Homework 4

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Question 1

Assume X has the standard normal distribution , $\mu = 0, \sigma^2 = 1$, $f_X(x) = (1/\sqrt{2\pi})e^{-x^2/2}$.
Find $E(X^2)$.

Solution:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \Big[-xe^{\frac{-x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx \Big] \tag{1}$$

To calculate the $\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$ by the definition of pdf function, we have:

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = 1 \Rightarrow \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx = \sqrt{2\pi}$$

So the answer of equation (1) would be:

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} = 1.$$

Question 2

A median of a distribution is a value m such that $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$. (If X is continuous, m satisfies $\int_{-\infty}^{m} f(x)dx = \int_{m}^{\infty} f(x)dx = \frac{1}{2}$.) Find the median of the following distribution.

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & x < 0 \end{cases}$$

Solution:

$$\int_0^m 3x^2 dx = m^3 = \frac{1}{2} \Rightarrow m = (\frac{1}{2})^{1/3} = .794.$$

Question 3

Prove that

$$\frac{d}{da}E((X)-a)^2 = 0 \Leftrightarrow E(X) = a$$

by differentiating the integral. Verify, using calculus, that $a = E(X)$ is indeed a minimum.
(*Hint: Use theorem 2.4.3 or this link to know how to calculate the derivative of an expectation)

Solution:

$$\frac{d}{da}E(X-a)^2 = \frac{d}{da}\int_{-\infty}^{\infty}(x-a)^2 f_X(x)dx = \int_{-\infty}^{\infty}\frac{d}{da}(x-a)^2 f_X(x)dx$$

$$\int_{-\infty}^{\infty} -2(x-a)f_X(x)dx = -2\Big[\int_{-\infty}^{\infty} x f_X(x)dx - a \underbrace{\int_{-\infty}^{\infty} f_X(x)dx}_{\text{definition of pdf function=1}}\Big]$$

$$= -2[E(X)-a].$$

First:

$\frac{d}{da}E((X)-a)^2 = 0$ then $-2[E(X)-a] = 0$ which implies that $E(X) = a$.

Second:

$EX = a$ then $\frac{d}{da}E(X-a)^2 = -2[E(X)-a] = -2[a-a] = 0$.

$E(X) = a$ is a minimum since $\frac{d^2}{da^2}E(X-a)^2 = 2 > 0$