Group Theory Notes

TeXstudio Team

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Chapter 1

The First Chapter

There is precisely one simply-connected Lie group corresponding to each Lie algebra.

1.1 Representation

A representation R of a group element g is a one to one map to an element of a vector space i.e. it is homeomorphic.

$$g \to R(g)$$

The following properties are preserved:

- $R(e) = I$
- $R(g^{-1}) = (R(g))^{-1}$
- $R(g) \circ R(h) = R(gh)$

A representation identifies with each point (abstract group eement) of the group manifold (the abstract group) a linear transformation of a vector space. Ggenerally if one accepts arbitrary (not linear) transformations of an arbitrary (not necessarily a vector) space. Such a map is called a realization.

1.1.1 Similartiy Transform

$$R \to R^{'} := S^{-1}RS$$

This means that if we have a representation, we can transform its elements wildly with literally any non-singular matrix S [1] to get nicer matrices.

[1] $det(s) \neq 0$

1.1.2 Invariant Subspaces

This means, if we have a vector in the subspace V' and we act on it with arbitrary group elements, the transformed vector will always be again part of the subspace V'. If we find such an invariant subspace we can define a representation R' of G on V', called a subrepresentation of R, by

$$R'(g)v = R'(g)v$$

This means that the representation R is actually composed of smaller building blocks called subrepresentations.

1.1.3 Irreducible Representation

An irrep is a rep of a Group G on a vector space V that has no invariant subspaces other than $\{0\}$ and V itself.
There are many possible representations for each group 86 , how do we know which one to choose to describe nature? There is an idea that is based on the Casimir elements. A Casimir element C is built from generators of the Lie algebra and its defining feature is that it commutes with every generator X of the group.

$$[C, X] = 0 \tag{1.1}$$

A famous Lemma, called Schur's Lemma tells us that if we have an irreducible representation, $R : \mathfrak{g} \to GL(V)$, any linear operator $T : V \to V$ that commutes with all operators $R(X)$ must be a scalar multiple of the identity operator. Therefore, the Casimir elements give us linear operators with constant values for each representation. As we will see, these values provide us with a way of labelling representations naturally. 88 We can therefore start to investigate the irreducible representations, by starting with the representation with the lowest possible scalar value for the Casimir element.

An irreducible representation cannot be rewritten, using a similarity transformation, in block diagonal form 84 . In contrast, a reducible representation can be rewritten in block-diagonal form through sim- ilarity transformations. These notions are important because we use irreducible representations to describe elementary particles 85 . We will see later that the behavior of elementary particles under transforma- tions is described by irreducible representations of the corresponding symmetry group.
An important observation that helps us to make sense of the vector space, is that for any Lie group, one or several of the corresponding generators can be diag- onalized using similarity transformations. In physics we use these diagonal generators to get labels for the basis vectors that span our vector space.

We use the eigenvectors of these diagonal generators as basis for our vector space and the corresponding eigenvalues as labels. This idea is incredibly important to actually understand the physical implications of a given group. If there is just one gener- ator that can be diagonalized simultaneously, each basis vector is labelled by just one number: the corresponding eigenvalue. If there are several generators that can be diagonalized simultaneously, we get several numbers as labels for each basic vector. Each such number is simply the eigenvalue of a given diagonal generator that belongs to this basic vector (=eigenvector). In particular this is where the "charge labels" for elementary particles: electric charge, weak charge, color charge, come from.

1.2 Poincare Group

$$\begin{aligned}\text{The Poincare Group} &= \text{Lorentz Group} + \text{Translations in time and space}\\ &= \mathbb{R}^{1,3} \rtimes ISO(1,3)^{\uparrow}\end{aligned}$$

Chapter 2

The Second Chapter

2.1 Noether's Theorem

$$\text{Symmetry i.e. invairiance of the Lagrangian under a generator} \rightarrow \text{Conserved quantity} \tag{2.1}$$

2.2 Quantum Mechanics

Everytime the aciton of a generator leaves the Lagrangian invariant it leads to the conservation of a quantity.

$$\text{momentum } \hat{P}_i \rightarrow \text{generator of spatial-translations: } -i\partial_i \tag{2.2}$$

$$\text{energy } \hat{E} \rightarrow \text{generator of time-translations: } i\partial_0 \tag{2.3}$$

$$\text{position } \hat{x}_i \rightarrow x_i \tag{2.4}$$

The Canonical commutation algebra is a realtionship between the position and momentum opearators that

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{2.5}$$

2.3 Spin and Angular Momentum

Analogously we identify the first part, called spin, with the corresponding finite dimensional representation of the generators as this was the part of the conserved quantity that resulted from the invariance under mixing of the field components. Hence the finite-dimensional representation.

$$\text{spin } \hat{S}_i \rightarrow \text{generators of rotations (fin. dim. rep.) } S_i \tag{2.6}$$

Where spin is defined by the relations:

$$S_i = \frac{1}{2}\epsilon_{ijk}S_{jk} \tag{2.7}$$

$$\hat{S}_i = i\frac{\sigma_i}{2} \tag{2.8}$$

2.4 Quantum Field Theory

Quantum Field Theory is about the dance between various quantizable fields i.e. functions of space and time $\phi(\vec{x}, t)$. We will therefore be dealing with points in spacetime and thus it natural to talk about the densities of our dynamical variables such as conjugate momentum $\pi = \pi(x)$ rather than the total quantities we get by integrating them over spacetime i.e. $\Pi = \int \pi(x) d^3x \neq \Pi(x)$.

Earlier we discovered that invariance under displacements of the field $\phi \to \phi - i\epsilon$ itself is a new conserved quantity called conjugate momentum Π. We now identify it with the corresponding generator:

$$\text{conj. mom. density } \pi(x) \to \text{gen. of displ. of the field itself} : \; -i\frac{\partial}{\partial\phi(x)} \tag{2.9}$$

Similar to Quantum Mechanics, in Quantum Field Theory, everything flows from this relationship:

$$[\phi_i(x), \pi_j(y)] = i\delta(x - y)\delta_{ij} \tag{2.10}$$