Probability and Statistics

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Section 1

Random Samples

Random Samples

Definition 1.1

The random variables $X_1, ..., X_n$ are called a random sample of size n from the population $f(x)$ if $X_1, ..., X_n$ are mutually independent random variables and the marginal pdf or pmf of each X_i is the same function $f(x)$. Alternatively, $X_1, ..., X_n$ are called independent and identically distributed random variable with pdf or pmf f(x). This is commonly abbreviated to iid random variables.

Random Samples

- In many experiments there are $n > 1$ repeated observations made on the variable, where X_1 is the first observation, X_2 is the second observation etc.
- In that case, each X_i has a marginal distribution given by $f(x)$.
- In addition, the value of one observation has no effect on or relationship with any of the other observations.
- $X_1, ..., X_n$ are mutually independent.

Random Samples

- Then, the joint pdf or pmf is giben by

$$f(x_1, ..., x_n|\theta) = f(x_1|\theta)f(x_2|\theta)...f(x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta),$$

  where we assume that the population pdf of pmf is a member of a parametric family, and θ is the vector of parameters.
- The random sampling model in Definition (1.1) is sometimes called sampling from an infinite population.
- Suppose we obtain the values of $X_1, ..., X_n$ sequentially.
- First, the experiment is performed and $X_1 = x_1$ is observed.
- Then, the experiment is repeated and $X_2 = x_2$ is observed.
- The assumption of independence implies that the probability distribution for X_2 is unaffected by the fact that $X_1 = x_1$ was observed first.

Sums of Random Variable from a Random Sample

- Suppose we have drawn a sample $(X_1, ..., X_n)$ from the population. We might want to obtain some summary statistics.
- This summary might be defined as a function

$$T(x_1, ..., x_n),$$

which might be real or vector-valued.

- So,

$$Y = T(X_1, ..., X_n),$$

is a random variable or a random vector.

- We can use similar techniques as those introduced for functions of random variables, to investigate the distributional properties of Y.
- Thanks to $(X_1, ..., X_n)$ possessing the iid property, the distribution of Y will be tractable.
- This distribution is usually derived from the distribution of the variables in the random sample. Hence, it is called the sampling distribution of Y.

Sums of Random Variable from a Random Sample

Definition 1.2

Let $X_1, ..., X_n$ be a random sample of size n from a population and let $T = (x_1, ..., x_n)$ be a real-valued or vector-valued function whose domain includes the sample space of $(X_1, ..., X_n)$. Then, the random variable or random vector $Y = T(X_1, ..., X_n)$ is called a statistic. The probability distribution of a statistic Y is called the sampling distribution of Y.

Definition 1.3

The sample mean is the arithmetic average of the values in a random sample. It is usually denoted by

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Sums of Random Variable from a Random Sample

Definition 1.4

The sample variance is the statistic defined by

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

The sample standard deviation is the statistic defined by

$$S = \sqrt{S^2}.$$

Sums of Random Variable from a Random Sample

Lemma 1.1

Let $X_1, ..., X_n$ be a random sample from a population and let $g(x)$ be a function such that $E[g(X_1)]$ and $\text{Var}(g(X_1))$ exist. Then

$$E[\sum_{i=1}^{n} g(X_i)] = nE[g(X_1)]$$

and

$$Var(\sum_{i=1}^{n} g(X_i)) = nVar(g(X_1)).$$

- **Proof**: Exercise

Sums of Random Variable from a Random Sample

Theorem 1.1

Let $X_1, ..., X_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Then,

1. $E[\bar{X}] = \mu$,
2. $\text{Var}(\bar{X}) = \sigma^2/n$,
3. $E[S^2] = \sigma^2$.

Section 2

Inequalities

Theorem 2.1

Chebychev's Inequality: Let X be a random variable and let $g(x)$ be a non-negative function. Then, for any $r > 0$,

$$P(g(X) \geq r) \leq \frac{E[g(X)]}{r}.$$

- **Proof**: Exercise!

Inequalities

- This result comes in very handy when we want to turn a probability statement into an expectation statement. For example, this would be useful if we already have some moment existence assumptions and we want to prove a result involving a probability statement.

Example 2.1

Let $g(x) = (x-\mu)^2/\sigma^2$, where $\mu = E[X]$ and $\sigma^2 = Var(X)$. Let, for convenience, $r = t^2$. Then,

$$P[\frac{(X-\mu)^2}{\sigma^2} \geq t^2] \leq \frac{1}{t^2} E[\frac{(X-\mu)^2}{\sigma^2}] = \frac{1}{t^2}.$$

- This implies that

$$P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2},$$

and, consequently,

$$P[|X-\mu| < t\sigma] \geq 1 - \frac{1}{t^2}.$$

Example 8.1 cont.

- Therefore, for instance for $t = 2$, we get

$$P[|X - \mu| \geq 2\sigma] \leq 0.25 \quad \text{or} \quad P[|X - \mu| < 2\sigma] \geq 0.75.$$

This says that, there is at least a 75% chance that a random variable will be within 2σ of its mean (independent of the distribution of X).

Example 2.2

Let $Z \sim N(0,1)$. Then,

$$P(Z \geq t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} dx$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t}.$$

The second inequality follows from the fact that for $x > t$, $x/t > 1$. Now, since Z has a symmetric distribution, $P(|Z| \geq t) = 2P(Z \geq t)$. Hence,

$$P(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}.$$

Example 8.2 cont.

- Set $t = 2$ and observe that

$$P(|Z| \geq 2) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-2}}{2} = 0.054.$$

and, consequently,

$$P[|X - \mu| < t\sigma] \geq 1 - \frac{1}{t^2}.$$

This is a much tighter bound compared to the one given by Chebychev's Inequality.

- Generally Chebychev's Inequality provides a more conservative bound.

Inequalities

- A related inequality is Markov's Inequality.

Lemma 2.1

If $P(Y \geq 0) = 1$ and $P(Y = 0) < 1$, then, for any $r > 0$,

$$P(Y \geq r) \leq \frac{E[Y]}{r},$$

and the relationship holds with equality if and only if $P(Y = r) = p = 1 - P(Y = 0)$, where $0 < p \leq 1$.

Section 3

Convergence Concepts

Convergence Concepts

- The underlying idea in this section is to understand what happens to sequence of random variables, or also summary statistics, when we let the sample size go to infinity.
- This is, of course, an idealistic concepts, if you like, because the sample size never goes to infinity. However, the idea is to attain a grasp of what happens when the sample size becomes large enough.
- This is important because, although important results are based on the case when the sample size approaches ∞, they are actually relevant for finite sample size, which are large enough.
- How large is "large enough" is very much related to the data, its dependence structure, the econometric model at hand etc, so it is difficult to give a proper rule of thumb.

Convergence Concepts

- The tools you will learn in this section are the fundamental building blocks of what is known as "asymptotic theory" or "large sample theory".
- The three important concepts we will consider in what follows are
  1. almost sure convergence,
  2. convergence in probability,
  3. convergence in distribution.
- In the first instance we will consider the case where a sequence of random variables $X_1, ..., X_n$ are independently and identically distribution (iid). This is one (and the simplest) of many possible dependence settings.

Almost Sure Convergence

- Remember that random variables are functions defined on the sample space, e.g. $X(\omega)$. Our interest will be on a sequence of random variables, indexed by sample size, i.e. $X_n(\omega)$.
- To motivate the following discussion, consider pointwise convergence:

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \text{for all } \omega \in \Omega,$$

  where Ω is, as before, the sample space.
- Notice that, convergence occurs for all ω! There is not much of probabilistic statement here.
- This is the strongest form of convergence we can have on the sample space. But it is not relevant for probabilistic statements.
- A slightly restricted version of pointwise convergence is almost sure convergence.

Almost Sure Convergence

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Almost Sure Convergence

Definition 3.1

Almost Sure Convergence: A sequence of random variables $X_1, X_2, \ldots$ defined on a probability space $(\Omega, \mathcal{F}, P)$ convergence almost surely to a random variable X if

$$\lim_{n\to\infty} X_n(\omega) = X(\omega),$$

for each ω, except for $\omega \in E$, where $P(E) = 0$.

- The idea is this: pointwise convergence fails for some points in Ω. However, the number of such points is so small that we can safely assign zero measure (or zero probability) to the set of these points.
- Other ways of expressing this definition are,

$$P(\lim_{n\to\infty} |X_n - X| > \varepsilon) = 0 \quad \text{for every } \varepsilon > 0,$$

or

$$P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1.$$

Almost Sure Convergence

- This time, we have a difference. Convergence fails on a very small set E such that $P(E) = 0$.
- That $P(E) = 0$ is due to the set being so small that we can safely assign zero probability to the set.
- Remember that in earlier lectures we have stated that for a continuously distributed random variable, the probability of a single point is always equal to zero. This is similar, in spirit, to the situation at hand.
- This type of convergence is also called convergence almost everywhere and convergence with probability 1.
- The following notation is common:

$$X_n \xrightarrow{a.s.} X$$

$$X_n \xrightarrow{wp1} X$$

$$\lim_{n\to\infty} X_n = X \, a.s.$$

- Note that, at the cost of notational sloppiness, the argument of $X_n(\omega)$ is usually dropped.
- Also, $X_n(\omega)$ need not converge to a function. It can also simply converge to some constant, say, a.

Convergence in Probability

- The next convergence type is convergence in probability . Its definition is similar to that of almost sure convergence but in essence it is a much weaker convergence concept.

Definition 3.2

Convergence in Probability: A sequence of random variables $X_1, X_2, ...$ converges in probability to a random variable X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0.$$

Convergence in Probability

- One could also write

$$\lim_{n\to\infty} P(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) = 0,$$

$$X_n \xrightarrow{P} X,$$

or

$$p \lim_{n\to\infty} X_n = X.$$

Convergence in Probability

- Almost sure convergence states that we have pointwise convergence for all $\omega \in \Omega$ except for a small, zero measure set E. Importantly, this set is independent of n.
- Convergence in probability states that as the sample size goes towards ∞, the probability that X_n will deviate from X by more that ε decreases towards zero.

Theorem 3.1

Weak Law of Large Numbers: If $X_1, X_2, \ldots$ are iid random variables with common mean $\mu < \infty$ and variance $\sigma^2 < \infty$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu,$$

as $n \to \infty$.

consistent estimator

- What the Weak and Strong LLNs are saying is that under certain conditions, the sample mean converges to the population mean as $n \to \infty$. This is known as consistency: one would say that the sample mean is a consistent estimator of the population mean.
- In actual applications, this means that if the sample size is large enough, then the sample mean is close to the population mean. So n does not have to be that close to infinity. On the other hand, as mentioned at the beginning ,"how large" the sample size should be in order to be considered a "large enough" sample is a different question in its own. We will not deal with this here.
- Sometimes, consistency is compared to unbiasedness.

Biased Estimator

- An estimator $\hat{\beta}$ of a population value β is an unbiased estimator

$$E[\hat{\beta}] = \beta.$$

- What are the things we might want to estimate? One example would be parameters of a distribution family. For example, we might know that the data are distributed with $N(\mu, \sigma^2)$, but we may not know the particular values of μ and σ^2. In this case, we would estimate these parameters.

Convergence in Probability

- The p lim operator has some nice properties that makes it much more convenient to deal with, compared to the expectation operator. In particular, let $X_1, X_2, ...$ and $Y_1, Y_2, ...$ be two random sequences and $a_1, a_2, ...$ be some non-stochastic sequence. Then, we have the following.

1. $$p \lim_{n\to\infty} \frac{X_n}{Y_n} = \frac{p\lim_{n\to\infty} X_n}{p\lim_{n\to\infty} Y_n},$$

   while we usually have

   $$E[\frac{X_n}{Y_n}] \neq \frac{E[X_n]}{E[Y_n]};$$

2. $$p \lim_{n\to\infty} (X_n + Y_n) = p \lim_{n\to\infty} X_n + p \lim_{n\to\infty} Y_n;$$

3. the p lim of a non-random sequence is equal to its limit:

   $$p \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

Convergence in Probability

- Note that, one concept does not usually imply another. In other words, a consistent estimator can be biased, while an unbiased estimator can be inconsistent.
- Suppose we are trying to estimate the population parameter β. Consider the following estimators.
  1. $\hat{\beta} = \beta + 20/n$: consistent but biased.
  2. $\hat{\beta} = X$ where $P(X = \beta + 100) = P(X = \beta - 100) = 0.5$: unbiased but inconsistent.

Convergence in Probability

- Returning to the discussion at hand, it is important to acknowledge that neither almost sure convergence nor convergence in probability (and nor any convergence type) says anything about the distribution of the sequence $X_1, X_2, \ldots$. For example, it might be such that the distribution of X_i changes as i varies. This is fine.
- So far, we have only considered LLNs that work when the sequence is drawn from an iid population. If this assumption is violated, we can still probably have convergence of the sample mean to the population mean, but we will have to find an appropriate LLN that works for the particular population distribution we have.
- A useful result relating almost sure convergence and convergence in probability is that

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X.$$

  Convergence in probability, however, does not imply almost sure convergence.
- Obviously, for some constant K

$$K \xrightarrow{a.s.} K \quad \text{and} \quad K \xrightarrow{p} K.$$

Convergence in Probability

- As far as economists and most of the econometricians are concerned, one would not care too much about whether convergence is achieved almost surely or in probability. As long as convergence is achieved, the rest is not important.
- However, in some cases it might be easier to prove the LLN for one of the two convergence types. This is no problem, *as* $\xrightarrow{a.s.}$ *implies* $\xrightarrow{p}$ *anyway*.
- In addition, convergence almost surely might be slower than convergence in probability in the sense that it might require a larger sample size before the sample mean is close enough to the population mean.

Convergence in Distribution

Definition 3.3

Convergence in Distribution: A sequence of random variables $X_1, X_2, \ldots$ converges in distribution to a random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

at every x where $F(x)$ is continuous.

- This is also called convergence in law. The following short hand notation is used to denote convergence in distribution:

$$X_n \xrightarrow{d} X,$$

$$X_n \xrightarrow{d} F_X,$$

$$X_n \xrightarrow{L} F_X.$$

- It is important to underline that it is not X_n that converges to a distribution. Instead, it is the distribution of X_n that converges to the distribution of X.

Convergence in Distribution

- As far as sequences of random vectors are concerned, a sequence of random vectors $X_n = (X_{1,n}, ..., X_{d,n})$ converges in distribution to a random vector X if

$$\lim_{n \to \infty} F_{X_n}(x_1, ..., x_n) = F_X(x_1, ..., x_2),$$

  at every $x = (x_1, ..., x_2)$ where $F(x_1, ..., x_d)$ is continuous.
- Importantly, convergence in probability implies convergence in distribution.

Theorem 3.2

If the sequence of random variables X1,X2, ... converges in probability to a random variable X, the sequence also converges in distribution to X.

- Consequently, almost sure convergence implies convergence in distribution, as well.

Convergence in Distribution

Theorem 3.3

The sequence of random variables X1,X2, ... converges in probability to a constant a if and only if the sequence also converges in distribution to a. Equivalently, the statement

$$P(|X_n - a| > \varepsilon) \to 0 \text{ for every } \varepsilon \to 0$$

is equivalent to

$$F_{X_n}(x) = P(X_n \leq x) \to \begin{cases} 0 & \text{if } x < a \\ \\ 1 & \text{if } x > a \end{cases}$$

.

Convergence in Distribution

- We now introduce one of the most useful theorems we have considered so far.

Theorem 3.4

Central Limit Theorem: Let $X_1, X_2, \ldots$ be a sequence of iid random variables with $E[X_i] = \mu < \infty$ and $0 < Var(X_i) = \sigma^2 < \infty$. Define

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Let $G_n(x)$ denote the cdf of $\sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma}$. Then, for any x, $-\infty < x < \infty$,

$$\lim_{n\to\infty} G_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

In other words,

$$\sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1).$$

Convergence in Distribution

- This is a powerful result! We start with the iid and finite mean and variance assumptions. In return, the Central Limit Theorem (CLT) promises us that the distribution of a properly standardised version of the sample mean given by

$$\sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma}$$

  will converge to the standard normal distribution as the sample size tends to infinity.
- As before, the sample size will never be equal to ∞. BUT, for large enough samples, $\sqrt{n}\frac{(\bar{X}_n-\mu)}{\sigma}$ will be approximately standard normal. As n becomes larger, this approximate result will become more accurate.
- As with LLNs, it is possible to obtain CLTs for non-iid data. However, this will require one to make stronger assumptions regarding the moments of the sequence of random variables. The trade-off between dependence and moment assumptions is always there.
- Two useful result are given next.

Convergence in Distribution

Theorem 3.5

If X_n is a sequence of random vectors each with support χ, $g(x)$ is continuous on χ and

$$X_n \xrightarrow{d} X,$$

then

$$g(X_n) \xrightarrow{d} g(X)$$

Theorem 3.6

Slutsky's Theorem: If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} k$, where k is a constant, then

1. $Y_n X_n \xrightarrow{d} kX$,
2. $X_n + Y_n \xrightarrow{d} X + k$.

Convergence in Distribution

Example 3.1

Suppose that

$$\sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1),$$

however the value of σ is unknown. What to do?

The Delta Method

- When talking about the CLT, our focus has been on the limiting distribution of some standardised random variable.
- There are many instances, however, when we are not specifically interested in the distribution of the standardised random variable itself, but rather of some function of it.
- The delta method comes in handy in such cases. This method utilises our knowledge on the limiting distribution of a random variable in order find the limiting distribution of a function of this random variable.
- In essence, this method is a combination of Slutsky's Theorem and Taylor's approximation.

The Delta Method

Theorem 3.7

Delta Method: Let Y_n be a sequence of random variables that satisfies

$$\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0, \sigma^2).$$

For a given function $g(.)$ and a specific value of θ, suppose that $g'(\theta)$ exist and $g'(\theta) \neq 0$. Then,

$$\sqrt{n}[g(Y_n) - g(\theta)] \xrightarrow{d} N(0, \sigma^2[g'(\theta)]^2).$$

Section 4

Reference

Reference

Casella, G., & Berger, R. (2002). *Statistical inference*. Cengage Learning. https://books.google.fr/books?id=FAUVEAAAQBAJ