Notes on Tensors

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Preface

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Chapter 1

Tensor-Algebra

1.1 Vector Transformation Rules

The rules:

- For basis vectors forward transformations brings us from old to new coordinate systems and backward brings us from new to old.
- However, with vector components it's the opposite.

Suppose we have a vector $\vec{v}$ in a basis $\vec{e}_j$. We now transform it to a basis $\tilde{\vec{e}}_i$ where it becomes $\tilde{v}$. We call the forward transformation as F_{ij} and the backward as B_{ij} which we define as:

$$\tilde{\vec{e}}_j = \sum_{i=1}^{n} F_{ji} \vec{e}_i$$

$$\vec{e}_j = \sum_{i=1}^{n} B_{ji} \tilde{\vec{e}}_i$$

We can try to derive the statements made previously,

$$\vec{v} = \sum_{j=1}^{n} v_j \vec{e}_j = \sum_{i=1}^{n} \tilde{v}_i \tilde{\vec{e}}_i$$

$$\vec{v} = \sum_{j=1}^{n} v_j \vec{e}_j = \sum_{j=1}^{n} v_j (\sum_{i=1}^{n} B_{ij} \tilde{\vec{e}}_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} (B_{ij} v_j) \tilde{\vec{e}}_i$$

Thus,

$$\tilde{v}_i = \sum_{j=1}^{n} B_{ij} v_j \tag{1.1}$$

Similarly,

$$\vec{v} = \sum_{j=1}^{n} v_j \vec{e}_j = \sum_{i=1}^{n} \tilde{v}_i \tilde{\vec{e}}_i$$

$$\vec{v} = \sum_{j=1}^{n} \tilde{v}_j \tilde{\vec{e}}_j = \sum_{j=1}^{n} \tilde{v}_j (\sum_{i=1}^{n} F_{ij} \vec{e}_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} (F_{ij} \tilde{v}_j) \vec{e}_i$$

Thus,

$$v_i = \sum_{j=1}^{n} F_{ij} \tilde{v}_j \tag{1.2}$$

Now because vector components beehave contrary to the basis vectors, they are said to be ***"Contravariant"***

1.2 Index Notation

1.2.1 Einstein Notation i.e. Summing convention

Let us consider the sum[1],

$$x_i = \sum_{j}^{n} \Lambda_{ij} \mathcal{X}^j$$

Is the same as,

$$x_i = \Lambda_{ij} \mathcal{X}^j$$

Here, we define i to be the free index and j to be the summing index or the dummy index that is repeated to signify so.

1.2.2 Index Convention

When we sum from 1 to 3 we use the symbols i, j and k i.e. the English alphabet to signify that we are only considering dimnesions that are spatial/that are not a timme dimension. However, when we use the symbols ν and μ i.e. Greek alphabets we are summing from 0 to 3, we also include the temporal dimension according to the tradition of special relativity in which we name components as $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$ in the Cartesian framework.

[1]Mind you there are no exponents there.

1.3 Covectors

- Covectors can be thought of as row vector or as functions that act on Vectors such that any covector $\alpha : \mathbb{V} \to \mathbb{R}$
- Covectors are linear maps i.e. $\beta(\alpha)\vec{v} = \beta\alpha\vec{v}$ and $(\beta + \alpha)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
- Covectors are elements of a Dual vector space $\mathbb{V}^*$ which has different rules for addition and scaling i.e. scalar multiplication
- You visualize covectors to be some sort of gridline on your vector space such that applying a covector to a vector is equivalent to projecting the vector along the gridline
- Covectors are invariant but their components are not
- The covectors that form the basis for the set of all covectors is called the ***"Dual Basis"***, because they are a basis for the Dual Space $\mathbb{V}^*$ i.e. any covector can be expressed as the linear combination of the dual basis
- However we are free to choose a dual basis
- For covector components, forward transformation brings us from old to new and backwards vice versa
- We can flip between row and column vectors for an orthonormal basis
- Vector components are measured by counting how many are used in the construction of a vector, but covector components are measured by counting the number of covector lines that the basis vector pierces
- The covector basis transforms contravariantly compared to the basis and it's components transform covariantly according to the basis
- The covector basis is denoted as ϵ^j

1.3.1 Contravariant Components

We denote contravariant components using the symbols

$$A^i$$

and their basis like

$$\overrightarrow{e}_i$$

1.3.2 Covariant Components

We denote convariant components using the symbols

$$A_i$$

and their basis like

$$\overrightarrow{e}^i$$

1.3.3 Relationship Between the Two Types of Components

$$|\overrightarrow{e}^1| = \frac{1}{|\overrightarrow{e}_1|\cos(\theta_1)}$$

and,

$$|\overrightarrow{e}_1| = \frac{1}{|\overrightarrow{e}^1|\cos(\theta_1)}$$

Or with 3 components we have: Since both types of components represent the same vector (as in same magnitude) only inn different bases, we can write

$$\overrightarrow{A} = A^i\overrightarrow{e}_i = A_i\overrightarrow{e}^i$$

1.3.4 Using Cramer's Method to find Components

1.4 Linear Maps

Linear maps to put it naively, Linear Maps transform input vectors but not the basis. Geometrically speaking, Linear Maps:

- Keep gridlines parallel
- Keep gridlines evenly spaced
- Keep the origin stationary

To put it more abstractly, Linear Maps:

- Maps vectors to vectors, $\mathbb{L} : \mathbb{V} \to \mathbb{V}$
- Adds inputs or outputs, $\mathbb{L}(\vec{V} + \vec{W}) = \mathbb{L}(\vec{V}) + \mathbb{L}(\vec{W})$
- Scale the inputs or outputs, $\mathbb{L}(\alpha\vec{V}) = \alpha\mathbb{L}(\vec{V})$
- i.e. They are Linear/Linearity

- When I transform the basis using a forward transformation, the transformed Linear map $\tilde{\mathbb{L}}_i^l$ can be written as:

$$\tilde{\mathbb{L}}_i^l = \mathbb{B}_k^l \mathbb{L}_j^k \mathbb{F}_i^j \tag{1.3}$$

1.5 Metric Tensor

- Pythagoras' theorem is a lie for non-orthonormal bases
- The metric Tensor is Tensor that helps us compute lengths and angles
- For two dimensions it can be written as:

$$g_{ij} = \begin{bmatrix} e_1e_1 & e_1e_2 \\ e_2e_1 & e_2e_2 \end{bmatrix}$$

- Or more abstractly

$$g_{ij} = e_ie_j$$

- The dot product between two vectors can be written as

$$||\vec{v}||||\vec{w}|| \cos\theta = v^i w^j g_{ij}$$

- we can see how this allows us to compute angles as well
- Moreover this formula works in all coordinates thus, the vector length stays constant
- To transform the components of the Metric Tensor we have to apply the transformation twice i.e.$g_{\mu\nu} = \mathbb{F}_\mu^\rho \mathbb{F}_\nu^\sigma \tilde{g}_{\rho\sigma}$ or $\tilde{g}_{\rho\sigma} = \mathbb{B}_\rho^\mu \mathbb{B}_\sigma^\nu g_{\mu\nu}$
- $\alpha g(\vec{V}, \vec{W}) = g(\alpha\vec{V}, \vec{W}) = g(\vec{V}, \alpha\vec{W})$
- $g(\vec{V} + \vec{U}, \vec{W}) = g(\vec{V}, \vec{W}) + g(\vec{U}, \vec{W})$
- $\alpha(\vec{V} + \vec{U})g = \alpha\vec{V}g + \alpha\vec{U}g$
- $g(\vec{V}, \vec{W}) = V^iW^jg_{ij} = V^iW^jg_{ji} = g(\vec{W}, \vec{V})$
- $g(\vec{V}, \vec{V}) = ||\vec{V}||^2 \geq 0 \; \forall \; \vec{V} \neq 0$
- In short, $g := \mathbb{V} \times \mathbb{V} \to \mathbb{R}$
- We can define a new quantity called ***"Scale-Factor"*** as, $h_i = \sqrt{g_{ii}}$

- Through this we can rewrite some of the Operators from Vector Calculus:
  - Gradient: $\boldsymbol{\nabla}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x^1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x^2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x^3}\hat{e}_3$
  - Divergence: $\nabla\circ\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(h_2h_3A_1) + \frac{\partial}{\partial x^2}(h_1h_3A_2) + \frac{\partial}{\partial x^3}(h_1h_2A_3)\right]$
  - Curl: $\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$
  - Laplacian:$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x^3}\right)\right]$

1.6 Bilinear Forms

- The metric tensor is an example of a Bilinear form
- We define a Bilinear form as:
  - $\mathcal{B} := \mathbb{V}\times\mathbb{V}\to\mathbb{R}$
  - $\alpha\mathcal{B}(\vec{V},\vec{W}) = \mathcal{B}(\alpha\vec{V},\vec{W}) = \mathcal{B}(\vec{V},\alpha\vec{W})$
  - $\mathcal{B}(\vec{V}+\vec{U},\vec{W}) = \mathcal{B}(\vec{V},\vec{W}) + \mathcal{B}(\vec{U},\vec{W})$
  - $\alpha(\vec{V}+\vec{U})\mathcal{B} = \alpha\vec{V}\mathcal{B} + \alpha\vec{U}\mathcal{B}$
  - $\mathcal{B}(\vec{V},\vec{U}) \to V^iW^j\mathcal{B}_{ij}$
- Bilinear forms are (0,2) Tensors, they transform using two covariant rules when we transform them
- A form is simply a function that takes vectors as inputs and outputs a number
- So covectors are sometimes called Linear forms/ 1-forms
- This structure is called a Bilinear form since each individual input is Linear while the other input is held constant
- $\mathcal{B}_{\mu\nu} = \mathbb{F}^{\rho}_{\mu}\mathbb{F}^{\sigma}_{\nu}\tilde{\mathcal{B}}_{\rho\sigma}$ and $\tilde{\mathcal{B}}_{\rho\sigma} = \mathbb{B}^{\mu}_{\rho}\mathbb{B}^{\nu}_{\sigma}\mathcal{B}_{\mu\nu}$
- $\mathcal{B}(\vec{V},\vec{W}) = \mathcal{B}_{\mu\nu}V^{\mu}W^{\nu} = \begin{bmatrix}\begin{bmatrix}\mathcal{B}_{11} & \mathcal{B}_{12}\end{bmatrix} & \begin{bmatrix}\mathcal{B}_{21} & \mathcal{B}_{22}\end{bmatrix}\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}\begin{bmatrix}w_1\\w_2\end{bmatrix}$

1.7 Clearer Definitions

1.7.1 Linear Maps

- A Tensor is a collection of vectors and covectors combined together using the Tensor Product

- Pure Matrices can be broken down into the product of row and column matrices.

- Each element of the same column in a pure matrix is a scalar product of each other

- Pure matrices as Linear maps only produce output vectors in the same direction due to the previous statement

- Any linear map $\mathbb{L}$ can be written as the Linear combination of the product of Vector-Co-vector pairs i.e. $\mathbb{L} = \mathbb{L}^{\mu}_{\nu}\vec{e}_{\mu}\epsilon^{\nu} := \mathbb{V} \to \mathbb{V}$

- Reminder: $\epsilon^i \otimes {}^2\vec{e}_j = \delta^i_j$

1.7.2 Bilinear Forms

- Bilinear forms are a Linear combination of covector-covector pairs

- $\mathcal{B} = \mathcal{B}_{ij}(\epsilon^i \otimes \epsilon^j)$

1.7.3 Tensors

- An object that is invriant under a change of coordinates and has components that change in a special and predictable way under a change of coordinates.

- A collection of vectors and covectors combined using the Tensor Product

- For a tensor $T^{i_m}_{j_n}$, we say it has a type (m, n)

[2] This symbol represents the Tensor product, see for more information

1.8 Tensor Addition and Subtraction

- Two tensors can be added provided they have the same structure i.e. the same number of vectors and covectors

- The resultant of tensor addition or subtraction is another tensor with the same structure i.e. $A^i_j \pm B^i_j = C^i_j$

1.9 Tensor Products

- The Tensor product and the Kronecker product are kind of doing the same thing, the Tensor product combines the abstract vector and the abstract covector and the Kronecker product combines 1 dimensional arrays

- However both products result in the same set of components

1.9.1 Tensor Product

- Combines 2 Tensors into a 3rd new Tensor

- The result is a Linear map

- Eg: $(\vec{e}_i \otimes \epsilon^j)\vec{v} = v^j \vec{e}_i$

1.9.2 Kronecker Product

- Combines 2 arrays into a 3rd new array

- Eg:- $\begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} & \alpha_2 \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} \end{bmatrix}$

1.9.3 Array Multiplication

$$\begin{bmatrix} W^1 \\ W^2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} L^1_1 \\ L^2_1 \end{bmatrix} & \begin{bmatrix} L^1_2 \\ L^2_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = \begin{bmatrix} L^1_1 V^1 + L^1_2 V^2 \\ L^2_1 V^1 + L^2_2 V^2 \end{bmatrix}$$

- This method isn't very useful for higher dimensional tensors, for them we stick to Einstein notation as the larger number of components, the more number of ways we can do the summation

1.9.4 Tensor Product Spaces

We have the following properties for a Tensor product $\forall\ \vec{U}, \vec{V}, \vec{W}\ \in \mathbb{V}$, $\forall\ \alpha, \beta \in \mathbb{V}^*$ and $\forall\ n \in \mathcal{F}$

- $n(\vec{V} \otimes \alpha) = (n\vec{V}) \otimes \alpha = \vec{V} \otimes (n\alpha)$ Here, n is a scalar and α a covector
- $\vec{V} \otimes \alpha + \vec{V} \otimes \beta = \vec{V} \otimes (\alpha + \beta)$
- $\vec{V} \otimes \alpha + \vec{W} \otimes \alpha = (\vec{V} + \vec{W}) \otimes \alpha$
- $\vec{U}\alpha\vec{V} \otimes \vec{U}\beta\vec{V} = \vec{U}(\alpha \otimes \beta)\vec{V}$
- $\mathbb{V} \otimes \mathbb{V}^* := \mathbb{V} \times \mathbb{V}^* \to \mathbb{R}$ for example, whose elements are (1,1) Tensors
- We can always do the summation differently and end up with different elements even if the map leads us to the same space
- We can combine any number of tensors provided the upstairs indices match the downstairs indices across the product
- For example,$T_{i\ kl}^{j}\alpha_j D^{kl} := \mathbb{V}^* \times (\mathbb{V} \otimes \mathbb{V}) \to \mathbb{V}^*$
- You can guess where the output of the operation by finding out where the free index lies
- **Multilinear map**, a function that's linear when all inputs except one are held constant
- A Tensor when used as a function is simply a Multilinear map

1.10 Raising and Lowering Indices

- To go from the vector space to the dual space we use the covariant metric g_{ij} i.e. this lowers the indices. Thus they are sometimes denoted using the symbol $\flat$ and called "flat operators"
- To go from the dual space to the vector space we use the covariant metric $\mathfrak{g}^{ij}$ i.e. this raises the indices. Thus they are sometimes denoted using the symbol $\sharp$ and called "sharp operators"
- These matrices are inverses of each other i.e. $g_{ki}\mathfrak{g}^{ij} = \delta_i^k$

Chapter 2

Tensor-Calculus

2.1 Derivatives

We can rewrite the gradient for a scalar field we know from vector calculus as,

$$(grad\ f)_\mu = \frac{\partial f}{\partial x^\mu} \tag{2.1}$$

we can write the gradient of a vector field as,

$$(grad\ \vec{v})^\mu{}_\nu = \frac{\partial x^\mu}{\partial x^\nu} \tag{2.2}$$

and of a Tensor field,

$$(grad\ t)_{\mu\nu\alpha} = \frac{\partial t_{\mu\nu}}{\partial x^\alpha} \tag{2.3}$$

We can use the following notation for simplicity,

$$v^\mu{}_{,\nu} = \partial_\nu v^\mu := \frac{\partial v^\mu}{\partial x^\nu} \tag{2.4}$$

and

$$v^{\mu,\nu} = \partial^\nu v^\mu := g^{\nu\rho}\frac{\partial v^\mu}{\partial x^\rho} \tag{2.5}$$

2.2 Covector Field

2.3 Differential Forms

2.4 Gradient

2.5 Geodesics

2.6 Covariant Derivative

The covariant derivative of a vector field v^μ is defined as,

$$\nabla_\mu v^\alpha = \partial_\mu v^\alpha + \Gamma^\alpha_{\mu\nu} v^\nu \tag{2.6}$$

where, the object $\Gamma^\alpha_{\mu\nu}$ is called the Christoffel symbol. The Christoffel symbol is not a tensor because it contains all the information about the curvature of the coordinate system and can therefore be transformed entirely to zero if the coordinates are straightened. Nevertheless we treat it as any ordinary tensor in terms of the index notation.
We define the Christoffel symbol in terms of the metric $g_{\mu\nu}$ and it's inverse $g^{\alpha\beta}$ as,

$$\Gamma^\alpha_{\ \mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\nu}}{\partial x^\mu} + \frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right) = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\nu,\mu} + g_{\beta\mu,\nu} - g_{\mu\nu,\beta} \right) \tag{2.7}$$

We can define the covariant derivative of a covector as,

$$\nabla_\mu w_\alpha = \partial_\mu w_\alpha + \Gamma^\alpha_{\mu\nu} w_\nu \tag{2.8}$$

The covariant derivative of a tensor $t^{\alpha\beta}$ is then,

$$\nabla_\mu t^{\alpha\beta} = \partial_\mu t^{\alpha\beta} + \Gamma^\alpha_{\mu\sigma} t^{\sigma\beta} + \Gamma^\beta_{\mu\sigma} t^{\alpha\sigma} \tag{2.9}$$

and of a tensor t^α_β,

$$\nabla_\mu t^\alpha_\beta = \partial_\mu t^\alpha_\beta + \Gamma^\alpha_{\mu\sigma} t^\sigma_\beta + \Gamma^\beta_{\mu\sigma} t^\alpha_\sigma \tag{2.10}$$

2.6.1 Properties

- The covariant derivative produces, as its name says, covariant expressions.
- $\nabla_\mu g_{\alpha\beta} = 0$

- $g^{\alpha\gamma}\nabla_\alpha t^{\mu\nu}_\gamma = \nabla_\alpha(t^{\mu\nu}_\gamma g^{\alpha\gamma}) = \nabla_\alpha t^{\mu\nu\alpha}$
- $\nabla^\alpha = g^{\alpha\beta}\nabla_\beta$

Therefore c is a perfectly valid tensor. We can also contract indices: ,or with help of the metric: . Since (as we saw in the above exercise) we can always bring the and/or inside or outside the operator. We can therefore write

2.7 Lie Brackets and Flow

2.8 Interesting Tensors

2.8.1 Kronecker delta

It simply has the 'function' of 'renaming' an index:

$$\delta^\mu_\nu x^\nu = x^\mu$$

it is in a sense simply the identity matrix.

2.8.2 Levi-Civita Pseudotensor

The Levi-Civita Pseudotensor i.e. Tensor density is a completely anti-symmetric i.e. $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$, we define it as:

$$\epsilon_{ijk} = \begin{cases} 1 \text{ if } ijk \text{ is an even permuation of } 123 \\ -1 \text{ if } ijk \text{ is an odd permuation of } 123 \\ 0 \text{ if two indices are equal} \end{cases} \tag{2.11}$$

Identities

$$\epsilon_{\alpha\beta\nu}\epsilon_{\alpha\beta\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \tag{2.12}$$

From this it follows that,

$$\epsilon_{\alpha\beta\nu}\epsilon_{\alpha\beta\sigma} = 2\delta_{\nu\sigma} \tag{2.13}$$

and

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma} = 6 \tag{2.14}$$

Cross-Product

Using these identities and the definition we can rewrite the cross-product of two vectors as,

$$\vec{a} = \vec{a} \times \vec{b} = \epsilon_{ijk} a_j b_k \tag{2.15}$$

Thus the expressions in vector product notation can be changed to index notation for example,

$$c = \nabla.(\nabla \times \vec{a}) = \nabla_i(\epsilon_{ijk}\nabla_j a_k) = \epsilon_{ijk}\partial_i\partial_j a_k$$

because,

$$\nabla_i = \frac{\partial}{\partial x_i} := \partial_i$$

Rot

Rot is defined as the generalized rotation of a covector,

$$(rot\tilde{w})_{\alpha\beta} = \partial_\alpha w_\beta - \partial_\beta w_\alpha \tag{2.16}$$

2.8.3 Inertia Tensor

2.8.4 Electromagnetic Field Tensor

$$F^{\mu\nu} = \tag{2.17}$$

and it's

$$F_{\mu\nu} = \tag{2.18}$$

and it's dual as,

$$G^{\mu\nu} = \tag{2.19}$$

We can now rewrite Maxwell's equations as:

2.8.5 Torsion Tensor

2.8.6 The Riemann Curvature Tensor

2.8.7 The Ricci Tensor