Probability and Statistics

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January 15, 2022

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***Acknowledgement**: This slide is prepared based on Casella and Berger, 2002

Sample Space

Definition 1.1

The set of all possible outcomes of a particular experiment is called the *sample space* for the experiment, which generally denoted by Ω.

Example 1.1

In tossing two fair coins the sample space is:

$$\Omega = \{HH, HT, TH, TT\}.$$

Sample space

Exercise:

- what is the sample space of a fair dice?
- Whats is the sample space of a credit risk problem ?
- Whats is the sample space of a Classification problem ?
- Define sample space of a user visiting a specific web page?
- Define sample space of a chat bot?

Event

Definition 1.2

The event space ($\mathscr{A}$) is the space of potential results of the experiment. In discrete case , $\mathscr{A}$ is the power set of Ω

Definition 1.3

An event is any collection of possible outcomes of an experiment, which is, any subset of Ω.

$(\Omega, \mathscr{A}, P)$ is called Probability Space.

Exercise: pick any Amazone's product page, define several events.

Axiomatic Foundations

Definition 2.1

A collection, $\mathbb{B}$, of subsets Ω is called sigma algebra, if it satisfies the following:

1. $\varnothing \in \mathbb{B}$
2. If $A \in \mathbb{B}$, then $A^c \in \mathbb{B}$ ($\mathbb{B}$ is closed under complement).
3. If $A_1, A_2, ... \in \mathbb{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathbb{B}$.

The pair $(\mathbb{B}, \Omega)$ is called a measurable space or Borel Space.
Interpretation: Sigma Algebra is the collection of events that can be assigned probabilities.

Exercise:

- Is the set of all subsets of $\mathbb{B}$ countable?
- Is $\mathbb{B}$ a set?
- Define $\mathbb{B}$ for a web page? Check if fits all 3 properties of sigma algebra.

Axiomatic Foundations

if $A_1, A_2, \ldots \in \mathbb{B}$ then $A_1^c, A_2^c, \ldots \in \mathbb{B}$, by(2). Now, by (3), $\cup_{i=1}^{\infty} A_i^c \in \mathbb{B}$. Use De Morgan's Law,

$$\left(\cup_{i=1}^{\infty} A_i^c\right)^c = \cap_{i=1}^{\infty} A_i.$$

- is $\cap_{i=1}^{\infty} A_i$ countable or uncountable?

Then, by (2), $\cap_{i=1}^{\infty} A_i \in \mathbb{B}$ and $\mathbb{B}$ is closed under countable intersections, as well.

Example 2.1

if Ω is finite or countable, then we can define for a given sample space Ω

$$\mathbb{B}=(\text{all subsets of } \Omega \text{ including } \Omega \text{ itself }).$$

- Take, for example, $\Omega=\{A,B,...,Z\}$. Then the sigma algebra is the power set of Ω.

Kolmogorov's Axioms

Definition 2.2

Given a sample space Ω and an associated sigma algebra $\mathbb{B}$, a probability function is a function P with domain $\mathbb{B}$ that satisfies

1. $P(A) \geq 0$ for all $A \in \mathbb{B}$.
2. $P(\Omega) = 1$.
3. If $A_1, A_2, \ldots \in \mathbb{B}$ are pairwise disjoint, then

$$P\Big(\cup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i).$$

Kolmogorov's Axioms

- These three points are usually referred to as the Axioms of Probability or the Kolmogorov's Axioms.
- Now, any function $P(.)$ that satisfies the Kolmogorov Axioms is a valid probability function.

Example 2.2

In tossing a fair coin.we have, $\Omega = \{H, T\}$. The probability function is

$$P(\{H\}) = P(\{T\}),$$

as the coin is fair.

- observe that $\Omega = \{H\} \cup \{T\}$. Then, from Axiom 2 we must have

$$P(\{H\} \cup \{T\}) = 1.$$

- Since $\{H\}$ and $\{T\}$ are disjoint,

$$P(\{H\} \cup \{T\}) = P(\{H\}) + P(\{T\}).$$

So,

$$P(\{H\} \cup \{T\}) = P(\{H\}) + P(\{T\}) = 1.$$

Axioms of Probability

- Our intuition and the Kolmogorov Axioms together tell us that $P(\{H\} = P(\{T\}) = 1/2$.
- However, any non-negative probabilities that add up to one would have been valid, say $P(\{H\}) = 1/4$ and $P(\{T\}) = 3/4$. The reason we chose equal probabilities is our knowledge that the coin is fair!

Exercise: Use above example and provide a similar approach for a web page (buttons, option boxes, popup boxes,...)

The Foundation of Probabilities Functions

Theorem 2.1

If P is a probability function and A is any set in $\mathbb{B}$, then

1. $P(\varnothing) = 0$ where $\varnothing$ is the empty set.
2. $P(A) \leq 1$.
3. $P(A^c) = 1 - P(A)$.

- **Proof: Exercise!**

The Foundation of Probabilities Functions

Theorem 2.2

If P is a probability function and A and B are any sets in $\mathbb{B}$, then

1. $P(B \cap A^c) = P(B) - P(A \cap B)$.
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
3. If $A \subset B$, then $P(A) \leq P(B)$.

- **Proof: Exercise!**

Independence

Definition 3.1

Two events, A and B, are statistically independent if

$$P(A \cap B) = P(A)P(B).$$

Theorem 3.1

If A and B are independent events, then the following pairs are also independent:

1. A and B^c
2. A^c and B
3. A^c and B^c

Conditional Probability

Definition 4.1

If A and B are events in S, and $P(B) > 0$, then the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{1}$$

- In words, given that B has occurred, what is the probability that A will occur?
- By definition,

$$P(B|B) = 1,$$

as B has already occurred.

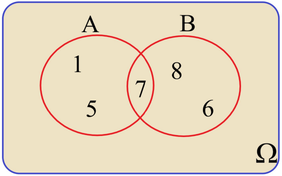
Conditional Probability

- If A and B are disjoint sets, $P(A \cap B) = P(\varnothing) = 0$ then:

$$P(A|B) = 0 = P(B|A)$$

- In fact, what happens in the conditional probability calculation is that B becomes the sample space.
- It is straightforward to verify that the probability function $P(.|B)$ satisfies Kolmogorov's Axioms, for any B for which $P(B) > 0$.

Conditional Probability



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{7}{7+8+6} = \frac{1}{3}$$

Figure: Conditional Probability.

NLP application

"You shall know a word by the company it keeps" (J. R. Firth 1957: 11)

Example 4.1

Omid is a ______ Data Scientist...

We want to predict ______ given other words!

$$P(w_{t+i}|w_t) =? \tag{2}$$

For each position t = 1, . . . , T, predict context words within a window of fixed size m, given center word.

Ref: Stanford CS224N course, Prof. Manning

Bayes Rule

- Observe that since

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(B|A) = \frac{P(A \cap B)}{P(A)},$$

we have

$$P(A|B)P(B) = P(A \cap B) = P(B|A)(PA) \Rightarrow P(A|B) = P(B|A)\frac{P(A)}{P(B)} \qquad (3)$$

- Which is known as Bayaes's Rule.

Bayes Theorem

Theorem 4.1

Let $A_1, A_2, \ldots$ be a partition of the sample, and let B be any set. Then, for each $i = 1, 2, \ldots$

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B|A_j)(PA_j)}.$$

This is actually not much more different that (2) since

$$\sum_{j=1}^{\infty} P(B|A_j)P(A_j) = \sum_{j=1}^{\infty} P(A_j \cap B) = P(B)$$

given that $A_1, A_2, \ldots$ is a partition of the sample space

Random Variables

Definition 5.1

A random variable is a function from a sample space Ω into the real numbers.

Experiment	**Random Variable**
Toss two dice	X = sum of the numbers
Toss a coin 10 times	X = number of heads in 10 tosses

Random Variables

- Suppose the sample space is $\Omega = \{\omega_1, ..., \omega_n\}$ and the original probability function is P. Define the new random variable

$$X : \Omega \to \mathcal{X}, \qquad \mathcal{X} = \{x_1, ..., x_m\}$$

- Define the new probability function for X ad P_X where

$$P_X(X = x_i) = P(\{\omega_j \in \Omega : X(\omega_j) = x_i\}).$$

- P_X is an induced probability function, as it is defined in terms of the original probability function, P.
- If $\mathcal{X}$ is uncountable, the induced probability function is defined in a slightly different way. Namely, for any set $A \subset \mathcal{X}$,

$$P_X(X \in A) = P(\{\omega \in \Omega : X(\omega) \in A\}).$$

Random Variables

- In both cases, it is possible to show that the induced probability function satisfies the Kolmogorov Axioms.
- Note that the general convention in the literature is to assign capital letters to random variables and lower case letters to the particular value they take. Hence, for example, the number of people who answer "yes" in the survey could be X while a particular value, say 12, would be x.

Probability Function

- Suppose the sample space is $\Omega = \{\omega_1, ..., \omega_n\}$and the original probability function is P. Define the new random variable

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$$P_X(X \in A) = P(\{\omega \in \Omega : X(\omega) \in A\}).$$

Probability Function

Example 6.1

Consider again the experiment of tossing a fair coin three times. Define the random variable X to be the number of heads obtained in the three tosses. A complete enumeration of the value of X for each point in the sample space is:

s	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
X(s)	3	2	2	2	1	1	1	0

Probability Function

The range for the random variable X is $\mathcal{X} = \{0, 1, 2, 3\}$. Assuming that all eight points Ω have probability $\frac{1}{8}$, by simply counting in the above display we see that the induced probability function on X is given by

x	0	1	2	3
P_X(X=x)	1/8	3/8	3/8	1/8

For example,

$$P_X(X = 1) = P(HTT, THT, TTH) = \frac{3}{8}$$

Distribution Functions

- All random variables are associated with a distribution function. This distribution function includes all information about the randomness of the variable.

Definition 6.1

The cumulative distribution function or CDF of a random variable X, denoted by $F_X(x)$, is defined by

$$F_X(x) = P_X(X \leq x), \qquad \text{for all } x.$$

- When we write $P_X(X \leq x)$, we mean the probability that the random variable X takes a value equal to or smaller than x. The subscript X in $P_X(.)$ denotes that this probability is obtained with respect to the probability distribution of X.

Distribution Functions

- In this particular case, this is too clear and so a bit redundant. However, if we consider $Y = f(X)$, then the notation provides clarification because we will have to deal with $P_X(Y \leq y)$.
- Note that the CDF is also generally denoted as simply the 'distribution."

Example 6.2

Consider the experiment of tossing three fair coins, and let X =number of heads observed. The CDF of X is

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0, \\ 1/8 & \text{if } 0 \leq x < 1, \\ 1/2 & \text{if } 1 \leq x < 2, \\ 7/8 & \text{if } 2 \leq x < 3, \\ 1 & \text{if } 3 \leq x < \infty. \end{cases}$$

- Note that, $F_X(x)$ is defined for all possible values of $x \in \chi$. Hence,

$$P_X(x \leq 2.5) = P(X = 0, 1 or 2) = 7/8.$$

Distribution Functions

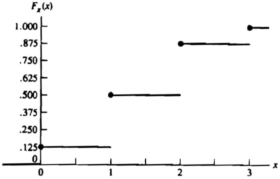

Figure: from Casella and Berger (2002, p.30). CDF of example 1.5.1

Distribution Functions

Theorem 6.1

The function $F_X(x)$ is a CDF id and only if the following three conditions hold:

1. $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to+\infty} F_X(x) = 1$.
2. $F_X(x)$ is a non-decreasing function of x.
3. $F_X(x)$ is right-continuous; that is, for every number x_0, $\lim_{x\downarrow x_0} F_x(x) = F(x_0)$.

- We can also have a continuous CDF.

Definition 6.2

A random variable X is continuous(discrete) if $F_X(x)$ is a continuous(step) function of x.

Example 6.3

An example of a continuous CDF is the function

$$F_X(x) = \frac{1}{1+e^{-x}}$$

observe that

$$\lim_{x \to -\infty} F_X(x) = 0 \quad \text{since} \quad \lim_{x \to -\infty} e^{-x} = \infty,$$

$$\lim_{x \to \infty} F_X(x) = 1 \quad \text{since} \quad \lim_{x \to \infty} e^{-x} = 0,$$

$$\frac{d}{dx} F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0,$$

where the final line proves that $F_X(x)$ is non-decreasing in x. Finally, by definition, $F_X(x)$ is right-continuous as it is continuous in the first place.

- This CDF is a special case of the logistic distribution.

To be a CDF, a function has to possess some key properties.

Theorem 6.2

The function $F_X(x)$ is a CDF id and only if the following three conditions hold:

1. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.
2. $F_X(x)$ is a non-decreasing function of x.
3. $F_X(x)$ is right-continuous; that is, for every number x_0, $\lim_{x \downarrow x_0} F_x(x) = F(x_0)$.

Definition 6.3

The random variables X and Y are identically disturbed if, for every set $A \in \mathbb{B}^1$, $P(X \in A) = P(Y \in A)$.

Theorem 6.3

The following two statements are equivalent:

1. The random variables X and Y are identically distributed.
2. $F_X(x) = F_Y(x)$ for every x.

Density Function

Discrete

Definition 6.4

The probability mass function of a discrete random variable is given by

$$f_X(x) = P(X = x) \quad \text{for all x.}$$

- **Notation convention**: for a given CDF F_X, the corresponding pdf is usually denoted by f_X, the corresponding lower-case letter.
- Similar to CDF being simply called the "distribution", the pdf is sometimes simply called the density.

Density and Mass

Definition 6.5

The probability density function or pdf, $f_X(x)$, of a continuous random variable X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

Using the Fundamental Theorem of Calculus, if f_X is continuous then

$$\frac{d}{dz}F_X(z)|_{z=x} = f_X(x).$$

Example 6.4

For the logistic distribution considered before, we have

$$F_X(x) = \frac{1}{1 + e^{-x}}$$

and, hence,

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

Then, for continuous random variables in general,

$$P(a < X < b) = F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X(x)dx - \int_{-\infty}^{a} f_X(x)dx$$

$$= \int_{a}^{b} f_X(x)dx.$$

Theorem 6.4

A function $f_X(x)$ is a pdf (or pmf) of a random variable X if and only of

1. $f_X(x) \geq 0$ for all x.
2. $\sum_x f_X(x) = 1$ (discrete) or $\int_{-\infty}^{\infty} f_X(x)dx = 1$ (continuous).

- Any non-negative function with a finite positive integral can be turned into a pdf or pmf. Take, for example, if

$$h(x) = \begin{cases} \geq 0 & \text{for} \quad x \in A \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\int_{x \in A} h(x)dx = \mathrm{K} < \infty, \quad \textit{where } \mathrm{K} > 0,$$

then $f_X(x) = h(x)/K$ is a pdf of a random variable X taking values in A.
- In some cases, although $F_X(x)$ exists, $f_X(x)$ may not exist because $F_X(x)$ can be continuous but not differentiable. Therefore, sometimes statistical analysis would be based on $F_X(x)$ and not $f_X(x)$.

Distribution of Functions of a Random Variable

- If X is a random variable with CDF F_X, then $Y = g(X)$ is also a random variable.
- Importantly, since Y is a function of X, we can determine its random behaviour in terms of the behaviour of X.
- Then, for any set A,

$$P(Y \in A) = P(g(X) \in A).$$

  This clearly shows that the distribution of Y depends on the function $g(.)$ and the CDF F_X.
- Formally,

$$g(x) : \mathscr{X} \to \mathscr{Y},$$

  where $\mathscr{X}$ and $\mathscr{Y}$ are the sample spaces of X and Y, respectively.

Distribution of Functions of a Random Variable

- Notice that the mapping $g(.)$ is associated with the inverse mapping $g^{-1}(.)$, a mapping from the subsets of $\mathscr{Y}$ to those $\mathscr{X}$:

$$g^{-1}(A) = \{x \in \mathscr{X} : g(x) \in A\}. \tag{4}$$

- Therefor, the mapping $g^{-1}(.)$ takes sets into sets, that is, $g^{-1}(A)$ is the set of points in $\mathscr{X}$ that $g(x)$ takes into the set A.
- If $A = \{y\}$, a point set, then

$$g^{-1}(\{y\}) = \{x \in \mathscr{X} : g(x) = y\}.$$

- Now, if $Y = g(X)$, then for all $A \in \mathscr{Y}$,

$$\begin{aligned} P(Y \in A) &= P(g(X) \in A) \\ &= P(\{x \in \mathscr{X} : g(x) \in A\}) \\ &= P(X \in g^{-1}(A)), \end{aligned} \tag{5}$$

where the last line follows from (1). This defines the probability distribution of Y.

Example 7.1

A discrete random variable X has binomial distribution if its pmf is of the form

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, ..., n,$$

where n is a positive integer and $0 \le p \le 1$. Values such as n and p are called parameters of a distribution. Different parameter values imply different distributions.

- The CDF of $Y = g(X)$ is

$$\begin{aligned} F_Y(y) = P(Y \leq y) = P(g(X) \leq y) \\ = P(\{x \in \mathscr{X} : g(x) \leq y\}) \\ = \int_{x \in \mathscr{X} : g(x) \leq y} f_X(x) dx. \end{aligned}$$

Distribution of Functions of a Random Variable

Theorem 7.1

Let X have CDF $F_X(x)$, let $Y = g(X)$ and let $\mathscr{X}$ and $\mathscr{Y}$ be defined as

$$\mathscr{X} = \{x : f_X(x) > 0\} \text{ and } \mathscr{Y} = \{y : y = g(x) \text{ for some } x \in \mathscr{X}\}. \quad (6)$$

1. If g is an increasing function on $\mathscr{X}$, $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathscr{Y}$.
2. If g is a decreasing function on $\mathscr{X}$ and X is a continuous random variable, $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathscr{Y}$.

Proof: Exercise!

Theorem 7.2

Let X have pdf $f_X(x)$ and let $Y = g(X)$,where g is a monotone function, Let $\mathscr{X}$ and $\mathscr{Y}$ be defined as in (3). Suppose that $f_X(x)$ is continuous on $\mathscr{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathscr{Y}$. The the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_x(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| & y \in \mathscr{Y} \\ \\ 0 & \text{otherwise} \end{cases}$$

Proof: Exercise!

Distribution of Functions of a Random Variable

Example 7.2

Suppose X $f_X(x) = 1$ for $0 < x < 1$ and 0 otherwise, which is the *uniform*$(0, 1)$ distribution. Observe that $F_X(x) = x$, $0 < x < 1$. We now make the transformation $Y = g(X) = -\log X$. Then,

$$g'(x) = \frac{d}{dx}(-\log x) = -\frac{1}{x} < 0 \quad \text{for } 0 < x < 1;$$

hence, $g(x)$ is monotone and has e continuous derivative on $0 < x < 1$. Also, $\mathscr{Y} = (0, \infty)$. Observe that $g^{-1}(y) = e^{-y}$. Then, using Theorem (1.2),

$$\begin{aligned} f_Y(y) &= 1 * |-e^{-y}| \quad \text{if } 0 < y < \infty \\ &= e^{-y} \quad \text{if } 0 < y < \infty. \end{aligned}$$

Distribution of Functions of a Random Variable

Example 7.3

Let X have the standard normal distribution,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad -\infty < X < \infty.$$

- Consider $Y = X^2$. The function $g(x) = x^2$ is monotone on $(-\infty, 0)$ and on $(0, \infty)$. The set $\mathscr{Y} = (0, \infty)$. Applying Theorem (1.2), we take

$$A_0 = \{0\};$$

$$A_1 = (-\infty, 0), \quad g_1(x) = x^2, \quad g_1^{-1} = -\sqrt{y};$$
$$A_2 = (0, \infty), \quad g_2(x) = x^2, \quad g_2^{-1} = -\sqrt{y}.$$

Example 1.3 cont.

- Then, the pdf of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2} \Big| -\frac{1}{2\sqrt{y}} \Big| + \frac{1}{\sqrt{2\pi}} e^{(-\sqrt{y})^2} \Big| \frac{1}{2\sqrt{y}} \Big|$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{y/2}, \qquad 0 < y < \infty.$$

- This is the pdf of a chi squared random variables with 1 degree of freedom. You will use this distribution frequently in your later econometrics courses.

Theorem 7.3

Let X have continuous CDF $F_X(x)$ and define the random variable Y as $Y = F_X(X)$. Then, Y is uniformly distributed on (0,1), that is

$$P(Y \leq y) = y, \quad 0 < y < 1.$$

- **Proof**: Exercise!

Distribution of Functions of a Random Variable

- This result connects any random variable with some CDF $F_X(x)$ with a uniformly distributed random variable. Hence, if we want to simulate random numbers from some distribution $F_X(x)$, all we have to do is to generate uniformly distributed random variables, Y, and then solve for $F_X(x) = y$. As long as we can compute $F_X^{-1}(y)$, we can generate random numbers from the distribution $F_X(x)$.

Casella, G., & Berger, R. (2002). *Statistical inference*. Cengage Learning.
https://books.google.fr/books?id=FAUVEAAAQBAJ