Tensors

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August 1, 2020

Abstract

1 Vector Transformation Rules

The rules:

- For basis vectors forward transformations brings us from old to new coordinate systems and backward brings us from new to old.
- However, with vector components it's the opposite.

Suppose we have a vector $\vec{v}$ in a basis $\vec{e}_j$. We now transform it to a basis $\tilde{\vec{e}}_i$ where it becomes $\tilde{v}$. We call the forward transformation as F_{ij} and the backward as B_{ij} which we define as:

$$\tilde{\vec{e}}_j = \sum_{i=1}^{n} F_{ij}\vec{e}_i$$

$$\vec{e}_j = \sum_{i=1}^{n} B_{ij}\tilde{\vec{e}}_i$$

We can try to derive the statements made previously,

$$\vec{v} = \sum_{j=1}^{n} v_j\vec{e}_j = \sum_{i=1}^{n} \tilde{v}_i\tilde{\vec{e}}_i$$

$$\vec{v} = \sum_{j=1}^{n} v_j\vec{e}_j = \sum_{j=1}^{n} v_j(\sum_{i=1}^{n} B_{ij}\tilde{\vec{e}}_i) = \sum_{i=1}^{n}\sum_{j=1}^{n}(B_{ij}v_j)\tilde{\vec{e}}_i$$

Thus,

$$\tilde{v}_i = \sum_{j=1}^{n} B_{ij}v_j \tag{1}$$

Similarly,

$$\vec{v} = \sum_{j=1}^{n} v_j \vec{e}_j = \sum_{i=1}^{n} \tilde{v}_i \tilde{\vec{e}}_i$$

$$\vec{v} = \sum_{j=1}^{n} \tilde{v}_j \tilde{\vec{e}}_j = \sum_{j=1}^{n} \tilde{v}_j (\sum_{i=1}^{n} F_{ij} \vec{e}_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} (F_{ij} \tilde{v}_j) \vec{e}_i$$

Thus,

$$v_i = \sum_{j=1}^{n} F_{ij} \tilde{v}_j \tag{2}$$

Now because vector components beehave contrary to the basis vectors, they are said to be ***"Contravariant"***

2 Index Notation

2.1 Einstein Notation i.e. Summing convention

Let us consider the sum[1],

$$x_i = \sum_{j}^{n} \Lambda_{ij} \mathcal{X}^j$$

Is the same as,

$$x_i = \Lambda_{ij} \mathcal{X}^j$$

Here, we define i to be the free index and j to be the summing index or the dummy index that is repeated to signify so.

2.2 Index Convention

When we sum from 1 to 3 we use the symbols i, j and k i.e. the English alphabet to signify that we are only considering dimnesions that are spatial/that are not a timme dimension. However, when we use the symbols ν and μ i.e. Greek alphabets we are summing from 0 to 3, we also include the temporal dimension according to the tradition of special relativity in which we name components as $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$ in the Cartesian framework.

3 Covectors

- Covectors can be thought of as row vector or as functions that act on Vectors such that any covector $\alpha : \mathbb{V} \to \mathbb{R}$
- Covectors are linear maps i.e. $\beta(\alpha)\vec{v} = \beta\alpha\vec{v}$ and $(\beta + \alpha)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

[1] Mind you there are no exponents there.

- Covectors are elements of a Dual vector space $\mathbb{V}^*$ which has different rules for addition and scaling i.e. scalar multiplication
- You visualize covectors to be some sort of gridline on your vector space such that applying a covector to a vector is equivalent to projecting the vector along the gridline
- Covectors are invariant but their components are not
- The covectors that form the basis for the set of all covectors is called the ***"Dual Basis"***, because they are a basis for the Dual Space $\mathbb{V}^*$ i.e. any covector can be expressed as the linear combination of the dual basis
- However we are free to choose a dual basis
- For covector components, forward transformation brings us from old to new and backwards vice versa
- We can flip between row and column vectors for an orthonormal basis
- Vector components are measured by counting how many are used in the construction of a vector, but covector components are measured by counting the number of covector lines that the basis vector pierces
- The covector basis transforms contravariantly compared to the basis and it's components transform covariantly according to the basis

3.1 Contravariant Components

We denote contravariant components using the symbols

$$A^i$$

and their basis like

$$\overrightarrow{e}_i$$

3.2 Covariant Components

We denote convariant components using the symbols

$$A_i$$

and their basis like

$$\overrightarrow{e}^i$$

3.3 Relationship Between the Two Types of Components

$$|\overrightarrow{e}^1| = \frac{1}{|\overrightarrow{e}_1|\cos(\theta_1)}$$

and,

$$|\overrightarrow{e}_1| = \frac{1}{|\overrightarrow{e}^1|\cos(\theta_1)}$$

Or with 3 components we have: Since both types of components represent the same vector (as in same magnitude) only inn different bases, we can write

$$\overrightarrow{A} = A^i \overrightarrow{e}_i = A_i \overrightarrow{e}^i$$

3.4 Using Cramer's Method to find Components

4 Linear Maps

Linear maps to put it naively, Linear Maps transform input vectors but not the basis. Geometrically speaking, Linear Maps:

- Keep gridlines parallel
- Keep gridlines evenly spaced
- Keep the origin stationary

To put it more abstractly, Linear Maps:

- Maps vectors to vectors, $\mathbb{L} : \mathbb{V} \to \mathbb{V}$
- Adds inputs or outputs, $\mathbb{L}(\vec{V} + \vec{W}) = \mathbb{L}(\vec{V}) + \mathbb{L}(\vec{W})$
- Scale the inputs or outputs, $\mathbb{L}(\alpha\vec{V}) = \alpha\mathbb{L}(\vec{V})$
- i.e. They are Linear/Linearity

When I transform the basis using a forward transformation, the transformed Linear map $\tilde{\mathbb{L}}_i^l$ can be written as:

$$\tilde{\mathbb{L}}_i^l = \mathbb{B}_k^l \mathbb{L}_j^k \mathbb{F}_i^j \tag{3}$$

5 Covariant Quantities

6 Contravariant Quantities

7 Clearer Definitions

8 Covariant Differentiation

9 Metric Tensor

- Pythagoras' theorem is a lie for non-orthonormal bases
- The metric Tensor is Tensor that helps us compute lengths and angles
- For two dimensions it can be written as:

$$g_{ij} = \begin{bmatrix} e_1e_1 & e_1e_2 \\ e_2e_1 & e_2e_2 \end{bmatrix}$$

- Or more abstractly

$$g_{ij} = e_ie_j$$

- The dot product between two vectors can be written as

$$||\vec{v}||||\vec{w}|| \cos\theta = v^i w^j g_{ij}$$

- we can see how this allows us to compute angles as well
- To transform the components of the Metric Tensor we have to apply the transformation twice i.e.$\tilde{g}_{\rho\sigma} = \mathbb{F}^{\mu}_{\rho}\mathbb{F}^{\nu}_{\sigma}\tilde{g}_{\mu\nu}$ or $g_{\rho\sigma} = \mathbb{B}^{\mu}_{\rho}\mathbb{B}^{\nu}_{\sigma}\tilde{g}_{\mu\nu}$