Gauge Symmetry

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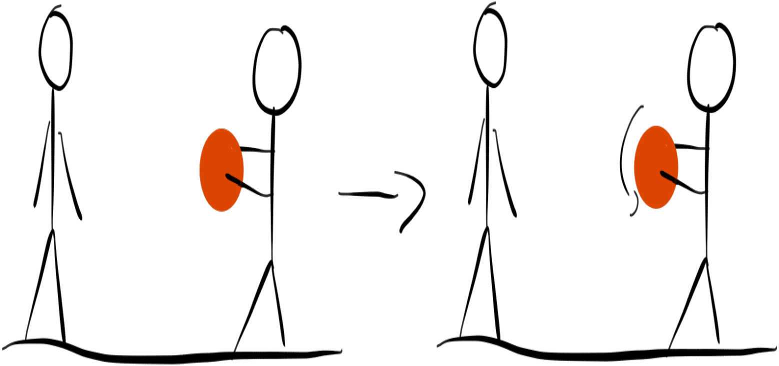
"In this talk we'll learn about symmetries intuitively. Gauge symmetry is introduced through a toy economic model. We then explore the implications of Gauge symmetry in Quantum Mechanics and Electromagnetism"

arXiv:1901.10420v1[physics.hist-ph] 29 Jan 2019

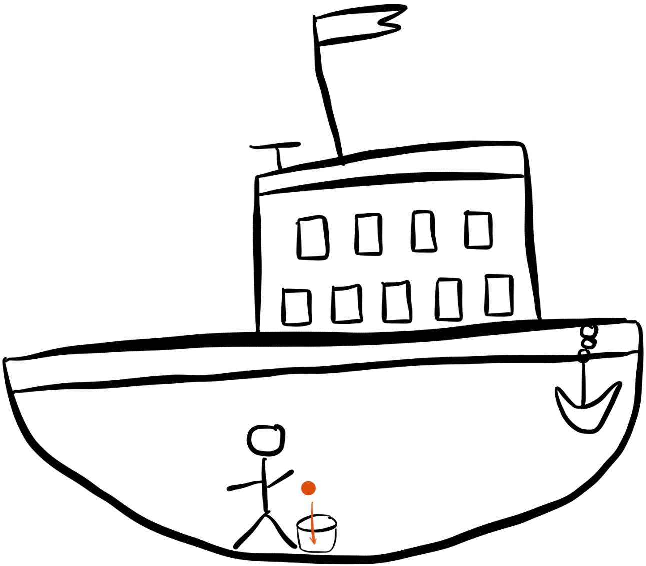
Overview

Symmetries Intuitively

What is a symmetry?



Thought Experiment: Galileo's ship



Definitions and assumptions

Definitions

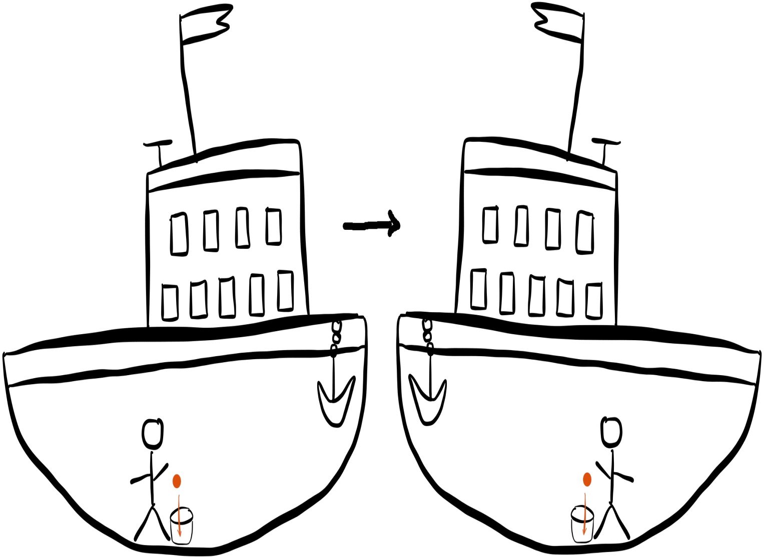
- *System:* The region within the arbitrary boundary and on which the attention is focused is called the system [a], i.e. the ship
- *Sub-System:* A self-contained system within a larger system. i.e. the dude playing with the ball

[a] [Zemanksky, Richard H. Dittman (1997)]

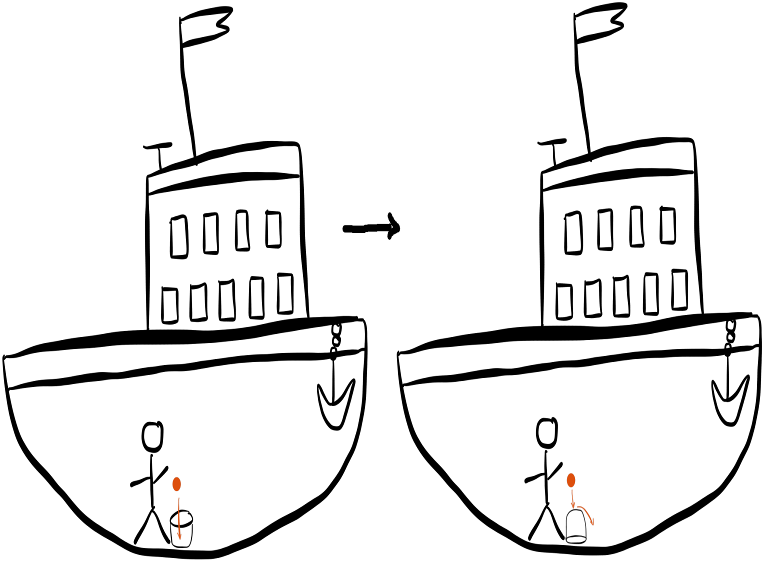
Assumptions

- We assumed the system is closed

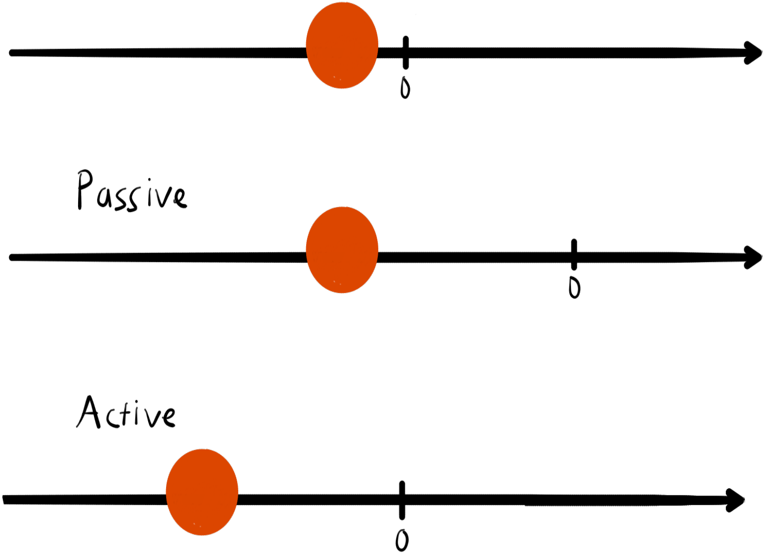
Global Transformations



Local Transformations



Active vs Passive Transformations



Symmetries vs Redundancies

Symmetry

- Invariance under *active transformations is called a symmetry*
- Symmetry is a *real feature of a system*

Redundancy

- Invariance under *passive transformations is called a redundancy*
- Redundancy is *only a feature of our description*

Gauge Symmetry

Definitions

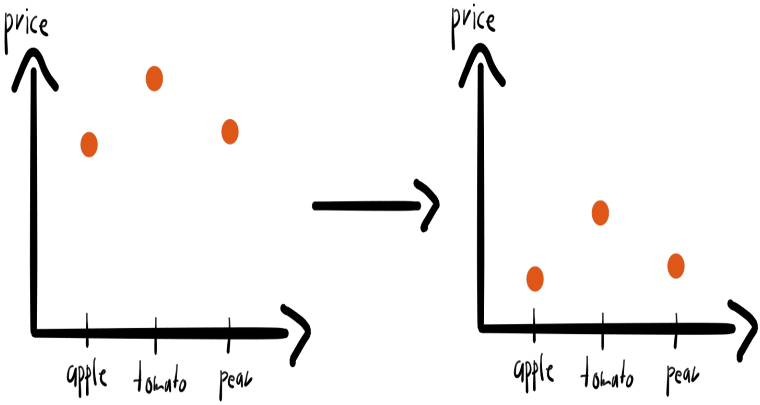
- Gauge symmetry: Invariance under the change of scale
- Gauge: The change of scale
- Gauge Transformation: A transformation that maps from the normal set of equations to the gauge
- Gauge Invariance: The equations of motion of the system are invariant under the application of a select family of gauges
- Gauge freedom: The freedom to pick our coordinate system

Toy-Economic model

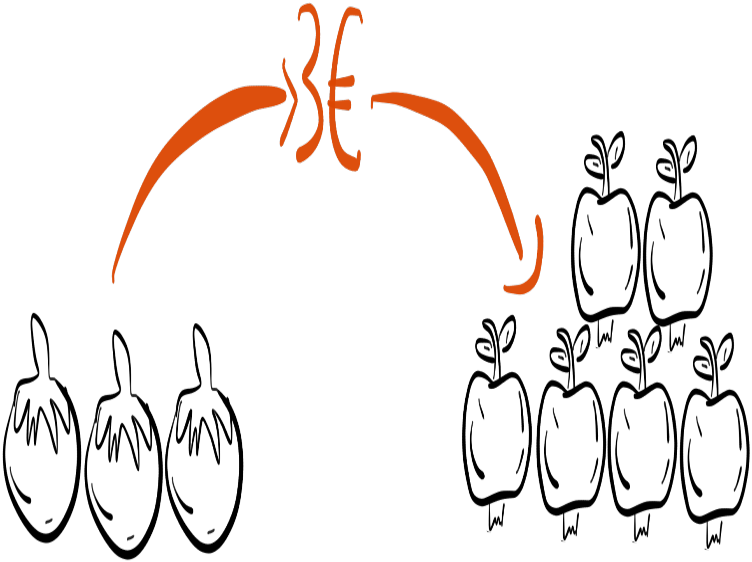
Toy-Economic model

- The toy model we will use in the following describes a simplified financial market.
- It consists of several countries and the basic process we try to describe is that money and goods can be traded and carried around.
- This setup is arguably the simplest setup where a gauge symmetry shows up.
- First of all, let's imagine that we have a common currency in several countries. For concreteness, we call this currency Euro and the countries Germany, France, the United Kingdom and Italy.
- In addition, we consider this subsystem of the whole world isolated and assume that there is a trader who only does business within these countries.

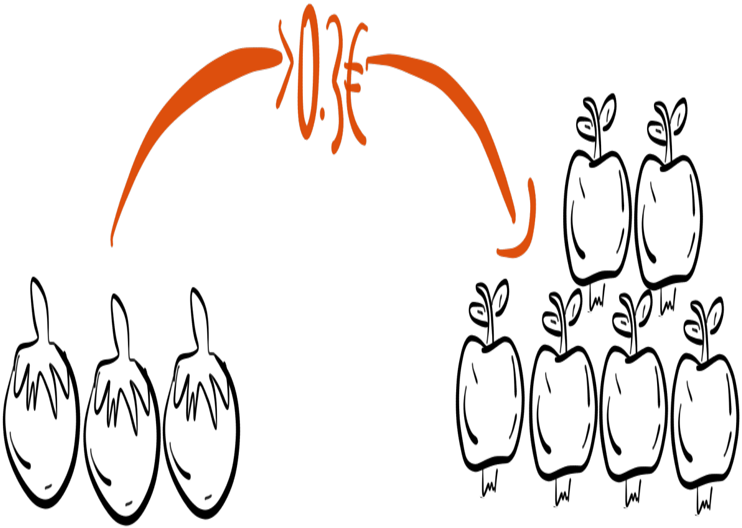
Global Gauge Symmetry



Naive Example



Scaling the currency



Local Gauge Symmetry

Setting things up

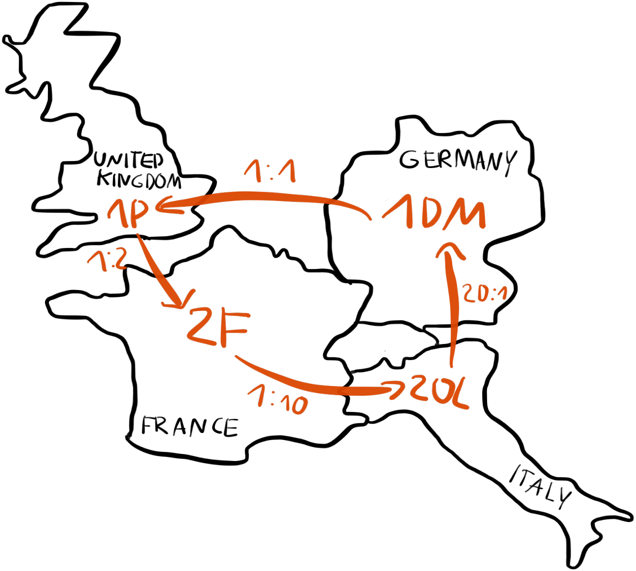
- For concreteness, we now introduce an independent local currencies in Germany, which we call Deutsche Mark (DM). Moreover, we introduce Francs (F) in France, in England Pounds (P) and Lira (L) in Italy.
- This is only possible if we introduce bookkeepers which keep track of the values of the local currencies and are able to exchange one currency for another.
- We can then imagine that the bookkeepers always adjust their exchange rates perfectly whenever the value of a local currency changes. If this the case such changes have no noticeable effect

Exchange rates example

For example, let's assume that the exchange rates are

- DM/P $= 1$
- P/F $= 2$
- F/L $= 10$
- DM/L $= 20$

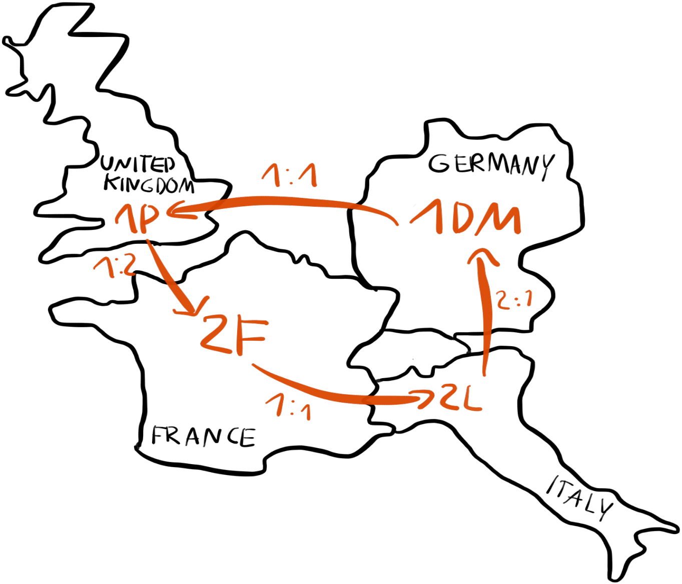
Exchange rates visually



Transforming the currency

- $DM/P = 1$
- $P/F = 2$
- $F/L = 10 \to F/\widetilde{L} = F/(L/10) = 1$
- $DM/L = 20 \to DM/\widetilde{L} = DM/(L/10) = 2$

Exchange rates visually



Gauge Theory Intuitively

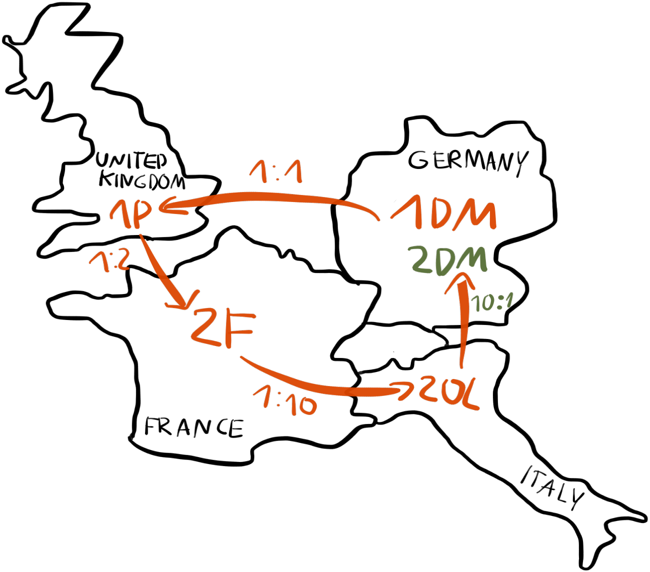
Exchange rates example

For example, let's assume that the exchange rates are

- $DM/P = 1$
- $P/F = 2$
- $F/L = 10$
- $DM/L = 20$

Now a trader is able to earn money simply by trading currencies. In the financial world this is known as an arbitrage opportunity.

Arbitrage opportunities visually



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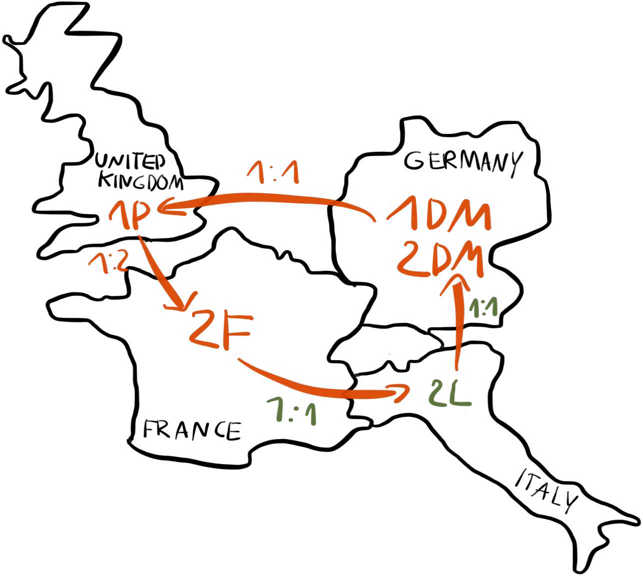
Arbitrage opportunities visually

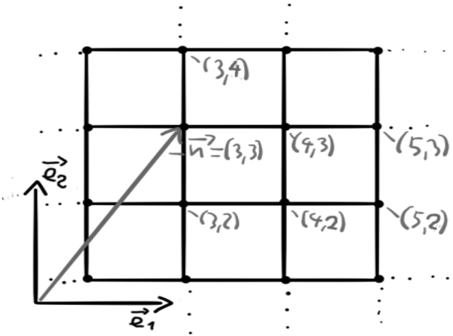


Mathematical Description of the Toy Model

Vectors in the lattice

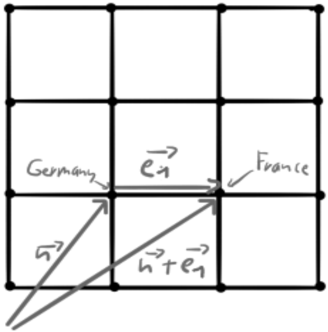
Vectors

Each point in the lattice can be identified using a vector, $\overrightarrow{n} = (n_1, n_2)$

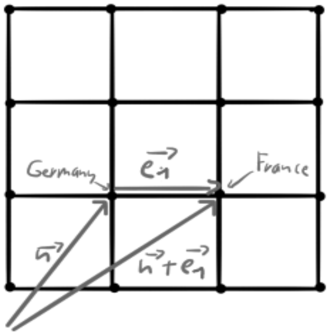


Basis

Every vector can be expressed as a linear combination of the basis vectors e_i



Exchange rates



We denote the Exchange rates between the country labelled by the vector $\vec{n}$ and its neighbour in the i-direction by $R_{\vec{n},i}$. For this we introduce a logarithm,

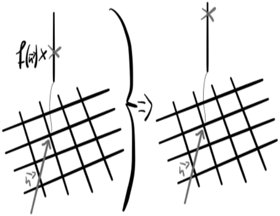
$$R_{\vec{n},i} = e^{A_i(\vec{n})} \tag{1}$$

where, $A_i(\vec{n}) = \ln(R_{\vec{n},i})$

Change of currency

We use the notation $f(\vec{n})$ to denote a change of the currency in the country at $\vec{n}$ by a factor f. We again introduce the corresponding logarithm,

$$f(\vec{n}) = e^{\epsilon(\vec{n})} \tag{2}$$



Gauge Transformation

Scaling the Exchange Rates

$$R_{\vec{n},i} \rightarrow \frac{f_{\vec{n}+\vec{e}_i}}{f_{\vec{n}}} R_{\vec{n},i} \tag{3}$$

Rewriting in terms of logarithms

$$R_{\vec{n},i} = e^{A_i(\vec{n})} \rightarrow \frac{f_{\vec{n}+\vec{e}_i}}{f_{\vec{n}}} R_{\vec{n},i}$$

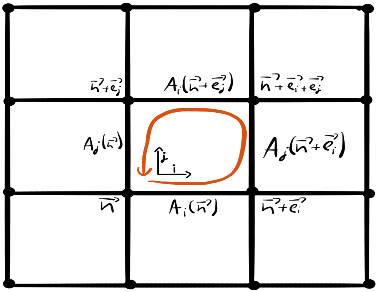
$$e^{A_i(\vec{n})} \rightarrow \frac{e^{\epsilon(\vec{n}+\vec{e}_i)}}{e^{\epsilon(\vec{n})}} e^{A_i(\vec{n})}$$

$$e^{A_i(\vec{n})} \rightarrow e^{A_i(\vec{n})+\epsilon(\vec{n}+\vec{e}_i)-\epsilon(\vec{n})}$$

Thus, we can conclude,

$$A_i(\vec{n}) \rightarrow A_i(\vec{n}) + \epsilon(\vec{n}+\vec{e}_i) - \epsilon(\vec{n}) \tag{4}$$

Gain Factor



The total gain we can earn by following a specific loop can be quantified by,

$$G = R_{\vec{n},i} R_{\vec{n}+\vec{e}_i,j} \frac{1}{R_{\vec{n}+\vec{e}_j,i}} \frac{1}{R_{\vec{n},j}} \tag{5}$$

Once again we use a logarithm,

$$G := e^{F_{ij}(\vec{n})} \tag{6}$$

Logartihm of the Gain Factor

- When this gain factor is larger than one, we can earn money by trading money following the loop, if it is smaller than one we loose money.
- We can rewrite the logarithm of the gain factor as:

$$F_{ij}(\vec{n}) = A_j(\vec{n} + \vec{e}_i) - A_j(\vec{n}) - [A_i(\vec{n} + \vec{e}_j) - A_i(\vec{n})] \tag{7}$$

Consistency Check

- A crucial consistency check is that G and F_{ij} are unchanged by gauge transformations
- We already argued above that an arbitrage opportunity is something real and thus cannot depend on local conventions
- Quantities like this are usually called gauge invariant
- So in words, G and $F_{ij}(\vec{n})$ encode what is physical in the structure of exchange rates.
- Moreover, an important technical observation is that $F_{ij}(\vec{n})$ is antisymmetric: $F_{ij}(\vec{n}) = -F_{ji}(\vec{n})$, which follows directly from the definition

Extending to a dynamical system

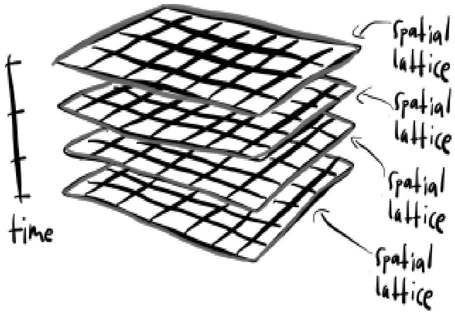
$2 + 1$ dimensions

Previously we had,

$$i, j \in \{1, 2\}$$

Now we include time but also switch notation,

$$\mu, \nu \in \{0, 1, 2\}$$



Logartihm of the Gain Factor

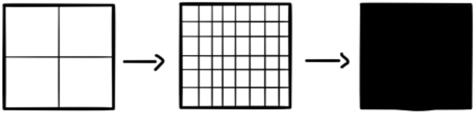
$$F_{\mu\nu}(\vec{n}) = A_\mu(\vec{n} + \vec{e}_\nu) - A_\mu(\vec{n}) - [A_\nu(\vec{n} + \vec{e}_\mu) - A_\nu(\vec{n})] \tag{8}$$

Continuum Limit

Taking Limits

- In the continuum limit, the lattice spacing goes to zero
- We're basically get in this limit the difference quotient

$$\text{Differences} \rightarrow \text{Differentials}$$

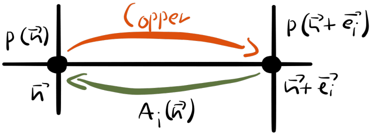


Logartihm of the Gain Factor

Taking equation (8) to the continuum limit

$$F_{\mu\nu}(x_\mu) = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{9}$$

Goods Perspective



The factor for such a process is given by,

$$g = \frac{p(\vec{n} + \vec{e}_i)}{p(\vec{n}) R_{\vec{n},i}} \tag{10}$$

Once again we use a logarithm,

$$g := e^{J_i(\vec{n})} \tag{11}$$

Goods Logarithm

The gain factor for such a process is given by,

$$g = \frac{p(\vec{n} + \vec{e}_i)}{p(\vec{n}) R_{\vec{n},i}} \tag{12}$$

Once again we use a logarithm,

$$g := e^{J_i(\vec{n})} \tag{13}$$

Rewriting equation (13),

$$\therefore \; e^{J_i(\vec{n})} = \frac{e^{\phi(\vec{n}+\vec{e}_i)}}{e^{\phi(\vec{n})} e^{A_i(\vec{n})}}$$

$$\therefore \; J_i(\vec{n}) = \phi(\vec{n} + \vec{e}_i) - \phi(\vec{n}) - A_i(\vec{n}) \tag{14}$$

Copper Current

- The amount of money we earn depends on the amount of copper we carry around. Thus, in general, we have
$$J_i(\vec{n}) = q(\phi(\vec{n} + \vec{e}_i) - \phi(\vec{n}) - A_i(\vec{n})) \tag{15}$$

- where q is related to the amount of copper involved in the trade.
- Generalizing to $d + 1$ dimensions
$$J_\mu(\vec{n}) = q\left(\phi(\vec{n} + \vec{e}_\mu) - \phi(\vec{n}) - A_\mu(\vec{n})\right) \tag{16}$$

- In the continuum limit, equation. (16)
$$J_\mu(\vec{n}) = q\left(\frac{\partial \phi}{\partial x_\mu} - A_\mu(\vec{n})\right) \tag{17}$$

Generalizing things

- An important idea in every gauge theory is that the equation of motions should not depend on local conventions.
- Using what we learned above, this means that the equation should only depend on J_μ and $F_{\mu\nu}$.
- If we then assume that the amount of copper is conserved within the system ($\partial_\mu J_\mu = 0$), we can derive the inhomogeneous Maxwell equations

$$\partial_\nu F_{\mu\nu} = \mu_0 J_\mu \tag{18}$$

- Where μ_0 is a constant which encodes how strongly the pattern of arbitrage possibilities react to the presence and flow of copper.
- This equation only contains the gauge invariant quantities J_μ and $F_{\mu\nu}$, has exactly one free index (μ) on both sides and we get zero if we calculate the derivative

$$\partial_\mu F_{\mu\nu} = 0 \tag{19}$$

Gauge Symmetry in Physics

Quantum Mechanics

Wave Function

$$\psi(x) = R(x)e^{i\phi(x)} \tag{20}$$

Operator

- Operators are basically matrices that correspond to measurements of things that we can observe i.e. observables
- In general we represent operators as $\hat{O}$
- To find out what we measure we use the equation

$$\psi^*(x)\hat{O}\psi(x) \tag{21}$$

- Where ψ^* is simply the complex conjugation
- An example of an operator is the momentum operator, $\hat{P} = -i\frac{\partial}{\partial x_i}$

Global Gauge Symmetry in Quantum Mechanics

Global Phase Shift

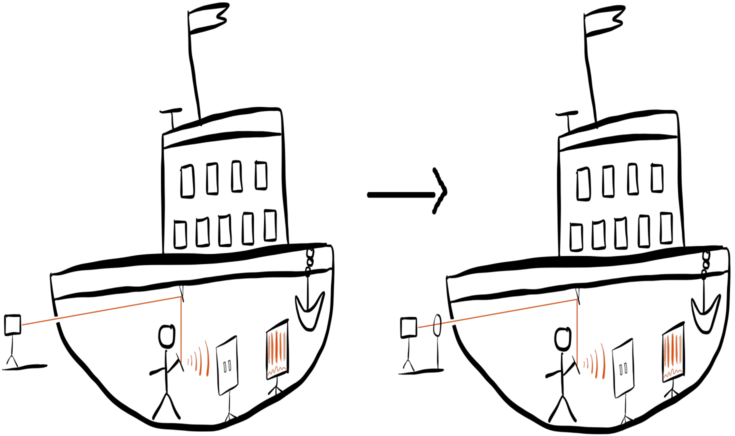
- We shift the phase of the entire system by means of the transformation

$$\psi(x) \to e^{i\epsilon}\psi(x) \tag{22}$$

Plugging in (22) into (21)

$$\psi^*(x)\hat{O}\psi(x) \to \psi^*(x)e^{-i\epsilon}\hat{O}e^{i\epsilon}\psi(x) = \psi^*(x)\hat{O}\psi(x) \tag{23}$$

Visualizing Global Gauge Symmetry in Quantum Mechanics



Local Gauge Symmetry in Quantum Mechanics

Local Phase Shift

- We shift the phase of the entire system by means of the transformation

$$\psi(x) \to e^{i\epsilon(x)}\psi(x) \tag{24}$$

- Now we use $\hat{P}$ instead of $\hat{O}$ to see the difference
- Plugging in (24) into (21)

$$\begin{aligned}
\psi^*(x)\hat{P}\psi(x) &\to \psi^*(x)e^{-i\epsilon(x)}(-i\frac{\partial}{\partial x_i})e^{i\epsilon(x)}\psi(x) \\
&= -i\psi^*(x)e^{-i\epsilon(x)}\partial_x e^{i\epsilon(x)}\psi(x) \\
&= -i\psi^*(x)\partial_x\psi(x) + \psi^*(x)(\partial_x\epsilon(x))\psi(x) \\
&\neq \psi^*(x)\hat{P}\psi(x)
\end{aligned} \tag{25}$$

Local Gauge Symmetry in Quantum Mechanics

- However, if we interpret the local transformation in a passive sense, it
- shouldn't make any difference.
- Analogous to what we did in our money toy model, we can achieve this by introducing a bookkeeper A_μ which keeps track of such local changes of the phase.
- In particular, we replace the momentum operator, with the so-called covariant momentum operator

$$\hat{P} = -i\partial_x - A_x \tag{26}$$

- This bookkeeper A_x becomes under a local phase shift

$$A_x \to A_x + \partial_x \epsilon(x) \tag{27}$$

Local Gauge Symmetry in Quantum Mechanics

- After the introduction of this bookkeeper our description is indeed invariant under local phase shifts.

$$\psi^*(x)\hat{P}\psi(x) \to \psi^*(x)e^{-i\epsilon(x)}\hat{\tilde{P}}e^{i\epsilon(x)}\psi(x)$$

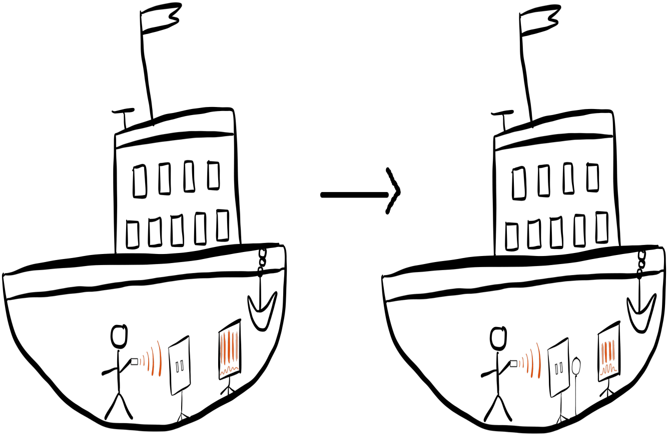
$$= \psi^*(x)e^{-i\epsilon(x)}\left(-i\partial_x - A_x - \partial_x\epsilon(x)\right)e^{i\epsilon(x)}\psi(x)$$

$$= -i\psi^*(x)\partial_x\psi(x) + \psi^*(x)(\partial_x\epsilon(x))\psi(x) \tag{28}$$

$$-\psi^*(x)A_x\psi(x) - \psi^*(x)(\partial_x\epsilon(x))\psi(x) \tag{29}$$

$$= \psi^*(x)\hat{P}\psi(x) \tag{30}$$

Visualizing Local Gauge Symmetry in Quantum Mechanics



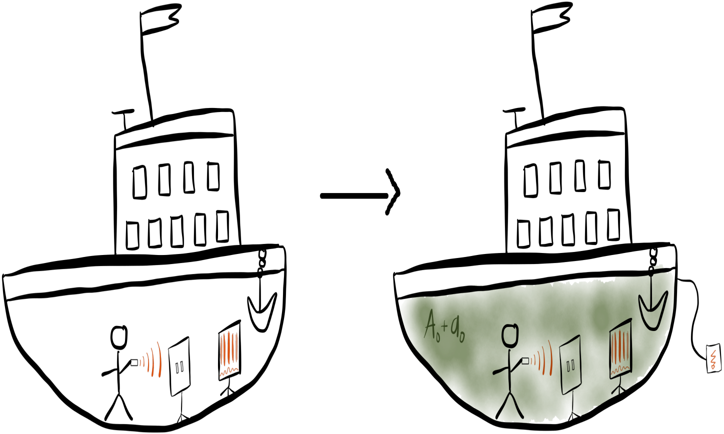
Gauge Symmetry in Electrodynamics

- The equations of Electrodynamics (Maxwell's equations) also posses a global symmetry

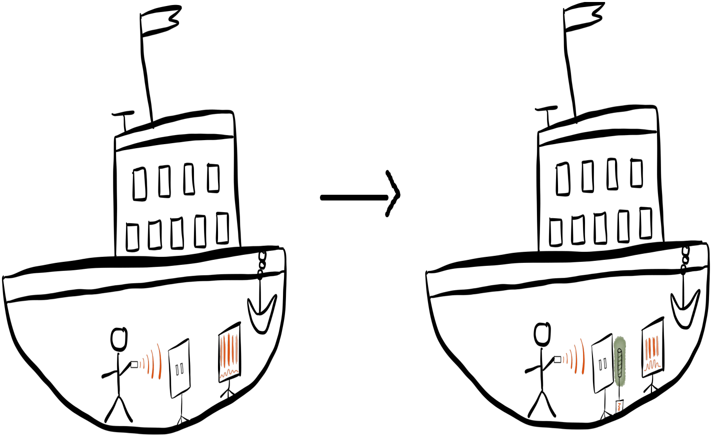
$$A_\mu(x) \to A_\mu(x) + a_\mu \tag{31}$$

- where a_μ are arbitrary real numbers. This comes about since we can only measure potential differences.
- Similarly to what we discussed above, we can now also argue that a local shift of the electromagnetic potential is not a symmetry of the system.
- To understand this, imagine that we only change the potential of a single object inside the ship.
- Clearly the physicist would have no problem finding this out.
- The most important point is that in Electrodynamics our bookkeepers A_μ are dynamical physical actors that can induce real physical changes.

Visualizing Global Gauge Symmetry in Electrodynamics



Visualizing Local Gauge Symmetry in Electrodynamics



Putting the pieces together

- We argued that the invariance under global transformations represents a real symmetry, while the invariance under local transformations is only a redundancy.
- Formulated differently, we have invariance under active global transformations and passive local transformations if we write formulate the theory appropriately.
- However, Quantum Mechanics is not invariant under active local phase shifts.
- Analogously to what we did in the money toy model, we then argued that our bookkepers A_μ can also appear as real dynamical parts of the system and not only as purely mathematical bookkeepers.
- The crucial point is then that as soon as we have a system where the bookkeepers are no longer purely mathematical objects but physical parts of it, they can induce measurable changes.
- What this implies immediately is that, in principle, it's possible to cancel any local phase shift using an electromagnetic potential A_μ.

Putting the pieces together

- However, it's important to take note that we still do not have a local symmetry when an electromagnetic potential is present, for example, in the ship in which our quantum physicist detects local phase shifts.
- It is instructive to reformulate this point using the notions "active transformation" and "passive transformation",
- A passive transformation is simply a change of the coordinate systems and therefore cannot lead to any physical change.
- All we achieve through a passive transformation is a different description of the same physical situation.
- Therefore, when we perform a passive transformation, we must be careful to keep our description consistent.
- This means in particular that whenever we perform a local passive transformation, we have to accompany
- When we perform a passive transformation these two transformations always go hand in hand.

Putting the pieces together

- In contrast, an active transformation means that a real physical change happens and therefore the physical situation doesn't need to remain unchanged.
- In particular, when we induce an active local phase shift the corresponding transformation of A_μ does not happen automatically.
- Otherwise we wouldn't be able to detect local phase shifts in experiments.
- In other words, the active shifts in ψ and A_μ are two separate transformations which can happen independently.

Summary

- We understand what symmetry is in the context of physics is and the different kinds of symmetries
- After constructing a toy-economic model
- Quantum Mechanics and Electrodynamics are invariant under active global gauge transformations. Hence, global gauge symmetry is a real symmetry.
- However, only our description is invariant under passive local transformations.
- Therefore, local gauge symmetry is a misnomer and should be better called local gauge redundancy.

What's next?

- A rigorous mathematical framework for symmetries i.e. Group Theory
- Curvature of spaces i.e. Tensors
- Connection to modern Physics i.e. the Standard Model

Addendum

From "Lie algebras in particle physics" by Howard Georgi[a]

[a] [Peter Woit (2006)]

"A symmetry principle should not be an end in itself. Sometimes the physics of a problem is so complicated that symmetry arguments are the only practical means of extracting information about the system. Then, by all means use them. But, do not stop looking for an explicit dynamical scheme that makes more detailed calculation possible. Symmetry is a tool that should be used to determine the underlying dynamics, which must in turn explain the success (or failure) of the symmetry arguments. Group theory is a useful technique, but it is no substitute for physics."

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