Summary of Quantum Mechanics

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"In this talk we review the basic aspects of quantum mechanics in relation to problems in Quantum Foundations. This by no means exhaustive but rather an overview."

Overview

Motivation

History

History of QM

Black-Body Problem

- At the end of the 19th century, physicists were unable to explain why the observed spectrum of black-body radiation
- By then it had been accurately measured, diverged significantly at higher frequencies from that predicted by existing theories.

History of QM

Planck's Law

$$B_\nu(\lambda, T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1} \qquad (1)$$

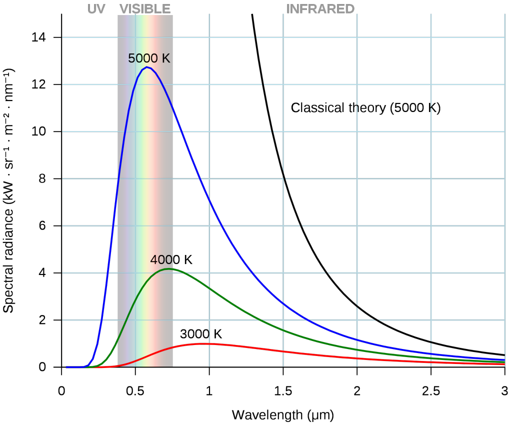


Figure: The Black Body distribution

History of QM

Photoelectric Effect

$$E = hf + V_0 \tag{2}$$

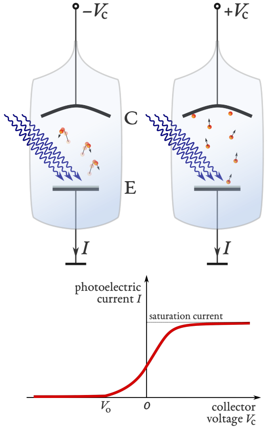


Figure: Newer theoretical constraints are being put on current models

History of QM

de Brogile Hypothesis

$$\lambda = \frac{h}{|p|} \tag{3}$$

History of QM

Davidsson-Germer Experiment

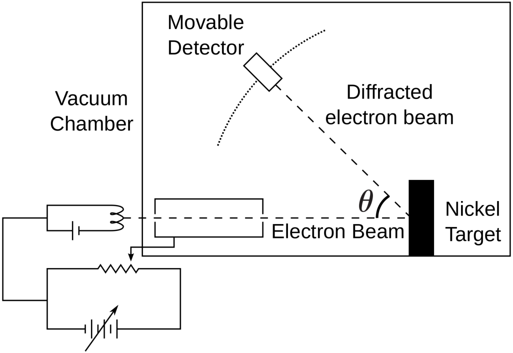


Figure: A depiction of the Davidsson-Germer Experiment

Mathematical Preliminaries

Vector Spaces

Axioms

A linear vector space or simply a vector space $\mathbb{V}$ is a set along with the multiplication (.) and addition $(+)$ operations over $\mathbb{C}$, such that the following axioms hold:

- **Commutativity:** $|U\rangle + |V\rangle = |V\rangle + |U\rangle$
- **Associativity:** $(|U\rangle + |V\rangle) + |W\rangle = |V\rangle + (|U\rangle + |W\rangle)$
- **Additive Identity:** $\exists\ |0\rangle \in \mathbb{V} \mid\ |V\rangle + |0\rangle = |0\rangle + |V\rangle = |V\rangle$
- **Additive Inverse:** $\forall\ |V\rangle\ \exists\ \left|V^{-1}\right\rangle\ \mid\ |V\rangle + \left|V^{-1}\right\rangle = 0$
- **Multiplicative identity:** $\exists\ 1 \in \mathbb{V} \mid 1.|V\rangle = |V\rangle$
- **Distributive properties:** $\alpha(|U\rangle + |V\rangle) = \alpha|U\rangle + \alpha|V\rangle$
- **Additive Identity:**
  - $(\alpha + \beta)|U\rangle = \alpha|U\rangle + \beta|U\rangle$
  - $(\alpha\beta)|V\rangle = \alpha(\beta|V\rangle)$

Here, $\alpha, \beta \in \mathbb{C}$ and $|U\rangle, |V\rangle$ and $|W\rangle \in \mathbb{V}$

Dual Spaces

Axioms

Every vector space $\mathbb{V}$ has a dual space $\mathbb{V}^*$

- $\forall \ |V\rangle \ \exists \ \langle V| := |V\rangle \to \mathbb{R} \ | \ \langle expression1|expression2\rangle$
- There exists a symmetric bilinear function g that maps i.e. $g := |V\rangle \to \langle V|$

Inner Product

Definition

Inner product is a generalization of the dot product that we are familiar with. It is defined as follows, the operation I,

- **Skew-symmetry:** $\langle V|W\rangle = \langle W|V\rangle^*$
- **Positive semidefiniteness:** $\langle V|V\rangle \leq 0$, if and only if $|V\rangle = |0\rangle$
- **Linearity for the vectors:**
  $\langle U| (\alpha |V\rangle + \beta |W\rangle) = \langle U| \alpha |V\rangle + \langle U| \beta |W\rangle = \alpha \langle U|V\rangle + \beta \langle V|W\rangle$

Where, $\alpha \in \mathbb{C}$,$|U\rangle, |V\rangle, |W\rangle \in \mathbb{V}$ and $|U\rangle, ketV, ketW \in \mathbb{V}^*$

Linear Maps

Definition

A linear map/transformation is simply transformation $\hat{O}$ that

- Adds inputs or outputs, $\mathbb{L}(\vec{V} + \vec{W}) = \mathbb{L}(\vec{V}) + \mathbb{L}(\vec{W})$
- Scale the inputs or outputs, $\mathbb{L}(\alpha\vec{V}) = \alpha\mathbb{L}(\vec{V})$

Here, $\alpha, \beta \in \mathbb{C}$ and $|U\rangle, |V\rangle$ and $|W\rangle \in \mathbb{V}$

Sums and Products

Cartesian Product

here the symbol $\times$ means **"Cartesian Product"** i.e. it's action w.r.t two sets is the set of all ordered pairs (a, b) where $a \in \mathbb{A}$ and $b \in \mathbb{B}$

Direct Sum

content...

Tensor Product

The tensor product $v \otimes w \ \forall \ v \in \mathbb{V}, w \in \mathbb{W}$ is an element of the set $\mathbb{V}^* \times \mathbb{W}^*$ i.e. the set of all Bilinear functions on which act on the pair $(h, g) \in \mathbb{V} \times \mathbb{W}$

Hilbert Spaces

A Hilbert space is a vector space $\mathcal{H}$ that

- has an norm defined as $|V| = \sqrt{\langle V|V\rangle}$
- is complete i.e. all of it's Cauchy sequences converge

Postulates of Quantum Mechanics

Postulate # 1

The State Vector

- In Quantum Mechanics, we start with an object called the state vector $|\psi\rangle \in \mathcal{H}$
- All the information about the system is contained in it
- The position basis representation of the state vector is called the wavefunction $\psi(\vec{x}, t) = \langle x|\psi\rangle$.

The Wavefunction

Admissibility Conditions for a Wavefunction

A physically relevant wavefunction must be:

- Continuous i.e. no singularities in it's topology
- Smooth i.e. a Taylor expansion for it exists
- Quadratically integrable with the integral being single valued i.e. finite everywhere and $\psi \rightarrow 0$ as $r \rightarrow \infty$
- Forming an orthonormal set
- Satisfying the boundary conditions of the quantum mechanical system it represents

Postulate # 2

Observables

- Variables are promoted to Linear maps
- The maps are Hermitian/Self-Adjoint

$$\hat{O} = \hat{O}^{\dagger} = \left(\hat{O}^{*}\right)^{T} \tag{4}$$

- The observable values are the eigenvalues from the eigenvalue equation of the map acting on a state vector

$$\hat{O}\left|\psi\right\rangle = \lambda\left|\psi\right\rangle \tag{5}$$

Observables: A few Corollaries

- The eigenvalues are real numbers
- The eigenstates form an orthonormal basis set

Postulate # 4

Time Evolution

- Time is Unitary
- That is to say if my time evolution was given by an operator $\hat{U}$, then

$$\hat{U}\hat{U}^{\dagger} = \left(\hat{U}^{*}\right)^{T} = \mathbb{I} \tag{6}$$

Time Evolution: The Different Pictures

The Schrödinger Picture

- State vectors evolve in time
- Operators don't

$$i\hbar\frac{\partial |\psi\rangle}{\partial t} = \hat{H}|\psi\rangle \tag{7}$$

The Heisenberg Picture

- Operators vectors evolve in time
- State vectors don't

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{O}, \hat{H}] + \frac{\partial \hat{O}}{\partial t} \tag{8}$$

Postulate # 4

Measurement

- Measurement is defined as a form of time-evolution that is non-unitary and non-deterministic
- It is irreversible
- The state is "transformed" to the eingenstate: $|\psi\rangle \rightarrow |\lambda\rangle$
- Born's rule: $P(\lambda) \propto \langle\lambda|\psi\rangle$

Summary of Postulates

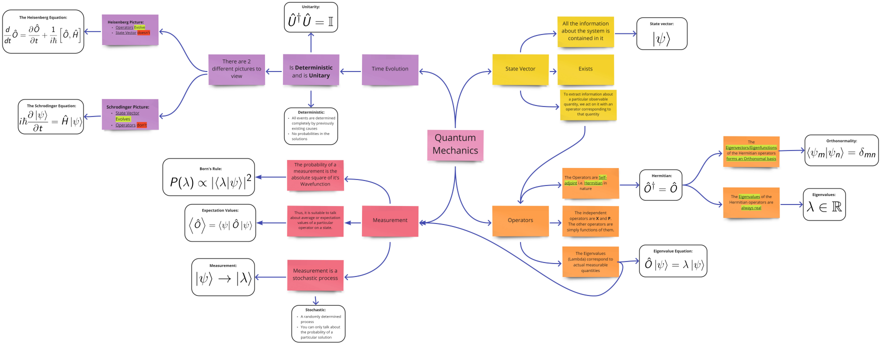


Figure: A Summary of all Postulates

Corollaries

- **The Continuity Equation:** $\frac{\partial \rho}{\partial t} = -\nabla . \vec{J}$
- **The Uncertainty Relation:** $\Delta \hat{A} \Delta \hat{B} \geq \frac{1}{2i} |\langle [\hat{A}, \hat{B}] \rangle|$

Spin

What is Spin?

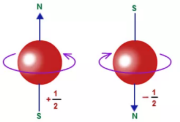
A particle can possess two kinds of angular momentum,

- Extrinsic angular momentum
- Intrinsic angular momentum

Spin is the property of a particle attributed to its intrinsic angular momentum.

A particle can possess a spin pair along a specific axis.

- up-down
- right-left
- in-out



Spin

Discovery of Spin

Atomic spectra measure radiation absorbed or emitted by electrons jumping from one state to another, where a state is represented by values of 'n', 'l', and 'm'.

In general, excitation or "transition" is allowed only if all three numbers change in the process. This is because a transition will be able to cause the emission or absorption of electromagnetic radiation only if it involves a change in the electromagnetic dipole of the atom.

But, later on, it was seen that the atomic spectra measured in an external magnetic field cannot be predicted with just 'n', 'l', and 'm'.

Pauli proposed a new quantum degree of freedom (or quantum number) with two possible values, in order to resolve inconsistencies between observed molecular spectra and the developing theory of quantum mechanics.

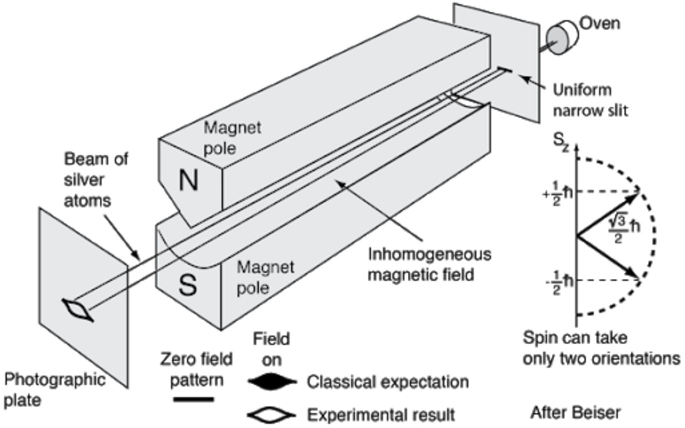
George Uhlenbeck and Samuel Goudsmit identified Pauli's new degree of freedom as electron spin.

Spin

Experimental evidence for Spin

Otto Stern and Walter Gerlach experimentally determined the spatial quantization of the spin moment of the momentum of electrons of atoms situated in the magnetic field in 1920.
The original experiment took the form of a collimated beam of silver atoms heading in, say, the y direction, and passing through a non-uniform magnetic field directed in the z-direction.

Spin



Oven
Uniform narrow slit
Magnet pole
N
Beam of silver atoms
Inhomogeneous magnetic field
S
Magnet pole
S_z
$+\frac{1}{2}\hbar$
$\frac{\sqrt{3}}{2}\hbar$
$-\frac{1}{2}\hbar$
Spin can take only two orientations
Photographic plate
Zero field pattern
Field on
Classical expectation
Experimental result
After Beiser

Spin

Assuming the silver atoms possess a non-zero magnetic moment μ, the magnetic field will have two effects. First, the magnetic field will exert a torque on the magnetic dipole, so that the magnetic moment vector will precess about the direction of the magnetic field. Secondly, and more importantly here, the non-uniformity of the field means that the atoms experience a sideways force given by

$$F_z = -\frac{\mathrm{d}U}{\mathrm{d}z} \tag{9}$$

$$U = -\mu . B \tag{10}$$

$$F_z = -\mu_z \frac{\mathrm{d}B}{\mathrm{d}z} \tag{11}$$

$$\mu_z = \pm \mu_B = \frac{e\hbar}{2m_e} \tag{12}$$

$$S_x = \pm \frac{\hbar}{2} = S_y = S_z \tag{13}$$

Spin

Spin matrices

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{14}$$

$$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{15}$$

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{16}$$

Spin

Pauli Spin Matrices

$$S_x = \frac{\hbar}{2}\sigma_x, \;\; S_y = \frac{\hbar}{2}\sigma_y, \;\; S_z = \frac{\hbar}{z}\sigma_z \tag{17}$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{18}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{19}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{20}$$

$$[\sigma_x, \sigma_y] = 2i\sigma_z \tag{21}$$

$$[\sigma_y, \sigma_z] = 2i\sigma_x \tag{22}$$

$$[\sigma_z, \sigma_x] = 2i\sigma_y \tag{23}$$

Entanglement

Tensor Product

The tensor product $A \otimes B$ of two vector spaces A and B (over the same field) is a vector space, endowed with a bilinear map from the Cartesian product $A \times B$ to $A \otimes B$.

$$S_{AB} = S_A \otimes S_B \tag{24}$$

Say system 'A' is based on rolling a dice and 'B' based on tossing a coin. The operator σ acts on the system and gives us the output.

	1	2	3	4	5	6
H	***H1***	***H2***	***H3***	***H4***	***H5***	***H6***
T	***T1***	***T2***	***T3***	***T4***	***T5***	***T6***

Entanglement

Combined Spaces

The tensor product space in the above case is twelve-dimensional. A superposition of two of these basis vectors might look like,

$$|H\rangle \otimes |4\rangle \text{ or } |H4\rangle \tag{25}$$

$$\alpha_{h3} |H3\rangle + \alpha_{t4} |T4\rangle \tag{26}$$

In each case, the first half of the state-label describes the state of the coin, and the second half describes the state of the die.

Entanglement

Consider another set of systems, a penny and a dime represented by 'A' and 'B'.
Let σ be the operator acting on the system to give the an output.
It was found that

$$< \sigma_A >= 0 \ , < \sigma_B >= 0 \tag{27}$$

But,

$$< \sigma_A \, \sigma_B >= -1 \tag{28}$$

$$< \sigma_A \, \sigma_B > \neq < \sigma_A >< \sigma_B > \tag{29}$$

This is called statistical correlation.

Entanglement

Two Spins

Let the spin of system 'A' be σ and the spin of system 'B' be τ
Our bases for these systems are $|uu\rangle$, $|ud\rangle$, $|du\rangle$, $|dd\rangle$

$$\sigma\,|uu\rangle = |uu\rangle\,,\;\; \tau\,|uu\rangle = |uu\rangle \tag{30}$$

$$\sigma\,|ud\rangle = |ud\rangle\,,\;\; \tau\,|ud\rangle = -\,|ud\rangle \tag{31}$$

$$\sigma\,|du\rangle = -\,|du\rangle\,,\;\; \tau\,|du\rangle = |du\rangle \tag{32}$$

$$\sigma\,|dd\rangle = -\,|dd\rangle\,,\;\; \tau\,|dd\rangle = -\,|dd\rangle \tag{33}$$

Entanglement

Product States

Product states of those two systems is given by,

$$|ProductState\rangle = \alpha_u |u\rangle + |\alpha_d |d\rangle\rangle \otimes \{\beta_u |u\rangle + |\beta_d |d\rangle\rangle \tag{34}$$

$$\alpha_u \beta_u |uu\rangle + |\alpha_d \beta_u |du\rangle\rangle + \beta_u \alpha_{du} |d\rangle + |\beta_d \alpha_u |du\rangle\rangle \tag{35}$$

The maximal entangled state or the singlet state can be written as

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \tag{36}$$

The singlet cannot be written as a product state.

Entanglement

The other maximally entangled states called the triplet are

$$\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \tag{37}$$

$$\frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \tag{38}$$

$$\frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \tag{39}$$

The spin polarization principle states that there is some direction at which the spin is '+1'. So ,

$$< \sigma_x >^2 + < \sigma_y >^2 + < \sigma_z >^2 = 1 \tag{40}$$

which tells us that not all the expectation values can be zero

Entanglement

This fact continues to hold for all product states. However, it does not hold for the entangled state $|sing\rangle$. In fact the RHS for the $|sing\rangle$ state becomes 0.

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \tag{41}$$

$$< \sigma_z >= \langle sing| \, \sigma_z \, |sing\rangle = \langle sing| \, \frac{1}{\sqrt{2}} \sigma_z(|ud\rangle - |du\rangle) \tag{42}$$

$$\langle sing| \, \sigma_z \, |sing\rangle = \langle sing| \, \frac{1}{\sqrt{2}} \sigma_z(|ud\rangle + |du\rangle) \tag{43}$$

$$< \sigma_z >= \frac{1}{2}(\langle ud| - \langle du|)(|ud\rangle + |du\rangle) = 0 \tag{44}$$

Similarly for others, it results to 0.

$$< \sigma_x >=< \sigma_y >=< \sigma_z >= 0 \tag{45}$$

The Trouble with Quantum Mechanics

- **Ontology:** Does the Wavefunction exist or is it simply a calculative device?
- **Measurement:** Why is measurement distinct from time evolution? Why is it stochastic?
- **Locality:** Why do non-local effects arise in Quantum Mechanics? Are they artifacts of our ignorance or are they real?

What's going on now?

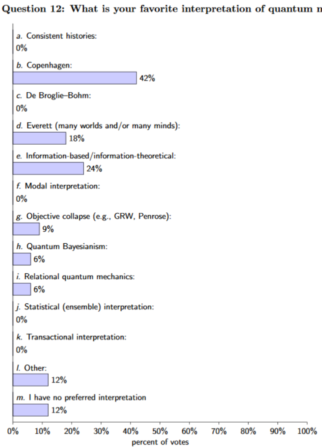


Figure: From a survey: arXiv:1301.1069 [quant-ph]

What's going on now?

A strong no-go theorem on the Wigner's friend paradox

Kok-Wei Bong, Aníbal Utreras-Alarcón, Farzad Ghafari, Yeong-Cherng Liang, Nora Tischler, Eric G. Cavalcanti, Geoff J. Pryde & Howard M. Wiseman

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Abstract

Does quantum theory apply at all scales, including that of observers? New light on this fundamental question has recently been shed through a resurgence of interest in the long-standing Wigner's friend paradox. This is a thought experiment addressing the quantum measurement problem—the difficulty of reconciling the (unitary, deterministic) evolution of isolated systems and the (non-unitary, probabilistic) state update after a measurement. Here, by building on a scenario with two separated but entangled friends introduced by Brukner, we prove that if quantum evolution is controllable on the scale of an observer, then one of 'No-Superdeterminism', 'Locality' or 'Absoluteness of Observed Events'—that every observed event exists absolutely, not relatively—must be false. We show that although the violation of Bell-type inequalities in such scenarios is not in general sufficient to demonstrate the contradiction between those three assumptions, new inequalities can be derived, in a theory-independent manner, that are violated by quantum correlations. This is demonstrated in a proof-of-principle experiment where a photon's path is deemed an observer. We discuss how this new theorem places strictly stronger constraints on physical reality than Bell's theorem.

Figure: Newer theoretical constraints are being put on current models

Addendum

"The aspiration to truth is more precious than its assured possession" - Gotthold Lessing