Notes of General Relativity

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From the lectures of
prof. Francesco D'Eramo.
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With additions and comparisons to Carroll's book.

These notes rely heavily on Carroll's textbook, but I think they could still be useful as they are tailored to prof. D'Eramo's lessons.

If you find these notes helpful, feel free to use and share them. On the other hand, if you come across any errors or inaccuracies, I'd appreciate it if you could let me know.

Disclaimer: I don't guarantee the accuracy of the content in these notes.

Source for these: `github.com/BelliLuigi/RG`
Mail: student unipd mail w/o numbers.

Have fun :)

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Chapter 1

Introduction

1.1 Lecture 1

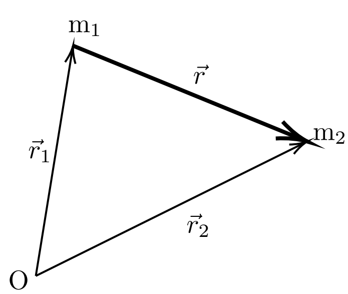
General Relativity describes *gravity* in terms of *curvature* of *space-time*.

We will define and describe those three words.

To understand *curvature*, let's think about a RF, reference frame, in a flat space, so that the sum of all internal angles of a triangle is 180°, as we add curvature, the sum increase its value.

Sphere is a 2D *manifold*. What is a manifold?

From Newton to Einstein



We got two masses, m_1, m_2, the origin, O, of the RF.
Each mass' position is identified by its own position vector.

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$

$$\vec{F}_{21} = -\frac{Gm_1 m_2}{r^2}\hat{r}$$

$$\text{with } \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

so, we see that m_2 is attracted.
P.S. $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

Introducing the second law of dynamics in the study, we have

$$m_2\vec{a}_2 = \vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

simplifying m_2

$$\vec{a}_2 = -\frac{Gm_1}{r^2}\hat{r}$$

We can express $\mathbf{a}_2$ as

$$\vec{a}_2 = -\nabla\phi \text{ Gradient of the Gravitational Potential}$$

$$\phi = -\frac{Gm_1}{r}$$

$$\nabla^2\phi = -4\pi G\rho$$

We will use the Minkowski metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{1.1}$$

We will see also other symbols, like the Christoffel one, or the Ricci Tensor... But in the end the central goal is to derive the *Einstein Equation*:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} \tag{1.2}$$

In GR particles move freely along *straight lines* of a curved space-time. These are called *geodesics*.

Example Two chalks, one on the desk, the other is launched in the air. Which one is accelerated? From a GR perspective, the one in the air is moving along a geodesic, so it is the one moving freely, while the other is stopped from doing that by some interference/force.
In GR gravity is *not* a force.

Chapter 2

Math tools

2.1 Lec 2 - A recap of SR

We will develop some of the necessary math on this framework.
Let's look at the Galilean Relativity.
Newtonian dynamics is based on three principles

1. inertia
2. $\vec{F} = m\vec{a}$
3. action-reaction

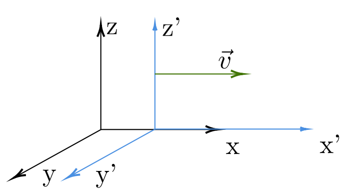
The first says something like *An object at rest remains at rest, and an object in motion remains in motion at constant speed and in a straight line unless acted on by an unbalanced force.*

The second one says:

$$(2) : \vec{F} = 0 \implies \vec{a} = 0 \implies (1)$$

So, it seems the first principle is contained by the second, but we know that $\vec{F} = m\vec{a}$ is valid only in Inertial Frames (IF).

Galilean Relativity: all the laws of *mechanics* take the same form in every IF. (You can not distinguish two IF just by doing experiments.)



$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

$$t = t' = 0 \implies O = O'$$

Taking the first derivative:

$$\begin{cases} v'_x = v_x - v \\ v'_y = v_y \\ v'_z = v_z \end{cases} \quad \text{and for the } 2^{\text{nd}} \text{ derivative: } \begin{cases} a'_x = a_x \\ a'_y = a_y \\ a'_z = a_z \end{cases} \implies \vec{a}' = \vec{a} \tag{2.1}$$

so also $\vec{F}' = \vec{F}$. And if m is independent on the frame, we got

$$\vec{F}' = m\vec{a}' = \vec{F} = m\vec{a} \tag{2.2}$$

Then there are Maxwell equations, people thanks to them found that EM-waves propagates with speed c in the void.

But they found also that these equations were not invariant in Galilean Boosts.

Things started to get better when the idea of a preferred IF was ditched and Einstein decided to use Lorentz Transformations.

There are two postulates:

- *Relativity principle*: same as before but with *physics* instead of *mechanics*. **All the laws of physics ...**
- *Speed of light*: in every IF, light propagates with constant speed, c.

So we see that Galilean transformation become inconsistent with this, meanwhile stays valid for $\vec{v} \ll \vec{c}$.

As mentioned before, updated version of G. Boosts are Lorentz transformations (or Lorentz Boosts.)

$$\begin{cases} x' = \frac{x - vt}{\sqrt{1-(\frac{v}{c})^2}} \\ y' = y \\ z' = z \\ t' = \frac{t - vx/c^2}{\sqrt{1-(\frac{v}{c})^2}} \end{cases} \tag{2.3}$$

To ensure the L.T. Is consistent we can perform three different checks:

- $v \ll c$
- v = 0
- dimensional check

People use a notation to make the L.T. easier to write: $\gamma(v) \equiv \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$, so it becomes

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{vx}{c^2}) \end{cases} \tag{2.4}$$

What happens to the transformation of velocity is: (v is fixed)

$$\begin{cases} dx' = \gamma(dx - vdt) \\ dy' = dy \\ dz' = dz \\ dt' = \gamma\left(dt - \frac{vdx}{c^2}\right) \end{cases} \tag{2.5}$$

so

$$\begin{cases} v'_x = \frac{dx'}{dt'} \\ v'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{vdx}{c^2}\right)} = \frac{v_y}{\gamma\left(1 - \frac{vv_x}{c^2}\right)} \\ v'_z = \frac{dz'}{dt'} = ... \end{cases} \tag{2.6}$$

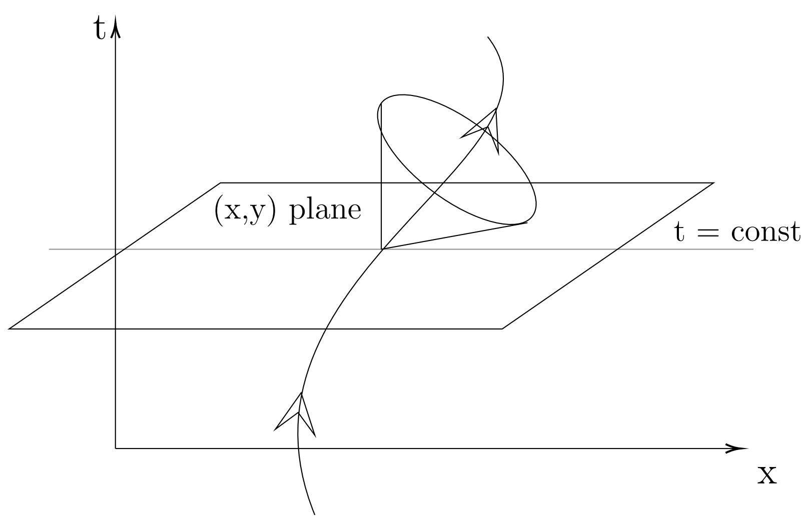
So we see that space-time changes also along other axes.

Now let's talk about space-time and its parts.

Space-time space-time is a manifold. For now it is a collection of (t,x,y,z), four dimensional set of all the possible values of the coordinates.

Event a point of space-time.

World line path of a particle in space-time.



There is no notion of absolute time anymore, because now it is dependent on the frame. Regarding the light-cone, after the event on the (x,y) plane, the particle can move *only* inside the light-cone, in the appropriate direction (time forward). Now let's talk about **Clock Synchronization.**

It is kinda easy if in in IF. In GR it is quite subtle instead.

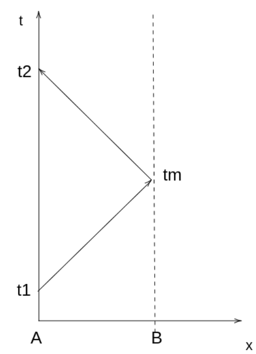


Figure 2.1: Reception and send of the signal

Example: Be me in Origin of a RF watching my clock (A). How to define t at another generic location (B)??

I send a light ray at time t_1 to B. I get the answer on t_2. There is symmetry between the two trajectories so

$$t_m = \frac{t_1 + t_2}{2}.$$

I say to my friend on B: "set your clock to $\mathrm{t_m}$ when you receive the signal." So, following this methodology, each point could have its own clock.

Proper time: How to define proper time?
t is the time coordinate. Let's introduce the metric tensor:

$$\text{the Minkowski metric tensor: } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

for a Lorentz Transformation if I have 2 events E,F.

$$\text{Frame 1: } x^{\mu}_F = (t_F, x_F, y_F, z_F)$$
$$x_E = (...)$$
$$\text{Frame 2: } x^{\mu'}_F = (t_{F'}, x_{F'}, y_{F'}, z_{F'})$$
$$x^{\mu'}_E = (...)$$

same events in 2 different frames.
A Lorentz Transformation connects these two events.

Be Δs^2 the Lorentz Invariant separation between E-F.

$$\Delta s^2 = -c\left(t_F - t_E\right)^2 + \left(x_F - x_E\right)^2 + \left(y_F - y_E\right)^2 + \left(z_F - z_E\right)^2 =$$
$$= -c(t_{F'} - t_{E'})^2 + (x_{F'} - x_{E'})^2 + (y_{F'} - y_{E'})^2 + (z_{F'} - z_{E'})^2$$
$$\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$
$$\text{we have defined } \Delta x^\mu \equiv x^\mu_F - x^\mu_F, \text{with } \mu = 0, 1, 2, 3.$$

From this point onward, we will use $c = 1$, not a big deal, it is just a rescaling.

So, repeating for clarity, the Lorentz Invariant separation is

$$\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu x^\nu = \eta_{\mu'\nu'}\Delta x^{\mu'}\Delta x^{\nu'} \tag{2.8}$$

Minkowski metric tensor does not change form if we change coordinates (Cartesian coordinates, meanwhile if we use like polar ones it changes for obvious reasons.)

if

$$\begin{aligned} \Delta s^2 &> 0, \text{ space-like separation} \\ &< 0, \text{ time-like, (it could be an actual WL for a massive particle)} \\ &= 0, \text{ light-like or null} \end{aligned}$$

Now we can define the *proper time* as

$$\Delta\tau^2 \equiv -\Delta s^2$$
$$\text{or}$$
$$\Delta\tau^2 = -\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

to be clear, if the proper time is *positive* it is time-like.

If the segment **EF** marks the begin and end of the trajectory of a massive particle, $\Delta\tau$ is the time elapsed on a clock sitting on a RF that moves with constant speed between E and F.

In the moving frame $\Delta\tau = \Delta t_*$ where $t*$ is the time coordinate of the moving frame. In a frame where I'm at rest this is how Δt^2 changes:

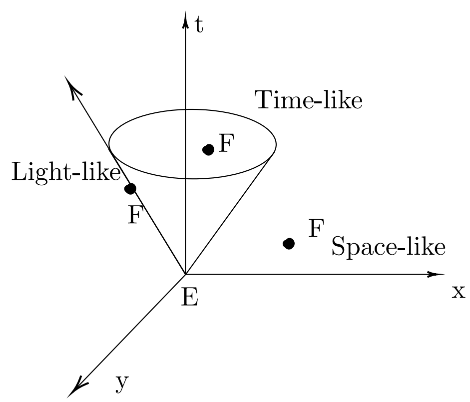
$$\Delta\tau^2 = +\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2. \tag{2.9}$$

2.2 Lecture 3

The meaning of the Lorentz Invariant is that **events**, like (E, F) exist before I define coordinates. It is a property of the two events.

So to recap what we did in the last lecture, be:

$$x_E^{\mu} \text{ and } x_E^{\mu'} \tag{2.10}$$



If I have two events and computing $\Delta\tau$ gives a positive result, the separation is **time-like**. This means that they could be on the WL of a massive particle moving at constant speed.

Physical meaning of $\Delta\tau$ It's the time elapsed on a clock of the observer moving between E and F at constant speed.

This means that if I compute $\Delta\tau$ on the frame where the observer it is at rest, i get

$$\Delta\tau = \Delta t'$$

Lets do an example:

Example In fig. 2.2 we see the straight line **ABC** that is the WL of a object not moving. Computing its proper time will be:

$$\Delta\tau_{ABC} = (t_c - t_A) \tag{2.11}$$

But for the other WL, of a object moving at constant speed between **AB'** and **B'C**, we see that

$$t_B = t_{B'}$$
$$\text{and so}$$
$$\Delta\tau_{AB'C} = 2\sqrt{(t_B - t_A)^2 - (\tilde{x} - \bar{x})^2} = \Delta\tau_{ABC}\sqrt{1 - \left(\frac{v}{c}\right)^2}$$
$$\implies \Delta\tau_{AB'C} < \Delta\tau_{ABC}$$

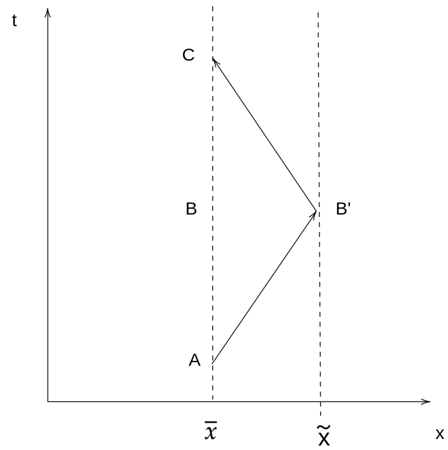


Figure 2.2: It is like the twin paradox.

This means that I have the longest **proper time** when I don't move.

We can do one more generalization: by parametrize the WL with a quantity λ we get

$$x^{\mu}(\lambda)$$

$$\Delta\tau = \int \sqrt{-\eta_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}}d\lambda \text{ that is a time like trajectory.}$$

Enough with proper time.

2.2.1 Tensor Calculus

Be a Lorentz Group, we want to look for the transformations.

$$x^{\mu} \to x^{\mu'} = \Lambda_{\mu}^{\mu'} x^{\mu} \tag{2.12}$$

we see that it is a linear transformation. An example to see better what are we doing could be

$$x^{0'} = \Lambda_0^{0'} x^0 + \Lambda_1^{0'} x^1 + \Lambda_2^{0'} x^2 + \Lambda_3^{0'} x^3 \tag{2.13}$$

What we need to know is that $\Lambda_{\mu}^{\mu'}$ is a constant matrix.

We see that Λ is a constant matrix.

We want to find linear transformations such that

$$\Delta s^2 = \eta_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu} = \eta_{\mu'\nu'}\Delta x^{\mu'}\Delta x^{\nu'} \tag{2.14}$$

So the Lorentz Invariant is still invariant. (WTF)

Now, because a SR[1] property: if I move from IF[2] to another, η is still unchanged. So

$$\eta_{\mu\nu} = \eta_{\mu'\nu'}$$

We have to say that Minkowski assumes cartesian coordinates.

The question now is: What trivial transformations leave Δs^2 unchanged?

Translations

$$\eta_{\mu\nu}\Delta x^\mu x^\nu = \eta_{\mu'\nu'}\left(\Lambda^{\mu'}_\mu \Delta x^\mu\right)\left(\Lambda^{\nu'}_\nu \Delta x^\nu\right)$$
$$\implies \eta_{\mu\nu} = \eta_{\mu'\nu'}\Lambda^{\mu'}_\mu \Lambda^{\nu'}_\nu$$
$$\text{this obviously needs to be valid } \forall \Delta x^\mu$$
$$\text{an alternative notation could be } \eta = \Lambda^T \eta \Lambda$$

We will use just the first notation, because we need to get good at tensors.

To be more concrete:

$$\Lambda^{\mu'}_\mu = \begin{pmatrix} \Lambda^{0'}_0 & \Lambda^{0'}_1 & \Lambda^{0'}_2 & \Lambda^{0'}_3 \\ \Lambda^{1'}_0 & ... & ... & ... \\ \Lambda^{2'}_0 & ... & ... & ... \\ \Lambda^{3'}_0 & ... & ... & ... \end{pmatrix} \tag{2.15}$$

Rotations Rotations are a kind of transformation of the type:

$$x_{i'} = R_{ii'} x_i$$
$$\text{or } R^T \mathbb{I} R = \mathbb{I}$$
$$\text{with } RR^T = R^T R = \mathbb{I}$$

it could be something like

$$\Lambda^{\mu'}_\mu = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.16}$$

this one is a boost along the x direction. If we do some computing we find that

$$\tanh\eta \equiv v$$

so this is the same of the L.T. we saw last week.

Rotations do not change the time coordinate. The point was to tell what L.T. is in this language.

[1]Special Relativity

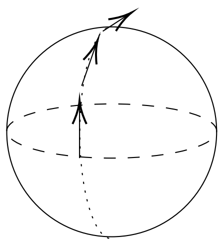
[2]Inertial frame

Vectors I have a generic vector, **do i need to specify about the RF** where it is defined, so in a specific spacetime location? yes

In newtonian mechanics parallel vectors are the same because I can superpose them, I can move them around, also to use the parallelogram rule to get a sum.

$\Longrightarrow$ If I have 3D euclidean space there is no ambiguities about where i move my vectors.

BUT in a sphere:



I have this vector at the equator tangent to the surface. If I transport it to the pole i get a different vector.

There are ambiguities. So in a non-flat space we need a **different** procedure.

A vector field is a map between:

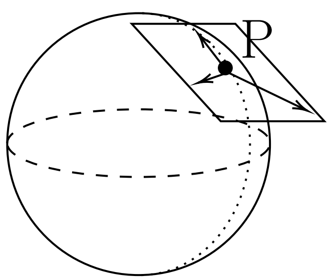
$$x^\mu \to v^\mu$$

where x^μ is an event and v^μ is a vector.

Let's define: **Tangent space $\mathbf{T_P}$**:

Given an event P we define the tangent space T_P as all the vectors in P.

Instead of having spacetime we have a sphere.



Define a plane tangent to the sphere only in P. All vectors that lie there $\in T_P$.

T_P is a **vector space**:

$$V, W \in T_P \implies \alpha V + \beta W, (\alpha, \beta \in \mathbb{R}) \in T_P$$

So if there is a vector there is also the inverse vector.

Whenever I have a vector space, I can define infinite basis independently on the coordinate choice. The number of elements in the basis is equal to the dimension of the space, in our case 4 elements.

Obviously if I define the basis its elements need to be Linearly Independent.

Basis Given a generic vector $V \in T_P$, I can define V regardless the coordinate system I'm using. So we can say *metaphorically* that V exists before I define coordinates.

Be our basis:

$$\hat{e}_{(\mu)}, \text{ with } \mu = 0, 1, 2, 3$$

those indices are label, does not mean "tensor". So my basis is made of

$$\hat{e}_{(0)}, \hat{e}_{(1)}, \hat{e}_{(2)}, \hat{e}_{(3)}$$

Now we can talk about

Components given a generic vector V

$$V = V^0 \hat{e}_{(0)} + V^1 \hat{e}_{(1)} + V^2 \hat{e}_{(2)} + V^3 \hat{e}_{(3)} = V^\mu \hat{e}_{(\mu)}$$

using repeating indices we get the last equivalence.

V^μ are components of the vector V in this specific frame.

In another frame $V^{\mu'}$ could not be the same:

$$V = V^\mu \hat{e}_{(\mu)} = V^{\mu'} \hat{e}_{(\mu')}$$

Question: how do components transform?

covariant vector : is a math object whose components transform based on position

$$V^{\mu'} = \Lambda^{\mu'}_{\mu} V^\mu$$

These are not the only covariant vectors (?).

If you have a generic WL or path, you can parametrize the position by a λ in this way:

$$x^\mu (\lambda)$$

And taking its first derivative you get something similar to the four-velocity

$$u^\mu \sim \frac{dx^\mu}{d\lambda}$$

(I say similar because four-velocity is defined like $u^\mu = \frac{dx^\mu}{d\tau}$).

If I do a L.T. x^μ will change but λ won't.

$$u^{\mu'} = \Lambda^{\mu'}_{\mu} u^\mu$$

I can get a more general definition of what a vector is by following this procedure: choose basis $\rightarrow$ find components $\rightarrow$ study how components change if i change position or basis.

Second definition : Transformation of the basis vectors. The question is "how to relate $\hat{e}_{(\mu)}$ to $\hat{e}_{(\mu')}$?"

We will take advantage of **invariance**.

$$V = V^{\mu}\hat{e}_{(\mu)} = V^{\mu'}\hat{e}_{(\mu')} = \left(\Lambda^{\mu'}_{\mu}V^{\mu}\right)\hat{e}_{(\mu')}$$

That's possible **only** if $\hat{e}_{(\mu)} = \Lambda^{\mu'}_{\mu}\hat{e}_{(\mu')}$.

An inverse of LT it is also a LT, so

$$\Lambda^{\mu'}_{\mu}\Lambda^{\mu}_{\nu'} = \delta^{\mu'}_{\nu'}$$
$$\Lambda^{\mu}_{\mu'}\Lambda^{\mu'}_{\nu} = \delta^{\mu}_{\nu}$$

Those are Kroneker's delta and they are an Identity matrix.

Now we can study how basis vectors change.

$$\hat{e}_{(\mu)} = \Lambda^{\mu'}_{\mu}\hat{e}_{(\mu')}$$
$$\Lambda^{\mu}_{\nu'}\hat{e}_{(\mu)} = \Lambda^{\mu'}_{\mu}\Lambda^{\mu}_{\nu'}\hat{e}_{\mu'}$$
$$\Lambda^{\mu}_{\nu'}\hat{e}_{(\mu)} = \delta^{\mu'}_{\nu'}\hat{e}_{(\mu')}$$
$$\Lambda^{\mu}_{\nu'}\hat{e}_{(\mu)} = \hat{e}_{(\nu')}$$
$$\text{so } \hat{e}_{(\nu')} = \Lambda^{\mu}_{\nu'}\hat{e}_{(\mu)}$$

2.3 Lecture 4

Brief recap of lec3

We defined vectors

- localized at each spacetime point
- for each event P we defined the tangent space T_P
- there is linear combination inside T_P
- it has a basis
- Vectors and basis transform under LT Group.

Dual vectors

Using old terminology they are covariant, so with lower indices. Meanwhile contravariant do have upper indices.

Let's start with defining the **dual space** of a vector space: *Given a vector space (for concreteness T_P)*, we define the **dual space** T_P^* as the space of linear maps between T_P and $\mathbb{R}$.

Example Being $\omega \in T_P^*$, $V \in T_P$ then

$$\omega\left(V\right) \in \mathbb{R}$$

Linearity tells me that

$$\omega\left(\alpha V + \beta W\right) = \alpha\omega\left(V\right) + \beta\omega\left(W\right)$$

1$^{\text{st}}$ statement : The dual space is a vector space.

$$\left(\alpha\omega + \beta\eta\right)\left(v\right) = \alpha\omega\left(v\right) + \beta\eta\left(v\right)$$

2$^{\text{nd}}$ statement : What is the dual of the dual?

$$\left(T_P^*\right)^* = T_P \implies v\left(\omega\right) = \omega\left(v\right) \in \mathbb{R}$$

Basis for $\mathbf{T_P}^*$: $\hat{o}^{(\mu)}$.
How to define this? Definition is

$$\hat{o}^{(\mu)}\left(\hat{e}_{(\nu)}\right) \equiv \delta^\mu_\nu$$

Now let's see if we can get how dual vectors work with vectors. If I have:

- generic item of T_P: $V = V^\nu \hat{e}_{(\nu)}$
- generic item of T_P^* : $\omega = \omega_\mu \hat{o^{(\mu)}}$

I can compute:

$$\omega(v) = \omega_\mu \hat{o}^{(\mu)}\left(v^\nu \hat{e}_{(\nu)}\right) =$$
$$= \omega_\mu v^\nu \hat{o}^{(\mu)}\left(\hat{e}_{(\nu)}\right) = \omega_\mu v^\nu \delta^\mu_\nu = \omega_\mu v^\mu$$

exercise: show the way $\omega_{\mu'}$ transform. What to do is to start from Λ equality.

What is the example of a dual vector?

$$A_{\mu'} = \Lambda^{\mu'}_{\mu} A_\mu$$

the gradient is a beautiful example of a *dual vector*.

$$A_\mu = \frac{\partial \phi}{\partial x^\mu} \; ; \; A_{\mu'} = \frac{\partial \phi}{\partial x^{\mu'}}$$

This is useful to define LTs, in this way

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial \phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}} \to A_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} A_\mu$$

the LT is the last partial derivative.

There is a *more compact* notation tho write partial derivatives that is

$$\partial_\mu \phi \equiv \frac{\partial \phi}{\partial x^\mu}$$

2.3.1 Tensors

Tensors are generalization of dual vectors and vectors.
They are *multilinear maps*, i.e. functions of several variables and linear for all of them. For each tensor of *rank* (k,l), we have

$$T^*_P \times \ldots \times T^*_P \times T_P \times \ldots \times T_P \to \mathbb{R}$$

Where each dual vector space is present **k**-times, and vector space **l**-times.

Now let's see what is multilinearity on the combat field.

Be a (1,1) tensor:

- $\alpha, \beta, \gamma, \delta \in \mathbb{R}$
- $\omega, \eta \in T^*_P$
- v,w $\in$ $\mathrm{T_P}$

Given these we have

$$T(\alpha\omega + \beta\eta, \gamma v + \delta w) = = \alpha\gamma T(\omega, v) + \beta\delta T(\eta, w) + \alpha\delta T(\omega, w) + \beta\gamma T(\eta, v) \tag{2.17}$$

Once we have this general definition, let's take one step back:

- Scalar $\rightarrow$ (0,0) tensor
- Vector $\rightarrow$ (1,0) tensor
- Dual vector $\rightarrow$ (0,1) tensor

Tensor product

Be:

- T, rank (k,l) tensor
- S, rank (m,n) tensor

We want to understand the action of $\otimes$.
So we know that $T \otimes S$ outputs (k+m, l+n) tensor. In particular,

$$T \otimes S \left[\omega^{(1)}, \ldots, \omega^{(k)}, \omega^{(k+1)}, \ldots, \omega^{(k+m)}, v^{(1)}, \ldots, v^{(l)}, v^{(l+1)}, \ldots, v^{(l+n)}\right] \equiv$$
$$\equiv T\left(\omega^{(1)}, \ldots, \omega^{(k)}, v^{(1)}, \ldots, v^{(l)}\right) \times S\left(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, v^{(l+1)}, \ldots, v^{(l+n)}\right)$$
$$\Longrightarrow T \otimes S \neq S \otimes T$$

so tensors do not commute.

Basis for a tensor

Let T be a generic tensor with rank (k,l), *basis* is given by

$$\hat{e}_{(\mu_1)} \otimes \ldots \otimes \hat{e}_{(\mu_k)} \otimes \hat{o}^{(\nu_1)} \otimes \ldots \otimes \hat{o}^{(\nu_l)}$$

A tensor can be written as

$$T = T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l}\left(\hat{e}_{(\mu_1)} \otimes \ldots\right) = T^{\mu_1^{'},\ldots,\mu_k^{'}}_{\nu_1^{\prime},\ldots,\nu_l^{\prime}}\left(\hat{e}_{(\mu_1^{\prime})} \otimes \ldots\right)$$

So the tensor is always the same, the thing that changes is its components, because a change of RF I think.

We will often write the components instead of the actual tensor, but it is our convention to think they are equivalent.

This is how the components are related:

$$\hat{e}_{(\mu')} = \Lambda^{\mu}_{\mu'}\hat{e}_{(\mu)}$$
$$\hat{o}^{(\mu')} = \Lambda^{\mu'}_{\mu}\hat{o}^{(\mu)}$$
$$\Longrightarrow T = T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l}\left(\Lambda^{\mu_1'}_{\mu_1}\hat{e}_{(\mu_1')} \otimes \ldots\right)$$

So we find, as result, that when I change frame

$$T^{\mu_1',\ldots,\mu_k'}_{\nu_1',\ldots,\nu_k'} = \Lambda^{\mu_1'}_{\mu_1}\ldots\Lambda^{\nu_1}_{\nu_1'}\ldots T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l} \tag{2.18}$$

2.4 Lec 5

2.4.1 Transformations

The goal is to find what is this $T^{\mu'_1,\dots,\mu'_k}_{\nu'_1,\dots,\nu'_l}$ =?.

$$T = T^{\mu_1,\dots,\mu_k}_{\nu_1,\dots,\nu_l}\left(\hat{e}_{(\mu_1)}\otimes\dots\right) = T^{\mu'_1,\dots,\mu'_k}_{\nu'_1,\dots,\nu'_l}\left(\hat{e}_{(\mu'_1)}\otimes\dots\right) \tag{2.19}$$

I know two facts:

$$\begin{cases}\hat{e}_{\mu'} = \Lambda^{\mu}_{\mu'}\hat{e}_{(\mu)}\\ \hat{\sigma}^{\mu'} = \Lambda^{\mu'}_{\mu}\hat{\sigma}^{\mu}\end{cases} \tag{2.20}$$

and also the *inverse*.

So i apply the Lambda transformation to each term of the basis and I get the following

$$T^{\mu'_1,\dots,\mu'_k}_{\nu'_1,\dots,\nu'_l} = \left(\Lambda^{\mu'_1}_{\mu_1}\dots\Lambda^{\mu'_k}_{\mu_k}\right)\left(\Lambda^{\nu_1}_{\nu'_1}\dots\Lambda^{\nu_l}_{\nu'_l}\right)\left(T^{\mu_1,\dots,\mu_k}_{\nu_1,\dots,\nu_l}\right) \tag{2.21}$$

that is something that was obvious by looking at indexes.

2.4.2 Tensor Manipulations / Operations

We defined (k, l) vectors as a multilinear map from dual spaces and vector spaces to real numbers, but it is not only that. For example a $(1, 1)$ tensor could be a map from vectors to vectors, in this way

$$V^{\mu} \to A^{\mu}_{\nu}V^{\nu} \tag{2.22}$$

so if i do not saturate all the indices, i get a tensor of rank made by what remains. If we saturate, we get real numbers or (0,0) tensors.

There are some objects that are well known in flat spacetime.

Particular Tensor in flat ST

These are

- $\eta_{\mu\nu}$ metric, or metric tensor
- $\eta^{\mu\nu}$, inverse metric
- δ^{μ}_{ν}, Kronecker's δ
- $\epsilon_{\mu\nu\rho\delta}$, totally anti-symmetric tensor of Levi-Civita

This last one is defined:

$$\begin{cases}+1 \text{ if } (0,1,2,3) \text{ or even permutations}\\ -1 \text{ if odd permutations}\\ 0 \text{ otherwise}\end{cases} \tag{2.23}$$

These are the only tensors of the flat spacetime that their components do not depend on the RF.

Other operations

Contraction

$$(k,l) \to (k-1,l-1)$$

Example: I have (3,2) tensor $T^{\mu\nu\rho}_{\delta\gamma} \to (2,1)$?? We contract:

$$T^{\mu\;\nu\;\rho}_{\;\delta\;\gamma} \to T^{\mu\;\nu\;\rho}_{\;\delta\;\nu} \equiv A^{\mu\rho}_{\delta}$$

Obviously I can *only* contract an upper with a lower index.
It is very important the order, and which indices we contract.

$$T^{\mu\;\nu\;\rho}_{\;\delta\;\nu} \neq T^{\mu\;\nu\;\rho}_{\;\nu\;\delta}$$

What is the actual operation we perform?

$$T^{\mu\nu\rho}_{\delta\gamma} = \delta^{\gamma}_{\nu} T^{\mu\nu\rho}_{\delta\gamma}$$

Raising/Lowering Indices To raise we use $\eta^{\mu\nu}$, to lower $\eta_{\mu\nu}$.

$$\eta^{\rho\alpha} T^{\mu\nu}_{\alpha\beta} \equiv T^{\mu\nu\rho}_{\beta}$$

$$\eta^{\rho\;\beta} T^{\mu\nu}_{\;\;\;\;\alpha\;\beta} \equiv T^{\mu\nu\;\beta}_{\alpha}$$

The order is important, and wring by hand one should be careful keeping the position moving up and down the indices.

Simple operations:

$$V^{\mu} \to V_{\mu} = \eta_{\mu\nu} V^{\nu}$$
$$V_{\mu} \to V^{\mu} = \eta^{\mu\nu} V_{\nu}$$

Inner Product

$$T_P \times T_P \to \mathbb{R}$$

$$(V,W) \to \eta_{\mu\nu} V^{\mu} V^{\nu}$$

Symmetry Properties

Let's consider a (0,2) tensor $T_{\mu\nu}$, or to be precise, its components. It is symmetric? Anti-symmetric? Both? None?

A tensor is *symmetric* if

$$T_{\mu\nu} = T_{\nu\mu}$$

it is *anti-symmetric* if

$$T_{\mu\nu} = -T_{\nu\mu}$$

It is **never** possible to have a tensor that is *both*. But really possible that is *none* of the above.

We can *symmetrize* a tensor:

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$$

We can *anti-symmetrize* a tensor:

$$T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$$

A tensor can be symmetric on all indices, so it *totally symmetric*, or just on some indices, like two, three etc. The general formula can be:

$$T_{(\mu\nu\rho)} = \frac{1}{3!}\left(T_{\mu\nu\rho} + \text{ all permutations }\right)$$

For anti-simmetrizing odd permutations get the minus in front.

Trace

$$x^{\mu}_{\mu}$$

given a (1,1) tensor $\to \mathbb{R}$ by summing all indices. For example the trace of metric tensor is 2. Of Kronecker delta is 4.

2.5 Lec 6

2.5.1 Energy & momentum

Since out goal is to get to the Einstein Equation, we know that in there there should be the *energy momentum tensor* $T^{\mu\nu}$.
As always we will study everything for a flat space-time but it will be useful for non flat ones.
We already saw the four-velocity u^μ:

$$u^\mu \equiv \frac{dx^\mu}{d\tau}$$

while the proper time is $\Delta\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$.
We need to make clear that we are talking about a time-like space-time trajectory, so $\Delta s^2 < 0$.
Let's start with the WL of a single particle, this is specified by a map $\mathbb{R} \to M$, where M is a manifold that represents spacetime. We usually think the path as a curve parameterized by λ so $x^\mu(\lambda)$.
We also use as parameter the τ so $x^\mu(\tau)$, this has some advantages because maybe it could be easier to switch to four-velocity.

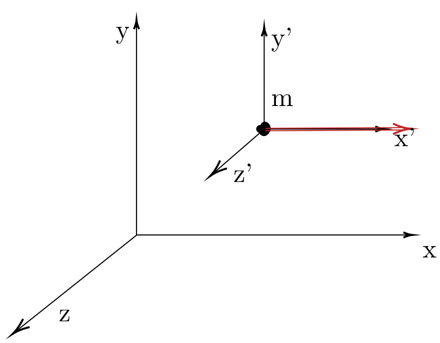
$$u^\mu u_\mu = u_\mu u^\mu = \eta_{\mu\nu}u^\mu u^\nu = -1 \tag{2.24}$$

By the way, four-velocity is what we need to find the *four momentum*:

$$p^\mu \equiv m u^\mu \tag{2.25}$$

where m is the rest mass that has the same values ∀ RF, and it's just a number.

So in rest frame (x',y',z'):



So in the rest frame (x',y',z'):
$p^\mu = (m, 0, 0, 0)$, because the four-velocity in the rest frame is $u^\mu = (1, 0, 0, 0)$.
What is the expression of p^μ in the (x,y.z) frame?
And what is the fastest way to compute it?

We can start from the rest frame and use a LT.
For a generic four vector we have:

$$\begin{cases} a^{0'} = \gamma\left(a^0 - va^1\right) \\ a^{1'} = \gamma\left(a^1 - va^0\right) \\ a^{2'} = a^2 \\ a^{3'} = a^3 \end{cases} \tag{2.26}$$

Now we find the inverse, we can search the inverse of the matrix or use an inverse LT,

$$\begin{cases} a^0 = \gamma \left(a^{0'} + va^{1'} \right) \\ a^1 = \gamma \left(a^{1'} + va^{0'} \right) \\ a^2 = a^{2'} \\ a^3 = a^{3'} \end{cases} \tag{2.27}$$

So for the four-momentum we have:

$$\begin{cases} p^0 = E = \gamma p^{0'} = \gamma m = \frac{m}{\sqrt{1-v^2}} \\ p^1 = m\gamma v = \frac{mv}{\sqrt{1-v^2}} \\ p^2 = 0 \\ p^3 = 0 \end{cases} \tag{2.28}$$

In the NR limit we should be able to recover Newton Mechanics:

$$E \approx m + \frac{mv^2}{2} + \dots$$
$$p^1 \approx mv + \dots$$

The four-momentum as we got it provides the description of a single particle but often we need to study a lot of particles as a continuum, like a *fluid*, characterized by quantities as density, pressure, entropy, viscosity...
A single momentum four-vector field is insufficient to describe the energy and the momentum of a fluid so we go further and define the *energy-momentum tensor*.

2.5.2 Energy-Momentum Tensor

$$T^{\alpha\beta}$$

For now it is just a tensor, and we are happy to see that it transform like a tensor:

$$T^{\alpha'\beta'} = \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} T^{\alpha\beta}$$

In words, it is defined like "*the flux of four-momentum p^α across the surface where x^β is constant*".

For a system of N particles we have:

$$p^\alpha = \sum_{j=1}^{N} p^\alpha_j$$

where j shows the *j-th* particle, not an index to contract. The *number density,* n for a system of N particles is:

$$n = \sum_j \delta \left(\vec{r} - \vec{r}_j \right)$$

So we have these components of the energy-momentum tensor:

$$T^{\alpha 0} = \sum_j p_j^\alpha \frac{dt}{dt} \delta\left(\vec{r} - \vec{r}_j\right)$$

$$T^{\alpha i} = \sum_j p_j^\alpha \frac{dx_j^i}{dt} \delta\left(\vec{r} - \vec{r}_j\right)$$

The $\frac{dt}{dt}$ is obvious that is simplified but we put it for clarity, while $\frac{dx_j^i}{dt}$ is the flux. The meaning is that the tensor is the output of many contribution, each contribute has the center around the *j-th* particle.

This gives me all the components of the E-M tensor. Now, what is a tensor? We used the word tensor because we know a priori what we are gonna find it, but without knowing and looking at the definition and components. We will do it by looking at LTs, and how they act on this object.

First thing we compute:

$$\frac{dx_j^i}{dt} = \frac{dx_j^i}{d\tau} \frac{d\tau}{dt} = \frac{dx_j^i}{d\tau} \frac{1}{\gamma_j}$$

because

$$\frac{d\tau^2}{dt^2} = \frac{dt^2 - dx^2}{dt^2} = 1 - v_j^2 = \frac{1}{\gamma_j^2}$$

why is it useful? Because it appears in components

$$\frac{dx_j^i}{d\tau} = \left(m^j \frac{dx_j^i}{d\tau}\right) \frac{1}{m_j \gamma_j} = \frac{p_j^i}{p_j^0}$$

so i can rewrite the energy-momentum tensor like:

$$T^{\alpha\beta} = \sum_j \frac{p_j^\alpha p_j^\beta}{p_j^0} \delta\left(\vec{r} - \vec{r}_j\right)$$

so if

- $\beta = 0 \to p_j^\alpha$
- $\beta = 1 \to \frac{p_j^i}{p_j^0}$

If i switch α with β, I find the same objects, because the tensor is symmetric.

$$T^{(\alpha\beta)} = T^{\alpha\beta} \; ; \; T^{[\alpha\beta]} = 0$$

Why is a tensor? If I change frame I can change

If I change frame I have to change α, β but also p_0 that is not Lorentz Invariant, It's the energy of the particle j, and with a boost it will be different. We have shown that

$$\frac{\delta\left(\vec{r} - \vec{r}_j\right)}{p_j^0}$$

is a Lorentz scalar. Writing

$$T_j^{\alpha\beta} = \frac{p_j^\alpha p_j^\beta}{p_j^0}\delta^{(3)}\left(\vec{r} - \vec{r}_j\right)$$

with a 3-d Dirac Delta function, and with this definition

$$T^{\alpha\beta} = \sum_j T_j^{\alpha\beta}$$

that is a tensor because sum of tensors is still a tensor.

Let's focus on the single contribution:

$$T_j^{\alpha\beta} = \frac{p_j^\alpha m u_j^\beta}{m\gamma_j}\delta^{(3)}\left(\vec{r} - \vec{r}_j\right) =$$

we see that mu^β is p^β, and $m\gamma$ is p^0. We can simplify the masses and we get:

$$\begin{aligned} T_j^{\alpha\beta} &= \frac{p_j^\alpha u_j^\beta}{\gamma_j}\delta^{(3)}\left(\vec{r} - \vec{r}_j\right) = \\ &= \int d\tau_j p_j^\alpha u_j^\beta \delta^{(4)}\left(x^\mu - x_j^\mu\left(\tau_j\right)\right) = \end{aligned}$$

The two expression are equivalent, to show it I have to compute the integral. I use the delta function of the 0 component: so the p,u, γ elements can be extracted from the integral and we compute just

$$= \int d\tau_j \delta\left(x^0 - x_j^0\left(\tau\right)\right) = \int \frac{1}{\left|dx_j^0/d\tau\right|_{d\overline{\tau}}} d\tau_j \delta\left(\tau_j - \overline{\tau_j}\right) = \tag{2.29}$$

$\overline{\tau}$ is where the argument of the delta function is 0. The integral is straightforward. We have to change variable of the delta function, that gets contributes only from the point that match the argument, I integrate $\tau_j -$ number , There is a jacobian factor: $\frac{1}{\left|dx_j^0/d\tau\right|_{d\overline{\tau}}}$ that is equal to $\gamma_j = \frac{dt}{d\tau_j}$.

This is a tensor, because I have objects with indices α, β, integral over τ_j which is a Lorentz Invariant Scalar and a δ over all space coordinates.

Delta function has the very useful property:

$$\delta\left(f\left(x\right)\right) = \frac{1}{\left|f'\left(x_0\right)\right|}\delta\left(x - x_0\right) \to \int_{x-\epsilon}^{x+\epsilon} dx \delta\left(x - x_0\right) = 1$$

Case I: Dust

The dust is defined as *generic ensemble of N particles that move very slowly.* So it's any set of particles with kinetic energy much smaller than rest mass energy.

SO the important part is that the relative velocity in some RF $\to 0$.

The total energy density ρ is described as

$$T^{00} = \sum_{j=1}^{N} p_j^0 \delta \left(x - \overline{x}_j \right) = \rho$$

So dust in the rest frame is

$$T^{\mu\nu} \begin{pmatrix} m \cdot n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.30}$$

where $\rho = m \cdot n$, all particles have the same mass, n is the number density. It's clear that if the particles are at rest, the flux is null.

What is $T^{\mu\nu}$ in a generic frame? I could apply LTs, (and we are invited to try it), but actually i can reason a little on the meaning of the tensor.

I can call u^μ *fluid four-velocity*, if you think about that there is a velocity field on a moving fluid like a river. In the rest frame

$$u^\mu = (1, 0, 0, 0)$$

so

$$T^{\mu\nu} = \rho u^\mu u^\nu$$

This is more generic way to find the tensor $T^{\mu\nu}$ in a generic frame, The strategy, as you may guess, is that I know the expression in a generic frame and I want to recover the tensor from this.

Dust is something very common in cosmology, and is a fluid with zero pressure. But is not always true that in a fluid the pressure is negligible. Photons for example do not have 0 pressure or relative velocity $\to 0$.

Case II: Fluid with pressure or *Perfect FLuid*

Be

$$T^{\mu\nu}_{rest} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.31}$$

Let's talk about the physics underlying this definition:

The parts $0i$ or $i0$ under rotations or LTs transform like *3-d* vectors, and since the values are 0s, the fluid is *isotropic* $\to$ no preference about any direction.

ij parts represent the flux of momentum p against surface of constant spatial coordinates. Momentum flux is the force, and force flux is the pressure. We could have different p_i along the diagonal, the fluid wouldn't be isotropic, but there will still be 0 sheer forces.

In the most generic frame:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p\eta^{\mu\nu} + pu^\mu u^\nu = (\rho + p) u^\mu u^\nu + p\eta^{\mu\nu} \tag{2.32}$$

Conservation of energy and momentum

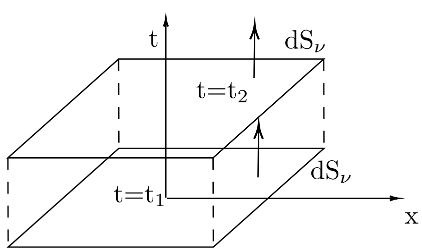
$p^{\mu\ \text{total}}$ has to be constant, so energy and momentum are constant. I want this condition to be local:

$$p^{\mu}{}_{\text{total}} = \int_V dx^3 T^{\mu 0} = \int_{4d} dS_\nu T^{\mu\nu}$$

If i set $\nu = 0$ I simplify it. I integrate over the entire V that is the entire space. It can be written as the flux of the ν components. The four dimensional integral has a surface S where $T^{\mu\nu}$ is constant. It's a flux integral. $dS_\nu = (1, 0, 0, 0)$.

$$\Delta p^{\mu}{}_{\text{total}} = p^{\mu}{}_{\text{total}}\ (t_2) - p^{\mu}{}_{\text{total}}\ (t_1)$$

So let's make a scheme to understand the situation.



This is an integral of the flux along the time direction.
So the expression of $\Delta p^{\mu}{}_{\text{total}}$ can be written as

$$= \int_{\text{total surface}} dS_\nu T^{\mu\nu} = \int_{Volume} dx^3 \partial_\nu T^{\mu\nu} = \int_{volume} dV \partial_\nu T^{\mu\nu}$$

So we have two surfaces and we closed it inside a solid, and it is like we are doing gauss theorem, We can use the divergence theorem in 4d. We state that the divergence of $T^{\mu\nu}$:

$$\partial_\mu T^{\mu\nu} = 0$$
$$\partial_\nu T^{\mu\nu} = 0$$

And that's a way to express conservation.

Results for perfect fluids

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\eta^{\mu\nu}$$

and taking the derivative

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \partial_\mu (\rho + p)\, u^\mu u^\nu + \partial_\mu p \eta^{\mu\nu} + (\rho + p)\,(\partial_\mu u^\mu)\, u^\nu + (\rho + p)\, u^\mu \partial_\mu u^\nu = \\ &= \partial_\mu (\rho + p)\, u^\mu u^\nu + \partial_\mu p \eta^{\mu\nu} + (\rho + p)\,[(\partial_\mu u^\mu u^\nu + u^\mu \partial_\mu u^\nu)]\end{aligned}$$

It is quite clear, just the thing that we neglect derivative of the metric because in flat spacetime it is a constant matrix, and so it's 0.

Let's take the projection of this identity along the direction of the four-velocity.

$$u_\nu \partial_\mu T^{\mu\nu} = -\partial_\mu \left(\rho + p\right) u^\mu + \left(\partial_\mu p\right) u^\mu + \left(\rho + p\right) \left[-\partial_\mu u^\mu + u_\nu u^\mu \partial_\mu u^\nu\right] =$$

and since,

$$\partial_\mu \left(u_\nu u^\nu\right) = 0 = \partial_\mu \left(-1\right)$$

we get

$$= -\partial_\mu \left(\rho + p\right) u^\mu + \partial_\mu p u^\mu - \left(\rho + p\right) \partial_\mu u^\mu = 0 \tag{2.33}$$

And in the NR limit, $p \ll \rho$:

$$-\partial_\mu \rho u^\mu + \rho \partial_\mu u^\mu + \partial_\mu p u^\mu = 0$$

we can neglect the last term since the hypothesis

$$\partial_\mu \left(\rho u^\mu\right) = 0 \implies \partial_t \rho + \partial_i \left(\rho u^i\right) = 0$$

we recover the continuity equation.

exercise : instead of projecting on u^ν, project orthogonal to u^μ,compute: $p^\alpha_\nu \partial_\mu T^{\mu\nu} = 0$. idk what it means: $p^\alpha_\beta \equiv \delta^\alpha_\beta + u^\alpha u_\beta$. Evaluate alpha = 1,2,3.

2.6 Lec 7

2.6.1 Equivalence Principle

The idea of the *universality* of the gravitational interaction, in the form of the *Equivalence principle* led Einstein to think that gravity is special, not just another field, but a metric tensor that describes the curvature of spacetime.

Weak Equivalence Principle, WEP

It states that *inertial* mass and *gravitational* mass of any object are equal.

From the Second Law of Mechanics:

$$\vec{F} = m_i \vec{a}$$

with m_i = inertial mass. While to quantify gravitational forces in Newtonian mechanics:

$$\vec{F} = -m_g \nabla \Phi_g$$

With $\nabla \Phi_g$ gradient of scalar field Φ_g, known as gravitational potential. From these formulas, we see no actual reason why $m_i = m_g$:

- The inertial mass has a universal character, it takes the same value no matter what kind of force is being exerted.
- The gravitational mass is a quantity specific to the gravitational force. One could think $\frac{m_g}{m_i}$ as the *gravitational charge*.

Galileo showed by rolling balls down the inclined plane, that the response of matter to gravitation is universal, and in Newtonian mechanics it translates in WEP:

$$m_i = m_g$$

This, for freely falling objects, becomes

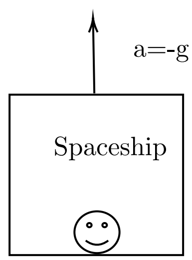
$$a = -\nabla \Phi$$

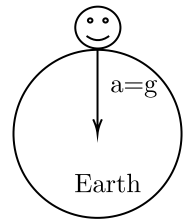
This led to think an equivalent formulation of WEP that is: *there exists a preferred class of trajectories through space time, called Inertial or Freely-Falling.* Freely falling is intended as "moving under the sole influence of gravity", these objects are unaccelerated.

The universality of gravitation can be stated in another form: If we consider a physicist in a spaceship that is accelerating at a constant rate, like

$$\vec{a} = -\vec{g}$$

he would be not able to distinguish by scientific experiments the situation in which he sits on Earth's surface. (Restricted to local observation).





If the spaceship would be sufficiently big, we would see that the effect of acceleration would always be in the same direction, while on the surface or the Earth we would see that it points towards the center of the earth, so radial vs straight parallel lines.

So WEP could be stated as *the motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, in small enough regions of spacetime.* In larger regions there would be inhomogenities, which will lead to tidal forces.

Einstein's Equivalence Principle, EEP

The Einstein Equivalence Principle is just a little generalization of the WEP:

> In small enough regions of spacetime, the laws of physics reduce to those of special relativity: it is impossible to detect the existence of gravitational field by means of local experiments.

Consider a hydrogen atom, a bound state of a proton and an electron. Its mass is actually less than the sum of the masses of the proton and electron considered individually, because there is a negative binding energy—you have to put energy into the atom to separate the proton and electron. According to the WEP, the gravitational mass of the hydrogen atom is therefore less than the sum of the masses of its constituents; the gravitational field couples to electromagnetism (which holds the atom together) in exactly the right way to make the gravitational mass come out right. This means that not only must gravity couple to rest mass universally, but also to all forms of energy and momentum—which is practically the claim of the EEP.

It is the EEP that implies that we should attribute the action of gravity the curvature of spacetime.

Strong Equivalence Principle, SEP

Is defined to include all of the laws of physics, gravitational and otherwise. We will define *unaccelerated* as *freely falling*, from here we decide that gravity is not a force, because a force leads to acceleration, and our definition of zero acceleration is *moving freely in the presence of whatever gravitational field happens to be around.*

We know that there is a class of preferred frames: Inertial Frames (where laws of dynamics are true). We introduce a new class of frames: Freely Falling Frames, where *unaccelerated particles move only due to gravity.*

Obviously these frames must be local frames, otherwise, due to inhomogenities on the gravitational field, particles initially at rest will begin to move with respect to such frame.

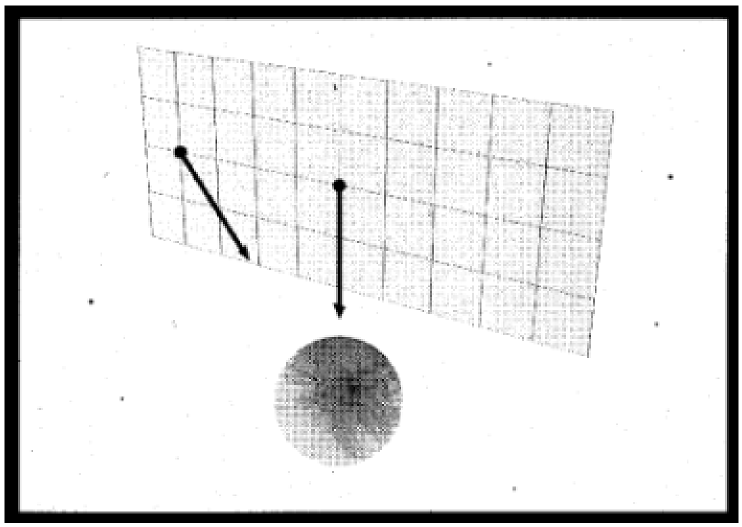


Figure 2.3: The failure of global frames.

After this we need a mathematical framework were what just said is consistent. The solution is to think that spacetime has a curved geometry and gravitation the manifestation of this curvature. Before jumping in what is a manifold, let's see if the consequences are of our world.

Gravitational Redshift

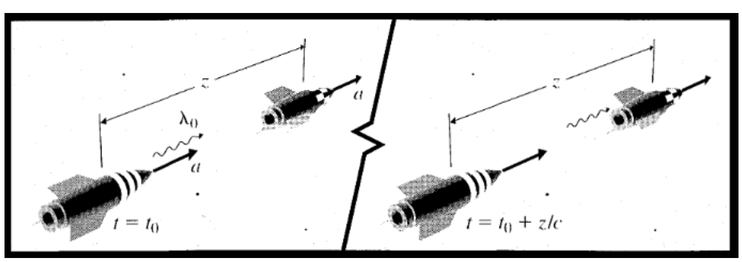


Figure 2.4: Doppler shift measured by two rockets, each feeling acceleration **a**.

Be two spaceships, separated by distance z, each moving with constant $\vec{a}$ acceleration in a region without gravitational fields.

At t_0 ship in the back emits a photos of λ_0.

The distance z stays constant, so the photon is received after $\Delta t = z/c$ in our reference frame. At $t = t_0 + \Delta t$ the boxes have picked up an additional velocity $\Delta v = \vec{a}\Delta t = \vec{a}z/c$. The photon reaching the front spaceship will be redshifted bu the Doppler Effect, by

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{az}{c^2}$$

And according to EEP this should happen also in a uniform gravitational field.

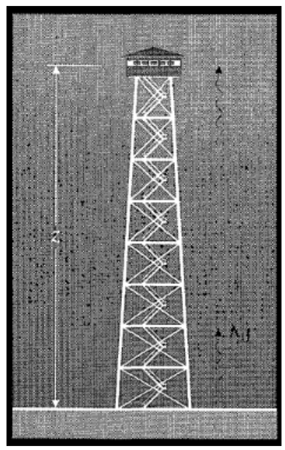


Figure 2.5: Gravitational redshift on Earth's surface

So if a photon is emitted from the ground with λ_0, it will be redshifted by

$$\frac{\Delta\lambda}{\lambda_0} = \frac{a_g z}{c^2}$$

To note that is a direct consequence of EEP, no details of GR were required. The thing is, if i try to represent this with Minkowski metric, I don't notice the redshift.

So now we really need to talk about Manifolds

2.6.2 Manifolds

A manifold is a space that may be curved and have a complicated topology, but in local regions looks just like $\mathbb{R}^n$. A crucial part is that the dimensionality n of the Euclidean Spaces being used must be the same in every patch of the manifold. For example are not manifolds, a line ending on a plane and two cones intersecting at their vertices.

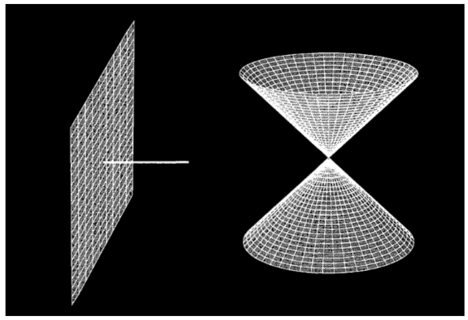


Figure 2.6: Not manifolds

Coordinate System

Be

$$\begin{cases} U \subset M \\ \phi : U \to \mathbb{R}^n \\ \phi(U) \text{ is open in } \mathbb{R}^n \end{cases} \tag{2.34}$$

These are a system of conditions to define a coordinate system or *chart.*

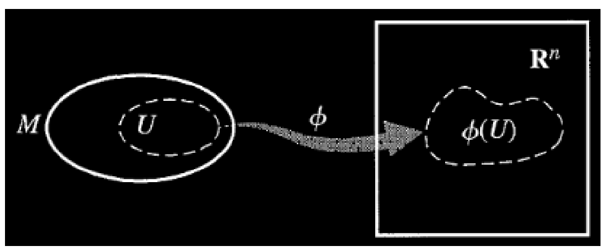


Figure 2.7: A coordinate chart covering an open subset U of M.

An *atlas* is a indexed collection of charts $\{(U_\alpha, \phi_\alpha)\}$.

Vectors again

One point we stressed was the notion of a tangent space, the set of all vectors at a single point in spacetime. A vector is not a thing that stretches from one point to another but is an object associated with a single point.

Be $f : M \to \mathbb{R}$. Each curve passing through a point P, defines an operator, the *directional derivative*, which maps $f \to df/d\lambda$ (at p).

We claim the tangent space T_P can be identified with the space of directional derivative operators along curves through P. And for any f we can write:

$$\frac{df}{d\lambda} = \frac{df}{dx^\mu}\frac{dx^\mu}{d\lambda} \implies \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

If i change the coordinates i can apply the chain rule.

2.7 Lec 8

2.7.1 Brief Recap

We saw the WEP, that states $m_i = m_g$, and as consequence we get that is is impossible distinguish a gravitational field from motion, at least locally.

With the EEP we were able to derive the expression that quantify the gravitational redshift. The SEP included gravity.

Focusing on EEP, we introduced *Locally Inertial Frames, LIF* (equivalent to Freely Falling Frames). Having accepted that gravity cannot be treated as a force, because it is impossible to disentangle acceleration due to gravity, we identified a preferred class of frames: LIFs.

In LIFs laws of physics are equal to the laws of SR and spacetime is Minkowskian.

We introduced Coordinates: with a generic set M, a chart, given a subset $U \subset M$, is a injective linear map ϕ, that

$$\phi : U \to \mathbb{R}^n$$

An *atlas* is an indexed $\{U_\alpha, \phi_\alpha\}$, in such a way that U_α cover M. A *manifold* is a set M along with an atlas. A manifold can be C^∞ if ϕ are differentiable an infinite amount of times, otherwise is C^p, differentiable p-times.

Vectors again again

We resurrected T_P, P generic M point. T_P is the vector space of all the vectors defined at that point.

> T_P is identified with the space of directional derivatives operators acting along the curves through P

Why this identification makes sense?

A generic curve through spacetime, that we call WL, is a parametric curve that is indicated by $x^\mu(\lambda)$: for a specific value of λ I have the x^μ point.

$$x^\mu\left(\overline{\lambda}\right) = P$$

How do I define the directional derivatives? Be

$$\frac{d}{d\lambda}$$

that acts on functions, *f*,

$$f : M \to \mathbb{R}$$

then

$$\frac{d}{d\lambda}(f) = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} f \tag{2.35}$$

And if true $\forall f$, I can identify the equality among the operators:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

We see that it is like a basis.

Basis vectors for $T_P \to \partial_\mu$. (we previously called them $\hat{e}_{(\mu)}$).

In conclusion a generic vector is

$$V = V^\mu \partial_\mu$$

where V^μ are the components, and ∂_μ are the basis elements.

It's very easy to show how vectors transform because we know how derivatives transform:

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \tag{2.36}$$

or in a tensor-like notation:

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \tag{2.37}$$

If V tensor is invariant, by definition, it's components transform anyway like

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

Example LTs

$$x^{\alpha'} = \Lambda^{\alpha'}_{\alpha} x^\alpha$$

I consider a specific change of coordinates: LTs, so

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \left(\Lambda^{\mu'}_{\alpha} x^\alpha \right) = \Lambda^{\mu'}_{\alpha} \frac{\partial x^\alpha}{\partial x^\mu} = \Lambda^{\mu'}_{\alpha} \delta^\alpha_\mu = \Lambda^{\mu'}_{\mu}$$

we get under more general transformations vectors components transform like this, and more, we recovered the LT transformation.

2.7.2 Dual Vectors

Be the *cotangent space* T^*_P. If $\omega \in T^*_P$, $\omega$is a linear map $\omega : T_P \to \mathbb{R}$. We want to define formally T^*_P (like we did for T_P). We know that the tangent space T_P holds directional derivatives, while cotangent space T^*_P holds gradients.

For a generic $f : M \to \mathbb{R}$:

$$\frac{d}{d\lambda} \in T_P$$

$$d \in T^*_P$$

and so

$$\begin{array}{ccc} df & \left(\frac{d}{d\lambda} \right) \equiv & \frac{df}{d\lambda} \\ \downarrow & \downarrow & \downarrow \\ \in T^*_P, & \in T_P, & \in \mathbb{R} \end{array}$$

Basis for $\mathrm{T_P}^*$: dx^μ.

$$dx^{\mu}\left(\frac{\partial}{\partial x^{\nu}}\right)=\frac{\partial x^{\mu}}{\partial x^{\nu}}=\delta^{\mu}_{\nu}$$

that is the same as the old $\hat{O}^{(\mu)}\left(\hat{e}_{\nu}\right)=\delta^{\mu}_{\nu}$.

A dual vector is

$$\omega=\omega_{\mu}dx^{\mu}$$

the basis component transform like

$$dx^{\mu'}=\frac{\partial x^{\mu'}}{\partial x^{\mu}}dx^{\mu}$$

and the vector components

$$\omega_{\mu'}=\frac{\partial x^{\mu}}{\partial x^{\mu'}}\omega_{\mu}$$

2.7.3 Tensors (k,l)

$$T:T^{*}_{P}\times\ldots\times T^{*}_{P}\times T_{P}\times\ldots\times T_{P}\rightarrow\mathbb{R}$$

Tensors can be expanded into components:

$$T=T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}\left(\partial_{\mu_1}\otimes\ldots\otimes\partial_{\mu_k}\otimes dx^{\nu_1}\otimes\ldots\otimes dx^{\nu_l}\right)$$

So the components of a generic tensor transform like

$$T^{\mu'_1\ldots\mu'_k}_{\nu'_1\ldots\nu'_l}=\left(\frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\ldots\frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}}\ldots\right)T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}$$

Now let's see a unusual tensor, it's a (2,1) tensor, how does transform?

$$T^{\alpha'\beta'}_{\gamma'}=\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}T^{\alpha\beta}_{\gamma}$$

Example/exercise (from 2.4 of Carroll) Be a tensor S_{ij}, with $i,j=1,2$, so it's a (0.2) tensor. We know that

- $S_{11}=1$
- $S_{12}=S_{21}=0$
- $S_{22}=x^2$

We get new coordinates:

$$x'=\frac{2x}{y}$$
$$y'=\frac{y}{2}$$

What are the expressions for $S_{i'j'}$?

One could think to compute each entry doing

$$S_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} S_{ij}$$

So, like the (1,1) one looks like

$$S_{1'1'} = \frac{\partial x^i}{\partial x^{1'}} \frac{\partial x^j}{\partial x^{1'}} S_{ij}$$

It is good exercise to do this. But it seems that there is a much faster way than this.

We can write the tensor S as

$$S = S_{\mu\nu} \left(dc^{\mu} \otimes dx^{\nu} \right)$$

and for our case

$$S = S_{ij} \left(dx^i \otimes dx^j \right)$$

The action of this tensor could be written as

$$S\left(dx^i, dx^j\right) = S_{11}dx^2 + S_{12}dxdy + S_{21}dydx + S_{22}dy^2 = dx^2 + x^2dy^2 \quad (2.38)$$

the two middle terms are not grouped because tensor product does not commute.

Now we can take the inverse coordinate transformation and write it down.

$$\begin{cases} x = x'y' \\ y = 2y' \end{cases} \rightarrow \begin{cases} dx = x'dy' + y'dx' \\ dy = 2dy' \end{cases} \quad (2.39)$$

and then we substitute inside eq. [2.38], getting

$$\left(x'dy' + y'dx'\right)^2 + 4dy'^2 =$$
$$x'^2dy'^2 + y'^2dx'^2 + x'y'\left(dx'dy' + dy'dx'\right) + 4dy'^2$$

so we get:

$$\begin{cases} S_{ii} = y'^2 \\ S_{ij} = S_{ji} = x'y' \\ S_{jj} = x'^2 + 4\left(x'y'\right)^2 \end{cases} \rightarrow \begin{pmatrix} y'^2 & x'y' \\ x'y' & x'^2 + 4\left(x'y'\right)^2 \end{pmatrix} = S_{i'j'} \quad (2.40)$$

2.7.4 Special Tensors

Special tensor that we will see are

- Derivative
- metric tensor
- Levi-Civita Symbol

Derivative

We will show that the derivative of a tensor is not a tensor anymore.

Derivative of a scalar is a (0,1) tensor.

$$\partial_\mu \phi \to \partial_{\mu'} \phi = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \phi$$

ϕ does not change under transformation, but the derivative does.

Derivative of a tensor $\neq$ tensor: **example** :

$$A_{\mu\nu} = \partial_\mu V_\nu$$

But why?

Let's transform it:

$$A_{\mu'\nu'} = \partial_{\mu'} V_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} V_\nu \right) =$$

now if we apply the partial derivative we obtain

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\mu V_\nu + \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \right) \left(\frac{\partial^2 x^\nu}{\partial x^{\nu'} \partial x^\mu} \right) V_\nu$$

What happened? We see there is a piece that we don't like. This second piece is 0 for LTs, because they are linear so second derivatives are null. This because in Euclidean space the derivative is independent on the coordinates, and on Minkowski space too, since it's flat.

The tensor itself is independent of the coordinate system, but the operation of taking a partial derivative is highly dependent on what coordinate system you're using.

We give importance to this anyway because GR is not a theory for just LTs. We will develop a covariant derivative that applied to a tensor will give back a tensor.

Metric tensor

The roles of the metric tensor are many:

- supplies a notion of *past* and *future*
- allows the computation of path length and proper time
- determines the *shortest distance* between two points
- replaces the Newtonian gravitational field ϕ
- provides a notion of locally inertial frames
- determines causality
- replaces the Euclidean 3D dot product of Newtonian mechanics.

We know the Minkowski's one:

$$\eta_{\mu\nu} = diag\,(-1,1,1,1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.41}$$

In general $g_{\mu\nu}$ will be the metric tensor.

We assume the determinant of $g_{\mu\nu}$: $det\,(g_{\mu\nu}) \equiv g \neq 0 \rightarrow$ the metric tensor is *invertible*.

$$g_{\mu\nu}g^{\nu\alpha} = \delta^{\alpha}_{\mu} \ ; \ g^{\mu\nu}g_{\nu\alpha} = \delta^{\mu}_{\alpha}$$

A constant metric tensor implies not curvature.

A metric tensor that depends explicitly on the coordinates must describe a non-flat space? **FALSE** e.g. the polar coordinates: writing $\eta_{\mu\nu}$ in them

$$\eta^{polar}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta, \end{pmatrix} \tag{2.42}$$

. Given a generic metric $g_{\mu\nu}$ and a given event P, it is always possible to find new coordinates:

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}}g_{\mu\nu}$$

And

$$g_{\hat{\mu}\hat{\nu}}\,(P) = \eta_{\hat{\mu}\hat{\nu}} \ ; \ \partial_{\hat{\sigma}}\,g_{\hat{\mu}\hat{\nu}}\,(P) = 0$$

This recalls the definition of LIFs.

2.8 Lec 9

Active and passive transformations

Active transformations change the physical position of a set of points relative to a fixed coordinate system.

Passive transformations leave the points fixed but change the coordinate system relative to which they are described. We prefer the first approach, we leave vectors untouched and transform RFs.

2.8.1 Still on Metric Tensor

We described a little this tensor in the previous lecture. I remember that:

$$det\,(g) = g \neq 0$$

in 3D Euclidean Space

$$g = \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and in spherical it would be like

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

In general the metric tensor has this form

$$g_{\mu\nu} = \operatorname{diag}\left(-1, \ldots, 1, \ldots, 0, \ldots\right)$$

If there are just '+1', the metric is Euclidean.

If I have one ' -1' and only ' $+1$' the metric is Lorentzian.

We will not discuss other combinations.

Today we will try to formalize EEP.

Let P be a spacetime point and

$$g_{\mu\nu}\left(P\right) = \text{ some generic matrix } \neq \eta_{\mu\nu}$$

I want to find new coordinates $x^{\hat{\mu}}$ such that $g_{\hat{\mu}\hat{\nu}}\left(P\right) = \eta_{\mu\nu}$ and $\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}}\left(P\right) = 0$. If you watch the last expression you see that it includes 40 different expressions $(64/2) + 8$

I choose $x_P^{\mu} = x_P^{\hat{\mu}} = 0$, so P is the origin of both frames. So I get

$$g_{\hat{\mu}\hat{\nu}}\left(x^{\hat{\alpha}}\right) = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} g_{\mu\nu}\left(x^{\alpha}\right)$$

I'm interested in transformations around P, so I see that spacetime is locally Minkoskian.

Doing a Taylor expansion I will see that first order is Minkoskian:

$$x^{\mu}\left(x^{\hat{\mu}}\right)=\left(\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\right)_{P}x^{\hat{\mu}}+\frac{1}{2}\left(\frac{\partial^{2}x^{\mu}}{\partial x^{\hat{\mu}}\partial x^{\hat{\nu}}}\right)_{P}x^{\hat{\mu}}x^{\hat{\nu}}+\frac{1}{6}\left(\frac{\partial^{3}x^{\mu}}{\partial x^{\hat{\mu}}\partial x^{\hat{\nu}}\partial x^{\hat{\rho}}}\right)_{P}x^{\hat{\mu}}x^{\hat{\nu}}x^{\hat{\rho}}+\dots \tag{2.43}$$

Why did he stop at 3^{rd} order? Listen to recording at $\sim 39:00$.

$$\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}=\left(\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\right)_{P}+\left(\frac{\partial^{2}x^{\mu}}{\partial x^{\hat{\mu}}\partial x^{\hat{\alpha}}}\right)_{P}x^{\hat{\alpha}}+\frac{1}{2}\left(\frac{\partial^{3}x^{\mu}}{\partial x^{\hat{\mu}}\partial x^{\hat{\alpha}}\partial x^{\hat{\beta}}}\right)_{P}x^{\hat{\alpha}}x^{\hat{\beta}} \tag{2.44}$$

$$g_{\mu\nu}\left(x^{\alpha}\right)=\left(g_{\mu\nu}\right)_{P}+\left(\partial_{\rho}g_{\mu\nu}\right)_{P}x^{\rho}+\frac{1}{2}\left(\partial_{\rho}\partial_{\sigma}g_{\mu\nu}\right)_{P}x^{\rho}x^{\sigma}+\dots \tag{2.45}$$

Now i just need to write down the metric tensor in new coordinates:

$$\hat{g}+\left(\hat{\partial}\hat{g}\right)_{P}\hat{x}+\left(\hat{\partial}\hat{\partial}\hat{g}\right)_{P}\hat{x}\hat{x}=\left(\frac{\partial x}{\partial\hat{x}}\frac{\partial x}{\partial\hat{x}}g\right)_{P}+\left(\frac{\partial x}{\partial\hat{x}}\frac{\partial^{2}x}{\partial\hat{x}\partial\hat{x}}+\frac{\partial x}{\partial\hat{x}}\frac{\partial x}{\partial\hat{x}}\partial g\right)_{P}\hat{x}+$$
$$+\left(\frac{\partial x}{\partial\hat{x}}\frac{\partial^{3}x}{\partial\hat{x}\partial\hat{x}\partial\hat{x}}g+\frac{\partial^{2}x}{\partial\hat{x}\partial\hat{x}}\frac{\partial^{2}x}{\partial\hat{x}\partial\hat{x}}g+\frac{\partial x}{\partial\hat{x}}\frac{\partial^{2}x}{\partial\hat{x}\partial\hat{x}}\partial\hat{g}+\frac{\partial x}{\partial\hat{x}}\frac{\partial x}{\partial\hat{x}}\hat{\partial}\hat{\partial}g\right)\hat{\partial}\hat{x}$$

This is the structure (with all indices suppressed) of the Taylor expansion (.rec..) around point P.

It's true to write:

$$\hat{g}=A+B\hat{x}+C\hat{x}\hat{x}+\dots$$

with A,B,C the three terms up there. So we can set terms of equal order in $\hat{x}$ on each side equal to each other. Therefore it's like having

$$\left(g_{\hat{\mu}\hat{\nu}}\right)_{P}=A=\left(\frac{\partial x}{\partial}\hat{x}\frac{\partial x}{\partial\hat{x}}g\right)_{P}$$

On the left we have 10 numbers in all to describe a symmetric two-index tensor, and they are determined by the matrix on the right, This is a 4×4 matrix without constraints, so we have enough freedom to put the 10 numbers of the left tensor into *canonical form*.

At first order we have, on the left, four derivatives of 10 components for a total of 40 numbers, while on the right side we have 10 choices of $\hat{\mu}$s and four choices of μs, for a total of 40 degrees of freedom. This is precisely the number of choices we need to determine all of the first derivatives of the metric, which we can therefore set to 0.

At second order with left side the item is symmetric on it's indices in pairs for a total of $10\times 10=100$ numbers, On the right-hand side we have symmetry in the three lower indices gaining 20 possibilities multiplied by four for the upper index we have 80 degrees of freedom, 20 fewer than we require to set the second derivative of the metric to 0.

2.8.2 Levi Civita symbol

We like tensors but sometimes we also like nontensorial objects. Let's remember the Levi Civita symbol

$$\tilde{\epsilon}_{\mu_1 \dots \mu_n} = \begin{cases} +1 \text{ if even permutations} \\ -1 \text{ if odd permutations} \\ 0 \text{ otherwise} \end{cases}$$

By definition this symbol has the components specified above in *any* coordinate system, and it is a symbol and not a tensor because it is defined to not change under coordinate transformations, We are able to treat him like tensor only in inertial coordinates in flat spacetime.

If $\epsilon_{\mu\nu\rho\gamma}$ was a tensor then it should transform like

$$\tilde{\epsilon}_{\mu'_1 \dots \mu'_n} = \frac{\partial x^{\mu_1}}{\partial x^{\mu_{1'}}} \cdots \frac{\partial x^{\mu_{n'}}}{\partial x^{\mu_{n'}}} \tilde{\epsilon}_{\mu_1 \dots \mu_n} \tag{2.46}$$

but this is **NOT TRUE**.

We are able to treat it as a tensor only in inertial coordinates of spacetime since LTs would have left the components invariant anyway.

It's behaviour can be related to the determinant of a generic matrix M:

$$\tilde{\epsilon}_{\mu_{1'} \dots \mu_{n'}} \|M\| = \tilde{\epsilon}_{\mu_1 \dots \mu_n} M^{\mu_1}_{\mu_{1'}} \dots M^{\mu_n}_{\mu_{n'}}.$$

For example, a 2×2 matrix:
M^{μ}_{ν} with $\mu, \nu = 1, 2$.

$$\tilde{\epsilon}_{12} \cdot det\,(M) = det\,(M) = \tilde{\epsilon}_{\mu_1 \mu_2} M^{\mu_1}_1 M^{\mu_2}_2 = \tag{2.47}$$

$$= \tilde{\epsilon}_{12} M^1_1 M^2_2 + \tilde{\epsilon}_{21} M^2_1 M^1_2 = M^1_1 M^2_2 - M^2_1 M^1_2 = ad - bc \tag{2.48}$$

Setting $M^{\mu}_{\mu'} = \partial x^{\mu} / \partial x^{\mu'}$, we get

$$\tilde{\epsilon}_{\mu'_1 \dots \mu'_n} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \tilde{\epsilon}_{\mu_1 \dots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}$$

If you notice, we moved the determinant from the left-hand side to the right-hand one, by reversing it.

$\tilde{\epsilon}$ is not a tensor because otherwise that determinant wouldn't be there. So it does not transform the way I want. Let's construct a Levi-Civita *tensor.*

Remember the transformation of the metric tensor?

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} g_{\mu'\nu'}$$

I can take the determinant and apply the Binet Rule [3]

$$\det g\,(x^{\mu}) = \det \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} \right)^2 \det g \left(x^{\mu'} \right)$$

[3] det(AB) = det(A)*det(B)

and rewrite it in

$$\det g\left(x^{\mu'}\right) = \frac{1}{\det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right)^2} \det g\left(x^{\mu}\right)$$

I see that g is not invariant if i change coordinates.

Weights Tensor densities, they almost transform like a tensor up to a given factor of a given power.

For example $\tilde{\epsilon}$ has weight $w = +1$

g has weight $w = -2$

Using this I get that

$$\sqrt{|g|}\tilde{\epsilon}_{\mu_1 \dots \mu_n} (\rightarrow w = 0) \equiv \epsilon_{\mu_1 \dots \mu_n}$$

So it is a tensor, because a tensor has $w = 0$. We introduce this distinction:

- $\tilde{\epsilon}$ is the **symbol**
- ϵ it the L-C **tensor**

For the L-C symbol with upper indices $\tilde{\epsilon}^{\mu_1 \cdots \mu_n}$ the values are

$$\epsilon^{\mu_1 \cdots \mu_n} = \operatorname{sgn}(g)\, \tilde{\epsilon}_{\mu_1 \dots \mu_n}$$

so we have a weight of -1 and so the relative tensor with upper indices is

$$\epsilon^{\mu_1 \cdots \mu_n} = \frac{1}{\sqrt{|g|}} \tilde{\epsilon}^{\mu_1 \cdots \mu_n}$$

2.9 Lec 10

2.9.1 Differential form

A *p-form* or *differential form* is a (0,p) that is completely antisymmetric. Examples of p-forms

- scalars are 0-forms
- dual vectors are 1-forms
- $\tilde{\epsilon}$ is a 4-form

P-forms do have a operation called *wedge product*:
be A a p-form and B a q-form, then $(A \wedge B)$ is a (p+q)-form, in detail

$$(A \wedge B)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots} B_{\mu_{p+1} \ldots \mu_{p+q}]}$$

So, for example, if A and B are both a 1-form,

$$(A \wedge B) = \frac{2!}{1!1!} A_{[\mu} B_{\nu]} = \frac{2!}{1!} \frac{1}{2!} \left(A_\mu B_\nu - A_\nu B_\mu\right) = A_\mu B_\nu - A_\nu B_\mu$$

Also note that

$$A \wedge B = (-1)^{pq} B \wedge A$$

There is also something about Integral over volume & principle of least action. To Be Filled

Exterior Derivative

There is also this operation, not really used, that is

$$d : p - \text{form} \to (p+1) - \text{form}$$

$$(dA)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \ldots \mu_n]}$$

It has a special property: dA is a tensor.

$$\partial_\alpha A_{\beta\gamma\delta\ldots} \tag{2.49}$$

$$\text{is not a tensor, we already saw that, because of the extra piece} \tag{2.50}$$

$$\text{that is symmetric and become 0 by anti-symmetrization} \tag{2.51}$$

$$\partial_{[\alpha} A_{\beta\gamma\delta\ldots]} \text{ is a tensor!} \tag{2.52}$$

Now let's see how this is related to integrals.

Be in 3D, can be cartesian coordinates (x,y,z) or spherical (r, θ, ϕ), and be the gravitational field $\Phi(x, y, z)$ or $\Phi(r, \theta, \phi)$. What's the integral over space of Φ?

$$\int_{\text{space}} \Phi dV$$

with

$$dV = dxdydz = \left(r^2 \sin^2\theta\right) dr d\theta d\phi$$

Thinking about our guidelines: we want to describe independently on the chosen coordinates. Φ is a scalar, so let's see it's integration

$$I = \int \Phi\left(x\right) d^n x \neq \int \phi\left(x'\right) d^n x'$$

because

$$d^n x^{\mu'} = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| d^n x^\mu$$

there is a Jacobian in there.

The integrand of an integral is a p-form, and the integral a real number.

$$d^n x = dx^0 \wedge \ldots \wedge dx^{n-1} = \frac{1}{n!}\tilde{\epsilon}_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}$$

This is the integration measure. When I change coordinate system I get

$$d^n x = \frac{1}{n!}\tilde{\epsilon}_{\mu_1 \ldots \mu_n}\left(dx^{\mu_1} \wedge \ldots dx^{\mu_n}\right) = \tag{2.53}$$

$$\frac{1}{n!}\tilde{\epsilon}_{\mu_1 \ldots \mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \times \left(dx^{\mu_1'} \wedge \ldots \wedge dx^{\mu_n'}\right) = \tag{2.54}$$

$$= \frac{1}{n!}\tilde{\epsilon}_{\mu_1' \ldots \mu_n'} det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(dx^{\mu_1} \ldots dx^{\mu_n}\right) \tag{2.55}$$

What did I get? That

$$d^n x = \left[det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\right]^{-1} d^n x'$$

$d^n x$ is not a tensor but it is a tensor density. We want an invariant integration measure $\sqrt{|g|}d^n x$, such that if Φ is a scalar also

$$\int d^n x \sqrt{|g|}\Phi$$

is a scalar.

Now why ∂_α does not give a tensor? $\partial_\mu A_\nu$ is not a tensor.

$$\partial_{\mu'} A_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\partial_\mu A_\nu + \frac{\partial^2 x^\nu}{\partial x^{\mu'} x^{\nu'}} A_\nu$$

This last piece is symmetric, so in the exterior derivative it cancels out.

2.9.2 Covariant Derivative

This is the only derivative that matters for real.

$$\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$$

the factor Γ is unknown and we need to construct it. Keep in mind that the combination of the two addends makes a tensor, but singularly they aren't.

The transformation need to be both *linear on V*

$$\nabla (T + S) = \nabla T + \nabla S$$

and needs to follow the *Leibniz product rule*[4]

$$\nabla (T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$$

$\Gamma^\nu_{\mu\alpha}$ is called *connection* and it is *not a tensor* nut we can think of it like a collection of numbers.

It is a Christoffel symbol.

Before diving in the deep of this symbol $\Gamma^\mu_{\nu\alpha}$ we want to know hot it transform. One could think the trivial way, because if we have two coordinates systems

$$\Gamma^{\alpha'}_{\mu'\nu'} \neq \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \Gamma^\alpha_{\mu\nu}$$

but this is not true since Γ is not a tensor.

Since we know that the covariant derivative is a tensor by construction, let's start from here

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu \tag{2.56}$$

How are the sides of the equality related? The left-hand side of eq.2.56 is

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\alpha'} V^{\alpha'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right) + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} V^\alpha \tag{2.57}$$

while the right hand side of eq.2.56 is

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha \right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\alpha} V^\alpha \tag{2.58}$$

we see that we have free indices μ', ν' on both sides. Now let's develop the left-hand side some more

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^\nu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} V^\alpha \tag{2.59}$$

Remember that for us partial derivatives commute.

Let's compare the sides:

[4] $(fg)' = f'g + fg'$ to have an example on things we already studied.

- the first on the right-side cancels out with the second on the left (green).
- Renaming the first term on the left to get V^α, lead to simplifying every vector on both sides.

We can *rename* the vector because everything is valid for a generic vector, and its indices were summed over.

We are left with

$$\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\alpha} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\Gamma^{\nu'}_{\mu'\alpha'} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\alpha}$$

And this is how Γ transforms.

If this derivation is still obscure to you and want to see it on paper, old-style, I did again the derivation of this result, see image 2.8. We see that Γ is not a tensor because of this extra piece after the transformation.

We saw that for a *vector* the covariant derivative acts like

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha}V^\alpha$$

Instead for a *dual vector* the derivative is defined as

$$\nabla_\mu \omega_\nu \equiv \partial_\mu \omega_\nu + \tilde{\Gamma}^\alpha_{\mu\nu}\omega_\alpha$$

The question that arises from this is obvious: *how* $\tilde{\Gamma}$ *is related to* Γ*?*

Let's compute

$$\nabla_\mu\left(\omega_\lambda V^\lambda\right) = \partial_\mu\left(\omega_\lambda V^\lambda\right)$$

This because the covariant derivative of a scalar *is* the derivative of a scalar. So, applying the Leibniz Rule:

$$\begin{gathered}
\nabla_\mu V^\lambda \cdot \omega_\lambda + V^\lambda \nabla_\mu \omega_\lambda = \partial_\mu V^\lambda \omega_\lambda + V^\lambda \partial_\mu \omega_\lambda \\
\left(\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\alpha}V^\alpha\right)\omega_\lambda + V^\lambda\left(\partial_\mu\omega_\lambda + \tilde{\Gamma}^\alpha_{\mu\lambda}\omega_\alpha\right) = \left(\partial_\mu V^\lambda\right)\omega_\lambda + \left(\partial_\mu \omega_\lambda\right)V^\lambda \\
\to \Gamma^\lambda_{\mu\alpha}V^\alpha\omega_\lambda + \tilde{\Gamma}^\alpha_{\mu\lambda}V^\lambda\omega_\alpha = 0 \\
\text{renaming some indices to get} \\
\Gamma^\lambda_{\mu\alpha}V^\alpha\omega_\lambda + \tilde{\Gamma}^\lambda_{\mu\alpha}V^\alpha\omega_\lambda \\
\implies \Gamma^\lambda_{\mu\alpha} + \tilde{\Gamma}^\lambda_{\mu\alpha} = 0
\end{gathered}$$

To conclude, the covariant derivative of a dual vector is actually

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\alpha_{\mu\nu}\omega_\alpha \tag{2.60}$$

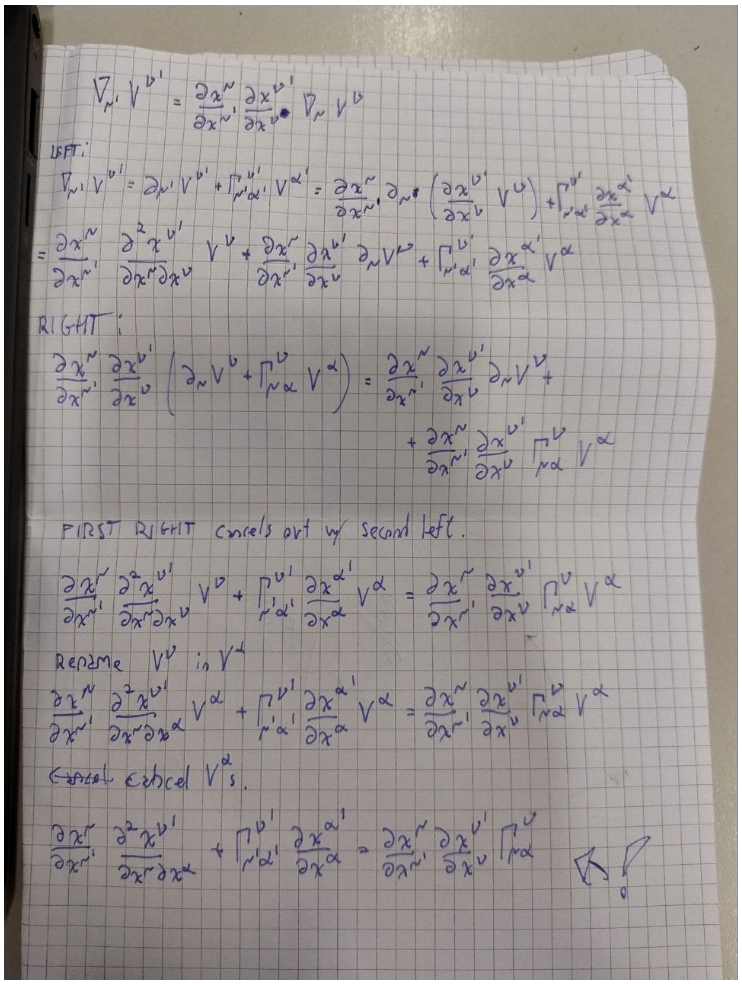


Figure 2.8: Same derivation but on paper. If you have any doubts compare with this one.

2.10 Lec 11

2.10.1 Covariant derivative - Connection

In the last lecture we talked about the covariant derivative and we saw the version for the vector, for the dual vector, and how it transform between two coordinates systems. We constructed it so the output is a tensor, and we saw that even after changes of coordinates we still get a tensor.

The question now is, how to do

$$\nabla_\rho T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = ?$$

The development is pretty boring but straight-forward:

$$= \partial_\rho T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} + \Gamma^{\mu_1}_{\rho\alpha} T^{\alpha\mu_2 \dots \mu_k}_{\nu_1 \dots \nu_l} + \Gamma^{\mu_2}_{\rho\alpha} T^{\mu_1\alpha\mu_3 \dots \mu_k}_{\nu_1 \dots \nu_l} + \dots - \Gamma^{\alpha}_{\mu\nu_1} T^{\mu_1 \dots \mu_k}_{\alpha\nu_2 \dots \nu_l} - \dots \quad (2.61)$$

These Γ connections are just tables of 64 entries of numbers, not tensors, and putting indices up and down to it it's abuse of notation.

Now we will make a couple of assumptions on the structure of Γ.

Torsion

Statement I Given two different connections $\Gamma^\mu_{\alpha\beta}$ and $\tilde{\Gamma}^\mu_{\alpha\beta}$, we define

$$S^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \tilde{\Gamma}^\mu_{\alpha\beta}$$

$\rightarrow S^\mu_{\alpha\beta}$ is a (1,2) tensor. Why? Since I have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$$

I can define a complement

$$\tilde{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \tilde{\Gamma}^\nu_{\mu\alpha} V^\alpha$$

so i get

$$\rightarrow \nabla_\mu V^\nu - \tilde{\nabla}_\mu V^\nu = \left(\Gamma^\nu_{\mu\alpha} - \tilde{\Gamma}^\nu_{\mu\alpha}\right) V^\alpha = S^\nu_{\mu\alpha} V^\alpha$$

and this is valid *only* if $S^\mu_{\nu\alpha}$ is a tensor.

Statement II if $\Gamma^\mu_{\alpha\beta}$ is a connection $\implies \Gamma^\mu_{\beta\alpha}$ is a connection.

That's why we define the *Torsion tensor*

$$T^\mu_{\alpha\beta} \equiv \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha} = 2\Gamma^\mu_{[\alpha\beta]}$$

The metric adopted in this course is a *Torsion-Free* metric, so the torsion tensor is vanishing.

How many entries do I have for a connection?

$$\Gamma^{\mu\rightarrow 4}_{\alpha\beta\rightarrow 10}$$

so in total I have 40 entries, 4 for the upper index and 10 for the lowers because symmetry,

As we will see later, the name *connection* comes from the fact that it is used to transport vectors from one tangent space to another.

Metric Compatibility

So, the torsion tensor is antisymmetric on its lower indices, and a connection that is symmetric on its lower indices is *torsion-free* We can define a unique connection on a manifold with metric $g_{\mu\nu}$ by introducing two additional properties, torsion-freeness and the metric compatibility. The metric compatibility is a property of the covariant derivative and it's expressed as follows

$$\nabla \rho g_{\mu\nu} = 0$$

A connection is *metric compatible* if the covariant derivative of the metric with respect to that connection is everywhere 0.

We want to see how this property works with the *inverse metric tensor*, so let's start from

$$\nabla_\rho \left(g^{\alpha\beta} g_{\beta\gamma}\right) = \nabla_\rho \left(\delta^\alpha_\gamma\right) = \Gamma^\alpha_{\rho\lambda}\delta^\lambda_\gamma - \Gamma^\sigma_{\rho\gamma}\delta^\alpha_\sigma + \partial_\rho \left(\delta^\alpha_\gamma\right) = \left(\Gamma^\alpha_{\rho\gamma} - \Gamma^\alpha_{\rho\gamma}\right) = 0 \quad (2.62)$$

the term with the partial derivative cancels out because δ is constant, and we equal everything to zero because the covariant derivative of the Kronecker delta is 0. On the right side we can apply the Leibniz rule so

$$g^{\alpha\beta}\nabla_\rho \left(g_{\beta\gamma}\right) + \nabla_\rho \left(g^{\alpha\beta}\right) g_{\beta\gamma} = 0 \quad (2.63)$$

the first term is 0, because we said so, the connection is metric compatible. We are left with

$$g_{\beta\gamma}\nabla_\rho \left(g^{\alpha\beta}\right) = 0$$

by multiplying on both sides $g^{\gamma\sigma}$

$$g^{\gamma\sigma} g_{\beta\gamma}\nabla_\rho \left(g^{\alpha\beta}\right) = 0$$

I get

$$\delta^\sigma_\beta \nabla_\rho \left(g^{\alpha\beta}\right) = \nabla_\rho \left(\delta^\sigma_\beta g^{\alpha\beta}\right) = \nabla_\rho \left(g^{\alpha\sigma}\right) = 0$$

So, in conclusion the covariant derivative of the inverse of metric tensor is null. It was not trivial.

After this we can see that a metric-compatible covariant derivative commutes with raising and lowering of indices, so for a generic vector V^ν

$$\nabla_\mu V^\nu = g_{\alpha\nu}\nabla_\mu \left(v^\mu\right) = \nabla_\mu \left(g_{\alpha\nu} V^\nu\right) = \nabla_\mu V_\nu$$

With non-metric compatible connections we would have to be very careful about index placement wen taking a covariant derivative.

There is exactly one torsion-free connection on a manifold that is compatible with some generic metric on that manifold.

We can demonstrate *existence* and *uniqueness* by deriving a manifestly unique expression for the connection coefficients in terms of the metric, so we will expand the equation of metric compatibility for three different permutations of the indices.

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \text{ (a)}$$
$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \text{ (b)}$$
$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0 \text{ (c)}$$

we see that *(a)-(b)-(c) = 0*, and it is obvious because individually they're equal to 0.

But let's see in detail what's happening

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \boxed{\Gamma^\alpha_{\rho\mu} g_{\alpha\nu}} - \boxed{\Gamma^\alpha_{\rho\nu} g_{\mu\alpha}} +$$
$$+\Gamma^\alpha_{\mu\nu} g_{\alpha\rho} + \boxed{\Gamma^\alpha_{\mu\rho} g_{\nu\alpha}} + \boxed{\Gamma^\alpha_{\nu\rho} g_{\alpha\mu}} + \Gamma^\alpha_{\nu\mu} g_{\rho\alpha} = 0$$

We subtract the highlighted ones and use the symmetry of the connection to get

$$2g_{\rho\alpha}\Gamma^\alpha_{\mu\nu} - \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}$$

We multiply both sides by $g^{\rho\sigma}$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} [\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}] \tag{2.64}$$

This is one of the most important expressions of the course.

We have seen that a flat spacetime has a Minkowski metric everywhere.

But how to define curvature?

The fact that $g_{\mu\nu}$ depends on the coordinates is not enough (e.g. spherical coordinates). But now we have a new object

Let's think a little bit: in flat spacetime if $\partial \leftrightarrow \nabla$ and this $\implies \Gamma = 0$. Is that true?

In 3D coordinates, using other coordinates, do connections have to be 0? Be

$$\overline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$

that's the gradient in cartesian coordinates, but in polar coordinates

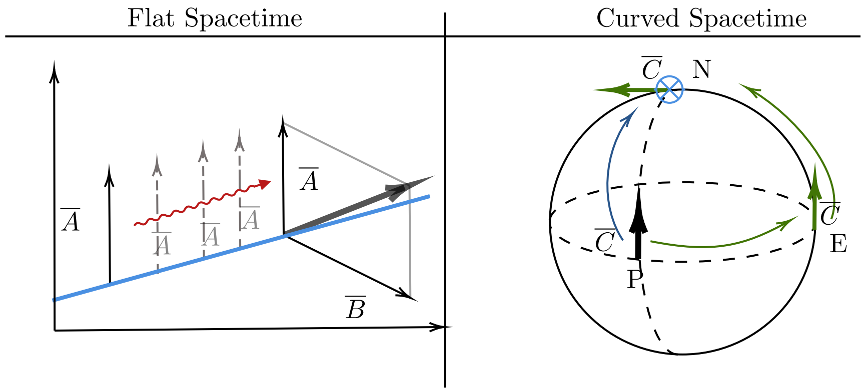
$$= \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta} \right)$$

I can compute the additional factor with christoffel symbols. Take $ds^2 = dr^2 + r^2 d\theta^2$, polar coordinates on plane, and compute the christoffel symbol. How many entries?

$$\Gamma^{\alpha \to 2}_{\beta\gamma \to 3}$$

I don't remember what the professor was trying to say but in doubt see Carroll.

2.10.2 Parallel Transport



In flat spacetime, if we want to add $\overline{A}+\overline{B}$, we move the $\overline{A}$ vector along the blue line. We *can* move $\overline{A}$ also along a non-straight line but $\overline{A}$ remains $\overline{A}$, and for the sum I get the same result.

In flat spacetime parallel transport does not depend on the path.

In curved spacetime, if I move $\overline{C}$ from P to N along the geodesic, we get that $\overline{C}$ points inside the sheet. But if I choose another path like P-E-N, then the outcome is a vector $\overline{C}$ that points toward left. It's not the same vector.

In curved spacetime parallel transport depends on the trajectory, we have to specify the path.

Be the path $x^{\mu}(\lambda):\mathbb{R}\rightarrow$ spacetime coordinates . In FST parallel transportation of a generic tensor is

$$\frac{d}{d\lambda}\left(T^{\mu_1\dots\mu_l}_{\nu_1\dots\nu_l}\right)=0$$

and the first factor is the directional derivative. Or

$$\frac{dx^{\sigma}}{d\lambda}\partial_{\sigma}\left(T^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_l}\right)=0$$

While in CST:

$$\frac{dx^{\sigma}}{d\lambda}\nabla_{\sigma}\left(T^{\mu_1\dots\mu_k}_{\nu_1\dots\nu_l}\right)=0$$

Let's see a simple example: imagine we have $x^{\mu}(\lambda)=(\lambda,0,0,0)$ so $x^{\mu}(0)=(0,0,0,0)$ and the four velocity is $V^{\mu}(0)=(1,0,0,0)$. In FST

$$\frac{dx^{\sigma}}{d\lambda}\left(\partial_{\sigma}V^{\mu}\right)=\frac{dx^{0}}{d\lambda}\partial_0V^{\mu}=\partial_0V^{\mu}=0$$

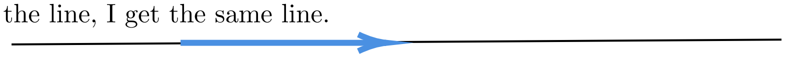
The vector does not change.

Why is it useful? Let's introduce

Geodesics

They are a straight line in curved space, the trajectory of a particle only subject to gravity. With *straight line* we mean the path that is parallel transported along it's tangent vector.

Example In FST we have a vector that have the same direction of a straight line, and we want to move it along that line. So if I transport the vector tangent to the line, I get the same line.



In CST, with path $x^{\mu}(\lambda)$, we have tangent

$$\frac{dx^{\mu}}{d\lambda}$$

and that gives

$$\frac{dx^{\sigma}}{d\lambda}\nabla_{\sigma}\left(\frac{dx^{\mu}}{d\lambda}\right)=0$$

It's a geodesics if this last condition is satisfied. Let's focus on this

$$\frac{dx^{\sigma}}{d\lambda}\left[\frac{\partial}{\partial x^{\sigma}}\frac{dx^{\mu}}{d\lambda}+\Gamma^{\mu}_{\sigma\alpha}\frac{dx^{\alpha}}{d\lambda}\right]=0$$

$$\rightarrow\frac{d^2x^{\mu}}{d\lambda}+\Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda}=0$$

the last one is the *geodesics equation*, and it's very important.

Suggested exercise: do that in polar coordinates in FST.

2.11 Lec 12

2.11.1 Geodesic Equation

Given a generic path (WL) $x^{\mu}(\lambda)$, parallel transport of a generic tensor is

$$\frac{D}{d\lambda}\left(T^{\mu_1\ldots\mu_k}\right)_{\nu_1\ldots\nu_l} = \frac{dx^{\sigma}}{d\lambda}\nabla_{\sigma}T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} = 0$$

while for a vector is

$$\frac{D}{d\lambda}\left(V^{\rho}\right) = \frac{dx^{\mu}}{d\lambda}\nabla_{\mu}V^{\rho} = \frac{dx^{\mu}}{d\lambda}\left[\partial_{\mu}V^{\rho} + \Gamma^{\rho}_{\mu\sigma}V^{\sigma}\right] = 0$$

We can look at the parallel transport equations as a first-order differential equation defining an initial-value problem: given a tensor at some point along the path, there will be a unique continuation of the tensor to other points along the path.

Probably was not said before but to make the parallel transport properly tensorial we need to replace the partial derivative by a covariant one, and define the *directional covariant derivative* as

$$\frac{D}{d\lambda} = \frac{dx^{\mu}}{d\lambda}\nabla_{\mu}$$

We will review *geodesic equation* and derive it again with another method.

Geodesic I

Geodesic is a path that transport its own tangent vector. That's the definition we will use for the first method. We schematized this in Lec 11, fig. 2.10.2.

$$\begin{aligned} &\frac{D}{d\lambda}\left(\frac{dx^{\mu}}{d\lambda}\right) = 0 \\ \to\ &\frac{dx^{\sigma}}{d\lambda}\nabla_{\sigma}\left(\frac{dx^{\mu}}{d\lambda}\right) = \frac{dx^{\sigma}}{d\lambda}\left[\partial_{\sigma}\left(\frac{dx^{\mu}}{d\lambda}\right) + \Gamma^{\mu}_{\sigma\rho}\frac{dx^{\rho}}{d\lambda}\right] = 0 \\ \to\ &\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0 \end{aligned}$$

so we get the *geodesic equation.*

Geodesic II

This time let's start from the concept of distance so

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$$

There is no point in trying to minimize it, because if it's the path of a photon it is 0, it's already minimized for each null-path.

Since the metric is compatible with the connection we are using, we take the parallel transportation of this quantity, that is a scalar, $g_{\mu\nu}V^{\mu}W^{\nu}$:

$$\frac{D}{d\lambda}\left(g_{\mu\nu}V^{\mu}W^{\nu}\right)=\left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^{\mu}W^{\nu}+g_{\mu\nu}\frac{D}{d\lambda}\left(V^{\mu}\right)W^{\nu}+g_{\mu\nu}V^{\mu}\left(W^{\nu}\right) \quad (2.65)$$

And we get that the first term on the right is 0 because of *metric compatibility* and the other two are 0 as well because we are parallel transporting a vector.

What does it mean? It means that scalar product is preserved by parallel transportation.

We will keep this lemma in mind while deriving the Geodesics Equation.

Now since in Lorentzian spacetime the definition of distance is kinda tricky let's stick to use proper time τ, instead. So for a time-like trajectory, like the one of a massive particle, we have

$$d\tau^2=-g_{\mu\nu}dx^{\mu}dx^{\nu} \text{ and } d\tau^2>0$$

Now, for every trajectory we fix the extremes so we get the so-called *proper time functional*

$$\tau\equiv\int_{path}\sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}}d\lambda$$

To search for the shortest distance one could do a thing called *calculus of variations* or, thinking about the twin paradox, recognize that the twin that experience more time is the one who stands still on the Earth. But we will try to check this out anyway. We can simplify the algebra writing the integral above as

$$\tau\equiv\int_{path}\sqrt{-f}d\lambda$$

$$\text{with } f=g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}.$$

Now we add perturbations to the path, and we pick the one with the maximum proper time.

Now, to extremize the proper time functional, we need that its variation is null. This implies that the tangent vector to the path $\frac{dx^{\mu}}{d\lambda}$, behaves consistently along the curve. This vector is normalized such that the scalar product f is constant, and the fact that parallel transport preserves the scalar product guarantees that this holds along the path.

So, extremes called A and B, are fixed, we took a generic trajectory $x^{\mu}(\lambda)$. We consider a *perturbation* of this trajectory such that

$$x^{\mu}\rightarrow x^{\mu}\delta x^{\mu}$$

$$g_{\mu\nu}\rightarrow g_{\mu\nu}+\partial_{\rho}g_{\mu\nu}\delta x^{\rho}$$

Where the second line comes from Taylor expansion in curved spacetime, which uses partial derivative not covariant one, because we are thinking of the components of $g_{\mu\nu}$ as functions of spacetime in some specific coordinates system.

Since we modify the trajectory, τ also changes

$$\delta\tau = \int -\frac{1}{2\sqrt{-f}}\delta f d\lambda$$

reminder/pro-tip: it's convenient to choose the proper time to parametrize trajectory because we chose it time-like.

As said before f is still kinda normalized (*-1*) even changing parameters.

Stationary points of the *functional of the proper time*, so path for which $\delta\tau = 0$, are equivalent to stationary points of this simpler integral

$$I = \frac{1}{2}\int f d\tau = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

adding the perturbations on I gets us

$$\begin{aligned}
\delta I &= \frac{1}{2}\int \left(\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + g_{\mu\nu}\frac{d\left(\delta x^\mu\right)}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx_\mu}{d\tau}\frac{d\left(x^\nu\right)}{d\tau}\right) d\tau \\
&= \frac{1}{2}\int \left(\partial_\rho g_{\mu\nu}\delta x^\rho\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} g_{\mu\nu}\frac{d}{d\tau}\delta\left(x^\mu\right)\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d}{d\tau}\delta\left(x^\nu\right)\right) d\tau
\end{aligned}$$

The last two terms can be integrated by parts,

$$\begin{aligned}
\int g_{\mu\nu}\frac{d}{d\tau}\left(\delta x^\mu\right)\frac{dx^\nu}{d\tau}d\tau &= \text{ boundary term } - \int \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\delta x^\mu d\tau \\
&= -\int \left(\partial_\rho g_{\mu\nu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right)\delta x^\mu d\tau
\end{aligned}$$

the boundary term vanishes because we take our variation δx^μ to vanish at the endpoints of the path.

So now we can insert back what we found in 2.11.1

$$\begin{aligned}
\delta I = \frac{1}{2}\int \partial_\rho g_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\rho d\tau - \frac{1}{2}\int \left(\partial_\rho g_{\mu\nu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right)\delta x^\mu d\tau + \\
-\frac{1}{2}\int \left(\partial_\rho g_{\mu\nu}\frac{dx^\rho}{d\tau}\frac{dx^\mu}{d\tau} + g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\right)\delta x^\nu d\tau
\end{aligned}$$

Now, we can make things easier if

- we use α instead of ρ in the first term
- α instead of μ in the second term
- α instead of ν in the third term
- ρ in second term is now μ
- ρ in third term is now ν
- ν in second term is now μ

so

$$\delta I = \int \left[\frac{(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\mu\alpha})}{2} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \frac{2}{2} g_{\mu\alpha}\frac{d^2x^\mu}{d\tau^2} \right] \delta x^\alpha d\tau \tag{2.66}$$

this has to be checked doing math by hand because probably there is a minus sign outside

Since we are searching for stationary points, we want δI to vanish for any variation δx^α, this implies

$$g_{\beta\alpha}\frac{d^2x^\beta}{d\tau^2} + \frac{1}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{2.67}$$

now if I multiply for the inverse metric $g^{\rho\alpha}$ the full expression

$$\frac{d^2x^\rho}{d\tau^2} + \frac{g^{\rho\alpha}}{2}\left[\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{2.68}$$

We see that we got the *Geodesic Equation* but with a specific choice for the Christoffel connection.

Some comments on geodesic I can change variables, like $\tau \to \alpha\lambda + \beta$ but the geodesic equation does not change.

The first derivation was hiding that λ need to be related to τ in a likear way. Since

$$u^\mu \nabla_\mu u^\nu = 0 \text{ or } p^\mu \nabla_\mu p^\nu = 0 \tag{2.69}$$

geodesics are trajectories defined by this equation, freely falling particles move like this.

2.12 Lec 13

This lecture is about establishing if spacetime is curved or not. We already attempt at this but failed:

- metric depends on the curvature
- $\Gamma^{\alpha}_{\beta\gamma} = 0$

Today we will develop an actual way to determine curvature of spacetime independently on coordinates chosen. This way is the *Riemann Tensor.*

Quick recall on LIC

Local Inertial coordinates. Given a generic spacetime with generic coordinates system x^{μ} and metric tensor $g_{\mu\nu}$, for a given spacetime point P we can find LIC $x^{\hat{\mu}}$.

So LIC at P

$$g_{\hat{\mu}\hat{\nu}}\left(P\right) = \eta_{\hat{\mu}\hat{\nu}} \tag{2.70}$$

obviously this is valid only in P. Also $\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}}\left(P\right) = 0$

Exponential Map

Geodesics provide a convenient way of mapping the tangent space T_P of a point P to a region of the manifold that contains P, called the **exponential map**. This map defines a set of coordinates for this region that are automatically the LIC, local inertial coordinates.

These is only one geodesic such that

$$x^{\mu}\left(\lambda = 0\right) \to P$$

Given a vector $k \in T_P$, it defines a unique geodesic passing through it, for which k is the tangent vector at P, and $\lambda\left(P\right) = 0$:

$$\frac{dx^{\mu}}{d\lambda}\left(\lambda = 0\right) \to k$$

The uniqueness is given from the fact that the geodesic equation is a second order differential equation, and specifying the initial data in the form as above determines a solution.

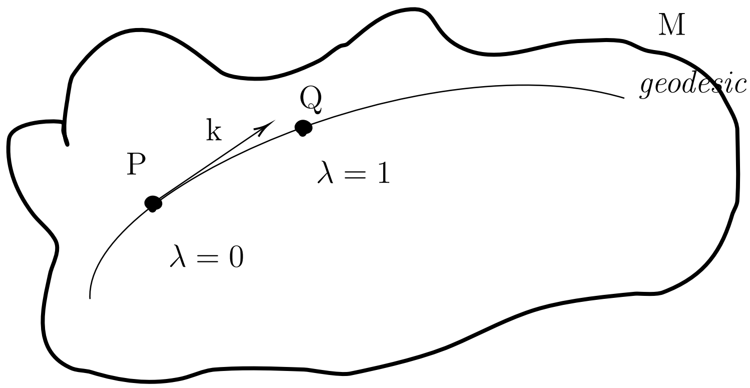
So the exponential map at P, so given a point in spacetime

$$\exp_P : T_P \to M \tag{2.71}$$

is a map from the tangent space to a point on the manifold M. And is defined as

$$\exp_P\left(k\right) = x^{\mu}\left(\lambda = 1\right) \to Q \tag{2.72}$$

The exponential map is invertible.



2.12.1 Riemann Normal Coordinates

Given a generic $g_{\mu\nu}$ and a point P, first I can find vectors $\hat{e}_{(\hat{\mu})}$, (where the hat on the e means that it is a basis vector and the hat on the index is means that is of inertial coordinates.)

Those are requirements to

$$g_{\hat{\mu}\hat{\nu}} = g\left(\hat{e}_{(\hat{\mu})}, \hat{e}_{(\hat{\nu})}\right) = \eta_{\hat{\mu}\hat{\nu}}$$

Where the *g(,)* denotes the metric thought as a multilinear map from $T_P \times T_P \to \mathbb{R}$. This is easy because starting with any set of components for $g_{\mu\nu}$ we can always diagonalize this matrix and rescale the basis vectors to satisfy the above relation.

Now we want to find a coordinates system and to do that we will use the exponential map. Let's define coordinates

$$x_P = 0$$
$$x_Q = x_Q^{\hat{\mu}} \hat{e}_{(\hat{\mu})} = k^{\hat{\mu}} \hat{e}_{(\hat{\mu})}$$

where $x_Q^{\hat{\mu}}$ is the inverse of exp_P, x_Q is the position vector and it is represented as a linear combination of basis vectors.

So, RNC are $x_Q = k^{\mu} \hat{e}_{(\hat{\mu})}$, and a particular property is that in this coordinates system

$$x^{\hat{\mu}}(\lambda) = \lambda k^{\hat{\mu}}$$

is a geodesic, only in these coordinates.

More in deep explanation

Now, I understand that if this is the first time approaching this it's kinda difficult to get the point. The part that makes the exponential map useful is that it's hard to find a coordinates system $x^{\hat{\mu}}$ for which the basis vectors $\{\hat{e}_{(h\mu)}\}$ are made of $\hat{e}_{(\hat{\mu})} = \partial_{\hat{\mu}}$m and such that the first partial derivatives of $g_{\hat{\mu}\hat{\nu}}$ vanish.

But the exponential map achieves that automatically. For any point Q sufficiently close to P, there is a unique geodesic path connecting P to Q, and a unique parametrization λ. At P the tangent vector can be written as a linear combination of our basis vectors $k = k^{\hat{\mu}}\hat{e}_{(\hat{\mu})}$. Then we define $x^{\hat{\mu}}$ to be these components $x^{\hat{\mu}}(Q) = k^{\hat{\mu}}$

Check on RNC

We want to verify that RNC satisfy $\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}}(P) = 0$. Noting that a ray in the tangent space gets mapped to a geodesic by the exponential map, leads to see that in RNC a curve $x^{\hat{\mu}}(\lambda)$ of the form

$$x^{\hat{\mu}}(\lambda) = \lambda k^{\hat{\mu}}$$

will solve the geodesic equation. Plugging this inside gives

$$\frac{d^2 x^{\hat{\mu}}}{d\lambda^2} + \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}} \frac{dx^{\hat{\alpha}}}{d\lambda}\frac{dx^{\hat{\beta}}}{d\lambda} = 0 \tag{2.73}$$

where the second derivative is null, and the other two partial derivatives correspond to $k^{\hat{\alpha}}, k^{\hat{\beta}}$. So we are left with

$$\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}} k^{\hat{\alpha}} k^{\hat{\beta}} = 0, \forall k \tag{2.74}$$

and since it true for every k

$$\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}(P) = 0$$

Now we apply the metric compatibility

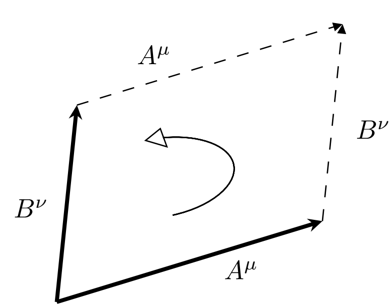
$$\nabla_{\hat{\rho}}(g_{\hat{\mu}\hat{\nu}}) = \partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}} - \Gamma - \Gamma \tag{2.75}$$

And each term is equal to zero. RNCs makes the LICs.

2.12.2 Riemann Curvature Tensor

As we already discussed, parallel transport of a vector around a closed loop in a curved space will result in a different vector than the one we started with. The transformation depends on the total curvature enclosed by the loop.

Since spacetime looks flat locally, if we define a loop made by two infinitesimal vectors A^{μ} and B^{ν} we can perform parallel transport on a vector V^{μ} by moving it anti-clockwise.



As we know the action of parallel transport is independent on the coordinates, so there should be a tensor that quantifies the change of the vector, so with an upper and a lower index. Depending also on vectors **A** and **B**, there are two additional lower indices. And this tensor must be anti-symmetric since we can walk the loop in the opposite direction, and the outcome would be the inverse. So we should get something like this

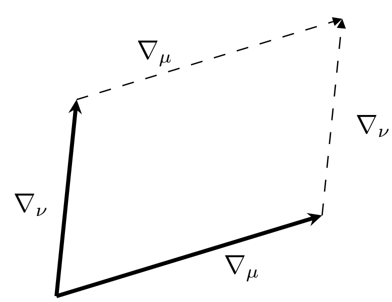
$$\delta V^\mu = R^\mu_{\alpha\beta\gamma} A^\alpha B^\beta V^\gamma \tag{2.76}$$

As decided above this tensor should be anti-symmetric on the *last two indices*.

$$R^\mu_{\alpha\beta\gamma} = -R^\mu_{\alpha\gamma\beta}$$

To compute what's inside the **R** tensor is kinda difficult so we take a related operation, the commutator of two covariant derivatives.

The relation between the two is because the covariant derivative of a tensor in a certain direction measures how much the tensor changes relative to what it would have been if it had been parallel transported, because the covariant derivative of a tensor in a direction along which is parallel transported is null. The commutator of two covariant derivatives measures the difference between parallel transporting the tensor first one way and then the other, and viceversa.



We could do the computation with a scalar but

$$[\nabla_\mu, \nabla_\nu]\phi = \nabla_\mu (\nabla_\nu \phi) - \nabla_\nu (\nabla_\mu \phi) = \partial_\mu (\partial_\nu \phi) - \partial_\nu (\partial_\mu \phi) = 0 \tag{2.77}$$

Not only is equal to zero, but it's trivially zero.

So we will use a vector

$$[\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu (\nabla_\nu V^\rho) - \nabla_\nu (\nabla_\mu V^\rho)$$

now we take just one of the terms

$$\nabla_\mu (\nabla_\nu V^\rho) = \partial_\mu (\nabla_\nu V^\rho) + \Gamma^\rho_{\mu\beta} \nabla_\nu V^\beta - \Gamma^\beta_{\mu\nu} \nabla_\beta V^\rho$$

again, we take the first term of the second part

$$\partial_\mu (\partial_\nu V^\rho + \Gamma^\rho_{\nu\alpha} V^\alpha) + \Gamma^\rho_{\mu\beta} \left(\partial_\nu V^\beta + \Gamma^\beta_{\nu\alpha} V^\alpha\right) - \Gamma^\beta_{\mu\nu} \nabla_\beta V^\rho$$

$$\partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\alpha}) V^\alpha + \Gamma^\rho_{\nu\alpha} \partial_\mu V^\alpha + \Gamma^\rho_{\mu\beta} \partial_\nu V^\beta + \Gamma^\rho_{\mu\beta} \Gamma^\beta_{\nu\alpha} V^\alpha - \Gamma^\beta_{\mu\nu} \nabla_\beta V^\rho$$

Now, let's look at this last row for some simplifications.

- The first term is symmetric in μ, ν.
- The sum of the third and the fourth is symmetric in μ, ν if we swap the dummy index $\beta \to \alpha$.
- The last term is symmetric but only with *zero torsion,*

The terms that are symmetric vanish since we are using the operator of anti-symmetry. So we are left, with

$$[\nabla_\mu, \nabla_\nu] V^\rho = \tag{2.78}$$

$$= \left(\partial_\mu \Gamma^\rho_{\nu\alpha} - \partial_\nu \Gamma^\rho_{\mu\alpha}\right) V^\alpha + \left(\Gamma^\rho_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\rho_{\nu\beta}\Gamma^\beta_{\mu\alpha}\right) V^\alpha - \left(\Gamma^\beta_{\mu\nu} - \Gamma^\beta_{\nu\mu}\right) \nabla_\beta V^\rho \tag{2.79}$$

This last piece has equivalent to the *torsion* that is

$$T^\beta_{\mu\nu} \nabla_\beta V^\rho$$

but since we work with torsion free connections, it's zero. We are left with

$$[\nabla_\mu, \nabla_\nu] V^\rho = [\partial_\mu \Gamma^\rho_{\nu\alpha} - \partial_\nu \Gamma^\rho_{\mu\alpha} + \Gamma^\rho_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\rho_{\nu\beta}\Gamma^\beta_{\mu\alpha}] V^\alpha \tag{2.80}$$

And this, can be written as

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma \tag{2.81}$$

that is the Riemann Tensor.

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Today we will talk about some properties of the Riemann tensor.

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma + \text{ torsion, (} \nabla_\rho V^\sigma)$$

and

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

The simplest way to derive the symmetries is to examine the Riemann tensor with all lower indices.

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda}R^\lambda_{\sigma\mu\nu}$$

There are four properties that can help us reduce the number of independent entries of this tensor from a number of n^4 to just 20 (in 4 dimensions spaces).

1. $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$
2. ...
3. ...
4. ...

So the Riemann tensor is anti-symmetric on the last two indices.

We now consider the components of this tensor in locally inertial coordinates $x^{\hat{\mu}}$ at some point P. Then the Christoffel symbols will vanish, but not their derivatives.

$$\begin{cases} g_{\hat{\mu}\hat{\nu}}(P) = \eta_{\hat{\mu}\hat{\nu}} \\ \partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}}(P) = 0 \to \Gamma^{\hat{\alpha}}_{\hat{\mu}\hat{\nu}}(P) = 0 \end{cases} \tag{2.82}$$

We are left with

$$\begin{aligned} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}(P) &= g_{\hat{\rho}\hat{\lambda}}\left(\partial_{\hat{\mu}}\Gamma^{\hat{\lambda}}_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\Gamma^{\hat{\lambda}}_{\hat{\mu}\hat{\sigma}}\right) = \\ &= g_{\hat{\rho}\hat{\lambda}}\frac{1}{2}\partial_{\hat{\mu}}[g^{\hat{\lambda}\hat{\alpha}}(\partial_{\hat{\nu}}g_{\hat{\alpha}\hat{\sigma}} + \partial_{\hat{\sigma}}g_{\hat{\alpha}\hat{\nu}} - \partial_{\hat{\alpha}}g_{\hat{\sigma}\hat{\nu}})] - (\hat{\mu} \leftrightarrow \hat{\nu}) \\ &= \frac{g_{\hat{\rho}\hat{\lambda}}}{2}g^{\hat{\lambda}\hat{\alpha}}[\partial_{\hat{\mu}}\partial_{\hat{\nu}}g_{\hat{\alpha}\hat{\sigma}} + \partial_{\hat{\mu}}\partial_{\hat{\sigma}}g_{\hat{\alpha}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\alpha}}g_{\hat{\sigma}\hat{\nu}}] - (\hat{\mu} \leftrightarrow \hat{\nu}) \end{aligned}$$

Now as usual let's look for some simplifications:

The first term inside the square brackets is symmetric so it goes away. The product of $g_{\hat{\rho}\hat{\lambda}}g^{\hat{\lambda}\hat{\alpha}} = \delta^{\hat{\alpha}}_{\hat{\rho}}$ so every α becomes a ρ. We are left with

$$R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} = \frac{1}{2}[\partial_{\hat{\mu}}\partial_{\hat{\sigma}}g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\rho}}g_{\hat{\sigma}\hat{\nu}} - \partial_{\hat{\nu}}\partial_{\hat{\sigma}}g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\rho}}g_{\hat{\sigma}\hat{\mu}}] \tag{2.83}$$

By looking at it, we see that the tensor is antisymmetric on it's first two indices

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$

also for exercise one can see that exchanging block, the tensor is invariant under interchange of the first pair of indices with the second:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

another thing that could be checked is that the complete anti-symmetrization of this tensor is null

$$R_{[\rho\sigma\mu\nu]} = 0$$

so the properties are

1. $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$
2. $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
3. $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$
4. $R_{[\rho\sigma\mu\nu]} = 0$

Now we have to count how many independent entries we are left with.

Starting from anti-symmetry on the first two and last two indices and symmetry on the exchange of this pairs, we can think of a symmetric matrix $R_{[\rho\sigma][\mu\nu]}$. The pairs are thought as individual indices. An $m \times m$ symmetric matrix has

$$\frac{m(m+1)}{2}$$

individual components, while the two anti-symmetric matrices have

$$\frac{n(n-1)}{2}$$

free components, so

$$\frac{1}{2}\left[\frac{1}{2}n(n-1)\right]\left[\frac{1}{2}n(n-1)+1\right] = \frac{1}{8}\left(n^4 - 2n^3 + 3n^2 - 2n\right)$$

Independent components, and putting $n = 4$ we get 21 independent entries.

But.

We know that a totally antisymmetric tensor with 4 indices has

$$\frac{n(n-1)(n-2)(n-3)}{4!}$$

terms and it helps reducing the number of independent components by 1.

So, in conclusion the Riemann tensor has 20 free components.

Bianchi Identity

It's a relation that tells us about the covariant derivative of the Riemann tensor. There is an algebraic version of the Bianchi Identity that is expressed as

$$R^{\rho}_{\sigma\mu\nu} + R^{\rho}_{\nu\sigma\mu} + R^{\rho}_{\mu\nu\sigma} = 0 \tag{2.84}$$

that states that the cyclic permutation of the lower three indices sum to zero.

There is a differential way to define the Bianchi Identity that is

$$\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0 \tag{2.85}$$

Ricci tensor

It is defined as

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{2.86}$$

Is it symmetric? Yes.

$$R_{\nu\mu} = R^{\lambda}_{\nu\lambda\mu} = g^{\rho\lambda} R_{\rho\nu\lambda\mu} = g^{\rho\lambda} R_{\lambda\mu\rho\nu} = R^{\lambda}_{\mu\lambda\nu} = R_{\mu\nu}$$

The trace of the Ricci tensor is the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \tag{2.87}$$

2.13.1 A kind of Einstein Equation

Professor gave us this equation, that is an arrival point

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2.88}$$

as we know $T_{\mu\nu}$ describe energy and momentum. We don't want to give up conservation of energy and momentum. We know in flat spacetime that

$$\partial_\mu T_{\mu\nu} = 0$$

and so in curved spacetime

$$\nabla_\mu T^{\mu\nu} = 0$$

Now we need a tensor in order to keep it null when we change frame. This means that also in the EE we need

$$\nabla_\mu \left(G_{\mu\nu} \right) = 0$$

where the tensor $G_{\mu\nu}$ called Einstein tensor is defined as

$$g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2.89}$$

let's prove this.

$$\begin{gathered} g^{\mu\lambda} g^{\nu\sigma} \left[\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} \right] = 0 \\ \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\mu R_{\rho\nu} = 0 \\ \text{or } \nabla^\mu R_{\rho\mu} = \frac{1}{2} \nabla_\rho R \end{gathered}$$

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2.14.1 Einstein Equation

Today we will derive the Einstein Equation. We will use a technique called *minimal coupling principle*. It has three steps

1. Take a physics law valid in flat spacetime (e.g. poisson equation for gravitational potential).
2. write it in a coordinates independent form (tensorial).
3. Assume it is valid for curved spacetime.

Let's start with the first step.

We consider the motion of freely-falling particles. From cartesian coordinates, x^μ, we have a straigt line

$$\frac{d^2x^\mu}{d\lambda^2} = 0$$

n.b. this is not a tensorial relation: $\frac{dx^\mu}{d\lambda}$ is a well-defined vector, but the second derivative is not. Now I change coordinates, but keeping spacetime flat. This is because a law that changes with coordinates is not coordinates independent.

$$\frac{d}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0$$

this one is composed by a vector inside the brackets, while the operator outside is not a tensor.

We could use the chain rule to write

$$\frac{d^2x^\mu}{d\lambda^2} = \frac{dx^\sigma}{d\lambda}\partial_\sigma \frac{dx^\mu}{d\lambda}$$

but the partial derivative is still a problem. Maybe we should use the covariant one. So we get

$$\frac{dx^\sigma}{d\lambda}\nabla_\sigma\left(\frac{dx^\mu}{d\lambda}\right) = \frac{dx^\sigma}{d\lambda}\left[\partial_\sigma\left(\frac{dx^\mu}{d\lambda}\right) + \Gamma^\mu_{\sigma\alpha}\frac{dx^\alpha}{d\lambda}\right] = 0$$

in the end

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\alpha}\frac{dx^\sigma}{d\lambda}\frac{dx^\alpha}{d\lambda} = 0$$

This is the geodesic equation. In general relativity we said freely-falling particles move along geodesics. Now we have generalized a small thing to curved spacetime but it is different from saying that this describes gravity.

So, let's show how results from the Newtonian limit fit in this picture. This limit has three requirements

- slowly moving particles $v \ll c$

- weak gravitational field, so the metric could be Minkowskian with a little perturbation
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, |h_{\mu\nu}| \ll 1$$
- the gravitational field is also static, it does not change in time ($\partial_0 g_{\mu\nu} = 0$)

We consider time-like geodesics, so trajectories of massive particles, so it is useful to parametrize them using the proper time τ.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

I want to use this to describe the motion under gravitational field, so we want to recover the $\vec{a} = -\vec{\nabla}\phi$. ϕ is hidden inside the connection, while $\vec{a}$ is the second derivative.

Since we required slow motion

$$\frac{dt}{d\tau} \gg \frac{dx^i}{d\tau}$$

So the geodesics equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0 \to \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{2.90}$$

In this situation I just need 4 entries for the connection, that is defined as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right]$$

so

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\alpha}\left[\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}\right] \tag{2.91}$$

since the condition of a static gravitational field, the time derivatives are equal to zero. We are left with

$$\partial_\alpha g_{00} = \partial_\alpha\left(\eta_{00} + h_{00}\right) = \partial_\alpha h_{00}$$

because the Minkowskian metric tensor is constant. Now, since the metric tensor is the Minkowskian one plus a little perturbation it could be assumed directly as minkowskian, so it is a first-order approximation.

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\alpha}\partial_\alpha h_{00} = -\frac{1}{2}\eta^{\mu\alpha}\partial_\alpha h_{00} = -\frac{1}{2}d^\mu h_{00} \tag{2.92}$$

At this point we can start to put together again the geodesic equation. We have

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\partial^\mu h_{00}\left(\frac{dt}{d\tau}\right)^2 \tag{2.93}$$

Now let's look what happens base on which coordinates we choose.

For $\mu = 0$ we are left with

$$\frac{d^2t}{d\tau^2} = 0 \implies \frac{dt}{d\tau} = \text{ const}$$

while for $\mu = i$

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial^i h_{00}\left(\frac{dt}{d\tau}\right)^2 \to \frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} \tag{2.94}$$

this because we divided both sides by $\left(\frac{dt}{d\tau}\right)^2$. As we said before the left term is exactly the acceleration

$$a^i = \frac{1}{2}\partial^i h_{00}$$

and if $h_{00} = -2\Phi$ we get

$$\vec{a} = -\vec{\nabla}\Phi \tag{2.95}$$

Also

$$g_{00} = -\left(1 + 2\Phi\right)$$

. Now, we would like to substitute the poissonian equation for Newtonian potential.

$$\nabla^2\Phi = 4\pi G\rho$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j = \partial_x^2 + \partial_y^2 + \partial_z^2$.

So we have a second-order differential operator acting on the gravitational potential and non the right side a measure of the mass distribution, but we want an equation between tensors. The tensor generalization of the mass density is the energy-momentum tensor $T_{\mu\nu}$. The gravitational potential should get replaced by the metric tensor, because as above we need perturbed the metric to reproduce gravity. So the first attempt could be like

$$\nabla^\sigma\nabla_\sigma\left(g_{\mu\nu}\right) = kT_{\mu\nu}$$

but it won't work because of metric compatibility $\nabla_\sigma\left(g_{\mu\nu}\right) = 0$.

There is a quantity that is constructed from second derivative of the metric that is the Riemann tensor, but since it has too many indices we will resort to the Ricci tensor $R_{\mu\nu}$. So it could be something like

$$R_{\mu\nu} = kT_{\mu\nu}$$

with some constant k. But there is a problem, with the conservation of energy. We like that

$$\nabla^\mu T_{\mu\nu} = 0$$

is true, this implies

$$\nabla^\mu R_{\mu\nu} = 0 \textbf{ BUT } \text{actually } \nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$$

That's an impasse. We will try to work it out writing the Bianchi Identity

$$
\begin{gathered}
g^{\nu\sigma} g^{\lambda\mu} \left[\nabla_\lambda R_{\rho\sigma\nu\mu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} \right] = 0 \\
\nabla^\mu R_{\mu\rho} - \nabla_\rho R + \nabla^\nu R_{\rho\nu} = 0 \\
2\nabla^\mu R_{\mu\rho} = \nabla_\rho R \to \nabla^\mu R_{\mu\rho} = \frac{1}{2} \nabla_\rho R \\
\nabla^\mu R_{\mu\rho} - \frac{1}{2} \nabla_\rho R = 0 \\
\nabla^\mu R_{\mu\rho} - \frac{1}{2} g_{\mu\rho} \nabla^\mu R = 0 \\
\nabla^\mu \left(R_{\mu\rho} - \frac{1}{2} g_{\mu\rho} R \right) = 0
\end{gathered}
$$

This is what we were looking for: a combination of Ricci tensor and scalar that has vanishing divergence!

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu} \tag{2.96}$$

It seems we are good about the left side, with

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu} \tag{2.97}$$

we still consider something that is slow moving in a weak gravitational field. So we want to see if it works in this setup, so we need to choose $T_{\mu\nu}$.

$$T_{\mu\nu} = \rho u_\mu u_\nu$$

with ρ is the rest frame energy, u^μ is the fluid four-velocity, where the fluid is a massive body such as the Earth. In flat spacetime: $u^\mu = (1, 0, 0, 0)$.

In curved spacetime: $u^\mu = \left(u^0, 0, 0, 0\right)$. The time component can be fixed resorting to $g_{\mu\nu} u^\mu u^\nu = -1$. So with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ we can get

$$
\begin{gathered}
g_{00} = -1 + h_{00} \\
g^{00} = -1 - h_{00}
\end{gathered}
$$

So in first order in $h_{\mu\nu}$

$$u^0 = 1 + \frac{1}{2} h_{00} \tag{2.98}$$

This means that

$$T_{00} = \rho u_0 u_0 \approx \rho \left(1 + h_{00} + \ldots\right) \tag{2.99}$$

Starting from the Ricci tensor we can see what happens to the Riemann tensor with 00 components

$$R_{00} = R^i_{0i0} \text{ roman indices since the others need to be different from 0}$$

$$R^i_{0j0} = \partial_j \Gamma^i_{00} - \partial_0 \Gamma^i_{j0} + \Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{j0}$$

$$R^i_{0j0} = \partial_j \Gamma^i_{00} = \partial_j \left[\frac{1}{2} g^{i_k} \left(\partial_0 g_{k0} + \partial_0 g_{k0} - \partial_k g_{00}\right)\right] =$$

$$= \frac{1}{2} \partial_j k^{ik} \left(-\partial_k k_{00}\right) = -\frac{1}{2} \partial_j \partial^i h_{00}$$

In the second step, the time derivative is null, since we assumed static gravitational field, while the third and fourth term are in the second order in the metric perturbation, so they can be neglected.

Now we would like to try to contract the proposed equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu}$$

$$R - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = k g^{\mu\nu} T_{\mu\nu}$$

$$R = -kT \rightarrow R_{\mu\nu} = kT_{\mu\nu} - \frac{1}{2} k g_{\mu\nu} T = k\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

So the 00 component of the Ricci tensor is

$$R_{00} = -\frac{1}{2} \nabla^2 h_{00} \tag{2.100}$$

for the energy-momentum tensor

$$T_{00} = \rho \tag{2.101}$$

, it's trace

$$T = g^{\mu\nu} \left(\rho u_\mu u_\nu\right) = -\rho \tag{2.102}$$

Regarding the second covariant derivative of h_{00}, since we have

$$G_{\mu\nu} = k T_{\mu\nu}$$

and eq.2.100, we get

$$\nabla^2 h_{00} = -k\rho \tag{2.103}$$

We also set before $h_{00} = -2\Phi$, this means that $k = (\pi G)$.

In conclusion we got the Einstein Equation for general relativity

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.104}$$

This equation is non-linear.

Chapter 3

Black Holes

3.1 Lec 16

3.1.1 Scharzschild's Metric

The most obvious application of a theory of gravity is the case of spherical symmetry, that is like the one of the Earth or the Sun. We will start with a solution of the vacuum outside them, because is both easier and more useful. A specific solution of the Einstein Equation is the Schwarzschild Solution, for a static (components of $g_{\mu\nu}$ do not depend on time) spacetime with $\left(S^2\right)$ symmetry.

There is more than one way to derive this solution, we will use the most boring one:

1. *Guess* a generic form for $g_{\mu\nu}$
2. Compute $\Gamma^{\mu}_{\alpha\beta}$, $R^{\alpha}_{\beta\gamma\delta}$, $R_{\alpha\beta}$
3. Solve $R_{\mu\nu} = 0$, since the space outside the sun is empty space, vacuum.

Let's start from the **guess** step. We want to *guess* the metric in spherical coordinates

$$ds^2 = -A\left(r\right)dt^2 + B\left(r\right)dr^2 + C\left(r\right)r^2 d\Omega^2 \tag{3.1}$$

where $d\Omega^2$ is the metric on a S^2 sphere.

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Since we are almost in the vacuum, we can start in fact from the Minkowski metric in polar coordinates: $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$. Now we would like simplify a little eq.3.1:

- $r^2 C\left(r\right) \to r^2$, this is a change of coordinates, nothing to do with spherical symmetry, but know we have to rescale $A\left(r\right), B\left(r\right)$
- $A \equiv e^{2\alpha(r)}$

- $B \equiv e^{2\beta(r)}$

. With this, the most general guess is:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{3.2}$$

We have now two unknown functions α, β of the radial coordinate, not time because of staticness, not angle because of isotropy.

Now it's time to **compute the metric tensor, Christoffel thing and Riemann tensor.**

Metric

$$\begin{array}{cccc} g_{tt} = -e^{2\alpha(r)} & g_{rr} = e^{2\beta(r)} & g_{\theta\theta} = r^2 & g_{\phi\phi} = r^2 \sin^2\theta \\ g^{tt} = -e^{-2\alpha(r)} & g^{rr} = e^{-2\beta(r)} & g^{\theta\theta} = \frac{1}{r^2} & g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta} \end{array} \tag{3.3}$$

other entries are null.

Christoffel Symbols In most generic spacetimes there are 40 independent Christoffel coefficients. As you may remember the Christoffel connection is defined as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right] \tag{3.4}$$

if $\rho = t$:

$$\Gamma^t_{\mu\nu} = \frac{1}{2} g^{t\sigma} \left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right] \tag{3.5}$$

from this I get contribution only for $\sigma = t$, because of how is defined the metric.

$$(\sigma = t) \to \Gamma^t_{\mu\nu} = \frac{1}{2} g^{tt} \left[\partial_\mu g_{t\nu} + \partial_\nu g_{t\mu} - \partial_t g_{\mu\nu}\right] \tag{3.6}$$

At this point, one immediately see that the last derivative is null since staticness.

If $\mu, \nu \neq t \to 0$ since components of the metric tensor with mixed indices are null.

$$\Gamma^t_{tr} = \frac{1}{2} g^{tt} \partial_r g_{tt} = \frac{1}{2}\left(-e^{-2\alpha}(r)\right)\left(-2\alpha' e^{+2\alpha(r)}\right) = \alpha' \to \Gamma^t_{tr} = \alpha' \tag{3.7}$$

Now for $\rho = r$, so σ must be r, too:

$$\Gamma^t_{\mu\nu} = \frac{1}{2} g^{rr} \left[\partial_\mu g_{r\nu} + \partial_\nu g_{r\mu} - \partial_r g_{\mu\nu}\right] \tag{3.8}$$

if also$\mu, \nu = r$

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \left[\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right] = \frac{1}{2} g^{rr} d_r g_{rr} = \beta' \to \Gamma^r_{rr} = \beta' \tag{3.9}$$

instead, if $\mu, \nu = t$:

$$\Gamma^r_{tt} = \frac{1}{2} g^{rr} \left(\partial_t g_{rt} + \partial_t g_{rt} - \partial_r g_{tt}\right) = -\frac{1}{2} g^{rr} \partial_r g_{tt} = \frac{1}{2} e^{-2\beta} e^{2\alpha} 2\alpha' = \alpha' e^{2(\alpha-\beta)} \tag{3.10}$$

The remaining components are given but it would be interesting retrieving them by yourself, since professor said that he could ask to compute one component of the Γ at the exam.

$$\begin{array}{ccc} \Gamma^r_{\theta\theta} = -re^{-2\beta} & \Gamma^r_{\phi\phi} = -r\sin^2\theta e^{-2\beta} & \Gamma^\theta_{r\theta} = \frac{1}{r} \\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta & \Gamma^\phi_{r\phi} = \frac{1}{r} & \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}, \end{array} \tag{3.11}$$

Riemann tensor As we know it is defined as

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

And since we know all the Γs we should know every Riemann tensor component. Eventually we care about $R_{\mu\nu} = 0$, so we will impose

$$R_{\alpha\alpha} = 0, \alpha \text{ fixed } \to R^\beta_{\alpha\beta\alpha}$$

- $R^t_{rtr} = -\alpha'' - \alpha'^2 + \alpha'\beta'$
- $R^t_{\theta t\theta} = -r\alpha' e^{-2\beta}$
- $R^t_{\phi t\phi} = -r\alpha'\sin^2\theta e^{-2\beta}$
- $R^r_{\theta r\theta} = r\beta' e^{-2\beta}$
- $R^r_{\phi r\phi} = r\beta'\sin^2\theta e^{-2\beta}$
- $R^\theta_{\phi\theta\phi} = \left(1 - e^{-2\beta}\right)\sin^2\theta$

How to compute the others?

$$R^r_{trt} = g^{rr}R_{rtrt} = g^{rr}R_{trtr} = g^{rr}g_{tt}T^t_{rtr} \tag{3.12}$$

So we can reconstruct them from elements we already have.

Ricci tensor We want $R_{\alpha\alpha} = 0$. Let's see:

$$R_{tt} = R^r_{trt} + R^\theta_{t\theta t} + R^\phi_{t\phi t} = e^{2(\alpha-\beta)}\left[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2\alpha'}{r}\right] \tag{3.13}$$

$$R_{\theta\theta} = \left[\left(\beta' - \alpha'\right)r - 1\right]e^{-2\beta} + 1 \tag{3.14}$$

$$R_{rr} = -\alpha'' - \alpha'^2 + \alpha'\beta' + 2\frac{\beta'}{r} \tag{3.15}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta \tag{3.16}$$

They are individually equal to zero.

I will use some of these Ricci tensor components to retrieve some informations about α, β.

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0$$
$$\frac{2}{r}\left(\alpha' - \beta'\right) = 0, r \neq 0$$
$$\alpha' + \beta' = 0 \rightarrow \alpha(r) + \beta(r) = \text{const}$$

it is possible to choose the time coordinates t such that the constant is null.

Now we can do some restrictions on the metric

$$\begin{cases} ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^{2d\Omega^2} \\ \alpha(r) + \beta(r) = \gamma \rightarrow \text{ constant} \end{cases} \tag{3.17}$$

this allows us to write

$$ds^2 = -e^{2(\gamma-\beta(r))} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 = \tag{3.18}$$
$$= -e^{-2\beta(r)} \left(e^{\gamma} dt\right)^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \tag{3.19}$$

we will operate this change of variables $dt' = e^{\gamma} dt$.

Now I can set $\gamma = 0$ and trying to fins the solution for $\alpha(r)$.

$$\alpha(r) = -\beta(r)$$

I use $R_{\theta\theta} = 0$ to find the missing function, solving for α:

$$\left[(-2\alpha\prime) r - 1\right] e^{2\alpha} + 1 = 0$$
$$\frac{\partial}{\partial r}\left(r e^{2\alpha}\right) = +1$$

this is a differential equation

$$r e^{2\alpha}(r) = r + D \text{ , D is a constant}$$
$$e^{2\alpha(r)} = 1 + \frac{D}{r}$$

I could call D radius, and introduce a new variable $R_S = -D$

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.20}$$

What is R_S? And what is his physical meaning? Newtonian gravitation helps us. This one is the most general solution of spherical symmetry and it has to reproduce something we already know.

In the weak field regime we discussed how

$$\begin{cases} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ h_{00} = -2\Phi \end{cases} \tag{3.21}$$

with Φ gravitational potential.

$$\Phi = -\frac{GM}{r}$$

so by comparison

$$R_S = 2GM$$

We need to adjust some dimensions still

$$[G]\,[M] = [F]\left[L^2\right]\left[M^{-2}\right][M] = [L]\left[T^{-2}\right]\left[L^2\right] = \left[L^3\right]\left[T^{-2}\right]$$

so if we divide for velocity squared we get a distance. This means

$$R_S = \frac{2GM}{c^2}$$

Now if one plugs in the values of the masses of Sun and Earth, he can find the relative R_S, that is something like 3km and 9mm. This is where the metric blows up.

Singularities The metric blows up for $r = R_S, 0$, but this shouldn't be a problem since the metric previously derived is valid well outside the Sun. It could be a problem if the object is bigger or if the radius of the object is smaller than the Schwarzschild radius,

What is a singularity? Be in polar coordinates in the plane

$$g^{\theta\theta} = \frac{1}{r^2}$$

the only issue here is with the coordinates chosen, not with the space itself. But if I focus on *scalars*, and I find one that blows up, then I have a singularity. Like

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 48\frac{G^2M^2}{r^6}$$

we see that it blows up at $r = 0$ but not at $r = R_S$

3.2 Lec 17

3.2.1 Geodesic of Schwarzschild

We derived the metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

now, the geodesic for test particles is

$$\frac{dx^2}{d\lambda^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

In the Newtonian limit there are two quantities that are the *effective potential*, that combines multiple effect into a single potential. It is defined as

$$V_{eff}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

where L is the angular momentum, r is the distance between the two masses, m is the mass of the orbiting body and the second term is the potential. There is the *energy, E*

$$E = \frac{1}{2} m\dot{r}^2 + V_{eff}(r)$$

we said about geodesic that they're a trajectory that parallel transport the momentum vector

$$p^\nu \nabla_\nu p^\mu = 0$$

lower one index, so

$$p^\nu \nabla_\nu p_\mu = 0$$

this is null also because of metric compatibility. I think this means that I can bring in the metric without spoiling the result. We remember that $p^\nu \equiv \frac{dx^\nu}{d\lambda}$. The equation will go in this way

$$\begin{aligned}
p^\nu \nabla_\nu p_\mu &= 0 \\
p^\nu \left[\partial_\nu p_\mu - \Gamma^\sigma_{\nu\mu} p_\sigma\right] &= 0 \\
\frac{dx^\nu}{d\lambda} \partial_\nu p_\mu &= \Gamma^\sigma_{\nu\mu} p^\nu p_\sigma \\
\frac{dx^\nu}{d\lambda} \frac{d}{dx^\nu} p_\mu &= \Gamma^\sigma_{\nu\mu} p^\nu p_\sigma \\
\frac{dp_\mu}{d\lambda} &= \Gamma^\sigma_{\nu\mu} p_\sigma p^\nu \\
\frac{dp_\mu}{d\lambda} &= \frac{1}{2} g^{\sigma\rho} \left[\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right] p_\sigma p^\nu \\
&= \frac{1}{2} p^\nu p^\rho \left[\partial_\mu g_{\rho\nu}\right] \\
\frac{dp_\mu}{d\lambda} &= \frac{1}{2} \partial_\mu \left(g_{\rho\nu}\right) p^\rho p^\nu \qquad (3.22)
\end{aligned}$$

$$(3.23)$$

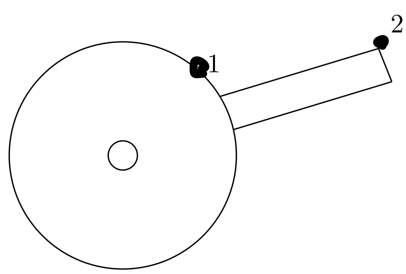
What does it mean? Imagine there is a coordinate that does not appear in $g_{\mu\nu}$. In fact looking at the expression of the metric at the start of the lecture, the metric does not explicitly depend on t. This corresponds to the stationarity of the Schwarzschild spacetime. The conserved quantity associated with this symmetry is the particle's energy E related to p_t.

Same for the azimuthal angle ϕ, the metric does not explicitly depend on it. It correspond to the spherical symmetry of this spacetime, invariance under rotation around the central mass. The conserved quantity associated with this symmetry is p_ϕ.

Still because of spherical symmetry we can focus on trajectories such that $\theta = \frac{\pi}{2}$, without loss of generality.

Gravitational Redshift

We put ourselves on Earth, far away from R_S.



We emit a photon from the bottom to the top of the tower, of frequency ν, we will find the relation between ω_1, ω_2.

This is a radial trajectory, this means that $d\phi = 0$. Now we will unravel the conserved quantity p_t

$$p_t = g_{t\alpha} p^\alpha = g_{tt} p^t$$
$$p_t = -\left(1 - \frac{2GM}{r}\right) \frac{dt}{d\lambda} = \text{ const } = \alpha$$

Now the goal is to find what ω_1 is. I am sitting on 1. I want to achieve the first frequency ω_1. In LIC we have

$$u^{\hat{\mu}} = (1, 0, 0, 0)$$
$$p^{\hat{\mu}} = (\omega, \omega \hat{n})$$

The angular frequency is given by

$$\omega = -u^{\hat{\mu}} p_{\hat{\mu}}$$

but this has to be true in every coordinate system so

$$\omega = -u^\mu p_\mu$$

Now, what we want is to know what is the angular frequency observed by observer. He has four velocity

$$u^\mu = (1,0,0,0)$$

and we know that

$$u^\mu u_\mu = -1$$

since almost all the components are null we get the same writing

$$g_{00}u^0u^0 = -1$$

and this leads us to find

$$\omega = -g_{00}u^0p^0 = \left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-\frac{1}{2}}\frac{dt}{d\lambda} = \alpha\left(1-\frac{2GM}{r}\right)^{-\frac{1}{2}}$$

so, as the photon moves radially, its energy will change.

$$\frac{\omega_2}{\omega_1} = \frac{\left(1-\frac{2GM}{r_1}\right)^{\frac{1}{2}}}{\left(1-\frac{2GM}{r_2}\right)^{\frac{1}{2}}} \approx 1-\frac{GM}{r_1}+\frac{GM}{r_2} = 1-\Delta\phi \tag{3.24}$$

or

$$\frac{\Delta\omega}{\omega} = -\Delta\phi$$

where this ϕ is the gravitational potential, not a coordinate.

Planets

Most of the experimental tests of GR involve the motion of test particles in the solar system, Einstein suggested three tests two of which are the gravitational redshift, we talked about it above and the precession of perihelia. Now we will study the latter. To start we need to understand how the motion in the plane of the ecliptic works and in particular how to describe orbits in this particular metric, the Schwarzschild one.

Since the orbits are elliptical we need to express the radius r in terms of the azimuthal angle ϕ.

We shall start from two conserved quantities along the geodesics we mentioned before, $p_t = -E, p_\phi = L$.

$$p_t = g_{tt}p^t = -\left(1-\frac{2GM}{r}\right)\frac{dt}{d\lambda} = -E$$

and

$$p_\phi = g_{\phi\phi}p^\phi = r^2 sin^2\theta\frac{d\phi}{d\lambda} = r^2\frac{d\phi}{d\lambda} = L$$

both quantities are expressed *per unit mass.*

We know that the four velocity satisfies the normalization condition

$$g_{\mu\nu}u^{\mu}u^{\nu} = -\epsilon_r$$

where ϵ_r is a place-holder that

$$\epsilon_r = \begin{cases} 0 \text{ if massless particle} \\ +1 \text{ if massive particle} \end{cases}$$

Now we apply the normalization in the Schwarzschild metric so

$$g_{tt}\left(\frac{dt}{d\lambda}\right)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon_r \tag{3.25}$$

by substituting the conserved quantities and their friends we got

$$-\left(1-\frac{2GM}{r}\right)\frac{E^2}{\left(1-\frac{2GM}{r}\right)^2} + \frac{1}{\left(1-\frac{2GM}{r}\right)}\left(\frac{dr}{d\lambda}\right)^2 + \frac{r^2L^2}{r^4} = -\epsilon_r \tag{3.26}$$

with some manipulation,

$$\frac{1}{2}E^2 = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)\left(\frac{\epsilon}{2}+\frac{L^2}{2r^2}\right) \tag{3.27}$$

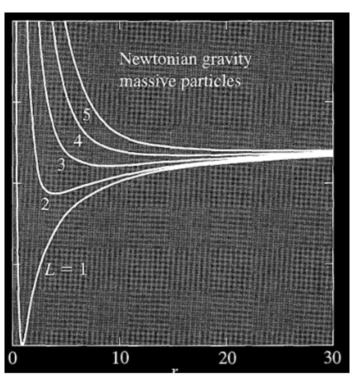
the second term of the right-hand side is $V_{eff}(r)$, the effective potential

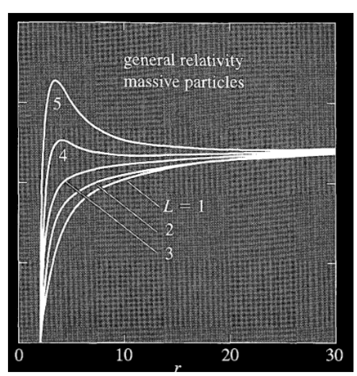
$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff}(r) = \frac{1}{2}E^2 = \mathcal{E} \tag{3.28}$$

Let's focus on this potential

$$V_{eff}(r) = \frac{\epsilon}{2} + \frac{L^2}{2r^2} - \frac{\epsilon GM}{r} - \frac{GML^2}{r^3} \tag{3.29}$$

the first term is constant, the second is centrifugal, the third is newtonian and the fourth is GR's son.





Effective potential in Newtonian (left) and in general relativity (right). GM is set equal to 1. On the vertical axis *V(r)*, on the horizontal one *r*. For Newtonian, for large enough energy every orbit reaches a turning point and returns to infinity. In GR there is an innermost circular orbit $\geq 3GM$, and any orbit that falls inside this radius continues to $r = 0$.

In fig. 3.2.1 we see different curves for different values of L. For all of this curves, the behaviour of the orbit of the particle woll be to move in the potential until it reaches a *turning point* where $V(r) = \mathcal{E}$, when it will begin to move in the other direction. Sometimes there is no turning point to hit, so the particle will keep going. In other cases the particle may simply move in a circular orbit at radius $r_c = const$. This happens where the potential is flat

$$\frac{dV}{dr} = 0$$

Differentiating eq.3.29 we find that this kind of orbit happens when

$$\epsilon GMr_c^2 - L^2 r_c + 3GML^2\gamma = 0$$

with

$$\gamma = \begin{cases} 0 \text{ for Newtonian gravity} \\ 1 \text{ for GR} \end{cases}$$

Circular orbits to be stable need to correspond to a minimum of the potential, viceversa unstable if they correspond to a maximum.

The critical radius r_c for a massive particle is

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM} \tag{3.30}$$

For large angular momenta L, there will be two circular orbits, one stable and one unstable. In the $L \to \infty$ limit their radii will be

$$r_c = \left(\frac{L^2}{GM}, 3GM\right) \tag{3.31}$$

the second solution is the solution for a circular orbit for a massless particle.

With decreasing L the two solutions come closer together , and coincide when

$$L = \sqrt{12}GM \text{ for which } r_c = 6GM$$

So this r_c is the smallest possible radius for a stable circular orbit in this metric.

Precession of Mercury's perihelium

With this we mean the rotation of the perihelium of the orbit of Mercury. It is measured as

$$\Delta\phi = \frac{43''}{\text{century}}$$

where ϕ is again the azimuthal angle.

This phenomenon reflects the fact that noncircular orbits in GR are not perfect closed ellipses. We will try to derive it on our own. The strategy is to describe the evolution of the radial coordinate r as function of ϕ.

We already talked about the fact that for a massive particle (e.g. Mercury), the symmetries of this spacetime, the Schwarzschild one, with metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} + r^2 d\Omega^2$$

lead to two conserved quantities, the momentum

$$p_\phi = L = g_{\phi\phi}\frac{d\phi}{d\lambda} = r^2\frac{d\phi}{d\lambda} \tag{3.32}$$

and the energy per unit mass

$$p_t = -E = g_{tt}\frac{dt}{d\lambda} = -\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} \tag{3.33}$$

. The normalization condition for the four-velocity rewritten with the values from the two above leads to

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff}(r) = \mathcal{E} \tag{3.34}$$

As said before, we would like to express the radius in function of ϕ, like $\frac{dr}{d\phi}$, so we can rewrite

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{dr}{d\phi}\frac{d\phi}{d\lambda}\right)^2 = \left(\frac{dr}{d\phi}\right)^2\frac{L^2}{r^4} \tag{3.35}$$

this allow us to write 3.34 as

$$\frac{1}{2}\left(\frac{dr}{d\phi}\right)^2\frac{L^2}{r^4} + V_{eff}(r) = \mathcal{E} \tag{3.36}$$

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{2r^4}{L^2}V_{eff}(r) = \frac{2r^4}{L^2}\mathcal{E} \tag{3.37}$$

We found a differential equation for $r(\phi)$, we will get to $\Delta\phi$.

3.3 Lec 18

3.3.1 Still about precession of perihelia

Last lecture we introduced the topic, dealing with conserved quantities along geodesics and expressing the radius of the orbit as $r(\phi)$.

We will introduce a new variable:

$$x \equiv \frac{L^2}{GMr} \propto \frac{1}{r}$$

that makes

$$\frac{dr}{d\phi} = \frac{d}{d\phi}\left(\frac{L^2}{GMx}\right) = \frac{L^2}{GM}\left(-\frac{1}{x^2}\right)\frac{dx}{d\phi}$$

and

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{L^4}{G^2M^2}\frac{1}{x^4}\left(\frac{dx}{d\phi}\right)^2 =$$

$$= \frac{L^4}{G^2M^2}\frac{G^4M^4}{L^8}r^4\left(\frac{dx}{d\phi}\right)^2$$

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 = \frac{G^2M^2}{L^4}\left(\frac{dx}{d\phi}\right)^2 \tag{3.38}$$

putting eq.3.38 in eq.3.36 yields:

$$\frac{1}{2}\frac{G^2M^2}{L^2}\left(\frac{dx}{d\phi}\right)^2 + V_{eff}\left(\frac{L^2}{GMx}\right) = \mathcal{E} \tag{3.39}$$

I want to express $V_{eff}(x)$, so

$$V_{eff\left(\frac{L^2}{GMx}\right)} = \frac{1}{2} - \frac{G^2M^2x}{L^2} + \frac{1}{2}\frac{G^2M^2x^2}{L^2} - GMl^2\left(\frac{GMx}{L^2}\right)^3$$

$$V_{eff}(x) = \frac{1}{2} - \left(\frac{GM}{L}\right)^2 x + \frac{1}{2}\left(\frac{GM}{L}\right)^2 x^2 - \left(\frac{GM}{L}\right)^4 x^3$$

$$\left(\frac{L}{GM}\right)^2 V_{eff}(x) = \frac{1}{2}\left(\frac{L}{GM}\right)^2 - x + \frac{x^2}{2} - \left(\frac{GM}{L}\right)^2 x^3 \tag{3.40}$$

This is useful since we can plug this in the equation for conservation of energy, 3.39:

$$\frac{1}{2}\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{L}{GM}\right)^2 V_{eff}(x) = \left(\frac{L}{GM}\right)^2 \mathcal{E}$$

$$\frac{1}{2}\left(\frac{dx}{d\phi}\right)^2 + \frac{1}{2}\left(\frac{L}{GM}\right)^2 - x + \frac{1}{2}x^2 - \left(\frac{GM}{L}\right)^2 x^3 = const$$

$$\rightarrow \frac{dx}{d\phi}\frac{d^2x}{d\phi^2} - \frac{dx}{d\phi} + \frac{2}{2}x\frac{dx}{d\phi} - 3\left(\frac{GM}{L}\right)^2 x^2\frac{dx}{d\phi} = 0$$

$$\frac{d^2x}{d\phi^2} = 1 - x + 3\left(\frac{GM}{L}\right)^2 x^2 \tag{3.41}$$

This last one, eq.3.41 is a differential equation that describes the orbit of planets. The one from Newtonian approach is

$$\frac{d^2x_0}{d\phi^2} = 1 - x_0$$

with $x_0 = 1 + ecos\phi$, and so $r_0 = \frac{L^2/GM}{1+ecos\phi}$.

We will treat the term that does not appear in the newtonian one as perturbation.

We can expand our variable x into the Newtonian solution plus a small correction

$$x = x_0 + x_1 \tag{3.42}$$

with $x_0 \ll x_1$ and we plug it into eq.3.41, that becomes

$$\begin{aligned}
\frac{d^2x_0}{d\phi^2} + \frac{d^2x^1}{d\phi^2} &= 1 - x_0 - x_1 + 3\left(\frac{GM}{L}\right)^2 (x_0 + x_1)^2 \\
\frac{d^2x_1}{d\phi^2} &\approx -x_1 + 3\left(\frac{GM}{L}\right)^2 x_0^2 \\
\frac{d^2x_1}{d\phi^2} + x_1 &= 3\left(\frac{GM}{L}\right)^2 \left[1 + 2ecos\phi + e^2cos^2\phi\right] \\
\frac{d^2x_1}{d\phi^2} + x_1 &= 3\left(\frac{GM}{L}\right)^2 \left[\left(1 + \frac{1}{2}e^2\right) + 2ecos\phi + \frac{1}{2}e^2cos2\phi\right] {}^{1}
\end{aligned} \tag{3.43}$$

While our dear Mr. Carrol tells us how one could solve this differential equation, our beloved professor D'Eramo, tells us directly the solution that is

$$x_1 = 3\left(\frac{GM}{L}\right)^2 \left[\left(1 + \frac{1}{2}e^2\right) + e\phi sin\phi - \frac{1}{6}e^2cos2\phi\right] \tag{3.44}$$

Inside the square brackets there are terms with different meaning:

- the first is just a constant
- the second has the most interesting effect
- the third has a periodic effect, it oscillates around zero.

We decide that we will deal with x_1 made just by this second term, so the expansion of x (3.42) becomes

$$x = 1 + ecos\phi + \frac{3G^2M^2e}{L^2}\phi sin\phi + \dots \tag{3.45}$$

as we don't care of those other terms since they have little use for us. To lighten up the notation we introduce

$$\alpha \equiv 3\left(\frac{GM}{L}\right)^2$$

[1] because $cos^2\phi = \frac{1+cos2\phi}{2}$.

Equation 3.45 can be rewritten as equation for an ellipse

$$x = 1 + ecos\left[(1-\alpha)\,\phi\right] \tag{3.46}$$

indeed

$$cos\left[(1-\alpha)\,\phi\right] = cos\phi + \alpha\frac{d}{d\alpha}cos\left[(1-\alpha)\,\phi\right]_{\alpha=0} = cos\phi + \alpha\phi sin\phi$$

Since the inverse proportionality, the minimum radius is when the x is max, so

$$x_P = 1 + e$$

and is achieved for $x = 0$. The angular shift per revolution is calculated by finding the difference between the actual angle and 2π:

$$\Delta\phi = 2\pi\alpha = 6\pi\frac{G^2M^2}{L^2}$$

Now we want to convert the angular momentum L to more usable quantities. To do so we will use some expressions from the Newtonian orbits, since the quantity we are looking for is already a small perturbation. An ordinary ellipses

$$r = \frac{\left(1-e^2\right)a}{1+ecos\phi}$$

with a is the semi-major axis. To find a

$$r_P + r_A = 2a = \frac{L^2}{GM}\left[\frac{1}{1-e} + \frac{1}{1+e}\right]$$

so

$$a = \frac{L^2}{GM\left(1-e^2\right)} \rightarrow L^2 = GM\alpha\left(1-e^2\right)$$

that leads us to

$$\Delta\phi = 6\pi\frac{GM}{a\left(1-e^2\right)c^2} \tag{3.47}$$

we added c so it is dimesionally good.

3.3.2 Finally Black Holes

I think it's beautiful starting every time from just the metric. Ours is obviously

$$ds^2 = -1\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2d\Omega$$

We see that there are two potential problems:

- $r = 0$

- $r = 2GM$

in particular the second solution is called *event horizon.* It seems that this solution is a coordinate singularity, but this singularity is not physical, it arises from the definition of the Schwarzschild coordinates. If we contract the Riemann tensor with itself

$$R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} = \frac{48G^2M^2}{r^6}$$

we see that indeed $r = 2GM$ is not a singularity.

The event horizon is the boundary beyond which nothing can escape, classically the escape velocity at this point equals the speed of light, beyond it increases.

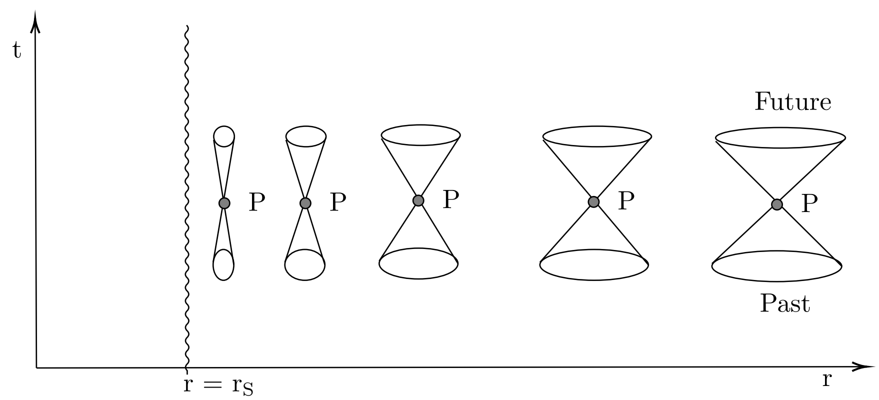
To analyze this we are interested in using light-like geodesics.

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega \tag{3.48}$$

Since spherical symmetry, variations on the angles do not care. This let us write

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1} \tag{3.49}$$

We can use this to measure the slope of a light-cone on a spacetime diagram.



We see that the closer to the event horizon the steeper becomes the slope of the light-cone, So a light ray that approaches $r = 2GM$ never seems to get there in this coordinates system, but asymptote to it.

This is just an illusion, a particle has no trouble in reaching the Schwarzschild radius, but an observer far away would never be able to tell. Indeed if *friend*[2] dives in the black hole while sending signals at regular pace, we would see signal become less frequent. Proper time of *friend* is

$$d\tau = \left(1 - \frac{2GM}{r}\right)^{1/2} dt$$

[2] if he was really your friend, you wouldn't have let him

The fact that his trajectory in the t-r plane never reaches r_S is not a statement we like, instead we never *see* him reach r_S. This is dependent on our coordinates system, so we want to change to a system that behaves better at r_S.

Tortoise coordinates

The time of *friend* grows infinite approaching r_s, The tortoise coordinate is intended to grow infinite at the appropriate rate such as to cancel out the singular behaviour. We start from eq.3.49,

$$\int dt = \pm \int \frac{dr}{1 - \frac{2GM}{r}} \tag{3.50}$$

$$t = \pm r^* + \text{ constant} \tag{3.51}$$

$$r^* = t - t_0 = \pm \left(r + 2GM \ln \left[\frac{r}{2GM} - 1 \right] \right) \tag{3.52}$$

So we want to transform from $(t, r) \to (t, r^*)$. So to rewrite the metric,

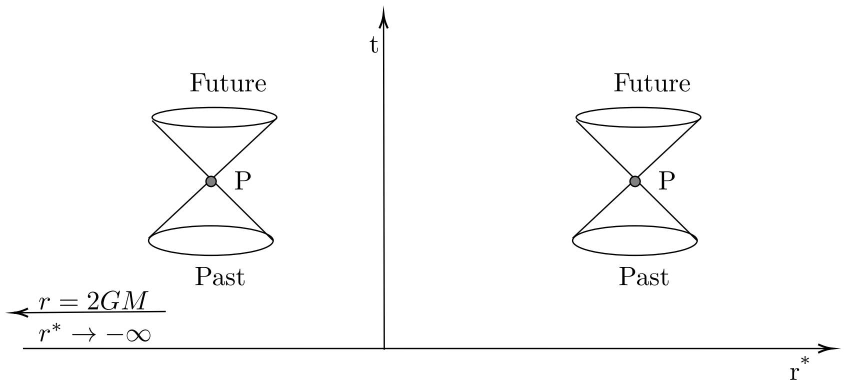
$$dr* = \frac{dr}{\left(1 - \frac{2GM}{r}\right)}$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\left(dt^2 - dr^{*2}\right) + r^2 d\Omega \tag{3.53}$$

with r as function of r^*.

3.4 Lec 19

We see that with this metric, the light cones do not close up.



The big *con* is that the surface of the r_S is positioned to infinity.

Next move is to define coordinates adapted to light-like geodesics

$$v \equiv t + r^* \tag{3.54}$$
$$u \equiv t - r^* \tag{3.55}$$

then *infalling* radial light-like geodesics are characterized by $v =$ constant, the *outgoing* ones $u =$ constant

3.4.1 Eddington-Finkelstein

Now we want to have a coordinates change from $(t, r) \to (v, r)$, these are EF coordinates. v is good to discuss infalling geodesics.

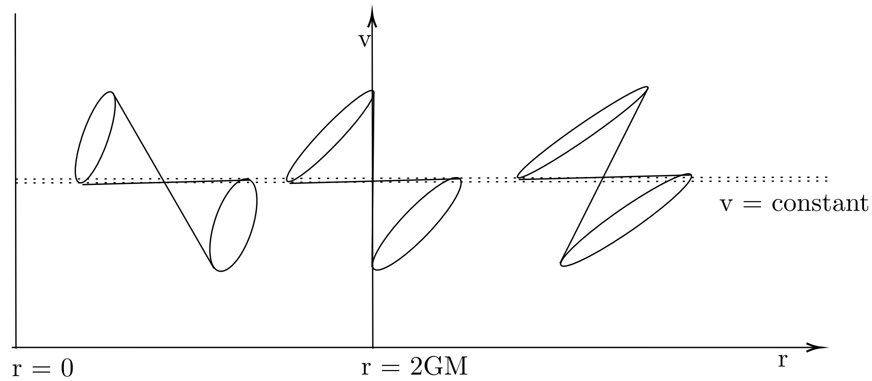
$$dv = dt + dr^* = dt + \frac{dr}{1 - \frac{2GM}{r}}$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = \tag{3.56}$$
$$= -\left(1 - \frac{2GM}{r}\right)\left(dv - \frac{dr}{1 - \frac{2GM}{r}}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = \tag{3.57}$$
$$= -\left(1 - \frac{2GM}{r}\right) dv^2 + (dvdr - drdv) + r^2 d\Omega. \tag{3.58}$$

In EF coordinates the condition for radial null curves is solved by

$$\frac{dv}{dr} = \begin{cases} 0, & \text{(infalling)} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1}, & \text{outgoing} \end{cases} \tag{3.59}$$

In this coordinates system the light cones remain well behaved at $r = r_S$. But we see that light-cones tilts, such that for $r < 2GM$ all-future directed paths are in the directions of decreasing r.



The surface at $r = r_S$, acts as a point of no return, once a test particle passes it, it can never come back, An event horizon is a surface past which particles can never escape to infinity. Since nothing can escape it, we cannot see inside, so it's a *black hole*, which is a region of spacetime separated from infinity by event horizon.

In (v, r) coordinates system we can cross the event horizon on future-directed paths, but not on past-directed ones. Well, wtf,

If we have chosen u instead of v the metric would have been

$$(t, r) \to (u, r)$$

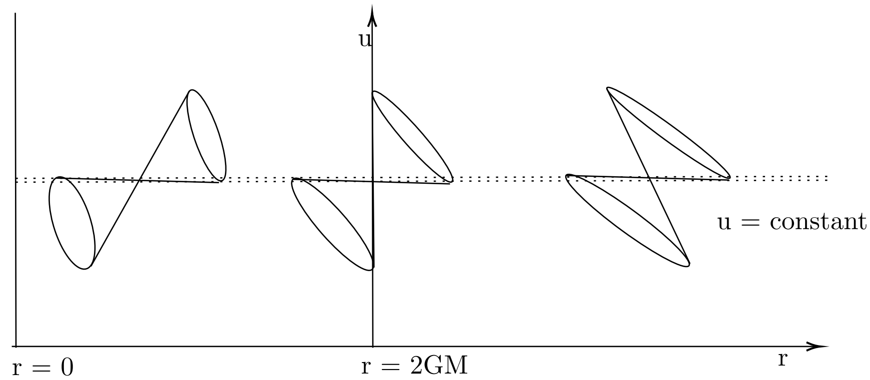
$$du = dt - dr^* = dt - \frac{dr}{1 - \frac{2GM}{r}}$$

$$\begin{cases} \frac{dv}{dr} = -2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{infalling} \\ du = 0 & \text{outgoing} \end{cases}$$

and the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - (dudr + drdu) + r^2 d\Omega \qquad (3.60)$$

Now we can pass through the event horizon but this time along past-directed curves.



So, we can follow either future or past directed geodesics through r_S, but we arrive at different places. If we keep v constant and decrease r we must have $t \to +\infty$, while for u constant we get $t \to -\infty$.

Something to digest BHs First, external geometry of a black hole is the Schwarzschild solution valid for any star or planet, this means BHs are not cosmic vacuum cleaners.

Second, if I did understand, it is technically possible escape a Newtonian BH, indeed c in here is just a number, and while light can't escape the BH travelling inertially, it could be accelerated by something and manage to escape. While in GR, inside the event horizon, the geometry of the spacetime is curved in such a way that all paths lead inward.

3.4.2 Kruskal-Szekeres coordinates

Enough with light-like geodesics, now we try with space-like geodesics. A first guess one could do is to use both u,v.

$$\begin{aligned} v &= t + r^* \\ u &= t - r^* \\ dv &= dt + dr^* = dt + \frac{dr}{1 - \frac{2GM}{r}} \\ du &= dt - dr^* = \dots \end{aligned}$$

we see

$$\begin{aligned} dt &= \frac{du + dv}{2} \\ dr &= \left(\frac{du - dv}{2}\right)\left(1 - \frac{2GM}{r}\right) \end{aligned}$$

so the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = \tag{3.61}$$

$$= -\left(1 - \frac{2GM}{r}\right) \left(\frac{du + dv}{2}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{+1} \left(\frac{du - dv}{2}\right)^2 = \tag{3.62}$$

$$= -\left(1 - \frac{2GM}{r}\right) dudv \tag{3.63}$$

so for $r \to 2GM$, $r^* \to -\infty$, $v \to -\infty$ and $u \to +\infty$.

We can make an improvement, since in this coordinates r_S is at infinity like at the start. We choose new coordinates that are

$$\begin{aligned} u' &= -e^{-\frac{u}{4GM}} \\ v' &= +e^{\frac{v}{4GM}} \\ du' &= \frac{1}{4GM} e^{-\frac{u}{4GM}} du \\ dv' &= \frac{1}{4GM} e^{+\frac{v}{4GM}} dv \end{aligned}$$

this makes

$$dudv = 16G^2M^2 e^{\frac{u-v}{4GM}} du'dv' \tag{3.64}$$

$$(u - v = r^*) \tag{3.65}$$

$$= 16G^2M^2 e^{-\frac{1}{2GM}\left(r + 2GM \ln\left(\frac{r}{2GM} - 1\right)\right)} du'dv' \tag{3.66}$$

the metric becomes

$$ds^2 = \frac{-32G^3M^3}{r} e^{-\frac{r}{2GM}} du'dv' \tag{3.67}$$

this does not explode at $r = r_S$.

Reminder: we used the notation $2dudv$ that should be instead $dudv + dvdu$, because tensor product does not commute.

Coordinates u' and v' are null coordinates, since their partial derivatives are null vectors. We have four partial derivatives vectors (two null and two space-like) as basis for the tangent space. We prefer work with a system where coordinates are one time-like and the others three space-like. We define

$$\begin{aligned} T &= \frac{1}{2} (v' + u') \\ R &= \frac{1}{2} (v' - u') \end{aligned}$$

these give us

$$ds^2 = \frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} \left(-dT^2 + dR^2\right) \tag{3.68}$$

where r is defined from

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{\frac{r}{2GM}} \tag{3.69}$$

These coordinates are the Kruskal-Szekres coordinates, (T, R, θ, ϕ).

Radial null curves, here, or light-like geodesics are

$$T = \pm R + const \tag{3.70}$$

while event horizon is at

$$T = \pm R \tag{3.71}$$

consistent with being a null surface.

Surfaces are $r =$ constant so

$$T^2 - R^2 = \text{ constant} \tag{3.72}$$

these looks like hyperbolae in R-T plane.

Surfaces of constant t are given by

$$\frac{T}{rR} = \tanh\left(\frac{t}{4GM}\right) \tag{3.73}$$

this defines straight lines through the origin with slope given by hyper-tangent. For $t \to \infty$ it becomes eq,3.71.

In this diagram each point on the diagram is a two-sphere, it represent the maximal extension of the Schwarzschild geometry. We see that $-\infty < R < +\infty$ and $T^2 < R^2 + 1$. It is convenient to divide the diagram in 4 regions:

- Region I corresponds to $r > 2GM$
- With future directed light-like geodesics we reach region II
- With past ones we reach region III
- With space-like geodesics one gets to region IV

Definitions that relate (T,R) to (t,r) are good only in region I.

Region II is a *black hole*, every future-directed path there hits $r = 0$.

Region III is the time reversed of region II, could be the *white hole*. A singularity in the past from which everything appears to spring.

Region IV, cannot be reached from our region and viceversa. It is a mirror image of region I and could be reached by a wormhole.

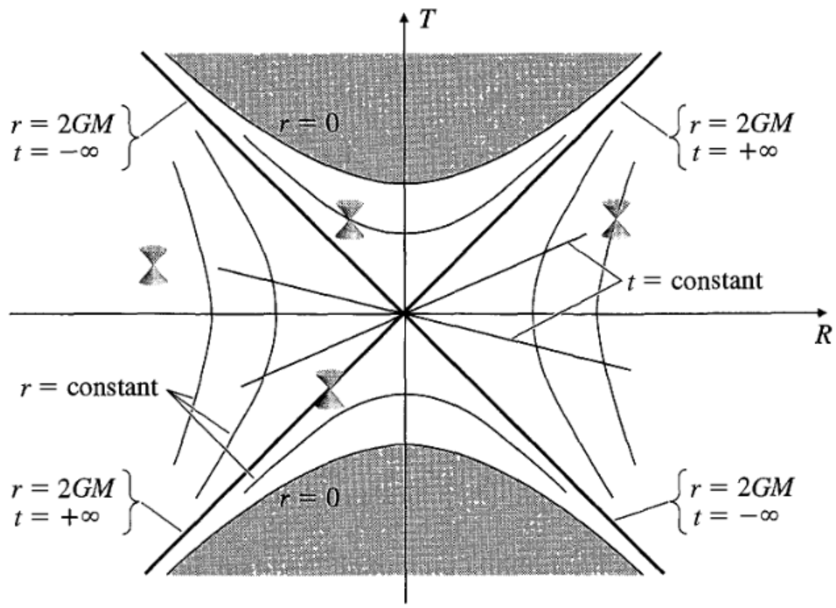


Figure 3.1: Kruskal diagram

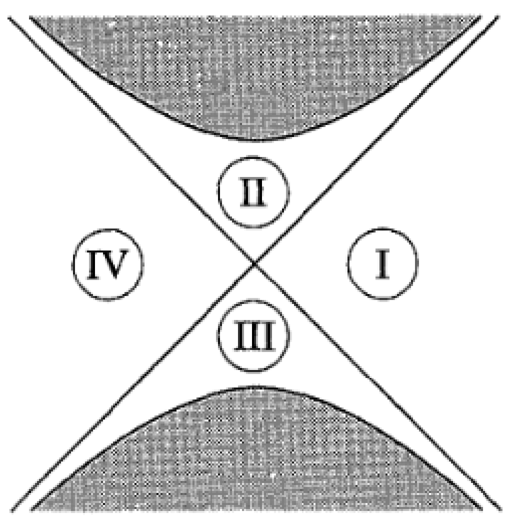


Figure 3.2: Regions of the kruskal diagram

Chapter 4

Gravitational Waves

4.1 Linearized gravity

Roughly speaking this is gravitational waves, while the macro area is *linearized gravity* which is the study of GR where $g_{\mu\nu}$ can be decomposed in

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, (|h_{\mu\nu}| << 1)$$

and $h_{\mu\nu}$ is a symmetric matrix. That's nothing new, right? What we assumed back then was

- weak field
- motion of slow test particle[1]
- static field.

We will give up on the last two assumptions for our derivation.

As just said, we will start from

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.1}$$

we want to find the *inverse metric* $g^{\mu\nu}$, we know

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$$

let's express $g^{\mu\nu}$ like

$$g^{\mu\nu} = \eta^{\mu\nu} + \Delta^{\mu\nu} \tag{4.2}$$

[1]A *test particle* is and idealized concept to study the properties of spacetime and gravitational fields without significantly disturbing them. Specifically

- Infinitesimally small, point particle
- Negligible mass and energy, no contribute to curvature of spacetime.
- Moves along geodesics
- Used to explore effects of spacetime curvature.

we will keep from now on only terms that are linear in $h_{\mu\nu}$, so $O(h)$ or that have the same order of $h_{\mu\nu}$.

$$\begin{aligned}
g^{\mu\nu} g_{\nu\rho} &= \delta^{\mu}_{\rho} \\
\left(\eta^{\mu\nu} + \Delta^{\mu\nu}\right)\left(\eta_{\nu\rho} + h_{\nu\rho}\right) &= \delta^{\mu}_{\rho} \\
\delta^{\mu}_{\rho} + \eta^{\mu\nu} h_{\nu\rho} + \Delta^{\mu\nu}\eta_{\nu\rho} + O\left(h^2\right) &= \delta^{\mu}_{\rho} \\
\eta^{\mu\nu} h_{\nu\rho} + \Delta^{\mu\nu}\eta_{\nu\rho} &= 0 \\
\eta^{\alpha\rho}\Delta^{\mu\nu}\eta_{\nu\rho} &= -\eta^{\mu\nu} h_{\nu\rho}\eta^{\alpha\rho} \\
\delta^{\alpha}_{\nu}\Delta^{\mu\nu} &= -\eta^{\mu\nu}\eta^{\alpha\rho} h_{\nu\rho} \\
\Delta^{\mu\alpha} &= -\eta^{\mu\nu}\eta^{\alpha\rho} h_{\nu\rho} \\
\Delta^{\mu\nu} &= -\eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}
\end{aligned} \tag{4.3}$$

in the last step we changed indices to insert it on eq.4.2:

$$g^{\mu\nu} = \eta^{\mu\nu} + \Delta^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta} \tag{4.4}$$

The theory we are studying is of a dynamical symmetric tensor that propagates in flat spacetime, where the word *propagates* gives us an hint about our goal. This is to take take h and rewrite the Einstein Equation that is a equation of motion.

How does h transform under LTs?

$$\begin{aligned}
g_{\mu'\nu'} &= \Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'} g_{\mu\nu} = \\
&= \Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'}\left(\eta_{\mu\nu} + h_{\mu\nu}\right) = \\
&= \eta_{\mu'\nu'} + \Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'} h_{\mu\nu} = \\
&= \eta_{\mu'\nu'} + h_{\mu'\nu'}
\end{aligned} \tag{4.5}$$

Christoffel symbol Now we can compute the Christoffel symbol

$$\begin{aligned}
\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right] = \\
&= \frac{1}{2}\eta^{\rho\sigma}\left[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right]
\end{aligned} \tag{4.6}$$

we kept just η outside the square brackets because the partial derivatives inside them are already $O(h)$, since derivative of η is null.

Riemann tensor

$$\begin{aligned}
R^{\rho}_{\sigma\mu\nu} &= \partial_\mu \Gamma^{\rho}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma\Gamma - \Gamma\Gamma = \\
= \frac{1}{2}\partial_\mu &\left[\eta^{\rho\alpha}\left(\partial_\sigma h_{\alpha\nu} + \partial_\nu h_{\alpha\sigma} - \partial_\alpha h_{\sigma\nu}\right)\right] - \\
&\frac{1}{2}\partial_\nu\left[\eta^{\rho\alpha}\left(\partial_\sigma h_{\alpha\mu} + \partial_\mu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\mu}\right)\right]
\end{aligned}$$

terms that are $\Gamma\Gamma$ as you can see are not linear in h.

Now we will lower the upper index of the Riemann tensor since it is easier to manipulate.

$$
\begin{aligned}
R_{\lambda\sigma\mu\nu} \simeq \eta_{\lambda\rho} R^{\rho}_{\sigma\mu\nu} = \\
= \eta_{\lambda\rho}\eta^{\rho\alpha} \left(\partial_\mu\partial_\sigma h_{\alpha\nu} + \partial_\mu\partial_\nu h_{\alpha\sigma} - \partial_\mu\partial_\alpha h_{\sigma\nu}\right) - \\
\frac{1}{2}\eta_{\lambda\rho}\eta^{\rho\alpha} \left(\partial_\nu\partial_\sigma h_{\alpha\mu} + \partial_\nu\partial_\mu h_{\sigma\alpha} - \partial_\nu\partial_\alpha h_{\sigma\mu}\right) = \\
= \frac{1}{2}\delta^{\alpha}_{\lambda}\left[(\ldots)(\ldots)\right] = \\
= \frac{1}{2}\left[\left(\partial_\mu\partial_\sigma h_{\lambda\nu} + \partial_\mu\partial_\nu h_{\lambda\sigma} - \partial_\nu\partial_\lambda h_{\sigma\nu}\right) - \left(\partial_\nu\partial_\sigma h_{\lambda\mu} + \partial_\nu\partial_\mu h_{\sigma\lambda} - \partial_\nu\partial_\lambda h_{\sigma\mu}\right)\right] \\
\text{Partial derivatives commute, } h \text{ is symmetric} \\
= \frac{1}{2}\left[\partial_\mu\partial_\sigma h_{\lambda\nu} - \partial_\mu\partial_\lambda h_{\sigma\nu} - \partial_\nu\partial_\sigma h_{\lambda\mu} + \partial_\nu\partial_\lambda h_{\sigma\mu}\right] \qquad (4.7)
\end{aligned}
$$

Ricci tensor

$$
\begin{aligned}
R^{\mu}_{\sigma\mu\nu} &= \frac{1}{2}\left[\partial^\mu\partial_\sigma h_{\mu\nu} - \partial^\mu\partial_\mu h_{\sigma\nu} - \partial_\nu\partial_\sigma h^{\mu}_{\mu} + \partial_\nu\partial^\mu h_{\sigma\mu}\right] \\
R_{\sigma\nu} &= \frac{1}{2}\left(\partial^\mu\partial_\sigma h_{\mu\nu} + \partial^\mu\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\nu h - \Box h_{\sigma\nu}\right) \qquad (4.8)
\end{aligned}
$$

with

$$
\begin{aligned}
h \equiv \eta^{\mu\nu} h_{\mu\nu}, \text{ the trace of } h \\
\Box = \partial_\mu\partial^\mu = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2
\end{aligned}
$$

Ricci scalar

$$
R = \eta^{\sigma\nu} R_{\sigma\nu} = \frac{1}{2}\left(\partial^\mu\partial^\sigma h_{\mu\sigma} + \partial^\mu\partial^\sigma h_{\sigma\mu} - \Box h - \Box h\right) = \partial^\mu\partial^\sigma h_{\mu\sigma} - \Box h \qquad (4.9)
$$

4.1.1 Gauge Invariant

In the context of linearized gravity, gauge invariance refers to the freedom to make certain transformations to the metric perturbation $h_{\mu\nu}$ without changing the physical content of the theory. In the linearized regime, this invariance implies that $h_{\mu\nu}$ can be transformed under an infinitesimal coordinate change

$$
x^\mu \to x^\mu + \xi^\mu
$$

ξ is dependent on coordinates, is not a constant, and is small.

Let's see how we write the metric tensor in new coordinate system.

$$
\begin{aligned}
g_{\mu'\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\left(\eta_{\mu\nu} + h_{\mu\nu}\right) = \\
&= \left(\delta^{\mu}_{\mu'} + \partial_{\mu'}\xi^\mu\right)\left(\partial^{\nu}_{\nu'} + \xi^\nu\right)\left(\eta_{\mu\nu} + h_{\mu\nu}\right) = \\
&= \eta_{\mu'\nu'} + h_{\mu'\nu'} + \partial_{\mu'}\xi_{\nu'} + \partial_{\nu'}\xi_{\mu'} \qquad (4.10)
\end{aligned}
$$

where the last three terms are

$$\partial^{\mu}_{\mu'}\partial^{\nu}_{\nu'}h_{\mu\nu} \equiv h_{\mu'\nu'} + \partial_{\mu'}\xi_{\nu'} + \partial_{\nu'}\xi_{\mu'} \tag{4.11}$$

we did this infinitesimal change of coordinates and we found that the metric is minkowskian plus a correction.

We have

$$\left.\begin{aligned} &h_{\mu\nu} \\ &h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} \end{aligned}\right\} \text{ that are equivalent}$$

they both describe the same physical perturbation. Indeed *gauge invariance* allows $h_{\mu\nu}$ to be shifted as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

Now we can compute the variation of the linearized Riemann tensor under this transformation

$$\begin{aligned} \delta R_{\lambda\sigma\mu\nu} &= \frac{1}{2}\left[\partial_\mu\partial_\sigma\delta h_{\nu\lambda} - \partial_\mu\partial_\lambda\delta h_{\sigma\nu} - \partial_\nu\partial_\sigma\delta h_{\lambda\mu} + \partial_\nu\partial_\lambda\delta h_{\sigma\mu}\right] \\ &= \frac{1}{2}[\,\underline{\partial_\mu\partial_\sigma}(\partial_\nu\xi_\lambda + \underline{\partial_\lambda\xi_\nu}) \underline{-\partial_\mu\partial_\lambda}(\underline{\partial_\sigma\xi_\nu} + \partial_\nu\xi_\sigma) + \\ &\quad -\partial_\nu\partial_\sigma\left(\partial_\lambda\xi_\mu + \partial_\mu\xi_\lambda\right) + \partial_\nu\partial_\lambda\left(\partial_\sigma\xi_\mu + \partial_\mu\xi_\sigma\right)] \end{aligned}$$

As you see by colors and underlining each term cancel with an opposite other. Since the variation of the Riemann tensor is null, this transformation leaves Riemann tensor, Ricci tensor and scalar unchanged.

If you have a gauge transformation, which is defined as the actual h, nothing changes at the level of the Riemann tensor.

4.1.2 Degrees of Freedom for $h_{\mu\nu}$

I can choose ξ in such a way that h takes the form that is particularly convenient for our purposes. Let's see more deeply the meaning of $h_{\mu\nu}$ tensor and its components.

The issue of gauge invariant will be postponed to the next lecture.

We assumed we work on a given coordinates system, we pick a specific gauge, and we will discuss what happens to $h_{\mu\nu}$.

As one may remember

$$h_{\mu'\nu'} = \Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'}h_{\mu\nu}$$

the components of h, transform exactly as a Lorents tensor. I change coordinates and this is a LT. Or I could say that the tensor transform as a Lorentz tensor, this sentence is stronger than the previous, and we already checked it. In particular when we discuss LTs we have rotations (3) and boosts (3). Let's focus on the rotations. It's convenient to decompose the metric h:

$$h_{\mu\nu} = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} = h_{01} & \dots & \dots & \dots \\ h_{02} & \dots & \dots & \dots \\ h_{03} & \dots & \dots & \dots \end{pmatrix}$$

because the $h_{\mu\nu}$ tensor is symmetric.

Under **spatial rotations**:

- h_{00} is a scalar, and it's invariant for spatial rotations, and $h_{00} = -2\phi$, (1 degree of freedom)
- h_{0i} is a vector, (3 degrees of freedom), $h_{0i} = w_i$
- h_{ij} is a symmetric rank-2 tensor, (6 degrees of freedom), $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$

where Ψ encodes the trace of h_{ij} and s_{ij} is traceless.

Let's what can we say about Ψ. The trace of h is

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}$$

so

$$\delta_{ij} h_{ij} = 2\left(\delta_{ij} s_{ij}\right) - 2\Psi\left(\delta_{ij}\delta_{ij}\right) = -6\Psi \rightarrow \Psi = -\frac{1}{6}\delta_{ij} h_{ij} \tag{4.12}$$

in this case we don't care about the position of the indices because in Minkowski spatial the metric is the identity matrix; you can see that in the second step the first term just disappear because the s is traceless.

In the context of **weak field**, **staticness** and **slow motion** the metric is

$$ds^2 = -\left(1 + 2\phi\right) dt^2 + dx^2 + dy^2 + dz^2 = -\left(1 + 2\phi\right) dt^2 + \delta_{ij} dx^i dx_j$$

while in the case of just **weak field**, like ours

$$ds^2 = -\left(1 + 2\phi\right) dt^2 + w_i \left(dt dx^i + dx^1 dt\right) + \left[2s_{ij} + \left(1 - 2\phi\right)\delta_{ij}\right] dx^i dx^j$$

This definitions of the parts of the perturbation tensor are not gauge or solving equations, we just defined some convenient notation. The traceless tensor s_{ij} is known as *strain*, and as we'll see later, it contains gravitational radiation.

Ψ is invariant under rotations because trace of a tensor is invariant under rotations.

How do the components of the metric tensor evolve? According to the Einstein Equations[2], in the linear regime, obviously. We can say that ϕ, Ψ, w_i are not propagating degrees of freedom, while s_{ij} is propagating

What does it mean? If you write the Einstein Equations, with the expressions for $R_{\mu\nu}$ and R, and instead of h you write its decomposition, you can find the left-hand side of the EEs in function of ϕ, Ψ, w_i, s_{ij}. The solution you find fo r ϕ, Ψ, w_i only involve derivatives of space, so they do not evolve with time and it means that if I know $T_{\mu\nu}$ and s I can infer the functional dependence of the others. The actual field that propagates is the field of the strain.

[2] The Einstein equations are not just one, we write one in tensorial form, but they are many

4.2 Lec 21

The Einstein tensor expressed with the components of $h_{\mu\nu}$ becomes

$$G_{00} = 2\nabla^2\Psi + \partial_k\partial_l s^{kl} \tag{4.13}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s^k_j \tag{4.14}$$

$$\begin{aligned} G_{ij} &= \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\phi - \Psi) + \delta_{ij}\partial_0\partial_k w^k - \partial_0\partial_{(i}w_{j)} \\ &\quad + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} + 2\partial_k\partial_{(i}s^k_{j)} - \delta_{ij}\partial_k\partial_l s^{kl} \end{aligned} \tag{4.15}$$

Let's remember, the Einstein Equations are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

When you try to solve this equation here, instead of the full Ricci tensor it's ok to use the linearized expression found yesterday, to put the minkowsky metric instead of $g_{\mu\nu}$ since otherwise we would have second order in h in that factor because the Ricci scalar is already first order in h. On the right-hand side $8\pi G$ is zero-th order in h and $T_{\mu\nu}$ is linear in h

$$G_{\mu\nu} = R_{\mu\nu} - \eta_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

If we know T_{00}[3] and s_{ij}, We can find Ψ from eq.4.13 and

$$G_{00} = 8\pi G T_{00}$$

also, since there is no time derivative, Ψ does not propagate.

The G_{0i} term specifies w_j in terms of Ψ, s_{ij}, T_{0i}. The G_{ij} term specifies ϕ. Always without a time derivative.

So as probably just said the propagating terms are all in the s_{ij} component of the metric.

Gauge transformations of components of $h_{\mu\nu}$

We already discussed the family of gauge transformations made from

$$h_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

What does it look like plugged in

- $h_{00} = -2\phi$
- $h_{0i} = w_i$
- $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$

[3] energy density, via the sources

Let's start from $h_{00} = -2\phi$

$$h_{00} \to h_{00} + \partial_0 \xi_0 + \partial_0 \xi_0 = h_{00} + 2\partial_0 \xi_0 = h_{00} - 2\partial_0 \xi^0$$
$$\to -2\phi = -2\phi - 2\partial_0 \xi^0 \to \phi = \phi + \partial_0 \xi^0$$

About $h_{0i} = w_i = w^i$:

$$h_{0i} \to h_{0i} + \partial_0 \xi_i + \partial_i \xi_0 \to w_i = w_i + \partial_0 \xi^i - \partial_i \xi^0$$

Lastly, about $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$, we will split in the two components

$$\Psi = -\frac{1}{6}\delta_{ij} h_{ij}$$
$$\Psi \to -\frac{1}{6}\delta_{ij}\left(h_{ij} + \partial_i \xi^j + \partial_j \xi^i\right) = \Psi - \frac{1}{3}\partial_i \xi^i$$

last step we used the delta to make j to i. While for s_{ij}

$$2s_{ij} = h_{ij} + 2\Psi\delta_{ij} \to s_{ij} = \frac{h_{ij}}{2} + \Psi\delta_{ij}$$
$$s_{ij} \to \frac{h_{ij} + \partial_i \xi^j + \partial_j \xi^i}{2} + \delta_{ij}\left(\Psi - \frac{1}{3}\partial_k \xi^k\right)$$
$$s_{ij} \to s_{ij} + \frac{\partial_i \xi^j + \partial_j \xi^i}{2} - \frac{1}{3}\partial_k \xi^k \delta_{ij}$$

Now, about degrees of freedom we see that

- $\phi \to 1$, it's a scalar
- $\Psi \to 1$, same reason
- $w_i \to 3$ it's a vector
- $s_{ij} \to 5$, because it's symmetric but also traceless.

We are ok. We foresaw 10 degrees of freedom and we got them. These will be used to simplify equations.

Transverse Gauge

The **transverse gauge** is a specific choice of gauge condition in linearized gravity that simplifies the equations by imposing constraints on the metric perturbations $h_{\mu\nu}$. We saw that the left hand side of the Einstein Equations can be written as function of ϕ, Ψ, w_i, s_{ij}, But they are a lot, what we can do with *gauge freedom* is to choose a specific gauge that makes EEs easier. The *transverse gauge* is defined by imposing this condition

$$\partial_i s^{ij} = 0$$

while it's gauge transformations is

$$\partial_i s^{ij} \to \partial_i s^{ij} + \frac{\partial_i \partial_j \xi^i + \partial_j \partial_i \xi^i}{2} - \frac{\partial_j \partial_k \xi^k}{3} = 0$$

Since setting $\partial_i s^{ij}$ only involves ξ^i and not ξ^0, we can use the latter to set

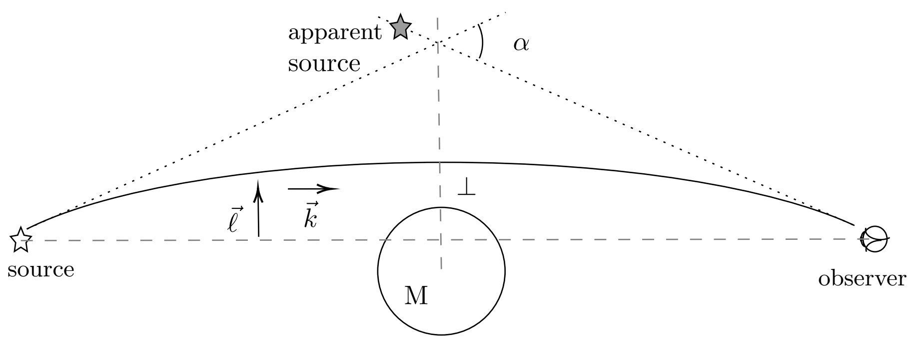
$$\partial_i w^i = 0$$

and the gauge transformations is

$$\partial_i v^i \to \partial_i w^i + \partial_0 \partial_i \xi^i - \partial_i \partial_i \xi^0 = 0$$

The transverse gauge is defined by imposing these conditions

4.2.1 Deflection of light



Deflection of light, or *gravitational lensing*, can be studied under two assumptions: static field, and transverse gauge (linearized gravity and weak field).

What we want is to write the Einstein Equations, because of transverse gauge and staticness (time derivative = 0), many terms are null.

$$G_{00} = 2\nabla^2\phi + \partial_k \partial_l s^{kl} = 2\nabla^2\phi \tag{4.16}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + \ldots + \ldots + \ldots = -\frac{1}{2}\nabla^2 w_j \tag{4.17}$$

$$G_{ij} = \left(\delta_{ij}\nabla^2 - \partial_i \partial_j\right)(\phi - \Psi) - \Box s_{ij} \tag{4.18}$$

Regarding $T_{\mu\nu}$, this is the source, and in our case, since the Sun (see fig. 4.2.1) is at rest, we can take the NR form of it so

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we stay in the frame where the object that bends is at rest. With this *energy-momentum tensor*

$$G_{00} = 2\nabla^2\phi = 8\pi G\rho \tag{4.19}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j = 0 \tag{4.20}$$

$$G_{ij} = \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\phi - \Psi) - \nabla^2 s_{ij} = 0 \tag{4.21}$$

note that the d'alembertian operator $\square$ has become ∇^2 in the third equation, this because of staticness. That laplacian is in spatial coordinates.[4] Now we claim that, since $\nabla^2 w_j = 0$, if this condition has to be well-behaved at ∞, then the only solution is $w_j = 0$, in general the solution of $\frac{\partial^2}{\partial x^2}$ is $w = Ax + B$, in one dimension.

About the third equation, taking its trace, we get

$$3\nabla^2 - \nabla^2(\phi - \Psi) = 2\nabla^2(\phi - \Psi) = 0$$

for the same reason, the solution is $\phi = \Psi$. Plugging this solution in the full equation gets us $\nabla^2 s_{ij} = 0$, that, again, implies $s_{ij} = 0$ for a well-behaved solution.

The perturbed metric for static Newtonian sources, with staticness and transverse gauge is

$$ds^2 = -(1 + 2\phi)\,dt^2 + (1 - 2\phi)\left(dx^2 + dy^2 + dz^2\right) \tag{4.22}$$

and the perturbed metric is

$$h_{\mu\nu} = \begin{pmatrix} -2\phi & 0 & 0 & 0 \\ 0 & -2\phi & 0 & 0 \\ 0 & 0 & -2\phi & 0 \\ 0 & 0 & 0 & -2\phi \end{pmatrix} \tag{4.23}$$

This is an improvement over the previous study on the motion of a slow particle, that allowed us to fill only the temporal part of the metric, while here, with a fast particle, we filled also the spatial section.

If I want to study the deflection of light, I have to study the trajectory of a photon. This can be written as

$$x^\mu(\lambda) = x^\mu_{(0)}(\lambda) + x^\mu_{(1)}(\lambda)$$

where we decomposed in two contributions:

- $x^\mu_{(0)}$ that solves the geodesic equation for a straight null path
- $x^\mu_{(1)}$ that solves for the perturbation.

[4] $\square = \nabla^2 - \frac{1}{v^2}\partial_t^2$.

practically, these contribution are respectively of zero-th and first order in h. Consequentially, the four momentum can also be split into two contributions

$$k^\mu = \frac{dx^\mu_{(0)}}{d\lambda}$$
$$\ell^\mu = \frac{dx^\mu_{(1)}}{d\lambda}$$

where we chose λ in such a way that these are the four-momentum.

Let's look now at the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

In the linear regime, Christoffel connection is

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\rho}\left[\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}\right]$$

and its only non-vanishing components are

$$\Gamma^0_{0i} = \Gamma^i_{00} = \partial_i\phi$$
$$\Gamma^i_{jk} = \delta_{jk}\partial_i\phi - \delta_{ik}\partial_j\phi - \delta_{ij}\partial_k\phi$$

The geodesic becomes

$$\frac{d}{d\lambda}\left(k^\mu + \ell^\mu\right) + \Gamma^\mu_{\alpha\beta}\left(k^\alpha + \ell^\alpha\right)\left(k^\beta + \ell^\beta\right) = 0 \tag{4.24}$$

we can split this equation based on the orders

$$\mathcal{O}\left(h^0\right) \rightarrow \frac{dk^\mu}{d\lambda}$$
$$\mathcal{O}\left(h^2\right) \rightarrow \frac{d\ell^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}k^\alpha k^\beta = 0$$

Geodesic Motion Be

$$k^\mu = \left(\omega, \vec{k}\right), k^\mu k_\mu = 0, \omega = |\vec{k}| = k$$

Let's see what happens with ℓ^0:

$$\frac{d\ell^0}{d\lambda} + \Gamma^0_{0i}\omega k^i + \Gamma^0_{i0}\omega k^i = 0$$
$$\left(\Gamma^0_{0i} = \Gamma^0_{i0} = \partial_i\phi\right)$$
$$\frac{d\ell^0}{d\lambda} + 2\partial_i\phi k^i k = 0$$
$$\frac{d\ell^0}{d\lambda} = -2k\left(\vec{k}\cdot\vec{\nabla}\phi\right)$$

while for the spatial part

$$\frac{d\vec{\ell}}{d\lambda} = -2\left(k^2\nabla\phi - \vec{k}\left(\vec{k}\cdot\vec{\nabla}\phi\right)\right) = -2k^2\vec{\nabla}_{\perp}\phi \tag{4.25}$$

This is the *gradient transverse*, obtained by subtracting the part that is parallel to k:

$$\vec{\nabla}_{\perp}\phi = \vec{\nabla}\phi - \frac{1}{k^2}\left(\vec{k}\cdot\vec{\nabla}\phi\right)\vec{k} \tag{4.26}$$

We found out that the motion of the photon is the composition of two four-vectors.

The product of them gives

$$\vec{k}\cdot\vec{\ell} = (k_\mu + \ell_\mu)(k^\mu + \ell^\mu) = 0$$

or better

$$(\eta_{\mu\nu} + h_{\mu\nu})(k+\ell)^\mu (k+\ell)^\nu = 0$$
$$\text{at first order}$$
$$\vec{k}\cdot\vec{\ell} = k\ell^0 + 2k^2\phi = 0 \rightarrow \ell^0 = -2k\phi$$

This means that there are two directions, ℓ, k, ℓ changes along the trajectory, it is max at the beginning and minimum at the end.

$$\alpha = \frac{|\Delta\vec{\ell}|}{k}$$

with

$$\begin{aligned}
\Delta\vec{\ell} = \int d\lambda\left(-2k^2\vec{\nabla}_{\perp}\phi\right) &= -2\int d(\lambda k)\, k\vec{\nabla}_{\perp}\left(-\frac{GM}{(x^2+b^2)^{1/2}}\right) \\
&= -2k\int dx \frac{GMb}{(x^2+b^2)^{3/2}} \\
\rightarrow \Delta\alpha = GMb\int_{-\infty}^{+\infty} dx \frac{1}{(x^2+b^2)^{3/2}} &= \frac{4GM}{b}
\end{aligned}$$