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Categorifying cardinal arithmetic

Plan

Goal: prove $a \times (b + c) = (a \times b) + (a \times c)$ for any natural numbers a, b, and c. by taking a tour of some deep ideas from category theory.

- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?

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Step 1: categorification

The idea of categorification

The first step is to understand the equation

$$a \times (b + c) = (a \times b) + (a \times c)$$

as expressing some deeper truth about mathematical structures.

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)\,?$$

Q: What is the role of the natural numbers a, b, and c?

Categorifying natural numbers

Q: What is the role of the natural numbers a, b, and c?

A: Natural numbers define the cardinalities, or sizes, of finite sets.

Natural numbers a, b, and c encode the sizes of finite sets A, B, and C.

$$a := |A|, \qquad b := |B|, \qquad c := |C|.$$

Categorifying equality

Natural numbers a, b, and c encode the sizes of finite sets A, B, and C.

$$a := |A|, \qquad b := |B|, \qquad c := |C|.$$

Q: What is true of A and B if $a = b$?

A: $a = b$ if and only if A and B are isomorphic, which means there exist functions $f \colon A \to B$ and $g \colon B \to A$ that are inverses in the sense that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. In this case, we write $A \cong B$.

For $a := |A|$ and $b := |B|$,
the equation $a = b$ asserts the existence of an isomorphism $A \cong B$.

Eugenia Cheng: "All equations are lies."

Categorification: the truth behind $a = b$ is $A \cong B$.

Categorification progress report

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)\,?$$

The story so far:

- The natural numbers a, b, and c encode the sizes of finite sets A, B, and C:

$$a := |A|, \qquad b := |B|, \qquad c := |C|.$$

- The equation "$=$" asserts the existence of an isomorphism "$\cong$".

Q: What is the deeper meaning of the symbols "$+$" and "$\times$"?

Categorifying $+$

Q: If $b := |B|$ and $c := |C|$ what set has $b + c$ elements?

A: The disjoint union $B + C$ is a set with $b + c$ elements.

$$B = \left\{ \begin{matrix} \sharp \\ \flat \\ \natural \end{matrix} \right\}, \qquad C = \left\{ \begin{matrix} \spadesuit & \heartsuit \\ \diamondsuit & \clubsuit \end{matrix} \right\}, \qquad B + C = \left\{ \begin{matrix} \sharp & \flat & \spadesuit & \heartsuit \\ & \natural & \diamondsuit & \clubsuit \end{matrix} \right\}$$

$$\boxed{b + c := |B + C|}$$

Categorifying $\times$

Q: If $a := |A|$ and $b := |B|$ what set has $a \times b$ elements?

A: The cartesian product $A \times B$ is a set with $a \times b$ elements.

$$A = \left\{ \; * \quad \star \; \right\}, \qquad B = \left\{ \begin{array}{c} \sharp \\ \flat \\ \natural \end{array} \right\}, \qquad A \times B = \left\{ \begin{array}{cc} (*, \sharp) & (\star, \sharp) \\ (*, \flat) & (\star, \flat) \\ (*, \natural) & (\star, \natural) \end{array} \right\}$$

$$\boxed{a \times b := |A \times B|}$$

Categorifying cardinal arithmetic

In summary:

- Natural numbers define cardinalities: there are sets A, B, and C so that $a := |A|$, $b := |B|$, and $c := |C|$.
- The equation $a = b$ encodes an isomorphism $A \cong B$.
- The disjoint union $B + C$ is a set with $b + c$ elements.
- The cartesian product $A \times B$ is a set with $a \times b$ elements.

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)\,?$$

A: It means that the sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$A \times (B + C) \cong (A \times B) + (A \times C)$$

Summary of Step 1

Q: What is the deeper meaning of the equation

$$a \times (b + c) = (a \times b) + (a \times c)\,?$$

A: The sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$\left\{ \begin{matrix} (*, \sharp) & (\star, \sharp) \\ (*, \flat) & (\star, \flat) \\ (*, \natural) & (\star, \natural) \\ (*, \spadesuit) & (\star, \spadesuit) \\ (*, \heartsuit) & (\star, \heartsuit) \\ (*, \diamondsuit) & (\star, \diamondsuit) \\ (*, \clubsuit) & (\star, \clubsuit) \end{matrix} \right\} \cong \left\{ \begin{matrix} (*, \sharp) \quad (*, \flat) \quad (*, \spadesuit) \quad (*, \heartsuit) \\ (*, \natural) \quad (*, \diamondsuit) \quad (*, \clubsuit) \\ (\star, \sharp) \quad (\star, \flat) \quad (\star, \spadesuit) \quad (\star, \heartsuit) \\ (\star, \natural) \quad (\star, \diamondsuit) \quad (\star, \clubsuit) \end{matrix} \right\}$$

$$A \times (B + C) \qquad \cong \qquad (A \times B) + (A \times C)$$

By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$

$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

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Step 2: the Yoneda lemma

The Yoneda lemma

The Yoneda lemma. Two sets A and B are isomorphic if and only if

- for all sets X, the sets of functions

$$\mathsf{Fun}(A, X) := \{h\colon A \to X\} \quad \text{and} \quad \mathsf{Fun}(B, X) := \{k\colon B \to X\}$$

  are isomorphic and moreover

- the isomorphisms $\mathsf{Fun}(A, X) \cong \mathsf{Fun}(B, X)$ are "natural" in the sense of commuting with composition with any function $\ell\colon X \to Y$.

? ? ?

Proof of the Yoneda lemma

The Yoneda lemma. A and B are isomorphic if and only if for any X the sets of functions $\mathsf{Fun}(A, X)$ and $\mathsf{Fun}(B, X)$ are "naturally" isomorphic.

Proof ($\Leftarrow$): Suppose $\mathsf{Fun}(A, X) \cong \mathsf{Fun}(B, X)$ for all X. Taking $X = A$ and $X = B$, use the bijections:

$$
\begin{tikzcd}
\mathsf{Fun}(A,A) \arrow[r, phantom, "\cong"] & \mathsf{Fun}(B,A) & \mathsf{Fun}(A,B) \arrow[r, phantom, "\cong"] & \mathsf{Fun}(B,B) \\
\arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & g \arrow[u, phantom, "\Cup"] & f \arrow[u, phantom, "\Cup"] & \mathrm{id}_B \arrow[l, mapsto] \arrow[u, phantom, "\Cup"]
\end{tikzcd}
$$

to define functions $g \colon B \to A$ and $f \colon A \to B$. By naturality:

$$
\begin{tikzcd}
\mathrm{id}_A \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & g \arrow[ddd, mapsto] \\
& \mathsf{Fun}(A,A) \arrow[r, phantom, "\cong"] \arrow[d, "f\circ -"'] & \mathsf{Fun}(B,A) \arrow[d, "f\circ -"] & \\
& \mathsf{Fun}(A,B) \arrow[r, phantom, "\cong"] & \mathsf{Fun}(B,B) & \\
f \arrow[rrr, mapsto] & & & \mathrm{id}_B = f \circ g
\end{tikzcd}
$$

and similarly $g \circ f = \mathrm{id}_A$. $\square$

Summary of Steps 1 and 2

By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$
$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

By the Yoneda lemma:

Step 2 summary: To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
$\rightsquigarrow$ we'll instead define a "natural" isomorphism

$$\mathsf{Fun}(A \times (B + C), X) \cong \mathsf{Fun}((A \times B) + (A \times C), X).$$

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Step 3: representability

The universal property of the disjoint union

Q: For sets B, C, and X, what is $\mathsf{Fun}(B+C,X)$?

Q: What is needed to define a function $f\colon B+C\to X$?

A: For each $b\in B$, we need to specify $f(b)\in X$, and for each $c\in C$, we need to specify $f(c)\in X$. So the function $f\colon B+C\to X$ is determined by *two* functions $f_B\colon B\to X$ and $f_C\colon C\to X$.

By "pairing"

$$\begin{array}{ccc} \mathsf{Fun}(B+C,X) & \cong & \mathsf{Fun}(B,X)\times\mathsf{Fun}(C,X) \\ \uplus & & \uplus \\ f & \leftrightsquigarrow & (f_B,f_C) \end{array}$$

A universal property of the cartesian product

Q: For sets A, B, and X, what is $\mathsf{Fun}(A \times B, X)$?

Q: What is needed to define a function $f\colon A \times B \to X$?

A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(-, b)\colon A \to X$ sending a to $f(a, b)$. So, altogether we need to define a function $f\colon B \to \mathsf{Fun}(A, X)$.

By "currying"

$$\begin{array}{ccc} \mathsf{Fun}(A \times B, X) & \cong & \mathsf{Fun}(B, \mathsf{Fun}(A, X)) \\ \Cup & & \Cup \\ f\colon A \times B \to X & \leftrightsquigarrow & f\colon B \to \mathsf{Fun}(A, X) \end{array}$$

Summary of Steps 1, 2, and 3

By categorification:

Step 1 summary: To prove $a \times (b + c) = (a \times b) + (a \times c)$
$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

By the Yoneda lemma:

Step 2 summary: To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
$\rightsquigarrow$ we'll instead define a "natural" isomorphism

$$\mathsf{Fun}(A \times (B + C), X) \cong \mathsf{Fun}((A \times B) + (A \times C), X).$$

By representability:

Step 3 summary:

- $\mathsf{Fun}(B + C, X) \cong \mathsf{Fun}(B, X) \times \mathsf{Fun}(C, X)$ by "pairing" and
- $\mathsf{Fun}(A \times B, X) \cong \mathsf{Fun}(B, \mathsf{Fun}(A, X))$ by "currying."

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Step 4: the proof

The proof

Theorem. For any natural numbers a, b, and c,

$$a \times (b + c) = (a \times b) + (a \times c).$$

Proof: To prove $a \times (b + c) = (a \times b) + (a \times c)$:

- pick sets A, B, and C so that $a := |A|$, and $b := |B|$, and $c := |C|$
- and show that $A \times (B + C) \cong (A \times B) + (A \times C)$.
- By the Yoneda lemma, this holds if and only if, "naturally,"
  $\mathrm{Fun}(A \times (B + C), X) \cong \mathrm{Fun}((A \times B) + (A \times C), X)$.
- Now

$$\begin{aligned}
\mathrm{Fun}(A \times (B+C), X) &\cong \mathrm{Fun}(B + C, \mathrm{Fun}(A, X)) \text{ by "currying"} \\
&\cong \mathrm{Fun}(B, \mathrm{Fun}(A, X)) \times \mathrm{Fun}(C, \mathrm{Fun}(A, X)) \text{ by "pairing"} \\
&\cong \mathrm{Fun}(A \times B, X) \times \mathrm{Fun}(A \times C, X) \text{ by "currying"} \\
&\cong \mathrm{Fun}((A \times B) + (A \times C), X) \text{ by "pairing."} \qquad \square
\end{aligned}$$

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Epilogue: what was the point of that?

Generalization to infinite cardinals

Note we didn't actually need the sets A, B, and C to be finite.

Theorem. For any cardinals α, β, γ,

$$\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma).$$

Proof: The one we just gave.

Exercise: Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \times \alpha^{\gamma} \quad \text{and} \quad (\alpha^{\beta})^{\gamma} = \alpha^{\beta\times\gamma} = (\alpha^{\gamma})^{\beta}.$$

Generalization to other mathematical contexts

In the discussion of representability or the Yoneda lemma, we didn't need A, B, and C to be sets at all!

Theorem.

- For vector spaces U, V, W,
$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).$$
- For nice topological spaces X, Y, Z,
$$X \times (Y \sqcup Z) \cong (X \times Y) \sqcup (X \times Z).$$
- For abelian groups A, B, C,
$$A \otimes_{\mathbb{Z}} (B \oplus C) \cong (A \otimes_{\mathbb{Z}} B) \oplus (A \otimes_{\mathbb{Z}} C).$$

Proof: The one we just gave.

The real point

The ideas of

- categorification (replacing equality by isomorphism),
- the Yoneda lemma (replacing isomorphism by natural isomorphism),
- representability (characterizing maps to or from an object),
- limits and colimits (like cartesian product and disjoint union), and
- adjunctions (such as currying)

are all over mathematics — so keep a look out!

Thank you!