The mathematics of voting:
on how to count the votes
so that your vote counts

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It makes sense that in a head-to-head race,
whoever receives the majority of the votes should win.

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This is the system used to determine the winner
in most states' presidential primaries:
whoever gets a plurality of votes wins.

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★ If polling identifies likely front runners, voters may choose to vote strategically for a candidate who is not their true top choice.

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As we shall discover, the voting system matters: the candidate who wins a plurality
election, might not win with vote-for-two or with anti-plurality.

In the GPA method, each voter ranks all n candidates.

★ Their first choice earns $n–1$ points.

★ Their second choice earns $n–2$ points.

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★ Their second to last choice earns 1 point.

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The candidate with the most points —"the highest GPA" — wins.

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In approval voting each voter may

★ vote for one candidate or

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★ vote for all but one of the candidates

…and each voter gets to decide how many candidates to vote for!

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And any candidate could win with approval voting depending on how many candidates each voter chooses to vote for!

Multi-round voting systems simulate elections with runoffs after eliminating some candidates — but voters only cast their ballots once.

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In ranked-choice voting each voter ranks all of the candidates:

★ If no candidate wins a majority of first-place votes, then the candidate with fewest first-place votes is eliminated.

★ For each voter whose top choice has been eliminated, their vote is re-allocated to their next choice.

★ If no candidate wins a majority when the re-allocated votes are included, then the candidate with fewest first-place votes is eliminated.

★ For each voter whose top choice has been eliminated, their vote is re-allocated to their next choice.

★ Eventually, there is a majority winner, who wins the election.

Maine's 2nd Congressional District 2018:
Bruce Poliquin v Jared Golden v Tiffany Bond v Will Hoar

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★ possible GPA method result: Golden > Bond > Hoar > Poliquin

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★ Any of Bell, Douglas, or Lincoln could have won with approval voting.

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★ Cory wins ranked-choice: Cory (20=7+4+9) > Beto (10=8+2+0)

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★ anonymous among the voters: all votes count equally

> (NOT satisfied by the Electoral College: outcome may change
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Non-desired properties:

★ dictatorship: Beyoncé gets to pick the winner

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2000 Florida vote totals		
George W. Bush	2,912,790	48.847%
Al Gore	2,912,253	48.838%
Ralph Nader	97,488	1.635%
Pat Buchanan	17,484	0.293%

If Nader voters had lied about their first choice,
they could have changed the result from Bush > Gore to Gore > Bush.

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cookie > vanilla > strawberry > chocolate > mint
mint > chocolate > strawberry > vanilla > cookie

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With the GPA method, each of these elections results in a tie — but none of the other methods award ties to opposing pairs of votes.

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Election reform matters at all levels:
if you think it is hopeless to reform the Electoral College,
perhaps you can amend your local election system
or improve the voting procedure for a volunteer organization?

Voting theory is just one of many topics in the mathematics of social choice:

★ council elections: which voting systems should be used to fill multiple vacant sets?

★ weighted voting systems: used for shareholder voting, the UN security council, and the Electoral College[*]

★ strategy-proof voting: what methods might encourage voters to vote their true preferences?

★ referenda: what's the best way to structure ballot questions when voters' opinions about one proposition might depend on the outcome of another?

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Thank you!