Scattering

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Barn

- One way of defining the size/radius of the proton is through its rate of interacting with itself or other particles.
- For example, if you strapped yourself on a proton that was traveling toward another proton for collision, then you would only see the cross sectional of the proton.
- The larger this area is, the more likely it is that you will interact with it. The smaller the area, the less likely to interact.
- This motivates a connection between proton size and scattering probability.

Barn

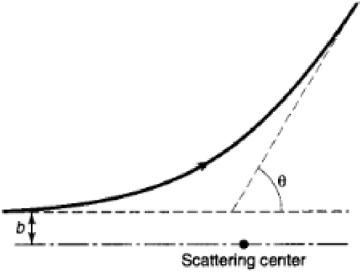
- In particle physics, a collision or interaction rate is expressed in effective cross-sectional area, typically just called cross section.
- As an "area," we can measure scattering cross sections as the square of some relevant length scale.
- The standard unit of cross section in particle physics is called the barn.
- The barn unit is approximately the cross-sectional area of a uranium nucleus, $10^{-28}m^2$.

Barn

In elementary particle physics, the cross sections that we consider are typically much smaller than a barn, so we often use nanobarns (nb, $10^{-37}m^2$), picobarns (pb, $10^{-40}m^2$), femtobarns (fb, $10^{-43}m^2$), or even attobarns (ab, $10^{-18}m^2$) to express them.

Classical Scattering

- Imagine a particle incident on some scattering center. It comes in with an energy $\boldsymbol{E}$ and an impact parameter $\boldsymbol{b}$, and it emerges at some scattering angle θ.
- The essential problem of classical scattering theory is this: *Given the impact parameter, calculate the scattering angle.*
- Ordinarily, of course, the smaller the impact parameter, the greater the scattering angle.

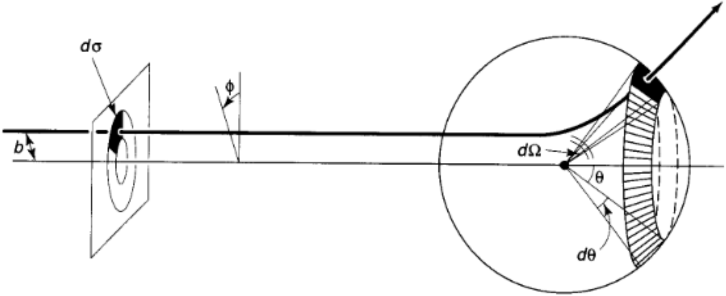


θ
b
Scattering center

- More generally, particles incident within an infinitesimal patch of cross-sectional area dσ will scatter into a corresponding infinitesimal solid angle dQ.
- The larger dσ is, the bigger dQ will be; the proportionality factor, D$(\theta) =$ d$\sigma/$dQ, is called the differential (scattering) cross-section:

Differential Cross Section:

$$d\sigma = D(\theta)d\Omega \tag{1}$$



dσ
ϕ
b
dΩ
θ
dθ

- In terms of the impact parameter and the azimuthal angle $\phi, d\sigma = b.db.d\phi$ and $d\Omega = \sin(\theta) d\theta d\phi$, so:

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{2}$$

- The total cross-section is the integral of D(θ) over all solid angles:

$$\sigma = \int D(\theta) d\Omega \tag{3}$$

- Roughly speaking, it is the total area of incident beam that is scattered by the target.
- Beams incident within this area will hit the target, and those farther out will miss it completely.
- Suppose we have a beam of incident particles, with uniform intensity/luminosity ($\mathcal{L}$), the number of particles entering area dσ (and hence scattering into solid angle dΩ), per unit time, is dN $=$ $\mathcal{L}$d$\sigma$$=$ $\mathcal{L}$D(θ)dΩ, so:

$$D(\theta) = \frac{1}{\mathcal{L}}\frac{dN}{d\Omega} \tag{4}$$

- This is often taken as the definition of the differential cross-section.
- If the detector accepts particles scattering into a solid angle dQ, we simply count the number recorded, per unit time, divide by dQ, and normalize to the luminosity of the incident beam.

Example-Hard Sphere Scattering

- Suppose the target is a billiard ball, of radius R, and the incident particle is a beam, which bounces off elastically. In terms of the angle α, the impact parameter is b $=$ R sin α, and the scattering angle is 0 $=$ n - 2α, so:

$$b = R \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R \cos\left(\frac{\theta}{2}\right) \tag{5}$$

- where,

$$\theta = \left\{ \begin{array}{ll} 2\cos^{-1}(b/R), & \text{for } b \leq r \\ 0, & \text{for } b \geq r \end{array} \right\} \tag{6}$$

- In the case of hard-sphere scattering:

$$\frac{dB}{d\theta} = -\frac{1}{2} R \sin\left(\frac{\theta}{2}\right) \tag{7}$$

- Therefore,

$$D(\theta) = \frac{R\cos(\theta/2)}{\sin(\theta)} . \left(\frac{R\sin(\theta/2)}{2}\right) = \frac{R^2}{4} \tag{8}$$

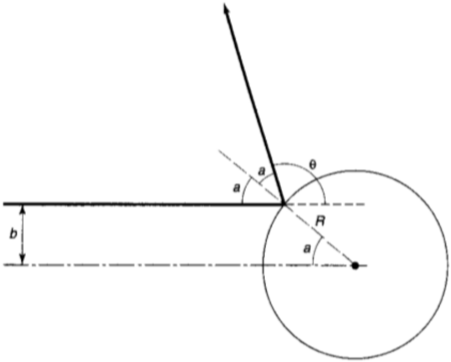
- This example is unusual in that the differential cross-section is actually independent of θ.

Scattering

Scattering

Differential cross section

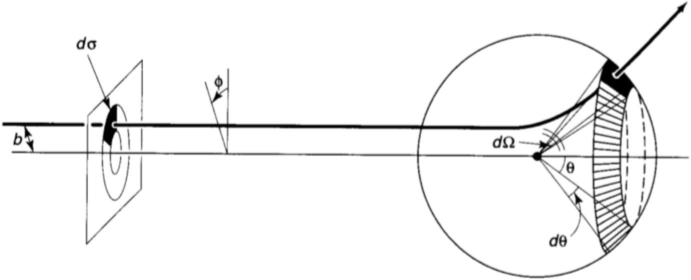
$$d\sigma = D(\theta)d\Omega \tag{9}$$



Scattering

Scattering cross section

$$D(\theta) = \frac{b}{sin\theta}\left|\frac{db}{d\theta}\right| \tag{10}$$



Scattering

Total cross section

$$\sigma \equiv \int D(\theta) d\Omega \tag{11}$$

Luminosity

L$\equiv$ no of incident particles per unit area, per unit time

$$D(\theta) = \frac{1}{L} \frac{dN}{d\Omega} \tag{12}$$

Scattering

Scattered spherical wave

For an incident wave of equation $\psi(z) = Ae^{ikz}$ travelling in z-direction, its outgoing spherical wave satisfies the Schrodinger equations of the form,

$$\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\} \tag{13}$$

For large r The relation between wave number k and energy of incident particles are,

$$k \equiv \frac{\sqrt{2mE}}{\hbar} \tag{14}$$

Scattering

Determining scattering amplitude

Probability of the incident particle travelling with speed v passing through infinitesimal area $d\sigma$ in time dt is,

$$dP = |\psi_{incident}|^2 dV = |A|^2 (vdt) d\sigma \tag{15}$$

This is equal to the probability that the particle later emerges into the corresponding solid angle $d\Omega$,

$$dP = |\psi_{scattered}|^2 dV = \frac{|A|^2 |f|^2}{r^2} (vdt) r^2 d\sigma \tag{16}$$

And $d\sigma = |f|^2 d\Omega$, so,

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{17}$$

Partial Wave Analysis

Partial Wave Analysis

Formalism

We know that the Schrodinger equation for a spherically symmetrical potential $V(r)$ admits the separable solutions,

$$\psi(r,\theta,\phi) = R(r)_l^m(\theta,\phi) \tag{18}$$

Where Y_l^m is a spherical harmonic and $u(R) = rR(r)$ satisfies the radial equation,

$$-\frac{\hbar^2}{2m}\frac{dl^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu \tag{19}$$

At very large r, the potential goes to zero, and the centrifugal term is negligible, so,

$$\frac{d^2u}{dr^2} \approx -k^2u \tag{20}$$

Partial Wave Analysis

Formalism (contd)

Whose general solution is,

$$u(r) = Ce^{ikr} + De^{-ikr} \tag{21}$$

The second term represents an incoming wave, so we dont need that $(D = 0)$,

$$R(r) \approx \frac{e^{ikr}}{r} \tag{22}$$

The radial equation then becomes,

$$\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}u = -k^2 u \tag{23}$$

Partial Wave Analysis

Spherical Hankel functions

The general solution for the radial equation is a linear combinations of spherical Bessel functions,

$$u(r) = Arj_l(kr) + Brn_l(kr) \tag{24}$$

We need solutions that are linear combinations analogous to e^ikr and e^-ikr, these are called the spherical Hankel functions,

$$h_l^{(1)} \equiv j_l(x) + in_l(x) \tag{25}$$

The Hankel function of the first kind becomes e^-ikr/r for large r, so we use these to get,

$$R(r) = Cj_l^{(1)}(kr) \tag{26}$$

Partial Wave Analysis

Exact wave fucntion

The exact wave function in the exterior region where $V(r) = 0$ is,

$$\psi(r, \theta, \phi) = A\left\{e^{ikz} + f(\theta, \phi)\frac{e^{ikr}}{r}\right\} \tag{27}$$

Where,

$$f(\theta, \phi) + \frac{1}{k}\sum_{l,m}(-i)^{l+1}C_{l,m}Y_l^m(\theta, \phi) \tag{28}$$

Partial Wave Analysis

Partial wave anplitudes

The $C_{l,m}$ are called the partial wave amplitudes. Now the cross section is,

$$D(\theta,\phi) = |f(\theta,\phi)|^2 = \frac{1}{k^2}\sum_{l,m}\sum_{l',m'}(i)^{l-l'}C^*_{l,m}C_{l'm'}(Y_l^m)^*Y_{l'}^{m'} \tag{29}$$

And the total cross-section is,

$$\sigma = \frac{1}{k^2}\sum_{l,m}\sum_{l',m'}(i)^{l-l'}C^*_{l,m}C_{l'm'}\int (Y_l^m)^*Y_{l'}^{m'}\,d\Omega = \frac{1}{k^2}\sum_{l,m}|C_{l,m}|^2 \tag{30}$$

Partial Wave Analysis

Exterior wave function

We know from the Legendre functions that,

$$Y_l^0(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(cos\theta) \tag{31}$$

where P_l is the lth Legendre Polynomial. Now the exact wave function in the exterior region is,

$$\psi(r, \theta) = A\left\{e^{ikz} + \sum_{l=0}^{\infty} \sqrt{\frac{2l+1}{4\pi}} C_l h_l^{(1)}(kr) P_l(cos\theta)\right\} \tag{32}$$

Partial Wave Analysis

Scattering amplitude and total-cross section

The scattering amplitude is given by,

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(-i)^{l+1}\sqrt{\frac{2l+1}{4\pi}}C_l P_l(cos\theta) \tag{33}$$

and the total cross-section is,

$$\sigma = \frac{1}{k^2}\sum_{l=0}^{\infty}|C_l|^2 \tag{34}$$

Partial Wave Analysis

Interior wave in spherical form

The general solution to the Schrodinger equation with $V = 0$ can be written in the form,

$$\sum_{l,m}[A_{l,m}j_l(kr) + B_{l,m}n_l(kr)]Y_l^m(\theta, \phi) \tag{35}$$

Expanding the plane wave in terms of spherical waves using Rayleigh's formula,

$$e^{ikz} = \sum_{l=0}^{\infty} i^l(2l + l)j_l(kr)P_l(cos\theta) \tag{36}$$

Partial Wave Analysis

Final wave function

The consistent exterior region wave function can be written as,
And hence the wave function in the exterior region can be written in the consistent form of,

$$\psi(r,\theta) = A\left[l(2l+l)j_l(kr) + \sum_{l=0}^{\infty}\sqrt{\frac{2l+1}{4\pi}}C_l h_l^{(1)}(kr)\right]P_l(cos\theta) \tag{37}$$

Phase Shift

Let's begin by considering a one-dimensional scattering problem with a localized potential on the half-line $x < 0$ and a brick wall at $x = 0$. So a wave incident from the left,

$$\psi_i(x) = Ae^{ikx} \tag{38}$$

is entirely reflected,

$$\psi_r(x) = Be^{-ikx} \tag{39}$$

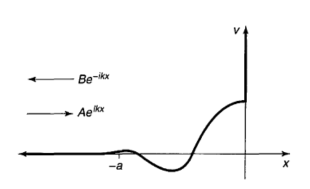


Figure: 1D scatterubg frin a localized potential bounded on the right by

Phase Shift

If there were no potential at all ($V(x) = 0$), but just at the wall ($x = 0$), then $B = -A$, since the total wave function, incident + reflected must vanish at the origin,

$$\psi_0 = A(e^{ikx} - e^{-ikx}) \tag{40}$$

If the potential is not zero ($V(x) \neq 0$), then the wave function ($x < -a$) takes the form:

$$\psi = A\left(e^{ikx} - e^{i(2\delta - kx)}\right) \tag{41}$$

Phase Shift

Why are we working with δ rather than the complex amplitude B? It makes the physics and math simpler:

- **Physically:** We only need to think of the conservation of probability. The potential merely shifts the phase
- **Mathematically:** We trade a complex number for a real one

Phase Shift

- Let's return to the 3D case. The incident plane wave carries no angular momentum in the z direction.
- Thus Rayleigh's formula contains no terms with $m \neq 0$ but insteaed it cfontains all values of the total angular momentum $(l = 0, 1, 2)$.
- Since angular momentum is conserved by a spherically symmetric potential each partial wave labelled by a particular l scatters independently with no change in amplitude (amplitude in this context refer to the amplitude of the wave no the probability amplitude) but differing in phase.
- If there is no potential then $\psi_0 = Ae^{ikx}$ and the lth partial wave is

$$\psi_0^l = Ai^l(2l+1)j_l(kr)P_l(\cos(\theta)) \tag{42}$$

Phase Shift

But from our previous considerations,

$$j_l(x) = \frac{1}{2}\left[h^{(1)}(x) + h_l^2(x)\right] \approx \frac{1}{2x}\left[(-i)^{l+1}e^{ix} + i^{l+1}e^{-ix}\right] \quad (43)$$

for $x >> 1$. So for large r,

$$\psi_0^{(l)} \approx A\frac{2l+1}{2ikr}\left[e^{ikr} - (-1)^l e^{-ikr}\right]P_l(\cos(\theta)) \quad (44)$$

The second term in square brackets corresponds to an incoming spherical wave. It is unchanged when we introduce the scattering potential. The first term is the outgoing wave. It picks up a phase shift δ_l:

$$\psi^{(l)} \approx A\frac{2l+1}{2ikr}\left[e^{i(kr+2\delta_l)} - (-1)^l e^{-ikr}\right]P_l(\cos(\theta)) \quad (45)$$

Phase Shift = Partial Wave

Well if we take the assymptotic i.e. large r limit of eq. (45):

$$\psi^{(l)} \approx A\left(\frac{(2l+1)}{2ikr}\left[e^{i(kr+2\delta_l)} - (-1)^l e^{-ikr}\right] + \frac{(2l+1)}{r} a_l e^{ikr}\right) P_l(\cos(\theta) \tag{46}$$

With the generic expression in terms of $e^{i\delta_l}$ we find

$$a = \frac{1}{2ik}(e^{2i\delta_l} - 1) = \frac{1}{k} e^{i\delta_l}\sin(\delta_l) \tag{47}$$

Phase Shift = Partial Wave

It follows in particular that,

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin(\delta_l)P_l(\cos(\theta)) \tag{48}$$

and,

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l) \tag{49}$$

Delta "Function"

The Dirac Delta is a functional [a] which we define as,

$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{cases} \tag{50}$$

$$\int_{-\infty}^{+\infty} \delta(x-a)dx = 1 \tag{51}$$

$\forall\ a \in \mathbb{R}$ We can visualize it as a sharp peak at a,

[a] An object that is a map between functions

Delta "Functional"

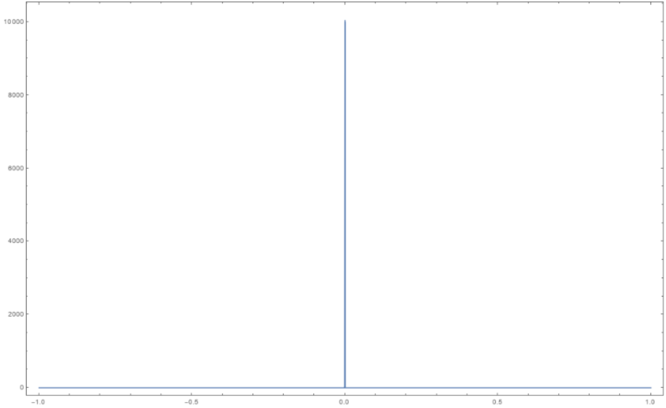


Figure: A Plot of $\delta(x)$

Delta Functional

We generalize equations (50) and (51) to three dimensions,

$$\delta^3(\vec{r}-\vec{a}) = \delta(x-a_x)\delta(y-a_y)\delta(z-a_z) \tag{52}$$

$$\int_{-\infty}^{+\infty} \delta^3(\vec{r}-\vec{a})dV = 1 \tag{53}$$

Integral Form of the Schrodinger Equation

Integral Form of the Schrodinger Equation is given as:

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0)\psi(\vec{r}_0) d^3\vec{r}_0 \qquad (54)$$

The First Born Approximation

So,

$$|\vec{r}-\vec{r}_0| = r^2 + r_0^2 - 2\vec{r}\vec{r}_0 \approxeq r^2\left(1 - 2\frac{\vec{r}.\vec{r}_0}{r^2}\right) \tag{55}$$

and hence,

$$|\vec{r}-\vec{r}_0| \approxeq r - \hat{r}.\vec{r}_0 \tag{56}$$

Let,

$$\vec{K} = k\hat{z} \tag{57}$$

then

$$e^{-i\vec{K}|\vec{r}-\vec{r}_0|} \approx e^{ikr}e^{-i\vec{K}.\vec{r}_0} \tag{58}$$

and therefore,

$$\frac{e^{-i\vec{K}|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \approx \frac{e^{ikr}}{r}e^{-i\vec{K}.\vec{r}_0} \tag{59}$$

The First Born Approximation

In the case of scattering, we want:

$$\psi_o(\vec{r}) = Ae^{ikz} \tag{60}$$

to represent an incident plane wave. For large r,

$$\psi \approxeq Ae^{ikz} - \frac{m}{2\pi\hbar^2 A}\int e^{-i\vec{K}.\vec{r}_0}V(\vec{r}_0)\psi(\vec{r}_0)d^3\vec{r}_0 \tag{61}$$

This is in the standard form. We can read off the scattering amplitude:

$$f(\theta,\phi) = \frac{m}{2\pi\hbar^2 A}\int e^{-i\vec{K}.\vec{r}_0}V(\vec{r}_0)\psi(\vec{r}_0)d^3\vec{r}_0 \tag{62}$$

The First Born Approximation

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$$f(\theta, \phi) = \frac{m}{2\pi\hbar^2 A} \int e^{-i\vec{K}.\vec{r}_0} V(\vec{r}_0)\psi(\vec{r}_0) d^3\vec{r}_0 \tag{65}$$

The First Born Approximation

So far this is exact. Now we invoke the Born approximation: "Suppose the incoming plane wave is not substantially altered by the potential; then we can say that

$$\psi(\vec{r}_0) \approxeq \psi_0(\vec{r}_0) = Ae^{ikz_o} = Ae^{i\vec{K}'\vec{r}_o} \tag{66}$$

where

$$K' = k\hat{z}$$

inside the integral.

The First Born Approximation

In the Born approximation then,

$$f(\theta, \phi) \approxeq -\frac{m}{2\pi\hbar^2} \int e^{i(k' - k).\vec{r}_0} V(r_0) d^3\vec{r}_0 \qquad (67)$$

In particular, for low energy scattering, the exponential factor is essentially constant over the scattering region and the Born approximation simplifies to:

$$f(\theta, \phi) \approxeq -\frac{m}{2\pi\hbar^2} \int V(\vec{r}) d^3 r \qquad (68)$$

The First Born Approximation

For a spherically symmetrical potential, $V(\vec{r}) = V(r)$ but not necessarily at low energy. The Born approximation reduces to a simpler form. First we define:

$$\mathcal{K} = k^{'} - k \tag{69}$$

and let the polar axis for the r_0, the integral lies along so that;

$$(k^{'} - k).r_0 = \mathcal{K} r_0 \cos(\theta_0) \tag{70}$$

Then,

$$f(\theta) \approxeq -\frac{m}{2\pi\hbar^2} \int e^{i\mathcal{K} r_0 \cos(\theta_0)} V(\theta_0) r_0^2 \sin(\theta_0) dr_0 d\theta_0 d\phi_0 \tag{71}$$

The First Born Approximation

The integral is trivial, 2π, and the integral θ_0 is on we have encountered before in equation (). Dropping the subscript on r, we are left with

$$f(\theta) \approxeq -\frac{2m}{\hbar^2 \mathcal{K}} \int_0^\infty rV(r)\sin(\mathcal{K}r)dr \tag{72}$$

The angular dependence of f is carried by $\mathcal{K}$. From our previous considerations we can see that:

$$\mathcal{K} = 2k\sin(\theta/2) \tag{73}$$

Note: We can't apply the Born approximationi to hard-sphere scattering as the integral blows up due to our assumption (i.e. potential does not affect the wave function) here.

The First Born Approximation

Suppose,

$$V(\vec{r}) = \begin{cases} V_0, & \text{if } r \leq a \\ 0, & \text{if } r > a \end{cases} \tag{74}$$

In this case the low-energy scattering amplitude is,

$$f(\theta, \phi) \approxeq -\frac{m}{2\pi\hbar^2} V_0 \left(\frac{4}{3} \pi a^3 \right) \tag{75}$$

The First Born Approximation

This is independent of θ and ϕ ! Thus, the differential cross-section is:

$$\frac{d\sigma}{d\Omega} = |f|^2 \approxeq \left[\frac{2mV_0a^3}{3\hbar^2}\right]^2 \tag{76}$$

and the total cross-section:

$$\sigma \approxeq 4\pi \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \tag{77}$$

Low-energy soft-sphere scattering

Suppose,

$$V(\vec{r}) = \begin{cases} V_0, & \text{if } r \leq a \\ 0, & \text{if } r > a \end{cases} \tag{78}$$

In this case the low-energy scattering amplitude is,

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This is independent of θ and ϕ !

Low-energy soft-sphere scattering

Thus, the differential cross-section is:

$$\frac{d\sigma}{d\Omega} = |f|^2 \approxeq \left[\frac{2mV_0a^3}{3\hbar^2}\right]^2 \tag{80}$$

and the total cross-section:

$$\sigma \approxeq 4\pi \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \tag{81}$$

Yukawa Scattering

The Yukawa potential is a toy-model for the binding force in the nucleus of an atom. It has the form,

$$V(r) = \beta \frac{e^{-\mu r}}{r} \tag{82}$$

where β and μ are constants. The Born approximation gives,

$$f(\theta) \approxeq -\frac{2m\beta}{\hbar^2 k} \int_0^\infty e^{-\mu r} \sin(kr) dr = -\frac{2m\beta}{\hbar^2(\mu^2 + k^2)} \tag{83}$$

Rutherford Scattering

If we substitute $\beta = q_1 q_2 / 4\pi\epsilon_0$ and $\mu = 0$. The scattering amplitude is given by,

$$f(\theta) \approx -\frac{2mq_1q_2}{4\pi\epsilon_0\hbar^2k^2} \tag{84}$$

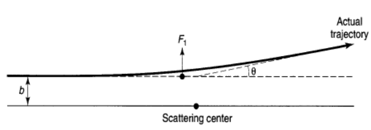
or,

$$f(\theta) \approx -\frac{q_1q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)} \tag{85}$$

The differential cross-section is the square of this:

$$\frac{d\sigma}{d\Omega} = \left[\frac{q_1q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right]^2 \tag{86}$$

The Born Series



From the impulse approximation we have,

$$I = \int F_{\perp} dt \tag{87}$$

If the deflection is small in comparism to the motion, it would then be a good approximation to the transverse momentum supplied to th particle. Thus we express the scattering angle as:

$$\theta = \arctan(I/p) \tag{88}$$

The Born Series

Let's recall the integral form of the Schrodinger equation:

$$\psi(\vec{r}) = \psi_0(\vec{r}) + \int g(\vec{r} - \vec{r}_0) V(\vec{r}_0) \psi(\vec{r}_0) d^3 r_0 \tag{89}$$

where is the incident wave and,

$$g(\vec{r}) = -\frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r}$$

is the Green's function with a factor $m/2\pi\hbar^2$ for convenience and V is the scattering potential. Suppose we take the equation for ψ and plug it back into (89),

$$\psi = \psi_0 + \int gV\psi_0 + \int \int gVgV\psi$$

The Born Series

Iterating this we obtain the series expasion for ψ,

$$\psi = \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_0 + \int\int\int gVgVgV\psi_0 ... \quad (90)$$

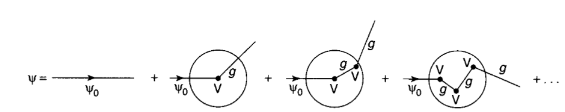


Figure: An example of the impulse approximation: the particle continues undeflected

Feynman Diagrams

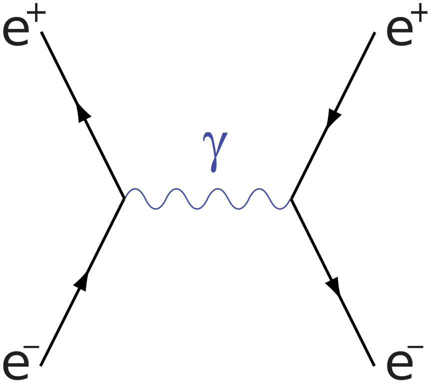


Figure: Bhabha scattering: Annihilation

Lifetime

The amount of time a particle exists before decaying

What precisely do we mean by the lifetime?
Lets imagine 2 muons; A stationary and moving one, From our perspective the moving one lasts longer due to time dilation. But even stationary muons don't all last the same amount of time, for there is an intrinsically random element in the decay process. We cannot hope to calculate the lifetime of any particular muon; rather, what we are after is the average (or "mean") lifetime of the muons in any large sample.

LifeTime

Elementary particles have no memories, so the probability of a given muon decaying in the next microsecond is independent of how long ago that muon was created

Old man analogy

An 80-year-old man is much more likely to die in the next year than is a 20- year-old, and his body shows the signs of eight decades of wear and tear. But all muons are identical, regardless of when they were produced; from an actuarial point of view they're all on an equal footing

Lifetime derivation

The critical parameter, then, is the decay rate, Γ, the probability per unit time that any given muon will disintegrate. If we had a large collection of muons, say, $N(t)$,at time t, then $N\Gamma dt$of them would decay in the next instant dt

$$dN = -\Gamma N dt \tag{91}$$

Integrating the above equation, we get

$$N(t) = N(0)e^{-\Gamma t} \tag{92}$$

LifeTime Derivation

the mean lifetime is simply the reciprocal of the decay rate

$$\tau = \frac{1}{\Gamma} \tag{93}$$

But a single particle can decay in different modes.Example a pion can decay into muon and muon neutrino and also it can decay into positron and electron neutrino and there are still some ways. Like this there are many ways a pion can decay. From this we can conclude a hadron have many ways to decay.

Therefore, in this circumstances the total decay rate is the sum of the individualised decay rates

Lifetime Derivation

$$\Gamma_{\text{tot}} = \sum_{i=1}^{n} \Gamma_i \tag{94}$$

The lifetime of the particle is the reciprocal of Γ_{tot}

$$\tau = 1/\Gamma_{\text{tot}} \tag{95}$$

This is the average lifetime of a hadron particle

Branching Ratio

In addition to lifetime ,we want to calculate the various branching ratios, that is, the fraction of all particles of the given type that decay by each mode. Branching ratios are determined by the decay rates

$$\text{Branching ratio for ith decay mode } = \Gamma_i/\Gamma_{\mathsf{tot}} \tag{96}$$

For decays, then, the essential problem is to calculate the decay rate for each mode; from there it is an easy matter to obtain the lifetime and branching ratios

Fermi's Golden Rule

Just a recap,

$$\sigma = \frac{1}{E_A}\frac{1}{E_B}\frac{1}{|v_A - v_B|}|_{out}\langle f|P_A P_B\rangle_{in}|^2$$

here, we are going to define the Probability Density term, i.e.

$$|_{out}\langle f|P_A P_B\rangle_{in}|^2,$$

more precisely.let's say we observe two positrons produced from proton collisions. For a consistent calculation of the cross section, we must sum over all possible momenta of the positrons which have positive energy, conserve fourmomentum, and correspond to real, on-shell positrons.

Fermi's Golden Rule

Consider the collsion of 2 paricles A and B resulting in a final state with n particles.

$$A + B \rightarrow 1 + 2 + 3 + ... + n$$

We denote the probability amplitude for the initial-state protons A and B with given momenta to collide and produce the final-state particles 1, 2, ..., n with particular momenta as,

$$M(A + B \rightarrow 1 + 2 + 3 + ... + n)$$

M is the lorentz invariant matrix element. Because we only care about the existence of the finalstate particles and not their momenta, we need to sum over all possible momenta, consistent with conservation laws. Lets begin constructing it.

Fermi's Golden Rule

Firstly,

$$|_{out}\langle f|P_A P_B\rangle_{in}|^2 = \sum_{Final Particle Momenta} |M(A+B \rightarrow 1+2+3+...+n)|^2$$

For a final-state particle i, its four-momentum is p_i and is four continuous components. so we can represent the sum as

$$\sum_{Final Particle Momenta} \rightarrow \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4}$$

This equation, though, is much too general. For now, the integrals includes all possible values of each momentum component: non-negative energies and three-momentum components that are any real number.

Fermi's Golden Rule

But we need the final state particles to be on shell. so the integral must go to zero whenever ${p_i}^2 \neq {m_i}^2$. This can be done by including the Dirac Delta δ like so,

$$\sum_{FinalParticleMomenta} \rightarrow \int \Big[\prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} 2\pi\delta\big({p_i}^2 = {m_i}^2\big) \Big]$$

There are still some unphysical configurations possible. The integrals must only be non-zero if the total four momentum of the initial protons is equal to the total four-momentum of all n final-state particles. This can also be enforced by a Dirac δ-function once again.

$$\sum \rightarrow \int \Big[\prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} 2\pi\delta\big({p_i}^2 = {m_i}^2\big) \Big] 2\pi^4 \delta^{(4)}\left(p_A + p_B - \sum_{i=1}^{n} p_i \right)$$

Fermi's Golden Rule

Here $\delta^{(4)}$ is just,

$$= \delta\left(E_A + E_B - \sum_{i=1}^{n} E_i\right)\delta\left(\vec{p_A} + \vec{p_B} - \sum_{i=1}^{n} \vec{p_i}\right)$$

This integral over the n final particle momenta is Lorentz invariant, and is called n-body Lorentz-invariant phase space, and is denoted by $d\prod_n$ or $dLIPS_n$ the final expresion is,

$$\sigma = \frac{1}{E_A}\frac{1}{E_B}\frac{1}{|v_A - v_B|}\int\Big[\prod_{i=1}^{n}\frac{d^4 p_i}{(2\pi)^4}2\pi\delta\big({p_i}^2 = {m_i}^2\big)\Big]$$

$$\times |M(A + B \to 1 + 2 + 3 + \ldots + n)|^2 (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum_{i=1}^{n} p_i\right)$$

Fermi's Golden Rule

This expression is called **Fermi's Golden Rule**.