Notes of General Relativity

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Chapter 1

Introduction

Lecture 1

General Relativity describes *gravity* in terms of *curvature* of *space-time*.

We will define and describe those three words.

To understand *curvature*, let's think about a RF in a flat space, so that the sum of all internal angles of a triangle is 180°, as we add curvature, the sum increase its value.

Sphere is a 2D *manifold*. What is a manifold?

From Newton to Einstein

We got two masses, m_1, m_2, the origin, O, of the RF.
Each mass' position is identified by its own position vector.

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$
$$\vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r}$$
$$\text{with } \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

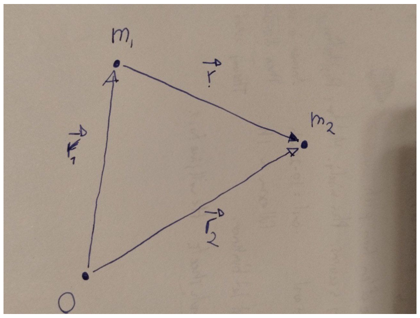
so, we see that m_2 is attracted.
P.S. $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

Introducing the second law of dynamics in the study, we have

$$m_2\vec{a}_2 = \vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

simplifying m_2 we obtain

$$\vec{a}_2 = -\frac{Gm_1}{r^2}\hat{r}$$



We can express $\mathbf{a}_2$ as

$$\vec{a}_2 = -\nabla\phi \text{ Gradient of the Gravitational Potential}$$
$$\phi = -\frac{Gm_1}{r}$$
$$\nabla^2\phi = -4\pi G\rho$$

We will use the Minkowski metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{1.1}$$

We will see also other symbols, like the Kristoffel one, or the Richie Tensor...
But in the end the central goal is to derive the *Einstein Equation*:

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.2}$$

In GR particles move freely along *straight lines* of a curved space-time. These are called *geodesics*.

Example Two chalks, one on the desk, the other is launched in the air. Which one is accelerated? From a GR perspective, the one in the air is moving along a geodesic, so it is the one moving freely, while the other is stopped from doing that by some interference/force.
In GR gravity is *not* a force.

Chapter 2

Math tools

2.1 A recap of SR

Lecture 2 We will develop some of the necessary math on this framework.
Let's look at the Galilean Relativity.
Newtonian dynamics is based on three principles

1. inertia
2. $\vec{F} = m\vec{a}$
3. action-reaction

The first says something like *An object at rest remains at rest, and an object in motion remains in motion at constant speed and in a straight line unless acted on by an unbalanced force.*

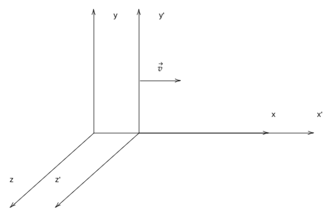
The second one says:

$$(2) : \vec{F} = 0 \implies \vec{a} = 0 \implies (1)$$

So, it seems the first principle is contained by the second, but we know that $\vec{F} = m\vec{a}$ is valid only in Inertial Frames (IF).

Galilean Relativity: all the laws of *mechanics* take the same form in every IF. (You can not distinguish two IF just by doing experiments.)

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$
$$t = t' = 0 \implies O = O'$$



Taking the first derivative:

$$\begin{cases} v'_x = v_x - v \\ v'_y = v_y \\ v'_z = v_z \end{cases} \quad \text{and for the second derivative:} \quad \begin{cases} a'_x = a_x \\ a'_y = a_y \\ a'_z = a_z \end{cases} \implies \vec{a}' = \vec{a} \tag{2.1}$$

so also $\vec{F}' = \vec{F}$. And if m is independent on the frame, we got

$$\vec{F}' = m\vec{a}' = \vec{F} = m\vec{a} \tag{2.2}$$

Then there are Maxwell equations, people thanks to them find that EM-waves propagates with speed c in the void.

But they found also that these equations were not invariant in Galilean Boosts.

Things started to go better when the idea of a preferred IF was ditched and Einstein decided to use Lorentz Transformations.

There are two postulates:

- *Relativity principle*: same as before but with *physics* instead of *mechanics*. **All the laws of physics ...**
- *Speed of light*: in every IF, light propagates with constant speed, c.

So we see that Galilean transformation become inconsistent with this, meanwhile stays valid for $\vec{v} \ll \vec{c}$.

As mentioned before, updated version of G. Boosts are Lorentz transformations (or Lorentz Boosts.)

$$\begin{cases} x' = \frac{x - vt}{\sqrt{1-(\frac{v}{c})^2}} \\ y' = y \\ z' = z \\ t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-(\frac{v}{c})^2}} \end{cases} \tag{2.3}$$

To ensure the L.T. Is consistent we can perform three checks:

- $v \ll c$
- v = 0
- dimensional check

People use a notation to make the L.T. easier to write: $\gamma(v) \equiv \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$, so it becomes

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{vx}{c^2}) \end{cases} \tag{2.4}$$

What happens to the transformation of velocity is: (v is fixed)

$$\begin{cases} dx' = \gamma(dx - vdt) \\ dy' = dy \\ dz' = dz \\ dt' = \gamma\left(dt - \frac{vdx}{c^2}\right) \end{cases} \tag{2.5}$$

so

$$\begin{cases} v'_x = \frac{dx'}{dt'} \\ v'_y = \frac{dy'}{dt} = \frac{dy}{\gamma\left(dt - \frac{vdx}{c^2}\right)} = \frac{v_y}{\gamma\left(1 - \frac{vv_x}{c^2}\right)} \\ v'_z = \frac{dz'}{dt} = ... \end{cases} \tag{2.6}$$

So we see that space-time changes also along other axes.

Now let's talk about space-time and its parts.

Space-time space-time is a manifold. For now it is a collection of (t,x,y,z), four dimensional set of all the possible values of the coordinates.

Event a point of space-time.

World line path of a particle in space-time.

There is no notion of absolute time anymore, because now it is dependent on the frame. Regarding the light-cone, after the event on the (x,y) plane, the particle can move *only* inside the light-cone, in the appropriate direction (time forward).

Now let's talk about **Clock Synchronization.**

It is kinda easy if in in IF. In GR it is quite subtle instead.

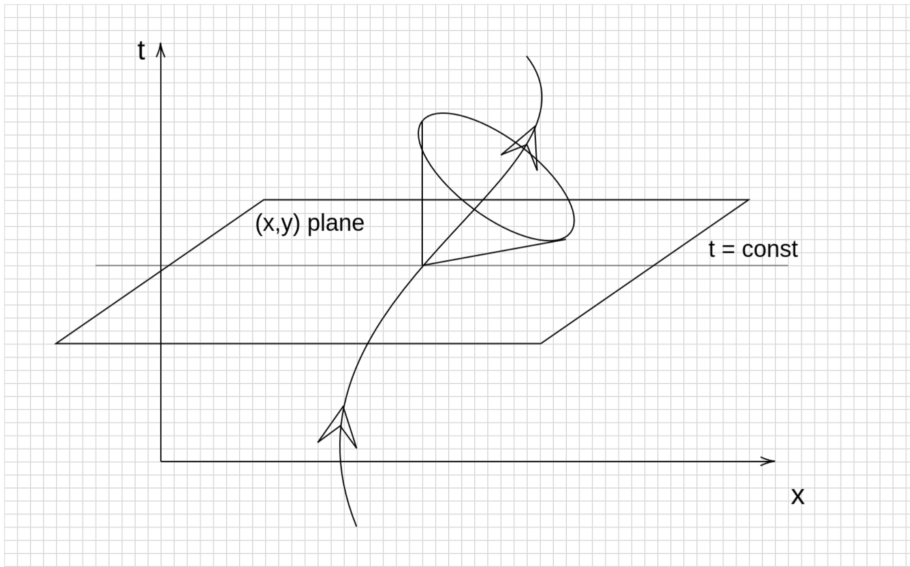


Figure 2.1: LL of a particle which moves forward in time, we see also a light cone

Example: Be me in Origin of a RF watching my clock (A). How to define t at another generic location (B)??

I send a light ray at time t_1 to B. I get the answer on t_2. There is symmetry between the two trajectories so

$$t_m = \frac{t_1 + t_2}{2}.$$

I say to my friend on B: "set your clock to $\mathrm{t_m}$ when you receive the signal." So, following this methodology, each point could have its own clock.

Proper time: How to define proper time?
t is the time coordinate. Let's introduce the metric tensor:

$$\text{the Minkowski metric tensor: } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

for a Lorentz Transformation if I have 2 events E,F.

$$\text{Frame 1: } x_F^\mu = (t_F, x_F, y_F, z_F)$$
$$x_E = (...)$$
$$\text{Frame 2: } x_F^{\mu'} = (t_{F'}, x_{F'}, y_{F'}, z_{F'})$$
$$x_E^{\mu'} = (...)$$

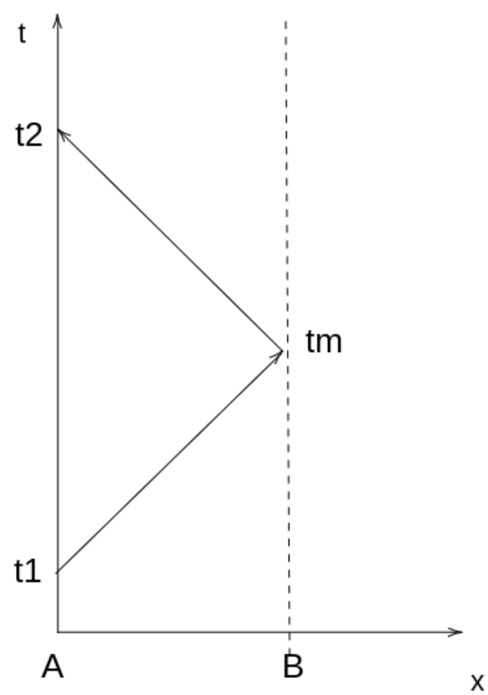


Figure 2.2: Reception and send of the signal

same events in 2 different frames.
A Lorentz Transformation cornets these two events.

Be Δs^2 the Lorentz Invariant separation between E-F.

$$\Delta s^2 = -c\left(t_F - t_E\right)^2 + \left(x_F - x_E\right)^2 + \left(y_F - y_E\right)^2 + \left(z_F - z_E\right)^2 =$$
$$= -c(t_{F'} - t_{E'})^2 + (x_{F'} - x_{E'})^2 + (y_{F'} - y_{E'})^2 + (z_{F'} - z_{E'})^2$$
$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$
$$\text{From this point we set } c = 1 \text{ just a rescaling}$$
$$\text{we have defined } \Delta x^\mu \equiv x_F^\mu - x_F^\mu, \text{with } \mu = 0, 1, 2, 3.$$

So, repeating for clarity, the Lorentz Invariant separation is

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu x^\nu = \eta_{\mu'\nu'} \Delta x^{\mu'} \Delta x^{\nu'} \tag{2.8}$$

Minkowski metric tensor does not change form if we change coordinates (Cartesian coordinates, meanwhile if we use like polar ones it changes for obvious reasons.)

if

$$\begin{aligned} \Delta s^2 &> 0 \text{space-like separation} \\ &< \text{time-like, (it could be an actual LL for a massive particle)} \\ &= \text{light-like or null} \end{aligned}$$

Now we can define the *proper time* as

$$\Delta \tau^2 \equiv -\Delta s^2 \text{ or } \Delta \tau^2 = -\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \tag{2.9}$$

So, if the proper time is *positive* it is time-like.

If the segment **EF** marks the begin and end of the trajectory of a massive particle, $\Delta\tau$, proper time, is the time elapsed on a clock sitting on a RF that moves with constant speed between E and F.
Int the moving frame $\Delta\tau = \Delta t_*$ where $t*$ is the time coordinate of the moving frame. In a frame where I'm at rest this is how Δt^2 changes:

$$\Delta \tau^2 = +\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2. \tag{2.10}$$

2.2 Lecture 3

The meaning of the Lorentz Invariant is that **events**, like (E, F) exist before I define coordinates. It is a property of the two events.

So to recap what we did in the last lecture, be:

$$x_E^\mu \text{ and } x_E^{\mu'} \tag{2.11}$$

If I have two events and computing $\Delta\tau$ gives a positive result, the separation is **time-like**. This means that they could be on the WL of a massive particle moving at constant speed.

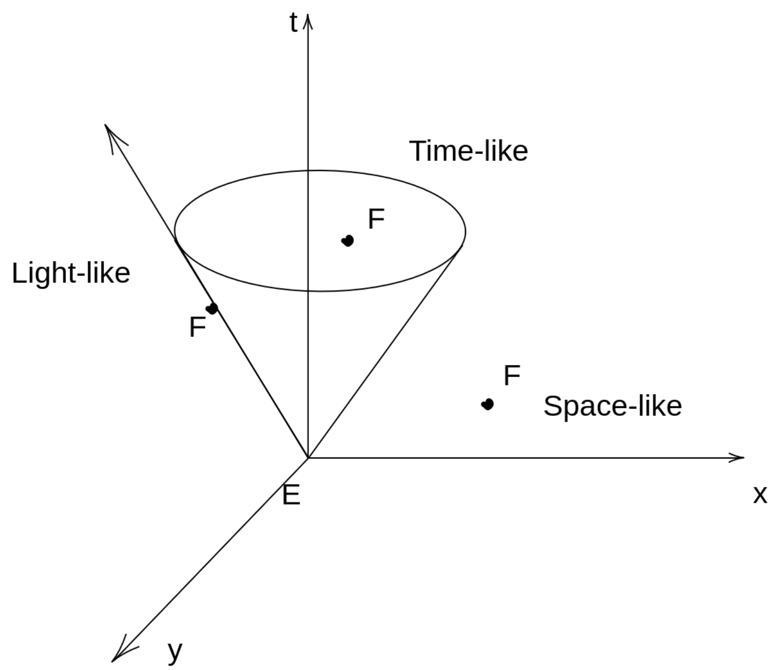


Figure 2.3: Given event E, the separation **EF** could be of different types based on the position respect the light cone

Physical meaning of $\Delta\tau$ It's the time elapsed on a clock of the observer moving between E and F at constant speed.

This means that if I compute $\Delta\tau$ on the frame where the observer it is at rest, i get

$$\Delta\tau = \Delta t'$$

Lets do an example:

Example In fig. 2.4 we see the straight line **ABC** that is the WL of a object not moving. Computing its proper time will be:

$$\Delta\tau_{ABC} = (t_c - t_A) \tag{2.12}$$

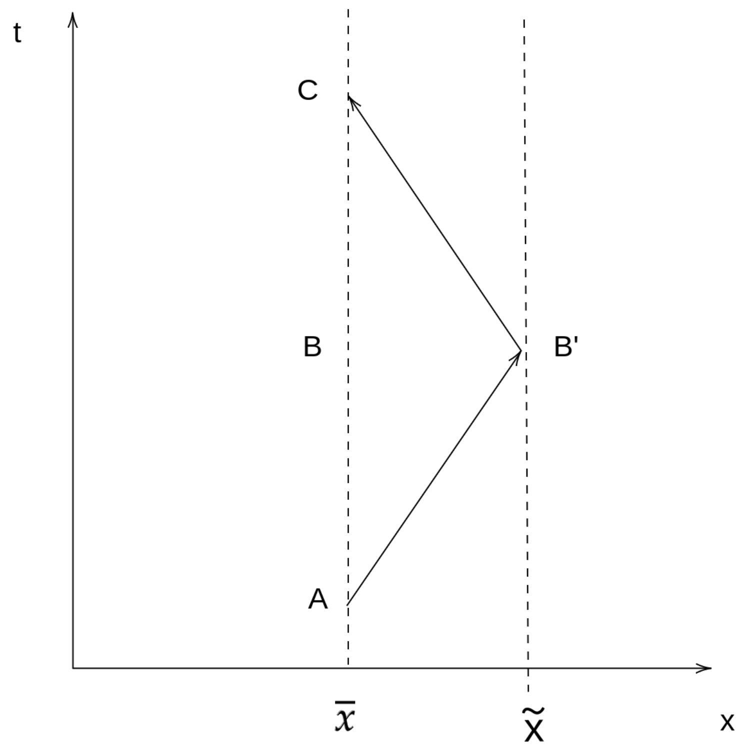


Figure 2.4: It is like the twin paradox.

But for the other WL, of a object moving at constant speed between **AB'** and **B'C**, first thing first, we see that

$$t_B = t_{B'}$$

and so

$$\Delta\tau_{AB'C} = 2\sqrt{\left(t_B - t_A\right)^2 - \left(\tilde{x} - \bar{x}\right)^2} = \Delta\tau_{ABC}\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

$$\implies \Delta\tau_{AB'C} < \Delta\tau_{ABC}$$

This means that I have the longest **proper time** when I don't move.

We can do one more generalization: by parametrize the WL with a quantity λ we get

$$x^\mu\left(\lambda\right)$$

$$\Delta\tau = \int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \text{ that is a time like trajectory.}$$

Enough with proper time.

2.2.1 Tensor Calculus

Be a Lorentz Group, we want to look for the transformations.

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}_\mu x^\mu \tag{2.13}$$

we see that it is a linear transformation. An example to see better what are we doing could be

$$x^{0'} = \Lambda^{0'}_0 x^0 + \Lambda^{0'}_1 x^1 + \Lambda^{0'}_2 x^2 + \Lambda^{0'}_3 x^3 \tag{2.14}$$

What we need to know is that $\Lambda^{\mu'}_\mu$ is a constant matrix.

We see that Λ is a constant matrix.

We want to find linear transformations such that

$$\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = \eta_{\mu'\nu'}\Delta x^{\mu'}\Delta x^{\nu'} \tag{2.15}$$

So the Lorentz Invariant is still invariant. (WTF)

Now, because a SR property: if I move from IF to another, η is still unchanged. So

$$\eta_{\mu\nu} = \eta_{\mu'\nu'}$$

We have to say that Minkowski assumes cartesian coordinates.

The question now is: What trivial transformations leave Δs^2 unchanged?

Translations

$$\eta_{\mu\nu}\Delta x^{\mu}x^{\nu} = \eta_{\mu'\nu'}\left(\Lambda^{\mu'}_{\mu}\Delta x^{\mu}\right)\left(\Lambda^{\nu'}_{\nu}\Delta x^{\nu}\right)$$
$$\implies \eta_{\mu\nu} = \eta_{\mu'\nu'}\Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}$$
$$\text{this obviously needs to be valid } \forall \Delta x^{\mu}$$
$$\text{an alternative notation could be } \eta = \Lambda^{T}\eta\Lambda$$

We will use just the first notation, because we need to get good at tensors.

To be more concrete:

$$\Lambda^{\mu'}_{\mu} = \begin{pmatrix} \Lambda^{0'}_{0} & \Lambda^{0'}_{1} & \Lambda^{0'}_{2} & \Lambda^{0'}_{3} \\ \Lambda^{1'}_{0} & ... & ... & ... \\ \Lambda^{2'}_{0} & ... & ... & ... \\ \Lambda^{3'}_{0} & ... & ... & ... \end{pmatrix} \tag{2.16}$$

Rotations Rotations are a kind of transformation of the type:

$$x_{i'} = R_{ii'}x_{i}$$
$$\text{or } R^{T}\mathbb{I}R = \mathbb{I}$$
$$\text{with } RR^{T} = R^{T}R = \mathbb{I}$$

it could be something like

$$\Lambda^{\mu'}_{\mu} = \begin{pmatrix} cosh\eta & -sinh\eta & 0 & 0 \\ -sinh\eta & cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.17}$$

this one is a boost along the x direction. If we do some computing we find that

$$tanh\eta \equiv v$$

so this is the same of the L.T. we saw last week.

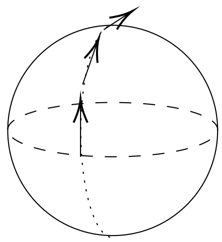
Rotations do not change the time coordinate. The point was to tell what L.T. is in this language.

Vectors I have a generic vector, **do i need to specify about the RF** where it is defined, so in a specific spacetime location? yes

In newtonian mechanics parallel vectors are the same because I can superpose them, I can move them around, also to use the parallelogram rule to get a sum.

$\implies$ If I have 3D euclidean space there is no ambiguities about where i move my vectors.

BUT in a sphere:

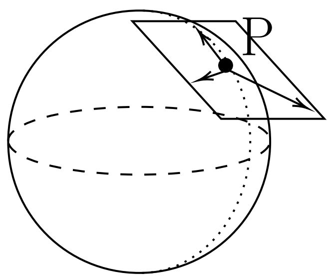


I have this vector at the equator tangent to the surface. If I transport it to the pole i get a different vector.

There are ambiguities. So in a non-flat space we need a **different** procedure.
A vector field is a map between:

$$x^\mu \to v^\mu$$

where x^μ is an event and v^μ is a vector.
Let's define: **Tangent space $\mathbf{T_P}$.**
Given an event P we define the tangent space T_P as all the vectors in P.
Instead of having spacetime we have a sphere.



Define a plane tangent to the sphere only in P. All vectors that lie there $\in T_P$.

T_P is a **vector space**:

$$V, W \in T_P \implies \alpha V + \beta W, (\alpha, \beta \in \mathbb{R}) \in T_P$$

So if there is a vector there is also the inverse vector.
Whenever i have a vector space, I can define infinite basis independently on the coordinate choice. The nUmber of elements in the basis is equal to the dimension of the space, in our case 4 elements.
Obviously if I define the basis its elements need to be Linearly Independent.

Basis Given a generic vector $V \in T_P$, I can define V regardless the coordinate system I'm using. So we can say *meTaphorically* that V exists before I define coordinates.
Be our basis:

$$\hat{e}_{(\mu)}, \text{ with } \mu = 0, 1, 2, 3$$

those indices are label, does not mean "tensor". So my basis is made of

$$\hat{e}_{(0)}, \hat{e}_{(1)}, \hat{e}_{(2)}, \hat{e}_{(3)}$$

Now we can talk about

Components given a generic vector v

$$V = V^0 \hat{e}_{(0)} + V^1 \hat{e}_{(1)} + V^2 \hat{e}_{(2)} + V^3 \hat{e}_{(3)} = V^\mu \hat{e}_{(\mu)}$$

using repeating indices we get the last equivalence.

V^μ are components of the vector V in this specific frame.

In another frame $V^{\mu'}$ could not be the same:

$$V = V^\mu \hat{e}_{(\mu)} = V^{\mu'} \hat{e}_{(\mu')}$$

Question: how do components transform?

covariant vector : is a math object whose components transform based on position

$$V^{\mu'} = \Lambda^{\mu'}_{\mu} V^\mu$$

These are not the only covariant vectors (?).

If you have a generic WL or path, you can parametrize the position by a λ in this way:

$$x^\mu (\lambda)$$

And taking its first derivative you get something similar to the four-velocity

$$u^\mu \sim \frac{dx^\mu}{d\lambda}$$

(I say similar because four-velocity is defined like $u^\mu = \frac{dx^\mu}{d\tau}$).

If I do a L.T. x^μ will change but λ won't.

$$u^{\mu'} = \Lambda^{\mu'}_{\mu} u^\mu$$

I can get a more general definition of what a vector is by following this procedure: choose basis $\rightarrow$ find components $\rightarrow$ study how components change if i change position or basis.

Second definition : Transformation of the basis vectors. The question is "how to relate $\hat{e}_{(\mu)}$ to $\hat{e}_{(\mu')}$?"

We will take advantage of **invariance**.

$$V = V^\mu \hat{e}_{(\mu)} = V^{\mu'} \hat{e}_{(\mu')} = \left(\Lambda^{\mu'}_{\mu} V^\mu \right) \hat{e}_{(\mu')}$$

That's possible **only** if $\hat{e}_{(\mu)} = \Lambda^{\mu'}_{\mu} \hat{e}_{(\mu')}$.

An inverse of LT it is also a LT, so

$$\Lambda^{\mu'}_{\mu} \Lambda^{\mu}_{\nu'} = \delta^{\mu'}_{\nu'}$$
$$\Lambda^{\mu}_{\mu'} \Lambda^{\mu'}_{\nu} = \delta^{\mu}_{\nu}$$

Those are Kroneker's delta and they are an Identity matrix.

Now we can study how basis vectors change.

$$
\begin{gathered}
\hat{e}_{(\mu)} = \Lambda_{\mu}^{\mu'} \hat{e}_{(\mu')} \\
\Lambda_{\nu'}^{\mu} \hat{e}_{(\mu)} = \Lambda_{\mu}^{\mu'} \Lambda_{\nu'}^{\mu} \hat{e}_{\mu'} \\
\Lambda_{\nu'}^{\mu} \hat{e}_{(\mu)} = \delta_{\nu'}^{\mu'} \hat{e}_{(\mu')} \\
\Lambda_{\nu'}^{\mu} \hat{e}_{(\mu)} = \hat{e}_{(\nu')} \\
\text{so } \hat{e}_{(\nu')} = \Lambda_{\nu'}^{\mu} \hat{e}_{(\mu)}
\end{gathered}
$$

2.3 Lecture 4

Brief recap of lec3

We defined vectors

- localized at each spacetime point
- for each event P we defined the tangent space T_P
- there is linear combination inside T_P
- it has a basis
- Vectors and basis transform under LT Group.

Dual vectors

Using old terminology they are covariant, so with lower indices. Meanwhile contravariant do have upper indices.

Let's start with defining the **dual space** of a vector space: *Given a vector space (for concreteness T_P)*, we define the **dual space** T_P^* as the space of linear maps between T_P and $\mathbb{R}$.

Example Being $\omega \in T_P^*$, $V \in T_P$ then

$$\omega(V) \in \mathbb{R}$$

Linearity tells me that

$$\omega(\alpha V + \beta W) = \alpha\omega(V) + \beta\omega(W)$$

1$^{\text{st}}$ statement : The dual space is a vector space.

$$(\alpha\omega + \beta\eta)(v) = \alpha\omega(v) + \beta\eta(v)$$

2$^{\text{nd}}$ statement : What is the dual of the dual?

$$(T_P^*)^* = T_P \implies v(\omega) = \omega(v) \in \mathbb{R}$$

Basis for $\mathbf{T_P}^*$: $\hat{o}^{(\mu)}$.
How to define this? Definition is

$$\hat{o}^{(\mu)}\left(\hat{e}_{(\nu)}\right) \equiv \delta^\mu_\nu$$

Now let's see if we can get how dual vectors work with vectors. If I have:

- generic item of T_P: $V = V^\nu \hat{e}_{(\nu)}$
- generic item of T_P^* : $\omega = \omega_\mu \hat{o^{(\mu)}}$

I can compute:

$$\omega(v) = \omega_\mu \hat{o}^{(\mu)}\left(v^\nu \hat{e}_{(\nu)}\right) =$$
$$= \omega_\mu v^\nu \hat{o}^{(\mu)}\left(\hat{e}_{(\nu)}\right) = \omega_\mu v^\nu \delta^\mu_\nu = \omega_\mu v^\mu$$

exercise: show the way $\omega_{\mu'}$ transform. What to do is to start from Λ equality.

What is the example of a dual vector?

$$A_{\mu'} = \Lambda^{\mu'}_{\mu} A_\mu$$

the gradient is a beautiful example of a *dual vector*.

$$A_\mu = \frac{\partial \phi}{\partial x^\mu} \; ; \; A_{\mu'} = \frac{\partial \phi}{\partial x^{\mu'}}$$

This is useful to define LTs, in this way

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial \phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}} \rightarrow A_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} A_\mu$$

the LT is the last partial derivative.

There is a *more compact* notation tho write partial derivatives that is

$$\partial_\mu \phi \equiv \frac{\partial \phi}{\partial x^\mu}$$

2.3.1 Tensors

Tensors are generalization of dual vectors and vectors.
They are *multilinear maps*, i.e. functions of several variables and linear for all of them. For each tensor of *rank* (k,l), we have

$$T^*_P \times \ldots \times T^*_P \times T_P \times \ldots \times T_P \rightarrow \mathbb{R}$$

Where each dual vector space is present **k**-times, and vector space **l**-times.

Now let's see what is multilinearity on the combat field.

Be a (1,1) tensor:

- $\alpha, \beta, \gamma, \delta \in \mathbb{R}$
- $\omega, \eta \in T^*_P$
- v,w $\in$ $\mathrm{T_P}$

Given these we have

$$T(\alpha\omega + \beta\eta, \gamma v + \delta w) = = \alpha\gamma T(\omega, v) + \beta\delta T(\eta, w) + \alpha\delta T(\omega, w) + \beta\gamma T(\eta, v) \quad (2.18)$$

Once we have this general definition, let's take one step back:

- Scalar $\rightarrow$ (0,0) tensor
- Vector $\rightarrow$ (1,0) tensor
- Dual vector $\rightarrow$ (0,1) tensor

Tensor product

Be:

- T, rank (k,l) tensor
- S, rank (m,n) tensor

We want to understand the action of $\otimes$.
So we know that $T \otimes S$ outputs (k+m, l+n) tensor. In particular,

$$T \otimes S \left[\omega^{(1)}, \ldots, \omega^{(k)}, \omega^{(k+1)}, \ldots, \omega^{(k+m)}, v^{(1)}, \ldots, v^{(l)}, v^{(l+1)}, \ldots, v^{(l+n)}\right] \equiv$$
$$\equiv T\left(\omega^{(1)}, \ldots, \omega^{(k)}, v^{(1)}, \ldots, v^{(l)}\right) \times S\left(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, v^{(l+1)}, \ldots, v^{(l+n)}\right)$$
$$\implies T \otimes S \neq S \otimes T$$

so tensors do not commute.

Basis for a tensor

Let T be a generic tensor with rank (k,l), *basis* is given by

$$\hat{e}_{(\mu_1)} \otimes \ldots \otimes \hat{e}_{(\mu_k)} \otimes \hat{o}^{(\nu_1)} \otimes \ldots \otimes \hat{o}^{(\nu_l)}$$

A tensor can be written as

$$T = T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l}\left(\hat{e}_{(\mu_1)} \otimes \ldots\right) = T^{\mu_1^{'},\ldots,\mu_k^{'}}_{\nu_1^{'},\ldots,\nu_l^{'}}\left(\hat{e}_{\left(\mu_1^{'}\right)} \otimes \ldots\right)$$

So the tensor is always the same, the thing that changes is its components, because a change of RF I think.

We will often write the components instead of the actual tensor, but it is our convention to think they are equivalent.

This is how the components are related:

$$\hat{e}_{(\mu')} = \Lambda^{\mu}_{\mu'}\hat{e}_{(\mu)}$$
$$\hat{o}^{\left(\mu'\right)} = \Lambda^{\mu'}_{\mu}\hat{o}^{(\mu)}$$
$$\implies T = T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l}\left(\Lambda^{\mu_1'}_{\mu_1}\hat{e}_{\left(\mu_1'\right)} \otimes \ldots\right)$$

So we find, as result, that when I change frame

$$T^{\mu_1',\ldots,\mu_k'}_{\nu_1',\ldots,\nu_k'} = \Lambda^{\mu_1'}_{\mu_1} \ldots \Lambda^{\nu_1}_{\nu_1'} \ldots T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_l} \tag{2.19}$$

2.4 Lec 5

2.4.1 Transformations

The goal is to find that is this $T^{\mu_1',\dots,\mu_k'}_{\nu_1',\dots,\nu_l'}$ =?.

$$T = T^{\mu_1,\dots,\mu_k}_{\nu_1,\dots,\nu_l}\left(\hat{e}_{(\mu_1)}\otimes\dots\right) = T^{\mu_1',\dots,\mu_k'}_{\nu_1',\dots,\nu_l'}\left(\hat{e}_{(\mu_1')}\otimes\right) \tag{2.20}$$

I know two facts:

$$\begin{cases}\hat{e}_{\mu'} = \Lambda^{\mu}_{\mu'}\hat{e}_{(\mu)} \\ \hat{o}^{\mu'} = \Lambda^{\mu'}_{\mu}\hat{o}^{\mu}\end{cases} \tag{2.21}$$

and also the *inverse.*

So i apply the Lambda transformation to each term of the basis and I get the following

$$T^{\mu_1',\dots,\mu_k'}_{\nu_1',\dots,\nu_l'} = \left(\Lambda^{\mu_1'}_{\mu_1}\dots\Lambda^{\mu_k'}_{\mu_k}\right)\left(\Lambda^{\nu_1}_{\nu_1'}\dots\Lambda^{\nu_l}_{\nu_l'}\right)\left(T^{\mu_1,\dots,\mu_k}_{\nu_1,\dots,\nu_l}\right) \tag{2.22}$$

that is something that was obvious by looking at indexes.

2.4.2 Tensor Manipulations / Operations

We defined (k,l) vectors as a multilinear map from dual spaces and vector spaces to real numbers, but it is not only that. For example a $(1,1)$ tensor could be a map from vectors to vectors, in this way

$$V^{\mu} \to A^{\mu}_{\nu}V^{\nu} \tag{2.23}$$

so if i do not saturate all the indices, i get a tensor of rank made by what remains. If we saturate, we get real numbers or (0,0) tensors.

There are some objects that are well known in flat spacetime.

Particular Tensor in flat ST

These are

- $\eta_{\mu\nu}$ metric, or metric tensor
- $\eta^{\mu\nu}$, inverse metric
- $\delta^{\mu}_{\nu}, kronecker's\delta$
- $\epsilon_{\mu\nu\rho\delta}$, totally anti-symmetric tensor of Levi-Civita

This last one is defined:

$$\begin{cases}+1 \text{ if } (0,1,2,3) \text{ or even permutations} \\ -1 \text{ if odd permutations} \\ 0 \text{ otherwise}\end{cases} \tag{2.24}$$

These are the only tensors of the flat spacetime that their components do not depend on the RF.

Other operations

Contraction

$$(k,l) \to (k-1, l-1)$$

Example: I have (3,2) tensor $T^{\mu\nu\rho}_{\delta\gamma} \to (2,1)$?? We contract:

$$T^{\mu\;\nu\;\rho}_{\;\delta\;\gamma} \to T^{\mu\;\nu\;\rho}_{\;\delta\;\nu} \equiv A^{\mu\rho}_{\delta}$$

Obviously I can *only* contract an upper with a lower index.
It is very important the order, and which indices we contract.

$$T^{\mu\;\nu\;\rho}_{\;\delta\;\nu} \neq T^{\mu\;\nu\;\rho}_{\;\;\nu\;\delta}$$

What is the actual operation we perform?

$$T^{\mu\nu\rho}_{\delta\gamma} = \delta^{\gamma}_{\nu} T^{\mu\nu\rho}_{\delta\gamma}$$

Raising/Lowering Indices To raise we use $\eta^{\mu\nu}$, to lower $\eta_{\mu\nu}$.

$$\eta^{\rho\alpha} T^{\mu\nu}_{\alpha\beta} \equiv T^{\mu\nu\rho}_{\beta}$$

$$\eta^{\rho\;\beta} T^{\mu\nu}_{\;\;\alpha\;\beta} \equiv T^{\mu\nu\;\beta}_{\alpha}$$

The order is important, and wring by hand one should be careful keeping the position moving up and down the indices.

Simple operations:

$$V^{\mu} \to V_{\mu} = \eta_{\mu\nu} V^{\nu}$$
$$V_{\mu} \to V^{\mu} = \eta^{\mu\nu} V_{\nu}$$

Inner Product

$$T_P \times T_P \to \mathbb{R}$$

$$(V, W) \to \eta_{\mu\nu} V^{\mu} V^{\nu}$$

Symmetry Properties

Let's consider a (0,2) tensor $T_{\mu\nu}$, or to be precise, its components. It is symmetric? Anti-symmetric? Both? None?

A tensor is *symmetric* if

$$T_{\mu\nu} = T_{\nu\mu}$$

it is *anti-symmetric* if

$$T_{\mu\nu} = -T_{\nu\mu}$$

It is **never** possible to have a tensor that is *both*. But really possible that is *none* of the above.

We can *symmetrize* a tensor:

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$$

We can *anti-symmetrize* a tensor:

$$T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$$

A tensor can be symmetric on all indices, so it *totally symmetric*, or just on some indices, like two, three etc. The general formula can be:

$$T_{(\mu\nu\rho)} = \frac{1}{3!}\left(T_{\mu\nu\rho} + \text{ all permutations }\right)$$

For anti-simmetrizing odd permutations get the minus in front.

Trace

$$x^{\mu}_{\mu}$$

given a (1,1) tensor $\rightarrow \mathbb{R}$ by summing all indices. For example the trace of metric tensor is 2. Of Kronecker delta is 4.

2.5 Lec 6

2.5.1 Energy & momentum

Since out goal is to get to the Einstein Equation, we know that in there there should be the *energy momentum tensor* $T^{\mu\nu}$.
As always we will study everything for a flat space-time but it will be useful for non flat ones.
We already saw the four-velocity u^{μ}:

$$u^{\mu} \equiv \frac{dx^{\mu}}{d\tau}$$

while the proper time is $\Delta\tau^2 = -\eta_{\mu\nu}dx^{\mu}dx^{\nu}$.
We need to make clear that we are talking about a time-like space-time trajectory, so $\Delta s^2 < 0$.
Let's start with the WL of a single particle, this is specified by a map $\mathbb{R} \to M$, where M is a manifold that represents spacetime. We usually think the path as a curve parameterized by λ so $x^{\mu}(\lambda)$.
We also use as parameter the τ so $x^{\mu}(\tau)$, this has some advantages because maybe it could be easier to switch to four-velocity.

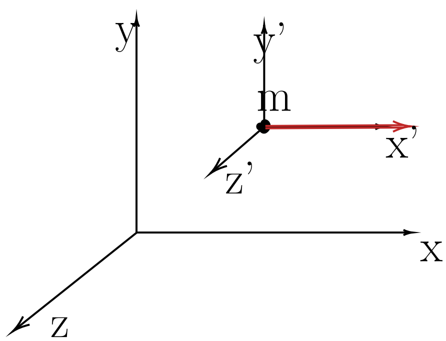
$$u^{\mu}u_{\mu} = u_{\mu}u^{\mu} = \eta_{\mu\nu}u^{\mu}u^{\nu} = -1 \tag{2.25}$$

By the way, four-velocity is what we need to find the *four momentum*:

$$p^{\mu} \equiv mu^{\mu} \tag{2.26}$$

where m is the rest mass that has the same values $\forall$ RF, and it's just a number.

So in rest frame (x',y',z'):



So in the rest frame (x',y',z'):
$p^{\mu} = (m, 0, 0, 0)$, because the four-velocity in the rest frame is $u^{\mu} = (1, 0, 0, 0)$.
What is the expression of p^{μ} in the (x,y.z) frame?
And what is the fastest way to compute it?

We can start from the rest frame and use a LT.
For a generic four vector we have:

$$\begin{cases} a^{0'} = \gamma\left(a^0 - va^1\right) \\ a^{1'} = \gamma\left(a^1 - va^0\right) \\ a^{2'} = a^2 \\ a^{3'} = a^3 \end{cases} \tag{2.27}$$

Now we find the inverse, we can search the inverse of the matrix or use an inverse LT,

$$\begin{cases} a^0 = \gamma \left(a^{0'} + v a^{1'} \right) \\ a^1 = \gamma \left(a^{1'} + v a^{0'} \right) \\ a^2 = a^{2'} \\ a^3 = a^{3'} \end{cases} \tag{2.28}$$

So for the four-momentum we have:

$$\begin{cases} p^0 = E = \gamma p^{0'} = \gamma m = \frac{m}{\sqrt{1-v^2}} \\ p^1 = m\gamma v = \frac{mv}{\sqrt{1-v^2}} \\ p^2 = 0 \\ p^3 = 0 \end{cases} \tag{2.29}$$

In the NR limit we should be able to recover Newton Mechanics:

$$E \approx m + \frac{mv^2}{2} + \dots$$
$$p^1 \approx mv + \dots$$

The four-momentum as we got it provides the description of a single particle but often we need to study a lot of particles as a continuum, like a *fluid*, characterized by quantities as density, pressure, entropy, viscosity...
A single momentum four-vector field is insufficient to describe the energy and the momentum of a fluid so we go further and define the *energy-momentum tensor.*

2.5.2 Energy-Momentum Tensor

$$T^{\alpha\beta}$$

For now it is just a tensor, and we are happy to see that it transform like a tensor:

$$T^{\alpha'\beta'} = \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta} T^{\alpha\beta}$$

In words, it is defined like "*the flux of four-momentum p^α across the surface where x^β is constant*".

For a system of N particles we have:

$$p^\alpha = \sum_{j=1}^{N} p^\alpha_j$$

where j shows the *j-th* particle, not an index to contract. The *number density,* n for a system of N particles is:

$$n = \sum_j \delta \left(\vec{r} - \vec{r}_j \right)$$

So we have these components of the energy-momentum tensor:

$$T^{\alpha 0} = \sum_j p_j^\alpha \frac{dt}{dt} \delta\left(\vec{r} - \vec{r}_j\right)$$

$$T^{\alpha i} = \sum_j p_j^\alpha \frac{dx_j^i}{dt} \delta\left(\vec{r} - \vec{r}_j\right)$$

The $\frac{dt}{dt}$ is obvious that is simplified but we put it for clarity, while $\frac{dx_j^i}{dt}$ is the flux. The meaning is that the tensor is the output of many contribution, each contribute has the center around the *j-th* particle.

This gives me all the components of the E-M tensor. Now, what is a tensor? We used the word tensor because we know a priori what we are gonna find it, but without knowing and looking at the definition and components. We will do it by looking at LTs, and how they act on this object.

First thing we compute:

$$\frac{dx_j^i}{dt} = \frac{dx_j^i}{d\tau} \frac{d\tau}{dt} = \frac{dx_j^i}{d\tau} \frac{1}{\gamma_j}$$

because

$$\frac{d\tau^2}{dt^2} = \frac{dt^2 - dx^2}{dt^2} = 1 - v_j^2 = \frac{1}{\gamma_j^2}$$

why is it useful? Because it appears in components

$$\frac{dx_j^i}{d\tau} = \left(m^j \frac{dx_j^i}{d\tau}\right) \frac{1}{m_j \gamma_j} = \frac{p_j^i}{p_j^0}$$

so i can rewrite the energy-momentum tensor like:

$$T^{\alpha\beta} = \sum_j \frac{p_j^\alpha p_j^\beta}{p_j^0} \delta\left(\vec{r} - \vec{r}_j\right)$$

so if

- $\beta = 0 \rightarrow p_j^\alpha$
- $\beta = 1 \rightarrow \frac{p_j^i}{p_j^0}$

If i switch α with β, I find the same objects, because the tensor is symmetric.

$$T^{(\alpha\beta)} = T^{\alpha\beta} \; ; \; T^{[\alpha\beta]} = 0$$

Why is a tensor? If I change frame I can change

If I change frame I have to change α, β but also p_0 that is not Lorentz Invariant, It's the energy of the particle j, and with a boost it will be different. We have shown that

$$\frac{\delta\left(\vec{r} - \vec{r}_j\right)}{p_j^0}$$

is a Lorentz scalar. Writing

$$T_j^{\alpha\beta} = \frac{p_j^\alpha p_j^\beta}{p_j^0} \delta^{(3)} \left(\vec{r} - \vec{r}_j\right)$$

with a 3-d Dirac Delta function, and with this definition

$$T^{\alpha\beta} = \sum_j T_j^{\alpha\beta}$$

that is a tensor because sum of tensors is still a tensor.

Let's focus on the single contribution:

$$T_j^{\alpha\beta} = \frac{p_j^\alpha m u_j^\beta}{m\gamma_j} \delta^{(3)} \left(\vec{r} - \vec{r}_j\right) =$$

we see that mu^β is p^β, and $m\gamma$ is p^0. We can simplify the masses and we get:

$$\begin{aligned} T_j^{\alpha\beta} &= \frac{p_j^\alpha u_j^\beta}{\gamma_j} \delta^{(3)} \left(\vec{r} - \vec{r}_j\right) = \\ &= \int d\tau_j p_j^\alpha u_j^\beta \delta^{(4)} \left(x^\mu - x_j^\mu \left(\tau_j\right)\right) = \end{aligned}$$

The two expression are equivalent, to show it I have to compute the integral. I use the delta function of the 0 component: so the p,u, γ elements can be extracted from the integral and we compute just

$$= \int d\tau_j \delta \left(x^0 - x_j^0 \left(\tau\right)\right) = \int \frac{1}{\left|\frac{dx_j^0}{d\tau}\right|_{d\overline{\tau}}} d\tau_j \delta \left(\tau_j - \overline{\tau_j}\right) = \tag{2.30}$$

$\overline{\tau}$ is where the argument of the delta function is 0. The integral is straightforward. We have to change variable of the delta function, that gets contributes only from the point that match the argument, I integrate $\tau_j -$ number , There is a jacobian factor: $\frac{1}{\left|\frac{dx_j^0}{d\tau}\right|_{d\overline{\tau}}}$ that is equal to $\gamma_j = \frac{dt}{d\tau_j}$.

This is a tensor, because I have objects with indices α, β, integral over τ_j which is a Lorentz Invariant Scalar and a δ over all space coordinates.

Delta function has the very useful property:

$$\delta \left(f \left(x\right)\right) = \frac{1}{\left|f' \left(x_0\right)\right|} \delta \left(x - x_0\right) \rightarrow \int_{x-\epsilon}^{x+\epsilon} dx \delta \left(x - x_0\right) = 1$$

Case I: Dust

The dust is defined as *generic ensemble of N particles that move very slowly.* So it's any set of particles with kinetic energy much smaller than rest mass energy.

SO the important part is that the relative velocity in some RF $\rightarrow 0$.

The total energy density ρ is described as

$$T^{00} = \sum_{j=1}^{N} p_j^0 \delta\left(x - \overline{x}_j\right) = \rho$$

So dust in the rest frame is

$$T^{\mu\nu} \begin{pmatrix} m \cdot n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.31}$$

where $\rho = m \cdot n$, all particles have the same mass, n is the number density. It's clear that if the particles are at rest, the flux is null.

What is $T^{\mu\nu}$ in a generic frame? I could apply LTs, (and we are invited to try it), but actually i can reason a little on the meaning of the tensor.

I can call u^μ *fluid four-velocity*, if you think about that there is a velocity field on a moving fluid like a river. In the rest frame

$$u^\mu = (1, 0, 0, 0)$$

so

$$T^{\mu\nu} = \rho u^\mu u^\nu$$

This is more generic way to find the tensor $T^{\mu\nu}$ in a generic frame, The strategy, as you may guess, is that I know the expression in a generic frame and I want to recover the tensor from this.

Dust is something very common in cosmology, and is a fluid with zero pressure. But is not always true that in a fluid the pressure is negligible. Photons for example do not have 0 pressure or relative velocity $\rightarrow 0$.

Case II: Fluid with pressure or *Perfect FLuid*

Be

$$T^{\mu\nu}_{rest} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.32}$$

Let's talk about the physics underlying this definition:

The parts $0i$ or $i0$ under rotations or LTs transform like *3-d* vectors, and since the values are 0s, the fluid is *isotropic* $\rightarrow$ no preference about any direction.

ij parts represent the flux of momentum p against surface of constant spatial coordinates. Momentum flux is the force, and force flux is the pressure. We could have different p_i along the diagonal, the fluid wouldn't be isotropic, but there will still be 0 sheer forces.

In the most generic frame:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p\eta^{\mu\nu} + pu^\mu u^\nu = (\rho + p)\, u^\mu u^\nu + p\eta^{\mu\nu} \tag{2.33}$$

Conservation of energy and momentum

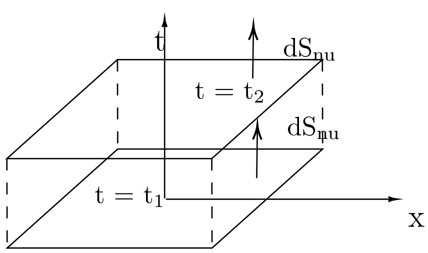
$p^{\mu \text{ total}}$ has to be constant, so energy and momentum are constant. I want this condition to be local:

$$p^\mu_{\text{total}} = \int_V dx^3 T^{\mu 0} = \int_{4d} dS_\nu T^{\mu\nu}$$

If i set $\nu = 0$ I simplify it. I integrate over the entire V that is the entire space. It can be written as the flux of the ν components. The four dimensional integral has a surface S where $T^{\mu\nu}$ is constant. It's a flux integral. $dS_\nu = (1, 0, 0, 0)$.

$$\Delta p^\mu_{\text{total}} = p^\mu_{\text{total}}\ (t_2) - p^\mu_{\text{total}}\ (t_1)$$

So let's make a scheme to understand the situation.



This is an integral of the flux along the time direction.
So the expression of $\Delta p^\mu_{\text{total}}$ can be written as

$$= \int_{\text{total surface}} dS_\nu T^{\mu\nu} = \int_{Volume} dx^3 \partial_\nu T^{\mu\nu} = \int_{volume} dV \partial_\nu T^{\mu\nu}$$

So we have two surfaces and we closed it inside a solid, and it is like we are doing gauss theorem, We can use the divergence theorem in 4d. We state that the divergence of $T^{\mu\nu}$:

$$\partial_\mu T^{\mu\nu} = 0$$
$$\partial_\nu T^{\mu\nu} = 0$$

And that's a way to express conservation.

Results for perfect fluids

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\eta^{\mu\nu}$$

and taking the derivative

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \partial_\mu (\rho + p)\, u^\mu u^\nu + \partial_\mu p\eta^{\mu\nu} + (\rho + p)\, (\partial_\mu u^\mu)\, u^\nu + (\rho + p)\, u^\mu \partial_\mu u^\nu = \\ &= \partial_\mu (\rho + p)\, u^\mu u^\nu + \partial_\mu p\eta^{\mu\nu} + (\rho + p)\, [(\partial_\mu u^\mu u^\nu + u^\mu \partial_\mu u^\nu)]\end{aligned}$$

It is quite clear, just the thing that we neglect derivative of the metric because in flat spacetime it is a constant matrix, and so it's 0.

Let's take the projection of this identity along the direction of the four-velocity.

$$u_\nu \partial_\mu T^{\mu\nu} = -\partial_\mu (\rho + p) u^\mu + (\partial_\mu p) u^\mu + (\rho + p) [-\partial_\mu u^\mu + u_\nu u^\mu \partial_\mu u^\nu] =$$

and since,

$$\partial_\mu (u_\nu u^\nu) = 0 = \partial_\mu (-1)$$

we get

$$= -\partial_\mu (\rho + p) u^\mu + \partial_\mu p u^\mu - (\rho + p) \partial_\mu u^\mu = 0 \tag{2.34}$$

And in the NR limit, $p \ll \rho$:

$$-\partial_\mu \rho u^\mu + \rho \partial_\mu u^\mu + \partial_\mu p u^\mu = 0$$

we can neglect the last term since the hypothesis

$$\partial_\mu (\rho u^\mu) = 0 \implies \partial_t \rho + \partial_i \left(\rho u^i\right) = 0$$

we recover the continuity equation.

exercise : instead of projecting on u^ν, project orthogonal to u^μ,compute: $p^\alpha_\nu \partial_\mu T^{\mu\nu} = 0$. idk what it means: $p^\alpha_\beta \equiv \delta^\alpha_\beta + u^\alpha u_\beta$. Evaluate alpha = 1,2,3.

2.6 Lec 7

2.6.1 Equivalence Principle

The idea of the *universality* of the gravitational interaction, in the form of the *Equivalence principle* led Einstein to think that gravity is special, not just another field, but a metric tensor that describes the curvature of spacetime.

Weak Equivalence Principle, WEP

It states that *inertial* mass and *gravitational* mass of any object are equal.

From the Second Law of Mechanics:

$$\vec{F} = m_i \vec{a}$$

with m_i = inertial mass. While to quantify gravitational forces in Newtonian mechanics:

$$\vec{F} = -m_g \nabla \Phi_g$$

With $\nabla \Phi_g$ gradient of scalar field Φ_g, known as gravitational potential. From these formulas, we see no actual reason why $m_i = m_g$:

- The inertial mass has a universal character, it takes the same value no matter what kind of force is being exerted.
- The gravitational mass is a quantity specific to the gravitational force. One could think $\frac{m_g}{m_i}$ as the *gravitational charge.*

Galileo showed by rolling balls down the inclined plane, that the response of matter to gravitation is universal, and in Newtonian mechanics it translates in WEP:

$$m_i = m_g$$

This, for freely falling objects, becomes

$$a = -\nabla \Phi$$

This led to think an equivalent formulation of WEP that is: *there exists a preferred class of trajectories through space time, called Inertial or Freely-Falling.* Freely falling is intended as "moving under the sole influence of gravity", these objects are unaccelerated.

The universality of gravitation can be stated in another form: If we consider a physicist in a spaceship that is accelerating at a constant rate, like

$$\vec{a} = -\vec{g}$$

he would be not able to distinguish by scientific experiments the situation in which he sits on Earth's surface. (Restricted to local observation).

If the spaceship would be sufficiently big, we would see that the effect of acceleration would always be in the same direction, while on the surface or the Earth we would see that it points towards the center of the earth, so radial vs straight parallel lines.

So WEP could be stated as *the motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, in small enough regions of spacetime.* In larger regions there would be inhomogenities, which will lead to tidal forces.

Einstein's Equivalence Principle, EEP

The Einstein Equivalence Principle is just a little generalization of the WEP:

> In small enough regions of spacetime, the laws of physics reduce to those of special relativity: it is impossible to detect the existence of gravitational field by means of local experiments.

Consider a hydrogen atom, a bound state of a proton and an electron. Its mass is actually less than the sum of the masses of the proton and electron considered individually, because there is a negative binding energy—you have to put energy into the atom to separate the proton and electron. According to the WEP, the gravitational mass of the hydrogen atom is therefore less than the sum of the masses of its constituents; the gravitational field couples to electromagnetism (which holds the atom together) in exactly the right way to make the gravitational mass come out right. This means that not only must gravity couple to rest mass universally, but also to all forms of energy and momentum—which is practically the claim of the EEP.

It is the EEP that implies that we should attribute the action of gravity the curvature of spacetime.

Strong Equivalence Principle, SEP

Is defined to include all of the laws of physics, gravitational and otherwise. We will define *unaccelerated* as *freely falling*, from here we decide that gravity is not a force, because a force leads to acceleration, and our definition of zero acceleration is *moving freely in the presence of whatever gravitational field happens to be around.*

We know that there is a class of preferred frames: Inertial Frames (where laws of dynamics are true). We introduce a new class of frames: Freely Falling Frames, where *unaccelerated particles move only due to gravity.*

Obviously these frames must be local frames, otherwise, due to inhomogenities on the gravitational field, particles initially at rest will begin to move with respect to such frame.

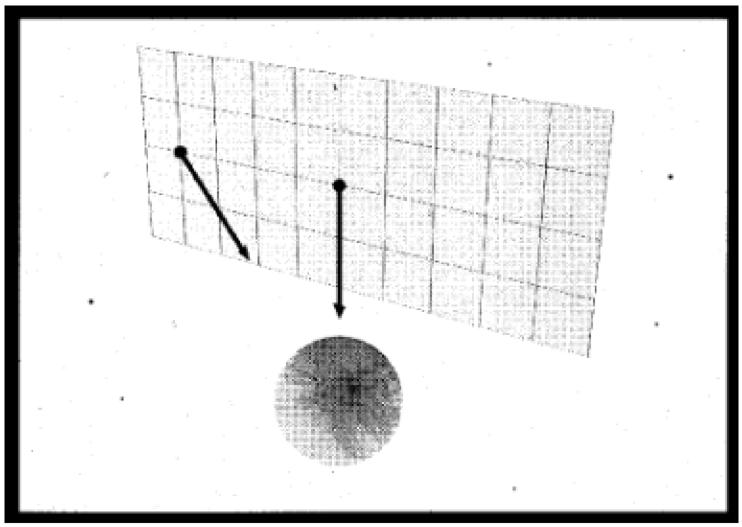


Figure 2.5: The failure of global frames.

After this we need a mathematical framework were what just said is consistent. The solution is to think that spacetime has a curved geometry and gravitation the manifestation of this curvature. Before jumping in what is a manifold, let's see if the consequences are of our world.

Gravitational Redshift

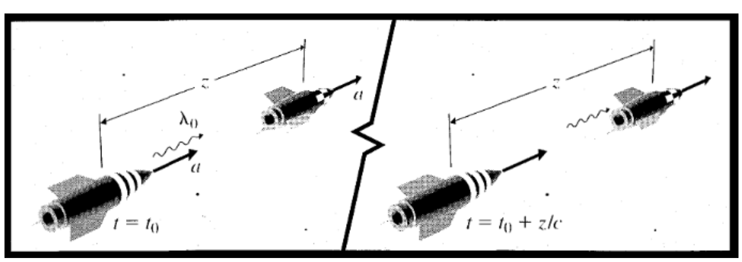


Figure 2.6: Doppler shift measured by two rockets, each feeling acceleration **a**.

Be two spaceships, separated by distance z, each moving with constant $\vec{a}$ acceleration in a region without gravitational fields.

At t_0 ship in the back emits a photos of λ_0.

The distance z stays constant, so the photon is received after $\Delta t = z/c$ in our reference frame. At $t = t_0 + \Delta t$ the boxes have picked up an additional velocity $\Delta v = \vec{a}\Delta t = \vec{a}z/c$. The photon reaching the front spaceship will be redshifted bu the Doppler Effect, by

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{az}{c^2}$$

And according to EEP this should happen also in a uniform gravitational field.

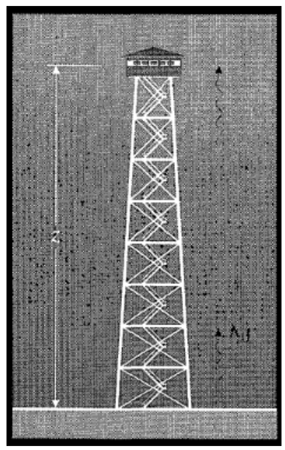


Figure 2.7: Gravitational redshift on Earth's surface

So i a photon e mitted from the ground with λ_0 will be redshifted by

$$\frac{\Delta\lambda}{\lambda_0} = \frac{a_g z}{c^2}$$

To note that is a direct consequence of EEP, no details of GR were required. The thing is, if i try to represent this with Minkowski metric, I don't notice the redshift.

So now we really need to talk about Manifolds

2.6.2 Manifolds

A manifold is a space that may be curved and have a complicated topology, but in local regions looks just like $\mathbb{R}^n$. A crucial part is that the dimensionality n of the Euclidean Spaces being used must be the same in every patch of the manifold. For example are not manifolds, a line ending on a plane and two cones intersecting at their vertices.

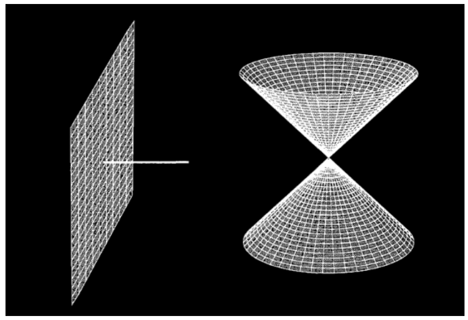


Figure 2.8: Not manifolds

Coordinate System

Be

$$\begin{cases} U \subset M \\ \phi : U \to \mathbb{R}^n \\ \phi\left(U\right) \text{ is open in } \mathbb{R}^n \end{cases} \tag{2.35}$$

These are a system of conditions to define a coordinate system or *chart.*

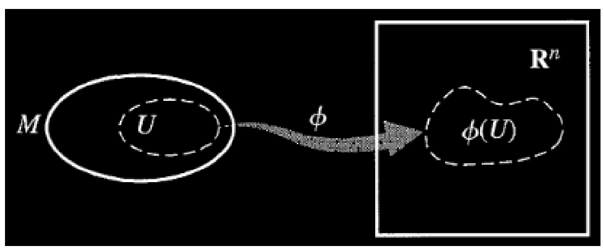


Figure 2.9: A coordinate chart covering an open subset U of M.

An *atlas* is a indexed collection of charts $\{(U_\alpha, \phi_\alpha)\}$.

Vectors again

One point we stressed was the notion of a tangent space, the set of all vectors at a single point in spacetime. A vector is not a thing that stretches from one point to another but is an object associated with a single point.

Be $f : M \to \mathbb{R}$. Each curve passing through a point P, defines an operator, the *directional derivative*, which maps $f \to df/d\lambda$ (at p).

We claim the tangent space T_P can be identified with the space of directional derivative operators along curves through P. And for any f we can write:

$$\frac{df}{d\lambda} = \frac{df}{dx^\mu}\frac{dx^\mu}{d\lambda} \implies \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

If i change the coordinates i can apply the chain rule.

2.7 Lec 8

2.7.1 Brief Recap

We saw the WEP, that states $m_i = m_g$, and as consequence we get that is is impossible distinguish a gravitational field from motion, at least locally.

With the EEP we were able to derive the expression that quantify the gravitational redshift. The SEP included gravity.

Focusing on EEP, we introduced *Locally Inertial Frames, LIF* (equivalent to Freely Falling Frames). Having accepted that gravity cannot be treated as a force, because it is impossible to disentangle acceleration due to gravity, we identified a preferred class of frames: LIFs.

In LIFs laws of physics are equal to the laws of SR and spacetime is Minkowskian.

We introduced Coordinates: with a generic set M, a chart, given a subset $U \subset M$, is a injective linear map ϕ, that

$$\phi : U \to \mathbb{R}^n$$

An *atlas* is an indexed $\{U_\alpha, \phi_\alpha\}$, in such a way that U_α cover M. A *manifold* is a set M along with an atlas. A manifold can be C^∞ if ϕ are differentiable an infinite amount of times, otherwise is C^p, differentiable p-times.

Vectors again again

We resurrected T_P, P generic M point. T_P is the vector space of all the vectors defined at that point.

> T_P is identified with the space of directional derivatives operators acting along the curves through P

Why this identification makes sense?

A generic curve through spacetime, that we call WL, is a parametric curve that is indicated by $x^\mu(\lambda)$: for a specific value of λ I have the x^μ point.

$$x^\mu\left(\overline{\lambda}\right) = P$$

How do I define the directional derivatives? Be

$$\frac{d}{d\lambda}$$

that acts on functions, *f*,

$$f : M \to \mathbb{R}$$

then

$$\frac{d}{d\lambda}(f) = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}f \tag{2.36}$$

And if true $\forall f$, I can identify the equality among the operators:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

We see that it is like a basis.

Basis vectors for $T_P \to \partial_\mu$. (we previously called them $\hat{e}_{(\mu)}$).

In conclusion a generic vector is

$$V = V^\mu \partial_\mu$$

where V^μ are the components, and ∂_μ are the basis elements.

It's very easy to show how vectors transform because we know how derivatives transform:

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \tag{2.37}$$

or in a tensor-like notation:

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \tag{2.38}$$

If V tensor is invariant, by definition, it's components transform anyway like

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

Example LTs

$$x^{\alpha'} = \Lambda^{\alpha'}_\alpha x^\alpha$$

I consider a specific change of coordinates: LTs, so

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \left(\Lambda^{\mu'}_\alpha x^\alpha \right) = \Lambda^{\mu'}_\alpha \frac{\partial x^\alpha}{\partial x^\mu} = \Lambda^{\mu'}_\alpha \delta^\alpha_\mu = \Lambda^{\mu'}_\mu$$

we get under more general transformations vectors components transform like this, and more, we recovered the LT transformation.

2.7.2 Dual Vectors

Be the *cotangent space* T^*_P. If $\omega \in T^*_P$, $\omega$is a linear map $\omega : T_P \to \mathbb{R}$. We want to define formally T^*_P (like we did for T_P). We know that the tangent space T_P holds directional derivatives, while cotangent space T^*_P holds gradients.

For a generic $f : M \to \mathbb{R}$:

$$\frac{d}{d\lambda} \in T_P$$

$$d \in T^*_P$$

and so

$$df \left(\frac{d}{d\lambda} \right) \equiv \frac{df}{d\lambda}$$

$$\downarrow \quad \downarrow \qquad \downarrow$$

$$\in T^*_P, \in T_P, \in \mathbb{R}$$

Basis for $\mathrm{T_P}^*$: dx^μ.

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$$

that is the same as the old $\hat{O}^{(\mu)}\left(\hat{e}_\nu\right) = \delta^\mu_\nu$.

A dual vector is

$$\omega = \omega_\mu dx^\mu$$

the basis component transform like

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$$

and the vector components

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$$

2.7.3 Tensors (k,l)

$$T : T_P^* \times \ldots \times T_P^* \times T_P \times \ldots \times T_P \to \mathbb{R}$$

Tensors can be expanded into components:

$$T = T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} \left(\partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_l} \right)$$

So the components of a generic tensor transform like

$$T^{\mu'_1 \ldots \mu'_k}_{\nu'_1 \ldots \nu'_l} = \left(\frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \right) T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l}$$

Now let's see a unusual tensor, it's a (2,1) tensor, how does transform?

$$T^{\alpha' \beta'}_{\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial x^\gamma}{\partial x^{\gamma'}} T^{\alpha\beta}_\gamma$$

Example/exercise (from 2.4 of Carroll) Be a tensor S_{ij}, with $i, j = 1, 2$, so it's a (0.2) tensor. We know that

- $S_{11} = 1$
- S_{12}=$S_{21} = 0$
- $S_{22} = x^2$

We get new coordinates:

$$x' = \frac{2x}{y}$$
$$y' = \frac{y}{2}$$

What are the expressions for $S_{i'j'}$?

One could think to compute each entry doing

$$S_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} S_{ij}$$

So, like the (1,1) one looks like

$$S_{1'1'} = \frac{\partial x^i}{\partial x^{1'}} \frac{\partial x^j}{\partial x^{1'}} S_{ij}$$

It is good exercise to do this. But it seems that there is a much faster way than this.

We can write the tensor S as

$$S = S_{\mu\nu} \left(dc^{\mu} \otimes dx^{\nu} \right)$$

and for our case

$$S = S_{ij} \left(dx^i \otimes dx^j \right)$$

The action of this tensor could be written as

$$S\left(dx^i, dx^j\right) = S_{11}dx^2 + S_{12}dxdy + S_{21}dydx + S_{22}dy^2 = dx^2 + x^2dy^2 \tag{2.39}$$

the two middle terms are not grouped because tensor product does not commute.

Now we can take the inverse coordinate transformation and write it down.

$$\begin{cases} x = x'y' \\ y = 2y' \end{cases} \rightarrow \begin{cases} dx = x'dy' + y'dx' \\ dy = 2dy' \end{cases} \tag{2.40}$$

and then we substitute inside eq. [2.39], getting

$$\left(x'dy' + y'dx'\right)^2 + 4dy'^2 = \\ x'^2dy'^2 + y'^2dx'^2 + x'y'\left(dx'dy' + dy'dx'\right) + 4dy'^2$$

so we get:

$$\begin{cases} S_{ii} = y'^2 \\ S_{ij} = S_{ji} = x'y' \\ S_{jj} = x'^2 + 4\left(x'y'\right)^2 \end{cases} \rightarrow \begin{pmatrix} y'^2 & x'y' \\ x'y' & x'^2 + 4\left(x'y'\right)^2 \end{pmatrix} = S_{i'j'} \tag{2.41}$$

2.7.4 Special Tensors

Special tensor that we will see are

- Derivative
- metric tensor
- Levi-Civita Symbol

Derivative

We will show that the derivative of a tensor is not a tensor anymore.

Derivative of a scalar is a (0,1) tensor.

$$\partial_\mu \phi \to \partial_{\mu'} \phi = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \phi$$

ϕ does not change under transformation, but the derivative does.

Derivative of a tensor $\neq$ tensor: **example** :

$$A_{\mu\nu} = \partial_\mu V_\nu$$

But why?

Let's transform it:

$$A_{\mu'\nu'} = \partial_{\mu'} V_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} V_\nu \right) =$$

now if we apply the partial derivative we obtain

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\mu V_\nu + \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \right) \left(\frac{\partial^2 x^\nu}{\partial x^{\nu'} \partial x^\mu} \right) V_\nu$$

What happened? We see there is a piece that we don't like. This second piece is 0 for LTs, because they are linear so second derivatives are null. This because in Euclidean space the derivative is independent on the coordinates, and on Minkowski space too, since it's flat.

The tensor itself is independent of the coordinate system, but the operation of taking a partial derivative is highly dependent on what coordinate system you're using.

We give importance to this anyway because GR is not a theory for just LTs. We will develop a covariant derivative that applied to a tensor will give back a tensor.

Metric tensor

The roles of the metric tensor are many:

- supplies a notion of *past* and *future*
- allows the computation of path length and proper time
- determines the *shortest distance* between two points
- replaces the Newtonian gravitational field ϕ
- provides a notion of locally inertial frames
- determines causality
- replaces the Euclidean 3D dot product of Newtonian mechanics.

We know the Minkowski's one:

$$\eta_{\mu\nu} = diag\left(-1,1,1,1\right) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.42}$$

In general $g_{\mu\nu}$ will be the metric tensor.

We assume the determinant of $g_{\mu\nu}$: $det\left(g_{\mu\nu}\right) \equiv g \neq 0 \rightarrow$ the metric tensor is *invertible*.

$$g_{\mu\nu}g^{\nu\alpha} = \delta_{\mu}^{\alpha}\ ;\ g^{\mu\nu}g_{\nu\alpha} = \delta_{\alpha}^{\mu}$$

A constant metric tensor implies not curvature.

A metric tensor that depends explicitly on the coordinates must describe a non-flat space? **FALSE** e.g. the polar coordinates: writing $\eta_{\mu\nu}$ in them

$$\eta_{\mu\nu}^{polar} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta, \end{pmatrix} \tag{2.43}$$

. Given a generic metric $g_{\mu\nu}$ and a given event P, it is always possible to find new coordinates:

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}}g_{\mu\nu}$$

And

$$g_{\hat{\mu}\hat{\nu}}\left(P\right) = \eta_{\hat{\mu}\hat{\nu}}\ ;\ \partial_{\hat{\sigma}}g_{\hat{\mu}\hat{\nu}}\left(P\right) = 0$$

This recalls the definition of LIFs.

2.8 Lec 9

Active and passive transformations

Active transformations change the physical position of a set of points relative to a fixed coordinate system.

Passive transformations leave the points fixed but change the coordinate system relative to which they are described. We prefer the first approach, we leave vectors untouched and transform RFs.

2.8.1 Still on Metric Tensor

We described a little this tensor in the previous lecture. I remember that:

$$det\left(g\right) = g \neq 0$$

in 3D Euclidean Space

$$g = \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and in spherical it would be like

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 sin^2\theta & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

In general the metric tensor has this form

$$g_{\mu\nu} = diag\left(-1, \ldots, 1, \ldots, 0, \ldots\right)$$

If there are just '+1', the metric is Euclidean.

If I have one ' -1' and only ' $+1$' the metric is Lorentzian.

We will not discuss other combinations.

Today we will try to formalize EEP.

Let P be a spacetime point and

$$g_{\mu\nu}\left(P\right) = \text{ some generic matrix } \neq \eta_{\mu\nu}$$

I want to find new coordinates $x^{\hat{\mu}}$ such that $g_{\hat{\mu}\hat{\nu}}\left(P\right) = \eta_{\mu\nu}$ and $\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}}\left(P\right) = 0$. If you watch the last expression you see that it includes 40 different expressions $(64/2) + 8$

I choose $x^{\mu}_P = x^{\hat{\mu}}_P = 0$, so P is the origin of both frames. So I get

$$g_{\hat{\mu}\hat{\nu}}\left(x^{\hat{\alpha}}\right) = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} g_{\mu\nu}\left(x^{\alpha}\right)$$

I'm interested in transformations around P, so I see that spacetime is locally Minkoskian.

Doing a Taylor expansion I will see that first order is Minkoskian:

$$x^{\mu}\left(x^{\hat{\mu}}\right)=\left(\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\right)_P x^{\hat{\mu}}+\frac{1}{2}\left(\frac{\partial^2 x^{\mu}}{\partial x^{\hat{\mu}}\partial x^{\hat{\nu}}}\right)_P x^{\hat{\mu}}x^{\hat{\nu}}+\frac{1}{6}\left(\frac{\partial^3 x^{\mu}}{\partial x^{\hat{\mu}}\partial x^{\hat{\nu}}\partial x^{\hat{\rho}}}\right)_P x^{\hat{\mu}}x^{\hat{\nu}}x^{\hat{\rho}}+\dots \tag{2.44}$$

Why did he stop at 3^{rd} order? Listen to recording at $\sim 39:00$.

$$\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}=\left(\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\right)_P+\left(\frac{\partial^2 x^{\mu}}{\partial x^{\hat{\mu}}\partial x^{\hat{\alpha}}}\right)_P x^{\hat{\alpha}}+\frac{1}{2}\left(\frac{\partial^3 x^{\mu}}{\partial x^{\hat{\mu}}\partial x^{\hat{\alpha}}\partial x^{\hat{\beta}}}\right)_P x^{\hat{\alpha}}x^{\hat{\beta}} \tag{2.45}$$

$$g_{\mu\nu}\left(x^{\alpha}\right)=\left(g_{\mu\nu}\right)_P+\left(\partial_{\rho}g_{\mu\nu}\right)_P x^{\rho}+\frac{1}{2}\left(\partial_{\rho}\partial_{\sigma}g_{\mu\nu}\right)_P x^{\rho}x^{\sigma}+\dots \tag{2.46}$$

Now i just need to write down the metric tensor in new coordinates:

$$\hat{g}+\left(\hat{\partial}\hat{g}\right)_P\hat{x}+\left(\hat{\partial}\hat{\partial}\hat{g}\right)_P\hat{x}\hat{x}=\left(\frac{\partial x}{\partial\hat{x}}\frac{\partial x}{\partial\hat{x}}g\right)_P+\left(\frac{\partial x}{\partial\hat{x}}\frac{\partial^2 x}{\partial\hat{x}\partial\hat{x}}+\frac{\partial x}{\partial\hat{x}}\frac{\partial x}{\partial\hat{x}}\partial g\right)_P\hat{x}+$$
$$+\left(\frac{\partial x}{\partial\hat{x}}\frac{\partial^3 x}{\partial\hat{x}\partial\hat{x}\partial\hat{x}}g+\frac{\partial^2 x}{\partial\hat{x}\partial\hat{x}}\frac{\partial^2 x}{\partial\hat{x}\partial\hat{x}}g+\frac{\partial x}{\partial\hat{x}}\frac{\partial^2 x}{\partial\hat{x}\partial\hat{x}}\partial\hat{g}+\frac{\partial x}{\partial\hat{x}}\frac{\partial x}{\partial\hat{x}}\hat{\partial}\hat{\partial}g\right)\hat{\partial}\hat{x}$$

This is the structure (with all indices suppressed) of the Taylor expansion (.rec..) around point P.

It's true to write:

$$\hat{g}=A+B\hat{x}+C\hat{x}\hat{x}+\dots$$

with A,B,C the three terms up there. So we can set terms of equal order in $\hat{x}$ on each side equal to each other. Therefore it's like having

$$\left(g_{\hat{\mu}\hat{\nu}}\right)_P=A=\left(\frac{\partial x}{\partial}\hat{x}\frac{\partial x}{\partial\hat{x}}g\right)_P$$

On the left we have 10 numbers in all to describe a symmetric two-index tensor, and they are determined by the matrix on the right, This is a 4×4 matrix without constraints, so we have enough freedom to put the 10 numbers of the left tensor into *canonical form*.

At first order we have, on the left, four derivatives of 10 components for a total of 40 numbers, while on the right side we have 10 choices of $\hat{\mu}$s and four choices of μs, for a total of 40 degrees of freedom. This is precisely the number of choices we need to determine all of the first derivatives of the metric, which we can therefore set to 0.

At second order with left side the item is symmetric on it's indices in pairs for a total of $10\times 10=100$ numbers, On the right-hand side we have symmetry in the three lower indices gaining 20 possibilities multiplied by four for the upper index we have 80 degrees of freedom, 20 fewer than we require to set the second derivative of the metric to 0.

2.8.2 Levi Civita symbol

We like tensors but sometimes we also like nontensorial objects. Let's remember the Levi Civita symbol

$$\tilde{\epsilon}_{\mu_1\ldots\mu_n} = \begin{cases} +1 \text{ if even permutations} \\ -1 \text{ if odd permutations} \\ 0 \text{ otherwise} \end{cases}$$

By definition this symbol has the components specified above in *any* coordinate system, and it is a symbol and not a tensor because it is defined to not change under coordinate transformations, We are able to treat him like tensor only in inertial coordinates in flat spacetime.

If $\epsilon_{\mu\nu\rho\gamma}$ was a tensor then it should transform like

$$\tilde{\epsilon}_{\mu'_1\ldots\mu'_n} = \frac{\partial x^{\mu_1}}{\partial x^{\mu_{1'}}}\cdots\frac{\partial x^{\mu_{n'}}}{\partial x^{\mu_{n'}}}\tilde{\epsilon}_{\mu_1\ldots\mu_n} \tag{2.47}$$

but this is **NOT TRUE**.

We are able to treat it as a tensor only in inertial coordinates of spacetime since LTs would have left the components invariant anyway.

It's behaviour can be related to the determinant of a generic matrix M:

$$\tilde{\epsilon}_{\mu_{1'}\ldots\mu_{n'}}\|M\| = \tilde{\epsilon}_{\mu_1\ldots\mu_n}M^{\mu_1}_{\mu_{1'}}\ldots M^{\mu_n}_{\mu_{n'}}.$$

For example, a 2×2 matrix:
M^{μ}_{ν} with $\mu,\nu = 1,2$.

$$\tilde{\epsilon}_{12}\cdot det\,(M) = det\,(M) = \tilde{\epsilon}_{\mu_1\mu_2}M^{\mu_1}_1 M^{\mu_2}_2 = \tag{2.48}$$

$$= \tilde{\epsilon}_{12}M^1_1M^2_2 + \tilde{\epsilon}_{21}M^2_1M^1_2 = M^1_1M^2_2 - M^2_1M^1_2 = ad - bc \tag{2.49}$$

Setting $M^{\mu}_{\mu'} = \partial x^{\mu}/\partial x^{\mu'}$, we get

$$\tilde{\epsilon}_{\mu'_1\ldots\mu'_n} = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|\tilde{\epsilon}_{\mu_1\ldots\mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}$$

If you notice, we moved the determinant from the left-hand side to the right-hand one, by reversing it.

$\tilde{\epsilon}$ is not a tensor because otherwise that determinant wouldn't be there. So it does not transform the way I want. Let's construct a Levi-Civita *tensor.*

Remember the transformation of the metric tensor?

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}g_{\mu'\nu'}$$

I can take the determinant and apply the Binet Rule [1]

$$detg\,(x^{\mu}) = det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right)^2 detg\left(x^{\mu'}\right)$$

[1] det(AB) = det(A)*det(B)

and rewrite it in

$$detg\left(x^{\mu'}\right) = \frac{1}{det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right)^2} detg\left(x^{\mu}\right)$$

I see that g is not invariant if i change coordinates.

Weights Tensor densities, they almost transform like a tensor up to a given factor of a given power.

For example $\tilde{\epsilon}$ has weight $w = +1$

g has weight $w = -2$

Using this I get that

$$\sqrt{|g|}\tilde{\epsilon}_{\mu_1 \ldots \mu_n}(\rightarrow w = 0) \equiv \epsilon_{\mu_1 \ldots \mu_n}$$

So it is a tensor, because a tensor has $w = 0$. We introduce this distinction:

- $\tilde{\epsilon}$ is the **symbol**
- ϵ it the L-C **tensor**

For the L-C symbol with upper indices $\tilde{\epsilon}^{\mu_1 \cdots \mu_n}$ the values are

$$\epsilon^{\mu_1 \cdots \mu_n} = sgn\left(g\right) \tilde{\epsilon}_{\mu_1 \ldots \mu_n}$$

so we have a weight of -1 and so the relative tensor with upper indices is

$$\epsilon^{\mu_1 \cdots \mu_n} = \frac{1}{\sqrt{|g|}} \tilde{\epsilon}^{\mu_1 \cdots \mu_n}$$

2.9 Lec 10

2.9.1 Differential form

A *p-form* or *differential form* is a (0,p) that is completely antisymmetric. Examples of p-forms

- scalars are 0-forms
- dual vectors are 1-forms
- $\tilde{\epsilon}$ is a 4-form

P-forms do have a operation called *wedge product*:

be A a p-form and B a q-form, then $(A \wedge B)$ is a (p+q)-form, in detail

$$(A \wedge B)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots} B_{\mu_{p+1} \ldots \mu_{p+q}]}$$

So, for example, if A and B are both a 1-form,

$$(A \wedge B) = \frac{2!}{1!1!} A_{[\mu} B_{\nu]} = \frac{2!}{1!} \frac{1}{2!} (A_\mu B_\nu - A_\nu B_\mu) = A_\mu B_\nu - A_\nu B_\mu$$

Also note that

$$A \wedge B = (-1)^{pq} B \wedge A$$

There is also something about Integral over volume & principle of least action. To Be Filled

Exterior Derivative

There is also this operation, not really used, that is

$$d : p - form \rightarrow (p+1) - form$$

$$(dA)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \ldots \mu_n]}$$

It has a special property: dA is a tensor.

$$\partial_\alpha A_{\beta\gamma\delta\ldots} \tag{2.50}$$

is not a tensor, we already saw that, because of the extra piece (2.51)

that is symmetric and become 0 by anti-symmetrization (2.52)

$$\partial_{[\alpha} A_{\beta\gamma\delta\ldots]} \text{ is a tensor!} \tag{2.53}$$

Now let's see how this is related to integrals.

Be in 3D, can be cartesian coordinates (x,y,z) or spherical (r, θ, ϕ), and be the gravitational field $\Phi(x, y, z)$ or $\Phi(r, \theta, \phi)$. What's the integral over space of Φ?

$$\int_{space} \Phi dV$$

with

$$dV = dxdydz = \left(r^2 sin^2\theta\right) drd\theta d\phi$$

Thinking about our guidelines: we want to describe independently on the chosen coordinates. Φ is a scalar, so let's see it's integration

$$I = \int \Phi\left(x\right) d^n x \neq \int \phi\left(x'\right) d^n x'$$

because

$$d^n x^{\mu'} = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| d^n x^\mu$$

there is a Jacobian in there.

The integrand of an integral is a p-form, and the integral a real number.

$$d^n x = dx^0 \wedge \ldots \wedge dx^{n-1} = \frac{1}{n!}\tilde{\epsilon}_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}$$

This is the integration measure. When I change coordinate system I get

$$d^n x = \frac{1}{n!}\tilde{\epsilon}_{\mu_1\ldots\mu_n}\left(dx^{\mu_1} \wedge \ldots dx^{\mu_n}\right) = \tag{2.54}$$

$$\frac{1}{n!}\tilde{\epsilon}_{\mu_1\ldots\mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \times \left(dx^{\mu_1'} \wedge \ldots \wedge dx^{\mu_n'}\right) = \tag{2.55}$$

$$= \frac{1}{n!}\tilde{\epsilon}_{\mu_1'\ldots\mu_n'} det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(dx^{\mu_1}\ldots dx^{\mu_n}\right) \tag{2.56}$$

What did I get? That

$$d^n x = \left[det\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\right]^{-1} d^n x'$$

$d^n x$ is not a tensor but it is a tensor density. We want an invariant integration measure $\sqrt{|g|}d^n x$, such that if Φ is a scalar also

$$\int d^n x\sqrt{|g|}\Phi$$

is a scalar.

Now why ∂_α does not give a tensor? $\partial_\mu A_\nu$ is not a tensor.

$$\partial_{\mu'} A_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\partial_\mu A_\nu + \frac{\partial^2 x^\nu}{\partial x^{\mu'} x^{\nu'}} A_\nu$$

This last piece is symmetric, so in the exterior derivative it cancels out.

2.9.2 Covariant Derivative

This is the only derivative that matters for real.

$$\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$$

the factor Γ is unknown and we need to construct it. Keep in mind that the combination of the two addends makes a tensor, but singularly they aren't.

The transformation need to be both *linear on V*

$$\nabla (T + S) = \nabla T + \nabla S$$

and needs to follow the *Leibniz product rule*[2]

$$\nabla (T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$$

$\Gamma^\nu_{\mu\alpha}$ is called *connection* and it is *not a tensor* nut we can think of it like a collection of numbers.

It is a Christoffel symbol.

Before diving in the deep of this symbol $\Gamma^\mu_{\nu\alpha}$ we want to know hot it transform. One could think the trivial way, because if we have two coordinates systems

$$\Gamma^{\alpha'}_{\mu'\nu'} \neq \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \Gamma^\alpha_{\mu\nu}$$

but this is not true since Γ is not a tensor.

Since we know that the covariant derivative is a tensor by construction, let's start from here

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu \tag{2.57}$$

How are the sides of the equality related? The left-hand side of eq.2.57 is

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\alpha'} V^{\alpha'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right) + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} V^\alpha \tag{2.58}$$

while the right hand side of eq.2.57 is

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha \right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\alpha} V^\alpha \tag{2.59}$$

we see that we have free indices μ', ν' on both sides. Now let's develop the left-hand side some more

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^\nu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^{\nu'}_{\mu'\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} V^\alpha \tag{2.60}$$

Remember that for us partial derivatives commute.

Let's compare the sides:

[2] $(fg)' = f'g + fg'$ to have an example on things we already studied.

- the first on the right-side cancels out with the second on the left (green).
- Renaming the first term on the left to get V^α, lead to simplifying every vector on both sides.

We can *rename* the vector because everything is valid for a generic vector, and its indices were summed over.

We are left with

$$\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\alpha} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\Gamma^{\nu'}_{\mu'\alpha'} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\alpha}$$

And this is how Γ transforms.

If this derivation is still obscure to you and want to see it on paper, old-style, I did again the derivation of this result, see image 2.10. We see that Γ is not a tensor because of this extra piece after the transformation.

We saw that for a *vector* the covariant derivative acts like

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$$

Instead for a *dual vector* the derivative is defined as

$$\nabla_\mu \omega_\nu \equiv \partial_\mu \omega_\nu + \tilde{\Gamma}^\alpha_{\mu\nu}\omega_\alpha$$

The question that arises from this is obvious: *how* $\tilde{\Gamma}$ *is related to* Γ*?*

Let's compute

$$\nabla_\mu \left(\omega_\lambda V^\lambda\right) = \partial_\mu \left(\omega_\lambda V^\lambda\right)$$

This because the covariant derivative of a scalar *is* the derivative of a scalar. So, applying the Leibniz Rule:

$$\begin{gathered}
\nabla_\mu V^\lambda \cdot \omega_\lambda + V^\lambda \nabla_\mu \omega_\lambda = \partial_\mu V^\lambda \omega_\lambda + V^\lambda \partial_\mu \omega_\lambda \\
\left(\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\alpha} V^\alpha\right)\omega_\lambda + V^\lambda\left(\partial_\mu \omega_\lambda + \tilde{\Gamma}^\alpha_{\mu\lambda}\omega_\alpha\right) = \left(\partial_\mu V^\lambda\right)\omega_\lambda + \left(\partial_\mu \omega_\lambda\right)V^\lambda \\
\to \Gamma^\lambda_{\mu\alpha} V^\alpha \omega_\lambda + \tilde{\Gamma}^\alpha_{\mu\lambda} V^\lambda \omega_\alpha = 0 \\
\text{renaming some indices to get} \\
\Gamma^\lambda_{\mu\alpha} V^\alpha \omega_\lambda + \tilde{\Gamma}^\lambda_{\mu\alpha} V^\alpha \omega_\lambda \\
\implies \Gamma^\lambda_{\mu\alpha} + \tilde{\Gamma}^\lambda_{\mu\alpha} = 0
\end{gathered}$$

To conclude, the covariant derivative of a dual vector is actually

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\alpha_{\mu\nu}\omega_\alpha \tag{2.61}$$

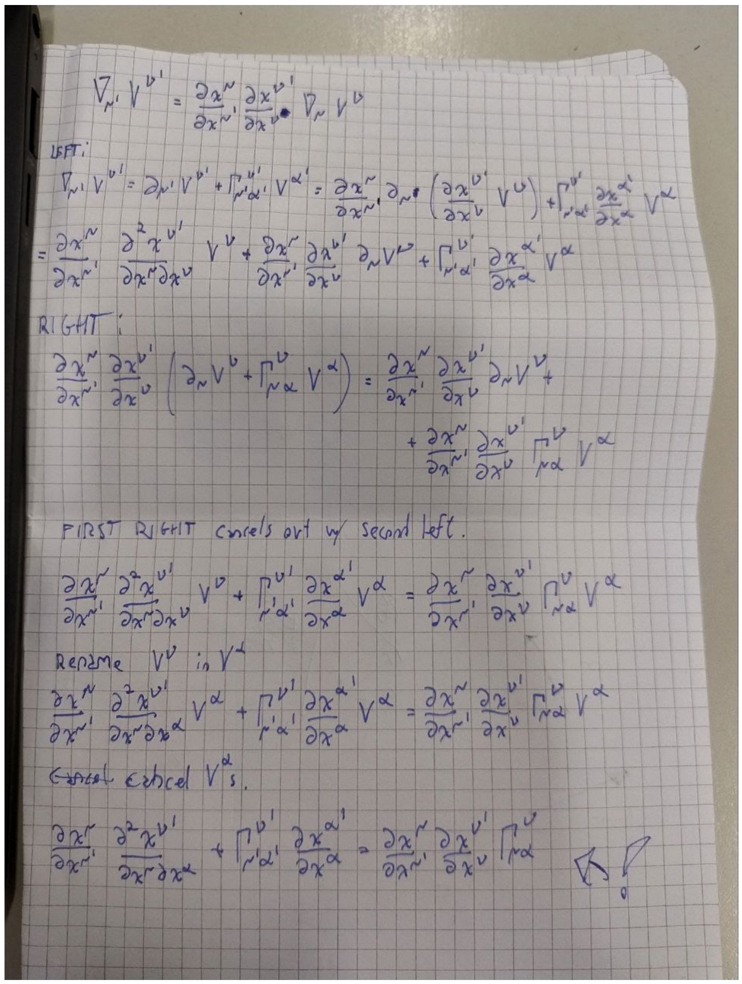


Figure 2.10: Same derivation but on paper. If you have any doubts compare with this one.

2.10 Lec 11

2.10.1 Covariant derivative - Connection

In the last lecture we talked about the covariant derivative and we saw the version for the vector, for the dual vector, and how it transform between two coordinates systems. We constructed it so the output is a vector, and we saw that even after changes of coordinates we still get a tensor.

The question now is, how to do

$$\nabla_\rho T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = ?$$

The development is pretty boring but straight-forward:

$$= \partial_\rho T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} + \Gamma^{\mu_1}_{\rho\alpha} T^{\alpha\mu_2 \dots \mu_k}_{\nu_1 \dots \nu_l} + \Gamma^{\mu_2}_{\rho\alpha} T^{\mu_1 \alpha \mu_3 \dots \mu_k}_{\nu_1 \dots \nu_l} + \dots - \Gamma^{\alpha}_{\mu\nu_1} T^{\mu_1 \dots \mu_k}_{\alpha\nu_2 \dots \nu_l} - \dots \quad (2.62)$$

These Γ connections are just tables of 64 entries of numbers, not tensors, and putting indices up and down to it it's abuse of notation.

Now we will make a couple of assumptions on the structure of Γ.

Torsion

Statement I Given two different connections $\Gamma^\mu_{\alpha\beta}$ and $\tilde{\Gamma}^\mu_{\alpha\beta}$, we define

$$S^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \tilde{\Gamma}^\mu_{\alpha\beta}$$

$\rightarrow S^\mu_{\alpha\beta}$ is a (1,2) tensor. Why? Since I have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$$

I can define a complement

$$\tilde{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \tilde{\Gamma}^\nu_{\mu\alpha} V^\alpha$$

so i get

$$\rightarrow \nabla_\mu V^\nu - \tilde{\nabla}_\mu V^\nu = \left(\Gamma^\nu_{\mu\alpha} - \tilde{\Gamma}^\nu_{\mu\alpha} \right) V^\alpha = S^\nu_{\mu\alpha} V^\alpha$$

and this is valid *only* if $S^\mu_{\nu\alpha}$ is a tensor.

Statement II if $\Gamma^\mu_{\alpha\beta}$ is a connection $\implies \Gamma^\mu_{\beta\alpha}$ is a connection.

That's why we define the *Torsion tensor*

$$T^\mu_{\alpha\beta} \equiv \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha} = 2\Gamma^\mu_{[\alpha\beta]}$$

The metric adopted in this course is a *Torsion-Free* metric, so the torsion tensor is vanishing.

How many entries do I have for a connection?

$$\Gamma^{\mu \rightarrow 4}_{\alpha\beta \rightarrow 10}$$

so in total I have 40 entries, 4 for the upper index and 10 for the lowers because symmetry,

As we will see later, the name *connection* comes from the fact that it is used to transport vectors from one tangent space to another.

Metric Compatibility

So, the torsion tensor is antisymmetric on its lower indices, and a connection that is symmetric on its lower indices is *torsion-free* We can define a unique connection on a manifold with metric $g_{\mu\nu}$ by introducing two additional properties, torsion-freeness and the metric compatibility. The metric compatibility is a property of the covariant derivative and it's expressed as follows

$$\nabla\rho g_{\mu\nu} = 0$$

A connection is *metric compatible* if the covariant derivative of the metric with respect to that connection is everywhere 0.

We want to see how this property works with the *inverse metric tensor*, so let's start from

$$\nabla_\rho \left(g^{\alpha\beta} g_{\beta\gamma}\right) = \nabla_\rho \left(\delta^\alpha_\gamma\right) = \Gamma^\alpha_{\rho\lambda}\delta^\lambda_\gamma - \Gamma^\sigma_{\rho\gamma}\delta^\alpha_\sigma + \partial_\rho \left(\delta^\alpha_\gamma\right) = \left(\Gamma^\alpha_{\rho\gamma} - \Gamma^\alpha_{\rho\gamma}\right) = 0 \quad (2.63)$$

the term with the partial derivative cancels out because δ is constant, and we equal everything to zero because the covariant derivative of the Kronecker delta is 0. On the right side we can apply the Leibniz rule so

$$g^{\alpha\beta}\nabla_\rho \left(g_{\beta\gamma}\right) + \nabla_\rho \left(g^{\alpha\beta}\right) g_{\beta\gamma} = 0 \quad (2.64)$$

the first term is 0, because we said so, the connection is metric compatible. We are left with

$$g_{\beta\gamma}\nabla_\rho \left(g^{\alpha\beta}\right) = 0$$

by multiplying on both sides $g^{\gamma\sigma}$

$$g^{\gamma\sigma} g_{\beta\gamma}\nabla_\rho \left(g^{\alpha\beta}\right) = 0$$

I get

$$\delta^\sigma_\beta \nabla_\rho \left(g^{\alpha\beta}\right) = \nabla_\rho \left(\delta^\sigma_\beta g^{\alpha\beta}\right) = \nabla_\rho \left(g^{\alpha\sigma}\right) = 0$$

So, in conclusion the covariant derivative of the inverse of metric tensor is null. It was not trivial.

After this we can see that a metric-compatible covariant derivative commutes with raising and lowering of indices, so for a generic vector V^ν

$$\nabla_\mu V^\nu = g_{\alpha\nu}\nabla_\mu \left(v^\mu\right) = \nabla_\mu \left(g_{\alpha\nu} V^\nu\right) = \nabla_\mu V_\nu$$

With non-metric compatible connections we would have to be very careful about index placement wen taking a covariant derivative.

There is exactly one torsion-free connection on a manifold that is compatible with some generic metric on that manifold.

We can demonstrate *existence* and *uniqueness* by deriving a manifestly unique expression for the connection coefficients in terms of the metric, so we will expand the equation of metric compatibility for three different permutations of the indices.

$$
\begin{aligned}
\nabla_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \text{ (a)}\\
\nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \text{ (b)}\\
\nabla_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0 \text{ (c)}
\end{aligned}
$$

we see that *(a)-(b)-(c) = 0*, and it is obvious because individually they're equal to 0.

But let's see in detail what's happening

$$
\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \Gamma^\alpha_{\rho\mu} g_{\alpha\nu} - \Gamma^\alpha_{\rho\nu} g_{\mu\alpha} + \Gamma^\alpha_{\mu\nu} g_{\alpha\rho} + \Gamma^\alpha_{\mu\rho} g_{\nu\alpha} + \Gamma^\alpha_{\nu\rho} g_{\alpha\mu} + \Gamma^\alpha_{\nu\mu} g_{\rho\alpha} = 0
$$