# Homework 6

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## Question 1

A random point $(X, Y)$ is distributed uniformly on the square with vertices $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$. That is, the jonit pdf is $f(x, y) = \frac{1}{4}$ on the square.
Determine the probabilities of the following events.

- $X^2 + Y^2 < 1$
- $2X - Y > 0$
- $|X + Y| < 2$

### Solution:

Since the distribution is uniform, the easiest way to calculate these probabilities is as the ratio of areas, the total are being 4.

- The circle $X^2 + Y^2 \leq 1$ has are $\pi$, so $P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}$. (Fig. 1)
- The are below the line $Y = 2X$ is half of the area of the square, so $P(2X - Y > 0) = \frac{2}{4}$. (Fig. 2)
- Clearly $P(|X + Y| < 2) = 1$. (Fig. 3)

## Question 2

A pdf is defined by

$$f(x, y) = \begin{cases} C(x + 2y) & \text{if} \quad 0 < y < 1 \text{ and } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

- Find the value of $C$.

### Solution:

Based on the pdf definition:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

So

$$\int_0^1\int_0^2 C(x+2y)dxdy = \int_0^1 C(\frac{x^2}{2}+2yx)\Big|_0^2$$
$$= \int_0^1 C(2+4y)dy = C(2y+4\frac{y^2}{2})\Big|_0^1 = 4C = 1$$

thus $C = \frac{1}{4}$

## Question 3

The random pair $(X, Y)$ has the distribution

| | | $X$ | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| | 2 | $\frac{1}{12}$ | $\frac{1}{6}$ | $\frac{1}{12}$ |
| $Y$ | 3 | $\frac{1}{6}$ | 0 | $\frac{1}{6}$ |
| | 4 | 0 | $\frac{1}{3}$ | 0 |

- Show that $X$ and $Y$ are dependent.
- Give a probability table of random variables $U$ and $V$ that have the same marginals as $X$ and $Y$ but are independent.

### Solution:

- The marginal distribution of $X$ is

$$P(X=1) = P(X=3) = \frac{1}{12}+\frac{1}{6}+0 = \frac{1}{4}$$
$$P(X=2) = \frac{1}{6}+0+\frac{1}{3} = \frac{1}{2}$$

The marginal distribution of $Y$ is

$$P(Y=2) = \frac{1}{12}+\frac{1}{6}+\frac{1}{12} = \frac{1}{3}$$
$$P(Y=3) = \frac{1}{6}+0+\frac{1}{6} = \frac{1}{3}$$
$$P(Y=4) = 0+\frac{1}{3}+0 = \frac{1}{3}$$

But

$$P(X=2,Y=3)=0 \neq (\frac{1}{2})(\frac{1}{3}) = P(X=2)P(Y=3).$$

Therefore the random variables are not independent.

- The distribution that satisfies $P(U=x,V=y)=P(U=x)P(V=y)$ where $U \sim X$ and $V \sim Y$ is

$$\begin{array}{cc|ccc} & & & U & \\ & & 1 & 2 & 3 \\ \hline & 2 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ V & 3 & \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ & 4 & \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \end{array}$$

# Question 4

A student from an elementary school in Kansas City is randomly selected and $X$ = the number of living parents of the student is recorded. Suppose the marginal distribution of $X$ is

$$f_X(0)=0.01 \qquad f_X(1)=0.09 \qquad f_X(2)=0.9$$

A retiree from Sun City is randomly selected and $Y$ = the number of living parents of the retiree is recorded. Suppose the marginal distribution of $Y$ is

$$f_Y(0)=0.7 \qquad f_Y(1)=0.25 \qquad f_Y(2)=0.05$$

Assume that these two random variables are independent and calculate:

- $f(0,2)$
- $f(1,0)$
- $P(X \geq Y)$

## Solution:

Since these two variables are independent:

$f(0,2)=f_X(0)f_Y(2)=0.01\times0.05=0.0005$ and $f(1,0)=f_X(1)f_Y(0)=0.09\times0.7=0.063$

And for the last part:

$$P(X \geq Y)=f(2,0)+f(2,1)+f(2,2)+f(1,0)+f(1,1)+f(0,0)$$

$$=0.9\times0.7+0.9\times0.25+0.9\times0.05+0.09\times0.7+0.09\times0.25+0.01\times0.7=0.9925$$

# Figures

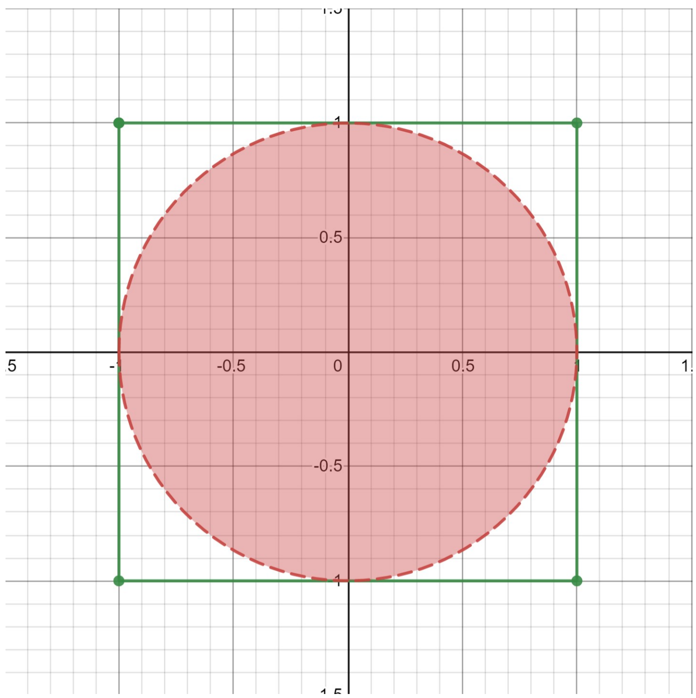


Figure 1: Diagram of $P(X^2 + Y^2 \leq 1) = \frac{\pi}{4}$.

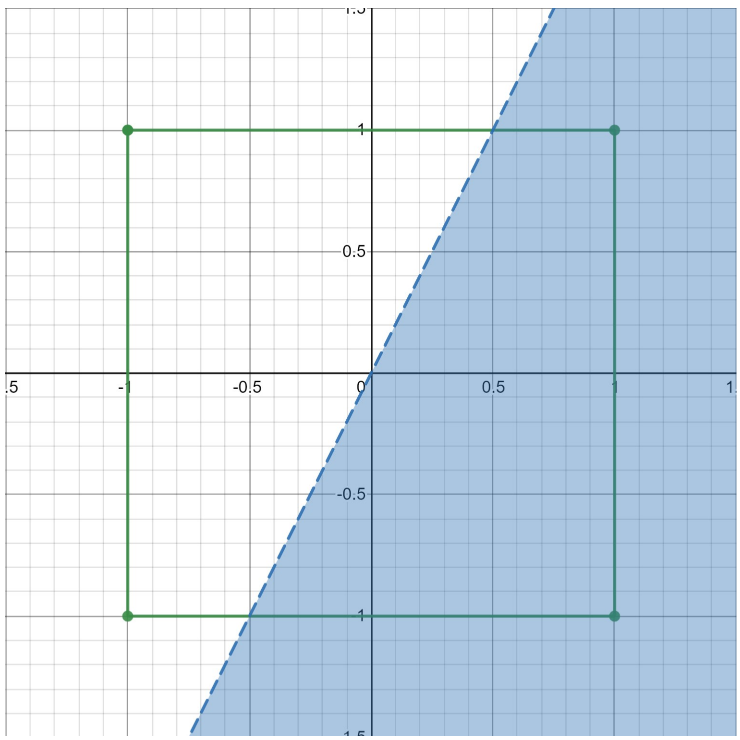


Figure 2: Diagram of $P(2X - Y > 0) = \frac{2}{4}$.

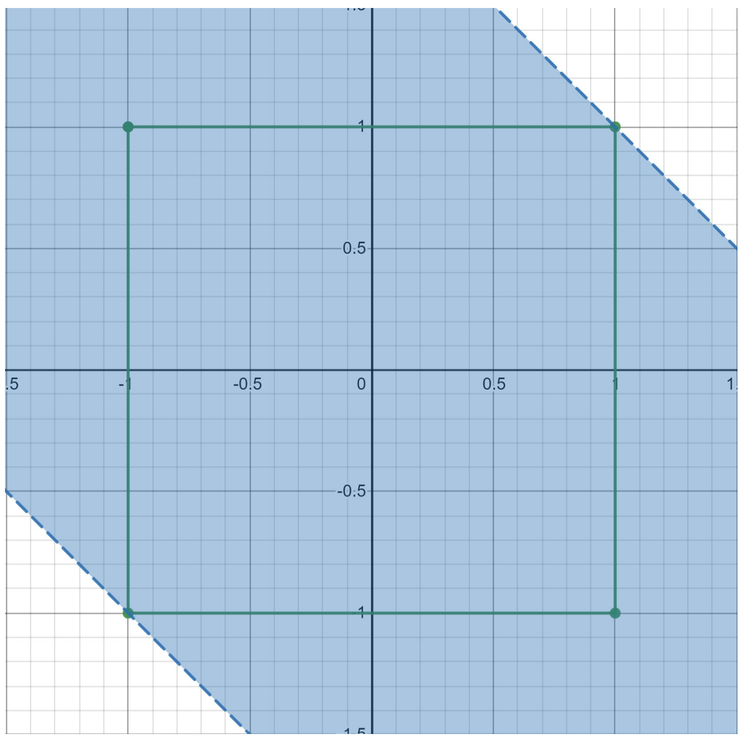


Figure 3: Diagram of $P(|X+Y|<2)=1$.