Homwrok 3,

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1 Question:

Build your own Probability Density Function using the theorem 7.2 we discussed at discussed.Here are two examples , you may get the idea.
Hint: You can use Sigmoid function as$F_X(x)$ and then use some properly defined Y=g(x) transformation function to build your desired pdf.

Example 1

Suppose X is a continuous random variable with a cdf in the form below:

$$F_X(x) = \frac{1}{1+e^{-x}}$$

Find the pdf and also cdf of $g(X) = sin(X)$.

Solution:

we have:

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}.$$

$Y = g(X) = sin(X)$, this function is monotone on $(0, \frac{\pi}{2})$:

$$g'(X) = cos(x) \geq 0 \quad X \in (0, \frac{\pi}{2})$$

So for $\mathcal{X} = (0, \frac{\pi}{2})$ and $\mathcal{Y} = (0, 1)$ we have:

$$g^{-1}(y) = \arcsin y \quad \text{ and } \quad \frac{d}{dy}g^{-1}(y) = \frac{1}{\sqrt{1-y^2}}.$$

From Theorem 7.2 we get:

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| & \text{if} \quad y \in (0,1) \\ 0 \quad \text{if} \quad \text{otherwise} \end{cases}$$

$$= \frac{e^{-\arcsin y}}{(1+e^{-\arcsin y})^2} \times \frac{1}{\sqrt{1-y^2}}.$$

To calculate $F_Y(y)$ we need to integrate $f_Y(y)$:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy = \int_{0}^{y} \frac{e^{-\arcsin y}}{(1+e^{-\arcsin y})^2} \times \frac{1}{\sqrt{1-y^2}}dy.$$

Using substitution technique:

$$u = \arcsin y \qquad \Rightarrow \qquad du = \frac{1}{\sqrt{1-y^2}}dy.$$

So:

$$\int_{-\infty}^{y} f_Y(y)dy = \int_{0}^{y} \frac{e^{-\arcsin y}}{(1+e^{-\arcsin y})^2} \times \frac{1}{\sqrt{1-y^2}}dy = \int \frac{e^{-u}}{(1+e^{-u})^2}du = \frac{1}{1+e^{-u}}$$

By substitution of $u = \arcsin y$, we have:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy = \frac{1}{1+e^{-\arcsin y}}\bigg|_0^y = \frac{1}{1+e^{-\arcsin y}} - \frac{1}{2}.$$

Example 2

Suppose X is a continuous random variable with a cdf in the form below:

$$F_X(x) = \frac{1}{1+e^{-x}}$$

Find the pdf and also cdf of $g(X) = \frac{2x+1}{6x+5}$.

Solution:

we have:

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}.$$

$Y = g(X) = \frac{2x+1}{6x+5}$, which is monotone on $\mathcal{R}$:

$$g'(X) = \frac{4}{(6x+5)^2} > 0$$

So for $\mathcal{X} = \mathcal{R}$ and $\mathcal{Y} = (1, \infty)$ we have:

$$g^{-1}(y) = \frac{1-5y}{6y-2} \qquad \text{and} \qquad \frac{d}{dy}g^{-1}(y) = \frac{1}{(3y-1)^2}.$$

From Theorem 7.2 we get:

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| & \text{if} \quad y \in (1,\infty) \\ 0 \quad \text{if} \quad \text{otherwise} \end{cases}$$

$$= \frac{e^{-\frac{1-5y}{6y-2}}}{(1+e^{-\frac{1-5y}{6y-2}})^2} \times \frac{1}{(3y-1)^2}.$$

Just like the previous example we need to integrate $f_Y(y)$ by substitution technique:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy = \int_{1}^{y} \frac{e^{-\frac{2x+1}{6x+5}}}{(1+e^{-\frac{2x+1}{6x+5}})^2} \times \frac{1}{(3y-1)^2}dy.$$

$$u = \frac{1-5y}{6y-2} \qquad \Rightarrow \qquad du = \frac{1}{(3y-1)^2}dy.$$

So:

$$\int_{-\infty}^{y} f_Y(y)dy = \int_{1}^{y} \frac{e^{-\frac{2x+1}{6x+5}}}{(1+e^{-\frac{2x+1}{6x+5}})^2} \times \frac{1}{(3y-1)^2}dy = \int \frac{e^{-u}}{(1+e^{-u})^2}du = \frac{1}{1+e^{-u}}$$

By substitution of $u = \frac{1-5y}{6y-2}$, we have:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy = \frac{1}{1+e^{-\frac{1-5y}{6y-2}}}\Big|_{1}^{y} = \frac{1}{1+e^{-\frac{1-5y}{6y-2}}} - \frac{1}{1+e}.$$