Probability and Statistics

Omid Safarzadeh

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Sample Space

Definition 1.1

The set of all possible outcomes of a particular experiment is called the *sample space* for the experiment, which generally denoted by Ω.

Example 1.1

In tossing two fair coins the sample space is:

$$\Omega = \{HH, HT, TH, TT\}.$$

Sample space

Exercise:

- what is the sample space of a fair dice?
- Whats is the sample space of a credit risk problem ?
- Whats is the sample space of a Classification problem ?
- Define sample space of a user visiting a specific web page?
- Define sample space of a chat bot?

Event

Definition 1.2

The event space ($\mathscr{A}$) is the space of potential results of the experiment. In discrete case , $\mathscr{A}$ is the power set of Ω

Definition 1.3

An event is any collection of possible outcomes of an experiment, which is, any subset of Ω.

$(\Omega, \mathscr{A}, P)$ is called Probability Space.
Exercise: pick any Amazone's product page, define several events.

Axiomatic Foundations

Definition 2.1

A collection, $\mathbb{B}$, of subsets Ω is called sigma algebra, if it satisfies the following:

1. $\varnothing \in \mathbb{B}$
2. If $A \in \mathbb{B}$, then $A^c \in \mathbb{B}$ ($\mathbb{B}$ is closed under complement).
3. If $A_1, A_2, ... \in \mathbb{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathbb{B}$.

The pair $(\mathbb{B}, \Omega)$ is called a measurable space or Borel Space.
Interpretation: Sigma Algebra is the collection of events that can be assigned probabilities.

Exercise:

- Is the set of all subsets of $\mathbb{B}$ countable?
- Is $\mathbb{B}$ a set?
- Define $\mathbb{B}$ for a web page? Check if fits all 3 properties of sigma algebra.

Axiomatic Foundations

if $A_1, A_2, ... \in \mathbb{B}$ then $A_1^c, A_2^c, ... \in \mathbb{B}$, by(2). Now, by (3), $\cup_{i=1}^{\infty} A_i^c \in \mathbb{B}$. Use De Morgan's Law,

$$\left(\cup_{i=1}^{\infty} A_i^c \right)^c = \cap_{i=1}^{\infty} A_i.$$

- is $\cap_{i=1}^{\infty} A_i$ countable or uncountable?

Then, by (2), $\cap_{i=1}^{\infty} A_i \in \mathbb{B}$ and $\mathbb{B}$ is closed under countable intersections, as well.

Axiomatic Foundations

Example 2.1

if Ω is finite or countable, then we can define for a given sample space Ω

$$\mathbb{B}\text{=(all subsets of } \Omega \text{ including } \Omega \text{ itself).}$$

- Take, for example, Ω=$\{$A,B,...,Z$\}$. Then the sigma algebra is the power set of Ω.

Kolmogorov's Axioms

Definition 2.2

Given a sample space Ω and an associated sigma algebra $\mathbb{B}$, a probability function is a function P with domain $\mathbb{B}$ that satisfies

1. $P(A) \geq 0$ for all $A \in \mathbb{B}$.
2. $P(\Omega) = 1$.
3. If $A_1, A_2, \ldots \in \mathbb{B}$ are pairwise disjoint, then

$$P\Big(\cup_{i=1}^{\infty} A_i \Big) = \sum_{i=1}^{\infty} P(A_i).$$

Kolmogorov's Axioms

- These three points are usually referred to as the Axioms of Probability or the Kolmogorov's Axioms.
- Now, any function $P(.)$ that satisfies the Kolmogorov Axioms is a valid probability function.

Example 2.2

In tossing a fair coin.we have, $\Omega = \{H, T\}$. The probability function is

$$P(\{H\}) = P(\{T\}),$$

as the coin is fair.

- observe that $\Omega = \{H\} \cup \{T\}$. Then, from Axiom 2 we must have

$$P(\{H\} \cup \{T\}) = 1.$$

- Since $\{H\}$ and $\{T\}$ are disjoint,

$$P(\{H\} \cup \{T\}) = P(\{H\}) + P(\{T\}).$$

So,

$$P(\{H\} \cup \{T\}) = P(\{H\}) + P(\{T\}) = 1.$$

Axioms of Probability

- Our intuition and the Kolmogorov Axioms together tell us that $P(\{H\} = P(\{T\}) = 1/2$.
- However, any non-negative probabilities that add up to one would have been valid, say $P(\{H\}) = 1/4$ and $P(\{T\}) = 3/4$. The reason we chose equal probabilities is our knowledge that the coin is fair!

Exercise: Use above example and provide a similar approach for a web page (buttons, option boxes, popup boxes,...)

The Foundation of Probabilities Functions

Theorem 2.1

If P is a probability function and A is any set in $\mathbb{B}$, then

1. $P(\varnothing) = 0$ where $\varnothing$ is the empty set.
2. $P(A) \leq 1$.
3. $P(A^c) = 1 - P(A)$.

- **Proof: Exercise!**

The Foundation of Probabilities Functions

Theorem 2.2

If P is a probability function and A and B are any sets in $\mathbb{B}$, then

1. $P(B \cap A^c) = P(B) - P(A \cap B)$.
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
3. If $A \subset B$, then $P(A) \leq P(B)$.

- **Proof: Exercise!**

Independence

Definition 3.1

Two events, A and B, are statistically independent if

$$P(A \cap B) = P(A)P(B).$$

Theorem 3.1

If A and B are independent events, then the following pairs are also independent:

1. A and B^c
2. A^c and B
3. A^c and B^c

Conditional Probability

Definition 4.1

If A and B are events in S, and $P(B) > 0$, then the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{1}$$

- In words, given that B has occurred, what is the probability that A will occur?
- By definition,

$$P(B|B) = 1,$$

  as B has already occurred.

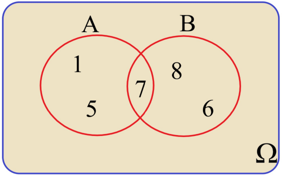
Conditional Probability

- If A and B are disjoint sets, $P(A \cap B) = P(\varnothing) = 0$ then:

$$P(A|B) = 0 = P(B|A)$$

- In fact, what happens in the conditional probability calculation is that B becomes the sample space.
- It is straightforward to verify that the probability function $P(.|B)$ satisfies Kolmogorov's Axioms, for any B for which $P(B) > 0$.

Conditional Probability



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{7}{7+8+6} = \frac{1}{3}$$

Figure: Conditional Probability.

NLP application

"You shall know a word by the company it keeps" (J. R. Firth 1957: 11)

Example 4.1

Omid is a ______ Data Scientist...

We want to predict ______ given other words!

$$P(w_{t+i}|w_t) = ? \tag{2}$$

For each position t = 1, . . . , T, predict context words within a window of fixed size m, given center word.

Ref: Stanford CS224N course, Prof. Manning

Bayes Rule

- Observe that since

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(B|A) = \frac{P(A \cap B)}{P(A)},$$

we have

$$P(A|B)P(B) = P(A \cap B) = P(B|A)(PA) \Rightarrow P(A|B) = P(B|A)\frac{P(A)}{P(B)} \quad (3)$$

- Which is known as Bayaes's Rule.

Bayes Theorem

Theorem 4.1

Let $A_1, A_2, \ldots$ be a partition of the sample, and let B be any set. Then, for each $i = 1, 2, \ldots$

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B|A_j)(PA_j)}.$$

This is actually not much more different that (2) since

$$\sum_{j=1}^{\infty} P(B|A_j)P(A_j) = \sum_{j=1}^{\infty} P(A_j \cap B) = P(B)$$

given that $A_1, A_2, \ldots$ is a partition of the sample space

Random Variables

Definition 5.1

A random variable is a function from a sample space Ω into the real numbers $\mathbb{R}$.

Experiment	**Random Variable**
Click on amazon product page	X $=$ click through BUY Button .
amazon product page visit	X $=$ total number of calling the page from server (GET API)

Random Variable example

Example 5.1

Consider tossing a fair coin three times. Define the random variable X to be the number of heads obtained in the three tosses. we have:

ω	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(\omega)$	3	2	2	2	1	1	1	0

Random Variables in $\mathbb{R}$

Suppose the sample space is $\Omega = \{\omega_1, ..., \omega_n\}$ and the original probability function is P. Define the new random variable

$$X : \Omega \to \mathbb{R}, \qquad \text{take A} \in \mathbb{R}, A = \{x_1, ..., x_m\}$$

$$P_X(X \in A) = P(\{\omega \in \Omega : X(\omega) \in A\}), A \in \mathbb{R}$$

Probability Function

- Suppose the sample space is $\Omega = \{\omega_1, ..., \omega_n\}$ and the original probability function is P. Define the new random variable

$$X : \Omega \to \mathscr{X}, \qquad \mathscr{X} = \{x_1, ..., x_m\}$$

- Define the new probability function for X as P_X where

$$P_X(X = x_i) = P(\{\omega_j \in \Omega : X(\omega_j) = x_i\}).$$

- P_X is an induced probability function, as it is defined in terms of the original probability function, P.
- If X is uncountable, the induced probability function is defined in a slightly different way. Namely, for any set $A \subset \mathscr{X}$,

$$P_X(X \in A) = P(\{\omega \in \Omega : X(\omega) \in A\}).$$

Probability Function

- Exercise : show that induced probability function satisfies the Kolmogorov Axioms.
- we assign capital letters to random variables and lower case letters to the particular value they take.

Distribution Functions

- All random variables are associated with a distribution function. This distribution function includes all information about the randomness of the variable.

Definition 6.1

The cumulative distribution function or CDF of a random variable X, denoted by $F_X(x)$, is defined by

$$F_X(x) = P_X(X \leq x), \qquad \text{for all } x.$$

- When we write $P_X(X \leq x)$, we mean the probability that the random variable X takes a value equal to or smaller than x. The subscript X in $P_X(.)$ denotes that this probability is obtained with respect to the probability distribution of X.

Example 6.1

Consider the experiment of tossing three fair coins, and let X =number of heads observed. The CDF of X is

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0, \\ 1/8 & \text{if } 0 \leq x < 1, \\ 1/2 & \text{if } 1 \leq x < 2, \\ 7/8 & \text{if } 2 \leq x < 3, \\ 1 & \text{if } 3 \leq x < \infty. \end{cases}$$

- Note that, $F_X(x)$ is defined for all possible values of $x \in \mathbb{R}$. Hence,

$$P_X(x \leq 2.5) = P(X = 0, 1 or 2) = 7/8.$$

Distribution Functions

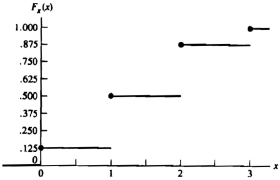


Figure: from Casella and Berger (2002, p.30). CDF of example 6.1

Distribution Functions

Theorem 6.1

The function $F_X(x)$ is a CDF if and only if the following three conditions hold:

1. $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to+\infty} F_X(x) = 1$.
2. $F_X(x)$ is a non-decreasing function of x.
3. $F_X(x)$ is right-continuous; that is, for every number x_0, $\lim_{x\downarrow x_0} F_x(x) = F(x_0)$.

- We can also have a continuous CDF.

Definition 6.2

A random variable X is continuous(discrete) if $F_X(x)$ is a continuous(step) function of x.

Example 6.2

take Sigmoid function:

$$F_X(x) = \frac{1}{1+e^{-x}}$$

observe that

$$\lim_{x\to-\infty} F_X(x) = 0 \quad \text{since} \quad \lim_{x\to-\infty} e^{-x} = \infty,$$

$$\lim_{x\to\infty} F_X(x) = 1 \quad \text{since} \quad \lim_{x\to\infty} e^{-x} = 0,$$

$$\frac{d}{dx}F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0,$$

so $F_X(x)$ is non-decreasing in x.
$F_X(x)$ is continuous everywhere.

Theorem 6.2

The following two statements are equivalent:

1. The random variables X and Y are identically distributed.
2. $F_X(x) = F_Y(x)$ for every x.

Density Function Discrete Case

Definition 6.3

The probability mass function of a discrete random variable is given by

$$f_X(x) = P(X = x) \text{ for all x.}$$

- CDF F_X, then pdf is denoted by f_X
- the pmf is called "density". ¿¿¿ for Discrete R.V.
- the pdf is also called "density" ¿¿¿ for Continous R.V.

Density and Mass Continous Case

Definition 6.4

The probability density function or pdf, $f_X(x)$, of a continuous random variable X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

A note on notation : The expression "X has a distribution given by $F_X(x)$" is abbreviated symbolically by "X $\sim F_X(x)$", where we read the symbol "$\sim$" as "is distributed as". We can similarly write X $\sim f_X(x)$ or, if X and Y have the same distribution, X $\sim$ Y

Example 6.3

For the logistic distribution considered before, we have

$$F_X(x) = \frac{1}{1 + e^{-x}}$$

and, hence,

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

Then, for continuous random variables in general,

$$P(a < X < b) = F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X(x)dx - \int_{-\infty}^{a} f_X(x)dx$$

$$= \int_{a}^{b} f_X(x)dx.$$

Theorem 6.3

A function $f_X(x)$ is a pdf (/ pmf) of a random variable X if and only of

1. $f_X(x) \geq 0$ for all x.
2. $\sum_x f_X(x) = 1$ (discrete) or $\int_{-\infty}^{\infty} f_X(x)dx = 1$ (continuous).

Important for Simulating

- Any non-negative function with a finite positive integral can be turned into a pdf or pmf. Take, for example, if

$$h(x) = \begin{cases} \geq 0 & \text{for} \quad x \in A \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\int_{x \in A} h(x) dx = \mathrm{K} < \infty, \quad \textit{where } \mathrm{K} > 0,$$

then $f_X(x) = h(x)/K$ is a pdf of a random variable X taking values in A.

- In some cases, although $F_X(x)$ exists, $f_X(x)$ may not exist because $F_X(x)$ can be continuous but not differentiable. Therefore, sometimes statistical analysis would be based on $F_X(x)$ and not $f_X(x)$.

Distribution of Functions of a Random Variable

- If X is a random variable with CDF F_X, then $Y = g(X)$ is also a random variable.
- Importantly, since Y is a function of X, we can determine its random behaviour in terms of the behaviour of X.
- Then, for any set A,

  $$P(Y \in A) = P(g(X) \in A).$$

  This clearly shows that the distribution of Y depends on the function $g(.)$ and the CDF F_X.
- Formally,

  $$g(x) : \mathscr{X} \to \mathscr{Y},$$

  where $\mathscr{X}$ and $\mathscr{Y}$ are the sample spaces of X and Y, respectively.

Distribution of Functions of a Random Variable

- Notice that the mapping $g(.)$ is associated with the inverse mapping $g^{-1}(.)$, a mapping from the subsets of $\mathscr{Y}$ to those $\mathscr{X}$:

$$g^{-1}(A) = \{x \in \mathscr{X} : g(x) \in A\}. \tag{4}$$

- Therefor, the mapping $g^{-1}(.)$ takes sets into sets, that is, $g^{-1}(A)$ is the set of points in $\mathscr{X}$ that $g(x)$ takes into the set A.
- If $A = \{y\}$, a point set, then

$$g^{-1}(\{y\}) = \{x \in \mathscr{X} : g(x) = y\}.$$

- Now, if $Y = g(X)$, then for all $A \in \mathscr{Y}$,

$$\begin{aligned} P(Y \in A) &= P(g(X) \in A) \\ &= P(\{x \in \mathscr{X} : g(x) \in A\}) \\ &= P(X \in g^{-1}(A)), \end{aligned} \tag{5}$$

where the last line follows from (1). This defines the probability distribution of Y.

Example 7.1

A discrete random variable X has binomial distribution if its pmf is of the form

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, ..., n,$$

where n is a positive integer and $0 \leq p \leq 1$. Values such as n and p are called parameters of a distribution. Different parameter values imply different distributions.

- The CDF of $Y = g(X)$ is

$$
\begin{aligned}
F_Y(y) = P(Y \leq y) &= P(g(X) \leq y) \\
&= P(\{x \in \mathscr{X} : g(x) \leq y\}) \\
&= \int_{x \in \mathscr{X} : g(x) \leq y} f_X(x) dx.
\end{aligned}
$$

Theorem 7.1

Let X have CDF $F_X(x)$, let $Y = g(X)$ and let $\mathscr{X}$ and $\mathscr{Y}$ be defined as

$$\mathscr{X} = \{x : f_X(x) > 0\} \text{ and } \mathscr{Y} = \{y : y = g(x) \text{ for some } x \in \mathscr{X}\}. \tag{6}$$

1. If g is an increasing function on $\mathscr{X}$, $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathscr{Y}$.
2. If g is a decreasing function on $\mathscr{X}$ and X is a continuous random variable, $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathscr{Y}$.

Proof: Exercise!

Distribution of Functions of a Random Variable

Theorem 7.2

Let X have pdf $f_X(x)$ and let $Y = g(X)$,where g is a monotone function, Let $\mathscr{X}$ and $\mathscr{Y}$ be defined as in (3). Suppose that $f_X(x)$ is continuous on $\mathscr{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathscr{Y}$. The the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_x(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & y \in \mathscr{Y} \\ \\ 0 & \text{otherwise} \end{cases}$$

Proof: Exercise!

Example 7.2

Suppose X $f_X(x) = 1$ for $0 < x < 1$ and 0 otherwise, which is the *uniform*$(0, 1)$ distribution. Observe that $F_X(x) = x$, $0 < x < 1$. We now make the transformation $Y = g(X) = -\log X$. Then,

$$g'(x) = \frac{d}{dx}(-\log x) = -\frac{1}{x} < 0 \quad \text{for } 0 < x < 1;$$

hence, $g(x)$ is monotone and has e continuous derivative on $0 < x < 1$. Also, $\mathscr{Y} = (0, \infty)$. Observe that $g^{-1}(y) = e^{-y}$. Then, using Theorem (1.2),

$$\begin{aligned} f_Y(y) &= 1 * |-e^{-y}| \qquad \text{if } 0 < y < \infty \\ &= e^{-y} \qquad \text{if } 0 < y < \infty. \end{aligned}$$

Distribution of Functions of a Random Variable

Theorem 7.3

Let X have continuous CDF $F_X(x)$ and define the random variable Y as $Y = F_X(X)$. Then, Y is uniformly distributed on (0,1), that is

$$P(Y \leq y) = y, \quad 0 < y < 1.$$

- **Proof**: Exercise!

Distribution of Functions of a Random Variable

- This result connects any random variable with some CDF $F_X(x)$ with a uniformly distributed random variable. Hence, if we want to simulate random numbers from some distribution $F_X(x)$, all we have to do is to generate uniformly distributed random variables, Y, and then solve for $F_X(x) = y$. As long as we can compute $F_X^{-1}(y)$, we can generate random numbers from the distribution $F_X(x)$.

Casella, G., & Berger, R. (2002). *Statistical inference*. Cengage Learning. https://books.google.fr/books?id=FAUVEAAAQBAJ