A Summary of Graph Theory

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December 5, 2020

Contents

Chapter 1

Basics

1.1 Definitions

- A Graph is a set of objects and the relationships between pairs of objects
- A Graph $G(V, E)$, is a set of V **Vertices/nodes** and E **Edges**

Figure 1.1: A visual representation of a simple Graph

- For the above figure we say that:
  - e **Connects** u and v
  - u and v are **End Points** of e
  - u and e are **Incident**
  - u and v are **Adjacent**
  - u and v are **Neighbors**
- Or in set theory lingo as $G(\{u, v\}, \{e\})$
- There also exist **directed Edges/Arcs** i.e. , they describe asymmetric relations

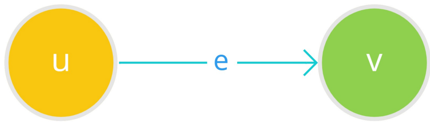


Figure 1.2: A visual representation of a simple directed Graph. Here u is called the **tail** and v the **head**

- Adding 1.4.1 to another directed graph with the same vertices but the edge pointing in the other direction results in a non-directed graph
- **Degree** of a vertex is the number of its incident edges i.e. neighbours denoted by $deg(v)$
- The degree of a graph is the maximum degree of its vertices

1.2 Types of Graph

- A **Regular graph** is a graph where each vertex has the same degree
- A regular graph of n degrees is called $n-$Regular
- The Complement of a graph $G = (V, E)$ is a graph $\bar{G} = (V, \bar{E})$ on the same set of vertices V and the following set of edges:
  - Two vertices are connected in $\bar{G}$ iff they are not connected in G i.e. $(u, v) \in \bar{E}\ iff\ (u, v) \notin E$
  - A **Path** is a continuous sequence of edges that connect two vertices
  - A **Walk** in a graph is a sequence of edges, such that each edge except for the first one starts with a vertex where the previous edge ended
  - The **Length** of a walk is the number of edges in it
  - A **Path** (rigorously) is a walk where all edges are distinct
  - A **Simple Path** is a walk where all vertices are distinct
- A **Cycle** in a graph is a path whose first vertex is the same as the last one; In particular, *all the edges in a Cycle are distinct*
- A **Simple Cycle** is a cycle where all vertices except for the first one are distinct and there first vertex is taken twice

- A graph is called **Connected** if there is a path between every pair of its vertices

- A **Connected Component** of a graph G is a maximal connected subgraph of G i.e., a connected subgraph of G which is not contained in a larger connected subgraph of G

- The **Indegree** of a vertex v is the number of edges ending at v

- The **Outdegree** of a vertex v is the number of edges leaving v

- A **Weighted Graph** associates a *weight* with every edge

- The **Weight** of a path is the sum of the weights of its edges

- A **Shortest Path** between two vertices is a path of the minimum weight

- The **Distance** between two vertices is the length of a shortest path between them

- A **Path Graph** P_n $\forall\ n \geq 2$, has n vertices labeled v_n and $n-1$ edges $\{v_{n-1}, v_n\}$

Figure 1.3: The Path Graph P_3

- A **Cycle Graph** C_n $\forall\ n \geq 3$, has n vertices labeled v_n and n edges $\{v_{n-1}, v_n\}, \{v_n, v_1\}$

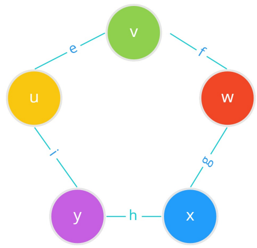


Figure 1.4: The Cycle Graph C_5

- A **Complete Graph (a.k.a Clique)** K_n $\forall\ n \geq 2$, has n vertices labeled v_n and all edges between them (i.e. $n(n-1)/2$ edges)

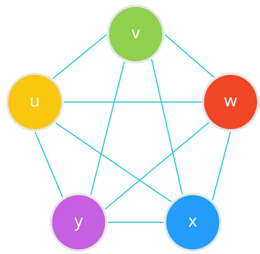


Figure 1.5: The Clique Graph K_5

1.3 Trees

- A tree is a connected graph without cycles
- A tree is a connected graph on n vertices with $n-1$ edges
- A graph is a tree if and only if there is a unique simple path between any pair of its vertices

1.3.1 How to make a tree?

- Remove any edge, keeping the Graph connected

- Stop only when $n-1$ edges are left

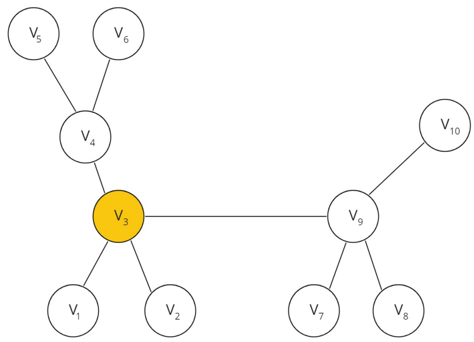


Figure 1.6: C_5 is not bipartite. In general, for odd $n > 2$, C_n is not bipartite. However,

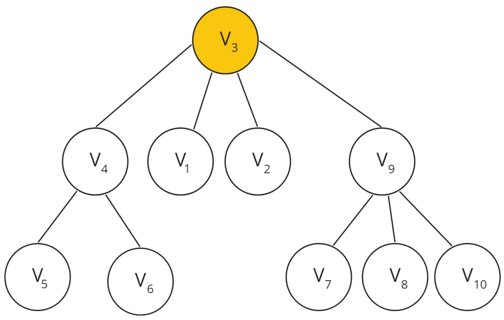
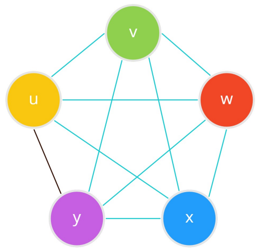


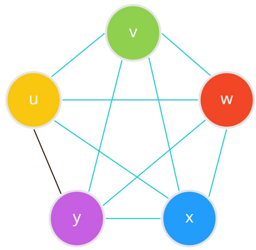
Figure 1.7: C_5 is not bipartite. In general, for odd $n > 2$, C_n is not bipartite. However,

1.4 Bipartite Graphs

- A graph G is **Bipartite** if its vertices can be partitioned into two disjoint sets (sets with no common elements) L and R such that every edge of G connects a vertex in L to a vertex in R i.e., no edge connects two vertices from the same part
- L and R are called the parts of G
- Trees are Bipartite Graphs

1.4.1 Examples and counter examples





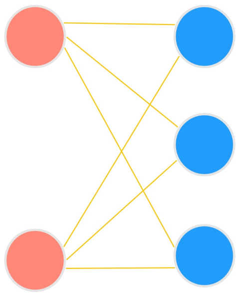


Figure 1.8: C_5 is not bipartite. In general, for odd $n > 2$, C_n is not bipartite. However,

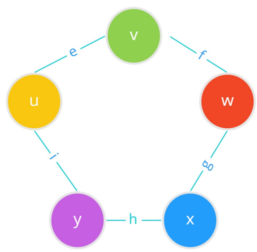


Figure 1.9: $K_{2,3}$ is bipartite

Chapter 2

Optimization