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Path induction and the indiscernibility of identicals

Algebraic Topology and Topological Data Analysis:
A Conference in Honor of Gunnar Carlsson

Plan

1. Induction over the natural numbers
2. Dependent type theory
3. Identity types
4. Path induction
5. Epilogue: what justifies path induction?

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Main takeaway:

- natural numbers induction: the natural numbers are freely generated by zero and the successor function
- path induction (substitution for equality): the identity type family is freely generated by the reflexivity proof

Ask me afterwards:

- arrow induction (Yoneda lemma): the hom type family is freely generated by the identity arrow

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Induction over the natural numbers

Peano's postulates

In Dedekind's 1888 book "Was sind und was sollen die Zahlen" and Peano's 1889 paper "Arithmetices principia, nova methodo exposita," the natural numbers $\mathbb{N}$ are characterized by:

- There is a natural number $0 \in \mathbb{N}$.
- Every natural number $n \in \mathbb{N}$ has a successor $\mathrm{succ}(n) \in \mathbb{N}$.
- 0 is not the successor of any natural number.
- No two natural numbers have the same successor.
- The principle of mathematical induction:

$$\forall P, P(0) \to (\forall k \in \mathbb{N}, P(k) \to P(\mathrm{succ}(k))) \to (\forall n \in \mathbb{N}, P(n))$$

By Dedekind's categoricity theorem, all triples given by a set $\mathbb{N}$, an element $0 \in \mathbb{N}$, and a function $\mathrm{succ} : \mathbb{N} \to \mathbb{N}$ satisfying the Peano postulates are isomorphic.

Natural numbers induction

In the statement of the principle of mathematical induction:

$$\forall P, P(0) \to (\forall k \in \mathbb{N}, P(k) \to P(\mathrm{succ}(k))) \to (\forall n \in \mathbb{N}, P(n))$$

the variable P is a predicate over the natural numbers.

A predicate over the natural numbers is a function

$$P : \mathbb{N} \to \{\top, \bot\}$$

that associates a truth value $\top$ or $\bot$ to each $n \in \mathbb{N}$.

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Thus, to prove a sentence of the form $\forall n \in \mathbb{N}, P(n)$ it suffices to:

- prove the base case, showing that $P(0)$ is true, and
- prove the inductive step, showing for each $k \in \mathbb{N}$ that $P(k)$ implies $P(\mathrm{succ}(k))$.

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- In the base case, when $n = 0$, $0^2 + 0 = 2 \times 0$, which is even.
- For the inductive step, assume for $k \in \mathbb{N}$ that $k^2 + k = 2 \times m$ is even. Then

$$
\begin{aligned}
(k+1)^2 + (k+1) &= (k^2 + k) + ((2 \times k) + 2) \\
&= (2 \times m) + (2 \times (k+1)) \\
&= 2 \times (m + k + 1) \qquad \text{is even.}
\end{aligned}
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By the principle of mathematical induction

$$\forall P, P(0) \to (\forall k \in \mathbb{N}, P(k) \to P(\mathrm{succ}(k))) \to (\forall n \in \mathbb{N}, P(n))$$

this proves that $n^2 + n$ is even for all $n \in \mathbb{N}$. $\square$

A construction by induction

The induction proof not only demonstrates for all $n \in \mathbb{N}$ that $n^2 + n$ is even but also defines a function $m : \mathbb{N} \to \mathbb{N}$ so that $n^2 + n = 2 \times m(n)$.

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$$\begin{aligned}(k+1)^2 + (k+1) &= (k^2 + k) + ((2 \times k) + 2)\\ &= (2 \times m(k)) + (2 \times (k+1))\\ &= 2 \times (m(k) + k + 1)\end{aligned}$$

  so we define $m(k+1) := m(k) + k + 1$.

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Construction: By induction on $n \in \mathbb{N}$:

- In the base case, $0^2 + 0 = 2 \times 0$, so we define $m(0) \coloneqq 0$.
- For the inductive step, assume for $k \in \mathbb{N}$ that $k^2 + k = 2 \times m(k)$. Then

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  so we define $m(k+1) \coloneqq m(k) + k + 1$.

By the principle of mathematical recursion, this defines a function $m : \mathbb{N} \to \mathbb{N}$ so that $n^2 + n = m(n)$ for all $n \in \mathbb{N}$.

Induction and recursion

Recursion can be thought of as the constructive form of induction

$$\forall P, P(0) \to (\forall k \in \mathbb{N}, P(k) \to P(\mathrm{succ}(k))) \to (\forall n \in \mathbb{N}, P(n))$$

in which the predicate

$$P : \mathbb{N} \to \{\top, \bot\} \quad \text{such as} \quad P(n) := \exists m \in \mathbb{N}, n^2 + n = 2 \times m$$

is replaced by an arbitrary family of sets

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The output of a recursive construction is a dependent function $p \in \prod_{n \in \mathbb{N}} P(n)$ which specifies a value $p(n) \in P(n)$ for each $n \in \mathbb{N}$.

$$\forall P, (p_0 \in P(0)) \to (p_s \in \prod_{k \in \mathbb{N}} P(k) \to P(\mathrm{succ}(k))) \to (p \in \prod_{n \in \mathbb{N}} P(n))$$

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The recursive function $p \in \prod_{n \in \mathbb{N}} P(n)$ satisfies computation rules:

$$p(0) := p_0 \qquad p(\mathrm{succ}(n)) := p_s(n, p(n)).$$

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- For any family of types $P : \mathbb{N} \to \mathsf{Type}$ there is a term

$$\mathbb{N}\text{-ind} : (p_0 : P(0)) \to (p_s : \prod_{k \in \mathbb{N}} P(k) \to P(\mathsf{succ}(k))) \to (p : \prod_{n \in \mathbb{N}} P(n))$$

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Note the final two postulates — that 0 is not a successor and succ is injective — are missing because they are provable.

Summary

We summarize the rules

- There is a type $\mathbb{N}$.
- There is a term $0 : \mathbb{N}$ and a function $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$.
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with the slogan:

The natural numbers type $\mathbb{N}$ is freely generated by the terms $0 : \mathbb{N}$ and $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$.

2

Dependent type theory

Types, terms, and contexts

Dependent type theory is a formal system for mathematical statements and proofs that has the following primitive notions:

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Types and terms can be defined in an arbitrary context of variables from previously-defined types, all of which are listed before the symbol "$\vdash$". Here Γ is shorthand for a generic context, which has the form

$$x_1 : A_1, x_2 : A_2(x_1), x_3 : A_3(x_1, x_2), \ldots, x_n : A_n(x_1, \ldots, x_{n-1})$$

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In a mathematical statement of the form "Let …be …then …" The stuff following the "let" likely declares the names of the variables in the context described after the "be", while the stuff after the "then" most likely describes a type or term in that context.

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- identity types $x, y : A \vdash x =_A y$

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Each type constructor comes with rules:

(i) formation: a way to construct new types

(ii) introduction: ways to construct terms of these types

(iii) elimination: ways to use them to construct other terms

(iv) computation: the way (ii) and (iii) relate

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The rules suggest a logical naming for certain types:

$A \times B$	"A and B"	$\sum_{x:A} B(x)$	"$\exists x.B(x)$"
$A + B$	"A or B"	$\prod_{x:A} B(x)$	"$\forall x.B(x)$"
$A \to B$	"A implies B"	$x =_A y$	"x equals y"

Product types and function types

Product types are governed by the rules

$\times$-form: given types A and B there is a type $A \times B$

$\times$-intro: given terms $a : A$ and $b : B$ there is a term $(a, b) : A \times B$

$\times$-elim: given $p : A \times B$ there are terms $\mathrm{pr}_1 p : A$ and $\mathrm{pr}_2 p : B$

plus computation rules that relate pairings and projections.

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Function types are governed by the rules

$^{\rightarrow}$-form: given types A and B there is a type $A \rightarrow B$

$^{\rightarrow}$-intro: if in the context of a variable $x : A$ there is a term $b_x : B$
there is a term $\lambda x.b_x : A \rightarrow B$

$^{\rightarrow}$-elim: given terms $f : A \rightarrow B$ and $a : A$ there is a term $f(a) : B$

plus computation rules that relate λ-abstractions and evaluations.

Mathematics in dependent type theory

${}^{\times}$-form: $A,\ B \rightsquigarrow A \times B$	${}^{\rightarrow}$-form: A and $B \rightsquigarrow A \rightarrow B$
${}^{\times}$-intro: $a : A,\ b : B \rightsquigarrow (a, b) : A \times B$	${}^{\rightarrow}$-intro: $x : A \vdash b_x : B \rightsquigarrow \lambda x.b_x : A \rightarrow B$
${}^{\times}$-elim: $p : A \times B \rightsquigarrow \mathrm{pr}_1 p : A,\ \mathrm{pr}_2 p : B$	${}^{\rightarrow}$-elim: $f : A \rightarrow B,\ a : A \rightsquigarrow f(a) : B$

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To prove a mathematical proposition in dependent type theory, one constructs a term in the type that encodes its statement.

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To prove a mathematical proposition in dependent type theory, one constructs a term in the type that encodes its statement.

Proposition. For any types A and B, $\text{modus-ponens} : (A \times (A \rightarrow B)) \rightarrow B$.

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$^{\times}$-elim: $p : A \times B \rightsquigarrow \mathrm{pr}_1 p : A$, $\mathrm{pr}_2 p : B$

$^{\to}$-form: A and $B \rightsquigarrow A \to B$
$^{\to}$-intro: $x : A \vdash b_x : B \rightsquigarrow \lambda x.b_x : A \to B$
$^{\to}$-elim: $f : A \to B$, $a : A \rightsquigarrow f(a) : B$

To prove a mathematical proposition in dependent type theory, one constructs a term in the type that encodes its statement.

Proposition. For any types A and B, modus-ponens $: (A \times (A \to B)) \to B$.

Construction: By $^{\to}$-intro, it suffices to assume given a term $p : (A \times (A \to B))$ and define a term of type B. By $^{\times}$-elim, p provides terms $\mathrm{pr}_1 p : A$ and $\mathrm{pr}_2 p : A \to B$. By $^{\to}$-elim, these combine to give a term $\mathrm{pr}_2 p(\mathrm{pr}_1 p) : B$. Thus we have

$$\lambda p.\mathrm{pr}_2 p(\mathrm{pr}_1 p) : (A \times (A \to B)) \to B. \quad \square$$

The natural numbers type, revisited

The natural numbers type is governed by the rules:

$\mathbb{N}$-form: $\mathbb{N}$ exists in the empty context

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Summary: the natural numbers type $\mathbb{N}$ is freely generated by $0 : \mathbb{N}$ and $\mathrm{succ} : \mathbb{N} \to \mathbb{N}$.

3

Identity types

The traditional view of equality

In first order logic, the binary relation "$=$" is governed by the following rules:

- Reflexivity: $\forall x,\ x = x$.
- Indiscernibility of Identicals:

  $\forall x, y,\ x = y$ implies that for all predicates P, $P(x) \leftrightarrow P(y)$

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As a consequence of these rules:

Principle of substitution: To prove that every x, y with $x = y$ has property $P(x, y)$:

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The following rules for identity types were developed by Martin-Löf:

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Summary: the identity type family is freely generated by the reflexivity terms.

The homotopical interpretation of dependent type theory

Note that identity types can be iterated:

$$\text{given } x, y : A \text{ and } p, q : x =_A y \text{ there is a type } p =_{x =_A y} q.$$

Does this type always have a term? In other words, are identity proofs unique?

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The total space of the identity type family $\sum_{x,y:A} x =_A y$ is interpreted as the path space of A and a term $p : x =_A y$ may be thought of as a path from x to y in A.

$$\begin{tikzcd} & \sum_{x,y:A} x =_A y \arrow[d, "\mathrm{pr}_1"] \\ A \arrow[ur, dashed, "\lambda x.\mathrm{refl}_x"] \arrow[r, "\lambda x.(x,x)"'] & A \times A \end{tikzcd}$$

4

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Now that terms $p : x =_A y$ are called paths, we re-brand $^=$-elim as:

Path induction: For any type family $x, y : A, p : x =_A y \vdash P(x, y, p)$, to prove $P(x, y, p)$ for all x, y, p it suffices to assume y is x and p is refl_x.

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$$\text{path-ind} : \Big(\prod_{x:A} P(x, x, \mathsf{refl}_x)\Big) \to \Big(\prod_{x,y:A} \prod_{p:x=_A y} P(x, y, p)\Big).$$

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The ∞-groupoid of paths

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The ∞-groupoid structure of A has

- terms $x : A$ as objects
- paths $p : x =_A y$ as 1-morphisms
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The required structures are proven from the path induction principle:

- constant paths (reflexivity) $\mathsf{refl}_x : x = x$
- reversal (symmetry) $p : x = y$ yields $p^{-1} : y = x$
- concatenation (transitivity) $p : x = y$ and $q : y = z$ yield $p * q : x = z$

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and furthermore concatenation is associative and unital, the associators are coherent ...

The higher coherences in path algebra

Path induction proves the (higher) coherences in the ∞-groupoid of paths:

Proposition. For any type A and terms $w, x, y, z : A$

$$\mathsf{assoc} : \prod_{p : w =_A x} \prod_{q : x =_A y} \prod_{r : y =_A z} (p * q) * r =_{w =_A z} p * (q * r).$$

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Construction: By path induction, it suffices to assume x is w and p is refl_w, reducing to the case

$$\prod_{q : w =_A y} \prod_{r : y =_A z} (\mathsf{refl}_w * q) * r =_{w =_A z} \mathsf{refl}_w * (q * r).$$

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$$\prod_{q:w=_A y} \prod_{r:y=_A z} q * r =_{w=_A z} q * r,$$

for which we have the proof $\mathsf{refl}_{q*r} : q * r =_{w=_A z} q * r$. □

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Let $x : A \vdash P(x)$ be any family of types over A.

Proposition. For any $x, y : A$ if $p : x =_A y$ then $\mathrm{tr}_{P,p} : P(x) \to P(y)$.

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5

Epilogue: what justifies path induction?

The Curry-Howard-Voevodsky Correspondence

type theory	logic	set theory	homotopy theory
A	proposition	set	space
$x : A$	proof	element	point
$\emptyset, 1$	$\bot, \top$	$\emptyset, \{\emptyset\}$	$\emptyset, *$
$A \times B$	A and B	set of pairs	product space
$A + B$	A or B	disjoint union	coproduct
$A \to B$	A implies B	set of functions	function space
$x : A \vdash B(x)$	predicate	family of sets	fibration
$x : A \vdash b : B(x)$	conditional proof	fam. of elements	section
$\prod_{x:A} B(x)$	$\forall x.B(x)$	product	space of sections
$\sum_{x:A} B(x)$	$\exists x.B(x)$	disjoint union	total space
$p : x =_A y$	proof of equality	$x = y$	path from x to y
$\sum_{x,y:A} x =_A y$	equality relation	diagonal	path space for A

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By Σ-elim a proof of contractibility provides:

- a term $c : A$ called the center of contraction and
- a dependent function $h : \prod_{x:A} c =_A x$ called the contracting homotopy, which can be thought of as a continuous choice of paths $h(x) : c =_A x$ for each $x : A$.

The hierarchy of types

Contractible types, those types A for which the type

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has a term, form the bottom level of Voevodsky's hierarchy of types.

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$$\text{is-succ}(n)\text{-type}(A) := \prod_{x,y:A} \text{is-}n\text{-type}(x =_A y)$$

Equivalences

Similarly, homotopy theory suggests definitions of when two types A and B are equivalent or when a function $f : A \to B$ is an equivalence:

An equivalence between types A and B is a term of type:

$$A \simeq B :\equiv \sum_{f:A\to B} \left(\sum_{g:B\to A} \prod_{a:A} g(f(a)) =_A a \right) \times \left(\sum_{h:B\to A} \prod_{b:B} f(h(b)) =_B b \right)$$

A term of type $A \simeq B$ provides functions $f : A \to B$ and $g, h : B \to A$ and homotopies α and β relating the composite functions $g \circ f$ and $f \circ h$ to the identities. Using this data, one can define a homotopy from g to h.

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This type is not a proposition and may have non-trivial higher structure.

What justifies the path induction principle?

Path induction: For any type family $x, y : A, p : x =_A y \vdash P(x, y, p)$, to prove $P(x, y, p)$ for all x, y, p it suffices to assume y is x and p is refl_x.

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The equivalence $A \simeq \left(\sum_{x,y:A} x =_A y\right)$ gives rise to an equivalence

$$\left(\prod_{x:A} P(x,x,\mathsf{refl}_x)\right) \simeq \left(\prod_{x,y:A} \prod_{p:x=_A y} P(x,y,p)\right).$$

The univalence axiom

Another notion of sameness between types is provided by the universe $\mathcal{U}$ of types, which has (small) types A, B, C as its terms.

Q: How do the types $A =_{\mathcal{U}} B$ and $A \simeq B$ compare?

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By path induction, there is a canonical function

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Voevodsky's univalence axiom — which is justified by the homotopical model of type theory — captures the common mathematical practice of applying results proven about one object to any other object that is equivalent to it.

Consequences of univalence

There are myriad consequences of the univalence axiom:

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- The structure-identity principle, which specializes to the statement that for set-based structures (monoids, groups, rings) isomorphic structures are identical.
- Function extensionality: for any $f, g : A \to B$, the canonical function defines an equivalence between the identity type and the type of homotopies:

$$\text{id-to-htpy} : (f =_{A \to B} g) \to \left(\prod_{a:A} f(a) =_B g(a)\right)$$

- By indiscernibility of identicals, if $x, y : A$ and $x =_A y$ then $P(x) \simeq P(y)$ for any $a : A \vdash P(a)$. By univalence, whenever $A \simeq B$ then $A =_{\mathcal{U}} B$ and thus any type constructed from A is equivalent to the corresponding type constructed from B.

References

Homotopy Type Theory: Univalent Foundations of Mathematics

https://homotopytypetheory.org/book/

Egbert Rijke, Introduction to Homotopy Type Theory

hott.zulipchat.com

github.com/HoTT-Intro/Agda

HoTTEST Summer School, July–August 2022

https://discord.gg/tkhJ9zCGs9

Thank you!