Introduction to Cosmology

V Mukund

18^{th} October,2020

Friedmann Equation

The key equation of general relativity is Einstein's field equation, which is the relativistic equivalent of Poisson's equation in Newtonian dynamics.
Poisson's equation :

$$\nabla^2 \phi = 4\pi G \rho \tag{1}$$

This equation gives a mathematical relation between the gravitational potential ϕ at a point in space and the mass density ρ at that point. By taking the gradient of the potential, you can determine the acceleration, and then can compute the trajectory of objects moving freely through space.
On the contrary, Einstein's field equation, gives a mathematical relation between the metric of space-time at a point and the energy and pressure at that space-time point.

Friedmann Equation

In a cosmology, Einstein's field equations can be used to find the linkage between a(t), k, R_0 which describe the curvature of the universe, and the energy density ϵ_t and pressure P(t) of the contents of the universe.
The equation that links together a(t), k, R_0 and ϵ_t is known as the Friedmann Equation named after Alexander Alexandrovich Friedmann. He derived his eponymous equation starting from Einsteins's field equations using general relativity.
Consider a homogeneous sphere of matter, with total mass M_s constant with time. The sphere is expanding or contracting isotropically, so that its radius $R_s(t)$ is increasing or decreasing with time. Place a test mass, of infinitesimal mass m, at the surface of the sphere.
From Newton's law of Gravitation, we know that

$$F = -\frac{GM_s m}{R_s(t)^2} \tag{2}$$

Friedmann Equation

Newtoninan Form of Friedmann Equation

The gravitational acceleration at the surface of the sphere will then be, from Newton's Second Law of Motion,

$$\frac{d^2 R_s}{dt^2} = -\frac{GM_s}{R_s(t)^2} \tag{3}$$

Multiply each side of the equation by $\frac{dr_s}{dt}$ and integrate to get,

$$\frac{1}{2}\left(\frac{\mathrm{d}R_s}{\mathrm{d}t}\right)^2 = \frac{GM_s}{R_s(t)} + U \tag{4}$$

Kinetic Energy per unit mass is,

$$E_{kin} = \frac{1}{2}\left(\frac{\mathrm{d}R_s}{\mathrm{d}t}\right)^2 \tag{5}$$

Friedmann Equation

Gravitation potential energy is given by,

$$E_{pot} = -\frac{GM_s}{R_s(t)} \tag{6}$$

It is constant for a bit of matter at the surface of a sphere, as the sphere expands or contracts under its own gravitational influence

The mass of the sphere is constant as it expands or contracts, so,

$$M_s = \frac{4\pi}{3}\rho_{(t)} R_s(t)^3 \tag{7}$$

Since the expansion is isotropic about the sphere's center, we can write the radius $R_s(t)$ in the form,

$$R_s(t) = a(t) r_s \tag{8}$$

where a(t) is the scale factor and r_s is the co-moving radius of the sphere.

Friedmann Equation

Equation (4) in terms of $\rho_{(t)}$ and a(t) can be written as,

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi}{3}Gr_s^2\rho_{(t)}a(t)^2 + U \tag{9}$$

By dividing both the sides of equation (9) by $\frac{r_(s)^2a^2}{2}$, we get,

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho_{(t)} + \frac{2U}{r_s^2}\frac{1}{a(t)^2} \tag{10}$$

Equation (10) gives the Newtonian Form of Friedmann Equation.

Friendmann Equation

Relativistic Form of Friedmann Equation

The Freidmann equation including all the general relativistic effects can be written as,

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3c^2}\epsilon_{(t)} - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2} \tag{11}$$

• The first change is that the mass density ρ has been replaced by an energy density ϵ divided by the square of the speed of light.
The peculiar motion of a particle is given by,

$$E = (m^2c^4 + p^2c^2)^{1/2} \tag{12}$$

where 'm' is the mass of the particle, 'c' is the speed of light and 'p' is the momentum of the particle.
If the particle is non-relativistic, its peculiar velocity 'v' is much less than 'c', hence its peculiar momentum will be $p \approx mv$ and its energy will be

$$E_{non-rel} \approx mc^2(1 + \frac{v^2}{c^2}) \approx mc^2 + \frac{1}{2}mv^2 \tag{13}$$

Friedmann Equation

Relativistic Friedmann Equation

• The second change that must be made is to make the relativistic substitution.

$$\frac{2U}{r_s^2} = -\frac{kc^2}{R_0^2} \tag{14}$$

The Friedmann equation is a very important equation in cosmology. But, if we want to apply the Friedmann equation to the real universe, we must have some way of relating it to observable properties. For example, the Friedmann equation can be related to the Hubble constant, H_0. We know that, in a universe whose expansion (or contraction) is described by a scale factor a(t), there's a linear relation between recession speed v and proper distance d:

$$v(t) = H(t)d(t) \tag{15}$$

where $H(t) \equiv \frac{\dot{a}}{a}$

Friedmann Equation

Thus the Friedmann equation can be written as,

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon_{(t)} - \frac{kc^2}{R_0^2 a(t)^2} \tag{16}$$

At the present moment,

$$H_0 = H(t_0) = (\frac{\dot{a}}{a}) = 70 \pm 7\ Km\, s^{-1} Mpc^{-1} \tag{17}$$

$$H_0^2 = \frac{8\pi G}{3c^2}\epsilon_0 - \frac{kc^2}{R_0^2} \tag{18}$$

Thus, the Friedmann equation gives a relation among H_0, which tells us the current rate of expansion, ϵ_0, which tells us the current energy density, and $\frac{k}{R_0^2}$, which tells us the current curvature.

Freidmann Equation

As we have seen, the general form of Friedmann equation can be written as

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon_{(t)} - \frac{kc^2}{R_0^2 a(t)} \tag{19}$$

For a flat space,

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon_{(t)} \tag{20}$$

Thus, for a given value of the Hubble parameter, there is a critical density,

$$\epsilon_{(c)} = \frac{3c^2}{8\pi G}H(t)^2 \tag{21}$$

$$\epsilon_{(c,0)} = (8.3 \pm 1.7) \times 10^{-10} J\ m^{-3} = 5200 \pm 1000\ MeV\ m^{-3} \tag{22}$$

Freidmann Equation

The critical density is frequently written as the equivalent mass density,

$$\rho_{(c,0)} = \frac{\epsilon_{(c,0)}}{c^2} = (9.2 \pm 1.8) \times 10^{-27} Kg\, m^{-3} = (1.4 \pm 0.3) \times 10^{11} M_0\, Mpc^{-3} \tag{23}$$

In discussing the curvature of the universe, it is more convenient to use not the absolute density ϵ, but the ratio of the density to the critical density $\epsilon_{(c)}$. Thus, when talking about the energy density of the universe, cosmologists often use the dimensionless density parameter

$$\Omega(t) = \frac{\epsilon(t)}{\epsilon_c(t)} \tag{24}$$

In terms of the density parameter, the Friedmann equation can be written in yet another form:

$$1 - \Omega(t) = \frac{kc^2}{R_0^2 a(t)^2 H(t)^2} \tag{25}$$

Fluid and Acceleration Equations

Energy conservation is a generally useful concept, so let's look at equation given by the first law of thermodynamics:

$$dQ = dE + PdV \tag{26}$$

If the universe is perfectly homogeneous, then for any volume dQ $= 0$; that is, there is no bulk flow of heat. Since dQ $= 0$ for a co-moving volume as the universe expands, the first law of thermodynamics, as applied to the expanding universe, reduces to the form

$$\dot{E} + P\dot{V} = 0 \tag{27}$$

Fluid and Acceleration Equations

Consider a sphere of co-moving radius rs expanding along with the universal expansion, so that its proper radius is $R_s(t) = a(t)r_s$. The volume of the sphere is

$$V(t) = \frac{4\pi}{3} r^3 a(t)^3 \tag{28}$$

The rate of change of the volume of the sphere is,

$$\dot{V} = \frac{4\pi}{3} r_s^3 (3a^2 \dot{a}) = V(3\frac{\dot{a}}{a}) \tag{29}$$

The internal energy of the sphere is

$$E(t) = V(t)\epsilon_{(t)} \tag{30}$$

Fluid and Acceleration Equations

$$\dot{E} = V\dot{\epsilon} + \dot{V}\epsilon = V(\dot{\epsilon} + 3\frac{\dot{a}}{a}\epsilon) \tag{31}$$

Combining (29) (30) and (31) we can say that the expansion or contraction of the universe can be represented as,

$$V(\dot{\epsilon} + 3\frac{\dot{a}}{a}\epsilon + 3\frac{\dot{a}}{a}P) = 0 \tag{32}$$

or,

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \tag{33}$$

The above equation is called the fluid equation and is the second key to describe the expansion of the universe.

Fluid and Acceleration Equations

The Friedmann equation and fluid equation are statements about energy conservation. By combining the two, we can derive an acceleration equation which tells how the expansion of the universe speeds up or slows down with time. The Friedmann equation, multiplied by a^2,becomes,

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\dot{\epsilon}a^2 - \frac{kc^2}{R_0^2} \tag{34}$$

$$2\ddot{a}\dot{a} = \frac{8\pi G}{3c^2}(\dot{\epsilon}a^2 + 2\epsilon a\dot{a}) \tag{35}$$

Dividing (35) by $2\epsilon a\dot{a}$ we get,

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}(\epsilon\frac{a}{\dot{a}} + 2\epsilon) \tag{36}$$

Fluid and Acceleration Equations

Using the fluid equation (33) we get,

$$\dot{\epsilon}\frac{a}{\dot{a}} = -3(\epsilon + P) \tag{37}$$

We get the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \tag{38}$$

A gas made of ordinary baryonic matter has a positive pressure P, resulting from the random thermal motions of the molecules, atoms, or ions of which the gas is made. A gas of photons also has a positive pressure, as does a gas of neutrinos or WIMPs. The positive pressure associated with these components of the universe will cause the expansion to slow down. Suppose, though, that the universe had a component with a pressure

$$P < -\frac{\epsilon}{3} \tag{39}$$

Equations of State

Equations of state

For substances of cosmological importance, the equation of state can be written in a simple linear form,

$$P = \omega\epsilon \tag{40}$$

where ω is a dimensionless number
Consider, a low-density gas of non-relativistic massive particles.
Non-relativistic, in this case, means that the random thermal motions of the gas particles have peculiar velocities which are tiny compared to the speed of light. Such a non-relativistic gas obeys the perfect gas law,

$$P = \frac{\rho}{\mu} kT \tag{41}$$

where μ is the mean mass of the gas particles.

Equations of State

The energy density ϵ of a non-relativistic gas is almost entirely contributed by the mass of the gas particles: $\epsilon = \rho c^2$. Thus, in terms of ϵ, the perfect gas law is

$$P \approx \frac{kT}{\mu c^2}\epsilon \tag{42}$$

For a non-relativistic gas, the temperature T and the root mean square thermal velocity $< \nu^2 >$ are associated by the relation

$$3kT = \mu < \nu^2 > \tag{43}$$

Equations of State

Thus, the equation of state for a non-relativistic gas can be written in the form,

$$P_{non-rel} = \omega \epsilon_{non-rel} \tag{44}$$

where,

$$\omega = \frac{< \nu^2 >}{3c^2} << 1 \tag{45}$$

Most of the gases we encounter in everyday life are non-relativistic. But, a gas of photons, or other massless particles, is guaranteed to be relativistic. Although photons have no mass, they have momentum, and hence exert pressure. The equation of state of photons, or of any other relativistic gas, is

$$P_{rel} = \frac{1}{3}\epsilon_{rel} \tag{46}$$

Equations of State

The equation-of-state parameter ω can't take on arbitrary values. Small perturbations in a substance with pressure P will travel at the speed of sound. For adiabatic perturbations in a gas with pressure P and energy density ϵ, the sound speed is given by the relation,

$$c_s^2 = c^2(\frac{\mathrm{d}P}{\mathrm{d}\epsilon}) \tag{47}$$

In a substance with $\omega > 0$, the sound speed is thus $c_s = \sqrt{\omega}c^2$. Sound waves cannot travel faster than the speed of light; if it did, you would be able to send a sound signal into the past, and violate causality. Thus, w is restricted to values $\omega \leq 1$.

Cosmological Constant

Cosmological Constant

If the mass density of the universe is ρ, then the gravitational potential ϕ is given by Poisson's equation:

$$\nabla^2 \phi = 4\pi G \rho \tag{48}$$

The gravitational acceleration a at any point in space is then found by taking the gradient of the potential

$$\vec{a} = -\nabla \phi \tag{49}$$

In a static universe, $\vec{a}$ must vanish everywhere in space. Thus, the potential ϕ must be constant in space. However, if ϕ is constant, then

$$\rho = \frac{1}{4\pi G} \nabla^2 \phi = 0 \tag{50}$$

Cosmological Constant

Einstein's Modification

$$\nabla^2 \phi + \Lambda = 4\pi G \rho \tag{51}$$

Where Lambda is called the cosmological constant. If the Friedmann equation is re-derived from Einstein's field equation, with the Λ term added, it becomes

$$(\frac{\dot{a}}{a}) = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda}{3} \tag{52}$$

The acceleration equation becomes,

$$(\frac{\ddot{a}}{a}) = \frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{\Lambda}{3} \tag{53}$$

Looking at the Friedmann equation (52), it tells us that adding the Λ term is equivalent to adding a new component to the universe with energy density

$$\epsilon_\Lambda \equiv \frac{c^2}{8\pi G}\Lambda \tag{54}$$

Cosmological Constant

The fluid equation tells us that to have Λ constant with time, the Λ term must have an associated pressure

$$P_{\Lambda} = -\epsilon_{\Lambda} = -\frac{c^2}{8\pi G}\Lambda \tag{55}$$

If $\ddot{a} = 0$, then in a universe with matter density ρ and cosmological constant Λ, the acceleration equation (53) reduces to

$$0 = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3} \tag{56}$$

If $\dot{a} = 0$, The Friedmann equation reduces to

$$0 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3} = 4\pi G\rho - \frac{kc^2}{R_0^2} \tag{57}$$

Cosmological Constant

Einstein's static model therefore had to be positively curved ($k = +1$), with a radius of curvature

$$R_0 = \frac{c}{2(\pi G\rho)^{1/2}} = \frac{c}{\Lambda^{1/2}} \tag{58}$$

Cosmological Constant

The total energy ΔE and the lifetime ΔT of these pairs of virtual particles must satisfy the relation

$$\Delta E \Delta T \leq h \tag{59}$$

The natural value for vacuum density is the Planck energy density

$$\epsilon_{vac} \approx \frac{E_P}{l_p^3} \tag{60}$$

This gives us the energy density,

$$\epsilon_{vac} = 3 \times 10^{133} \; eV \; m^{-3} \tag{61}$$