University of Padua, Academic Year 2024/25
General Relativity for Astrophysics and Cosmology
Equation Sheet for the Exam (Final version: 18 January 2025)

Christoffel Symbols:
$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$

Riemann Tensor:
$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta}$

Einstein's Equations (with Cosmological Constant):
$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

Action of the commutator on an arbitrary tensor (torsion free):
$[\nabla_\rho, \nabla_\sigma] X^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = R^{\mu_1}_{\lambda\rho\sigma} X^{\lambda\mu_2\ldots\mu_k}_{\nu_1\ldots\nu_l} + R^{\mu_2}_{\lambda\rho\sigma} X^{\mu_1\lambda\ldots\mu_k}_{\nu_1\ldots\nu_l} + \ldots - R^{\lambda}_{\nu_1\rho\sigma} X^{\mu_1\ldots\mu_k}_{\lambda\nu_2\ldots\nu_l} - R^{\lambda}_{\nu_2\rho\sigma} X^{\mu_1\ldots\mu_k}_{\nu_1\lambda\ldots\nu_l} + \ldots$

Action of the covariant derivative on an arbitrary tensor:
$\nabla_\rho T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} = \partial_\rho T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} + \Gamma^{\mu_1}_{\rho\alpha} T^{\alpha\mu_2\ldots\mu_k}_{\nu_1\ldots\nu_l} + \Gamma^{\mu_2}_{\rho\alpha} T^{\mu_1\alpha\ldots\mu_k}_{\nu_1\ldots\nu_l} \ldots - \Gamma^{\alpha}_{\rho\nu_1} T^{\mu_1\ldots\mu_k}_{\alpha\nu_2\ldots\nu_l} - \Gamma^{\alpha}_{\rho\nu_2} T^{\mu_1\ldots\mu_k}_{\nu_1\alpha\ldots\nu_l}$

Geodesic Equation:
$\frac{d^2 x^\sigma}{d\lambda^2} + \Gamma^{\sigma}_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$

Schwarzschild Metric:
$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$

Conserved quantities in an orbit in Schwarzschild:
$E = \left(1 - \frac{2GM}{rc^2}\right) \frac{dt}{d\lambda}, \quad L = r^2 \frac{d\phi}{d\lambda}$

Energy Conservation and Effective Potential:
$\frac{1}{2}E^2 = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{\text{eff}}(r), \quad V_{\text{eff}}(r) = \frac{\epsilon}{2} + \frac{L^2}{2r^2} - \frac{\epsilon GM}{c^2 r} - \frac{GML^2}{c^2 r^3}$
$\epsilon = 1$ for a massive particle, $\epsilon = 0$ for massless one.

Gauge Transformation:
For a vector field A_μ: $A_\mu \to A_\mu + \partial_\mu \lambda$.
For the metric under coordinate transformation $x^\mu \to x^\mu - \xi^\mu$: $g_{\mu\nu} \to g_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$

Linearized Einstein Tensor:
$G_{00} = 2\nabla^2\Psi + \partial_k\partial_l s^{kl}$
$G_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j{}^k$
$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) + \delta_{ij}\partial_0\partial_k w^k - \partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}\partial_k\partial_l s^{kl}$

Expression of the Metric Correction Linearized Gravity:
$ds^2 = -(1 + 2\Phi) c^2 dt^2 + (2w_i) dt dx^i + [2s_{ij} + (1 - 2\Psi)\delta_{ij}] dx^i dx^j$

Deflection of Light:
$\Delta\alpha = \frac{4GM}{c^2 b}$

Corrected Metric in the Transverse-Traceless Case:
$ds^2 = -c^2 dt^2 + (2s_{ij} + \delta_{ij}) dx^i dx^j$

Gravitational Waves:
$h^{TT}_{\mu\nu} = C_{\mu\nu} e^{ik_\sigma x^\sigma}$, $C_{11} = h_+$, $C_{22} = -h_+$, $C_{12} = C_{21} = h_\times$, $k^\mu = (\omega, k^1, k^2, k^3)$

Friedmann–Lemaître–Robertson–Walker (FLRW) Metric:
$ds^2 = -c^2 dt^2 + a^2(t)\gamma_{ij}$, $\gamma_{ij} = \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$, $a(t)$ being dimensionless.

Friedmann Equations:
$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$ and $\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P)$ where $H(t) = \frac{\dot a(t)}{a(t)}$, $P = w\rho$
$\dot\rho = -3H(\rho + P)$
$\rho = \sum_{i=1} \rho_i + \rho_\Lambda$, $P = \sum_{i=1} P_i + P_\Lambda$, $\rho_\Lambda = -P_\Lambda = \frac{\Lambda}{8\pi G}$, $\Omega_i = \frac{\rho_i}{\rho_{cr}}$, $\rho_{cr} = \frac{3H^2}{8\pi G}$

Tortoise coordinate:
$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$

Kruskal coordinates:
$T = \frac{1}{2}(v' + u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right)$
$R = \frac{1}{2}(v' - u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right)$
$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dR^2)$, with u' and v' $ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM}(dv'du' + du'dv')$