Homework 7

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One generalization of the binomial distribution is to allow the success probability to vary according to a distribution. A standard model for this situation is

$$X|P \sim \text{binomial(P)}, \quad i = 1, \cdots, n,$$
$$P \sim \text{beta}(\alpha, \beta).$$

By iterating the expectation, we calculate the mean of X as

$$E_X[X] = E_P\{E_{X|P}[X|P]\} = E_P[nP] = n\frac{\alpha}{\alpha+\beta}$$

Question:

For the hierarchy shown above, show that the variance of X can be written

$$Var(X) = nE[P](1 - E[P]) + n(n-1)Var(P).$$

Solution:

$Var(X) = E[Var(X|P)] + Var(E[X|P])$. Therefore,

$$Var(X) = E[nP(1-P)] + Var(nP)$$

$$= n\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} + n^2Var(P)$$

$$= n\frac{\alpha\beta(\alpha+\beta+1-1)}{(\alpha+\beta)^2(\alpha+\beta+1)} + n^2Var(P)$$

$$= n\frac{\alpha\beta(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} - n\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + n^2Var(P)$$

$$= n\frac{\alpha}{\alpha+\beta}\frac{\beta}{\alpha+\beta} - nVar(P) + n^2Var(p)$$

$$= nE[P](1 - E[P]) + n(n-1)Var(P)$$