Introduction to Cosmology

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Single Component Universe

Single Component Universe

In a spatially homogeneous and isotropic universe, the relation among the energy density $\epsilon(t)$, the pressure P(t), and the scale factor a(t) is given by the Friedmann equation,

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3c^2}\epsilon_{(t)} - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2} \tag{1}$$

and the fluid equation,

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \tag{2}$$

and the equation of state,

$$P = \omega\epsilon \tag{3}$$

Single Component Universe

Evolution of Energy Density

We know that the universe contains non-relativistic matter and radiation. Thus, the universe contains components with both $\omega = 0$ and $\omega = \frac{1}{3}$. It may well contain a cosmological constant, with $\omega = -1$. Moreover, the possibility exists that it may contain still more exotic components, with different values of ω We may write the total energy density ϵ as the sum of the energy density of the different components as,

$$\epsilon = \sum_{\omega} \epsilon_{\omega} \tag{4}$$

where ϵ_{ω} represents the energy density of the component with equation of state parameter ω

Single Component Universe

The total pressure P is the sum of the pressures of the different components

$$P = \sum_{\omega} P_{\omega} = \sum_{\omega} \omega \epsilon_{\omega} \tag{5}$$

Because the energy densities and pressures add in this way, the fluid equation (2) must hold for each component separately, as long as there is no interaction between the different components. If this is so, then the component with equation-of-state parameter w obeys the equation

$$\dot{\epsilon_{\omega}} + 3\frac{\dot{a}}{a}(\epsilon_{\omega} + P_{\omega}) = 0 \tag{6}$$

$$\dot{\epsilon_{\omega}} + 3\frac{\dot{a}}{a}(1 + \omega)\epsilon_{\omega} = 0 \tag{7}$$

Single Component Universe

(7) can be written as

$$\frac{d\epsilon_\omega}{\epsilon_\omega} = -3(1+\omega)\frac{da}{a} \tag{8}$$

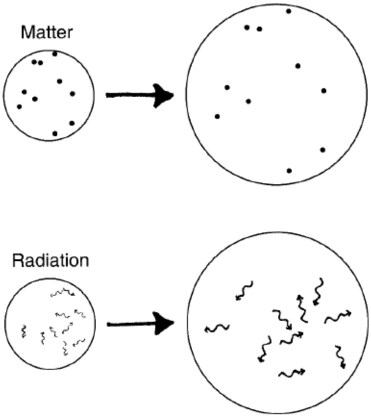
If ω is constant,

$$\epsilon_\omega(a) = \epsilon_{\omega,0}\ a^{-3(1+\omega)} \tag{9}$$

From equation (9), we can conclude that the energy density ϵ_m for non-relativistic matter decreases as the universe expands by,

$$\epsilon_m(a) = \frac{\epsilon_{m,0}}{a^3} \tag{10}$$

Single Component Universe



Single Component Universe

The energy density in radiation, ϵ_r, drops at a more faster rate,

$$\epsilon_r(a) = \frac{\epsilon_{r,0}}{a^4} \tag{11}$$

For both relativistic and non-relativistic particles, the number density has the dependence $n \propto \alpha^{-3}$ as the universe expands, assuming that particles are neither created nor destroyed But, we also know that $E = \frac{hc}{\lambda} \propto \alpha^{-1}$. Hence, in the case of radiation,

$$\epsilon_r \propto \alpha^{-3} \times \ \alpha^{-1} \tag{12}$$

$$\epsilon_r \propto \alpha^{-4} \tag{13}$$

Single Component Universe

The Cosmic Microwave Background, remember, is a relic of the time when the universe was hot and dense enough to be opaque to photons. If we go further back in time, we reach a time when the universe was hot and dense enough to be opaque to neutrinos. As a consequence, there should be a Cosmic Neutrino Background today, similar to the Cosmic Microwave Background. The energy density in neutrinos should be comparable to, but not exactly equal to, the energy density in photons. A detailed calculation indicates that the energy density of each neutrino flavor should be,

$$\epsilon = \frac{7}{8}(\frac{4}{11})^{4/3}\epsilon_{CMB} \approx 0.227\epsilon_{CMB} \tag{14}$$

The density parameter of the Cosmic Neutrino Background, considering all three flavors of neutrino, should then be,

$$\Omega_\nu = 0.681\Omega_{CMB} \tag{15}$$

Single Component Universe

The mean energy per neutrino will be comparable to, but not exactly equal to, the mean energy per photon:

$$E_\nu = \frac{5 \times 10^{-4}}{a} eV \tag{16}$$

If all neutrino species are effectively massless today, with $m_\nu c^2 << 5 \times 10^{-4} eV$ then the present density parameter in radiation is

$$\Omega_{r,0} = \Omega_{CMB,0} + \Omega_{\nu,0} = 5 \times 10^{-4} + 3.4 \times 10^{-5} = 8.4 \times 10^{-5} \tag{17}$$

The Benchmark model has $\Omega_{r,0} = 8.4 \times 10^{-5}$ in radiation $\Omega_{m,0} = 0.3$ in non-relativistic matter and $\Omega_{\Lambda,0} = 1 - \Omega_{r,0} - \Omega_{m,0} \approx 0.7$ in a cosmological constant.

Single Component Universe

In the Benchmark Model, at the present moment, the ratio of the energy density in Λ to the energy density in matter is

$$\frac{\epsilon_{\Lambda}(a)}{\epsilon_m(a)} = \frac{\epsilon_{\Lambda,0}}{\epsilon_{m,0}/a^3} = \frac{\epsilon_{\Lambda,0}}{\epsilon_{m,0}} a^3 \tag{18}$$

The moment of matter-Λ equality occurred when the scale factor was

$$a_{m\Lambda} = (\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}})^{1/3} \approx (\frac{0.3}{0.7})^{1/3} \approx 0.75 \tag{19}$$

the ratio of the energy density in matter to the energy density in radiation is currently

$$\frac{\epsilon_m(a)}{\epsilon_{r,0}(a)} = \frac{\epsilon_{m,0}}{\epsilon_{r,0}} a \tag{20}$$

Single Component Universe

The moment of radiation-matter equality took place when the scale factor was

$$a_{rm} = \frac{\epsilon_{m,0}}{\epsilon_{r,0}} = \frac{1}{3600} = 2.8 \times 10^{-4} \tag{21}$$

In a universe which is continuously expanding, the scale factor a is a monotonically increasing function of t. Thus, in a continuously expanding universe, the scale factor a can be used in the place of cosmic time t. We can refer, for instance, to the moment when $a = 2.8 \times 10^4$ that we are referring to a unique moment in the history of the universe. In addition, because of the simple relation between scale factor and redshift

$$a = \frac{1}{1+z} \tag{22}$$

Single Component Universe

A universe with many components can be written in the form,

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_{\omega} \epsilon_{\omega,0} a^{-1-3\omega} - \frac{kc^2}{R_0^2} \tag{23}$$

Single Component Universe

Curvature Only

For this type of universe, the Friedmann equation takes the form,

$$\dot{a}^2 = -\frac{kc^2}{R_0^2} \tag{24}$$

One solution to this equation has $k = 0$ and $\dot{a} = 0$ An empty, static, spatially flat universe is a permissible solution to the Friedmann equation. This is the universe whose geometry is described by the Minkowski metric, and in which all the transformations of special relativity hold true. Equation (24) tells us that it is also possible to have an empty universe with $k = -1$. A negatively curved empty universe must be expanding or contracting, with

$$\dot{a}^2 = \pm\frac{c}{R_0} \tag{25}$$

Single Component Universe

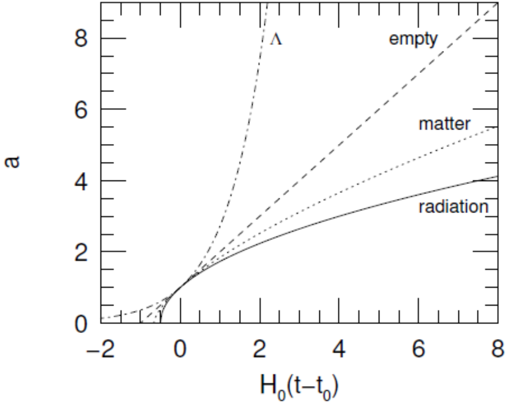
In an expanding empty universe, integration of this relation yields a scale factor of the form

$$a(t) = \frac{t}{t_0} \tag{26}$$

where $t_0 = \frac{R_0}{c}$

In Newtonian terms, if there's no gravitational force, then the relative velocity of any two points is constant, and thus the scale factor a simply increases linearly with time in an empty universe.

Single Component Universe



Single Component Universe

The scale factor in an empty, expanding universe is shown as the dashed line in the graph below. In an empty universe, $t_0 = H^{-1}$ with nothing to speed or slow the expansion, the age of the universe is exactly equal to the Hubble time. If a universe has a density ϵ which is very small compared to the critical density ϵ_c, then the linear scale factor is a good approximation to the true scale factor.

Assume if we were in an expanding universe with negligibly small value for the density parameter Ω, we can approximate it as an empty, negatively curved universe, with $t_0 = H_0^{-1} = \frac{R_0}{c}$. We observe a distant light source, such as a galaxy which has a redshift z. the light which we would observe now, at $t = t_0$, was emitted at an earlier time, $t = t_e$. In an empty expanding universe,

$$1 + z = \frac{1}{a(t_e)} = \frac{t_0}{t_e} \tag{27}$$

Single Component Universe

So it is easy to compute the time when the light you observe from the source was emitted

$$t_e = \frac{t_0}{1+z} = \frac{H_0^{-1}}{1+z} \tag{28}$$

in any universe described by a Robertson-Walker metric, the current proper distance from an observer at the origin to a galaxy at coordinate location (r , θ, ϕ) is,

$$d_p(t_0) = a(t_0)\int_0^r dr = r \tag{29}$$

if light is emitted by the galaxy at time t_e and detected by the observer at time t_0, the null geodesic followed by the light satisfies equation

$$c\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r \tag{30}$$

Single Component Universe

Thus, the current proper distance from the observer to the galaxy (the light source) is

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)} \tag{31}$$

Equation (31) holds true in any universe whose geometry is described by a Robertson-Walker metric. In the specific case of an empty expanding universe, $a(t) = t/t_0$, and thus

$$d_p(t_0) = ct_0\int_{t_e}^{t_0} \frac{dt}{t} = ct_0 \ln\left(\frac{t_0}{t_e}\right) \tag{32}$$

In terms of redshift z,

$$d_p(t_0) = \frac{c}{H_0}\ln(1+z) \tag{33}$$

Single Component Universe

In the limit $z << 1$, there is a linear relation between dp and z, as seen observationally in Hubble's law. In the limit $z >> 1$, however, $dp \propto \ln(z)$ in an empty expanding universe.

In an empty expanding universe, we can see objects which are currently at an arbitrarily large distance. However, at distances $d_p(t_0) >> \frac{c}{H_0}$, the redshift increases exponentially with distance. At first it may seem counterintuitive that you can see a light source at a proper distance much greater than $c = H_0$ when the age of the universe is only $\frac{1}{H_0}$

In an empty expanding universe, the proper distance at the time of emission was,

$$d_p(t_e) = \frac{c}{H_0}\frac{\ln(1+z)}{1+z} \tag{34}$$

Single Component Universe

Spatially Flat Universe

We can also simplify the Friedmann equations by setting $k = 0$ and demand that the universe can contain only a single component, with a single value of ω. In such a spatially flat, single-component universe, the Friedmann equation takes the simple form

$$\dot{a}^2 = \frac{8\pi G \epsilon_0}{3c^2} a^{-(1+3w)} \tag{35}$$

To solve this equation, we first make the assumption that the scale factor has the power law form $a \propto t^q$

$$q = \frac{2}{3 + 3\omega} \tag{36}$$

With the restriction $w \neq -1$, with the proper normalization, the scale factor in a spatially flat, single-component universe is

$$a(t) = (\frac{t}{t_0})^{\frac{2}{3+3\omega}} \tag{37}$$

Single Component Universe

The age of the universe t_0 is linked to the present energy density by the relation

$$t_0 = \frac{1}{1+w}(\frac{c^2}{6\pi G\epsilon_0})^{1/2} \quad (38)$$

The Hubble's constant for such a universe is

$$H_0 \equiv (\frac{\dot{a}}{a})_{t=t_0} = \frac{2}{3(1+\omega)} t_0^{-1} \quad (39)$$

The age of the universe, in the terms of the Hubble time is,

$$t_0 = \frac{2}{3(1+\omega)} H_0^{-1} \quad (40)$$

In a spatially flat universe, if $\omega > -1/3$, the universe is younger than the Hubble time. If $w < -1/3$, the universe is older than the Hubble time.

Single Component Universe

As a function of scale factor, the energy density of a component with equation-of-state parameter ω is

$$\epsilon(a) = \epsilon_0 a^{-3(1+\omega)} \tag{41}$$

So in a spatially flat universe with only a single component, the energy density as a function of time is(combining equations (37) and (41))

$$\epsilon(t) = \epsilon_0 (\frac{t}{t_0})^{-2} \tag{42}$$

Thus we can make the further substitutions,

$$\epsilon_0 = \epsilon_{c,0} = \frac{3c^2}{8\pi G} H_0^2 \tag{43}$$

and

$$t_0 = \frac{2}{3(1+\omega)} H_0^{-1} \tag{44}$$

Single Component Universe

So, equation (42) can be written as,

$$\epsilon_{(t)} = \frac{1}{6\pi(1+\omega)^2}\frac{c^2}{G}t^{-2} \tag{45}$$

Imagine that you are in a spatially flat, single component universe. If we see a galaxy or other distant light source, with a redshift z, we can use the relation

$$1 + z = \frac{a(t_0)}{a(t_e)} = (\frac{t_0}{t_e})^{2/(3+3\omega)} \tag{46}$$

To compute the time t_e at which the light from the distant galaxy was emitted,

$$t_e = \frac{t_0}{(1+z)^{3(1+\omega)/2}} \frac{1}{H_0}\frac{1}{(1+z)^{3(1+\omega)/2}} \tag{47}$$

Single Component Universe

The current proper distance to the galaxy is

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)} = ct_0\frac{3(1+\omega)}{1+3\omega}(1-(\frac{t_e}{t_0})^{(1+3\omega)/(3+3\omega)}) \qquad (48)$$

when $\omega \neq -\frac{1}{3}$.
In terms of H_0 and z rather than t_0 and t_e , the current proper distance is ,

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3\omega}(1-(1+z)^{-(1+3\omega)/2}) \qquad (49)$$

The most distant object that we can see is one for which the light emitted at $t = 0$ is just now reaching us at $t = t_0$. The proper distance (at the time of observation) to such an object is called the "Horizon distance".

Single Component Universe

In a universe governed by the Robertson-Walker metric, the current horizon distance is,

$$d_{hor}(t_0) = c \int_0^{t_0} \frac{dt}{a(t)} \tag{50}$$

In a spatially flat universe, the horizon distance has a finite value if $\omega > -\frac{1}{3}$. In such a case, computing the value of $d_p(t_0)$ in the limit $t_e -> 0$ gives us,

$$d_{hor}(t_0) = ct_0 \frac{3(1+\omega)}{1+3\omega} = \frac{c}{H_0} \frac{2}{1+3\omega} \tag{51}$$

In a flat universe dominated by matter ($\omega = 0$) or by radiation ($\omega = 1/3$), an observer can see only a finite portion of the infinite volume of the universe. The portion of the universe lying within the horizon for a particular observer is referred to as the visible universe for that observer.

Single Component Universe

Matter Only Universe

Let's consider a spatially flat universe containing only non-relativistic matter ($\omega = 0$) The age of such a universe is,

$$t_0 = \frac{2}{3H_0} \tag{52}$$

and the horizon distance,

$$d_{hor}(t_0) = 3ct_0 = 2\frac{c}{H_0} \tag{53}$$

The scale factor, as a function of time, is

$$a_m(t) = (\frac{t}{t_0})^{2/3} \tag{54}$$

Single Component Universe

If we see a galaxy with redshift z in a flat, matter-only universe, the proper distance to that galaxy, at the time of observation is ,

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{(\frac{t}{t_0})} = 3ct_0(1 - (\frac{t_e}{t_0})^{1/3}) = \frac{2c}{H_0}(1 - \frac{1}{\sqrt{1+z}}) \qquad (55)$$

In a flat, matter-only universe, $d_P(t_e)$ has a maximum for galaxies with a redshift $z = \frac{5}{4}$, where $d_p(t_e) = (\frac{8}{27})\frac{c}{H_0}$

Single Component Universe

Radiation Only Universe

In an expanding, flat universe containing only radiation, the age of the universe is,

$$t_0 = \frac{1}{2H_0} \tag{56}$$

and the horizon distance at t_0 is,

$$d_{hor}(t_0) = 2ct_0 = \frac{c}{H_0} \tag{57}$$

The scale factor of the radiation-only universe is,

$$a(t) = (\frac{t}{t_0})^{1/2} \tag{58}$$

Single Component Universe

If, at a time t_0, you observe a distant light source with redshift z in a flat, radiation-only universe, the proper distance to the light source will be

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{(\frac{t}{t_0})^{1/2}} = 2ct_0(1-(\frac{t_e}{t_0})^{1/2}) = \frac{c}{H_0}(1-\frac{1}{1+z}) \tag{59}$$

The proper distance at the time the light was emitted was,

$$d_p(t_e) = \frac{c}{H_0(1+z)}(1-\frac{1}{1+z}) = \frac{c}{H_0}\frac{z}{(1+z)^2} \tag{60}$$

The energy density in a flat, radiation-only universe is ,

$$\epsilon_r = \epsilon_0(\frac{t}{t_0})^{-1/2} \approx 0.094\frac{E_p}{l_p^3}(\frac{t}{t_p})^{-2} \tag{61}$$

Single Component Universe

Thus, in the early stages of our universe, when radiation was strongly dominant, the energy density measured in units of the Planck density ($\frac{E_p}{l_p^3} \approx 3 \times 10^{133} eV\ m^{-3}$), was comparable to one over the square of the cosmic time, measured in units of the Planck time ($t_p \approx 5 \times 10^{44} s$). Using blackbody equation,

$$T(t) \approx 0.61\, T_p(\frac{t}{t_p})^{-1/2} \tag{62}$$

$$E_{mean} \approx 1.66 E_p(\frac{t}{t_p})^{-1/2} \tag{63}$$

Single Component Universe

Lambda Only Universe

Let's consider a case with $\omega = -1$, that is, a universe in which the energy density is contributed by cosmological constant Λ

For such a flat, Lambda-dominated universe, the Friedmann equation takes the form,

$$\dot{a}^2 = \frac{8\pi G \epsilon_\Lambda}{3c^2} a^2 \tag{64}$$

where ϵ_Λ is constant with time. Hence (64) can be written as

$$\dot{a} = H_0 a \tag{65}$$

where ,

$$H_0 = \left(\frac{8\pi G \epsilon_\Lambda}{3c^2}\right)^{1/2} \tag{66}$$

Therefore the solution to eq(65) in an expanding universe is,

$$a(t) = e^{H_0(t-t_0)} \tag{67}$$

Single Component Universe

A spatially flat universe with nothing but a cosmological constant is exponentially expanding. In a Steady State universe, the density ϵ of the universe remains constant because of the continuous creation of real particles. If the cosmological constant Λ is provided by the vacuum energy, then the density ϵ of a lambda-dominated universe remains constant because of the continuous creation and annihilation of virtual particle-antiparticle pairs.

A flat universe containing nothing but a cosmological constant is infinitely old, and has an infinite horizon distance d_{hor}. If, in a flat, lambda-only universe, we see a light source with a redshift z, the proper distance to the light source, at the time we observe it, is

$$d_p(t_0) = c \int_{t_e}^{t_0} e^{H_0(t_0 - t_e) - 1)} = \frac{c}{H_0} z \tag{68}$$

Single Component Universe

The proper distance at the time the light was emitted was,

$$d_p(t_e) = \frac{c}{H_0}\frac{z}{1+z} \tag{69}$$

An exponentially growing universe, such as the flat lambda-dominated model, is the only universe for which $d_p(t_0)$ is linearly proportional to z for all values of z. In other universes that we have seen, $d_p(t_0) \propto z$ only holds true for the limit $z << 1$. In a flat, lambda-dominated universe, highly redshifted objects $(z >> 1)$ are at very large distances $(d_p(t_0) >> \frac{c}{H_0})$ at the time of observation; the observer sees them as they were just before they reached a proper distance $\frac{c}{H_0}$. Once a light source is more than a Hubble distance from the observer, their recession velocity is greater than the speed of light, and photons from the light source can no longer reach the observer