Lie Theory
Lecture 1: Linear Algebra

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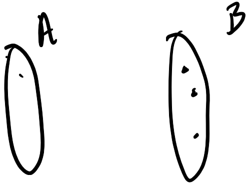
Preliminaries

- If a is an element of the set A, it is denoted by $a \in A$
- **One-to-one maps:** For a map $F : A \to B$, $F(a_1) = F(a_2) \implies a_1 = a_2 \ \forall \ a_1, a_2 \in A$
- **Onto maps:** $\forall \ b \in B$ there exists $a \in A$ such that $F(a) = b$



Vector Spaces

$\{\mathbb{V}, \cdot, +\}$

A linear vector space or simply a vector space $\mathbb{V}$ is a set along with the multiplication (.) and addition (+) operations over a field $\mathcal{F}$, such that the following axioms hold:

$\mathbb{R}$ or $\mathbb{C}$

- **Commutativity:** $|u\rangle + |v\rangle = |v\rangle + |u\rangle$
- **Associativity:** $(|u\rangle + |v\rangle) + |w\rangle = |v\rangle + (|u\rangle + |w\rangle)$
- **Additive Identity:** $\exists \ |0\rangle \in \mathbb{V} \mid \ |v\rangle + |0\rangle = |0\rangle + |v\rangle = |v\rangle$
- **Additive Inverse:** $\forall \ |v\rangle \ \exists \ \left|v^{-1}\right\rangle \mid \ |v\rangle + \left|v^{-1}\right\rangle = 0$
- **Multiplicative identity:** $\exists \ 1 \in \mathbb{V} \mid 1.|v\rangle = |v\rangle$
- **Multiplicative Associativity:** $(\alpha\beta)|v\rangle = \alpha(\beta|v\rangle)$
- **Distributive Properties:**
  - $(\alpha + \beta)|u\rangle = \alpha|u\rangle + \beta|u\rangle$
  - $\alpha(|u\rangle + |v\rangle) = \alpha|u\rangle + \alpha|v\rangle$

$\mathbb{R}^n, \mathbb{C}, M_n(\mathbb{R}, \mathbb{C})$

Here, $\alpha, \beta \in \mathcal{F}$ and $|u\rangle, |v\rangle$ and $|w\rangle \in \mathbb{V}$

= scalars = vector

Span, Bases, Components and Linear Independence

- If $S = \{|v\rangle_1, |v\rangle_1, ..., |v\rangle_k\} \subset \mathbb{V}$ is a set of k vectors in $\mathbb{V}$, then the span of S, denoted $Span\{|v\rangle_1, |v\rangle_2, ..., |v\rangle_k\}$ or Span S, is defined to be just the set of all vectors of the form $\{c^1 |v\rangle_1 + c^2 |v\rangle_2 + ... + c^k |v\rangle_k\}$
- Such vectors are known as linear combinations of the v_i, so Span S is just the set of all linear combinations of the vectors in S
- A basis for a vector space $\mathbb{V}$ is a linearly independent set $\mathcal{B} \subset \mathbb{V}$ whose span is all of $\mathbb{V}$

  $\hat{i}, \hat{j}, \hat{k}$ = Grahm-Schmit
- The dimension of a vector space $\mathbb{V}$, denoted $dim\ \mathbb{V}$, is the number of elements of any finite basis
- If no finite basis exists, then we say that $\mathbb{V}$ is infinite dimensional. x
- Components are simply the scalar coefficients with respect to a specific basis

  $|\psi\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
- Vectors exist independently of any chosen basis

Examples

$$M_n(\mathbb{R}/\mathbb{C}) \to H_n(\mathbb{R}/\mathbb{C}) \qquad H = (H^*)^T$$

$$Y^l_m(\theta) \to H_l(\mathbb{R})$$

$$f = \sum_{m=-\infty}^{\infty} \underline{\underline{c_m}}\, \underline{\underline{e^{\frac{im\pi x}{l}}}}$$

$$L^2([a,b]) \ni \int_a^b |f(x)|^2\, dx < \infty$$

$$L^2(\mathbb{R})$$

Linear Maps

A linear map/transformation is simply transformation L that

- Adds inputs or outputs, $L(|v\rangle + |w\rangle) = L(|v\rangle) + L(|w\rangle)$
- Scale the inputs or outputs, $L(\alpha\,|v\rangle) = \alpha L(|v\rangle)$

Here, $\alpha \in \mathcal{F}$ and $|v\rangle$ and $|w\rangle \in \mathbb{V}$

$\mathcal{L}(\mathbb{V}) := V \to V$ Automorphism

$M_n(\mathbb{R}/\mathbb{C}) \to$ Adjoint rep

$\frac{dB}{dt} = \frac{i}{\hbar}\, ad_H(B) + \frac{\partial B}{\partial t}$ $\leftarrow$ $ad_A(B) = [A, B]$

Dual Spaces

$\langle w| = \alpha_1 e^1 + \alpha_2 e^2$

$|v\rangle = \alpha^1 e_1 + \alpha^2 e_2$

Every vector space $\mathbb{V}$ has a dual space $\mathbb{V}^*$

- $\forall \ |v\rangle \ \exists \ \langle w| := |v\rangle \to \mathbb{R}$ linear map
- Much like a vector space, one can assign the dual space a Basis set e^i upstairs
- It is common to set the basis up in a way that

$$e^i(e_j) = \delta^i_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Orthonormal

Length = 1

Hermitian Forms

$\langle | := V \to \mathcal{F}$

$\langle \cdot | \cdot \rangle := V \times V \to \mathcal{F}$

A non-degenerate Hermitian form on a vector space $\mathbb{V}$ is a function $\langle \cdot | \cdot \rangle$ which assigns to an ordered pair of vectors $|v\rangle, |w\rangle \in \mathbb{V}$ a scalar, denoted $\langle v|w\rangle$, having the following properties: $\mathcal{C}$

- **Linearity for the vectors:** $\langle U| (\alpha |V\rangle + \beta |W\rangle) = \langle U| \alpha |V\rangle + \langle U| \beta |W\rangle = \alpha \langle U|V\rangle + \beta \langle V|W\rangle$
- **Hermicity/Skew-symmetry:** $\langle V|W\rangle = (\langle W|V\rangle)^*$
- **Non-degeneracy:** For each $|v\rangle \neq 0 \in \mathbb{V}$, there exists $w \in \mathbb{V}$ such that $\langle v|w\rangle \neq 0$
- **Positive-Definiteness:** $\langle V|V\rangle > 0$, for all $|V\rangle \neq |0\rangle$

Forms and Dual Spaces

$$\langle v | w \rangle = g(|v\rangle)|w\rangle$$

- There exists a symmetric bilinear function g that maps Vectors to Duals i.e. $g := |v\rangle \rightarrow \langle v|$ metric
- $\langle v|$ is termed the metric dual

Examples

$M_n(\mathbb{R}/\mathbb{C})$: $Tr(M_n)$

Q.M

$\mathbb{V} := \mathcal{H}$ Hilbert

$\|v\| = \sqrt{\underbrace{\langle v | v \rangle}_{\text{Norm}}}$

$\int_a^b |f(x)|^2 dx < \infty$

"complete"

S.R

x^μ : 4-Vecs / Events

$\Delta s = x_\mu x^\mu$

$\eta_{\mu\nu} x^\mu = x_\nu$

Metric $= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

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