Notes of General Relativity

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Contents

Chapter 1

Introduction

Lecture 1

General Relativity describes *gravity* in terms of *curvature* of *space-time*.

We will define and describe those three words.

To understand *curvature*, let's think about a RF in a flat space, so that the sum of all internal angles of a triangle is 180°, as we add curvature, the sum increase its value.

Sphere is a 2D *manifold*. What is a manifold?

From Newton to Einstein

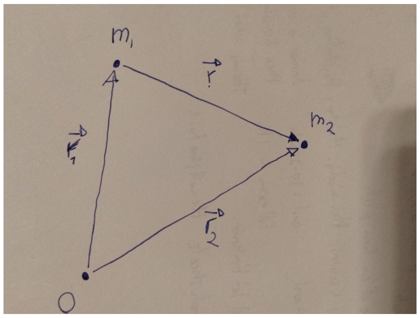
We got two masses, m_1, m_2, the origin, O, of the RF.
Each mass' position is identified by its own position vector.

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$
$$\vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r}$$
$$\text{with } \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

so, we see that m_2 is attracted.
P.S. $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

Introducing the second law of dynamics in the study, we have

$$m_2\vec{a}_2 = \vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r}$$
simplifying m_2 we obtain
$$\vec{a}_2 = -\frac{Gm_1}{r^2}\hat{r}$$



We can express $\mathbf{a}_2$ as

$$\vec{a}_2 = -\nabla\phi \text{ Gradient of the Gravitational Potential}$$
$$\phi = -\frac{Gm_1}{r}$$
$$\nabla^2\phi = -4\pi G\rho$$

We will use the Minkowski metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{1.1}$$

We will see also other symbols, like the Kristoffel one, or the Richie Tensor...
But in the end the central goal is to derive the *Einstein Equation*:

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.2}$$

In GR particles move freely along *straight lines* of a curved space-time. These are called *geodesics*.

Example Two chalks, one on the desk, the other is launched in the air. Which one is accelerated? From a GR perspective, the one in the air is moving along a geodesic, so it is the one moving freely, while the other is stopped from doing that by some interference/force.
In GR gravity is *not* a force.

Chapter 2

Math tools

2.1 A recap of SR

Lecture 2 We will develop some of the necessary math on this framework.
Let's look at the Galilean Relativity.
Newtonian dynamics is based on three principles

1. inertia
2. $\vec{F} = m\vec{a}$
3. action-reaction

The first says something like *An object at rest remains at rest, and an object in motion remains in motion at constant speed and in a straight line unless acted on by an unbalanced force.*
The second one says:

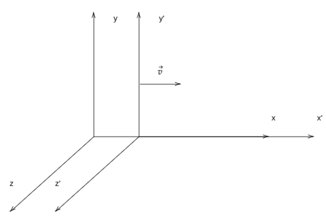
$$(2) : \vec{F} = 0 \implies \vec{a} = 0 \implies (1)$$

So, it seems the first principle is contained by the second, but we know that $\vec{F} = m\vec{a}$ is valid only in Inertial Frames (IF).

Galilean Relativity: all the laws of *mechanics* take the same form in every IF. (You can not distinguish two IF just by doing experiments.)

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

$t = t' = 0 \implies O = O'$



Taking the first derivative:

$$\begin{cases} v'_x = v_x - v \\ v'_y = v_y \\ v'_z = v_z \end{cases} \quad \text{and for the second derivative:} \quad \begin{cases} a'_x = a_x \\ a'_y = a_y \\ a'_z = a_z \end{cases} \implies \vec{a}' = \vec{a} \tag{2.1}$$

so also $\vec{F}' = \vec{F}$. And if m is independent on the frame, we got

$$\vec{F}' = m\vec{a}' = \vec{F} = m\vec{a} \tag{2.2}$$

Then there are Maxwell equations, people thanks to them find that EM-waves propagates with speed c in the void.
But they found also that these equations were not invariant in Galilean Boosts. Things started to go better when the idea of a preferred IF was ditched and Einstein decided to use Lorentz Transformations.

There are two postulates:

- *Relativity principle*: same as before but with *physics* instead of *mechanics*. **All the laws of physics ...**
- *Speed of light*: in every IF, light propagates with constant speed, c.

So we see that Galilean transformation become inconsistent with this, meanwhile stays valid for $\vec{v} \ll \vec{c}$.
As mentioned before, updated version of G. Boosts are Lorentz transformations (or Lorentz Boosts.)

$$\begin{cases} x' = \frac{x - vt}{\sqrt{1-(\frac{v}{c})^2}} \\ y' = y \\ z' = z \\ t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-(\frac{v}{c})^2}} \end{cases} \tag{2.3}$$

To ensure the L.T. Is consistent we can perform three checks:

- $v \ll c$

- v = 0
- dimensional check

People use a notation to make the L.T. easier to write: $\gamma(v) \equiv \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$, so it becomes

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{vx}{c^2}) \end{cases} \tag{2.4}$$

What happens to the transformation of velocity is: (v is fixed)

$$\begin{cases} dx' = \gamma(dx - vdt) \\ dy' = dy \\ dz' = dz \\ dt' = \gamma\left(dt - \frac{vdx}{c^2}\right) \end{cases} \tag{2.5}$$

so

$$\begin{cases} v'_x = \frac{dx'}{dt'} \\ v'_y = \frac{dy'}{dt} = \frac{dy}{\gamma\left(dt - \frac{vdx}{c^2}\right)} = \frac{v_y}{\gamma\left(1 - \frac{vv_x}{c^2}\right)} \\ v'_z = \frac{dz'}{dt} = ... \end{cases} \tag{2.6}$$

So we see that space-time changes also along other axes.

Now let's talk about space-time and its parts.

Space-time space-time is a manifold. For now it is a collection of (t,x,y,z), four dimensional set of all the possible values of the coordinates.

Event a point of space-time.

World line path of a particle in space-time.

There is no notion of absolute time anymore, because now it is dependent on the frame. Regarding the light-cone, after the event on the (x,y) plane, the particle can move *only* inside the light-cone, in the appropriate direction (time forward).

Now let's talk about **Clock Synchronization.**
It is kinda easy if in in IF. In GR it is quite subtle instead.

Example: Be me in Origin of a RF watching my clock (A). How to define t at another generic location (B)??

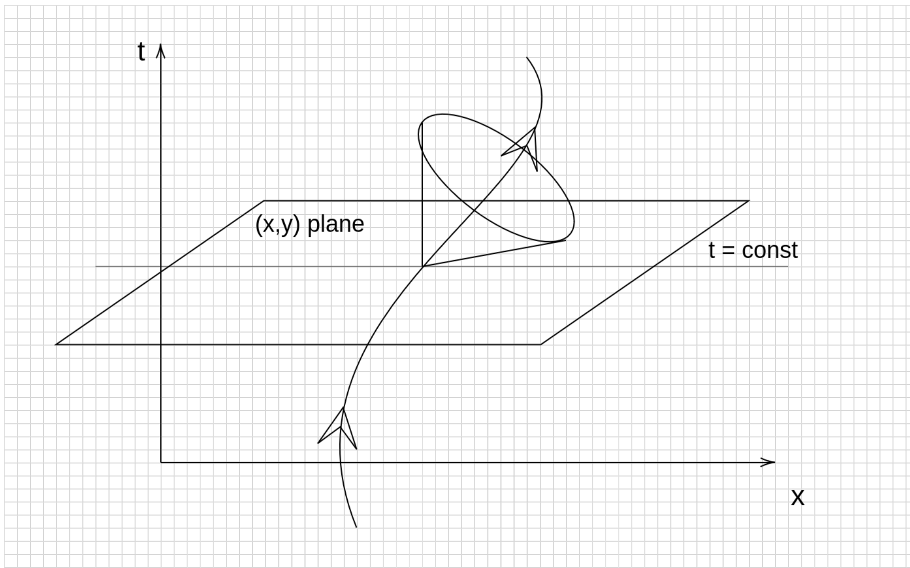


Figure 2.1: LL of a particle which moves forward in time, we see also a light cone

I send a light ray at time t_1 to B. I get the answer on t_2. There is symmetry between the two trajectories so

$$t_m = \frac{t_1 + t_2}{2}.$$

I say to my friend on B: "set your clock to t_m when you receive the signal." So, following this methodology, each point could have its own clock.

Proper time: How to define proper time?
t is the time coordinate. Let's introduce the metric tensor:

$$\text{the Minkowski metric tensor: } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

for a Lorentz Transformation if I have 2 events E,F.

$$\text{Frame 1: } x_F^\mu = (t_F, x_F, y_F, z_F)$$
$$x_E = (...)$$
$$\text{Frame 2: } x_F^{\mu'} = (t_{F'}, x_{F'}, y_{F'}, z_{F'})$$
$$x_E^{\mu'} = (...)$$

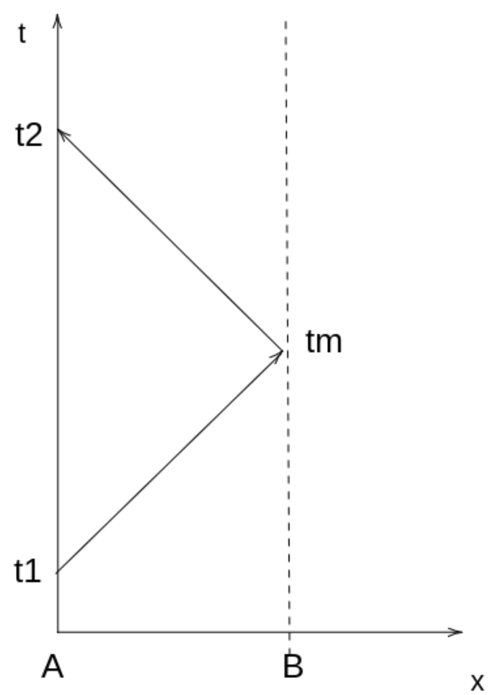


Figure 2.2: Reception and send of the signal

same events in 2 different frames.
A Lorentz Transformation cornets these two events.

Be Δs^2 the Lorentz Invariant separation between E-F.

$$\Delta s^2 = -c\,(t_F - t_E)^2 + (x_F - x_E)^2 + (y_F - y_E)^2 + (z_F - z_E)^2 =$$
$$= -c(t_{F'} - t_{E'}) + (x_{F'} - x_{E'}) + (y_{F'} - y_{E'}) + (z_{F'} - z_{E'})$$
$$\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$
$$\text{From this point we set } c = 1 \text{ just a rescaling}$$
$$\text{we have defined } \Delta x^\mu \equiv x_F^\mu - x_F^\mu, \text{with } \mu = 0, 1, 2, 3.$$

So, repeating for clarity, the Lorentz Invariant separation is

$$\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu x^\nu = \eta_{\mu'\nu'}\Delta x^{\mu'}\Delta x^{\nu'} \tag{2.8}$$

Minkowski metric tensor does not change form if we change coordinates (Cartesian coordinates, meanwhile if we use like polar ones it changes for obvious reasons.)

if

$\Delta s^2 >$ 0space-like separation

$<$ time-like, (it could be an actual LL for a massive particle)

$=$ light-like or null

Now we can define the *proper time* as

$$\Delta\tau^2 \equiv -\Delta s^2 \text{ or } \Delta\tau^2 = -\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu \tag{2.9}$$

So, if the proper time is *positive* it is time-like.

If the segment **EF** marks the begin and end of the trajectory of a massive particle, $\Delta\tau$, proper time, is the time elapsed on a clock sitting on a RF that moves with constant speed between E and F.
Int the moving frame $\Delta\tau = \Delta t_*$ where $t*$ is the time coordinate of the moving frame. In a frame where I'm at rest this is how Δt^2 changes:

$$\Delta\tau^2 = +\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2. \tag{2.10}$$

2.2 Lecture 3

The meaning of the Lorentz Invariant is that **events**, like (E, F) exist before I define coordinates. It is a property of the two events.

So to recap what we did in the last lecture, be:

$$x_E^\mu \text{ and } x_E^{\mu'} \tag{2.11}$$

If I have two events and computing $\Delta\tau$ gives a positive result, the separation is **time-like**. This means that they could be on the WL of a massive particle moving at constant speed.

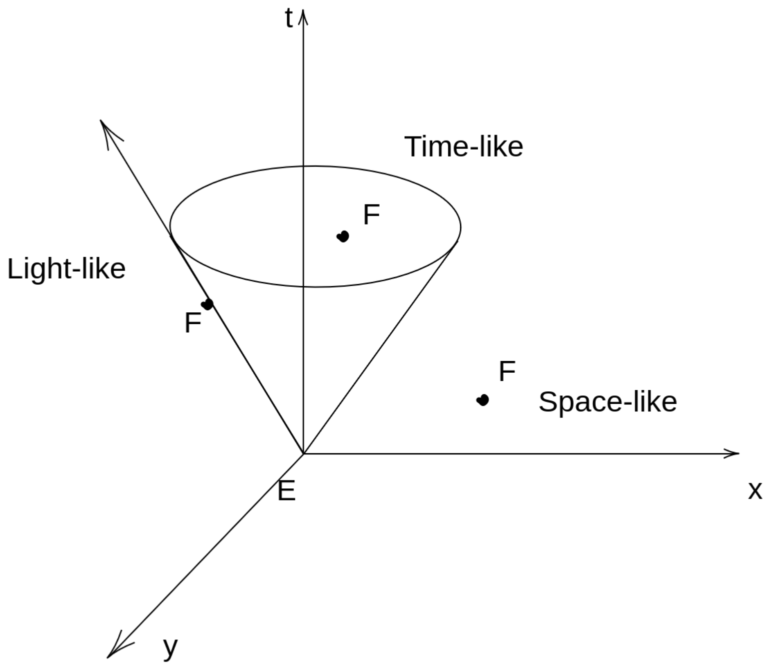


Figure 2.3: Given event E, the separation **EF** could be of different types based on the position respect the light cone

Physical meaning of $\Delta\tau$ It's the time elapsed on a clock of the observer moving between E and F at constant speed.

This means that if I compute $\Delta\tau$ on the frame where the observer it is at rest, i get

$$\Delta\tau = \Delta t'$$

Lets do an example:

Example In fig. 2.4 we see the straight line **ABC** that is the WL of a object not moving. Computing its proper time will be:

$$\Delta\tau_{ABC} = (t_c - t_A) \tag{2.12}$$

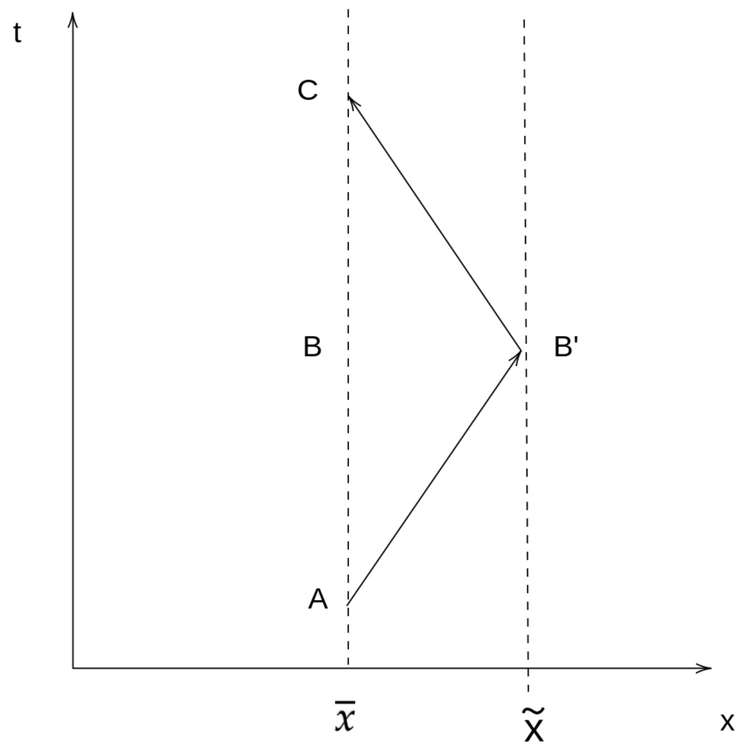


Figure 2.4: It is like the twin paradox.

But for the other WL, of a object moving at constant speed between **AB'** and **B'C**, first thing first, we see that

$$t_B = t_{B'}$$

and so

$$\Delta\tau_{AB'C} = 2\sqrt{(t_B - t_A)^2 - (\tilde{x} - \bar{x})^2} = \Delta\tau_{ABC}\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

$$\implies \Delta\tau_{AB'C} < \Delta\tau_{ABC}$$

This means that I have the longest **proper time** when I don't move.

We can do one more generalization: by parametrize the WL with a quantity λ we get

$$x^\mu(\lambda)$$

$$\Delta\tau = \int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \text{ that is a time like trajectory.}$$

Enough with proper time.

2.2.1 Tensor Calculus

Be a Lorentz Group, we want to look for the transformations.

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}_\mu x^\mu \tag{2.13}$$

we see that it is a linear transformation. An example to see better what are we doing could be

$$x^{0'} = \Lambda^{0'}_0 x^0 + \Lambda^{0'}_1 x^1 + \Lambda^{0'}_2 x^2 + \Lambda^{0'}_3 x^3 \tag{2.14}$$

What we need to know is that $\Lambda^{\mu'}_\mu$ is a constant matrix.

We see that Λ is a constant matrix.

We want to find linear transformations such that

$$\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = \eta_{\mu'\nu'}\Delta x^{\mu'}\Delta x^{\nu'} \tag{2.15}$$

So the Lorentz Invariant is still invariant. (WTF)

Now, because a SR property: if I move from IF to another, η is still unchanged. So

$$\eta_{\mu\nu} = \eta_{\mu'\nu'}$$

We have to say that Minkowski assumes cartesian coordinates.

The question now is: What trivial transformations leave Δs^2 unchanged?

Translations

$$\eta_{\mu\nu}\Delta x^{\mu}x^{\nu} = \eta_{\mu'\nu'}\left(\Lambda^{\mu'}_{\mu}\Delta x^{\mu}\right)\left(\Lambda^{\nu'}_{\nu}\Delta x^{\nu}\right)$$
$$\implies \eta_{\mu\nu} = \eta_{\mu'\nu'}\Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}$$
$$\text{this obviously needs to be valid } \forall \Delta x^{\mu}$$
$$\text{an alternative notation could be } \eta = \Lambda^T \eta \Lambda$$

We will use just the first notation, because we need to get good at tensors.

To be more concrete:

$$\Lambda^{\mu'}_{\mu} = \begin{pmatrix} \Lambda^{0'}_{0} & \Lambda^{0'}_{1} & \Lambda^{0'}_{2} & \Lambda^{0'}_{3} \\ \Lambda^{1'}_{0} & ... & ... & ... \\ \Lambda^{2'}_{0} & ... & ... & ... \\ \Lambda^{3'}_{0} & ... & ... & ... \end{pmatrix} \tag{2.16}$$

Rotations Rotations are a kind of transformation of the type:

$$x_{i'} = R_{ii'}x_i$$
$$\text{or } R^T \mathbb{I} R = \mathbb{I}$$
$$\text{with } RR^T = R^T R = \mathbb{I}$$

it could be something like

$$\Lambda^{\mu'}_{\mu} = \begin{pmatrix} cosh\eta & -sinh\eta & 0 & 0 \\ -sinh\eta & cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.17}$$

this one is a boost along the x direction. If we do some computing we find that

$$tanh\eta \equiv v$$

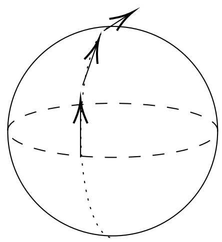
so this is the same of the L.T. we saw last week.

Rotations do not change the time coordinate. The point was to tell what L.T. is in this language.

Vectors I have a generic vector, **do i need to specify about the RF** where it is defined, so in a specific spacetime location? yes

In newtonian mechanics parallel vectors are the same because I can superpose them, I can move them around, also to use the parallelogram rule to get a sum. $\implies$ If I have 3D euclidean space there is no ambiguities about where i move my

vectors.
BUT in a sphere:



I have this vector at the equator tangent to the surface. If I transport it to the pole i get a different vector.

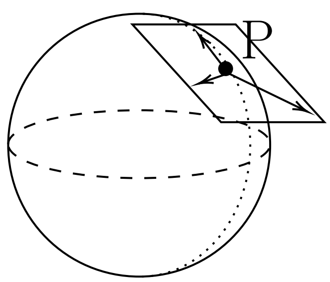
There are ambiguities. So in a non-flat space we need a **different** procedure.
A vector field is a map between:

$$x^\mu \to v^\mu$$

where x^μ is an event and v^μ is a vector.

Let's define: **Tangent space $\mathbf{T_P}$.**
Given an event P we define the tangent space T_P as all the vectors in P.

Instead of having spacetime we have a sphere.



Define a plane tangent to the sphere only in P. All vectors that lie there $\in T_P$.

T_P is a **vector space**:

$$V, W \in T_P \implies \alpha V + \beta W, (\alpha, \beta \in \mathbb{R}) \in T_P$$

So if there is a vector there is also the inverse vector.
Whenever i have a vector space, I can define infinite basis independently on the coordinate choice. The nUmber of elements in the basis is equal to the dimension of the space, in our case 4 elements.
Obviously if I define the basis its elements need to be Linearly Independent.

Basis Given a generic vector $V \in T_P$, I can define V regardless the coordinate system I'm using. So we can say *meTaphorically* that V exists before I define coordinates.

Be our basis:

$$\hat{e}_{(\mu)}, \text{ with } \mu = 0, 1, 2, 3$$

those indices are label, does not mean "tensor". So my basis is made of

$$\hat{e}_{(0)}, \hat{e}_{(1)}, \hat{e}_{(2)}, \hat{e}_{(3)}$$

Now we can talk about

Components given a generic vector v

$$V = V^0 \hat{e}_{(0)} + V^1 \hat{e}_{(1)} + V^2 \hat{e}_{(2)} + V^3 \hat{e}_{(3)} = V^\mu \hat{e}_{(\mu)}$$

using repeating indices we get the last equivalence.

V^μ are components of the vector V in this specific frame.
In another frame $V^{\mu'}$ could not be the same:

$$V = V^\mu \hat{e}_{(\mu)} = V^{\mu'} \hat{e}_{(\mu')}$$

Question: how do components transform?

covariant vector : is a math object whose components transform based on position

$$V^{\mu'} = \Lambda^{\mu'}_{\mu} V^\mu$$

These are not the only covariant vectors (?).

If you have a generic WL or path, you can parametrize the position by a λ in this way:

$$x^\mu (\lambda)$$

And taking its first derivative you get something similar to the four-velocity

$$u^\mu \sim \frac{dx^\mu}{d\lambda}$$

(I say similar because four-velocity is defined like $u^\mu = \frac{dx^\mu}{d\tau}$).

If I do a L.T. x^μ will change but λ won't.

$$u^{\mu'} = \Lambda^{\mu'}_{\mu} u^\mu$$

I can get a more general definition of what a vector is by following this procedure: choose basis $\rightarrow$ find components $\rightarrow$ study how components change if i change position or basis.

Second definition : Transformation of the basis vectors. The question is "how to relate $\hat{e}_{(\mu)}$ to $\hat{e}_{(\mu')}$?"

We will take advantage of **invariance**.

$$V = V^\mu \hat{e}_{(\mu)} = V^{\mu'} \hat{e}_{(\mu')} = \left(\Lambda^{\mu'}_{\mu} V^\mu \right) \hat{e}_{(\mu')}$$

That's possible **only** if $\hat{e}_{(\mu)} = \Lambda^{\mu'}_{\mu} \hat{e}_{(\mu')}$.

An inverse of LT it is also a LT, so

$$\Lambda^{\mu'}_{\mu}\Lambda^{\mu}_{\nu'} = \delta^{\mu'}_{\nu'}$$
$$\Lambda^{\mu}_{\mu'}\Lambda^{\mu'}_{\nu} = \delta^{\mu}_{\nu}$$

Those are Kroneker's delta and they are an Identity matrix.

Now we can study how basis vectors change.

$$\hat{e}_{(\mu)} = \Lambda^{\mu'}_{\mu}\hat{e}_{(\mu')}$$
$$\Lambda^{\mu}_{\nu'}\hat{e}_{(\mu)} = \Lambda^{\mu'}_{\mu}\Lambda^{\mu}_{\nu'}\hat{e}_{\mu'}$$
$$\Lambda^{\mu}_{\nu'}\hat{e}_{(\mu)} = \delta^{\mu'}_{\nu'}\hat{e}_{(\mu')}$$
$$\Lambda^{\mu}_{\nu'}\hat{e}_{(\mu)} = \hat{e}_{(\nu')}$$
$$\text{so } \hat{e}_{(\nu')} = \Lambda^{\mu}_{\nu'}\hat{e}_{(\mu)}$$

2.3 Lecture 4

Brief recap of lec3

We defined vectors

- localized at each spacetime point
- for each event P we defined the tangent space T_P
- there is linear combination inside T_P
- it has a basis
- Vectors and basis transform under LT Group.

Dual vectors

Using old terminology they are covariant, so with lower indices. Meanwhile contravariant do have upper indices.

Let's start with defining the **dual space** of a vector space: *Given a vector space (for concreteness T_P)*, we define the **dual space** T_P^* as the space of linear maps between T_P and $\mathbb{R}$.

Example Being $\omega \in T_P^*$, $V \in T_P$ then

$$\omega\left(V\right) \in \mathbb{R}$$

Linearity tells me that

$$\omega\left(\alpha V + \beta W\right) = \alpha\omega\left(V\right) + \beta\omega\left(W\right)$$

1$^{\text{st}}$ statement : The dual space is a vector space.

$$\left(\alpha\omega + \beta\eta\right)\left(v\right) = \alpha\omega\left(v\right) + \beta\eta\left(v\right)$$

2$^{\text{nd}}$ statement : What is the dual of the dual?

$$\left(T_P^*\right)^* = T_P \implies v\left(\omega\right) = \omega\left(v\right) \in \mathbb{R}$$

Basis for $\mathbf{T_P}^*$: $\hat{o}^{(\mu)}$.
How to define this? Definition is

$$\hat{o}^{(\mu)}\left(\hat{e}_{(\nu)}\right) \equiv \delta^\mu_\nu$$

Now let's see if we can get how dual vectors work with vectors. If I have:

- generic item of T_P: $V = V^\nu \hat{e}_{(\nu)}$
- generic item of T_P^* : $\omega = \omega_\mu \hat{o^{(\mu)}}$

I can compute:

$$\omega(v) = \omega_\mu \hat{o}^{(\mu)} \left(v^\nu \hat{e}_{(\nu)}\right) =$$
$$= \omega_\mu v^\nu \hat{o}^{(\mu)} \left(\hat{e}_{(\nu)}\right) = \omega_\mu v^\nu \delta^\mu_\nu = \omega_\mu v^\mu$$

Once we know this we can do an **exercise**: show the way $\omega_{\mu'}$ transform. What to do is to start from Λ equality.

What is the example of a dual vector?

$$A_{\mu'} = \Lambda^{\mu'}_{\mu} A_\mu$$

the gradient is a beautiful example of a *dual vector*.

$$A_\mu = \frac{\partial \phi}{\partial x^\mu} \; ; \; A_{\mu'} = \frac{\partial \phi}{\partial x^{\mu'}}$$

This is useful to define LTs, in this way

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial \phi}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^{\mu'}} \to A_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} A_\mu$$

the LT is the last partial derivative.

There is a *more compact* notation tho write partial derivatives that is

$$\partial_\mu \phi \equiv \frac{\partial \phi}{\partial x^\mu}$$

2.3.1 Tensors

Tensors are generalization of dual vectors and vectors.
They are *multilinear maps*, i.e. functions of several variables and linear for all of them. For each tensor of *rank* (k,l), we have

$$T^*_P \times \ldots \times T^*_P \times T_P \times \ldots \times T_P \to \mathbb{R}$$

Where each dual vector space is present **k**-times, and vector space **l**-times.
Now let's see what is multilinearity on the combat field.

Be a (1,1) tensor:

- $\alpha, \beta, \gamma, \delta \in \mathbb{R}$
- $\omega, \eta \in T^*_P$
- v,w $\in$ $\mathrm{T_P}$

Given these we have

$$T(\alpha\omega + \beta\eta, \gamma v + \delta w) = = \alpha\gamma T(\omega, v) + \beta\delta T(\eta, w) + \alpha\delta T(\omega, w) + \beta\gamma T(\eta, v) \tag{2.18}$$

Once we have this general definition, let's take one step back:

- Scalar $\rightarrow$ (0,0) tensor
- Vector $\rightarrow$ (1,0) tensor
- Dual vector $\rightarrow$ (0,1) tensor

Tensor product

Be:

- T, rank (k,l) tensor
- S, rank (m,n) tensor

We want to understand the action of $\otimes$.
So we know that $T \otimes S$ outputs (k+m, l+n) tensor. In particular,

$$T \otimes S \left[\omega^{(1)}, \ldots, \omega^{(k)}, \omega^{(k+1)}, \ldots, \omega^{(k+m)}, v^{(1)}, \ldots, v^{(l)}, v^{(l+1)}, \ldots, v^{(l+n)}\right] \equiv$$
$$\equiv T\left(\omega^{(1)}, \ldots, \omega^{(k)}, v^{(1)}, \ldots, v^{(l)}\right) \times S\left(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, v^{(l+1)}, \ldots, v^{(l+n)}\right)$$
$$\implies T \otimes S \neq S \otimes T$$

so tensors do not commute.

Basis for a tensor

Let T be a generic tensor with rank (k,l), *basis* is given by

$$\hat{e}_{(\mu_1)} \otimes \ldots \otimes \hat{e}_{(\mu_k)} \otimes \hat{o}^{(\nu_1)} \otimes \ldots \otimes \hat{o}^{(\nu_l)}$$

A tensor can be written as

$$T = T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l}\left(\hat{e}_{(\mu_1)} \otimes \ldots\right) = T^{\mu_1',\ldots,\mu_k'}_{\nu_1',\ldots,\nu_l'}\left(\hat{e}_{\left(\mu_1'\right)} \otimes \ldots\right)$$

So the tensor is always the same, the thing that changes is its components, because a change of RF I think.

We will often write the components instead of the actual tensor, but it is our convention to think they are equivalent.

This is how the components are related:

$$\hat{e}_{(\mu')} = \Lambda^{\mu}_{\mu'}\hat{e}_{(\mu)}$$
$$\hat{o}^{\left(\mu'\right)} = \Lambda^{\mu'}_{\mu}\hat{o}^{(\mu)}$$
$$\implies T = T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l}\left(\Lambda^{\mu_1'}_{\mu_1}\hat{e}_{\left(\mu_1'\right)} \otimes \ldots\right)$$

So we find, as result, that when I change frame

$$T^{\mu_1',\ldots,\mu_k'}_{\nu_1',\ldots,\nu_k'} = \Lambda^{\mu_1'}_{\mu_1} \ldots \Lambda^{\nu_1}_{\nu_1'} \ldots T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l} \tag{2.19}$$

2.4 Lec 5

2.4.1 Transformations

The goal is to find that is this $T^{\mu'_1,\ldots,\mu'_k}_{\nu'_1,\ldots,\nu'_l}$ =?.

$$T = T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l}\left(\hat{e}_{(\mu_1)} \otimes \ldots\right) = T^{\mu'_1,\ldots,\mu'_k}_{\nu'_1,\ldots,\nu'_l}\left(\hat{e}_{(\mu'_1)} \otimes\right) \tag{2.20}$$

I know two facts:

$$\begin{cases} \hat{e}_{\mu'} = \Lambda^{\mu}_{\mu'}\hat{e}_{(\mu)} \\ \hat{o}^{\mu'} = \Lambda^{\mu'}_{\mu}\hat{o}^{\mu} \end{cases} \tag{2.21}$$

and also the *inverse.*

So i apply the Lambda transformation to each term of the basis and I get the following

$$T^{\mu'_1,\ldots,\mu'_k}_{\nu'_1,\ldots,\nu'_l} = \left(\Lambda^{\mu'_1}_{\mu_1} \ldots \Lambda^{\mu'_k}_{\mu_k}\right)\left(\Lambda^{\nu_1}_{\nu'_1} \ldots \Lambda^{\nu_l}_{\nu'_l}\right)\left(T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l}\right) \tag{2.22}$$

that is something that was obvious by looking at indexes.

2.4.2 Tensor Manipulations / Operations

We defined (k, l) vectors as a multilinear map from dual spaces and vector spaces to real numbers, but it is not only that. For example a $(1, 1)$ tensor could be a map from vectors to vectors, in this way

$$V^{\mu} \to A^{\mu}_{\nu}V^{\nu} \tag{2.23}$$

so if i do not saturate all the indices, i get a tensor of rank made by what remains. If we saturate, we get real numbers or (0,0) tensors.

There are some objects that are well known in flat spacetime.

Particular Tensor in flat ST

These are

- $\eta_{\mu\nu}$ metric, or metric tensor
- $\eta^{\mu\nu}$, inverse metric
- $\delta^{\mu}_{\nu}, kronecker's\delta$
- $\epsilon_{\mu\nu\rho\delta}$, totally anti-symmetric tensor of Levi-Civita

This last one is defined:

$$\begin{cases} +1 \text{ if } (0,1,2,3) \text{ or even permutations} \\ -1 \text{ if odd permutations} \\ 0 \text{ otherwise} \end{cases} \tag{2.24}$$

These are the only tensors of the flat spacetime that their components do not depend on the RF.

Other operations

Contraction

$$(k,l) \to (k-1, l-1)$$

Example: I have (3,2) tensor $T^{\mu\nu\rho}_{\delta\gamma} \to (2,1)$?? We contract:

$$T^{\mu}{}_{\delta}{}^{\nu}{}_{\gamma}{}^{\rho} \to T^{\mu}{}_{\delta}{}^{\nu}{}_{\nu}{}^{\rho} \equiv A^{\mu\rho}_{\delta}$$

Obviously I can *only* contract an upper with a lower index.
It is very important the order, and which indices we contract.

$$T^{\mu}{}_{\delta}{}^{\nu}{}_{\nu}{}^{\rho} \neq T^{\mu\nu}{}_{\nu}{}^{\rho}{}_{\delta}$$

What is the actual operation we perform?

$$T^{\mu\nu\rho}_{\delta\gamma} = \delta^{\gamma}_{\nu} T^{\mu\nu\rho}_{\delta\gamma}$$

Raising/Lowering Indices To raise we use $\eta^{\mu\nu}$, to lower $\eta_{\mu\nu}$.

$$\eta^{\rho\alpha} T^{\mu\nu}_{\alpha\beta} \equiv T^{\mu\nu\rho}_{\beta}$$

$$\eta^{\rho\beta} T^{\mu\nu}{}_{\alpha\beta} \equiv T^{\mu\nu}_{\alpha}{}^{\beta}$$

The order is important, and wring by hand one should be careful keeping the position moving up and down the indices.

Simple operations:

$$V^{\mu} \to V_{\mu} = \eta_{\mu\nu} V^{\nu}$$
$$V_{\mu} \to V^{\mu} = \eta^{\mu\nu} V_{\nu}$$

Inner Product

$$T_P \times T_P \to \mathbb{R}$$

$$(V, W) \to \eta_{\mu\nu} V^{\mu} V^{\nu}$$

Symmetry Properties

Let's consider a (0,2) tensor $T_{\mu\nu}$, or to be precise, its components. It is symmetric? Anti-symmetric? Both? None?

A tensor is *symmetric* if

$$T_{\mu\nu} = T_{\nu\mu}$$

it is *anti-symmetric* if

$$T_{\mu\nu} = -T_{\nu\mu}$$

It is **never** possible to have a tensor that is *both*. But really possible that is *none* of the above.

We can *symmetrize* a tensor:

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$$

We can *anti-symmetrize* a tensor:

$$T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$$

A tensor can be symmetric on all indices, so it *totally symmetric*, or just on some indices, like two, three etc. The general formula can be:

$$T_{(\mu\nu\rho)} = \frac{1}{3!}\left(T_{\mu\nu\rho} + \text{ all permutations }\right)$$

For anti-simmetrizing odd permutations get the minus in front.

Trace

$$x^{\mu}_{\mu}$$

given a (1,1) tensor $\to \mathbb{R}$ by summing all indices. For example the trace of metric tensor is 2. Of Kronecker delta is 4.

2.5 Lec 6

2.5.1 Energy & momentum

Since out goal is to get to the Einstein Equation, we know that in there there should be the *energy momentum tensor* $T^{\mu\nu}$.
As always we will study everything for a flat space-time but it will be useful for non flat ones.
We already saw the four-velocity u^μ:

$$u^\mu \equiv \frac{dx^\mu}{d\tau}$$

while the proper time is $\Delta\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$.
We need to make clear that we are talking about a time-like space-time trajectory, so $\Delta s^2 < 0$.
Let's start with the WL of a single particle, this is specified by a map $\mathbb{R} \to M$, where M is a manifold that represents spacetime. We usually think the path as a curve parameterized by λ so $x^\mu(\lambda)$.
We also use as parameter the τ so $x^\mu(\tau)$, this has some advantages because maybe it could be easier to switch to four-velocity.

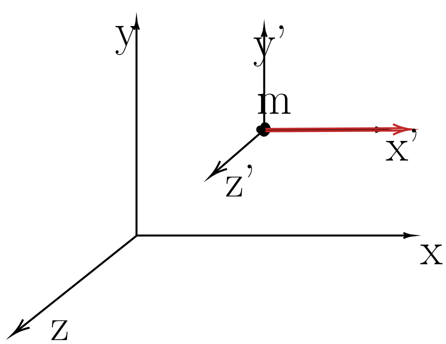
$$u^\mu u_\mu = u_\mu u^\mu = \eta_{\mu\nu}u^\mu u^\nu = -1 \tag{2.25}$$

By the way, four-velocity is what we need to find the *four momentum*:

$$p^\mu \equiv m u^\mu \tag{2.26}$$

where m is the rest mass that has the same values $\forall$ RF, and it's just a number.

So in rest frame (x',y',z'):



So in the rest frame (x',y',z'): $p^\mu = (m, 0, 0, 0)$, because the four-velocity in the rest frame is $u^\mu = (1, 0, 0, 0)$.
What is the expression of p^μ in the (x,y.z) frame?
And what is the fastest way to compute it?

We can start from the rest frame and use a LT.
For a generic four vector we have:

$$\begin{cases} a^{0'} = \gamma\left(a^0 - va^1\right) \\ a^{1'} = \gamma\left(a^1 - va^0\right) \\ a^{2'} = a^2 \\ a^{3'} = a^3 \end{cases} \tag{2.27}$$

Now we find the inverse, we can search the inverse of the matrix or use an inverse LT,

$$\begin{cases} a^0 = \gamma \left(a^{0'} + v a^{1'} \right) \\ a^1 = \gamma \left(a^{1'} + v a^{0'} \right) \\ a^2 = a^{2'} \\ a^3 = a^{3'} \end{cases} \tag{2.28}$$

So for the four-momentum we have:

$$\begin{cases} p^0 = E = \gamma p^{0'} = \gamma m = \frac{m}{\sqrt{1-v^2}} \\ p^1 = m \gamma v = \frac{mv}{\sqrt{1-v^2}} \\ p^2 = 0 \\ p^3 = 0 \end{cases} \tag{2.29}$$

In the NR limit we should be able to recover Newton Mechanics:

$$E \approx m + \frac{mv^2}{2} + \dots$$
$$p^1 \approx mv + \dots$$

The four-momentum as we got it provides the description of a single particle but often we need to study a lot of particles as a continuum, like a *fluid*, characterized by quantities as density, pressure, entropy, viscosity...
A single momentum four-vector field is insufficient to describe the energy and the momentum of a fluid so we go further and define the *energy-momentum tensor*.

2.5.2 Energy-Momentum Tensor

$$T^{\alpha\beta}$$

For now it is just a tensor, and we are happy to see that it transform like a tensor:

$$T^{\alpha'\beta'} = \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} T^{\alpha\beta}$$

In words, it is defined like "*the flux of four-momentum p^α across the surface where x^β is constant*".

For a system of N particles we have:

$$p^\alpha = \sum_{j=1}^{N} p^\alpha_j$$

where j shows the *j-th* particle, not an index to contract. The *number density,* n for a system of N particles is:

$$n = \sum_j \delta \left(\vec{r} - \vec{r}_j \right)$$

So we have these components of the energy-momentum tensor:

$$T^{\alpha 0} = \sum_j p_j^\alpha \frac{dt}{dt} \delta\left(\vec{r} - \vec{r}_j\right)$$
$$T^{\alpha i} = \sum_j p_j^\alpha \frac{dx_j^i}{dt} \delta\left(\vec{r} - \vec{r}_j\right)$$