Notes on Tensors

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Chapter 1

Tensor-Algebra

1.1 Vector Transformation Rules

The rules:

- For basis vectors forward transformations brings us from old to new coordinate systems and backward brings us from new to old.
- However, with vector components it's the opposite.

Suppose we have a vector $\vec{v}$ in a basis $\vec{e}_j$. We now transform it to a basis $\tilde{\vec{e}}_i$ where it becomes $\tilde{v}$. We call the forward transformation as F_{ij} and the backward as B_{ij}[1] which we define as:

$$\tilde{\vec{e}}_j = \sum_{i=1}^{n} F_{ji}\vec{e}_i$$

$$\vec{e}_j = \sum_{i=1}^{n} B_{ji}\tilde{\vec{e}}_i$$

We can try to derive the statements made previously,

$$\vec{v} = \sum_{j=1}^{n} v_j\vec{e}_j = \sum_{i=1}^{n} \tilde{v}_i\tilde{\vec{e}}_i$$

$$\vec{v} = \sum_{j=1}^{n} v_j\vec{e}_j = \sum_{j=1}^{n} v_j(\sum_{i=1}^{n} B_{ij}\tilde{\vec{e}}_i) = \sum_{i=1}^{n}\sum_{j=1}^{n}(B_{ij}v_j)\tilde{\vec{e}}_i$$

[1] We will follow this notation throughout this document

Thus,

$$\tilde{v}_i = \sum_{j=1}^{n} B_{ij} v_j \tag{1.1}$$

Similarly,

$$\vec{v} = \sum_{j=1}^{n} v_j \vec{e}_j = \sum_{i=1}^{n} \tilde{v}_i \tilde{\vec{e}}_i$$

$$\vec{v} = \sum_{j=1}^{n} \tilde{v}_j \tilde{\vec{e}}_j = \sum_{j=1}^{n} \tilde{v}_j (\sum_{i=1}^{n} F_{ij} \vec{e}_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} (F_{ij} \tilde{v}_j) \vec{e}_i$$

Thus,

$$v_i = \sum_{j=1}^{n} F_{ij} \tilde{v}_j \tag{1.2}$$

Now because vector components beehave contrary to the basis vectors, they are said to be ***"Contravariant"***

1.2 Index Notation

1.2.1 Einstein Notation i.e. Summing convention

Let us consider the sum[2],

$$x_i = \sum_{j}^{n} \Lambda_{ij} \mathcal{X}^j$$

Is the same as,

$$x_i = \Lambda_{ij} \mathcal{X}^j$$

Here, we define i to be the free index and j to be the summing index or the dummy index that is repeated to signify so.

1.2.2 Index Convention

When we sum from 1 to 3 we use the symbols i, j and k i.e. the English alphabet to signify that we are only considering dimnesions that are spatial/that are not a timme dimension. However, when we use the symbols ν and μ i.e. Greek alphabets we are summing from 0 to 3, we also include the temporal dimension according to the tradition of special relativity in which we name components as $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$ in the Cartesian framework.

[2]Mind you there are no exponents there.

1.3 Covectors

- Covectors can be thought of as row vector or as functions that act on Vectors such that any covector $\alpha : \mathbb{V} \to \mathbb{R}$
- Covectors are linear maps i.e. $\beta(\alpha)\vec{v} = \beta\alpha\vec{v}$ and $(\beta + \alpha)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
- Covectors are elements of a Dual vector space $\mathbb{V}^*$ which has different rules for addition and scaling i.e. scalar multiplication
- You visualize covectors to be some sort of gridline on your vector space such that applying a covector to a vector is equivalent to projecting the vector along the gridline
- Covectors are invariant but their components are not
- The covectors that form the basis for the set of all covectors is called the ***"Dual Basis"***, because they are a basis for the Dual Space $\mathbb{V}^*$ i.e. any covector can be expressed as the linear combination of the dual basis
- However we are free to choose a dual basis
- For covector components, forward transformation brings us from old to new and backwards vice versa
- We can flip between row and column vectors for an orthonormal basis
- Vector components are measured by counting how many are used in the construction of a vector, but covector components are measured by counting the number of covector lines that the basis vector pierces
- The covector basis transforms contravariantly compared to the basis and it's components transform covariantly according to the basis
- The covector basis is denoted as ϵ^j

1.3.1 Contravariant Components

We denote contravariant components using the symbols

$$A^i$$

and their basis like

$$\vec{e}_i$$

1.3.2 Covariant Components

We denote convariant components using the symbols

$$A_i$$

and their basis like

$$\overrightarrow{e}^i$$

1.3.3 Relationship Between the Two Types of Components

$$|\overrightarrow{e}^1| = \frac{1}{|\overrightarrow{e}_1| \cos(\theta_1)}$$

and,

$$|\overrightarrow{e}_1| = \frac{1}{|\overrightarrow{e}^1| \cos(\theta_1)}$$

Or with 3 components we have: Since both types of components represent the same vector (as in same magnitude) only inn different bases, we can write

$$\overrightarrow{A} = A^i \overrightarrow{e}_i = A_i \overrightarrow{e}^i$$

1.3.4 Using Cramer's Method to find Components

Camer's rule involves

- First find the simultaneous equations that relate the two different types of basis
- Now rewrite them in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{1.3}$$

- Use Cramer's rule

$$A_1 = \frac{\begin{vmatrix} x_1 & M_{12} \\ x_2 & M_{22} \end{vmatrix}}{\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix}} \tag{1.4}$$

$$A_2 = \frac{\begin{vmatrix} M_{11} & x_1 \\ M_{21} & x_2 \end{vmatrix}}{\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix}} \tag{1.5}$$

1.4 Linear Maps

Linear maps to put it naively, Linear Maps transform input vectors but not the basis. Geometrically speaking, Linear Maps:

- Keep gridlines parallel
- Keep gridlines evenly spaced
- Keep the origin stationary

To put it more abstractly, Linear Maps:

- Maps vectors to vectors, $\mathbb{L} : \mathbb{V} \rightarrow \mathbb{V}$
- Adds inputs or outputs, $\mathbb{L}(\vec{V} + \vec{W}) = \mathbb{L}(\vec{V}) + \mathbb{L}(\vec{W})$
- Scale the inputs or outputs, $\mathbb{L}(\alpha\vec{V}) = \alpha\mathbb{L}(\vec{V})$
- i.e. They are Linear/Linearity
- When I transform the basis using a forward transformation, the transformed Linear map $\tilde{\mathbb{L}}^l_i$ can be written as:

$$\tilde{\mathbb{L}}^l_i = \mathbb{B}^l_k \mathbb{L}^k_j \mathbb{F}^j_i \tag{1.6}$$

1.5 Metric Tensor

- Pythagoras' theorem is a lie for non-orthonormal bases
- The metric Tensor is Tensor that helps us compute lengths and angles
- For two dimensions it can be written as:

$$g_{ij} = \begin{bmatrix} e_1 e_1 & e_1 e_2 \\ e_2 e_1 & e_2 e_2 \end{bmatrix}$$

- Or more abstractly

$$g_{ij} = e_i e_j$$

- Or,

$$g = g_{ij}(\epsilon^i \otimes \epsilon^j)$$

- The dot product between two vectors can be written as

$$||\vec{v}||||\vec{w}||\cos\theta = v^i w^j g_{ij}$$

- we can see how this allows us to compute angles as well
- Moreover this formula works in all coordinates thus, the vector length stays constant
- To transform the components of the Metric Tensor we have to apply the transformation twice i.e.$g_{\mu\nu} = \mathbb{F}^{\rho}_{\mu}\mathbb{F}^{\sigma}_{\nu}\tilde{g}_{\rho\sigma}$ or $\tilde{g}_{\rho\sigma} = \mathbb{B}^{\mu}_{\rho}\mathbb{B}^{\nu}_{\sigma}g_{\mu\nu}$
- $\alpha g(\vec{V},\vec{W}) = g(\alpha\vec{V},\vec{W}) = g(\vec{V},\alpha\vec{W})$
- $g(\vec{V}+\vec{U},\vec{W}) = g(\vec{V},\vec{W}) + g(\vec{U},\vec{W})$
- $\alpha(\vec{V}+\vec{U})g = \alpha\vec{V}g + \alpha\vec{U}g$
- $g(\vec{V},\vec{W}) = V^iW^jg_{ij} = V^iW^jg_{ji} = g(\vec{W},\vec{V})$
- $g(\vec{V},\vec{V}) = ||\vec{V}||^2 \geq 0 \;\forall\; \vec{V} \neq 0$
- In short, $g := \mathbb{V}\times\mathbb{V}\to\mathbb{R}$
- We can define a new quantity called ***"Scale-Factor"*** as, $h_i = \sqrt{g_{ii}}$
- Through this we can rewrite some of the Operators from Vector Calculus:
  - Gradient: $\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x^1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x^2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x^3}\hat{e}_3$
  - Divergence: $\nabla\circ\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(h_2h_3A_1) + \frac{\partial}{\partial x^2}(h_1h_3A_2) + \frac{\partial}{\partial x^3}(h_1h_2A_3)\right]$
  - Curl: $\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$
  - Laplacian:$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x^3}\right)\right]$

1.6 Bilinear Forms

- The metric tensor is an example of a Bilinear form
- We define a Bilinear form as:
  - $\mathcal{B} := \mathbb{V}\times\mathbb{V}\to\mathbb{R}$
  - $\alpha\mathcal{B}(\vec{V},\vec{W}) = \mathcal{B}(\alpha\vec{V},\vec{W}) = \mathcal{B}(\vec{V},\alpha\vec{W})$

  - $\mathcal{B}(\vec{V}+\vec{U},\vec{W}) = \mathcal{B}(\vec{V},\vec{W}) + \mathcal{B}(\vec{U},\vec{W})$
  - $\alpha(\vec{V}+\vec{U})\mathcal{B} = \alpha\vec{V}\mathcal{B} + \alpha\vec{U}\mathcal{B}$
  - $\mathcal{B}(\vec{V},\vec{U}) \to V^iW^j\mathcal{B}_{ij}$

- Bilinear forms are (0,2) Tensors, they transform using two covariant rules when we transform them

- A form is simply a function that takes vectors as inputs and outputs a number

- So covectors are sometimes called Linear forms/ 1-forms

- This structure is called a Bilinear form since each individual input is Linear while the other input is held constant

- $\mathcal{B}_{\mu\nu} = \mathbb{F}^{\rho}_{\mu}\mathbb{F}^{\sigma}_{\nu}\tilde{\mathcal{B}}_{\rho\sigma}$ and $\tilde{\mathcal{B}}_{\rho\sigma} = \mathbb{B}^{\mu}_{\rho}\mathbb{B}^{\nu}_{\sigma}\mathcal{B}_{\mu\nu}$

- $\mathcal{B}(\vec{V},\vec{W}) = \mathcal{B}_{\mu\nu}V^{\mu}W^{\nu} = \begin{bmatrix}\begin{bmatrix}\mathcal{B}_{11} & \mathcal{B}_{12}\end{bmatrix} & \begin{bmatrix}\mathcal{B}_{21} & \mathcal{B}_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}\begin{bmatrix}w_1\\w_2\end{bmatrix}$

1.7 Clearer Definitions

1.7.1 Linear Maps

- A Tensor is a collection of vectors and covectors combined together using the Tensor Product

- Pure Matrices can be broken down into the product of row and column matrices.

- Each element of the same column in a pure matrix is a scalar product of each other

- Pure matrices as Linear maps only produce output vectors in the same direction due to the previous statement

- Any linear map $\mathbb{L}$ can be written as the Linear combination of the product of Vector-Co-vector pairs i.e. $\mathbb{L} = \mathbb{L}^{\mu}_{\nu}\vec{e}_{\mu}\epsilon^{\nu} := \mathbb{V} \to \mathbb{V}$

- Reminder: $\epsilon^i \otimes$[3] $\vec{e}_j = \delta^i_j$

[3]This symbol represents the Tensor product, see for more information

1.7.2 Bilinear Forms

- Bilinear forms are a Linear combination of covector-covector pairs

- $\mathcal{B} = \mathcal{B}_{ij}(\epsilon^i \otimes \epsilon^j)$

1.7.3 Tensors

- An object that is invariant under a change of coordinates and has components that change in a special and predictable way under a change of coordinates i.e. products of jacobians

- A collection of vectors and covectors combined using the Tensor Product

- For a tensor $T^{i_m}_{j_n}$, we say it has a type (m, n) and rank $m + n$

1.8 Tensor Addition and Subtraction

- Two tensors can be added provided they have the same structure i.e. the same number of vectors and covectors

- The resultant of tensor addition or subtraction is another tensor with the same structure i.e. $A^i_j \pm B^i_j = C^i_j$

1.9 Tensor Products

- The tensor product $v \otimes w \ \forall \ v \in \mathbb{V}, w \in \mathbb{W}$ is an element of the set $\mathbb{V}^* \times \mathbb{W}^*$[4] i.e. the set of all Bilinear functions on which act on the pair $(h, g) \in \mathbb{V}^* \times \mathbb{W}^*$

- The Tensor product and the Kronecker product are kind of doing the same thing, the Tensor product combines the abstract vector and the abstract covector and the Kronecker product combines 1 dimensional arrays

- However both products result in the same set of components

[4]here the symbol $\times$ means **"Cartesian Product"** i.e. it's action w.r.t two sets is the set of all ordered pairs (a, b) where $a \in \mathbb{A}$ and $b \in \mathbb{B}$

1.9.1 Tensor Product

- Combines 2 Tensors into a 3rd new Tensor
- The result is a Linear map
- Eg: $(\vec{e}_i \otimes \epsilon^j)\vec{v} = v^j \vec{e}_i$
- $(\epsilon^j \otimes \epsilon^j)(\vec{v}, \vec{w}) = \epsilon^i(\vec{v})\epsilon^j(\vec{w})$

1.9.2 Kronecker Product

- Combines 2 arrays into a 3rd new array
- Eg:- $\begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} & \alpha_2 \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \end{bmatrix}$

1.9.3 Array Multiplication

$$\begin{bmatrix} W^1 \\ W^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} L^1_1 \\ L^2_1 \end{bmatrix} & \begin{bmatrix} L^1_2 \\ L^2_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = \begin{bmatrix} L^1_1 V^1 + L^1_2 V^2 \\ L^2_1 V^1 + L^2_2 V^2 \end{bmatrix}$$

- This method isn't very useful for higher dimensional tensors, for them we stick to Einstein notation as the larger number of components, the more number of ways we can do the summation

1.9.4 Tensor Product Spaces

We have the following properties for a Tensor product $\forall\ \vec{U}, \vec{V}, \vec{W} \in \mathbb{V}$, $\forall\ \alpha, \beta \in \mathbb{V}^*$ and $\forall\ n \in \mathcal{F}$

- $n(\vec{V} \otimes \alpha) = (n\vec{V}) \otimes \alpha = \vec{V} \otimes (n\alpha)$ Here, n is a scalar and α a covector
- $\vec{V} \otimes \alpha + \vec{V} \otimes \beta = \vec{V} \otimes (\alpha + \beta)$
- $\vec{V} \otimes \alpha + \vec{W} \otimes \alpha = (\vec{V} + \vec{W}) \otimes \alpha$
- $\vec{U}\alpha\vec{V} \otimes \vec{U}\beta\vec{V} = \vec{U}(\alpha \otimes \beta)\vec{V}$
- $\mathbb{V} \otimes \mathbb{V}^* := \mathbb{V} \times \mathbb{V}^* \to \mathbb{R}$ for example, whose elements are (1,1) Tensors
- We can always do the summation differently and end up with different elements even if the map leads us to the same space
- We can combine any number of tensors provided the upstairs indices match the downstairs indices across the product

- For example,$T_i{}^j{}_{kl}\alpha_j D^{kl} := \mathbb{V}^* \times (\mathbb{V} \otimes \mathbb{V}) \to \mathbb{V}^*$
- You can guess where the output of the operation by finding out where the free index lies
- **Multilinear map**, a function that's linear when all inputs except one are held constant
- A Tensor when used as a function is simply a Multilinear map

1.10 Raising and Lowering Indices

- To go from the vector space to the dual space we use the covariant metric g_{ij} i.e. this lowers the indices. Thus they are sometimes denoted using the symbol $\flat$ and called "flat operators"
- To go from the dual space to the vector space we use the contravariant metric $\mathfrak{g}^{ij}$ i.e. this raises the indices. Thus they are sometimes denoted using the symbol $\sharp$ and called "sharp operators"
- These matrices are inverses of each other i.e. $g_{ki}\mathfrak{g}^{ij} = \delta_i^k$

Chapter 2

Tensor-Calculus

2.1 Derivatives

A derivative can be broken down as,

$$\frac{d\vec{R}}{d\lambda} = \frac{dc^i}{d\lambda}\frac{\partial\vec{R}}{\partial c^i} \tag{2.1}$$

where $\frac{dc^i}{d\lambda}$ represents the coefficients and $\frac{\partial\vec{R}}{\partial c^i}$ represents the basis vectors. We can rewrite the gradient for a scalar field we know from vector calculus as,

$$(grad\ f)_\mu = \frac{\partial f}{\partial x^\mu} \tag{2.2}$$

we can write the gradient of a vector field as,

$$(grad\ \vec{v})^\mu{}_\nu = \frac{\partial x^\mu}{\partial x^\nu} \tag{2.3}$$

and of a Tensor field,

$$(grad\ t)_{\mu\nu\alpha} = \frac{\partial t_{\mu\nu}}{\partial x^\alpha} \tag{2.4}$$

We can use the following notation for simplicity,

$$v^\mu{}_{,\nu} = \partial_\nu v^\mu := \frac{\partial v^\mu}{\partial x^\nu} \tag{2.5}$$

and

$$v^{\mu,\nu} = \partial^\nu v^\mu := g^{\nu\rho}\frac{\partial v^\mu}{\partial x^\rho} \tag{2.6}$$

2.2 Covector Field

df is called a differential or a differential form of f. We can think of the operators d as something that take in a scalar field f (function/0-form) and outputs a covector field df (level sets/1-form).
A 1-form is defined as a **linear-map**,

$$\vec{V} \to \mathbb{R}$$

That is,

$$df(\vec{V}) \in \mathbb{R}$$

Geometrically speaking, long arrows of the vector field match up with the densly packed areas of the covector field and vice versa. df can be thought of being proportional to the steepness of f in the direction of $\vec{V}$ and to the length of $\vec{V}$. df tells us the rate of change of f moving at a velocity $\vec{V}$. Simply put, $df(\vec{V})$ is the directional derivative.

2.2.1 Covector Field Components

dx and dy form the dual-basis of differential forms.[1]

$$df = Adx + Bdy$$

Where $A = \frac{\partial f}{\partial x}$ and $B = \frac{\partial f}{\partial y}$. [2] Or more eloquently,

$$df = \frac{\partial f}{\partial c^i} dc^i$$

2.2.2 Covector Field Transformation Rules

- Covector fields are invaraint to the choice of coordinates
- Mind you basis covectors are contravariant and their components are covariant

[1]The proof for this can be found when you apply df to $\frac{\partial}{\partial x}$ and find that it is 1. Thus generally,

[2]This can be derived from trying to take the derivative of f with respect to a tangent vector or by simply expanding the differential df

2.3 Integration with Differential Forms

$$\int_P df$$

- Ever single integral involves a path P and a covector field i.e. a differential form
- The result is simply the number of covector curves in the alligned direction pierced by the path
- There are two types of geometry: intrinsic i.e. (d-1 basis vectors) and extrinsic taking a bird's eye view of the entire plane (d basis vectors), where the entire structure has d dimensions

2.4 Gradient

$$df(\vec{v}) = \nabla f \cdot \vec{v} \tag{2.7}$$

So df is the dual covector field of ∇f. It follows this relation,

$$df = (\nabla f)^i g_{ij} dc^j \tag{2.8}$$

where dc^j are the basis covectors. We know that,

$$df = \frac{df}{dc^j} dc^j$$

Thus,

$$\frac{df}{dc^j} dc^j = (\nabla f)^i g_{ij} dc^j$$

$$\frac{df}{dc^j} = (\nabla f)^i g_{ij} \tag{2.9}$$

Now if we multiply by the contravariant metric tensor on both the sides,

$$\frac{df}{dc^j} \mathfrak{g}^{ij} = (\nabla f)^i g_{ij} \mathfrak{g}^{ij}$$

$$\frac{df}{dc^j} \mathfrak{g}^{ij} = (\nabla f)^i \delta_i^k$$

we find that

$$\frac{df}{dc^j} \mathfrak{g}^{ij} = (\nabla f)^j \tag{2.10}$$

the inverese metric takes us from the Gradient to the differential. We can also expand the components out as alinear combinations of the basis or in vector notation. It is quite common in this context to use $\flat$ and $\sharp$ instead of $\mathfrak{g}^{ij}$ and g_{ij} .

2.5 Geodesics

- A geodesic is the straightest possible path in a curved surface/space.
- A Geodesic curve has zero tangential acceleration when we travel along it at constant speed.
- This is the Geodesic equation, solving it for a particular u will tell you about the path a particle with zero velocity will have along the line λ

$$\frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda} = 0 \tag{2.11}$$

- Parallel Transport of a vector along a curve is the closest we can get to a "constant" vector field i.e. a vector field whose vectors are independent of any parameter
- Parallel Transport provides a connection between tangent spaces in a curved space
- The Geodesic is a curve resulting from parallel transporting a vector along itself i.e. $\nabla_{\vec{V}}\vec{V}$
- It is in a sense the "straightest possible path" in a curved space
- The covariant Derivative helps us find parallel transported vector fields
- Covariant Derivative provides a connection between tangent spaces in a curved space

2.6 Covariant Derivative

The covariant derivative of a vector field v^μ is defined as,

$$\nabla_\mu v^\alpha = v^\alpha{}_{;\mu} = \partial_\mu v^\alpha + \Gamma^\alpha_{\mu\nu}v^\nu \tag{2.12}$$

where, the object $\Gamma^\alpha_{\mu\nu}$ is called the Christoffel symbol. The Christoffel symbol is not a tensor because it contains all the information about the curvature of the coordinate system and can therefore be transformed entirely to zero if the coordinates are straightened. Nevertheless we treat it as any ordinary tensor in terms of the index notation. It can be defined in terms for derivatives as:

$$\Gamma^k_{ij}\vec{e}_k = \frac{\partial \vec{e}_i}{\partial x^j} \tag{2.13}$$

Where, the index i specifies the basis vector for which the derivative is being taken, the index j denotes the coordinate being varied to induce this change in the ith basis vector, and the index k identifies the direction in which this component of the derivative points. We define the Christoffel symbol in terms of the metric $g_{\mu\nu}$ and it's inverse $g^{\alpha\beta}$ as,

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\nu}}{\partial x^{\mu}} + \frac{\partial g_{\beta\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right) = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\nu,\mu} + g_{\beta\mu,\nu} - g_{\mu\nu,\beta} \right) \quad (2.14)$$

The connection coefficients defined here are of the Levi-Civita Connection. It a unique connection (covariant derivative) that is Torsion-Free and has Metric Compatibility. We can then define the covariant derivative of a covector as,

$$\nabla_{\mu} w_{\alpha} = w^{\alpha}{}_{;\mu} = \partial_{\mu} w_{\alpha} + \Gamma^{\alpha}_{\mu\nu} w_{\nu} \quad (2.15)$$

The covariant derivative of a tensor $t^{\alpha\beta}$ is then,

$$\nabla_{\mu} t^{\alpha\beta} = \partial_{\mu} t^{\alpha\beta} + \Gamma^{\alpha}_{\mu\sigma} t^{\sigma\beta} + \Gamma^{\beta}_{\mu\sigma} t^{\alpha\sigma} \quad (2.16)$$

and of a tensor t^{α}_{β},

$$\nabla_{\mu} t^{\alpha}_{\beta} = \partial_{\mu} t^{\alpha}_{\beta} + \Gamma^{\alpha}_{\mu\sigma} t^{\sigma}_{\beta} + \Gamma^{\beta}_{\mu\sigma} t^{\alpha}_{\sigma} \quad (2.17)$$

2.6.1 Properties

- The covariant derivative produces, as its name says, covariant expressions.
- $\nabla_{\mu} g_{\alpha\beta} = 0$
- $g^{\alpha\gamma} \nabla_{\alpha} t^{\mu\nu}_{\gamma} = \nabla_{\alpha} (t^{\mu\nu}_{\gamma} g^{\alpha\gamma}) = \nabla_{\alpha} t^{\mu\nu\alpha}$
- $\nabla^{\alpha} = g^{\alpha\beta} \nabla_{\beta}$
- $\nabla_{\partial_i}(a) = \frac{\partial a}{\partial u^i}$ i.e. for a scalar
- $\nabla_{\partial_i}(\vec{V}) = \left(\frac{\partial V^k}{\partial u^i} + V^j \Gamma^k_{ij} \right) \vec{e}_k$ i.e. for a vector
- $\nabla_{\partial_i}(\alpha) = \left(\frac{\partial \alpha_k}{\partial u^i} - \alpha_j \Gamma^j_{ik} \right) \epsilon^k$ i.e. for a covector
- $\nabla_{\partial_i}(T) = \left(\frac{\partial T_{rs}}{\partial u^i} - T_{ks} \Gamma^k_{ir} - T_{rk} \Gamma^k_{is} \right) (\epsilon^r \otimes \epsilon^s)$ i.e. a tensor
- $\nabla_{\vec{W}}(T \otimes S) = (\nabla_{\vec{W}} T) \otimes S + T \otimes (\nabla_{\vec{W}} S)$ i.e. for a tensor product

2.7 Interesting Tensors

2.7.1 Kronecker delta

It simply has the 'function' of 'renaming' an index:

$$\delta^\mu_\nu x^\nu = x^\mu$$

it is in a sense simply the identity matrix.

2.7.2 Levi-Civita Pseudotensor

The Levi-Civita Pseudotensor i.e. Tensor density is a completely anti-symmetric i.e. $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$, we define it as:

$$\epsilon_{ijk} = \begin{cases} 1 \text{ if } ijk \text{ is an even permuation of } 123 \\ -1 \text{ if } ijk \text{ is an odd permuation of } 123 \\ 0 \text{ if two indices are equal} \end{cases} \tag{2.18}$$

Identities

$$\epsilon_{\alpha\beta\nu}\epsilon_{\alpha\beta\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \tag{2.19}$$

From this it follows that,

$$\epsilon_{\alpha\beta\nu}\epsilon_{\alpha\beta\sigma} = 2\delta_{\nu\sigma} \tag{2.20}$$

and

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma} = 6 \tag{2.21}$$

Cross-Product

Using these identities and the definition we can rewrite the cross-product of two vectors as,

$$\vec{a} = \vec{a} \times \vec{b} = \epsilon_{ijk} a_j b_k \tag{2.22}$$

Thus the expressions in vector product notation can be changed to index notation for example,

$$c = \nabla.(\nabla \times \vec{a}) = \nabla_i(\epsilon_{ijk}\nabla_j a_k) = \epsilon_{ijk}\partial_i\partial_j a_k$$

because,

$$\nabla_i = \frac{\partial}{\partial x_i} := \partial_i$$

Rot

Rot is defined as the generalized rotation of a covector,

$$(rot\tilde{w})_{\alpha\beta} = \partial_\alpha w_\beta - \partial_\beta w_\alpha \tag{2.23}$$

2.7.3 Electromagnetic Field Tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{2.24}$$

and it's covariant version (by lowering the indices),

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{2.25}$$

and it's dual as,

$$G^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x/c & 0 & E_z/c & -E_y/c \\ B_y/c & -E_z & 0 & E_x/c \\ B_z/c & E_y & -E_x & 0 \end{pmatrix} \tag{2.26}$$

Along with the 4-current (the four-dimensional analogue of the electric current density),

$$J^\mu = (c\rho, j^1, j^2, j^3)$$

we can now rewrite Maxwell's equations as:

$$\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = \mu_0 J^\beta \tag{2.27}$$

$$\frac{\partial G^{\alpha\beta}}{\partial x^\alpha} = 0 \tag{2.28}$$

2.7.4 The Riemann Curvature Tensor

Derivation

To begin this process we first take the covariant derivative of a vector V_α, with respect to x^β,

$$V_{\alpha;\beta} = \frac{\partial V_\alpha}{\partial x^\beta} - \Gamma^\sigma_{\alpha\beta} V_\sigma \tag{2.29}$$

Now call this result $V_{\alpha\beta}$ and take another covariant derivative, this time with respect to x^{γ},

$$V_{\alpha\beta;\gamma} = \frac{\partial V_{\alpha\beta}}{\partial x^{\gamma}} - \Gamma^{\tau}_{\alpha\gamma} V_{\tau\beta} - \Gamma^{\eta}_{\beta\gamma} V_{\alpha\eta} \tag{2.30}$$

Substituting the expression from Eq. 6.21 into this equation gives,

$$\begin{aligned} V_{\alpha\beta;\gamma} = & \frac{\partial^2 V_{\alpha}}{\partial x^{\gamma} \partial x^{\beta}} - \frac{\partial \Gamma^{\sigma}_{\alpha\beta}}{\partial x^{\gamma}} V_{\sigma} - \Gamma^{\sigma}_{\alpha\beta} \frac{\partial V_{\sigma}}{\partial x^{\gamma}} \\ & - \Gamma^{\tau}_{\alpha\gamma} \left(\frac{\partial V_{\tau}}{\partial x^{\beta}} - \Gamma^{\sigma}_{\tau\beta} V_{\sigma} \right) \\ & - \Gamma^{\eta}_{\beta\gamma} \left(\frac{\partial V_{\alpha}}{\partial x^{\eta}} - \Gamma^{\sigma}_{\alpha\eta} V_{\sigma} \right) \end{aligned} \tag{2.31}$$

What we've done is ask what's the incremental change in V when I head in the xmu direction and then ask what happens to this quantity when I take a step in the x^{γ} direction. Now we try to do the opposite of that and try to find the difference between the two by using the commutator.

$$V_{\alpha;\gamma} = \frac{\partial V_{\alpha}}{\partial x^{\gamma}} - \Gamma^{\sigma}_{\alpha\gamma} V_{\sigma} \tag{2.32}$$

Call this result $V_{\alpha\gamma}$ and take another covariant derivative, this time with respect to x_{β}:

$$V_{\alpha;\gamma} = \frac{\partial V_{\alpha}}{\partial x^{\gamma}} - \Gamma^{\sigma}_{\alpha\gamma} V_{\sigma} \tag{2.33}$$

As before, you can substitute the expression from Eq. 6.24 into this equation to get,

$$\begin{aligned} V_{\alpha\gamma;\beta} = & \frac{\partial^2 V_{\alpha}}{\partial x^{\beta} \partial x^{\gamma}} - \frac{\partial \Gamma^{\sigma}_{\alpha\gamma}}{\partial x^{\beta}} V_{\sigma} - \Gamma^{\sigma}_{\alpha\gamma} \frac{\partial V_{\sigma}}{\partial x^{\beta}} \\ & - \Gamma^{\tau}_{\alpha\beta} \left(\frac{\partial V_{\tau}}{\partial x^{\gamma}} - \Gamma^{\sigma}_{\tau\gamma} V_{\sigma} \right) \\ & - \Gamma^{\eta}_{\gamma\beta} \left(\frac{\partial V_{\alpha}}{\partial x^{\eta}} - \Gamma^{\sigma}_{\alpha\eta} V_{\sigma} \right) \end{aligned} \tag{2.34}$$

In flat space, the order of covariant differentiation should make no difference, so 2.34 should be identical to 2.31. Any differences between these equations can therefore be attributed to the curvature of the space. If we examine these equations term by term, we can see that the first two terms are equal,

$$\frac{\partial^2 V_{\alpha}}{\partial x^{\gamma} \partial x^{\beta}} = \frac{\partial^2 V_{\alpha}}{\partial x^{\gamma} \partial x^{\beta}}$$

These two terms are equal because the order of applying partial derivatives doesn't matter when the field we consider is continuous. Hence these terms cancel out in the commutator. Now if we compare the second terms, we see that these terms don't cancel out,

$$-\frac{\partial\Gamma^{\sigma}_{\alpha\beta}}{\partial x^{\gamma}}V_{\sigma} \neq -\frac{\partial\Gamma^{\sigma}_{\alpha\gamma}}{\partial x^{\beta}}V_{\sigma}$$

Comparing the third and fourth terms, we can see that they are equal

$$-\Gamma^{\sigma}_{\alpha\beta}\frac{\partial V_{\tau}}{\partial x^{\gamma}} = -\Gamma^{\tau}_{\alpha\beta}\frac{\partial V_{\sigma}}{\partial x^{\gamma}}$$

because the symbols used for dummy indices σ and τ are irrelevant. The fourth term of 2.31 equals the third term of 2.34:

$$-\Gamma^{\tau}_{\alpha\gamma}\frac{\partial V_{\tau}}{\partial x^{\beta}} = -\Gamma^{\sigma}_{\alpha\gamma}\frac{\partial V_{\sigma}}{\partial x^{\beta}}$$

for the same reason. The fifth terms are not equal:

$$\Gamma^{\tau}_{\alpha\gamma}\Gamma^{\sigma}_{\tau\beta}V_{\sigma} \neq \Gamma^{\tau}_{\alpha\beta}\Gamma^{\sigma}_{\tau\gamma}V_{\sigma}$$

But the sixth terms are equal:

$$-\Gamma^{\eta}_{\beta\gamma}\frac{\partial V_{\alpha}}{\partial x^{\eta}} = -\Gamma^{\eta}_{\gamma\beta}\frac{\partial V_{\alpha}}{\partial x^{\eta}}$$

because Christoffel symbols are symmetric in their lower indices. The seventh terms are equal for the same reason:

$$\Gamma^{\eta}_{\beta\gamma}\Gamma^{\sigma}_{\alpha\eta}V_{\sigma} = \Gamma^{\eta}_{\gamma\beta}\Gamma^{\sigma}_{\alpha\eta}V_{\sigma}$$

So when the commutator is formed, most of the terms cancel out, but the second and fifth terms remain after subtraction. Those terms are

$$\begin{aligned} V_{\alpha\beta;\gamma} - V_{\alpha\gamma;\beta} &= \frac{\partial\Gamma^{\sigma}_{\alpha\gamma}}{\partial x^{\beta}}V_{\sigma} - \frac{\partial\Gamma^{\sigma}_{\alpha\beta}}{\partial x^{\gamma}}V_{\sigma} + \Gamma^{\tau}_{\alpha\gamma}\Gamma^{\sigma}_{\tau\beta}V_{\sigma} - \Gamma^{\tau}_{\alpha\beta}\Gamma^{\sigma}_{\tau\gamma}V_{\sigma} \\ &= \left(\frac{\partial\Gamma^{\sigma}_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial\Gamma^{\sigma}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\tau}_{\alpha\gamma}\Gamma^{\sigma}_{\tau\beta} - \Gamma^{\tau}_{\alpha\beta}\Gamma^{\sigma}_{\tau\gamma}\right)V_{\sigma} \end{aligned} \tag{2.35}$$

The terms within the parentheses define the Riemann curvature tensor:

$$R^{\sigma}_{\alpha\beta\gamma} = \frac{\partial\Gamma^{\sigma}_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial\Gamma^{\sigma}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\tau}_{\alpha\gamma}\Gamma^{\sigma}_{\tau\beta} - \Gamma^{\tau}_{\alpha\beta}\Gamma^{\sigma}_{\tau\gamma} \tag{2.36}$$

The reason we have derivatives . Thus, the necessary and sufficient condition for flat space is,

$$R^{\sigma}_{\alpha\beta\gamma} = 0 \tag{2.37}$$

2.7.5 The Ricci Tensor

A tensor related to the Riemann curvature tensor is the Ricci tensor, which you can find by contracting the Riemann tensor along the σ and β indices, written in four dimensions as,

$$R_{\alpha\gamma} = R^{\sigma}_{\alpha\sigma\gamma} = R^{1}_{\alpha 1\gamma} + R^{2}_{\alpha 2\gamma} + R^{3}_{\alpha 3\gamma} + R^{4}_{\alpha 4\gamma} \tag{2.38}$$

2.7.6 Einstein Tensor

The Einstein Tensor is constructed using a combination of the Ricci tensor, Ricci scalar and the metric

$$G_{\alpha\gamma} = R_{\alpha\gamma} - \frac{1}{2} R g_{\alpha\gamma} \tag{2.39}$$

This appears in Einstein's field equation for General Relativity, which is commonly written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.40}$$

where Λ, is the "Cosmological Constant" (which in a very loose sense denote the expansion of the universe), c the speed of light and G the gravitational constant.

2.8 Interesting Scalars

2.8.1 g

g is the determinant of the metric tensor, given by

$$g = det(g_{\mu\nu}) \tag{2.41}$$

A straightforward calculation shows that under a coordinate transformation $x^{\mu} \to x^{\mu^{'}}$, this doesn't transform by the tensor transformation law (under which it would have to be invariant, since it has no indices), but instead as

$$g \to \left[det \left(\frac{\partial x^{\mu'}}{x^{\mu}} \right) \right]^{-2} g \tag{2.42}$$

Here, the factor $det(\partial x^{\mu^{'}} / \partial x^{\mu})$ is the Jacobian of the transformation. Objects with this kind of transformation law which involve powers of the Jacobian are known as **tensor densities**; the determinant g is sometimes called a "scalar density."

2.8.2 Volume Element

The Volume element is a tensor density which can be exemplified as $dx^4 = dx^0 dx^1 dx^2 dx^3$. It transforms like,

$$dx^4 \rightarrow det\left(\frac{x^{\mu'}}{x^{\mu}}\right) dx^4 \tag{2.43}$$

2.8.3 Ricci Scalar

This is the Trace (sum of the elements in the leading diagonal) of the Ricci Tensor

$$R = R^{\lambda}_{\lambda} = g^{\mu\nu} R_{\mu\nu} = Tr(R_{\mu\nu}) \tag{2.44}$$

This plays an important role in General Relativity

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