Linear Alegbra

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3.1 Fields

A field $\mathcal{F}$ is an abstract algebraic object. Throughout these notes $\mathcal{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. [1]

3.2 Complex Numbers

A complex number is an order pair $\in \mathbb{C}$ where $a, b \in \mathbb{R}$ where we can denote it as $z = a + ib$ where $i = \sqrt{-1}$

3.2.1 Addition

$z_1 = a_1 + ib_1,\ z_2 = a_2 + ib_2$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

3.2.2 Multiplication

$z_1 = a_1 + ib_1,\ z_2 = a_2 + ib_2$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

3.2.3 Properties

[2]

[1] Many of theorems and definitions work even if replace $\mathcal{F}$ with an arbitrary field.
[2] $\mathcal{W}, \mathcal{Z}, \lambda \in \mathbb{C}$

3.2.3.1 Commutativity

$$\mathcal{W} + \mathcal{Z} = \mathcal{Z} + \mathcal{W}$$

$$\mathcal{W}\mathcal{Z} = \mathcal{Z}\mathcal{W}$$

3.2.3.2 Associativity

$$(\mathcal{Z}_1 + \mathcal{Z}_2) + \mathcal{Z}_3 = \mathcal{Z}_1 + (\mathcal{Z}_2 + \mathcal{Z}_3)$$

$$(\mathcal{Z}_1\mathcal{Z}_2)\mathcal{Z}_3 = \mathcal{Z}_1(\mathcal{Z}_2\mathcal{Z}_3)$$

3.2.3.3 Identities

$$\mathcal{Z} + 0 = \mathcal{Z}$$

$$\mathcal{Z}1 = \mathcal{Z}$$

3.2.3.4 Additive Inverse

$$\forall\ \mathcal{Z}\ \exists\ \mathcal{Z}^{-1} \mid \mathcal{Z} + \mathcal{Z}^{-1} = 0$$

3.2.3.5 Multiplicative Inverse

$$\forall\ \mathcal{Z} \neq 0\ \exists\ \mathcal{W} \mid \mathcal{Z}\mathcal{W} = 1$$

3.2.3.6 Distributive Property

$$\lambda(\mathcal{W} + \mathcal{Z}) = \lambda\mathcal{W} + \lambda\mathcal{Z}$$

3.3 Notation

n-tuple refers to an ordered set of n numbers over a field $\mathcal{F}$.

3.4 Definition of a Vector Space

A vector space $\mathbb{V}$ is a set along with the regular multiplication and addition operations over a field $\mathcal{F}$, such that the following axioms hold: [3]

[3] Here, $\alpha, \beta \in \mathcal{F}$ and $\mathcal{U}, \mathcal{V}$ and $\mathcal{W} \in \mathbb{V}$

3.4.1 Commutativity

$$\mathcal{U} + \mathcal{V} = \mathcal{V} + \mathcal{U}$$

3.4.2 Associativity

$$(\mathcal{U} + \mathcal{V}) + \mathcal{W} = \mathcal{V} + (\mathcal{U} + \mathcal{W})$$

$$(\alpha\beta)\mathcal{V} = \alpha(\beta\mathcal{V})$$

3.4.3 Additive Identity

$$\exists\ 0 \in \mathbb{V} \mid \mathcal{V} + 0 = 0 + \mathcal{V} = \mathcal{V}$$

3.4.4 Additive Inverse

$$\forall\ \mathcal{V}\ \exists\ \mathcal{V}^{-1} \mid \mathcal{V} + \mathcal{V} = 0$$

3.4.5 Multiplicative identity

$$\exists\ 1 \in \mathbb{V} \mid 1\mathcal{V} = \mathcal{V}$$

3.4.6 Distributive properties

$$\alpha(\mathcal{U} + \mathcal{V}) = \alpha\mathcal{U} + \alpha\mathcal{V}$$

$$(\alpha + \beta)\mathcal{U} = \alpha\mathcal{U} + \beta\mathcal{U}$$

3.5 Properties of a Vector Space

3.5.1 *A vector space has a unique additive identity*

Suppose there exist two additive identities 0 and $0^{'}$ for the vector space $\mathbb{V}$, we can say that

$$0 = 0 + 0^{'} = 0^{'}$$

Thus,

$$\boxed{0 = 0^{'} \qquad (3.1)}$$

3.5.2 Ever element in a vector space has a unique additive inverse

Suppose where $\mathcal{W}$ and $\mathcal{W}'$ are the additive inverses of $\mathcal{V}$, then

$$\mathcal{W} = \mathcal{W}' \tag{3.2}$$

3.5.3 $0\mathcal{V} = 0 \ \forall \ \mathcal{V} \in \mathbb{V}$

$\forall \ \mathcal{V} \in \mathbb{V}$,

$$0\mathcal{V} = (0+0)\mathcal{V} = 0\mathcal{V} + 0\mathcal{V}$$

$$0\mathcal{V} - 0\mathcal{V} = 0 = 0\mathcal{V}$$

Thus,

$$0 = 0\mathcal{V} \tag{3.3}$$

3.5.4 $0\alpha = 0 \ \forall \ \alpha \in \mathcal{F}$

$\forall \ \alpha \in \mathbb{F}$,

$$0\alpha = (0+0)\alpha = 0\alpha + 0\alpha$$

$$0\alpha - 0\alpha = 0 = 0\alpha$$

Thus,

$$0\alpha = 0 \tag{3.4}$$

3.5.5 $(-1)\mathcal{V} = -\mathcal{V} \ \forall \ \mathcal{V} \in \mathcal{F}$

$\forall \ \mathcal{V} \in \mathbb{V}$,

$$0\mathcal{V} = (0+0)\mathcal{V} = 0\mathcal{V} + 0\mathcal{V}$$

$$0\mathcal{V} - 0\mathcal{V} = 0 = 0\mathcal{V}$$

Thus,

$$0 = 0\mathcal{V} \tag{3.5}$$

3.6 Subspaces

3.6.1 Definition

A $\mathbb{U} \subset \mathbb{V}$ is called a ***subspace*** of $\mathbb{V}$ if $\mathbb{U}$ is also a vector space as defined in Sec 1.3

3.6.2 Properties

If $\mathbb{U} \subset \mathbb{V}$ then to check whether $\mathbb{U}$ is a subspace of $\mathbb{V}$, we simply need to check for the following properties

3.6.2.1 Additive identity

$$0 \in \mathbb{U}$$

3.6.2.2 Closed under addition

$$\mathcal{U}, \mathcal{V} \in \mathbb{U} \implies \mathcal{U} + \mathcal{V} \in \mathbb{U}$$

3.6.2.3 Closed under scalar multiplication

$$\forall\ \alpha \in \mathcal{F}\ and\ \mathcal{U} \in \mathbb{U} \implies \alpha\mathcal{U} \in \mathbb{U}$$

3.7 Sums

The sum of $\mathcal{U}$ and $\mathcal{V}$ which are subspaces of $\mathbb{V}$ is defined to be the set of all poissible sums of the elements is denoted in the RHS as,

$$\mathcal{U} + \mathcal{V} = \{u + v : u \in \mathcal{U}, v \in \mathcal{V}\}$$

3.8 Direct Sums

A direct sum of sub-spaces is a special type of sum in which

3.8.1 Proposition 1

Suppose $\mathbb{U}_1, \mathbb{U}_2$ are subspaces of $\mathbb{V}$. Then $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2$ if and only if both the following conditions hold:

- $\mathbb{V} = \mathbb{U}_1 + \mathbb{U}_2$

- the only wayt to write - as a sum $\mathcal{U}_1 + \mathcal{U}_2$, where each $\mathcal{U}_j \in \mathbb{U}_j$, is by taking all the $\mathcal{U}_j = 0$

3.8.1.1 Proof

First suppose that $\mathbb{V} = \mathbb{U}_1 \oplus \mathbb{U}_2$. Clearly the first condition holds because of how sum and direct sum are defined. To prove the latter suppose $\mathcal{U}_1 \in \mathbb{U}_1, \mathcal{U}_2 \in \mathbb{U}_2$ and

$$0 = \mathcal{U}_1 + \mathcal{U}_2$$

Then each $\mathcal{U}_i$ must be, as this follows from the uniqueness part of the definition of direct sum because $0 = 0 + 0$ and $0 \in \mathbb{U}_1, 0 \in \mathbb{U}_2$. Now suppose that both the conditions hold. Let $\mathcal{V} \in \mathbb{V}$. By the first condition we can write:

$$\mathcal{V} = \mathcal{U}_1 + \mathcal{U}_1$$

for some $\mathcal{U}_1 \in \mathbb{U}_1$ and $\mathcal{U}_2 \in \mathbb{U}_2$. To show that this representation is unique, suppose we also have:

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$$

where $\mathcal{V}_1 \in \mathbb{U}_1$ and $\mathcal{V}_2 \in \mathbb{U}_2$. Subtracting these two equations we have

$$0 = (\mathcal{U}_1 - \mathcal{V}_1) + (\mathcal{U}_2 - \mathcal{V}_2)$$

Clearly $\mathcal{U}_i - \mathcal{V}_i \in \mathbb{U}_i$, so the equation above and the second condition imply that each $\mathcal{U}_i - \mathcal{V}_i = 0$. Thus, $\mathcal{U}_i = \mathcal{V}_i$ as desired.

3.8.2 Proposition 2

Suppose that $\mathbb{U}$ and $\mathbb{W}$ are subspaces of $\mathbb{V}$. Then $\mathbb{V} = \mathbb{U}+\mathbb{W}$ i.f.f. $\mathbb{V} = \mathbb{U}+\mathbb{W}$ and $\mathbb{V} \cap \mathbb{W} = 0$.

3.8.2.1 Proof

First suppose that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$. Then $\mathbb{V} = \mathbb{U} + \mathbb{W}$, by the definition of a direct sum. Also, if $\mathcal{V} \in \mathbb{U} \cap \mathbb{W}$, then $0 = \mathcal{V} + (-\mathcal{V})$, where $\mathcal{V} \in \mathbb{U}$ and $-\mathcal{V} \in \mathbb{W}$. By the unique reppresentation of as the sum of a vector $\mathbb{U}$ in and a vector in $\mathbb{W}$, we must have $\mathcal{V} = 0$.Thus, $\mathbb{U} \cap \mathbb{W} = \{0\}$. This is one way to prove it.

To prove the other way, now suppose that $\mathbb{V} = \mathbb{U} + \mathbb{W}$ and $\mathbb{U} \cap \mathbb{W} = 0$. To prove that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$, suppose that

$$0 = \mathcal{U} + \mathcal{V}$$

where $\mathcal{U} \in \mathbb{U}$ and $\mathcal{W} \in \mathbb{W}$. To complete the proof, wee only need to show that $\mathcal{U} = \mathcal{W} = 0$. The equation above implies that $\mathcal{U} = -\mathcal{W} \in \mathbb{W}$. from axiom 4. Thus, $\mathcal{U} \in \mathbb{U} \cap \mathbb{W}$, and hence $\mathcal{U} = 0$

3.9 Note

- Sums of subspaces are analogous to unions of subsets
- Similarly, direct sums of subspaces are analogous to disjoint unions of subset i.e. they have no element in common
- No two subspaces of a vector space can be disjoint because both must contain 0 as per the axioms stated in
- Thus, disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals 0

Chapter 4

Finite Dimensional Vector Spaces

4.1 Span and Linear Independence

Linear Combination: Is a vector formed from a list $(\mathcal{V}_1, ..., \mathcal{V}_i)$ of vectors with the structure

$$\sum_{i}^{n} a_i \mathcal{V}_i \tag{4.1}$$

$\forall\ a_i \in \mathcal{F}$

4.1.1 Span

The **Span** of is the set of all linear combinations denoted as

$$span(\mathcal{V}_1, ..., \mathcal{V}_i) = \{\sum_{i}^{n} a_i \mathcal{V}_i : a_i \in \mathcal{F}\} \tag{4.2}$$

The span of any list of vectors in $\mathbb{V}$ is a subspace of $\mathbb{V}$ until and unless we consider an empty list, in that case $span() = \{\}$

If $span(\mathcal{V}_1, ..., \mathcal{V}_i) = \mathbb{V}$, then we say that $span(\mathcal{V}_1, ..., \mathcal{V}_i)$ **spans** $\mathbb{V}$.

A vector space is said to be finite dimensional if it spanned by a list of vectors in it. This follows from the definition that a list must be finite. A vector space that is not finite dimensional is said to be infinite dimensional, a good example of are the elements of Polynomials over a field i.e. $\mathbb{P}(\mathcal{F})$

4.2 Bases

4.3 Dimension

Chapter 5

Infinite Dimensional Vector Spaces