WELCOME

TOPICS DISCUSSED :-

- Central forces.
- Angular momentum in Central force motion.
- Energy of system in central force motion.
- Differential equation of Orbit.
- Explanation of Kepler's laws
- Derivation of the Kepler's laws by using Newton's formulation.

**<u>CENTRAL FORCE</u> :-

A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the centre, and whose magnitude only depends on the distance of the object from the centre.

In central force, the potential V is only a function of “r”. A central force is always a conservative force; the magnitude F of a central force can always be expressed as the derivative of a time- independent potential energy.

$$\bar{\nabla} \times \bar{F} = \frac{1}{r sin\theta}\left(\frac{\partial F}{\partial \phi}\right)\hat{\theta} - \left(\frac{1}{r}\right)\left(\frac{\partial F}{\partial \theta}\right)\bar{\hat{\phi}} = 0$$ (is spherical coordinates)

And the force is defined as $F = -\frac{\partial V}{\partial r}\ \hat{r}$ (force is only in the radial direction).

Angular Momentum :-

The equation of motion in polar coordinate is given by $m(\ddot{r} - r\dot{\theta}^2) = F_r \; and \; m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \; F_\theta$.

but for central force.

$$\bar{\tau} = \bar{r} * \bar{\boldsymbol{F}} \qquad \Rightarrow \bar{\tau} = \mathrm{r}\bar{\mathrm{r}} * -\frac{\delta V}{\delta r} r^{-}$$

External torque $\tau = 0$ so angular momentum is conserved.

but for central force, $F_\theta = m(r\ddot{\theta} - 2\dot{r}\dot{\theta}) = 0 \Rightarrow \frac{\mathrm{d}(mr^2\dot{\theta})}{dt} = 0$ means

Angular momentum : $J = |\vec{r} \times \vec{p}| = \mathrm{m}r^2\dot{\theta} \qquad \dot{\theta} = \frac{J}{mr^2}$ (to be used later)

$\bar{r}.\bar{J} = \bar{r}.(\bar{r} \times \bar{p}) = 0 \qquad \Rightarrow \quad \bar{r} \perp \bar{J}$ hence the position vector $\bar{r}$ is perpendicular to angular momentum vector $\bar{J}$.

** **Total Energy of the System** :-

$E = \frac{1}{2}mv^2 + V(r)$ and Velocity $\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

So the total energy $E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$

Substituting $\dot{\theta} = \frac{J}{mr^2}$ *we get :-*

$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) \quad r > 0 \qquad E = \frac{1}{2}m\dot{r}^2 + V_{eff}$

Where : $v_{eff} = \frac{J^2}{2mr^2} + V(r)$ *is identified as the effective potential* $V_{effective}$

Equation of motion and differential Equation of Orbit :-

From equation of motion in radial part : $m(\ddot{r} - r\dot{\theta}^2) = f(r)$

$$\Rightarrow m\frac{d^2r}{dt^2} - \frac{J^2}{mr^3} = f(r) \quad ------ \; 1$$

Where $J = mr^2\dot{\theta} \Rightarrow \mathrm{d}\theta = \frac{J}{mr^2}dt \Rightarrow \frac{\mathrm{d}}{\mathrm{dt}} = \frac{J}{mr^2}\frac{d}{\mathrm{d}\theta}$

Substitute in equation (1)

$$\frac{J^2}{m}\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{J^2}{mr^3} = f(r) \Rightarrow \frac{J^2}{m}\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{d\left(-\frac{1}{r}\right)}{d\theta}\right) - \frac{J^2}{mr^3} = f(r)$$

$$\Rightarrow \; -\left(\frac{J^2}{m}\frac{1}{r^2}\frac{d^2\left(\frac{1}{r}\right)}{d\theta^2} + \frac{J^2}{mr^3}\right) = f(r) \Rightarrow -\frac{J^2}{mr^2}\left(\frac{d^2\left(\frac{1}{r}\right)}{d\theta^2} + \frac{1}{r}\right) = f(r)$$

Let $\frac{1}{r} = u \Rightarrow -\frac{J^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right)$ (differential equation of orbit) [to be used later]

Copernicus, in 1543 proposed that the way to think about our solar system is to put the sun at the centre.

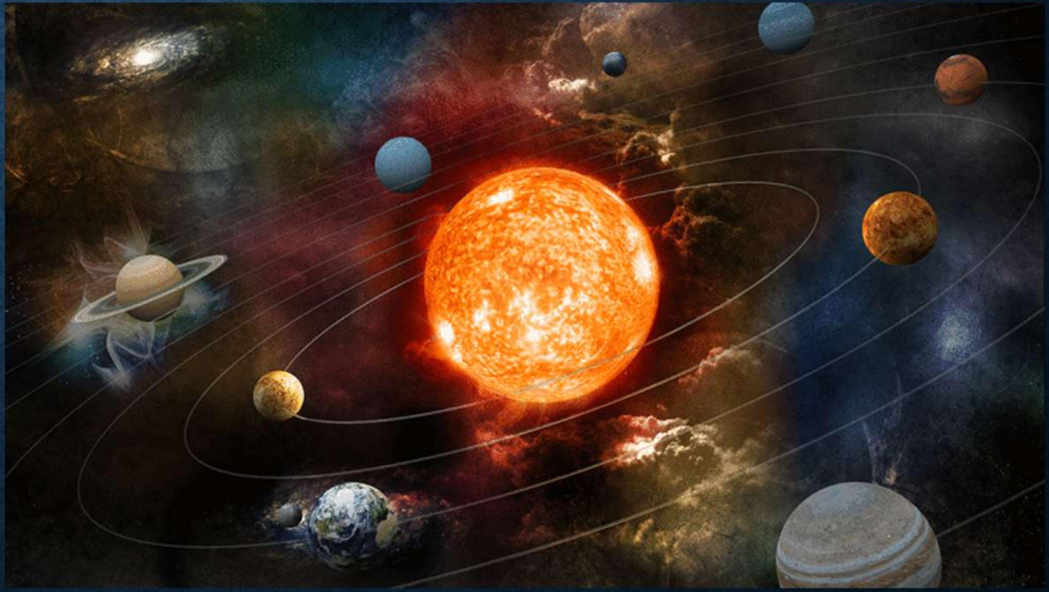
This is a very remarkable incident (considering the social risks included).

Johannes Kepler :

- German astronomer, mathematician, astrologer (YES !!)
- born on 27th of December 1571
- assistant to Tycho Brahe (a "RICH GUY" - had his own lab)
- carried out his research and produced 3 laws in 40 long years

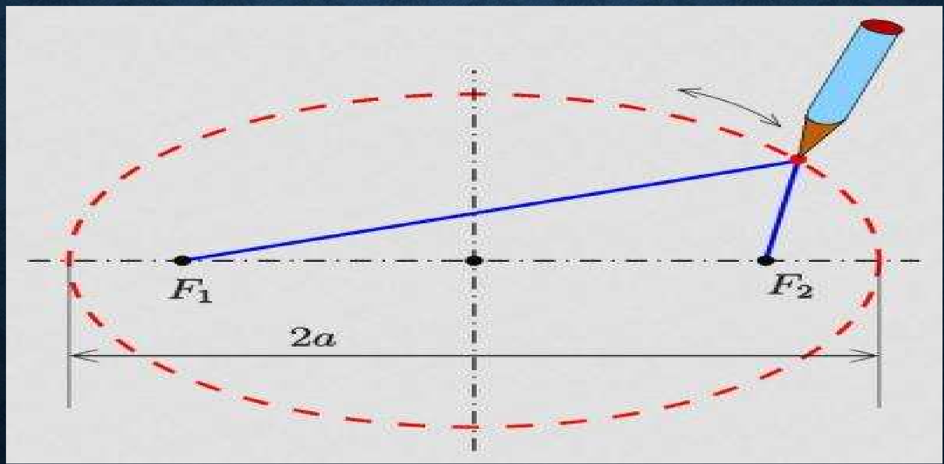
Kepler's Problem :-

Kepler discussed the orbital motion of the sun and Earth system under the potential **V (r) = -** $\frac{k}{r}$; where **k =** $\boldsymbol{Gm_s m_e}$ it is given m_e and m_s is mass of earth and Sun. Although Kepler discussed the sun and earth system but this method can be used for any system which is interacting with potential **V**$= -\frac{k}{r}$ **.**



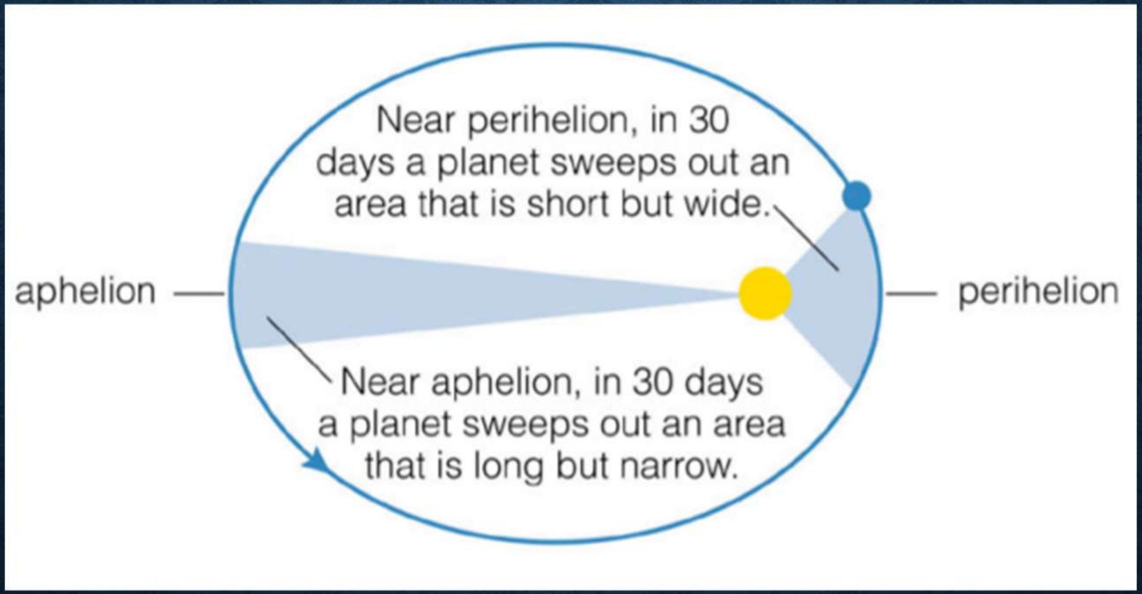
KEPLER'S FIRST LAW (THE LAW OF ORBITS) :

- Statement : "all planets revolved around the sun in elliptical orbits."
- This modified the idea of Copernicus' heliocentric theory which stated that "all planets went around the sun in circular orbits" by restating that the orbits of the planets were actually elliptical.



KEPLER'S SECOND LAW (THE LAW OF AREAS) :

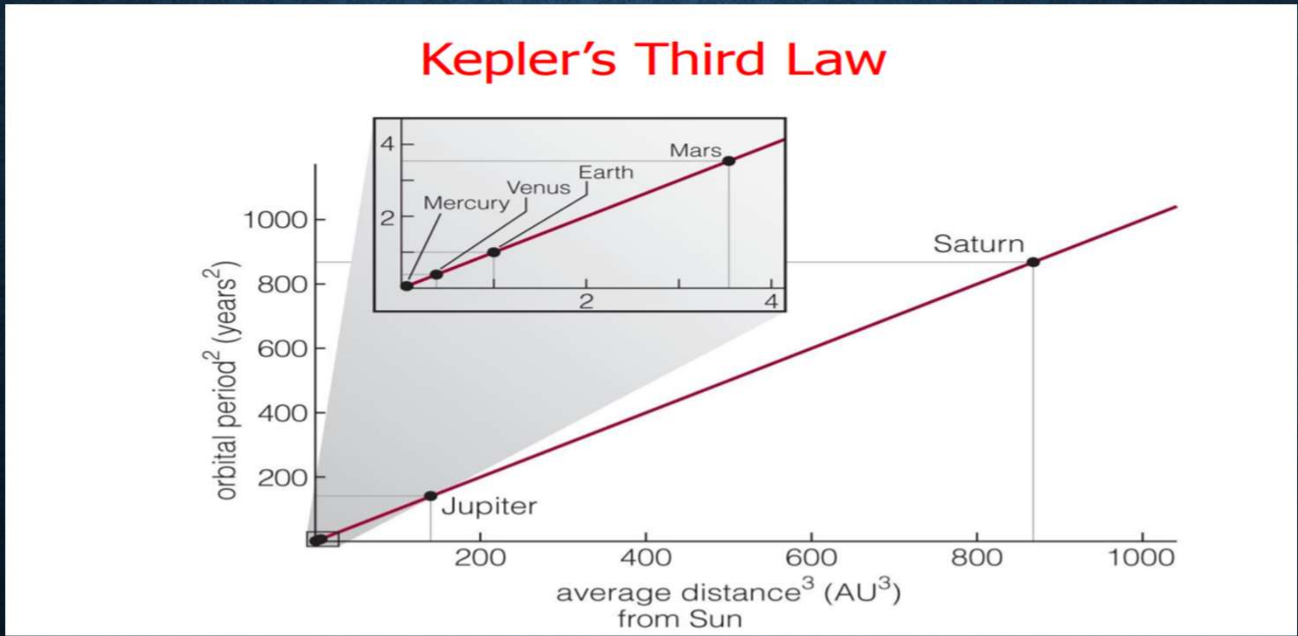
- Statement : " The line joining the sun and the planet sweeps equal areas in equal amounts of time. "



The area swept out in 30-day period of time are all equal.

KEPLER'S THIRD LAW (THE LAW OF PERIODS) :

- Statement : " The square of the time period of revolution of a planet around the sun is directly proportional to the cube of the semi-major axis of the orbit. "



DERIVATIONS

<u>Kepler's first Law</u> :-

From Newton's first law : $F = ma \Rightarrow F = m(\ddot{r} - r\theta^2)$

And the gravitational force is given by : $F = -\frac{GmM}{r^2}$ or $F = -\frac{k}{r^2}$

Equation of motion

$m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2}$ put $\dot{\theta} = \frac{J}{mr^2}$

$$m\ddot{r} - \frac{J^2}{mr^3} = -\frac{k}{r^2}$$

Equation of orbit is : $\frac{J^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -f\left(\frac{1}{u}\right)$

$$f(r) = -\frac{k}{r^2} \;\; ; \;\; f\left(\frac{1}{u}\right) = -ku^2 \Rightarrow \frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = +ku^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{ku^2n}{J^2u^2}$$

$$\frac{d^2u}{d\theta^2} + \left[u - \frac{km}{J^2}\right] = 0 \quad put \;\; u - \frac{km}{J^2} = y \quad so \quad \frac{d^2u}{d\theta^2} = \frac{d^2y}{d\theta^2}$$

Hence the equation reduces to :

$$\frac{d^2y}{d\theta^2} + y = 0$$

The solution of the equation reduces to $y = A\cos\theta = u - \frac{km}{J2} = A\cos\theta$

$$\Rightarrow \;\; u = \frac{km}{J^2} + A\cos\theta$$

$$\frac{1}{r} = \frac{km}{J^2} + A\cos\theta \Rightarrow \frac{\left(\frac{J^2}{km}\right)}{r} = 1 + \left[\frac{AJ^2}{km}\right]\cos\theta$$

Put $\frac{J^2}{km} = l$ and $e = \frac{AJ^2}{km}$ the equation reduces to $\frac{1}{l} = 1 + e\cos\theta$

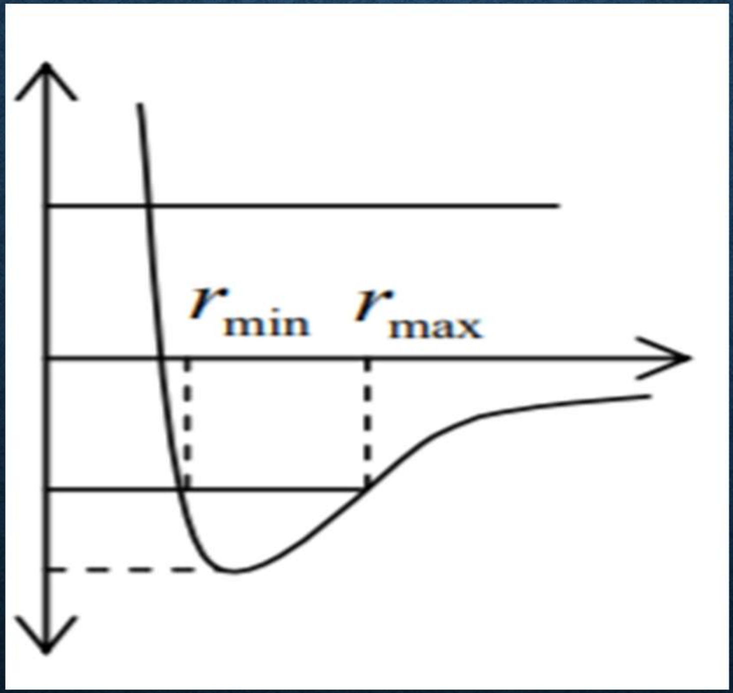
which is the equation of conics where l is the latus rectum and e is the eccentricity.

Now, we shall see the case specifically of elliptical orbit as Kepler discuss for planetary motion.

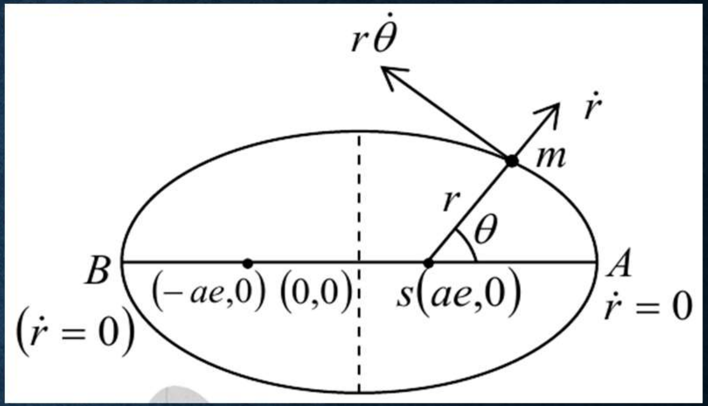
Total Energy : $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2m\ ^2} - \frac{k}{r}$ where $V_{eff} = \frac{J^2}{2m\ ^2} - \frac{k}{r}$

With constant angular momentum J.

If one will plot V_{eff} vs r it is clear that for negative energy the orbit is elliptical which is shown below.



Earth is orbiting in elliptical path with sun as focus as in figure.



Let the equation of ellipse is : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b=a\sqrt{1-e^2}$

minimum and minimum values of ‘r’ are (a-ae) and (a+ae)

$$r_{\max}+r_{\min}=2a$$

from the plot of effective potential it is identified that $r_{\max}$ *and* $r_{\min}$ are the turning points. So at these points the radial velocity will be “0”.

$$E=\frac{J^2}{2mr^2}-\frac{k}{r}\Rightarrow 2mEr^2+2mkr-J^2=0$$

This is a quadratic equation. Solving this we get $r_{\max}+r_{\min}=-\frac{2mk}{2mE}$

$$\Rightarrow E=-\frac{k}{2a}$$

Relation between the Energy and the Eccentricity :

The energy is given by $E = \frac{1}{2}m\dot{r}^2 + \left(\frac{J^2}{2mr^2} - \frac{k}{r}\right)$

For central potential $V(r) = -\frac{k}{r}$ the solution of orbit is :

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{where } l = \frac{J^2}{km}$$

$\frac{-l}{r^2}\dot{r} = -e\sin\theta\,\dot{\theta}$ where $\dot{\theta} = \frac{J}{mr^2}$ so $\dot{r} = \frac{eJ\sin\theta}{ml}$

After putting the value of $\frac{l}{r} = 1 + e\cos\theta$ and $\dot{r} = \frac{eJ\sin\theta}{ml}$

We will get : $e = \sqrt{1 + \frac{2EJ^2}{mk^2}}$

The condition on energy for possible nature of orbit for potential

$E > 0$	$e > 1$	Hyperbola
$E = 0$	$e = 1$	Parabola
$E < 0$	$e < 1$	Ellipse

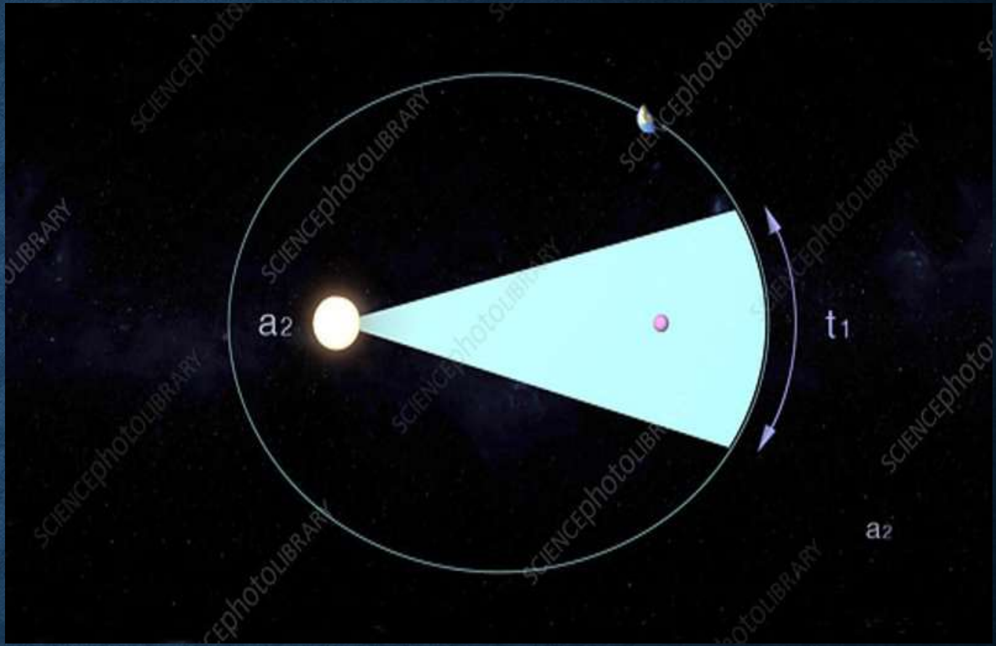
$E = -\frac{mk^2}{2J^2}$	$e = 0$	circle

Kepler's second law of Motion :-

Now, the area of this areal part is

$$A = \frac{1}{2} r \cdot r\, \boldsymbol{d\theta}$$

For areal velocity,



$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

But we've seen that : $\dot{\theta} = \frac{J}{mr^2}$ so areal velocity becomes $\frac{dA}{dt} = \frac{J}{2m}$

We see that the areal velocity is only dependent on the angular momentum and is independent of time i.e. it is a constant

Kepler's third law of Motion :-

The square of the time period (T) of revolution in elliptical orbit is proportional to the cube of the length of semi-major axis (a) i.e. $T^2 \propto a^3$.

$$\frac{dA}{\boldsymbol{dt}} = \frac{J}{2m} \Rightarrow \int \boldsymbol{d}A = \frac{J}{2m}\int \boldsymbol{dt} \Rightarrow \pi ab = \frac{J}{2m}T$$

$$\pi a \cdot a\sqrt{1-\boldsymbol{e}^2} = \frac{J}{2m}T \textit{ it is given } \boldsymbol{e} = \sqrt{1+\frac{2EJ^2}{mk^2}} \text{ and } E = -\frac{k}{2a}$$

$$e^2 = 1 - \frac{2kJ^2}{2amk^2} \Rightarrow 1 - e^2 = \frac{2kJ^2}{2amk^2}$$

$$T^2 = \frac{4m^2}{J2}\pi^{2^{a^2 \cdot a^2}}(1-e^2) \text{ put the value of } 1 - e^2 = \frac{2kJ^2}{2amk^2} = \frac{J^2}{amk}$$

$T^2 = \frac{4m^2}{J2}\pi^{2^{a^2 \cdot a^2}}(1 - e^2)$ put the value of $1 - e^2 = \frac{2kJ^2}{2am^{\ \ 2}} = \frac{J^2}{amk}$

$$T^2 = \frac{4n^2}{J^2}\pi^2 a^4(1 - e^2) \Rightarrow T^2 = \frac{4m^2}{J^2}\pi^2 a^4 \cdot \frac{J^2}{mak} = \frac{4\pi^2 ma^3}{k}$$

$\Rightarrow T^2 = \frac{4\pi ma^3}{k}$ if $k = Gm_S m$ $\ then$ $\Rightarrow T^2 = \frac{4\pi^2 a^3}{Gm_s}$

$$T^2 \propto a^3$$

THANK YOU !!