Recall from Calculus

Omid Safarzadeh

January 11, 2022

Set theory Operations

- **Union**: The union of A and B is the set of elements that belong to either A or B or both.
- **Intersection**: The intersection of A and B is the set of elements that belong to both A and B.
- **Complement**: The complement of A is the set of all elements that are not in A.

$$A^c = \{x : x \notin A\}.$$

Introduction

Derivative

Definition 1.1

Derivative of $f(x)$ at $x = a$ defined as:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

- Higher-order derivatives are defined as:

$$f'(x) = \frac{df(x)}{dx}, \quad f''(x) = f^2(x) = \frac{d}{dx}\left(\frac{df(x)}{d(x)}\right) = \frac{d^2 f(x)}{dx^2}, \ \ldots,$$

And so

$$f^n(x) = \frac{d^n f(x)}{dx^n}$$

Derivative

Some important derivative formula is summarized as bellow:

$$(f+g)' = f' + g' \tag{1}$$

$$(a.f)' = a.f' \tag{2}$$

$$(f.g)' = f'g + f.g' \tag{3}$$

$$(\frac{1}{f})' = \frac{-f'}{f^2} \tag{4}$$

$$f'(a) = 0 \tag{5}$$

$$f'(ax) = a \tag{6}$$

$$(\frac{1}{af(x)})' = \frac{-f'(x)}{(af(x))^2} \tag{7}$$

Derivative

$$(asin(bx))' = abcos(bx) \tag{8}$$

$$(acos(bx))' = -absin(bx) \tag{9}$$

$$(atan(bx))' = ab(1 + tan^2(x)) \tag{10}$$

$$(acot(bx))' = -ab(1 + cot^2(x)) \tag{11}$$

$$(e^{ax})' = ae^{ax} \tag{12}$$

Derivative

$$(asinh(bx))' = abcosh(bx) \tag{13}$$

$$(acosh(bx))' = absinh(bx) \tag{14}$$

$$(\sqrt{f(x)})' = \frac{f'(x)}{2\sqrt{f(x)}} \tag{15}$$

Chain rule

The chain rule provides us a technique for finding the derivative of composite functions, with the number of functions that make up the composition determining how many differentiation steps are necessary.

Definition 1.2

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

For example

$$f(g(x))' = g'(x).f'(g(x))$$

Integral

- Integration can be used to find areas, volumes, central points and many useful things.
- It is necessary to mention that an Integral is the **reverse** of finding a Derivative

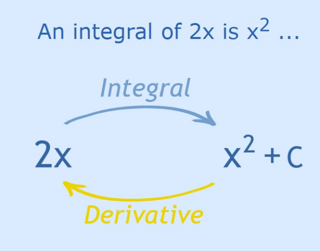

Figure: Integral vs Derivative

Some important integration formula is summarized as bellow:

$$\int u^n du = \frac{1}{n+1} u^{n+1} + c \tag{16}$$

$$\int \sin x dx = -\cos x + c \tag{17}$$

$$\int \cos x dx = \sin x + c \tag{18}$$

$$\int \frac{du}{u} = \ln |u| + c \tag{19}$$

$$\int a^u du = \frac{a^u}{\ln a} + c \tag{20}$$

Integral

$$\int \tan x dx = -\ln|\cos x| + c = \ln|\sec x| + c \tag{21}$$

$$\int \cot x dx = \ln|\sin x| \tag{22}$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + c \tag{23}$$

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c \tag{24}$$

Substitution

Needless to say, most problems we encounter will not be so simple. Here's a slightly more complicated example: find

$$\int 2x \cos(x^2) dx.$$

This is not a "simple" derivative, but a little thought reveals that it must have come from an application of the chain rule. Multiplied on the "outside" is $2x$, which is the derivative of the "inside" function x^2. Checking:

$$\frac{d}{dx} \sin(x^2) = \cos(x^2) \frac{d}{dx} x^2 = 2x \cos(x^2),$$

so

$$\int 2x \cos(x^2) dx = \sin(x^2) + C$$

Substitution

Example 2.1

Evaluate $\int x \sin(x^2) dx$**.**

- First we compute the antiderivative, then evaluate the definite integral. Let $u = x^2$ so $du = 2xdx$ or $xdx = du/2$. Then

$$\int x \sin(x^2) dx = \int \frac{1}{2} \sin u du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2) + C.$$

Integration by parts

Definition 2.1

integration by parts is a process that finds the integral of a product of functions in terms of the integral of the product of their derivative and antiderivative. It is frequently used to transform the antiderivative of a product of functions into an antiderivative for which a solution can be more easily found.

- Start with the product rule:

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

- We can rewrite this as

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx,$$

- and then

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Integration by parts

- If we let $u = f(x)$ and $v = g(x)$ then $du = f'(x)dx$ and $dv = g'(x)dx$ and
$$\int u dv = uv - \int v du.$$
- To use this technique we need to identify likely candidates for $u = f(x)$ and $dv = g'(x)dx$.

Example 2.2

Evaluate $\int x \sin x dx$.

- Let
$$u = x \xrightarrow{'} du = dx$$
$$dv = \sin x dx \xrightarrow{\int} v = -\cos x$$
- so
$$\int x \sin x dx = -x \cos x - \int -\cos x dx = -x \cos x + \sin x + C.$$

Newton-Raphson

$$\mathbf{x} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix} \quad \pi = \begin{bmatrix} \pi_1(x_1;\beta^{old}) \\ \pi_1(x_2;\beta^{old}) \\ \vdots \\ \pi_1(x_n;\beta^{old}) \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

W=

$$\begin{bmatrix} \pi(x_1;\beta^{old})(1-\pi(x_1;\beta^{old})) & 0 & \cdots & 0 \\ 0 & \pi(x_2;\beta^{old})(1-\pi(x_2;\beta^{old})) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi(x_n;\beta^{old})(1-\pi(x \end{bmatrix}$$

Newton-Raphson

$$\begin{aligned}\beta^{new} &= \beta^{old} + (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T(\mathbf{y}-\pi)\\ &= \underbrace{(\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}}_{\text{weighted least squares}}\mathbf{z}\end{aligned}$$

where $\mathbf{z} = \mathbf{X}\beta^{old} + \mathbf{W}^{-1}(\mathbf{y}-\pi)$ is the adjusted response.
$\beta^{new} = (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}\mathbf{z}$ is the solution to

$$\beta^{new} \leftarrow \arg\min_{\beta}(\mathbf{z}-\mathbf{X}\beta)^T\mathbf{W}(\mathbf{z}-\mathbf{X}\beta)$$

This is called *iteratively reweighted least squares*.

- **How to start?** Ex: take $\beta = [0, 0, ..., 0]^T$ initially
- Update β until convergence