Vector Analysis

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1 Vector Algebra

2 Differential Calculus

2.1 "Ordinary" Derivatives

What is the derivative of a function $f(x)$? It tells us how quickly f(x) changes when we make a small change dx in it's argument x,

$$df = \left(\frac{df}{dx}\right) dx \tag{1}$$

If we change x by an amount dx then $f(x)$ changes by an amount df, the derivative is a proportionality factor. Geometrically speaking, the derivative df/dx is the slope/gradient of the graph of $f(x)$ versus x.

2.2 Gradient

The gradient, geometrically speaking points in the diretion of maximum increase/ascent for the function

2.3 The Del Operator

2.4 The Divergence

2.5 The Curl

2.6 Product Rules

2.7 Second Derivatives

3 Integral Calculus

4 Curvilinear Coordinates

4.1 Spherical Coordinates

4.2 Cynlindrical Coordinates

5 The Dirac Delta Function

5.1 The Divergence of $\frac{\hat{r}}{r^2}$

$$\nabla.\frac{\hat{r}}{r^2} = 0 \tag{2}$$

5.2 The One-Dimensional Dirac Delta Function

The Dirac Delta is a functional [1] which we define as,

$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{cases} \tag{3}$$

$$\int_{-\infty}^{+\infty} \delta(x-a)dx = 1 \tag{4}$$

$\forall\ a \in \mathbb{R}$ We can visualize it as a sharp peak at a, We can interpret 4 as saying "the area of the delta distribution is always 1".

$$f(x)\delta(x-a) = f(a) \tag{5}$$

We can combine these to get,

$$\int_{-\infty}^{+\infty} \delta(x-a)f(x)dx = f(a) \tag{6}$$

[1] An object that is a map between functions

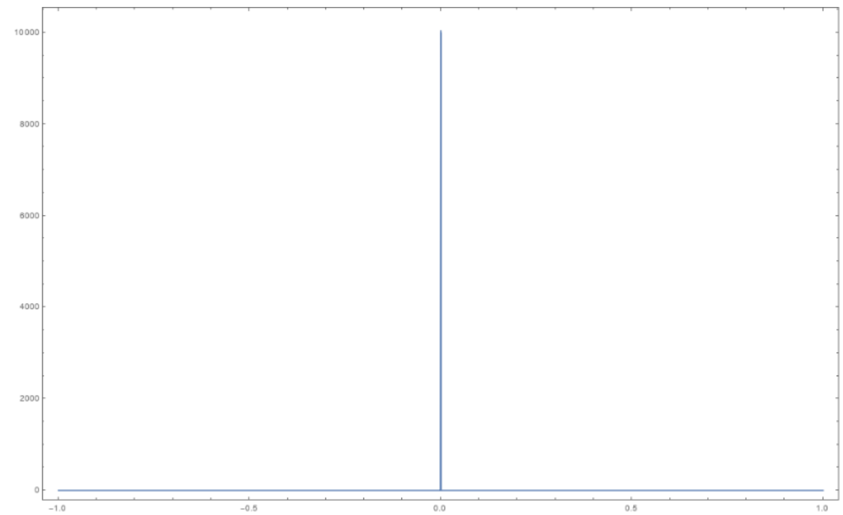

Figure 1: A Plot of $\delta(x)$

5.2.1 A few interesting properties

$$\delta(kx) = \frac{1}{|k|}\delta(x) \tag{7}$$

$$\frac{d}{dx}(\delta(x)) = -\delta(x) \tag{8}$$

where k is a constant

$$\frac{d\theta}{dx} = \delta(x) \tag{9}$$

Where θ is the step function defined as,

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ o, & \text{if } x \leq 0 \end{cases} \tag{10}$$

5.3 The Three-Dimensional Dirac Delta Function

We generalize () to three dimensions,

$$\delta^3(\vec{r} - \vec{a}) = \delta(x - a_x)\delta(y - a_y)\delta(z - a_z) \tag{11}$$

$$\int_{-\infty}^{+\infty} \delta^3(\vec{r} - \vec{a})dV = 1 \tag{12}$$

We can also define the three-dimensional delta function as

$$\delta^3(\boldsymbol{\mathscr{r}}) = \frac{1}{4\pi}\left[\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right] \tag{13}$$

Since,

$$\nabla \left(\frac{1}{\mathscr{r}} \right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

We can rewrite as,

$$\delta^3(\boldsymbol{\mathscr{r}}) = -\frac{1}{4\pi} \left[\nabla^2 \left(\frac{1}{\mathscr{r}} \right) \right] \tag{14}$$

References