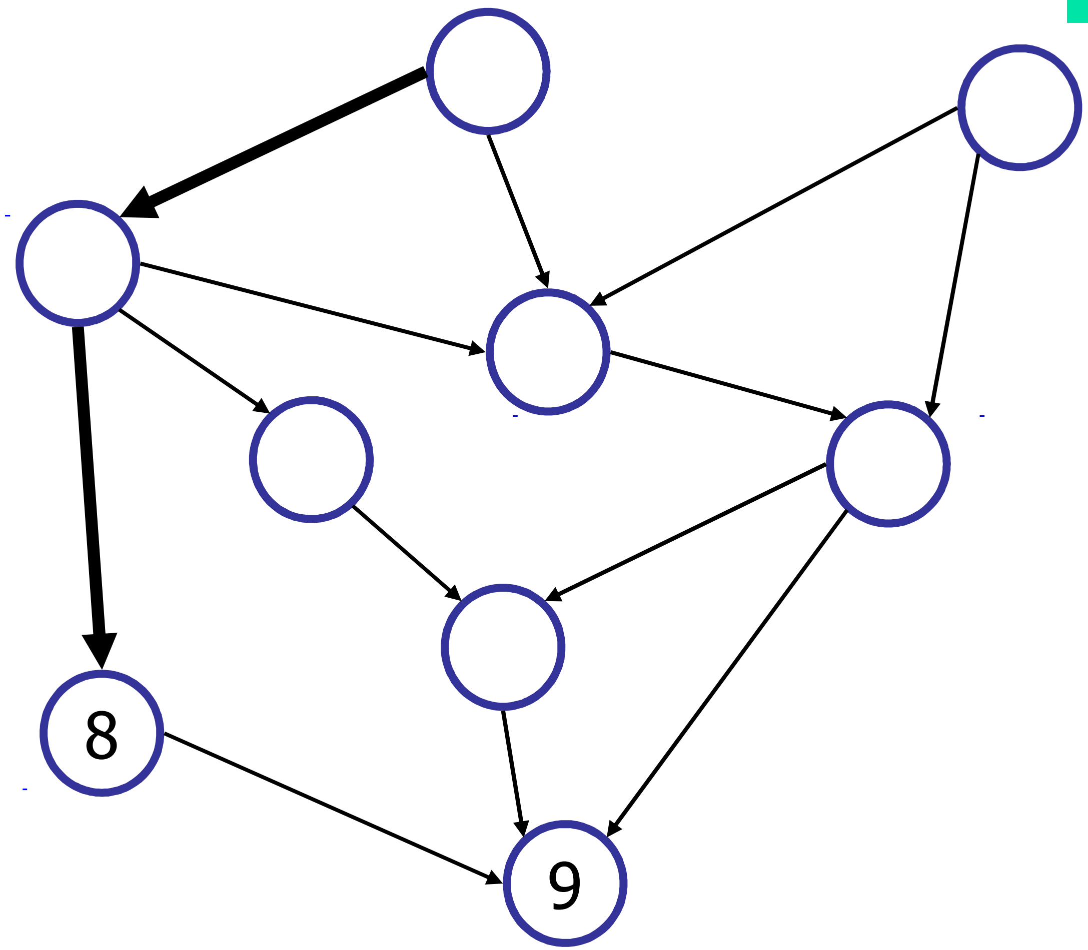
Topological Sorting Example (1/10)



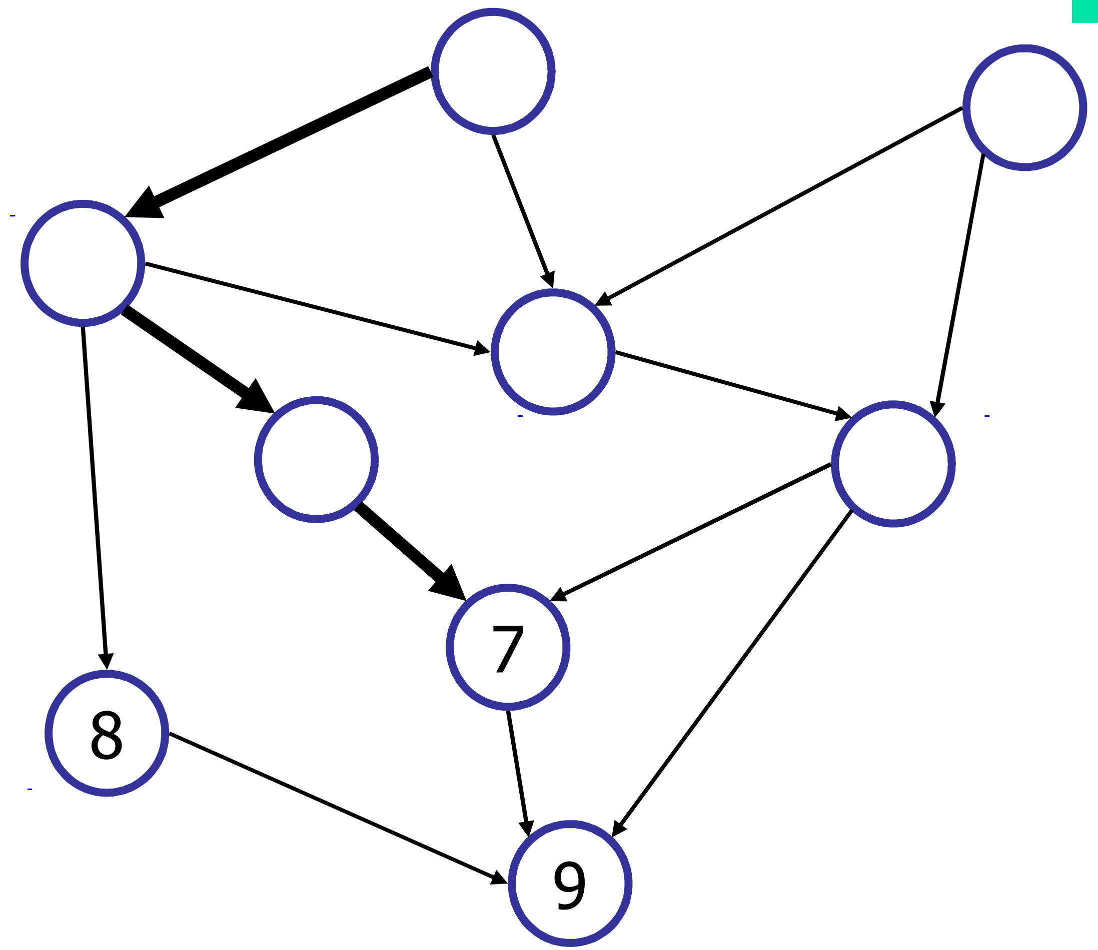
Topological Sorting Example (2/10)



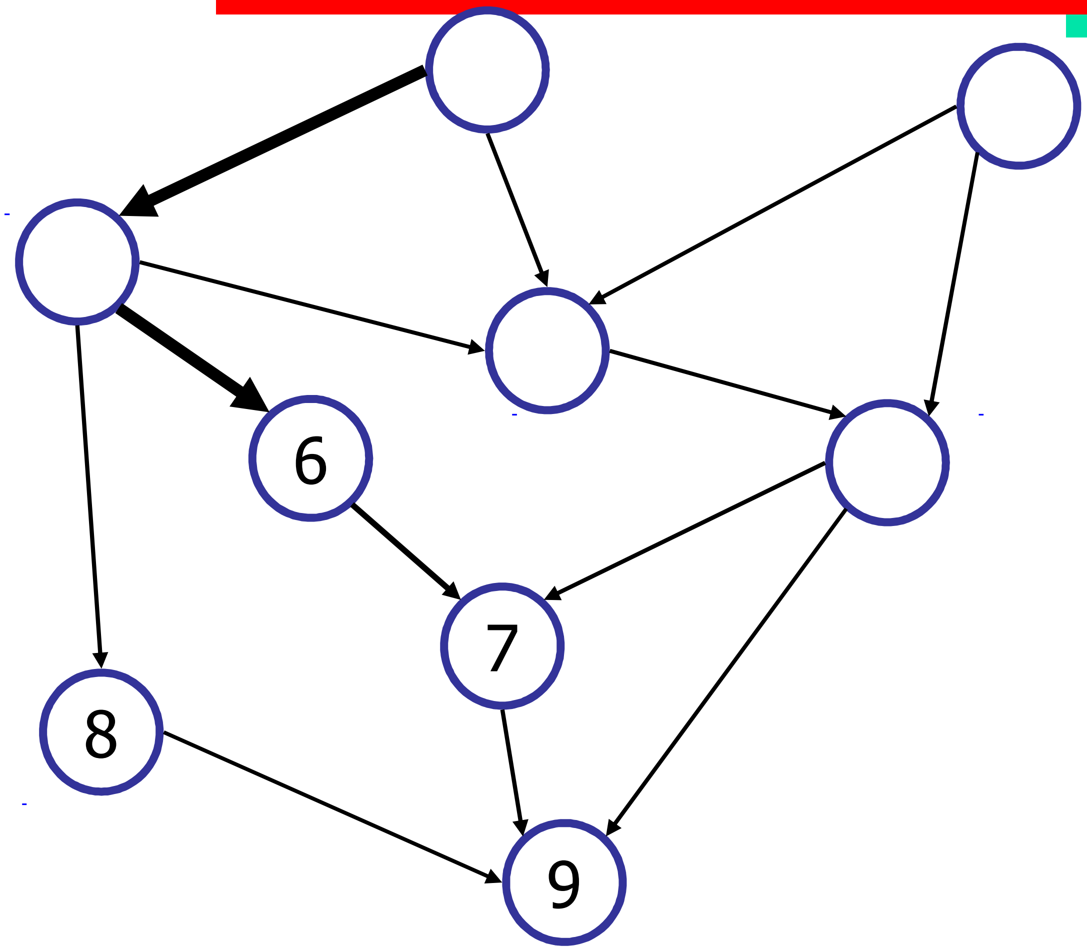
Topological Sorting Example (3/10)



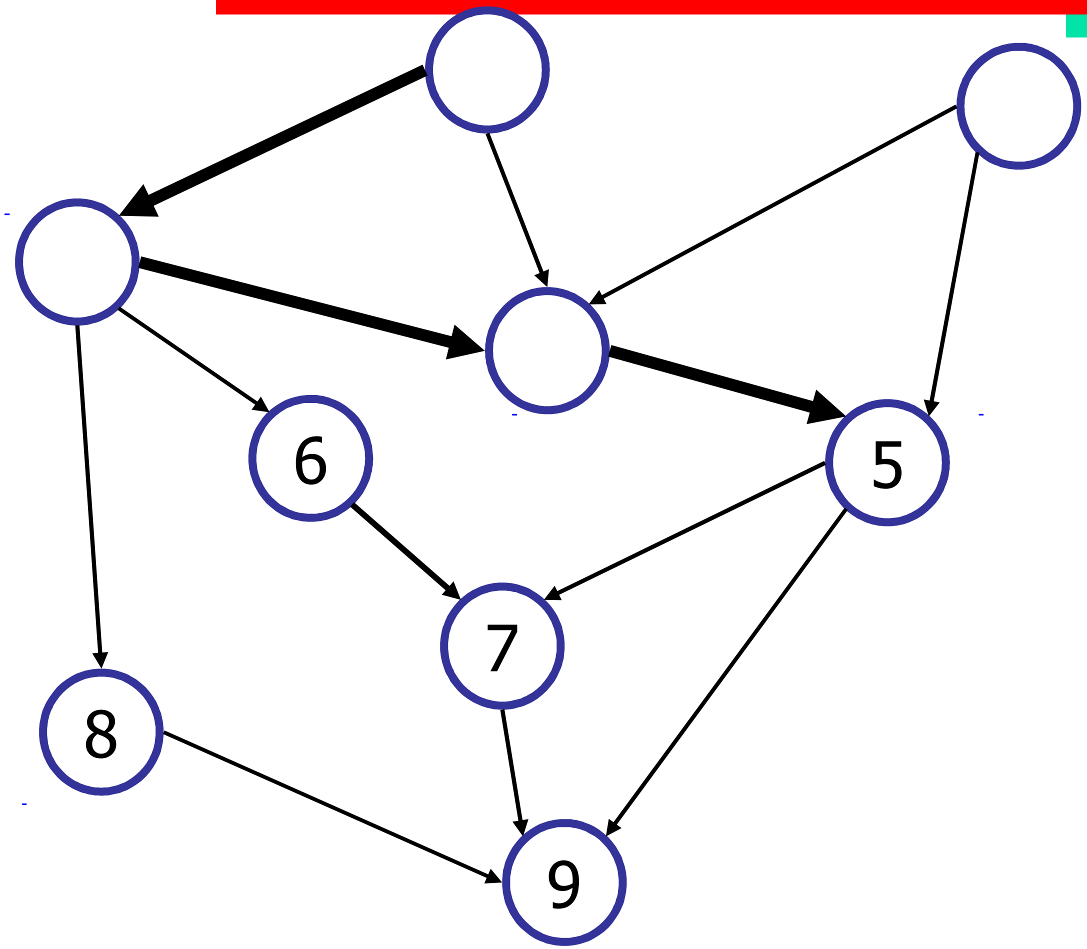
Topological Sorting Example (4/10)



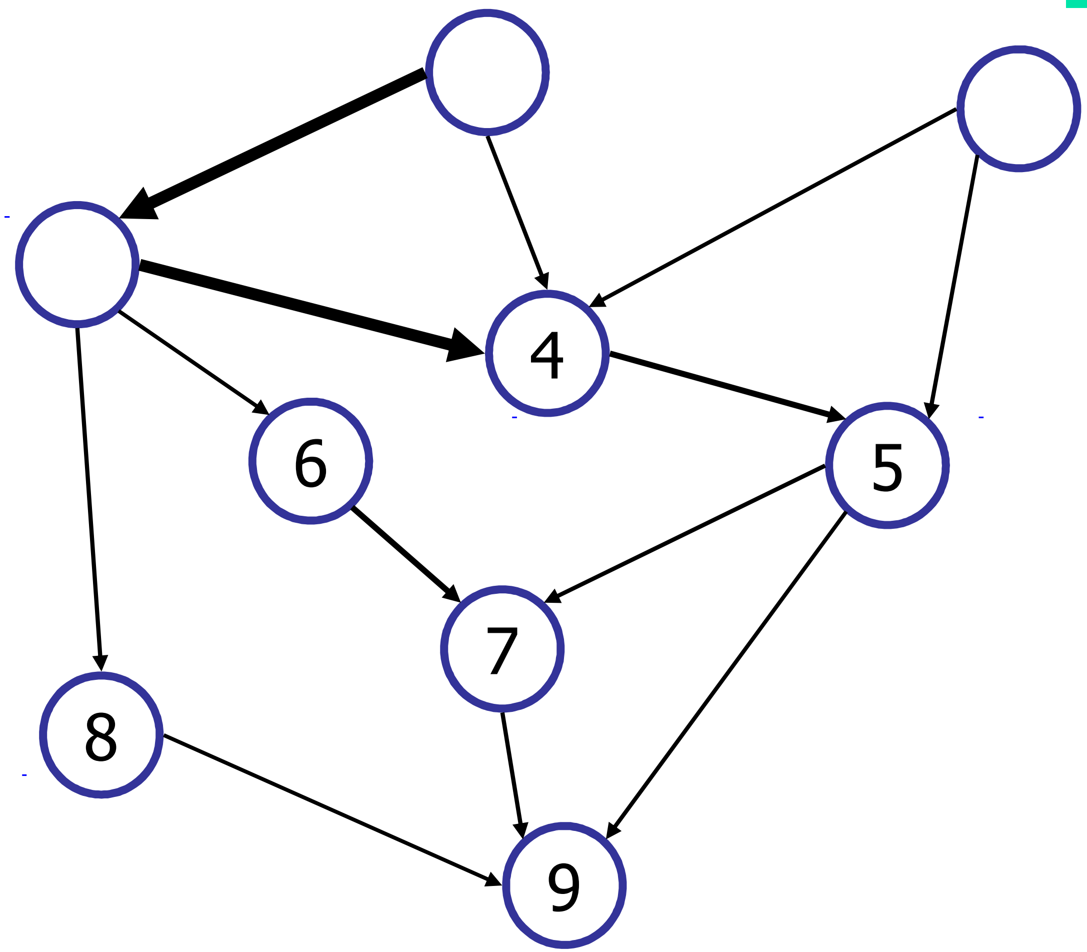
Topological Sorting Example (5/10)



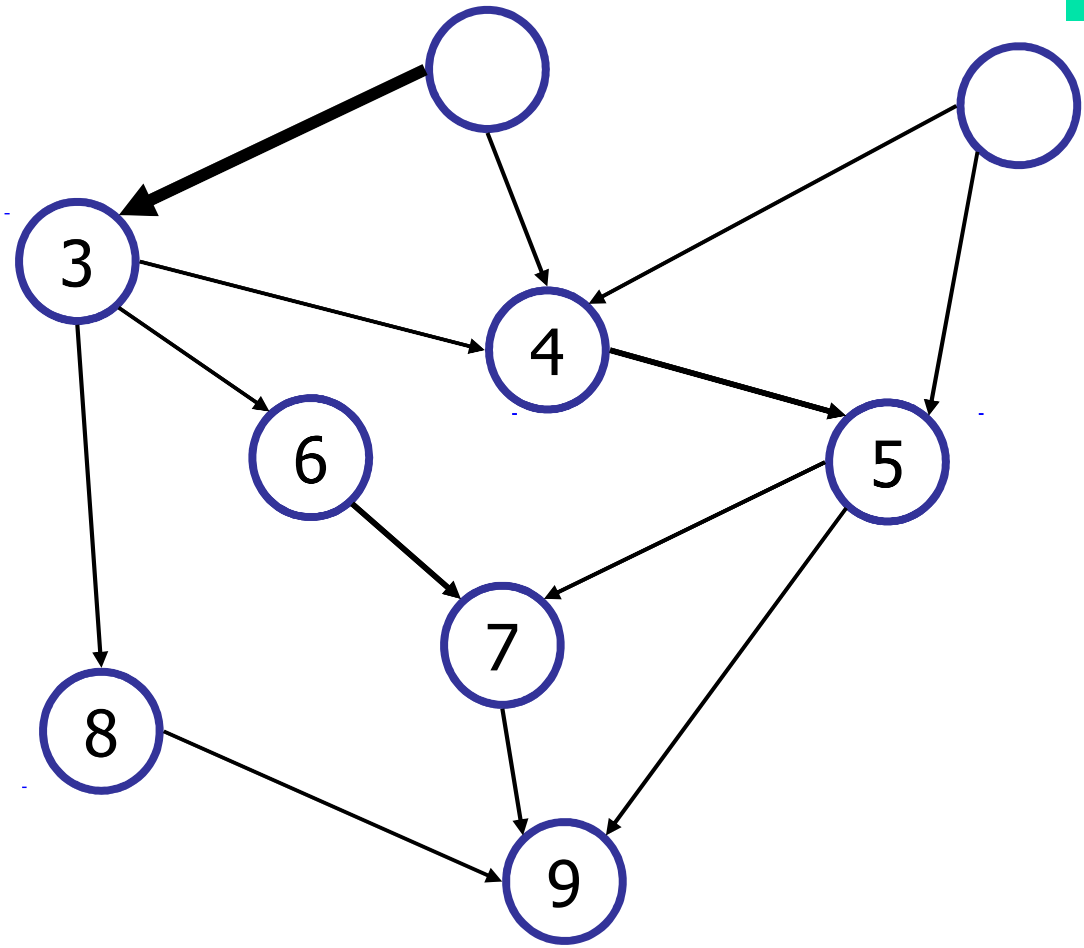
Topological Sorting Example (6/10)



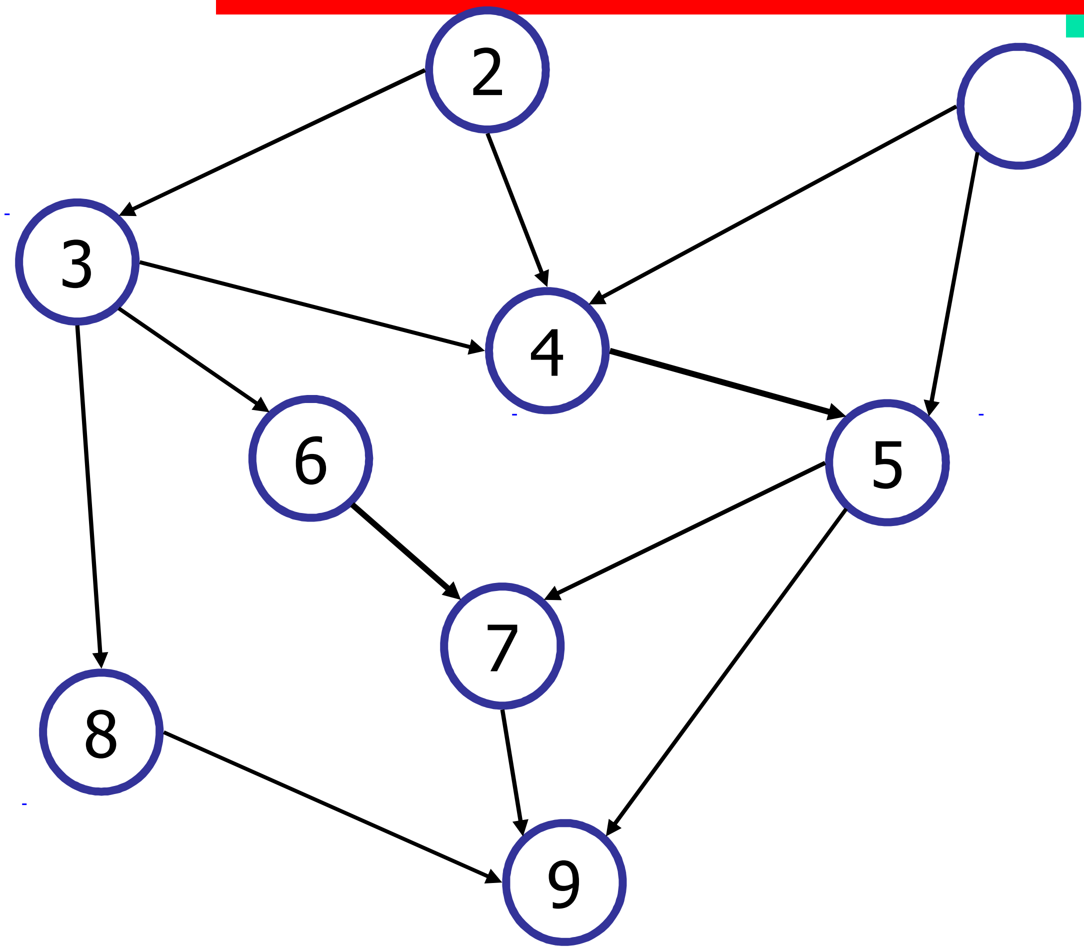
Topological Sorting Example (7/10)



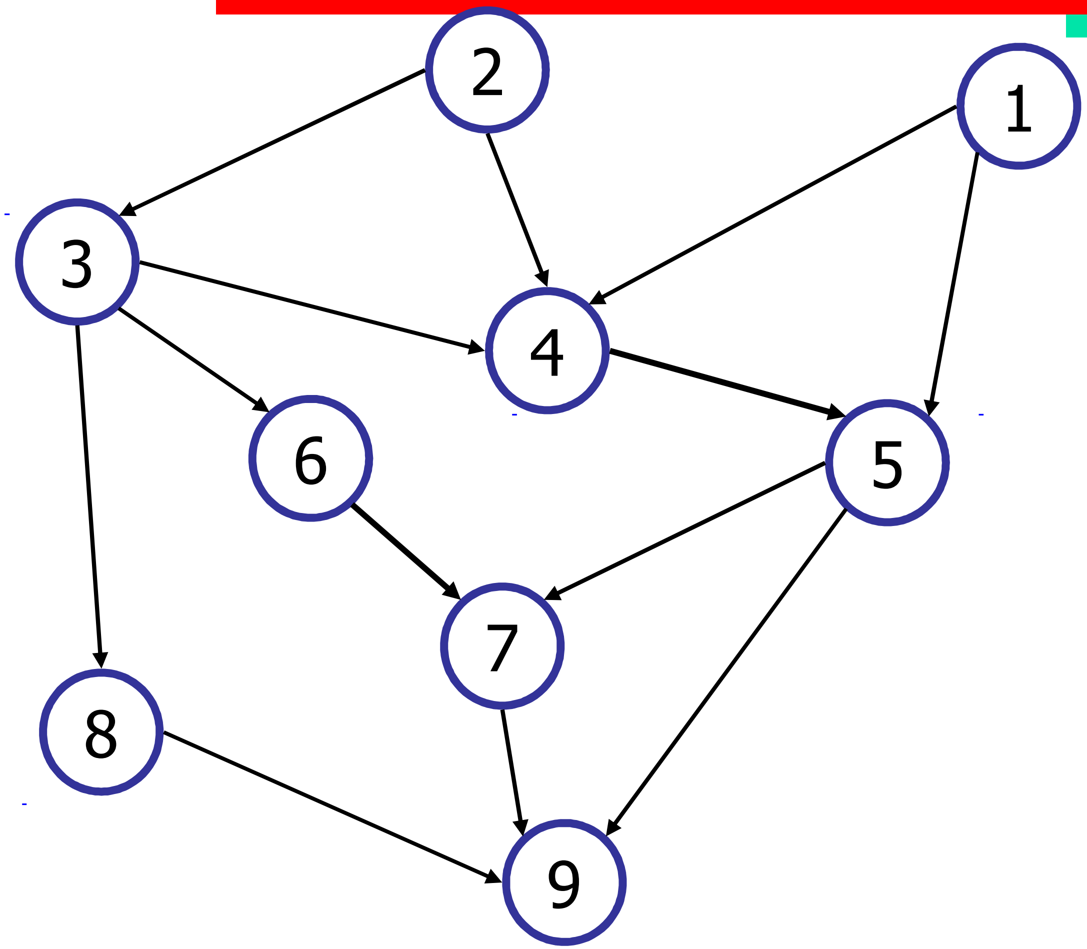
Topological Sorting Example (8/10)



Topological Sorting Example (9/10)



Topological Sorting Example (10/10)



Summary

- Depth-First Search
- Breadth-First Search
- Transitive Closure
- Topological Sorting
- Suggested reading (Sedgewick):
  - Ch.18
  - Ch.19