Variational Calculus

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Abstract:

"In this talk we'll explore the principles of variational calculus, it's special cases and applications. The emphasis will be placed on those aspects that concern classical systems, thus a few proof are omitted."

Overview

1. Introduction
2. Statement of the Problem
3. The Euler-Lagrange Equation
4. Degrees of freedom not present explicitly
5. More general variations
   - Several dependent variables
   - Several independent variables
   - Higher order derivatives
6. Variation with constraints: Constrained Minima/Maxima
7. δ notation
8. Calculus of Variations in Physical Principles
   - Fermat's principle in optics
   - Hamilton's principle in mechanics

Bibliography & Image Credits

Classical Dynamics of Particles and Systems
Bradley W. Carroll and Jerry B. Marion
Addison Wesley Publishing Company

Mathematical Methods for Physics and Engineering
K.F. Riley and Michael P. Hobson
Cambridge University Press

Intro.

Statement of the Problem

Statement of the Problem

$x \to y$ $\quad f \to g$

Functional

$$J = \int_{x_2}^{x_1} f\{\underbrace{y(x), y'(x)}_{\text{Dependent}}; x\} dx \qquad (1)$$

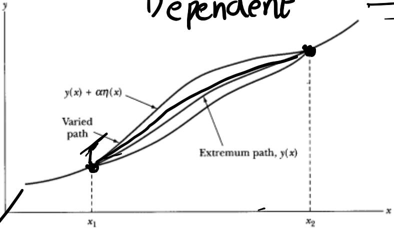
"Independent"

"Physical"

Configuration y

y'

Parameterize



$\alpha' = 0$

$\eta''(x)$

$\eta(x_i) = 0 \quad \forall i \in \{1,2\}$

η_{ext}

$$y(\alpha, x) = y(0, x) + \alpha\eta(x) \qquad (2)$$

Statement of the Problem

∇f

$$J(\alpha) = \int_{x_2}^{x_1} f\{y(\alpha, x), y'(\alpha, x); x\} dx \tag{3}$$

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{4}$$

$\forall\ \eta(x)$

Extrema

The Euler-Lagrange Equation

The Euler-Lagrange Equation

∂

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_2}^{x_1} f\{y(\alpha, x), y^{'}(\alpha, x); x\}dx \tag{5}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_2}^{x_1}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y^{'}}\frac{\partial y^{'}}{\partial \alpha}\right)dx$$

- From Equation (2) we have

$y = y(0) + \alpha\eta(x)$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \; ; \; \frac{\partial y^{'}}{\partial \alpha} = \frac{d\eta}{dx} = \frac{d}{dx}\left(\frac{\partial y}{\partial \alpha}\right) \tag{6}$$

The Euler-Lagrange Equation

General! → Cases/Conditions

$$\frac{\partial J}{\partial \alpha} = \int_{x_2}^{x_1} \left(\frac{\partial f}{\partial y}\eta(x) + \underbrace{\frac{\partial f}{\partial y'}\frac{\partial \eta}{\partial x}} \right) dx$$

$\int u\,dv = uv - \int v\,du$

- The second term in the integrand can be integrated by parts:

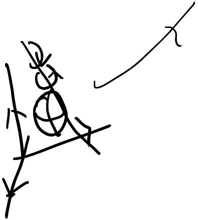
$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d\eta}{dx}dx = \frac{\partial f}{\partial y'}\cancel{\eta(x)}_{0}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \eta(x)dx \quad = 0$$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \qquad (7)$$

✓

Degrees of freedom not present explicitly



$f(y(x), y'(x), x)$

Jargon! → Succint

f does not contain y explicitly

$$0\,\cancel{\frac{\partial f}{\partial y}} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial f}{\partial y'} = constant \tag{8}$$

f does not contain x explicitly

Multiplying the EL Eq. by y',

$$\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = y' \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right) + y'' \frac{\partial f}{\partial y'} \tag{9}$$

$\frac{d}{dx}\left(f\right.$ $\qquad \frac{\partial y}{\partial x}$

$$y' \frac{\partial f}{\cancel{\partial y}} + y'' \frac{\partial f}{\partial y'} = \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) \tag{10}$$

$$\boxed{f - y' \frac{\partial f}{\partial y'} = constant} \tag{11}$$

More general variations

Several Dependent Variables

$i = \{0, 1, 2, 3\}$

Spatial

μ, ν, σ

$x = \begin{bmatrix} x_{11} & x_{22} & x_{3} \end{bmatrix} n$

$\sum_{i}^{n} x_i x^i$

Trace

$$f = f\{y_i(x), y_i'(x); x\} \tag{12}$$

$$y_i(\alpha, x) = y_i(0, x) + \alpha\eta_i(x) \tag{13}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} \right) \eta_i(x) dx \tag{14}$$

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0} \tag{15}$$

Several Independent Variables

$$I = \int \int \int ... \int f\left(y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, ..., \frac{\partial y}{\partial x_n}, x_1, x_2, ..., x_n\right) dx_1 dx_2 ... dx_n \quad (16)$$

$$\boxed{\frac{\partial f}{\partial y} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial y_{x_i}} = 0} \quad (17)$$

$$y_{x_i} = \sum_{i}^{n} \frac{\partial y}{\partial x_i}$$

$$=$$

$$y'$$

Higher order derivatives

n

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \dots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right) = 0 \quad (18)$$

Variation with constraints: Constrained Minima/Maxima

$$f = f\{y_i, y_i^{'}; x\} \tag{19}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} \right) \frac{\partial z}{\partial \alpha} \right] dx \tag{20}$$

$$g\{y_i; x\} = g\{y, z, ; x\} = 0 \tag{21}$$

$$dg = \left(\frac{\partial g}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial \alpha} \right) d\alpha = 0 \tag{22}$$

$$y(\alpha, x) = y(x) + \alpha\eta_1(x) \tag{23}$$

$$z(\alpha, x) = z(x) + \alpha\eta_2(x) \tag{24}$$

Variation with constraints: Constrained Minima/Maxima

$$\frac{\partial g}{\partial y}\eta_1(x) = -\frac{\partial g}{\partial z}\eta_2(x) \tag{25}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} \right) \eta_2(x) \right] dx \tag{26}$$

$$\frac{\eta_2(x)}{\eta_1(x)} = -\frac{\partial g/\partial y}{\partial g/\partial z} \tag{27}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} \right) \frac{\partial g/\partial y}{\partial g/\partial z} \right] \eta_1(x) dx \tag{28}$$

Variation with constraints: Constrained Minima/Maxima

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1} \tag{29}$$

$$\begin{cases} \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} = 0 \\ \frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} = 0 \end{cases} \tag{30}$$

Variation with constraints: Constrained Minima/Maxima

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_j \lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0 \tag{31}$$

$$g_j\{y_i; x\} = 0 \tag{32}$$

This is equivalent to,

$$\sum_i \frac{\partial g_j}{\partial y_i} dy_i = 0 \tag{33}$$

δ notation

We have the equation

$$\frac{\partial J}{\partial \alpha} d\alpha = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} d\alpha dx = 0 \tag{34}$$

It can be written as

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \delta y \; dx = 0 \tag{35}$$

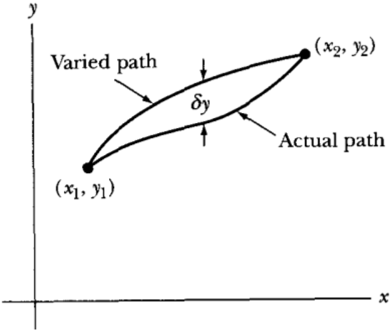
where $\delta \quad f$

$$\begin{cases} \frac{\partial J}{\partial \alpha} d\alpha = \delta J \\ \frac{\partial y}{\partial \alpha} d\alpha = \delta y \end{cases} \tag{36}$$

The condition of extremum then becomes,

$$\delta J = \delta \int_{x_1}^{x_2} f\{y, y^{'}; x\} dx = 0 \tag{37}$$

δ notation



$$\delta J = \int_{x_1}^{x_2} \delta f \, dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx \tag{38}$$

But,

$$\delta y' = \delta \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\delta y) \tag{39}$$

δ notation

So,

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} \delta y \right) dx \tag{40}$$

Integrating the second term by parts as before, we find,

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \delta y \; dx = 0 \tag{41}$$

Calculus of Variations in Physical Principles

Fermat's principle in optics

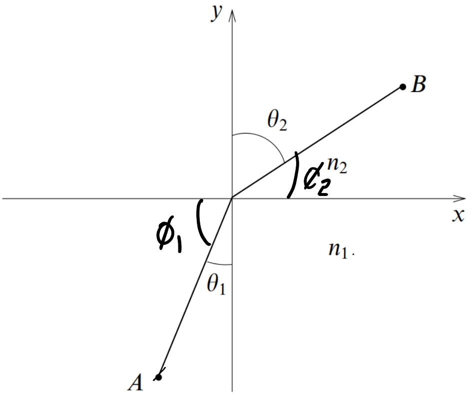
Mimisation

'Least time' → Dimensionally

↳ Hero: Reflection

Refraction

$\eta(y)$ [



$\eta(y)$

Fermat's principle in optics

'No holes'

$$\frac{1}{\sqrt{1+\tan^2\phi}} = \text{Distance} \frac{1}{\sqrt{1+y'^2}}$$

$y' \to$ continuous

$$P = \int_B^A n(y)(1+y'^2)^{-1/2} dx \tag{42}$$

$$n(y)(1+y'^2)^{-1/2} = k \tag{43}$$

$$n\cos\phi = constant \tag{44}$$

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{45}$$

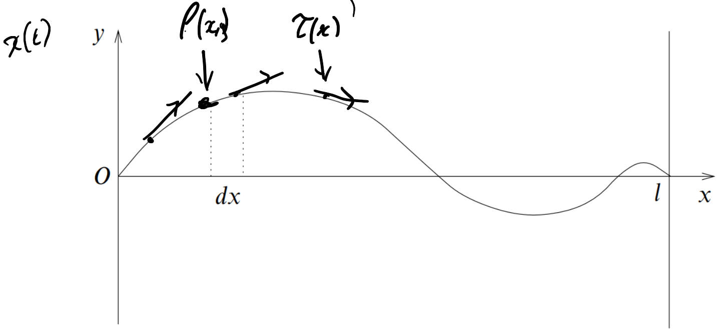
Snell's Law

Hamilton's principle in mechanics

$f = L = T - V = P \cdot E - K \cdot E$ 'scalar'

Physical System

$L[x, \dot{x}, t] = \frac{m\dot{x}^2}{2} - mgx$ $\qquad \mathcal{L} = \delta \int_{t_1}^{t_0} L(q_i, \dot{q}_i, t)\,dt = 0$ (46)



Hamilton's principle in mechanics

$\ddot{x}$

$L = \frac{1}{2} m \dot{x}^2 \quad - \quad g x$

$\delta \int L\,dt = 0$

$$T = \int_0^l \frac{\rho}{2} \left(\frac{\partial y}{\partial t} \right)^2 dx, \quad V = \int_0^l \frac{\tau}{2} \left(\frac{\partial y}{\partial x} \right)^2 dx \tag{47}$$

$$\mathcal{L} = \delta \frac{1}{2} \int_{t_2}^{t_1} dt \int_0^l \left[\rho \left(\frac{\partial y}{\partial t} \right)^2 - \tau \left(\frac{\partial y}{\partial x} \right)^2 \right] dx = 0 \tag{48}$$

$$\frac{\partial}{\partial t} \left(\rho \frac{\partial y}{\partial t} \right) - \frac{\partial}{\partial x} \left(\tau \frac{\partial y}{\partial x} \right) = 0$$

Wave Eq.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \quad // \tag{49}$$