QUESTION 1. Consider a two-dimensional Cartesian coordinate system (x, y) with the infinitesimal line element $ds^2 = dx^2+dy^2$. We then introduce new coordinates u and v, defined by $u = (x+y)/2$ and $v = (x - y)/2$. Find the components of the metric tensor in the new coordinates (u, v) using the transformation rule for a $(0, 2)$ tensor, which states that $g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$. You should use this method exclusively, without relying on any alternative approaches.

QUESTION 2. Consider the two-dimensional plane in polar coordinates, where the infinitesimal line element is given by $ds^2 = dr^2 + r^2 d\phi^2$. (i) How many independent Christoffel symbols are there in total in two dimensions? (ii) How many independent and non-vanishing Christoffel symbols are there for this particular case? (iii) Compute the explicit form of one non-vanishing Christoffel symbol at your choice.

QUESTION 3. Consider the two-dimensional spacetime where the infinitesimal line element is given by $ds^2 = -(1 + x)^2 dt^2 + dx^2$. (i) How many independent Christoffel symbols are there in principle in two dimensions? (ii) How many independent and non-vanishing Christoffel symbols are there for this example? (iii) Compute the explicit form of Γ^t_{tx} for this example.

QUESTION 4. Consider the two-dimensional space where the infinitesimal line element is given by $ds^2 = (1 + x^2)dx^2 + (1 + y^2)dy^2$. Compute the Christoffel symbols Γ^x_{xx} and Γ^x_{yy}.

QUESTION 5. A good approximation for the metric outside the Earth's surface is given by $ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$, where $\Phi = -GM_E/r$ is the Newtonian potential. Consider two clocks: one located at the Earth's surface (R_E is the Earth's radius) and another on top of a building of height h. Calculate the time elapsed on each clock as a function of the coordinate t. Provide an approximate ratio of these two times in the $h \ll R_E$ limit.

QUESTION 6. Consider the metric for a two-dimensional sphere with unit radius, given by the expression $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. The only non-vanishing Christoffel symbols are $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^\phi_{\theta\phi} = \cos\theta/\sin\theta$ (and the ones related by symmetry). Write down the geodesic equations for this case and verify explicitly that: (i) lines at constant longitude are geodesics; (ii) the only geodesic at constant latitude is the equator.

QUESTION 7. Consider the metric for a two-dimensional sphere with unit radius, given by the expression $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. The only non-vanishing Christoffel symbols are $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^\phi_{\theta\phi} = \cos\theta/\sin\theta$ (and the ones related by symmetry). Consider a vector $V^\mu = (V^\theta, V^\phi)$. Compute the covariant derivatives $\nabla_\mu V^\nu$ and $\nabla_\mu V_\nu$ and give the explicit expressions. Then use these expressions to check explicitly that $\nabla_\mu(V^\nu V_\nu) = \partial_\mu(V^\nu V_\nu)$.

QUESTION 8. Given an arbitrary $(2, 0)$ tensor $W^{\mu\nu}$, what conditions, if any, must be satisfied in order for the identity $[\nabla_\mu, \nabla_\nu]W^{\mu\nu} = 0$ to hold?

QUESTION 9. Consider a photon moving in a two-dimensional spacetime described by the metric $ds^2 = -(1+x^2/\ell^2)dt^2+dx^2$, where ℓ is a constant. The photon follows a geodesic $x^\mu(\lambda)$ parameterized by an affine parameter λ, with its four-momentum given by $p^\mu = dx^\mu/d\lambda$. Using the geodesic equation, derive all conserved quantities for this system. Use this result to express for $dx/d\lambda$ as a function of the radial coordinate and conserved quantities.

QUESTION 10. Starting from the expression of the Schwarzschild metric, compute the explicit expressions for the Christoffel symbols Γ^r_{rr} and $\Gamma^\phi_{r\phi}$.

QUESTION 11. For geodesic motion in the Schwarzschild metric, the following two quantities are conserved: $E = (1 - 2GM/r)\, dt/d\lambda$ and $L = r^2\, d\phi/d\lambda$. Determine the smallest allowed radius for a stable circular orbit for a massive object.

QUESTION 12. For geodesic motion in the Schwarzschild metric, the following two quantities are conserved: $E = (1 - 2GM/r)\, dt/d\lambda$ and $L = r^2\, d\phi/d\lambda$. Consider the circular orbits for a massive object and determine the radius of the stable orbit in the $L \to \infty$ limit. Compare this result with the corresponding Newtonian result.

QUESTION 13. For geodesic motion in the Schwarzschild metric, the following two quantities are conserved: $E = (1 - 2GM/r)\, dt/d\lambda$ and $L = r^2\, d\phi/d\lambda$. Discuss all possible circular orbits for a massless object and determine their stability.

QUESTION 14. For geodesic motion in the Schwarzschild metric, the following two quantities are conserved: $E = (1 - 2GM/r)\, dt/d\lambda$ and $L = r^2\, d\phi/d\lambda$. For this problem, we set our units such that $GM = 1$ and consider the motion of a photon with angular momentum $L = 10$. Find the minimum value of E such that the photon can reach the singularity at $r \to 0$.

QUESTION 15. Consider the linear regime where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. First, define the vector $V_\mu = \partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h^\nu_\nu$. Consider the gauge transformation described by the function $\xi_\alpha(x^\mu)$. (i) What happens to the vector V_μ after such a gauge transformation? (ii) Is it possible to find a gauge where V_μ is vanishing?

QUESTION 16. Consider a plane wave solution in the TT gauge $h^{TT}_{\mu\nu} = \mathrm{Re}\,(C_{\mu\nu} \exp[ik_\sigma x^\sigma])$ propagating along the positive direction of the z-axis. (i) What are the explicit components of k^μ? (ii) What is the explicit form of $C_{\mu\nu}$ for the + polarization? (iii) Take the plane wave solution for the + polarization and determine its form after a gauge transformation parameterized by the function $\xi^\mu = (\alpha(x^\mu), \beta(x^\mu), 0, 0)$ with $\alpha(x^\mu)$ and $\beta(x^\mu)$ generic functions of the spacetime coordinates.

QUESTION 17. The general TT gauge plane wave solution results in $h^{TT}_{\mu\nu} = \mathrm{Re}\,(C_{\mu\nu} \exp[ik_\sigma x^\sigma])$. We start by considering a Cartesian coordinate system (x, y, z) and $k^\mu = (\omega, 0, k, k)$. (i) What is the relation between ω and k? (ii) Define now a new Cartesian coordinate system (x', y', z') where $x = x'$, z' is the propagation direction, and y' is defined such that the new system is a right-handed system. We consider the plane wave solution characterized by a superposition of a "+ polarization" with amplitude h'_+ and a "× polarization" with amplitude $h'_\times$ defined in the new coordinate system. Write down the explicit expression for the amplitude tensor $C_{\mu\nu}$ valid in the original Cartesian system.

QUESTION 18. Assume that the Friedmann equation $H^2 = 8\pi G\rho/3 - \kappa/a^2$ and the fluid equation $\dot\rho + 3H(\rho + p) = 0$ are valid. Without any further assumption, derive the result $\ddot a/a = -4\pi G(\rho + 3p)/3$.

QUESTION 19. Assume that the universe is spatially flat ($\kappa = 0$) and that is filled with a single fluid with equation of state $p = (2/3)\rho$. Let t_0 be the current time, $a_0 = 1$ the current value of the scale factor, ρ_0 the current value of the energy density. Derive the following quantities and provide the answers in terms of the variables defined in this exercise : (i) how the energy density $\rho(a)$ depends on the scale factor; (ii) how the scale factor $a(t)$ depends on time; (iii) how the Hubble parameter $H(t)$ depends on time; (iv) how the energy density $\rho(t)$ depends on time.

QUESTION 20. Consider a spatially flat universe ($\kappa = 0$) with a non-vanishing cosmological constant Λ and containing only non-relativistic matter with energy density $\rho_M(a)$. Let t_0 be the current time, $a_0 = 1$ the current value of the scale factor, ρ_0 the current value of the energy density, and H_0 the current value of the Hubble parameter. Define $\Omega_M(a) \equiv \rho_M(a)/\rho_{\rm cr}(a)$, where $\rho_{\rm cr}(a)$ is the critical energy density. (i) Derive an expression for $\Omega_M(a)$ as a function of the scale factor a in terms of the data of the problem. (ii) With the additional information $\Omega_0 = 0.3$, derive the value of the scale factor a_Λ that corresponds to the moment when the universe began accelerating.