Introduction to Cosmology

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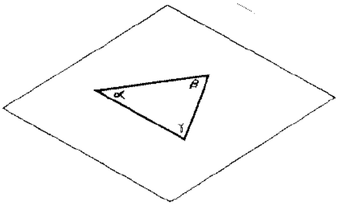
Curvature

Flat space

On a plane, a geodesic is a straight line. If a triangle is constructed on a plane by connecting three points with geodesics,the angles at its vertices will obey the equation,



$$\alpha + \beta + \gamma = \pi$$

The equation for the distance between two points is given by

$$ds^2 = dx^2 + dy^2$$

Curvature

Flat Space

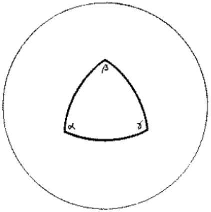
In polar coordinates,

$$ds^2 = dr^2 + r^2 d\theta^2$$

Positive Curved spave

If a triangle is constructed on the surface of the sphere by connecting three points with geodesics, the angles at its vertices, it obeys the equation,

$$\alpha + \beta + \gamma = \pi + \frac{A}{R^2}$$



Curvature

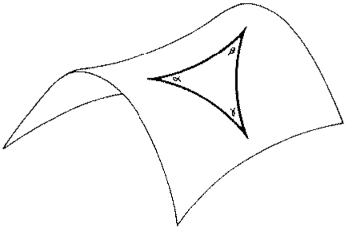
Positive Curved Space

In polar coordinates,

$$ds^2 = dr^2 + R^2 sin^2(\frac{r}{R})d\theta^2$$

This space which satisfies the above condition is called a positively curved space.

Negative Curved space



Negative curved space

If a triangle is constructed on this surface by connecting three points with geodesics, the angles at its vertices, then it obeys the equation,

$$\alpha + \beta + \gamma = \pi - (\frac{A}{R^2})$$

This space which satisfies the above condition is called a negatively curved space. In polar coordinates,

$$ds^2 = dr^2 + R^2 sinh^2(\frac{r}{R})d\theta^2$$

Curvature

Curvature

The curvature of a space is denoted by "K".

- If K $=$ 1, the space is called a positively curved space
- If K $=$ -1, the space is called a negatively curved space
- If K $=$ 0, the space is called a flat or euclidean space.

Curvature

3-D Flat Space

For a flat space in 3 dimensions,

$$ds^2 = dx^2 + dy^2 + dz^2$$

In polar coordinates,

$$ds^2 = dr^2 + R^2 sin^2(\frac{r}{R})[d\theta^2 + sin^2\theta d\phi]$$

Curvature

3-D positively Curved Space

For a positively curved space in 3 dimensions,

$$ds^2 = dr^2 + R^2 sin^2(\frac{r}{R})[d\theta^2 + sin^2\theta d\phi^2]$$

3-D negatively Curved Space

For a negatively curved space in 3 dimensions,

$$ds^2 = dr^2 + R^2 sinh^2(\frac{r}{R})[d\theta^2 + sin^2\theta d\phi^2]$$

Curved Space

The three possible metrics for a homogeneous, isotropic, 3-dimensional space can be written more compactly in the form

$$ds^2 = dr^2 + S_k(r^2) d\Omega^2$$

Curvature

where

$$\Omega \equiv d\theta^2 + sin^2\theta d\phi^2$$

and

$$S_\kappa(r) = \begin{cases} R\sin(r/R) & (\kappa = +1) \\ r & (\kappa = 0) \\ R\sinh(r/R) & (\kappa = -1)\ . \end{cases}$$

In radial coordinates,

$$ds^2 = \frac{dx^2}{1 - \frac{kx^2}{R^2}} + x^2 d\Omega^2$$

Robertson Walker Metric

Robertson Walker Metric

According to the laws of special relativity, the space time separation between two coordinates, (t,r,θ,ϕ) and (dt,r+dr,θ+dθ,ϕ+dϕ) is

$$ds^2 = -c^2dt^2 + dr^2 + r^2 d\Omega^2 \quad (1)$$

This is called as the Minkowski metric.

A photon's path through space-time is a four-dimensional geodesic and not just any geodesic, but a special variety called a null geodesic. A null geodesic is one for which, along every infinitesimal segment of the photon's path, ds = 0. In Minkowski space-time, then, a photon's trajectory obeys the relation

$$ds^2 = 0 = -c^2dt^2 + dr^2 + r^2 d\Omega^2 \quad (2)$$

Robertson Walker Metric

If the photon is moving along a radial path, towards or away from the origin, this means, since θ and ϕ are constant

$$c^2 dt^2 = dr^2 \tag{3}$$

$$\frac{dr}{dt} = \pm c \tag{4}$$

In the 1930's, the physicists Howard Robertson and Arthur Walker asked "What form can the metric of space-time assume if the universe is spatially homogeneous and isotropic at all time, and if distances are allowed to expand(or contract) as a function of time?" The general form of Robertson Walker metric can be written as

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dx^2}{1 - \frac{kx^2}{R_0^2}} + x^2 d\Omega^2\right] \tag{5}$$

Robertson Walker Metric

It can also be written in the form of

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + S_k(r)^2d\Omega^2] \tag{6}$$

Proper Distance

Proper Distance

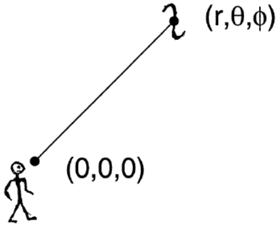
Consider a galaxy which is far away from us, sufficiently far away that we may ignore the small scale perturbations of space-time and adopt the Robertson-Walker metric. One question we can ask is, "Exactly how far away is this galaxy?" In an expanding universe, the distance between two objects is increasing with time. Thus, if we want to assign a spatial distance D between two objects, we must specify the time t at which the distance is the correct one.

$$ds^2 = a(t)^2[dr^2 + S_k(r)^2 d\Omega^2] \tag{7}$$

Along the spatial geodesic between the observer and galaxy, the angle (θ, ϕ) is constant, and thus,

$$ds = a(t)dr \tag{8}$$

Proper Distance



The proper distance dp is found by integrating over the radial co-moving coordinate r,

$$d_p(t) = a(t) \int_0^r dr = a(t) r \tag{9}$$

Proper Distance

The rate of change of proper distance is given by,

$$\dot{d}_p = \dot{a}r = \frac{\dot{a}}{a}d_p \tag{10}$$

Thus, at the current time (t $= t_0$), there is a linear relation between the proper distance to a galaxy and its recession speed,

$$v_p(t_0) \equiv H_0 d_p(t_0) \tag{11}$$

where

$$v_p(t_0) = \dot{d}_p t_0 \tag{12}$$

and

$$H_0 = (\frac{\dot{a}}{a})_{t=t_0} \tag{13}$$

Proper Distance

As the distance between galaxies increases, the radius of curvature of the universe, $R(t) = a(t)R_0$, increases at the same rate

The linear velocity-distance relation given in equation (11) implies that points separated by a proper distance greater than a critical value,

$$d_H(t_0) \equiv \frac{c}{H_0} \tag{14}$$

Which is generally called the Hubble distance, will have

$$v_p = \dot{d}_p > c \tag{15}$$

But we know that $H_0 = 70 \pm 7\ Kms^{-1}Mpc^{-1}$, So the current value of the Hubble distance in our universe is,

$$d_H(t_0) = 4300 \pm 400 Mpc \tag{16}$$

Proper Distance

When we observe a distant galaxy, we know its angular position very well, but not its distance. That is, we can point in its direction, but we don't know its current proper distance $d_p(t_0)$ or, for that matter, its co-moving coordinate distance r. We can, however, measure the redshift 'z' of the light we receive from the galaxy.
Although the redshift doesn't tell us the proper distance to the galaxy, it does tell us what the scale factor a was at the time the light from that galaxy was emitted.
Light that was emitted by the galaxy at a time te is observed by us at a time t_0. During its travel from the distant galaxy to us, the light traveled along a null geodesic, with ds = 0. The null geodesic has θ and ϕ constant Thus, along the light's null geodesic,

$$c^2dt^2 = a(t)^2dr^2 \tag{17}$$

Proper Distance

The wave crest is emitted at a time t_e and observed at a time t_0, such that,

$$c\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_r^0 dr = r \tag{18}$$

The next wave crest of light is emitted at a time '$te + \frac{\lambda_e}{c}$', and is observed at a time '$t_0 + \frac{\lambda_0}{c}$', where, in general, $\lambda_0 \neq \lambda_e$ For the second wave crest,

$$\int_{te+\frac{\lambda_e}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} = \int_0^r dr = r \tag{19}$$

Comparing (18) and (19) we can say,

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{te+\frac{\lambda_e}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} \tag{20}$$

Proper Distance

If we subtract the integral (20) with $\int_{t_e+\frac{\lambda_e}{c}}^{t_0} \frac{dt}{a(t)}$, we get,

$$\int_{t_e}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} \tag{21}$$

This relation becomes still simpler when we realize that during the time between the emission or observation of two wave crests, the universe doesn't have time to expand by a significant amount. The time scale for expansion of the universe is the Hubble time $H_0^{-1} \approx 14 Gyr$, The time between wave crests, for visible light, is $\frac{\lambda}{c} \approx 2 \times 10^{-}15\, s \approx 10^{-32} H_0^{-1}$ Thus, a(t) is effectively constant in the integrals of equation (21). Thus, we may write

$$\frac{1}{a(t_e)} \int_{t_e}^{t_e+\frac{\lambda_e}{c}} dt = \frac{1}{a(t_0)} \int_{t_0}^{t_0+\frac{\lambda_0}{c}} dt \tag{22}$$

Proper Distance

This can be simplified to,

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)} \tag{23}$$

Using the definition of redshift $z = \frac{(\lambda_0 - \lambda_e)}{\lambda_e}$ we find that the redshift of light from a distant object is related to the expansion factor at the time it was emitted to be ,

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} \tag{24}$$

Thus, if we observe a galaxy with a redshift z $= 2$, we are observing it as it was when the universe had a scale factor $a(t_e) = \frac{1}{3}$. The redshift we observe for a distant object doesn't depend on how the transition between $a(t_e)$ and $a(t_0)$ was made. It doesn't matter if the expansion was gradual or abrupt, it doesn't matter if the transition was monotonic or oscillatory. All that matters is the scale factors at the time of emission and the time of observation.