ON THE ∞-TOPOS SEMANTICS OF HOMOTOPY TYPE THEORY

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Abstract. Many introductions to *homotopy type theory* and the *univalence axiom* neglect to explain what any of it means, glossing over the semantics of this new formal system in traditional set-based foundations. This series of talks will attempt to survey the state of the art, first presenting Voevodsky's simplicial model of univalent foundations and then touring Shulman's vast generalization, which provides an interpretation of homotopy type theory with strict univalent universes in any ∞-topos. As we will explain, this achievement was the product of a community effort to abstract and streamline the original arguments as well as develop new lines of reasoning.

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Lecture I: A categorical semantics of dependent type theory

1. Introduction

1.1. **Why simplicial sets might model homotopy type theory.** For any Kan complex A, there is a natural **path space factorization** of the diagonal map defined by exponentiation with the simplicial interval

$$\Delta^0 + \Delta^0 \rightarrowtail \Delta^1 \xrightarrow{\sim} \Delta^0 \qquad \rightsquigarrow \qquad A \overset{r}{\underset{\sim}{\rightarrowtail}} A^{\Delta^1} \xrightarrow{(e_0,e_1)} A \times A$$

In particular, the left map, the inclusion of constant paths, is a trivial cofibration in Quillen's model structure, and as such has the right lifting property with respect to an arbitrary Kan fibration:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, tail, "r"', "\wr"] & E \arrow[d, two heads, "p"] \\
A^{\Delta^1} \arrow[r, "b"'] \arrow[ur, dashed] & B
\end{tikzcd}
\qquad \leftrightsquigarrow \qquad
\begin{tikzcd}
A \arrow[r, "d"] \arrow[rr, bend left, "e"] \arrow[d, tail, "r"', "\wr"] & P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\
A^{\Delta^1} \arrow[r, equal] \arrow[ur, dashed, "J"'] & A^{\Delta^1} \arrow[r, "b"'] & B
\end{tikzcd}$$

In particular, by pulling back along the codomain b it suffices to consider right lifting problems against Kan fibrations over the path space A^{Δ^1}, as displayed in the left-hand square of the rectangle above-right. This is the semantic interpretation of the homotopy type theoretic principle of **path induction**: given a type family $x : A, y : A, p : x =_A y \vdash P(x,y,p)$ and a family of terms $a : A \vdash d(a) : P(a,a,r_a)$ over the constant paths, there exists a section

$$x : A, y : A, p : x =_A y \vdash J_d(x,y,p) : P(x,y,p)$$

such that $a : A \vdash J_d(a,a,r_a) \equiv d(a) : P(a,a,r_a)$.

The homotopy type theoretic principle of path induction is stronger than this, because A might be a dependent type $\Gamma \vdash A$ in an arbitrary context Γ. But we can perform an analogous construction starting from an arbitrary Kan fibration $p \colon \Gamma.A \twoheadrightarrow \Gamma$ between Kan complexes. First form the path space factorizations for both $\Gamma.A$ and Γ

(1.1.1)
$$\begin{tikzcd}
\Gamma.A \arrow[rrr, tail, "\sim"] \arrow[dr, tail, dashed, "\sim"] \arrow[dddd, two heads] & & & (\Gamma.A)^{\Delta^1} \arrow[r, two heads] \arrow[dddd, two heads] & \Gamma.A \times \Gamma.A \arrow[dddd, two heads] \\
& \Delta^1 \pitchfork_\Gamma (\Gamma.A) \arrow[urr, dotted] \arrow[dr, dashed, two heads] \arrow[dddl, dotted, two heads] & & & \\
& & (\Gamma.A) \times_\Gamma (\Gamma.A) \arrow[uurr, dotted] \arrow[ddll, dotted, two heads] & & \\
& & & & \\
\Gamma \arrow[rrr, tail, "\sim"] & & & \Gamma^{\Delta^1} \arrow[r, two heads] & \Gamma \times \Gamma
\end{tikzcd}$$

and then pull back the top factorization so that it lies in the slice over Γ. This constructs a factorization of the fibered diagonal, in the slice over Γ, using the cotensor with the simplicial interval in the slice over Γ. Once more the inclusion of constant paths is a trivial fibration so we have a lifting property exactly as above. Moreover, since pullback is a simplicially enriched right adjoint, this construction is stable under pullback along any $f \colon \Delta \to \Gamma$ between Kan complexes.

$$\begin{tikzcd}
\Delta.f^*A \arrow[rr, dotted] \arrow[dr, tail, "\sim"] \arrow[dddr, two heads] & & \Gamma.A \arrow[dr, tail, "\sim"] \arrow[dddr, two heads] & & \\
& \Delta^1 \pitchfork_\Delta (\Delta.f^*A) \arrow[rr, dotted] \arrow[dr, two heads] \arrow[dd, two heads] & & \Delta^1 \pitchfork_\Gamma (\Gamma.A) \arrow[dr, two heads] \arrow[dd, two heads] & \\
& & (\Delta.f^*A) \times_\Delta (\Delta.f^*A) \arrow[rr, dotted] \arrow[dl, two heads] & & (\Gamma.A) \times_\Gamma (\Gamma.A) \arrow[dl, two heads] \\
& \Delta \arrow[rr, dotted, "f"'] & & \Gamma &
\end{tikzcd}$$

1.2. **Martin-Löf's Dependent Type Theory.** Here we use "homotopy type theory" and "univalent foundations" as synonyms for Martin-Löf's dependent type theory, with intensional identity types, together with Voevodsky's univalence axiom.

This type theory has four basic judgment forms:

$$\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash A \equiv B \qquad \Gamma \vdash a \equiv b : A$$

asserting that A is a **type**, a is a **term** of type A, and that A and B or a and b are **judgmentally equal** types or terms, all in context Γ. Here Γ is shorthand for an arbitrary **context**, given by a sequence of variables of previously-defined types. Some types will be defined in the empty-context

$$\cdot \vdash \mathbb{N}$$

Such types then form contexts of length one $[n : \mathbb{N}]$. A type that is defined in a context of length one

$$n : \mathbb{N} \vdash \mathbb{R}^n$$

can then be used to **extend** the context to a context of length two $[n : \mathbb{N}, v : \mathbb{R}^n]$. Types definable using any subset of the previously-defined variables

$$n : \mathbb{N}, v : \mathbb{R}^n \vdash T_v\mathbb{R}^n$$

may then extend the context further $[n : \mathbb{N}, v : \mathbb{R}^n, \ell : T_v\mathbb{R}^n]$. Fully elaborated, the first two judgments take the form

$$x_1 : A_1, x_2 : A_1(x_1), \ldots, x_n : A_n(x_1, \ldots, x_{n-1}) \vdash B(x_1, \ldots, x_n)$$
$$x_1 : A_1, x_2 : A_1(x_1), \ldots, x_n : A_n(x_1, \ldots, x_{n-1}) \vdash b(x_1, \ldots, x_n) : B(x_1, \ldots, x_n)$$

We may think of such judgments as functions that take as input a sequence of typed variables and produce either a type or a term as output.

The **structural rules** of dependent type theory govern the variables and the judgmental equality relations. For instance, the **substitution rule** takes the form

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}$$

where $\mathcal{J}$ can be the conclusion of any of the four judgment forms. This says that given any term $a : A$ defined in context Γ and any judgment whose context involves a variable $x : A$, there is a corresponding judgment that no longer involves a variable $x : A$ but instead substitutes the term a for x whenever that variable occurred.

The **variable rule** asserts the existence of projection functions

$$\frac{}{\Gamma, x : A, \Delta \vdash x : A}$$

that take any variable occuring in a context and providing it as a term of the corresponding type.

This base type theory then inherits a richer structure on account of various additional **logical rules** that can be used to form new types. These are called "logical rules" because the types themselves are used to encode mathematical statements, which are "grammatically correct" but are not necessarily provable or true. Such statements are then proven by constructing a term of the corresponding type.

For instance, given a context $\Gamma, x : A \vdash B(x)$ there is a **dependent pair type** and a **dependent function type** which in this case have the same **formation rules**:

$$\frac{\Gamma, x : A \vdash B(x)}{\Gamma \vdash \Sigma_{x:A} B(x)} \qquad \frac{\Gamma, x : A \vdash B(x)}{\Gamma \vdash \Pi_{x:A} B(x)}$$

These types are distinguished by their **introduction rules**, which provide terms:

$$\frac{\Gamma, x : A \vdash B(x)}{\Gamma, x : A, b(x) : B(x) \vdash (x, b(x)) : \Sigma_{x:A} B(x)} \qquad \frac{\Gamma, x : A \vdash B(x) \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x.b(x) : \Pi_{x:A} B(x)}$$

The **elimination rules** explain how generic terms $z : \Sigma_{x:A} B(x)$ and $f : \Pi_{x:A} B(x)$ can be used, while the **computation rules** describe the composites of the functions encoded by the introduction and elimination rules.

1.3. **Identity types.** Of the various logical rules that annimate dependent type theory, Martin-Löf's rules for identity types are of paramount importance.

The **formation rule**

$$\frac{}{\Gamma, x : A, y : A \vdash x =_A y}$$

says that it is meaningful to inquire whether two terms belonging to a common type may be identifiable. The **introduction rule**

$$\frac{}{\Gamma, a : A \vdash \mathrm{refl}_a : a =_A a}$$

guarantees that any term is automatically identifiable with itself, via "reflexivity."

The **elimination rule** provides a version of Leibniz' indiscernability of identicals:

$$\frac{\Gamma, x : A, y : A, p : x =_A y \vdash P(x, y, p) \qquad \Gamma, a : A \vdash d(a) : P(a, a, \mathrm{refl}_a)}{\Gamma, x : A, y : A, p : x =_A y \vdash J_d(x, y, p) : P(x, y, p)}$$

Here P can be be thought of as a predicate involving two terms of the same type and an identification between them, though the power of this rule derives from its applicability to arbitrary type families, not only the mere propositions. See [Ho, Ri] for a survey of its myriad consequences.

Finally, the **computation rule**

$$\frac{\Gamma, x : A, y : A, p : x =_A y \vdash P(x, y, p) \qquad \Gamma, a : A \vdash d(a) : P(a, a, \mathrm{refl}_a)}{\Gamma, a : A \vdash J_d(a, a, \mathrm{refl}_a) \equiv d(a) : P(a, a, \mathrm{refl}_a)}$$

is analogous to the computation rule for a recursively defined function.

Digression 1.3.1 (extensionality vs intensionality). By the structural rules for judgmental equality, if

$$\Gamma, x : A, y : A \vdash x \equiv y : A$$

then

$$\Gamma, x : A, y : A \vdash \mathrm{refl}_x : x =_A y$$

Thus judgmentally equal terms are always identifiable.

The **extensional** form of Martin-Löf's identity types adds a converse implication:

$$\Gamma, x : A, y : A, p : x =_A y \vdash x \equiv y : A$$

while the **intensional** form of Martin-Löf's identity types does not add this implication.

2. A categorical semantics of dependent type theory

2.1. **The category of contexts.** To interpret syntactic expressions in type theory as data in a category we must first build a category out of type theoretic syntax. A standard way to do this involves Cartmell's **contextual categories**.

Definition 2.1.1. Let $\mathbf{T}$ be a type theory with the standard structural rules: variable, substitution, weakening, and the usual rules concerning judgmental equality of types and terms. Then the **category of contexts** $\mathcal{C}(\mathbf{T})$ is a category in which:

- objects are contexts $[x_1 : A_1, \ldots, x_n : A_n]$ of arbitrary finite length up to definitional equality and renaming of free variables;
- maps of $\mathcal{C}(\mathbf{T})$ are **context morphisms** or **substitutions**, considered up to definitional equality and renaming of variables: a map

$$f = [f_1, \ldots, f_n] \colon [y_1 : B_1, \ldots, y_m : B_m] \to [x_1 : A_1, \ldots, x_n : A_n] \tag{2.1.2}$$

is an equivalent class of term judgements

$$\begin{array}{c} y_1 : B_1, \ldots, y_m : B_m \vdash f_1 : A_1 \\ \vdots \qquad \vdash \qquad \vdots \\ y_1 : B_1, \ldots, y_m : B_m \vdash f_n : A_n(f_1, \ldots, f_{n-1}) \end{array}$$

- composition is given by substitution and the identity is defined by using variables as terms.

The category of contexts has the following additional structures:

- $\mathcal{C}(\mathbf{T})$ has a terminal object corresponding to the empty context.
- The objects are partitioned according to the length of the context

$$\mathrm{ob}\,\mathcal{C}(\mathbf{T}) = \coprod_{n:\mathbb{N}} \mathrm{ob}_n \mathcal{C}(\mathbf{T})$$

  with the empty context being the only object of length 0.
- Each object of positive length has a **display map**

$$[x_1, \ldots, x_n] \colon [x_1 : A_1, \ldots, x_{n+1} : A_{n+1}] \to [x_1 : A_1, \ldots, x_n : A_n]$$

  namely the canonical projection away from the final type in the context. Writing Γ for the codomain context, we abbreviate this display map as

$$p_{A_{n+1}} \colon \Gamma.A_{n+1} \to \Gamma$$

  and refer the domain of any display map over Γ as a **context extension** of Γ. We refer to composites

$$\Gamma.A.B \xrightarrow{p_B} \Gamma.A \xrightarrow{p_A} \Gamma \qquad \text{or} \qquad \Gamma.\Delta \xrightarrow{p} \Gamma$$

  of display maps as **dependent projections**.
- For each substitution (2.1.2) and any context extension of its codomain, there is a **canonical pullback** defined by the context

$$[y_1 : B_1, \ldots, y_m : B_m, y_{m+1} : A_{n+1}(f_1(\vec{y}), \ldots, f_n(\vec{y}))]$$

  together with a canonical pullback square

$$\begin{tikzcd} \Delta.f^*A \arrow[r, "q(f)"] \arrow[d, "p_{f^*A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd}$$

  Moreover, these canonical pullbacks are *strictly stable under substitution*. Given any context morphism $g \colon \Theta \to \Delta$, $\Theta.g^* f^* A = \Theta.(f \cdot g)^* A$ and $q_f \cdot q_g = q_{f \cdot g}$; similarly $\Gamma.1^* A = \Gamma.A$.

Remark 2.1.3. A term $\Gamma \vdash a : A$ in a type A in context Γ is given by a section to the display map $p_A \colon \Gamma.A \to \Gamma$. Thus, the phrase "section" is typically reserved to mean section of a display map.

2.2. **Homotopy theoretic models of identity types.** The title of this section aludes to a famous paper of [AW] that described the homotopical semantics of identity types, but we instead focus on a result of Gambino and Garner inspired by it, constructing what they call the **identity type weak factorization system** on the category of contexts.

Theorem 2.2.1 ([GG])**.** *Let* $\mathbf{T}$ *be a dependent type theory with intensional identity types. Then the category of contexts* $\mathcal{C}(\mathbf{T})$ *admits a weak factorization system whose left class is comprised of those maps that lift against the display maps.*

In the proof, Gambino and Garner construct the factorization of an arbitrary context morphism, but the main idea is illustrated effectively by the simplest non-trivial case: factoring the identity map on a context $[x : A]$ of length 1 as

$$\begin{tikzcd} {[a : A]} \arrow[rr, equal] \arrow[dr, "{[a,a,\mathrm{refl}_a]}"'] & & {[a : A]} \\ & {[x : A, y : A, p : x =_A y]} \arrow[ur, "{[y]}"'] & \end{tikzcd}$$

The right factor is isomorphic in the category of contexts to a composite of display maps and hence belongs to the right class. To see that the left factor is in the left class, we must show it has the left lifting property with respect to an arbitrary display map $p_E \colon \Phi.E \to E$. By pulling back along the codomain of the lifting problem, we may assume that the codomain of this display map is the context $[x : A, y : A, p : x =_A y]$ presenting us with a lifting problem of the form below:

$$\begin{tikzcd}
{[a:A]} \arrow[r, "{[x,y,p,d(a)]}"] \arrow[d, "{[a,a,\mathrm{refl}_a]}"'] & {[x:A, y:A, p:x =_A y, z:P(x,y,p)]} \arrow[d, "p_P"] \\
{[x:A, y:A, p:x =_A y]} \arrow[r, equal] \arrow[ur, dashed, "{[x,y,p,J_d(x,y,p)]}"'] & {[x:A, y:A, p:x =_A y]}
\end{tikzcd}$$

By the elimination rule for identity types, we can use the data provided by the lifting problem to define the term

$$x : A, y : A, p : x =_A y \vdash J_d(x,y,p) : P(x,y,p)$$

required to define a section of the display map associated to the type family

$$x : A, y : A, p : x =_A y \vdash P(x,y,p).$$

The top triangle commutes by the computation rule.

Remark 2.2.2. Unusually, this factorization is not *functorial*. Given a commutative rectangle involving the identity functions between three contexts of length one:

$$\begin{tikzcd}
{[a:A]} \arrow[r, "g"] \arrow[d, equal] & {[b:B]} \arrow[r, "f"] \arrow[d, equal] & {[c:C]} \arrow[d, equal] \\
{[a:A]} \arrow[r, "g"] & {[b:B]} \arrow[r, "f"] & {[c:C]}
\end{tikzcd}$$

functoriality would demand that the induced substitutions between identity types commute. But as the maps induced by f, g, and $f \cdot g$ are defined by path induction (i.e., by the elimination rule of identity types) they only commute up to pointwise identification, not up to judgmental equality.

However, as these structures on the category of contexts are defined using the rules of type theory, they are strictly stable under substitution. The central components of the identity type weak factorization system equip the contextual category $\mathcal{C}(\mathbf{T})$ with what is called an "identity-type structure."

Definition 2.2.3. An **identity type structure** on a contextual category consists of:

- for each object $\Gamma.A$, an object $\Gamma.A.A.\mathrm{Id}_A$ extending the context defined by the canonical pullback:

$$\begin{tikzcd}
\Gamma.A.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\
\Gamma.A \arrow[r, "p_A"'] & \Gamma
\end{tikzcd}$$

- for each $\Gamma.A$, a morphism

$$\begin{tikzcd}
 & \Gamma.A.A.\mathrm{Id}_A \arrow[d, "p_{\mathrm{Id}_A}"] \\
\Gamma.A \arrow[r, "{(1,1)}"'] \arrow[ur, dashed, "\mathrm{refl}_A"] & \Gamma.A.A
\end{tikzcd}$$

  lifting the fibered diagonal; and
- for each $\Gamma.A.A.\mathrm{Id}_A.P$ and solid-arrow commutative diagram

$$\begin{tikzcd}
\Gamma.A \arrow[r, "d"] \arrow[d, "\mathrm{refl}_A"'] & \Gamma.A.A.\mathrm{Id}_A.P \arrow[d, "p_P"] \\
\Gamma.A.A.\mathrm{Id}_A \arrow[r, equal] \arrow[ur, dashed, "J"'] & \Gamma.A.A.\mathrm{Id}_A
\end{tikzcd}$$

  a section J of p_P so that $J \cdot \mathrm{refl}_A = d$

so that all this structure is strictly stable under pullback along any substitution $f\colon \Delta \to \Gamma$.

2.3. **Initiality.** A **contextual functor** is a functor between contextual categories preserving all of the structure on the nose. In particular, if the contextual categories have additional logical structure, such as the identity type structure just defined, or the corresponding structures for Σ- or Π-types, then the contextual functor must preserve this as well, up to equality.

The category of contexts is intended to be the universal contextual category built from a type theory in the sense made precise by the following conjecture:

Conjecture 2.3.1 (initiality conjecture)**.** *Let* $\mathbf{T}$ *be a type theory given by the standard structural rules together with some collection of the logical rules alluded to above. Then the category of contexts* $\mathcal{C}(\mathbf{T})$ *is the initial contextual category with the corresponding extra structure.*

Then further models of $\mathbf{T}$ in a category $\mathcal{E}$ can be defined by turning $\mathcal{E}$ into a contextual category with the appropriate logical structure: by initiality, this would induce a unique functor $\mathcal{C}(\mathbf{T}) \to \mathcal{E}$ providing an interpretation of the syntax of type theory in the category $\mathcal{E}$.

The difficulties with the initiality conjecture are multifaceted. On the one hand:

> The trouble with syntax is that it is very tricky to handle rigorously. Any full presentation must account for (among other complications) variable binding, capture-free substitution, and the possibility of multiple derivations of a judgment; and so any careful construction of an interpretation must deal with all of these, at the same time as tackling the details of the particular model in question. Contextual categories, by contrast, are a purely algebraic notion, with no such subtleties. [KL, 2075].

Another challenge is that there are many versions of type theory, determined by various collections of rules. Moreover, it's difficult to give a precise definition of what constitutes a type theory that is sufficiently general to cover all of the desired examples. An early version of initiality was proven for a type theory called the *calculus of constructions* by Streicher [Str1]. This has often been cited as evidence that initiality should be true for a much broader class of type theories, but at the time the simplicial "model" of homotopy type theory was being developed, Voevodsky argued that it was unacceptably unrigorous to assume initiality without proof.

Since [KL] was written, there was have been efforts to at least prove initiality for the "Book-HoTT" version of homotopy type theory appearing in [Ho]. This was achieved in a 2020 PhD thesis of Menno de Boer [dB] based on parallel formalization efforts undertaken by Brunerie and de Boer in Agda and Lumsdaine and Mörtberg in Coq.

Alternatively, the question of what constitutes a type theory can be answered in such a way as to render initiality automatic. Many practitioners equate type theories with their corresponding suitably structured categories, whether these be *contextual categories*, or *comprehension categories*, or *categories with families*, or *natural models* or something else. Each of these notions defines an *essentially algebraic theory* and thus the category of all such admits an initial object. From that point of view, the only question is whether a particular syntax presents the initial object, and this may be viewed as a practical consideration moreso than a theoretic one as the utility of a particular syntactic presentation derives from its use in proofs.

2.4. **Contextual categories from universes.** Even assuming initiality, it seems daunting to construct a model of type theory in a category $\mathcal{E}$, for one must still define a contextual category with suitable structures for all the type forming operations. By analyzing the type-forming operations, one can predict what sort of category might be suitable. For instance, a Π-type structure on a contextual category requires an operation that takes a composable pair of display maps as below-left to a display map as below-right:

$$\Gamma.A.B \xrightarrow{p_B} \Gamma.A \xrightarrow{p_A} \Gamma \qquad \rightsquigarrow \qquad \Gamma.\Pi_A B \xrightarrow{p_{\Pi_A B}} \Gamma$$

This, together with the fact that contextual categories must admit pullbacks of display maps, suggests that it would be reasonable to restrict to categories $\mathcal{E}$ that are **locally cartesian closed**, meaning that for any

$f: \Delta \to \Gamma$ the composition functor has two right adjoints

$$\begin{tikzcd} \mathcal{E}_{/\Delta} \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathcal{E}_{/\Gamma} \arrow[l, "f^*"'] \end{tikzcd} \tag{2.4.1}$$

defined by pullback along f and pushforward along f, respectively. Note when $\mathcal{E}$ has a terminal object 1—as is required to model the empty context—this implies that $\mathcal{E} \cong \mathcal{E}_{/1}$ and indeed any slice $\mathcal{E}_{/\Gamma}$ is cartesian closed and has all finite limits.

Note, however, that the definition of a contextual category, implicitly given in Definition 2.1.1, requires strict stability of the canonical pullback squares. In addition, each logical structure, such as the identity type structure of Definition 2.2.3, requires the pullbacks to preserve everything on the nose, up to equality of objects. Thus, the more categorically-natural requirement, that pullbacks preserve the various logical structures up to isomorphism, is not enough.

This leads to a massive coherence problem. It's not generally possible to choose strictly functorial pullbacks in a category with pullbacks, though it is possible to replace a category with pullbacks by an equivalent category with strictly functorial pullbacks.[1] But even after this is achieved, there remains the task of ensuring strict stability of all the logical structures.

Voevodsky's approach to the coherence problem makes use of a "universe" in a category, defined as follows:

Definition 2.4.2. A **universe** in a category $\mathcal{E}$ consists of an object U, together with a morphism $\pi\colon \tilde{U} \to U$, and, for each map $A\colon \Gamma \to U$, a choice of pullback square

$$\begin{tikzcd} (\Gamma; A) \arrow[r, "q_A"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ \Gamma \arrow[r, "A"'] & U \end{tikzcd}$$

Given a sequence of maps $A\colon \Gamma \to U$, $B\colon (\Gamma; A) \to U$, we write $(\Gamma; A, B)$ for $((\Gamma; A); B)$.

Construction 2.4.3 ([V1])**.** Given a category $\mathcal{E}$ with a universe U and a terminal object 1, define a contextual category $\mathcal{E}_U$ as follows:

- The objects of $\mathcal{E}_U$ are finite lists of morphisms as follows:

$$\mathrm{ob}_n \mathcal{E}_U = \{(A_1, \ldots, A_n) \in (\mathrm{mor}\mathcal{E})^n \mid A_i\colon (1; A_1, \ldots, A_{i-1}) \to U, \forall i\}.$$

- The morphisms between a pair of objects are defined by

$$\mathcal{E}_U((B_1, \ldots, B_m), (A_1, \ldots, A_n)) := \mathcal{E}((1; B_1, \ldots B_m), (1; A_1, \ldots, A_n)).$$

- The terminal object is the empty sequence of morphisms.
- For any object $(A_1, \ldots, A_{n+1})$ of positive length, its display map is provided by the universe structure:

$$\begin{tikzcd} (1; A_1, \ldots, A_{n+1}) \arrow[r] \arrow[d, "p_{A_{n+1}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ (1; A_1, \ldots, A_n) \arrow[r, "A_{n+1}"'] & U \end{tikzcd} \tag{2.4.4}$$

- Finally the canonical pullback associated to a morphism $f: (B_1, \ldots, B_m) \to (A_1, \ldots, A_n)$ and a display map (2.4.4) is defined by factoring the chosen pullback square for the composite $A_{n+1} \cdot f$ through the pullback

[1] This can be achieved by applying the general techniques for replacing a fibration by a split fibration to the codomain-projection fibration $\mathrm{cod}\colon \mathcal{E}^{\mathbf{2}} \to \mathcal{E}$; see mathoverflow.net/questions/144619/can-we-always-make-a-strictly-functorial-choice-of-pullbacks-re-indexing for a discussion.

that defines the context extension $(A_1,\dots,A_{n+1})$:

$$\begin{tikzcd}
(1;B_1,\dots,B_m,A_{n+1}\cdot f) \arrow[r] \arrow[d, "p_{f^*A_{n+1}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (1;A_1,\dots,A_n,A_{n+1}) \arrow[r] \arrow[d, "p_{A_{n+1}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\
(1;B_1,\dots,B_m) \arrow[r, "f"'] & (1;A_1,\dots,A_n) \arrow[r, "A_{n+1}"'] & U
\end{tikzcd}$$

Indeed every small contextual category arises this way:

Proposition 2.4.5 ([V1]). *Let $\mathcal{C}$ be a small contextual category. Consider the universe U in the presheaf category $[\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et}]$ given by*

$$U(\Gamma) = \{\Gamma.A\} \qquad \tilde{U}(\Gamma) = \{\text{sections } s:\Gamma\to\Gamma.A\}$$

with the evident projection map and any choice of pullbacks. Then $[\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et}]_U$ is isomorphic, as a contextual category, to $\mathcal{C}$.

When the original category $\mathcal{E}$ is locally cartesian closed, it is possible to specify additional structure on the universe U that would equip the contextual category $\mathcal{E}_U$ with the corresponding logical structure. The idea is that this introduces the desired structure in the universal case. To describe an identity structure on a universe, we require the following notion.

Digression 2.4.6. Recall that for maps $i\colon A\to B$ and $f: X\to Y$ in $\mathcal{E}$, a **lifting operation** witnessing a lifting property as below-left is a section to the map below-right:

$$\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B \arrow[r, "y"'] \arrow[ur, dashed] & Y
\end{tikzcd}
\qquad
\mathcal{E}(B,X) \xrightarrow{(-\cdot i,\, f\cdot -)} \mathcal{E}(A,X)\times_{\mathcal{E}(A,Y)}\mathcal{E}(B,Y)$$

If $\mathcal{E}$ is cartesian closed, an **internal lifting operation** is a section to the map

$$X^B \xrightarrow{(-\cdot i,\, f\cdot -)} X^A \times_{Y^A} Y^B$$

We may obtain an analogous structure on the contextual category $\mathcal{C}_U$ built from a universe $\pi\colon \tilde{U}\to U$ as follows. Firstly, note that since all display maps are pullbacks of π it suffices to consider orthogonality against π in the slice over U.

Definition 2.4.7. An **identity structure** on a universe $\pi\colon \tilde{U}\to U$ consists of a map

$$\mathrm{Id}\colon \tilde{U}\times_U\tilde{U}\to U$$

together with a specified lift $r\colon \tilde{U}\to \mathrm{Id}^*\tilde{U}$ of the relative diagonal:

$$\begin{tikzcd}
 & \mathrm{Id}^*\tilde{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\
\tilde{U} \arrow[r, "{(1,1)}"'] \arrow[ur, dashed, "r"] & \tilde{U}\times_U\tilde{U} \arrow[r, "\mathrm{Id}"'] & U
\end{tikzcd}$$

together with an internal lifting operation J for r against $\pi\times U$ in $\mathcal{E}_{/U}$.

Theorem 2.4.8 ([V2]). *An identity structure on a universe U in $\mathcal{E}$ induces an identity type structure on the contextual category $\mathcal{E}_U$.*

Proof. An identity structure on U defines a factorization of the fibered diagonal $\pi\colon \tilde{U} \to U$, which pulls back to define the factorization required by an identity type structure on $\mathcal{E}_U$:

$$
\begin{tikzcd}
(\Gamma;A) \arrow[rrr, dotted] \arrow[dr, "\mathrm{refl}_A"] \arrow[dddr, two heads, "p_A"'] & & & \tilde U \arrow[dr, "r"] \arrow[dddr, two heads, "\pi"'] & & & & \\
& (\Gamma;A,A,\mathrm{Id}_A) \arrow[rrr, dotted] \arrow[dr, "p_{\mathrm{Id}_A}"] \arrow[dd, two heads] & & & \mathrm{Id}^*\tilde U \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \tilde U \arrow[dr, "\pi"] & \\
& & (\Gamma;A,A) \arrow[rrr, dotted] \arrow[dl, two heads] & & & \tilde U\times_U \tilde U \arrow[rr, "\mathrm{Id}"] \arrow[dl, two heads] & & U \\
& \Gamma \arrow[rrr, dotted, "A"'] & & & U & & &
\end{tikzcd}
$$

Internal homs in a locally cartesian closed category are stable under pullback: given $f\colon \Delta \to \Gamma$ and $A, X \in \mathcal{E}_{/\Gamma}$,

$$f^*\mathsf{map}_\Gamma(A,X) \cong \mathsf{map}_\Delta(f^*A, f^*X).$$

Consequently, an internal lifting operation in a slice pulls back to an define internal lifting operation between the pullbacks of the original maps. For any morphism $A\colon \Gamma \to U$ in $\mathcal{E}$, the map $\mathrm{refl}_A\colon (\Gamma;A) \to (\Gamma;A,A,\mathrm{Id}_A)$ is defined to be the pullback of $r \in \mathcal{E}_{/U}$ along $A\colon \Gamma \to U$, while $\pi \times U \in \mathcal{E}_{/U}$ pulls back to the map $\pi \times \Gamma \in \mathcal{E}_{/\Gamma}$, which is the "generic display map" in the slice over Γ. The internal lifting operation between r and $\pi \times U$ in $\mathcal{E}_{/U}$ then defines an internal lifting operation between refl_A and $\pi\times\Gamma$ in $\mathcal{E}_{/\Gamma}$, which in particular specifies a solution to any lifting problem in $\mathcal{E}$ of the form below-left and hence also below-center and below-right:

$$
\begin{tikzcd}
(\Gamma;A) \arrow[r] \arrow[d, "\mathrm{refl}_A"'] & \tilde U\times\Gamma \arrow[d, "\pi\times\Gamma"] \\
(\Gamma;A,A,\mathrm{Id}_A) \arrow[r, "{(P,p_A\cdot p_A\cdot p_{\mathrm{Id}})}"'] \arrow[ur, dashed] & U\times\Gamma
\end{tikzcd}
\quad
\begin{tikzcd}
(\Gamma;A) \arrow[r] \arrow[d, "\mathrm{refl}_A"'] & \tilde U\times\Gamma \arrow[r] \arrow[d, "\pi\times\Gamma"'] & \tilde U \arrow[d, "\pi"] \\
(\Gamma;A,A,\mathrm{Id}_A) \arrow[r] \arrow[rr, bend right, "P"'] \arrow[urr, dotted] & U\times\Gamma \arrow[r] & U
\end{tikzcd}
\quad
\begin{tikzcd}
(\Gamma;A) \arrow[r] \arrow[d, "\mathrm{refl}_A"'] & (\Gamma;A,A,\mathrm{Id}_A,P) \arrow[r] \arrow[d, "p_P"] & \tilde U \arrow[d, "\pi"] \\
(\Gamma;A,A,\mathrm{Id}_A) \arrow[r, equal] \arrow[ur, dashed] & (\Gamma;A,A,\mathrm{Id}_A) \arrow[r, "P"'] & U
\end{tikzcd}
$$

When interpreted in the contextual category $\mathcal{E}_U$, this right-hand lift provides specified solutions to the lifting problems in Definition 2.2.3. The construction of this lift, as the pullback of a universal specified lift to a map between objects defined as strict pullbacks in the contextual category $\mathcal{E}_U$, this data is strictly stable under substitution as required. ☐

Of course, it remains to construct a suitable universe in our locally cartesian closed category. In part II, we'll explain how this is done in the category of simplicial sets, leading to the simplicial model of univalent foundations.

Lecture II: The simplicial model of univalent foundations

3. A universal fibration

Recall that Voevodsky defined a **universe** in a category $\mathcal{E}$ to be a morphism $\pi\colon \tilde{U} \to U$ together with a chosen pullback along any $A\colon \Gamma \to U$. In the resulting contextual category, such a map encodes an extension of the context Γ by A with the chosen pullback of π defining the corresponding display map.

When $\mathcal{E}$ is the category of simplicial sets, the intuitions developed in §1.1 suggest that we would like the display maps to be Kan fibrations, which we refer to simply as "fibrations" whenever there is no competing notion in sight. Thus, we would like the universe to be a simplicial set U that classifies fibrations. By the Yoneda lemma, this suggests that U_n should be defined to be the set of fibrations over Δ^n, but there are two problems with this naïve construction:

(i) Size: We must restrict to "small" fibrations—whose fibers are κ-presentable for some regular cardinal κ—and then ensure that we are taking large sets of all such, rather than proper classes.
(ii) Coherence: The action of the simplicial operators $\alpha\colon \Delta^m \to \Delta^n$ by pullback is only functorial up to isomorphism, not on the nose.

We largely sweep the size issues under the rug—referring to a "*set* of *small* fibrations" without delving too deeply into its construction[2]—to reserve our attention for the coherence issues, which are quite interesting.

3.1. **What do fibrations form?** The coherence issues derive from the fact that chosen pullbacks are not typically strictly functorial but rather *pseudofunctorial*. Put more precisely, in a category $\mathcal{E}$ with pullbacks, there is a groupoid-valued pseudofunctor

$$\begin{tikzcd} \mathcal{E}^{\mathrm{op}} \arrow[r, "\mathbb{E}"] & \mathcal{GPD} \\ B \arrow[r, mapsto] & (\mathcal{E}_{/B})^{\cong} \arrow[d, "f^*"] \\ A \arrow[u, "f"] \arrow[r, mapsto] & (\mathcal{E}_{/A})^{\cong} \end{tikzcd}$$

that sends an object B to the groupoid of morphisms with codomain B and acts on arrows by pullback. The pseudofunctor $\mathbb{E}$ is called the **core of self-indexing** of $\mathcal{E}$, the "self-indexing" being the corresponding pseudofunctor $B \mapsto \mathcal{E}_{/B} : \mathcal{E}^{\mathrm{op}} \to \mathcal{CAT}$. When $\mathcal{E}$ has a pullback stable class of fibrations $\mathscr{F}$, they similarly assemble into a pseudofunctor

$$\begin{tikzcd} \mathcal{E}^{\mathrm{op}} \arrow[r, "\mathbb{F}"] & \mathcal{GPD} \\ B \arrow[r, mapsto] & (\mathscr{F}_{/B})^{\cong} \arrow[d, "f^*"] \\ A \arrow[u, "f"] \arrow[r, mapsto] & (\mathscr{F}_{/A})^{\cong} \end{tikzcd}$$

defined by restricting to the full subgroupoid $(\mathscr{F}_{/B})^{\cong} \subset (\mathcal{E}_{/B})^{\cong}$ of fibrations over B. For any regular cardinal κ, there are analogous pseudofunctors $\mathbb{F}_\kappa \hookrightarrow \mathbb{E}_\kappa$ defined by restricting to small fibrations or small maps as appropriate.

By the bicategorical Yoneda lemma, a small fibration $p\colon E \twoheadrightarrow B$ defines a pseudonatural transformation

$$\mathcal{E}(-,B) \xrightarrow{p} \mathbb{F}_\kappa.$$

We might imagine that the *universal fibration* $\pi\colon \tilde{U} \twoheadrightarrow U$ is a representing element, defining a pseudonatural isomorphism $\mathcal{E}(B,U) \cong \mathbb{F}_\kappa(B)$. But this can't be, at least not if $\mathcal{E}$ is very interesting:

- Since $\mathcal{E}$ is a 1-category $\mathcal{E}(B,U)$ is a set, while $\mathbb{F}_\kappa(B) := (\mathscr{F}_{/B})^{\cong,\kappa}$ is the groupoid of small fibrations over B.
- Even ignoring this, we can see that the same fibration might be classified by multiple maps to U and the same map might classify multiple fibrations:

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, two heads, "\pi"] \\ B \arrow[r, "\cong"] & B \arrow[r, "\ulcorner p \urcorner"'] & U \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[r, "\cong"] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, two heads, "\pi"] \\ B \arrow[r, equal] & B \arrow[r, "\ulcorner p \urcorner"'] & U \end{tikzcd}$$

In a contextual category built from a universe, the display maps arose as specified pullbacks of the universe: a type in context Γ was encoded by the data of a morphism $A\colon \Gamma \to U$. The corresponding display map was then the pullback axiomatized by the universe structure

$$\begin{tikzcd} (\Gamma; A) \arrow[r, "q_A"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ \Gamma \arrow[r, "A"'] & U \end{tikzcd}$$

This guarantees that each display map may be realized as some pullback of $\pi\colon \tilde{U} \to U$ but does not imply that such pullbacks are unique. Thus, we ask the universe to *weakly* classify the small fibrations, demanding that the pseudonatural transformation $\pi\colon \mathcal{E}(-,U) \to \mathbb{F}_\kappa$ associated to $\pi\colon \tilde{U} \to U$ is surjective. Actually, we demand a bit more.

[2] The original sources [KL] and [S] provide a precise accounting.

Definition 3.1.1. In a model category $\mathcal{E}$ with all objects cofibrant, a small fibration $\pi\colon \tilde{U} \to U$ is a **small fibration classifier** if the pseudonatural transformation

$$\pi\colon \mathcal{E}(-,U) \to \mathbb{F}_\kappa$$

is a trivial fibration, meaning that for all cofibrations $i\colon A \rightarrowtail B$ in $\mathcal{E}$ any strictly commutative square of pseudonatural transformations admits a lift:

$$\begin{tikzcd}
\mathcal{E}(-,A) \arrow[r, "h\circ-"] \arrow[d, "i\circ-"'] & \mathcal{E}(-,U) \arrow[d, "\pi"] \\
\mathcal{E}(-,B) \arrow[r, "p"'] \arrow[ur, dashed, "k"'] & \mathbb{F}_\kappa
\end{tikzcd}$$

The condition amounts to the following property: given any pair of pullback squares between small fibrations as below

$$\begin{tikzcd}
D \arrow[rr] \arrow[dr] \arrow[dd, two heads, "q"'] \arrow[ddrr, phantom, "\lrcorner", very near start] & & \tilde{U} \arrow[dd, two heads, "\pi"] \\
& E \arrow[ur, dashed] \arrow[dd, two heads, "p"' near start] \arrow[dr, phantom, "\lrcorner", very near start] & \\
A \arrow[rr, "h"' near end] \arrow[dr, tail, "i"'] & & U \\
& B \arrow[ur, dashed, "k"'] &
\end{tikzcd} \tag{3.1.2}$$

there exists an extension k of h along i factoring the back pullback square as a composite of pullbacks. This has been referred to as **strong gluing** by Angiuli and Sterling [AS], the **stratification property** by [Ste, 2.3.3], and **acyclicity** by Shulman [S1, S].

Remark 3.1.3. As our terminology suggests, a "fibration classifier" weakly classifies fibrations, at least when all objects are cofibrant. Taking $A = \emptyset$, (3.1.2) implies that any $p\colon E \twoheadrightarrow B$ is classified by a pullback square into $\pi\colon \tilde{U} \twoheadrightarrow U$.

We'll explain the further relevance of this property below but first note that we've entirely sidestepped the question of how a fibration classifier might be constructed. It is to that subject that we now turn.

3.2. **Hofmann-Streicher universes.** There is a general method, due to Hofmann and Streicher [HS1], that defines a weak classifier for small maps in any presheaf category. Rather than introduce general notation for presheaf categories, we work in the category $\mathcal{E} = \hat{\mathbb{\Delta}}$ of simplicial sets.

Let κ be an infinite regular cardinal,[3] and write $\mathcal{S}\mathrm{et}_\kappa \subset \mathcal{S}\mathrm{et}$ for a full subcategory containing at least one set of each cardinality $\lambda < \kappa$ and at most κ many sets of each cardinality. Write $\mathrm{s}\mathcal{S}\mathrm{et}^\kappa \subset \mathrm{s}\mathcal{S}\mathrm{et}$ for the category of κ-small simplicial sets. By regularity, a map $f\colon A \to B$ of simplicial sets lies in $\mathrm{s}\mathcal{S}\mathrm{et}^\kappa$ just when its base and fibers are κ-small.

The goal is to define $\pi\colon \tilde{V} \to V \in \mathcal{E} := \mathrm{s}\mathcal{S}\mathrm{et}$ so that any small map $p\colon E \to B$ arises as a pullback of π along some classifying map $\ulcorner p \urcorner\colon B \to V$. As noted above, it's insufficient to define V_n to be the set of κ-small maps over Δ^n because the action of a simplicial operators $\alpha\colon \Delta^m \to \Delta^n$, by pullback, is only

[3] As the title of their article suggests, Hofmann and Streicher prefer to work with an inaccessible cardinal [HS1], but for the purposes of obtaining a κ-small map classifier it suffices to use a regular cardinal κ larger than the cardinality of the morphisms in the indexing category.

pseudofunctorial. However, under the equivalence of categories[4]

$$\begin{tikzcd} \mathsf{s}\mathcal{S}\mathsf{et}^{\kappa}_{/\Delta^n} \arrow[d, "\alpha^*"'] \arrow[r, phantom, "\simeq"] & \mathcal{S}\mathsf{et}_{\kappa}^{\mathbb{\Delta}^{\mathrm{op}}_{/[n]}} \arrow[d, "-\cdot\alpha_!"] \\ \mathsf{s}\mathcal{S}\mathsf{et}^{\kappa}_{/\Delta^m} \arrow[r, phantom, "\simeq"] & \mathcal{S}\mathsf{et}_{\kappa}^{\mathbb{\Delta}^{\mathrm{op}}_{/[m]}} \end{tikzcd}$$

the pullback action is replaced by precomposition with the composition functor $\alpha_! \colon \mathbb{\Delta}_{/[m]} \to \mathbb{\Delta}_{/[n]}$, which is strictly functorial. Thus

$$V_n := \mathrm{ob}(\mathcal{S}\mathsf{et}_{\kappa}^{\mathbb{\Delta}^{\mathrm{op}}_{/[n]}})$$

defines a large simplicial set $V \in \mathsf{s}\mathcal{S}\mathsf{et}$. An element $A \colon \Delta^n \to V$ defines a small map $\pi_A \colon \int A \to \Delta^n$ whose fiber over $\alpha \in (\Delta^n)_m$ is the set A_α defined by the functor $A \colon \mathbb{\Delta}^{\mathrm{op}}_{/[n]} \to \mathcal{S}\mathsf{et}_\kappa$. This functor also provides specified pullback squares

$$\begin{tikzcd} \int A \cdot \alpha \arrow[r] \arrow[d, "\pi_{A\cdot\alpha}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \int A \arrow[d, "\pi_A"] \\ \Delta^m \arrow[r, "\alpha"'] & \Delta^n \end{tikzcd}$$

that are strictly functorial in $\mathbb{\Delta}^{\mathrm{op}}$.

The simplicial set $\tilde{V}$ is defined by

$$\tilde{V}_n := \mathrm{ob}(\mathcal{S}\mathsf{et}_{*,\kappa}^{\mathbb{\Delta}^{\mathrm{op}}_{/[n]}})$$

where the presheaves now take values in small *pointed* sets. Since the indexing category $\mathbb{\Delta}_{/[n]}$ has a terminal object $\mathrm{id}_{[n]}$, an element $(A, a) \colon \Delta^n \to \tilde{V}$ is equally given by the data of $A \colon \Delta^n \to V$ together with a global section $a \in A_{\mathrm{id}_{[n]}}$ or equivalently by a section as below-left.

$$\begin{tikzcd} \int A \arrow[d, "\pi_A"] \\ \Delta^n \arrow[u, dashed, bend left, "a"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \int A \arrow[r, "a"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{V} \arrow[d, "\pi"] \\ \Delta^n \arrow[r, "A"'] & V \end{tikzcd}$$

Writing $\pi \colon \tilde{V} \to V$ for the evident forgetful functor, we see that any small $A \colon \Delta^n \to V$ with fits into a pullback as above-right where the top map is determined by an arbitrary choice of global section.

Note that even though V is large, the evident forgetful map $\pi \colon \tilde{V} \to V$ is κ-small since this means that the pullback to any κ-small object is κ-small. We prove that $\pi \colon \tilde{V} \to V$ weakly classifies small maps in $\mathcal{E} = \hat{\mathbb{\Delta}}$ with arbitrary codomain by proving that it has the acyclicity property (3.1.2) with respect to monomorphisms $i \colon A \rightarrowtail B$ and small maps over A and B.

Lemma 3.2.1 ([Ci, 3.9],[OP, 8.4]). *The Hofmann Streicher universe $\pi \colon \tilde{V} \to V$ defines a small map classifier:*

$$\begin{tikzcd} \mathcal{E}(-, A) \arrow[r, "h\circ-"] \arrow[d, "i\circ-"'] & \mathcal{E}(-, V) \arrow[d, "\pi"] \\ \mathcal{E}(-, B) \arrow[r, "p"'] \arrow[ur, dashed, "k"'] & \mathbb{E}_\kappa \end{tikzcd}$$

[4] More generally, the slice category over any presheaf X is equivalent to the category of presheaves on the category of elements of X. In particular, "$\mathbb{\Delta}^{\mathrm{op}}_{/[n]}$" should be read as an abbreviation for $(\mathbb{\Delta}_{/[n]})^{\mathrm{op}}$, the slice category $\mathbb{\Delta}_{/[n]}$ being the category of elements of the simplicial set Δ^n.

over the κ-small maps $\mathbb{E}_\kappa \subset \mathbb{E}$ in the core of self indexing. That is, given any pair of pullback squares between maps with κ-small fibers as below

$$\begin{tikzcd}
D \arrow[rr] \arrow[dr] \arrow[dd, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \tilde{V} \arrow[dd, "\pi"] \\
& E \arrow[ur, dashed] \arrow[dd, "p" near start] \arrow[dr, phantom, "\lrcorner", very near start] & \\
A \arrow[rr, "h" near end] \arrow[dr, tail, "i"'] & & V \\
& B \arrow[ur, dashed, "k"'] &
\end{tikzcd}$$

where $i \colon A \rightarrowtail B$ is a monomorphism, there exists an extension k of h along i factoring the back pullback square as a composite of pullbacks.

Proof. Since presheaf categories are locally cartesian closed, colimits in $\mathcal{E}$ are *universal*, preserved by pullback along any map such as $\pi^* \colon \mathcal{E}_{/V} \to \mathcal{E}_{/\tilde{V}}$. Since any simplicial set, such as B, is expresed canonically as a colimit of its simplices, this allows us to reduce to the case

$$\begin{tikzcd}
D \arrow[rr] \arrow[dr] \arrow[dd, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \tilde{V} \arrow[dd, "\pi"] \\
& E \arrow[ur, dashed] \arrow[dd, "p" near start] \arrow[dr, phantom, "\lrcorner", very near start] & \\
C \arrow[rr, "h" near end] \arrow[dr, tail, "i"'] & & V \\
& \Delta^n \arrow[ur, dashed, "k"'] &
\end{tikzcd}$$

where the monomorphism is given by a subcomplex of a simplex. Since E is a small map over Δ^n, we know that there is a classifying map $k' \colon \mathbb{\Delta}^{\mathrm{op}}_{/[n]} \to \mathcal{S}\mathrm{et}_{*,\kappa}$ with the property that for each $\alpha \colon [m] \to [n]$ in C, the restriction $k' \cdot \alpha \colon \mathbb{\Delta}^{\mathrm{op}}_{/[m]} \to \mathcal{S}\mathrm{et}_{*,\kappa}$ is naturally isomorphic to $h \cdot \alpha \colon \mathbb{\Delta}^{\mathrm{op}}_{/[m]} \to \mathcal{S}\mathrm{et}_{*,\kappa}$. Using excluded middle, we define $k \colon \mathbb{\Delta}^{\mathrm{op}}_{/[n]} \to \mathcal{S}\mathrm{et}_{*,\kappa}$ by

$$k(\alpha) = \begin{cases} (h \cdot \alpha)_{\mathrm{id}_{[m]}} & \alpha \colon [m] \to [n] \in S \\ k'_\alpha & \alpha \colon [m] \to [n] \notin S \end{cases}$$

transporting the functorial action in k' along the isomorphism to k. This defines an extension of the classifying square for q to a classifying square for p, as desired. □

4. The simplicial model of univalent foundations

The technique of Hofmann-Streicher universes can be used to define a universal Kan fibration, which is the key ingredient in the simplicial model of univalent foundations. This is not the approach taken to defining the universe in [KL] but was quickly noted as alternative possible route; see [Ci] or [Str2].

4.1. **A Kan fibration classifier.** The Hofmann-Streicher universe

$$\tilde{V}_n := \mathrm{ob}(\mathcal{S}\mathrm{et}_{*,\kappa}^{\mathbb{\Delta}^{\mathrm{op}}_{/[n]}}) \qquad V_n := \mathrm{ob}(\mathcal{S}\mathrm{et}_{\kappa}^{\mathbb{\Delta}_{/[n]}})$$

may be restricted to define the universal Kan fibration by specifying

$$U_n := \left\{ A \in \mathrm{ob}(\mathcal{S}\mathrm{et}_{\kappa}^{\mathbb{\Delta}^{\mathrm{op}}_{/[n]}}) \;\middle|\; \begin{array}{c} \int A \\ \big\downarrow {\scriptstyle \pi_A} \\ \Delta^n \end{array} \text{ is a Kan fibration} \right\},$$

and by defining $\pi\colon \tilde{U} \to U$ to be the pullback

(4.1.1)
$$\begin{tikzcd} \tilde{U} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{V} \arrow[d, "\pi"] \\ U \arrow[r, hook] & V \end{tikzcd}$$

Crucially, the Kan fibrations of simplicial sets are **local**: a map $p\colon E \twoheadrightarrow B$ is a Kan fibration if and only if for all n and all $b\colon \Delta^n \to B$, the pullback defines a Kan fibration. This follows from the fact that the fibrations are characterized by a right lifting property against maps with representable codomains:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, tail, "\wr"'] & \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, two heads, "p"] \\ \Delta^n \arrow[r, equal] \arrow[ur, dashed] & \Delta^n \arrow[r, "b"'] & B \end{tikzcd}$$

Consequently:

Corollary 4.1.2. *$\pi\colon \tilde{U} \twoheadrightarrow U$ is a Kan fibration.*

Proof. By construction, each pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{U} \arrow[d, two heads, "\pi"] \\ \Delta^n \arrow[r, "\ulcorner p \urcorner"'] & U \end{tikzcd}$$

is a Kan fibration. ◻

Another reflection of this locality is the following:

Lemma 4.1.3. *Let $p\colon E \to B$ be a small map and consider any classifying square*

$$\begin{tikzcd} E \arrow[r] \arrow[d, two heads, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{V} \arrow[d, "\pi"] \\ B \arrow[r, "\ulcorner p \urcorner"'] & V \end{tikzcd}$$

Then p is a Kan fibration if and only if the classifying square factors through (4.1.1).

Proof. If p is a pullback of $\pi\colon \tilde{U} \twoheadrightarrow U$ it is clearly a Kan fibration. Conversely, if p is a Kan fibration then so is its restriction along any $b\colon \Delta^n \to B$. Recall U is defined as a simplicial subset of V, so $\ulcorner p \urcorner\colon B \to V$ factors through $U \hookrightarrow V$ just when each for each $b\colon \Delta^n \to B$ the corresponding map $\ulcorner p \urcorner \cdot b\colon \Delta^n \to V$ so factors. But since a Kan fibration pulls back to Kan fibrations, this means the corresponding elements are in the simplicial subset. ◻

Put more abstractly, locality provides a pullback square

$$\begin{tikzcd} \mathcal{E}(-, U) \arrow[r, hook] \arrow[d, "\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E}(-, V) \arrow[d, "\pi"] \\ \mathbb{F}_\kappa \arrow[r, hook] & \mathbb{E}_\kappa \end{tikzcd}$$

in the category of groupoid-valued pseudofunctors and pseudonatural transformations between them. Thus the acyclicity condition of Definition 3.1.1 follows from the corresponding result for Hofmann-Streicher universes

Corollary 4.1.4. *The Kan fibration $\pi\colon \tilde{U} \to U$ is a κ-small Kan fibration classifier.*

Proof. For any lifting problem as presented by the left-hand square, the dashed lift exists by Lemma 3.2.1.

$$\begin{tikzcd} \mathcal{E}(-,A) \arrow[r] \arrow[d, "i\circ -"'] & \mathcal{E}(-,U) \arrow[r, hook] \arrow[d, "\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E}(-,V) \arrow[d, "\pi"] \\ \mathcal{E}(-,B) \arrow[r] \arrow[ur, dotted] \arrow[urr, dashed] & \mathbb{F}_\kappa \arrow[r, hook] & \mathbb{E}_\kappa \end{tikzcd}$$

On account of the pullback established by Lemma 4.1.3, this induces the required dotted lift. □

Since the cofibrations in the category of simplicial sets are the monomorphisms, all simplicial sets are cofibrant. Thus taking $A = \emptyset$ we see that:

Corollary 4.1.5. *Any Kan fibration $p\colon E \twoheadrightarrow B$ with small fibers is classified by a pullback square*

$$\begin{tikzcd} E \arrow[r] \arrow[d, two heads, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{U} \arrow[d, two heads, "\pi"] \\ B \arrow[r, "\ulcorner p \urcorner"'] & U \end{tikzcd} \quad \square$$

4.2. **Fibrancy of the universe.** In the simplicial model of univalent foundations, types are meant to correspond to Kan complexes. One use of the acyclity property is in proving that U is a Kan complex. Recall that Kan complexes of simplicial sets are characterized by right lifting against the **acyclic cofibrations**, monomorphisms that also define weak homotopy equivalences

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, tail, "j"', "\wr"] & U \\ B \arrow[ur, dashed] \end{tikzcd}$$

If the universe U strictly classified small fibrations, this would unravel to the following:

Definition 4.2.1. A model category $\mathcal{E}$ satisfy the **fibration extension property** if for all small fibrations $q\colon D \twoheadrightarrow A$ and all trivial cofibrations $j\colon A \overset{\sim}{\rightarrowtail} B$, there exists a small fibration $p\colon E \twoheadrightarrow B$ that pulls back along j to q.

$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d, two heads, "q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, dashed, two heads, "p"] \\ A \arrow[r, tail, "\sim", "j"'] & B \end{tikzcd}$$

Note that Definition 4.2.1 makes no reference to a particular universe U. Nevertheless, when $\pi\colon \tilde{U} \twoheadrightarrow U$ is a fibration classifier, satisfying the acyclicity property (3.1.2), the fibration extension property encodes the fibrancy of U.

Lemma 4.2.2. *Suppose $\pi\colon \tilde{U} \to U$ is a fibration classifier in a model category $\mathcal{E}$. Then U is fibrant if and only if $\mathcal{E}$ satisfies the fibration extension property.*

Proof. Suppose $\mathcal{E}$ satisfies the fibration extension property and consider a lifting problem:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, tail, "j"', "\wr"] & U \\ B \arrow[ur, dashed, "k"'] \end{tikzcd}$$

Define a small fibration $q\colon D \twoheadrightarrow A$ by pulling back π along h. Then use the fibration extension property to extend this to a small fibration $p\colon E \twoheadrightarrow B$ that pulls back along j to q. By acyclicity property, it follows that the classifying map h extends along j to a classifying map k for p so that $k \cdot j = h$, solving the lifting problem. This proves that the fibration extension property implies fibrancy of the universe.

Conversely, if U is fibrant and we are given a small fibration $q\colon D \twoheadrightarrow A$, we may use Corollary 4.1.5 to choose a classifying pullback square. In particular, this choice defines a classifying map and thus a lifting problem

$$\begin{tikzcd} A \arrow[r, "\ulcorner q \urcorner"] \arrow[d, tail, "j"', "\wr"] & U \\ B \arrow[ur, dashed, "\ulcorner p \urcorner"'] \end{tikzcd}$$

which admits a solution. The pullback of π along this map, displayed below-right, defines a small fibration over B. The pullback square for q factors through the one for p defining the desired extension square:

$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left] & E \arrow[r] \arrow[d, dashed, two heads, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, two heads, "\pi"] \\ A \arrow[r, tail, "j", "\sim"'] \arrow[rr, bend right, "\ulcorner q \urcorner"'] & B \arrow[r, "\ulcorner p \urcorner"] & U \end{tikzcd}$$

∎

Thus, to show that U is a Kan complex, we need only verify the fibration extension property in the category of simplicial sets. Since the Kan complexes are detected by right lifting against the simplicial horn inclusions $\Lambda^n_k \overset{\sim}{\rightarrowtail} \Delta^n$, it suffices to consider extensions of fibrations along such maps, as the proof just given reveals.

Theorem 4.2.3. *The category of simplicial sets satisfies the fibration extension property.*

Proof. By Lemma 4.2.2, it suffices to extend a Kan fibration over a horn Λ^n_k to a Kan fibration over the simplex Δ^n in such a way that the latter pulls back to the former. To do so, we use a result of Quillen [Q2] to factor $q\colon E \twoheadrightarrow \Lambda^n_k$ as a trivial fibration followed by a minimal fibration. Since the base of the minimal fibration is contractible, it is isomorphic to a projection map $F \times \Lambda^n_k \to \Lambda^n_k$ for some Kan complex F and thus extends as displayed:

$$\begin{tikzcd} E \arrow[r, dashed, tail, "\sim"] \arrow[d, two heads, "\wr"] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, dashed, two heads, "\wr"] \\ F \times \Lambda^n_k \arrow[r, tail, "\sim"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & F \times \Delta^n \arrow[d, two heads] \\ \Lambda^n_k \arrow[r, tail, "\sim"] & \Delta^n \end{tikzcd}$$

Thus it remains only to extend the trivial fibration, but this can be achieved defining $E' \overset{\sim}{\twoheadrightarrow} F \times \Delta^n$ to be the pushforward of $E \overset{\sim}{\twoheadrightarrow} F \times \Lambda^n_k$ along the monomorphism $F \times \Lambda^n_k \rightarrowtail F \times \Delta^n$. Since monomorphisms are stable under pullback, the trivial fibrations are stable under pushforward. Finally, if a map is postcomposed with a monomorphism and then pulled back along that monomorphism, the result is isomorphic to the map we started with; thus, taking right adjoints, the pushforward along a monomorphism pulls back along that monomorphism to a map isomorphic to the one we started with, which identifies E as the pullback of E'. ∎

4.3. **Internal universes.** By Construction 2.4.3 a universe U in $\mathcal{E}$ gives rise to a contextual category $\mathcal{E}_{/U}$. An internal universe structure on U will provide a universe, in the sense of Definition 2.4.2, inside the category $\mathcal{E}_{/U}$.

Definition 4.3.1. An **internal universe** in U consists of arrows $u_0\colon 1 \to U$ and $i\colon U_0 \to U$ where U_0 is defined by the canonical pullback

$$\begin{tikzcd} U_0 \arrow[r] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ 1 \arrow[r, "u_0"'] & U \end{tikzcd}$$

Note that the map u_0 defines an object $(1; u_0) \in \mathcal{E}_U$ in the contextual category built from the universe U. At the same time, the canonical pullback along the map $i\colon U_0 \to U$ in $\mathcal{E}$ defines a square

$$\begin{tikzcd} \tilde{U}_0 \arrow[r] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ U_0 \arrow[r, "i"'] & U \end{tikzcd}$$

via which the universe U induces a universe structure on U_0 as an object in $\mathcal{E}$. Thus, we say that the internal universe U_0 is **closed under identity types** if it carries an identity structure, as in Definition 2.4.7, commuting with i.

Theorem 2.4.8 proves that an identity structure on U induces an identity type structure on $\mathcal{E}_U$. Moreover:

Theorem 4.3.2 ([V2])**.** *Suppose $\mathcal{E}$ has a universe U with an identity structure and an internal universe U_0 in U closed under identity types. Then $\mathcal{E}_U$ has an identity type structure as well as a universe à la Tarski that is closed under identity types.*

In the category of simplicial sets, we constructed a universe U_κ that is a κ-small fibration classifier, for any regular cardinal κ. Now suppose κ is an inaccessible cardinal and $\lambda < \kappa$ is also inaccessible. Since $\lambda < \kappa$, U_λ is itself κ-small and so is representable as a pullback along some $u_\lambda\colon 1 \to U_\kappa$ in the category of simplicial sets

$$\begin{tikzcd} U_\lambda \arrow[r] \arrow[d, "p_\lambda"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U}_\kappa \arrow[d, "\pi_\kappa"] \\ 1 \arrow[r, "u_\lambda"'] & U_\kappa \end{tikzcd}$$

By construction of these universes, there is a natural inclusion $i\colon U_\lambda \to U_\kappa$. In summary:

Theorem 4.3.3 ([KL, 2.3.4])**.** *The universe U_κ carries an identity type structure, while U_λ give an internal universe that is closed under identity types.*

As discussed in [KL, 2.3.4], the universe U_κ also bears structure corresponding to the other logical rules and U_λ is closed under the corresponding logical structure.

4.4. **Univalence.** We now explain the interpretation of the univalence axiom in a model category $\mathcal{E}$, for instance the model category of simplicial sets that presents the ∞-topos of spaces. Our exposition follows [S1], which explains univalence as follows:

> The univalence axiom, when interpreted in a model category, is a statement about a "universe object" U, which is fibrant and comes equipped with a fibration $\pi\colon \tilde{U} \twoheadrightarrow U$ that is generic, in the sense that any fibration with "small fibers" is a pullback of π. ...In homotopy theory, it would be natural to ask for the stronger property that U is a classifying space for small fibrations, i.e. that homotopy classes of maps $A \to U$ are in bijection with (rather than merely surjecting onto) equivalence classes of small fibrations over A. The univalence axiom is a further strengthening of this: it says that the path space of U is equivalent to the "universal space of equivalences" between fibers of π ...In particular, therefore, if two pullbacks of π are equivalent, then their classifying maps are homotopic. [S1]

The univalence axiom concerns the object $\mathsf{Eq}(\tilde{U}) \twoheadrightarrow U \times U$ of equivalences defined for the universal fibration $\pi\colon \tilde{U} \to U$. The idea is that a generalized element

$$\begin{tikzcd} & \mathsf{Eq}(\tilde{U}) \arrow[d, two heads] \\ B \arrow[ur, dashed, "\ulcorner \epsilon \urcorner"] \arrow[r, "(\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner)"'] & U \times U \end{tikzcd}$$

should represent a fibered equivalence $\epsilon\colon E_1 \simeq E_2$ between the fibrations over B classified by these maps.

We explain how to construct the simplicial set $\mathsf{Eq}(\tilde{U})$ via a construction that makes sense in any simplicially locally cartesian closed, simplicial model category $\mathcal{E}$ that is right proper and whose cofibrations are the monomorphisms. Recall from (2.4.1), that any morphism f in a locally cartesian closed category gives rise to an adjoint triple $\Sigma_f \dashv f^* \dashv \Pi_f$ defined respectively by composition, pullback, and pushforward. Since the cofibrations are the monomorphisms, pushforward along any map preserves trivial fibrations. Since the model category is right proper, pushforward along a fibration also preserves fibrations. Since the model category is simplicial, we may form the **fibered path space** factorization (1.1.1) of any fibration $p\colon E \twoheadrightarrow B$ between fibrant objects through the **fibered path space** P_BE defined by the pullback:

(4.4.1)
$$\begin{tikzcd}
E \arrow[rrr, tail, "\sim"] \arrow[dr, tail] \arrow[ddd, two heads, "p"'] & & & E^{\Delta^1} \arrow[r, two heads] \arrow[ddd, two heads, "p^{\Delta^1}"] & E \times E \arrow[ddd, two heads, "p\times p"] \\
& P_BE \arrow[urr, dotted, tail] \arrow[dr, two heads] \arrow[ddl, dotted, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & & & \\
& & E \times_B E \arrow[uurr, dotted, tail] \arrow[dll, dotted, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & & \\
B \arrow[rrr, tail, "\sim"] & & & B^{\Delta^1} \arrow[r, two heads] & B \times B
\end{tikzcd}$$

As noted above, this construction has the following properties:

Proposition 4.4.2. *For any fibration between fibrant objects $p\colon E \twoheadrightarrow B$, the natural maps*

$$E \overset{r}{\underset{\sim}{\rightarrowtail}} P_BE \xrightarrow{(e_0,e_1)} E \times_B E$$

give a factorization of the fibered diagonal map $(1,1)\colon E \to E \times_B E$ over B as a trivial cofibration followed by a fibration, and this is construction is weakly stable over B, meaning that the pullback along any $f : A \to B$ is again such a factorization.

Proof. Since $\mathcal{E}$ is a simplicial model category and $p\colon E \twoheadrightarrow B$ is a fibration between fibrant objects, the maps depicted as fibrations in (4.4.1) are all fibrations. The map r is a monomorphism, since it is the left factor of a monomorphism, and is a weak equivalence by the 2-of-3 property and right properness. For the pullback stability of this construction, it's helpful to note that $P_BE \cong \Delta^1 \pitchfork_B E$ is the simplicial cotensor of $p\colon E \twoheadrightarrow B$ by Δ^1 in the slice $\mathcal{E}_{/B}$. Simplicial cartesian closure demands that the adjunctions (2.4.1) are simplicially enriched, and in particular that the right adjoint $f^*\colon \mathcal{E}_{/B} \to \mathcal{E}_{/A}$ preserves these simplicial cotensors. $\square$

Digression 4.4.3. In homotopy type theory, the object of equivalences between types in the universe $\mathcal{U}$ is

$$\mathsf{Eq}(\mathcal{U}) := \Sigma_{(A,B):\mathcal{U}\times\mathcal{U}}\Sigma_{f:A\to B}\mathrm{isEquiv}(f)$$

where

$$\mathrm{isEquiv}(f) := \Sigma_{b:B}\mathrm{isContr}(Pf_b)$$

where

$$\mathrm{isContr}(C) := \Sigma_{c:C}\Pi_{x:C}\, c =_C x$$

and

$$Pf_b := \Sigma_{a:A} f a =_B b$$

is the homotopy fiber of f over b.

A fibration $p\colon E \twoheadrightarrow B$ is a weak equivalence if and only if it is a trivial fibration. Between cofibrant objects this is the case if and only if it admits a section $s\colon B \to E$ together with a homotopy $\gamma\colon sp \sim 1_E$ over B. Such data in particular chooses a center of contraction $s(b) \in E_b$ in the fiber over each $b\colon 1 \to B$ and also provides a contracting homotopy, proving the contractibility of the fibers of p.

The data of such a homotopy is encoded by a dashed lift

$$\begin{tikzcd} & P_BE \arrow[d, two heads] \\ E \arrow[ur, dashed, bend left, "\gamma"] \arrow[r, "{(sp,1_E)}"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \times_B E \arrow[d, two heads, "\pi_1"] \\ B \arrow[r, "s"'] & E \end{tikzcd}$$

Since the map $(sp, 1_E)$ is a pullback of s along π_1, this lift is equally encoded by a lift of $s\colon B \to E$ along $\Pi_{\pi_1} P_B E \twoheadrightarrow E$. And thus a section of the composite $\Sigma_p \Pi_{\pi_1} P_B E \twoheadrightarrow B$ provides the data of both the section s and the homotopy γ. Thus we define

$$\mathsf{isContr}_B(E) := \Sigma_p \Pi_{\pi_1} P_B E \ \in \ \mathcal{E}_{/B},$$

which is the B-indexed object internalizing the assertion that for each $b \in B$, the fiber E_b is contractible. By construction:

Lemma 4.4.4. *A fibration $p\colon E \twoheadrightarrow B$ between fibrant objects is a trivial fibration if and only if* $\mathsf{isContr}_B(E) \twoheadrightarrow B$ *has a section.* □

Now a map $f\colon E_1 \to E_2$ over B is a weak equivalence if and only if its replacement by a weakly equivalent fibration over E_2 is a trivial fibration. This replacement is the map $p\colon P_b f \twoheadrightarrow E_2$ defined by the pullback of the fibered path space of E_2 over B in the following diagram.

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[dr, dashed, tail, "e"', "\sim"] \arrow[ddr, bend right, "{\langle 1,f \rangle}"'] & E_2 \arrow[dr, tail, "\sim"] & \\ & P_B f \arrow[r] \arrow[d, two heads, "{\langle q,p \rangle}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_B E_2 \arrow[d, two heads] \\ & E_1 \times_B E_2 \arrow[r, "f \times 1"'] & E_2 \times_B E_2 \end{tikzcd}$$

Note by construction, the map $q\colon P_B f \overset{\sim}{\twoheadrightarrow} E_1$ is a trivial fibration, and thus its section $e\colon E_1 \overset{\sim}{\rightarrowtail} P_B f$ is a trivial cofibration.

By Lemma 4.4.4, the map $p\colon P_B f \twoheadrightarrow E_2$ is a trivial fibration if and only if $\mathsf{isContr}_{E_2}(P_B f) \twoheadrightarrow E_2$ has a section, or equivalently if and only if the pushforward along $p_2\colon E_2 \twoheadrightarrow B$ has a section over B. Thus we define

$$\mathsf{isEquiv}_B(f) := \Pi_{p_2} \mathsf{isContr}_{E_2}(P_B f) \ \in \ \mathcal{E}_{/B}.$$

By construction:

Lemma 4.4.5. *A map $f\colon E_1 \to E_2$ between fibrations over a fibrant object B is a weak equivalence if and only if* $\mathsf{isEquiv}_B(f) \twoheadrightarrow B$ *has a section.* □

Finally the slice category $\mathcal{E}_{/B}$ has an internal hom $\mathsf{map}_B(E_1, E_2)$. The counit $\epsilon\colon \mathsf{map}_B(E_1,E_2) \times_B E_1 \to E_2$ pulls back to define a map

$$\mu\colon \mathsf{map}_B(E_1,E_2) \times_B E_1 \to \mathsf{map}_B(E_1,E_2) \times_B E_2$$

over $\mathsf{map}_B(E_1, E_2)$. The map μ is the "universal map from E_1 to E_2 over B" in the sense that the fiber of μ over an element $f : X \to \mathsf{map}_B(E_1, E_2)$ over $b\colon X \to B$ is a map $f_b\colon b^*E_1 \to b^*E_2$ between the fibers over b.

$$\begin{tikzcd}
b^*E_1 \arrow[rr, dotted] \arrow[dr, dotted, "f_b"'] \arrow[dd, dotted, two heads] & & \mathsf{map}_B(E_1,E_2)\times_B E_1 \arrow[rr, two heads] \arrow[dr, dashed, "\mu"] \arrow[drrr, dashed, "\epsilon"] \arrow[dd] & & E_1 \arrow[dd, two heads] & \\
& b^*E_2 \arrow[rr, dotted] \arrow[dl, dotted, two heads] & & \mathsf{map}_B(E_1,E_2)\times_B E_2 \arrow[rr] \arrow[dl] & & E_2 \arrow[dl, two heads] \\
X \arrow[rr, "f"] \arrow[rrrr, bend right, "b"'] & & \mathsf{map}_B(E_1,E_2) \arrow[rr, two heads] & & B &
\end{tikzcd}$$

Thus we define

$$\mathsf{Equiv}_B(E_1, E_2) := \Sigma_{\mathsf{map}_B(E_1,E_2)}\mathsf{isEquiv}_{\mathsf{map}_B(E_1,E_2)}(\mu) \in \mathcal{E}_{/B}$$

so that a section provides a map $E_1 \to E_2$ over B that is an equivalence. More generally:

Lemma 4.4.6. *Given fibrations $p_1 : E_1 \twoheadrightarrow B$ and $p_2 : E_2 \twoheadrightarrow B$ between fibrant objects over B, any lift of $b\colon X \to B$ to $\mathsf{Equiv}_B(E_1, E_2)$ provides a map $b^*E_1 \to b^*E_2$ over X that is an equivalence.* □

Finally, given a universe $\pi\colon \tilde{U} \to U$ define

$$\mathsf{Eq}(\tilde{U}) := \mathsf{Equiv}_{U\times U}(\pi_1^*\tilde{U}, \pi_2^*\tilde{U}) \in \mathcal{E}_{/U\times U}.$$

By construction:

Lemma 4.4.7. *Given a fibrant object B and a map $(\ulcorner E_1\urcorner, \ulcorner E_2\urcorner)\colon B \to U \times U$ classifying fibrations $p_1 : E_1 \twoheadrightarrow B$ and $p_2 : E_2 \twoheadrightarrow B$ over B, a lift to $\mathsf{Eq}(\tilde{U})$ provides an equivalence $E_1 \simeq E_2$ over B.*

$$\begin{tikzcd}
& \mathsf{Eq}(\tilde{U}) \arrow[d, two heads] \\
B \arrow[ur, dashed, "\ulcorner\epsilon\urcorner"] \arrow[r, "(\ulcorner p_1\urcorner, \ulcorner o_2\urcorner)"'] & U \times U
\end{tikzcd}$$

Proof. By Lemma 4.4.6, a lift of $(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)\colon B \to U \times U$ to $\mathsf{Eq}(\tilde{U})$ provides map from $(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)^*\pi_1^*\tilde{U} \cong \ulcorner p_1\urcorner^*\tilde{U} \cong E_1$ to $(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)^*\pi_2^*\tilde{U} \cong \ulcorner p_2\urcorner^*\tilde{U} \cong E_2$ over B that is an equivalence. □

In particular, there is a lift

$$\begin{tikzcd}
& \mathsf{Eq}(\tilde{U}) \arrow[d, two heads] \\
U \arrow[ur, dashed, "\mathrm{id}"] \arrow[r, "(1,1)"'] & U \times U
\end{tikzcd}$$

classifying the identity equivalence $\tilde{U} \cong \tilde{U}$ over U. We may now state the model categorical version of Voevodksy's univalence axiom.

Definition 4.4.8. The **univalence axiom**, for the universal fibration $\pi\colon \tilde{U} \to U$, asserts that the comparison map defined by any solution to the lifting problem

$$\begin{tikzcd}
U \arrow[r, "\mathrm{id}"] \arrow[d, tail, "\mathrm{refl}"', "\wr"] & \mathsf{Eq}(\tilde{U}) \arrow[d, two heads] \\
U^{\Delta^1} \arrow[ur, dashed, "\text{id-to-equiv}"] \arrow[r, two heads] & U \times U
\end{tikzcd}$$

from the path object to the object of equivalences is an equivalence.

By the 2-of-3 property, id-to-equiv: $U^{\Delta^1} \to \mathsf{Eq}(\tilde{U})$ is a weak equivalence if and only if id: $U \to \mathsf{Eq}(\tilde{U})$ is a weak equivalence, which is the case if and only if either projection $\mathsf{Eq}(\tilde{U}) \to U$ is a trivial fibration:

$$\begin{tikzcd} A \arrow[r, "w"] \arrow[d, tail, "i"'] & \mathsf{Eq}(\tilde{U}) \arrow[d, two heads, "p_2"] \\ B \arrow[r, "\ulcorner D_2 \urcorner"'] \arrow[ur, dashed] & U \end{tikzcd}$$

As with the fibrancy of the universe, this can be expressed as a property of the model category $\mathcal{E}$ that does not refer explicitly to the universal Kan fibration $\pi\colon \tilde{U} \twoheadrightarrow U$.

Definition 4.4.9. The **equivalence extension property** states that given a cofibraton $i\colon A \rightarrowtail B$, an equivalence $w\colon E_1 \xrightarrow{\sim} E_2$ of small fibrations over A and a fibration D_2 over B that pulls back along i to E_2, there exists a small fibration D_1 over B and an equivalence $v\colon D_1 \xrightarrow{\sim} D_2$ over B that pulls back along i to the equivalence w.

$$\begin{tikzcd} E_1 \arrow[rrr, dashed] \arrow[dr, "w", "\sim"'] \arrow[ddr, two heads] \arrow[drrr, phantom, "\lrcorner", very near start] & & & D_1 \arrow[dr, dashed, "v", "\sim"'] \arrow[dd, dashed, two heads] & \\ & E_2 \arrow[rrr] \arrow[d, two heads] \arrow[drr, phantom, "\lrcorner", very near start] & & & D_2 \arrow[dl, two heads] \\ & A \arrow[rr, tail, "i"'] & & B & \end{tikzcd}$$

As with Lemma 4.2.2, there is a slightly delicate argument necessary to prove that the equivalence extension property implies univalence of the fibration $\pi\colon \tilde{U} \to U$ as stated in Definition 4.4.8. Suppose the equivalence extension property holds and consider a commutative square

$$\begin{tikzcd} A \arrow[r, "w"] \arrow[d, tail, "i"'] & \mathsf{Eq}(\tilde{U}) \arrow[d, two heads, "p_1"] \\ B \arrow[r, "D_2"'] \arrow[ur, dashed, "v"'] & U \end{tikzcd}$$

By the equivalence extension property, $w\colon E_1 \simeq E_2$ extends to an equivalence $v\colon D_1 \simeq D_2$ between fibrations over B. By the acyclicity property, there is a classifying map for D_1 extending the classifying map for E_1.

$$\begin{tikzcd} E_1 \arrow[rr] \arrow[dr] \arrow[dd, two heads, "p_1"'] \arrow[ddr, phantom, "\lrcorner", very near start] \arrow[drr, phantom, "\lrcorner", very near start] & & \tilde{U} \arrow[dd, two heads, "\pi"] \\ & D_1 \arrow[ur, dashed] \arrow[dd, two heads, "q_1", near start] \arrow[ddr, phantom, "\lrcorner", very near start] & \\ A \arrow[rr, "p_0 w", near end] \arrow[dr, tail, "i"'] & & U \\ & B \arrow[ur, dashed, "D_1"'] & \end{tikzcd}$$

This defines a new lifting problem as below-left together with the data of an equivalence $v\colon D_1 \simeq D_2$ over B extending $w\colon E_1 \simeq E_2$. Unraveling the definitions, this gives us the lifting problem below-center and then below-right:

$$\begin{tikzcd} A \arrow[r, "w"] \arrow[d, tail, "i"'] & \mathsf{Eq}(\tilde{U}) \arrow[d, two heads] \\ B \arrow[r, "{(D_1, D_2)}"'] & U \times U \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "w = i^* v"] \arrow[d, tail, "i"'] & \mathsf{Equiv}_B(D_1, D_2) \arrow[d, two heads] \\ B \arrow[r, "v"'] & \mathsf{map}_B(D_1, D_2) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "w"] \arrow[d, tail, "i"'] & \mathsf{isEquiv}_B(v) \arrow[d, two heads] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Since v is a fibration, the map $\mathsf{isEquiv}_B(v) \twoheadrightarrow B$ is a trivial fibration.[5] so there exists a lift in the right-hand square which induces one in the middle square and then in the left-hand square. In particular, this final lift solves the original lifting problem, proving univalence.

Thus, we have reduced the verification of the univalence axiom to proving the equivalence extension property for Kan fibrations of simplicial sets. We define D_1 and v by the pullback

$$\begin{tikzcd} D_1 \arrow[r, dashed] \arrow[d, dashed, "v"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Pi_i E_1 \arrow[d, "\Pi_i w"] \\ D_2 \arrow[r, "\eta"'] & \Pi_i E_2 \end{tikzcd}$$

and leave the rest of the verification to the original sources [KL, S1].

Remark 4.4.10. Note that fibrancy of U was not required to prove the equivalence extension property, and in fact follows from it by an argument due to Stenzel [Ste, 2.4.3]. We must construct a lift

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, tail, "i"', "\wr"] & U \\ B \arrow[ur, dashed, "k"'] & \end{tikzcd}$$

that is, we must extend a fibration as indicated:

$$\begin{tikzcd} E_1 \arrow[r, dashed] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D_1 \arrow[d, dashed, two heads] \\ A \arrow[r, tail, "\sim", "j"'] & B \end{tikzcd}$$

Factor the composite $j \cdot p_1$ into an acyclic cofibration followed by a fibration and pull the fibration back:

$$\begin{tikzcd} E_1 \arrow[dr, dotted] \arrow[drr, tail, bend left, "\sim"] \arrow[ddr, two heads, bend right, "p_1"'] & & \\ & E_2 \arrow[r, tail, "\sim"] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & D_2 \arrow[d, two heads] \\ & A \arrow[r, tail, "\sim", "j"'] & B \end{tikzcd}$$

By right properness, j pulls back to a weak equivalence and thus the induced map to the pullback is an equivalence between fibrations over A. Now the equivalence extension property can be used to extend the fibration E_1 as desired.

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