

Notes of General Relativity

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Chapter 1

Introduction

Lecture 1

General Relativity describes *gravity* in terms of *curvature* of *space-time*.

We will define and describe those three words.

To understand *curvature*, let's think about a RF in a flat space, so that the sum of all internal angles of a triangle is 180° , as we add curvature, the sum increase its value.

Sphere is a 2D *manifold*. What is a manifold?

From Newton to Einstein

We got two masses, m_1, m_2 , the origin, O, of the RF.

Each mass' position is identified by its own position vector.

$$\begin{aligned}\vec{r} &= \vec{r}_1 + \vec{r}_2 \\ \vec{F}_{21} &= -\frac{Gm_1m_2}{r^2}\hat{r} \\ \text{with } \hat{r} &= \frac{\vec{r}}{|\vec{r}|}\end{aligned}$$

so, we see that m_2 is attracted.

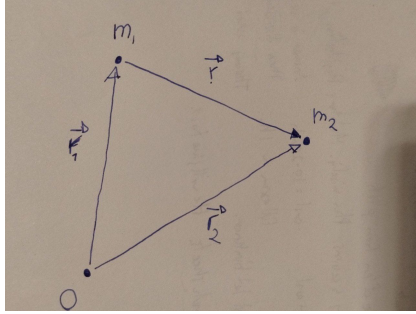
P.S. $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

Introducing the second law of dynamics in the study, we have

$$m_2\vec{a}_2 = \vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

simplifying m_2 we obtain

$$\vec{a}_2 = -\frac{Gm_1}{r^2}\hat{r}$$



We can express \mathbf{a}_2 as

$\vec{a}_2 = -\nabla\phi$ Gradient of the Gravitational Potential

$$\phi = -\frac{Gm_1}{r}$$

$$\nabla^2\phi = -4\pi G\rho$$

We will use the Minkowski metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \quad (1.1)$$

We will see also other symbols, like the Kristoffel one, or the Richie Tensor...
But in the end the central goal is to derive the *Einstein Equation*:

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} \quad (1.2)$$

In GR particles move freely along *straight lines* of a curved space-time. These are called *geodesics*.

Example Two chalks, one on the desk, the other is launched in the air. Which one is accelerated? From a GR perspective, the one in the air is moving along a geodesic, so it is the one moving freely, while the other is stopped from doing that by some interference/force.

In GR gravity is *not* a force.

Chapter 2

Math tools

2.1 A recap of SR

Lecture 2 We will develop some of the necessary math on this framework.

Let's look at the Galilean Relativity.

Newtonian dynamics is based on three principles

1. inertia
2. $\vec{F} = m\vec{a}$
3. action-reaction

The first says something like *An object at rest remains at rest, and an object in motion remains in motion at constant speed and in a straight line unless acted on by an unbalanced force.*

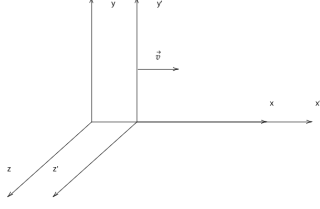
The second one says:

$$(2) : \vec{F} = 0 \implies \vec{a} = 0 \implies (1)$$

So, it seems the first principle is contained by the second, but we know that $\vec{F} = m\vec{a}$ is valid only in Inertial Frames (IF).

Galilean Relativity: all the laws of *mechanics* take the same form in every IF. (You can not distinguish two IF just by doing experiments.)

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$
$$t = t' = 0 \implies O = O'$$



Taking the first derivative:

$$\begin{cases} v'_x = v_x - v \\ v'_y = v_y \\ v'_z = v_z \end{cases} \quad \text{and for the second derivative:} \quad \begin{cases} a'_x = a_x \\ a'_y = a_y \\ a'_z = a_z \end{cases} \implies \vec{a}' = \vec{a} \quad (2.1)$$

so also $\vec{F}' = \vec{F}$. And if m is independent on the frame, we got

$$\vec{F}' = m\vec{a}' = \vec{F} = m\vec{a} \quad (2.2)$$

Then there are Maxwell equations, people thanks to them find that EM-waves propagates with speed c in the void.

But they found also that these equations were not invariant in Galilean Boosts. Things started to go better when the idea of a preferred IF was ditched and Einstein decided to use Lorentz Transformations.

There are two postulates:

- *Relativity principle*: same as before but with *physics* instead of *mechanics*.
All the laws of physics ...
- *Speed of light*: in every IF, light propagates with constant speed, c .

So we see that Galilean transformation become inconsistent with this, meanwhile stays valid for $\vec{v} \ll \vec{c}$.

As mentioned before, updated version of G. Boosts are Lorentz transformations (or Lorentz Boosts.)

$$\begin{cases} x' = \frac{x-vt}{\sqrt{1-(\frac{v}{c})^2}} \\ y' = y \\ z' = z \\ t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-(\frac{v}{c})^2}} \end{cases} \quad (2.3)$$

To ensure the L.T. Is consistent we can perform three checks:

- $v \ll c$

- $v = 0$
- dimensional check

People use a notation to make the L.T. easier to write: $\gamma(v) \equiv \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$, so it becomes

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{vx}{c^2}) \end{cases} \quad (2.4)$$

What happens to the transformation of velocity is: (v is fixed)

$$\begin{cases} dx' = \gamma(dx - vdt) \\ dy' = dy \\ dz' = dz \\ dt' = \gamma(dt - \frac{vdx}{c^2}) \end{cases} \quad (2.5)$$

so

$$\begin{cases} v'_x = \frac{dx'}{dt'} \\ v'_y = \frac{dy'}{dt} = \frac{dy}{\gamma(dt - \frac{vdx}{c^2})} = \frac{v_y}{\gamma(1 - \frac{vv_x}{c^2})} \\ v'_z = \frac{dz'}{dt} = \dots \end{cases} \quad (2.6)$$

So we see that space-time changes also along other axes.

Now let's talk about space-time and its parts.

Space-time space-time is a manifold. For now it is a collection of (t, x, y, z) , four dimensional set of all the possible values of the coordinates.

Event a point of space-time.

World line path of a particle in space-time.

There is no notion of absolute time anymore, because now it is dependent on the frame. Regarding the light-cone, after the event on the (x, y) plane, the particle can move *only* inside the light-cone, in the appropriate direction (time forward).

Now let's talk about **Clock Synchronization**.

It is kinda easy if in in IF. In GR it is quite subtle instead.

Example: Be me in Origin of a RF watching my clock (A). How to define t at another generic location (B)??

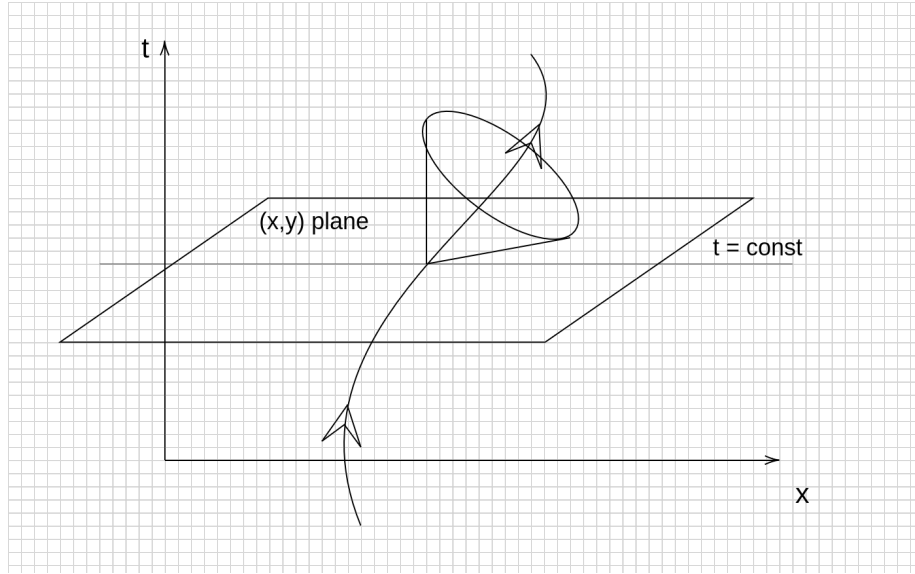


Figure 2.1: LL of a particle which moves forward in time, we see also a light cone

I send a light ray at time t_1 to B. I get the answer on t_2 . There is symmetry between the two trajectories so

$$t_m = \frac{t_1 + t_2}{2}.$$

I say to my friend on B: "set your clock to t_m when you receive the signal." So, following this methodology, each point could have its own clock.

Proper time: How to define proper time?

t is the time coordinate. Let's introduce the metric tensor:

$$\text{the Minkowski metric tensor: } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.7)$$

for a Lorentz Transformation if I have 2 events E,F.

$$\text{Frame 1: } x_F^\mu = (t_F, x_F, y_F, z_F)$$

$$x_E = (...)$$

$$\text{Frame 2: } x_F^{\mu'} = (t_{F'}, x_{F'}, y_{F'}, z_{F'})$$

$$x_E^{\mu'} = (...)$$

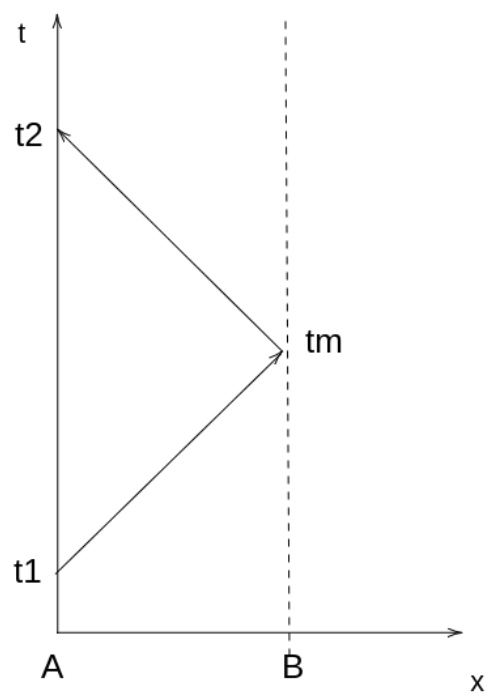


Figure 2.2: Reception and send of the signal

same events in 2 different frames.

A Lorentz Transformation connects these two events.

Be Δs^2 the Lorentz Invariant separation between E-F.

$$\begin{aligned}\Delta s^2 &= -c(t_F - t_E)^2 + (x_F - x_E)^2 + (y_F - y_E)^2 + (z_F - z_E)^2 = \\ &= -c(t_{F'} - t_{E'})^2 + (x_{F'} - x_{E'})^2 + (y_{F'} - y_{E'})^2 + (z_{F'} - z_{E'})^2 = \\ &\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu\end{aligned}$$

From this point we set $c = 1$ just a rescaling

we have defined $\Delta x^\mu \equiv x_F^\mu - x_E^\mu$, with $\mu = 0, 1, 2, 3$.

So, repeating for clarity, the Lorentz Invariant separation is

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = \eta_{\mu'\nu'} \Delta x^{\mu'} \Delta x^{\nu'} \quad (2.8)$$

Minkowski metric tensor does not change form if we change coordinates (Cartesian coordinates, meanwhile if we use like polar ones it changes for obvious reasons.)

if

$$\begin{aligned}\Delta s^2 &> 0 \text{ space-like separation} \\ &< 0 \text{ time-like, (it could be an actual LL for a massive particle)} \\ &= 0 \text{ light-like or null}\end{aligned}$$

Now we can define the *proper time* as

$$\Delta \tau^2 \equiv -\Delta s^2 \text{ or } \Delta \tau^2 = -\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \quad (2.9)$$

So, if the proper time is *positive* it is time-like.

If the segment **EF** marks the begin and end of the trajectory of a massive particle, $\Delta \tau$, proper time, is the time elapsed on a clock sitting on a RF that moves with constant speed between E and F.

In the moving frame $\Delta \tau = \Delta t_*$ where t_* is the time coordinate of the moving frame. In a frame where I'm at rest this is how Δt^2 changes:

$$\Delta \tau^2 = +\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2. \quad (2.10)$$

2.2 Lecture 3

The meaning of the Lorentz Invariant is that **events**, like (E, F) exist before I define coordinates. It is a property of the two events.

So to recap what we did in the last lecture, be:

$$x_E^\mu \text{ and } x_E^{\mu'} \quad (2.11)$$

If I have two events and computing $\Delta \tau$ gives a positive result, the separation is **time-like**. This means that they could be on the WL of a massive particle moving at constant speed.

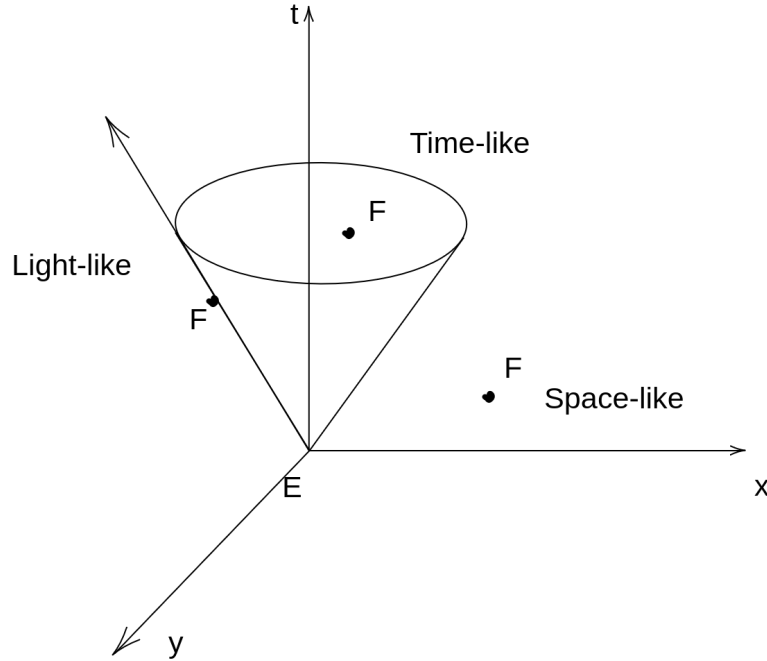


Figure 2.3: Given event E, the separation \mathbf{EF} could be of different types based on the position respect the light cone

Physical meaning of $\Delta\tau$ It's the time elapsed on a clock of the observer moving between E and F at constant speed.

This means that if I compute $\Delta\tau$ on the frame where the observer it is at rest, i get

$$\Delta\tau = \Delta t'$$

Lets do an example:

Example In fig. 2.4 we see the straight line \mathbf{ABC} that is the WL of a object not moving. Computing its proper time will be:

$$\Delta\tau_{ABC} = (t_c - t_A) \quad (2.12)$$

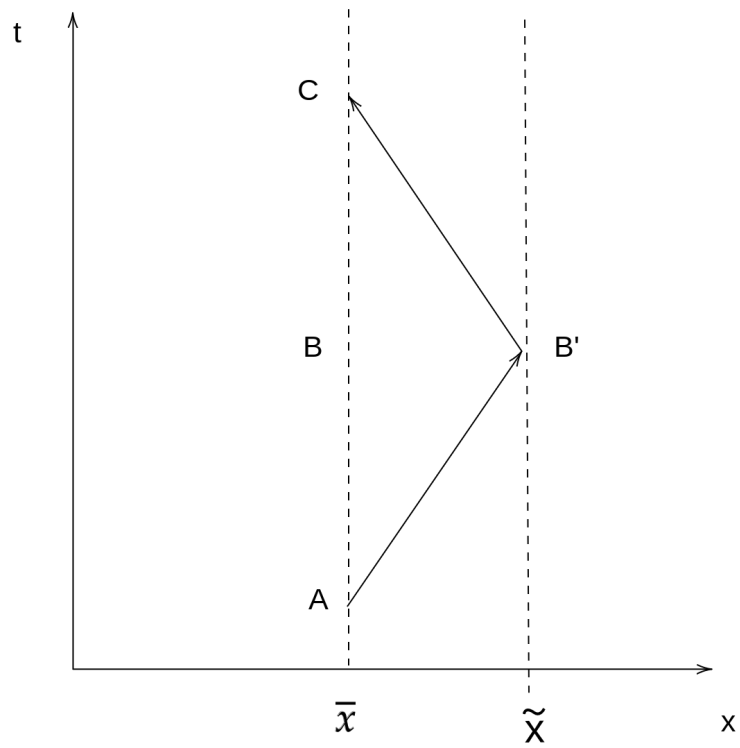


Figure 2.4: It is like the twin paradox.

But for the other WL, of a object moving at constant speed between **AB'** and **B'C**, first thing first, we see that

$$\begin{aligned} t_B &= t_{B'} \\ \text{and so} \\ \Delta\tau_{AB'C} &= 2\sqrt{(t_B - t_A)^2 - (\tilde{x} - \bar{x})^2} = \Delta\tau_{ABC}\sqrt{1 - \left(\frac{v}{c}\right)^2} \\ \implies \Delta\tau_{AB'C} &< \Delta\tau_{ABC} \end{aligned}$$

This means that I have the longest **proper time** when I don't move.

We can do one more generalization: by parametrize the WL with a quantity λ we get

$$\Delta\tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \text{ that is a time like trajectory.}$$

Enough with proper time.

2.2.1 Tensor Calculus

Be a Lorentz Group, we want to look for the transformations.

$$x^\mu \rightarrow x^{\mu'} = \Lambda_{\mu}^{\mu'} x^\mu \quad (2.13)$$

we see that it is a linear transformation. An example to see better what are we doing could be

$$x^{0'} = \Lambda_0^{0'} x^0 + \Lambda_1^{0'} x^1 + \Lambda_2^{0'} x^2 + \Lambda_3^{0'} x^3 \quad (2.14)$$

What we need to know is that $\Lambda_{\mu}^{\mu'}$ is a constant matrix.

We see that Λ is a constant matrix.

We want to find linear transformations such that

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = \eta_{\mu'\nu'} \Delta x^{\mu'} \Delta x^{\nu'} \quad (2.15)$$

So the Lorentz Invariant is still invariant. (WTF)

Now, because a SR property: if I move from IF to another, η is still unchanged. So

$$\eta_{\mu\nu} = \eta_{\mu'\nu'}$$

We have to say that Minkowski assumes cartesian coordinates.

The question now is: What trivial transformations leave Δs^2 unchanged?

Translations

$$\begin{aligned}\eta_{\mu\nu}\Delta x^\mu x^\nu &= \eta_{\mu'\nu'} \left(\Lambda_{\mu'}^{\mu} \Delta x^\mu \right) \left(\Lambda_{\nu'}^{\nu} \Delta x^\nu \right) \\ \implies \eta_{\mu\nu} &= \eta_{\mu'\nu'} \Lambda_{\mu'}^{\mu} \Lambda_{\nu'}^{\nu}\end{aligned}$$

this obviously needs to be valid $\forall \Delta x^\mu$

an alternative notation could be $\eta = \Lambda^T \eta \Lambda$

We will use just the first notation, because we need to get good at tensors.

To be more concrete:

$$\Lambda_{\mu}^{\mu'} = \begin{pmatrix} \Lambda_0^{0'} & \Lambda_1^{0'} & \Lambda_2^{0'} & \Lambda_3^{0'} \\ \Lambda_0^{1'} & \dots & \dots & \dots \\ \Lambda_0^{2'} & \dots & \dots & \dots \\ \Lambda_0^{3'} & \dots & \dots & \dots \end{pmatrix} \quad (2.16)$$

Rotations Rotations are a kind of transformation of the type:

$$\begin{aligned}x_{i'} &= R_{ii'} x_i \\ \text{or } R^T \mathbb{I} R &= \mathbb{I} \\ \text{with } R R^T &= R^T R = \mathbb{I}\end{aligned}$$

it could be something like

$$\Lambda_{\mu}^{\mu'} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.17)$$

this one is a boost along the x direction. If we do some computing we find that

$$\tanh\eta \equiv v$$

so this is the same of the L.T. we saw last week.

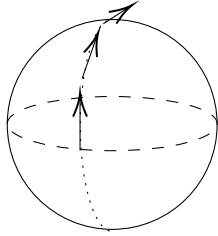
Rotations do not change the time coordinate. The point was to tell what L.T. is in this language.

Vectors I have a generic vector, **do i need to specify about the RF** where it is defined, so in a specific spacetime location? yes

In newtonian mechanics parallel vectors are the same because I can superpose them, I can move them around, also to use the parallelogram rule to get a sum.
 \implies If I have 3D euclidean space there is no ambiguities about where i move my

vectors.

BUT in a sphere:



I have this vector at the equator tangent to the surface. If I transport it to the pole I get a different vector.

There are ambiguities. So in a non-flat space we need a **different** procedure. A vector field is a map between:

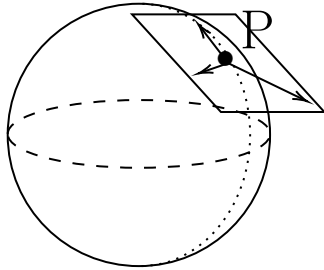
$$x^\mu \rightarrow v^\mu$$

where x^μ is an event and v^μ is a vector.

Let's define: **Tangent space T_P** .

Given an event P we define the tangent space T_P as all the vectors in P .

Instead of having spacetime we have a sphere.



Define a plane tangent to the sphere only in P . All vectors that lie there $\in T_P$.

T_P is a **vector space**:

$$V, W \in T_P \implies \alpha V + \beta W, (\alpha, \beta \in \mathbb{R}) \in T_P$$

So if there is a vector there is also the inverse vector.

Whenever I have a vector space, I can define infinite basis independently on the coordinate choice. The number of elements in the basis is equal to the dimension of the space, in our case 4 elements.

Obviously if I define the basis its elements need to be Linearly Independent.

Basis Given a generic vector $V \in T_P$, I can define V regardless the coordinate system I'm using. So we can say *metaphorically* that V exists before I define coordinates.

Be our basis:

$$\hat{e}_{(\mu)}, \text{ with } \mu = 0, 1, 2, 3$$

those indices are label, does not mean "tensor". So my basis is made of

$$\hat{e}_{(0)}, \hat{e}_{(1)}, \hat{e}_{(2)}, \hat{e}_{(3)}$$

Now we can talk about

Components given a generic vector v

$$V = V^0 \hat{e}_{(0)} + V^1 \hat{e}_{(1)} + V^2 \hat{e}_{(2)} + V^3 \hat{e}_{(3)} = V^\mu \hat{e}_{(\mu)}$$

using repeating indices we get the last equivalence.

V^μ are components of the vector V in this specific frame.

In another frame $V^{\mu'}$ could not be the same:

$$V = V^\mu \hat{e}_{(\mu)} = V^{\mu'} \hat{e}_{(\mu')}$$

Question: how do components transform?

covariant vector : is a math object whose components transform based on position

$$V^{\mu'} = \Lambda_{\mu}^{\mu'} V^\mu$$

These are not the only covariant vectors (?).

If you have a generic WL or path, you can parametrize the position by a λ in this way:

$$x^\mu(\lambda)$$

And taking its first derivative you get something similar to the four-velocity

$$u^\mu \sim \frac{dx^\mu}{d\lambda}$$

(I say similar because four-velocity is defined like $u^\mu = \frac{dx^\mu}{d\tau}$).

If I do a L.T. x^μ will change but λ won't.

$$u^{\mu'} = \Lambda_{\mu}^{\mu'} u^\mu$$

I can get a more general definition of what a vector is by following this procedure: choose basis \rightarrow find components \rightarrow study how components change if i change position or basis.

Second definition : Transformation of the basis vectors. The question is "how to relate $\hat{e}_{(\mu)}$ to $\hat{e}_{(\mu')}$?"

We will take advantage of **invariance**.

$$V = V^\mu \hat{e}_{(\mu)} = V^{\mu'} \hat{e}_{(\mu')} = \left(\Lambda_{\mu}^{\mu'} V^\mu \right) \hat{e}_{(\mu')}$$

That's possible **only** if $\hat{e}_{(\mu)} = \Lambda_{\mu}^{\mu'} \hat{e}_{(\mu')}$.

An inverse of LT it is also a LT, so

$$\Lambda_{\mu}^{\mu'} \Lambda_{\nu'}^{\mu} = \delta_{\nu'}^{\mu'}$$

$$\Lambda_{\mu'}^{\mu} \Lambda_{\nu}^{\mu'} = \delta_{\nu}^{\mu}$$

Those are Kroneker's delta and they are an Identity matrix.

Now we can study how basis vectors change.

$$\hat{e}_{(\mu)} = \Lambda_{\mu}^{\mu'} \hat{e}_{(\mu')}$$

$$\Lambda_{\nu'}^{\mu} \hat{e}_{(\mu)} = \Lambda_{\mu}^{\mu'} \Lambda_{\nu'}^{\mu} \hat{e}_{\mu'}$$

$$\Lambda_{\nu'}^{\mu} \hat{e}_{(\mu)} = \delta_{\nu'}^{\mu'} \hat{e}_{(\mu')}$$

$$\Lambda_{\nu'}^{\mu} \hat{e}_{(\mu)} = \hat{e}_{(\nu')}$$

$$\text{so } \hat{e}_{(\nu')} = \Lambda_{\nu'}^{\mu} \hat{e}_{(\mu)}$$

2.3 Lecture 4

Brief recap of lec3

We defined vectors

- localized at each spacetime point
- for each event P we defined the tangent space T_P
- there is linear combination inside T_P
- it has a basis
- Vectors and basis transform under LT Group.

Dual vectors

Using old terminology they are covariant, so with lower indices. Meanwhile contravariant do have upper indices.

Let's start with defining the **dual space** of a vector space: *Given a vector space (for concreteness T_P)*, we define the **dual space** T_P^* as the space of linear maps between T_P and \mathbb{R} .

Example Being $\omega \in T_P^*$, $V \in T_P$ then

$$\omega(V) \in \mathbb{R}$$

Linearity tells me that

$$\omega(\alpha V + \beta W) = \alpha \omega(V) + \beta \omega(W)$$

1st statement : The dual space is a vector space.

$$(\alpha \omega + \beta \eta)(v) = \alpha \omega(v) + \beta \eta(v)$$

2nd statement : What is the dual of the dual?

$$(T_P^*)^* = T_P \implies v(\omega) = \omega(v) \in \mathbb{R}$$

Basis for T_P^* : $\hat{o}^{(\mu)}$.

How to define this? Definition is

$$\hat{o}^{(\mu)}(\hat{e}_{(\nu)}) \equiv \delta_{\nu}^{\mu}$$

Now let's see if we can get how dual vectors work with vectors. If I have:

- generic item of T_P : $V = V^{\nu} \hat{e}_{(\nu)}$
- generic item of T_P^* : $\omega = \omega_{\mu} \hat{o}^{(\mu)}$

I can compute:

$$\begin{aligned}\omega(v) &= \omega_\mu \hat{o}^{(\mu)}(v^\nu \hat{e}_{(\nu)}) = \\ &= \omega_\mu v^\nu \hat{o}^{(\mu)}(\hat{e}_{(\nu)}) = \omega_\mu v^\nu \delta_\nu^\mu = \omega_\mu v^\mu\end{aligned}$$

Once we know this we can do an **exercise**: show the way $\omega_{\mu'}$ transform. What to do is to start from Λ equality.

What is the example of a dual vector?

$$A_{\mu'} = \Lambda_{\mu}^{\mu'} A_{\mu}$$

the gradient is a beautiful example of a *dual vector*.

$$A_{\mu} = \frac{\partial \phi}{\partial x^{\mu}} ; A_{\mu'} = \frac{\partial \phi}{\partial x^{\mu'}}$$

This is useful to define LTs, in this way

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial \phi}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \rightarrow A_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} A_{\mu}$$

the LT is the last partial derivative.

There is a *more compact* notation to write partial derivatives that is

$$\partial_{\mu} \phi \equiv \frac{\partial \phi}{\partial x^{\mu}}$$

2.3.1 Tensors

Tensors are generalization of dual vectors and vectors.

They are *multilinear maps*, i.e. functions of several variables and linear for all of them. For each tensor of *rank* (k,l), we have

$$T_P^* \times \dots \times T_P^* \times T_P \times \dots \times T_P \rightarrow \mathbb{R}$$

Where each dual vector space is present **k**-times, and vector space **l**-times.

Now let's see what is multilinearity on the combat field.

Be a (1,1) tensor:

- $\alpha, \beta, \gamma, \delta \in \mathbb{R}$
- $\omega, \eta \in T_P^*$
- $v, w \in T_P$

Given these we have

$$T(\alpha\omega + \beta\eta, \gamma v + \delta w) = \alpha\gamma T(\omega, v) + \beta\delta T(\eta, w) + \alpha\delta T(\omega, w) + \beta\gamma T(\eta, v) \quad (2.18)$$

Once we have this general definition, let's take one step back:

- Scalar $\rightarrow (0,0)$ tensor
- Vector $\rightarrow (1,0)$ tensor
- Dual vector $\rightarrow (0,1)$ tensor

Tensor product

Be:

- T, rank (k,l) tensor
- S, rank (m,n) tensor

We want to understand the action of \otimes .

So we know that $T \otimes S$ outputs (k+m, l+n) tensor. In particular,

$$\begin{aligned} T \otimes S \left[\omega^{(1)}, \dots, \omega^{(k)}, \omega^{(k+1)}, \dots, \omega^{(k+m)}, v^{(1)}, \dots, v^{(l)}, v^{(l+1)}, \dots, v^{(l+n)} \right] &\equiv \\ \equiv T \left(\omega^{(1)}, \dots, \omega^{(k)}, v^{(1)}, \dots, v^{(l)} \right) \times S \left(\omega^{(k+1)}, \dots, \omega^{(k+m)}, v^{(l+1)}, \dots, v^{(l+n)} \right) & \\ \implies T \otimes S \neq S \otimes T & \end{aligned}$$

so tensors do not commute.

Basis for a tensor

Let T be a generic tensor with rank (k,l), *basis* is given by

$$\hat{e}_{(\mu_1)} \otimes \dots \otimes \hat{e}_{(\mu_k)} \otimes \hat{o}^{(\nu_1)} \otimes \dots \otimes \hat{o}^{(\nu_l)}$$

A tensor can be written as

$$T = T_{\nu_1, \dots, \nu_l}^{\mu_1, \dots, \mu_k} (\hat{e}_{(\mu_1)} \otimes \dots) = T_{\nu'_1, \dots, \nu'_l}^{\mu'_1, \dots, \mu'_k} (\hat{e}_{(\mu'_1)} \otimes \dots)$$

So the tensor is always the same, the thing that changes is its components, because a change of RF I think.

We will often write the components instead of the actual tensor, but it is our convention to think they are equivalent.

This is how the components are related:

$$\begin{aligned} \hat{e}_{(\mu')} &= \Lambda_{\mu'}^{\mu} \hat{e}_{(\mu)} \\ \hat{o}^{(\mu')} &= \Lambda_{\mu}^{\mu'} \hat{o}^{(\mu)} \\ \implies T &= T_{\nu_1, \dots, \nu_l}^{\mu_1, \dots, \mu_k} \left(\Lambda_{\mu'_1}^{\mu_1} \hat{e}_{(\mu'_1)} \otimes \dots \right) \end{aligned}$$

So we find, as result, that when I change frame

$$T_{\nu'_1, \dots, \nu'_k}^{\mu'_1, \dots, \mu'_k} = \Lambda_{\mu_1}^{\mu'_1} \dots \Lambda_{\nu'_1}^{\nu_1} \dots T_{\nu_1, \dots, \nu_l}^{\mu_1, \dots, \mu_k} \quad (2.19)$$

2.4 Lec 5

2.4.1 Transformations

The goal is to find that is this $T_{\nu'_1, \dots, \nu'_l}^{\mu'_1, \dots, \mu'_k} = ?$.

$$T = T_{\nu_1, \dots, \nu_l}^{\mu_1, \dots, \mu_k} (\hat{e}_{(\mu_1)} \otimes \dots) = T_{\nu'_1, \dots, \nu'_l}^{\mu'_1, \dots, \mu'_k} (\hat{e}_{(\mu'_1)} \otimes) \quad (2.20)$$

I know two facts:

$$\begin{cases} \hat{e}_{\mu'} = \Lambda_{\mu'}^{\mu} \hat{e}_{(\mu)} \\ \hat{o}^{\mu'} = \Lambda_{\mu}^{\mu'} \hat{o}^{\mu} \end{cases} \quad (2.21)$$

and also the *inverse*.

So i apply the Lambda transformation to each term of the basis and I get the following

$$T_{\nu'_1, \dots, \nu'_l}^{\mu'_1, \dots, \mu'_k} = \left(\Lambda_{\mu_1}^{\mu'_1} \dots \Lambda_{\mu_k}^{\mu'_k} \right) \left(\Lambda_{\nu_1}^{\nu'_1} \dots \Lambda_{\nu_l}^{\nu'_l} \right) (T_{\nu_1, \dots, \nu_l}^{\mu_1, \dots, \mu_k}) \quad (2.22)$$

that is something that was obvious by looking at indexes.

2.4.2 Tensor Manipulations / Operations

We defined (k, l) vectors as a multilinear map from dual spaces and vector spaces to real numbers, but it is not only that. For example a $(1, 1)$ tensor could be a map from vectors to vectors, in this way

$$V^{\mu} \rightarrow A_{\nu}^{\mu} V^{\nu} \quad (2.23)$$

so if i do not saturate all the indices, i get a tensor of rank made by what remains. If we saturate, we get real numbers or $(0, 0)$ tensors.

There are some objects that are well known in flat spacetime.

Particular Tensor in flat ST

These are

- $\eta_{\mu\nu}$ metric, or metric tensor
- $\eta^{\mu\nu}$, inverse metric
- δ_{ν}^{μ} , *kroncker's* δ
- $\epsilon_{\mu\nu\rho\delta}$, totally anti-symmetric tensor of Levi-Civita

This last one is defined:

$$\begin{cases} +1 \text{ if } (0, 1, 2, 3) \text{ or even permutations} \\ -1 \text{ if odd permutations} \\ 0 \text{ otherwise} \end{cases} \quad (2.24)$$

These are the only tensors of the flat spacetime that their components do not depend on the RF.

Other operations

Contraction

$$(k, l) \rightarrow (k - 1, l - 1)$$

Example: I have (3,2) tensor $T_{\delta\gamma}^{\mu\nu\rho} \rightarrow (2,1)??$ We contract:

$$T_{\delta}^{\mu} \overset{\nu}{\underset{\gamma}{}} \rightarrow T_{\delta}^{\mu} \overset{\nu}{\underset{\nu}{}} \equiv A_{\delta}^{\mu\rho}$$

Obviously I can *only* contract an upper with a lower index. It is very important the order, and which indices we contract.

$$T_{\delta}^{\mu} \overset{\nu}{\underset{\nu}{}} \neq T_{\nu}^{\mu} \overset{\nu}{\underset{\delta}{}}$$

What is the actual operation we perform?

$$T_{\delta\gamma}^{\mu\nu\rho} = \delta_{\nu}^{\gamma} T_{\delta\gamma}^{\mu\nu\rho}$$

Raising/Lowering Indices To raise we use $\eta^{\mu\nu}$, to lower $\eta_{\mu\nu}$.

$$\eta^{\rho\alpha} T_{\alpha\beta}^{\mu\nu} \equiv T_{\beta}^{\mu\nu\rho}$$

$$\eta^{\rho} \overset{\beta}{} T_{\alpha}^{\mu\nu} \underset{\beta}{} \equiv T_{\alpha}^{\mu\nu} \overset{\beta}{}$$

The order is important, and wring by hand one should be careful keeping the position moving up and down the indices.

Simple operations:

$$V^{\mu} \rightarrow V_{\mu} = \eta_{\mu\nu} V^{\nu}$$

$$V_{\mu} \rightarrow V^{\mu} = \eta^{\mu\nu} V_{\nu}$$

Inner Product

$$T_P \times T_P \rightarrow \mathbb{R}$$

$$(V, W) \rightarrow \eta_{\mu\nu} V^{\mu} V^{\nu}$$

Symmetry Properties

Let's consider a (0,2) tensor $T_{\mu\nu}$, or to be precise, its components. It is symmetric? Anti-symmetric? Both? None?

A tensor is *symmetric* if

$$T_{\mu\nu} = T_{\nu\mu}$$

it is *anti-symmetric* if

$$T_{\mu\nu} = -T_{\nu\mu}$$

It is **never** possible to have a tensor that is *both*. But really possible that is *none* of the above.

We can *symmetrize* a tensor:

$$T_{(\mu\nu)} = \frac{1}{2} (T_{\mu\nu} + T_{\nu\mu})$$

We can *anti-symmetrize* a tensor:

$$T_{[\mu\nu]} = \frac{1}{2} (T_{\mu\nu} - T_{\nu\mu})$$

A tensor can be symmetric on all indices, so it *totally symmetric*, or just on some indices, like two, three etc. The general formula can be:

$$T_{(\mu\nu\rho)} = \frac{1}{3!} (T_{\mu\nu\rho} + \text{all permutations})$$

For anti-symmetrizing odd permutations get the minus in front.

Trace

$$x^\mu_\mu$$

given a (1,1) tensor $\rightarrow \mathbb{R}$ by summing all indices. For example the trace of metric tensor is 2. Of Kronecker delta is 4.