ON THE ∞-TOPOS SEMANTICS OF HOMOTOPY TYPE THEORY

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Abstract. Many introductions to *homotopy type theory* and the *univalence axiom* neglect to explain what any of it means, glossing over the semantics of this new formal system in traditional set-based foundations. This series of talks will attempt to survey the state of the art, first presenting Voevodsky's simplicial model of univalent foundations and then touring Shulman's vast generalization, which provides an interpretation of homotopy type theory with strict univalent universes in any ∞-topos. As we will explain, this achievement was the product of a community effort to abstract and streamline the original arguments as well as develop new lines of reasoning.

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Lecture I: A categorical semantics of dependent type theory

1. Introduction

1.1. Why simplicial sets might model homotopy type theory. For any Kan complex A, there is a natural **path space factorization** of the diagonal map defined by exponentiation with the simplicial interval

$$\Delta^0 + \Delta^0 \rightarrowtail \Delta^1 \xrightarrow{\sim} \Delta^0 \qquad \rightsquigarrow \qquad A \overset{r}{\underset{\sim}{\rightarrowtail}} A^{\Delta^1} \xrightarrow{(e_0,e_1)} A \times A$$

In particular, the left map, the inclusion of constant paths, is a trivial cofibration in Quillen's model structure, and as such has the right lifting property with respect to an arbitrary Kan fibration:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, tail, "r"', "\wr"] & E \arrow[d, two heads, "p"] \\ A^{\Delta^1} \arrow[r, "b"'] \arrow[ur, dashed] & B \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} A \arrow[r, "d"] \arrow[rr, bend left, "e"] \arrow[d, tail, "r"', "\wr"] & P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A^{\Delta^1} \arrow[r, equal] \arrow[ur, dashed, "J"'] & A^{\Delta^1} \arrow[r, "b"'] & B \end{tikzcd}$$

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In particular, by pulling back along the codomain b it suffices to consider right lifting problems against Kan fibrations over the path space A^{Δ^1}, as displayed in the left-hand square of the rectangle above-right. This is the semantic interpretation of the homotopy type theoretic principle of **path induction**: given a type family $x : A, y : A, p : x =_A y \vdash P(x, y, p)$ and a family of terms $a : A \vdash d(a) : P(a, a, r_a)$ over the constant paths, there exists a section

$$x : A, y : A, p : x =_A y \vdash J_d(x, y, p) : P(x, y, p)$$

such that $a : A \vdash J_d(a, a, r_a) \equiv d(a) : P(a, a, r_a)$.

The homotopy type theoretic principle of path induction is stronger than this, because A might be a dependent type $\Gamma \vdash A$ in an arbitrary context Γ. But we can perform an analogous construction starting from an arbitrary Kan fibration $p\colon \Gamma.A \twoheadrightarrow \Gamma$ between Kan complexes. First form the path space factorizations for both $\Gamma.A$ and Γ

$$\begin{tikzcd}
\Gamma.A \arrow[rrr, tail, "\sim"] \arrow[dr, dashed, tail, "\sim"] \arrow[ddd, two heads] & & & (\Gamma.A)^{\Delta^1} \arrow[r, two heads] \arrow[ddd, two heads] & \Gamma.A \times \Gamma.A \arrow[ddd, two heads] \\
& \Delta^1 \pitchfork_\Gamma (\Gamma.A) \arrow[urr, dotted] \arrow[dr, dashed, two heads] \arrow[ddl, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & \\
& & (\Gamma.A) \times_\Gamma (\Gamma.A) \arrow[uurr, dotted] \arrow[dll, dotted, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & \\
\Gamma \arrow[rrr, tail, "\sim"] & & & \Gamma^{\Delta^1} \arrow[r, two heads] & \Gamma \times \Gamma
\end{tikzcd} \tag{1.1.1}$$

and then pull back the top factorization so that it lies in the slice over Γ. This constructs a factorization of the relative diagonal, in the slice over Γ, using the cotensor with the simplicial interval in the slice over Γ. Once more the inclusion of constant paths is a trivial fibration so we have a lifting property exactly as above. Moreover, since pullback is a simplicially enriched right adjoint, this construction is stable under pullback along any $f : \Delta \to \Gamma$ between Kan complexes.

$$\begin{tikzcd}
\Delta.f^*A \arrow[rrr, dotted] \arrow[dr, tail, "\sim"] \arrow[dddr, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & \Gamma.A \arrow[dr, tail, "\sim"] \arrow[dddr, two heads] & & \\
& \Delta^1 \pitchfork_\Delta (\Delta.f^*A) \arrow[rrr, dotted] \arrow[dr, two heads] \arrow[dd, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & & \Delta^1 \pitchfork_\Gamma (\Gamma.A) \arrow[dr, two heads] \arrow[dd, two heads] & \\
& & (\Delta.f^*A) \times_\Delta (\Delta.f^*A) \arrow[rrr, dotted] \arrow[dl, two heads] \arrow[dl, phantom, "\lrcorner", very near start] & & & (\Gamma.A) \times_\Gamma (\Gamma.A) \arrow[dl, two heads] \\
& \Delta \arrow[rrr, dotted, "f"'] & & & \Gamma &
\end{tikzcd}$$

1.2. **Martin-Löf's Dependent Type Theory.** Here we use "homotopy type theory" and "univalent foundations" as synonyms for Martin-Löf's dependent type theory, with intensional identity types, together with Voevodsky's univalence axiom.

This type theory has four basic judgment forms:

$$\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash A \equiv B \qquad \Gamma \vdash a \equiv b : A$$

asserting that A is a **type**, a is a **term** of type A, and that A and B or a and b are **judgmentally equal** types or terms, all in context Γ. Here Γ is shorthand for an arbitrary **context**, given by a sequence of variables of previously-defined types. Some types will be defined in the empty-context

$$\cdot \vdash \mathbb{N}$$

Such types then form contexts of length one $[n : \mathbb{N}]$. A type that is defined in a context of length one

$$n : \mathbb{N} \vdash \mathbb{R}^n$$

can then be used to **extend** the context to a context of length two $[n : \mathbb{N}, v : \mathbb{R}^n]$. Types definable using any subset of the previously-defined variables

$$n : \mathbb{N}, v : \mathbb{R}^n \vdash T_v\mathbb{R}^n$$

may then extend the context further $[n : \mathbb{N}, v : \mathbb{R}^n, \ell : T_v\mathbb{R}^n]$. Fully elaborated, the first two judgments take the form

$$x_1 : A_1, x_2 : A_1(x_1), \ldots, x_n : A_n(x_1, \ldots, x_{n-1}) \vdash B(x_1, \ldots, x_n)$$
$$x_1 : A_1, x_2 : A_1(x_1), \ldots, x_n : A_n(x_1, \ldots, x_{n-1}) \vdash b(x_1, \ldots, x_n) : B(x_1, \ldots, x_n)$$

We may think of such judgments as functions that take as input a sequence of typed variables and produce either a type or a term as output.

The **structural rules** of dependent type theory govern the variables and the judgmental equality relations. For instance, the **substitution rule** takes the form

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}$$

where $\mathcal{J}$ can be the conclusion of any of the four judgment forms. This says that given any term $a : A$ defined in context Γ and any judgment whose context involves a variable $x : A$, there is a corresponding judgment that no longer involves a variable $x : A$ but instead substitutes the term a for x whenever that variable occurred.

The **variable rule** asserts the existence of projection functions

$$\frac{}{\Gamma, x : A, \Delta \vdash x : A}$$

that take any variable occuring in a context and providing it as a term of the corresponding type.

This base type theory then inherits a richer structure on account of various additional **logical rules** that can be used to form new types. These are called "logical rules" because the types themselves are used to encode mathematical statements, which are "grammatically correct" but are not necessarily provable or true. Such statements are then proven by constructing a term of the corresponding type.

For instance, given a context $\Gamma, x : A \vdash B(x)$ there is a **dependent pair type** and a **dependent function type** which in this case have the same **formation rules**:

$$\frac{\Gamma, x : A \vdash B(x)}{\Gamma \vdash \Sigma_{x:A} B(x)} \qquad \frac{\Gamma, x : A \vdash B(x)}{\Gamma \vdash \Pi_{x:A} B(x)}$$

These types are distinguished by their **introduction rules**, which provide terms:

$$\frac{\Gamma, x : A \vdash B(x)}{\Gamma, x : A, b(x) : B(x) \vdash (x, b(x)) : \Sigma_{x:A} B(x)} \qquad \frac{\Gamma, x : A \vdash B(x) \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x.b(x) : \Pi_{x:A} B(x)}$$

The **elimination rules** explain how generic terms $z : \Sigma_{x:A} B(x)$ and $f : \Pi_{x:A} B(x)$ can be used, while the **computation rules** describe the composites of the functions encoded by the introduction and elimination rules.

1.3. **Identity types.** Of the various logical rules that annimate dependent type theory, Martin-Löf's rules for identity types are of paramount importance.

The **formation rule**

$$\frac{}{\Gamma, x : A, y : A \vdash x =_A y}$$

says that it is meaningful to inquire whether two terms belonging to a common type may be identifiable. The **introduction rule**

$$\frac{}{\Gamma, a : A \vdash \mathrm{refl}_a : a =_A a}$$

guarantees that any term is automatically identifiable with itself, via "reflexivity."

The **elimination rule** provides a version of Leibniz' indiscernability of identicals:

$$\frac{\Gamma, x : A, y : A, p : x =_A y \vdash P(x, y, p) \qquad \Gamma, a : A \vdash d(a) : P(a, a, \mathrm{refl}_a)}{\Gamma, x : A, y : A, p : x =_A y \vdash J_d(x, y, p) : P(x, y, p)}$$

Here P can be be thought of as a predicate involving two terms of the same type and an identification between them, though the power of this rule derives from its applicability to arbitrary type families, not only the mere propositions. See [Ho, Ri] for a survey of its myriad consequences.

Finally, the **computation rule**

$$\frac{\Gamma, x : A, y : A, p : x =_A y \vdash P(x,y,p) \qquad \Gamma, a : A \vdash d(a) : P(a,a,\mathrm{refl}_a)}{\Gamma, a : A \vdash J_d(a,a,\mathrm{refl}_a) \equiv d(a) : P(a,a,\mathrm{refl}_a)}$$

is analogous to the computation rule for a recursively defined function.

Digression 1.3.1 (extensionality vs intensionality). By the structural rules for judgmental equality, if

$$\Gamma, x : A, y : A \vdash x \equiv y : A$$

then

$$\Gamma, x : A, y : A \vdash \mathrm{refl}_x : x =_A y$$

Thus judgmentally equal terms are always identifiable.

The **extensional** form of Martin-Löf's identity types adds a converse implication:

$$\Gamma, x : A, y : A, p : x =_A y \vdash x \equiv y : A$$

while the **intensional** form of Martin-Löf's identity types does not add this implication.

2. A categorical semantics of dependent type theory

2.1. **The category of contexts.** To interpret syntactic expressions in type theory as data in a category we must first build a category out of type theoretic syntax. A standard way to do this involves Cartmell's **contextual categories**.

Definition 2.1.1. Let $\mathbf{T}$ be a type theory with the standard structural rules: variable, substitution, weakening, and the usual rules concerning judgmental equality of types and terms. Then the **category of contexts** $\mathcal{C}(\mathbf{T})$ is a category in which:

- objects are contexts $[x_1 : A_1, \ldots, x_n : A_n]$ of arbitrary finite length up to definitional equality and renaming of free variables;
- maps of $\mathcal{C}(\mathbf{T})$ are **context morphisms** or **substitutions**, considered up to definitional equality and renaming of variables: a map

(2.1.2) $$f = [f_1, \ldots, f_n] \colon [y_1 : B_1, \ldots, y_m : B_m] \to [x_1 : A_1, \ldots, x_n : A_n]$$

  is an equivalence class of term judgements

$$\begin{array}{c} y_1 : B_1, \ldots, y_m : B_m \vdash f_1 : A_1 \\ \vdots \qquad \vdash \qquad \vdots \\ y_1 : B_1, \ldots, y_m : B_m \vdash f_n : A_n(f_1, \ldots, f_{n-1}) \end{array}$$

- composition is given by substitution and the identity is defined by using variables as terms.

The category of contexts has the following additional structures:

- $\mathcal{C}(\mathbf{T})$ has a terminal object corresponding to the empty context.
- The objects are partitioned according to the length of the context

$$\mathrm{ob}\,\mathcal{C}(\mathbf{T}) = \coprod_{n:\mathbb{N}} \mathrm{ob}_n \mathcal{C}(\mathbf{T})$$

  with the empty context being the only object of length 0.
- Each object of positive length has a **display map**

$$[x_1, \ldots, x_n] \colon [x_1 : A_1, \ldots, x_{n+1} : A_{n+1}] \to [x_1 : A_1, \ldots, x_n : A_n]$$

  namely the canonical projection away from the final type in the context. Writing Γ for the codomain context, we abbreviate this display map as

$$p_{A_{n+1}} \colon \Gamma.A_{n+1} \to \Gamma$$

and refer the domain of any display map over Γ as a **context extension** of Γ. We refer to composites

$$\Gamma.A.B \xrightarrow{p_B} \Gamma.A \xrightarrow{p_A} \Gamma \qquad \text{or} \qquad \Gamma.\Delta \xrightarrow{p} \Gamma$$

of display maps as **dependent projections**.

- For each substitution (2.1.2) and any context extension of its codomain, there is a **canonical pullback** defined by the context

$$[y_1 : B_1, \ldots, y_m : B_m, y_{m+1} : A_{n+1}(f_1(\vec{y}), \ldots, f_n(\vec{y}))]$$

together with a canonical pullback square

$$\begin{tikzcd} \Delta.f^*A \arrow[r, "q(f)"] \arrow[d, "p_{f^*A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd}$$

Moreover, these canonical pullbacks are *strictly stable under substitution*. Given any context morphism $g \colon \Theta \to \Delta$, $\Theta.g^* f^* A = \Theta.(f \cdot g)^* A$ and $q_f \cdot q_g = q_{f \cdot g}$; similarly $\Gamma.1^* A = \Gamma.A$.

Remark 2.1.3. A term $\Gamma \vdash a : A$ in a type A in context Γ is given by a section to the display map $p_A \colon \Gamma.A \to \Gamma$. Thus, the phrase "section" is typically reserved to mean section of a display map.

2.2. **Homotopy theoretic models of identity types.** The title of this section aludes to a famous paper of [AW] that described the homotopical semantics of identity types, but we instead focus on a result of Gambino and Garner inspired by it, constructing what they call the **identity type weak factorization system** on the category of contexts.

Theorem 2.2.1 ([GG])**.** *Let* $\mathbf{T}$ *be a dependent type theory with intensional identity types. Then the category of contexts* $\mathcal{C}(\mathbf{T})$ *admits a weak factorization system whose left class is comprised of those maps that lift against the display maps.*

In the proof, Gambino and Garner construct the factorization of an arbitrary context morphism, but the main idea is illustrated effectively by the simplest non-trivial case: factoring the identity map on a context $[x : A]$ of length 1 as

$$\begin{tikzcd} {[a : A]} \arrow[rr, equal] \arrow[dr, "{[a,a,\mathrm{refl}_a]}"'] & & {[a : A]} \\ & {[x : A, y : A, p : x =_A y]} \arrow[ur, "{[y]}"'] & \end{tikzcd}$$

The right factor is isomorphic in the category of contexts to a composite of display maps and hence belongs to the right class. To see that the left factor is in the left class, we must show it has the left lifting property with respect to an arbitrary display map $p_E \colon \Phi.E \to E$. By pulling back along the codomain of the lifting problem, we may assume that the codomain of this display map is the context $[x : A, y : A, p : x =_A y]$ presenting us with a lifting problem of the form below:

$$\begin{tikzcd} {[a : A]} \arrow[r, "{[x,y,p,d(a)]}"] \arrow[d, "{[a,a,\mathrm{refl}_a]}"'] & {[x : A, y : A, p : x =_A y, z : P(x,y,p)]} \arrow[d, "p_P"] \\ {[x : A, y : A, p : x =_A y]} \arrow[r, equal] \arrow[ur, dashed, "{[x,y,p,J_d(x,y,p)]}"'] & {[x : A, y : A, p : x =_A y]} \end{tikzcd}$$

By the elimination rule for identity types, we can use the data provided by the lifting problem to define the term

$$x : A, y : A, p : x =_A y \vdash J_d(x, y, p) : P(x, y, p)$$

required to define a section of the display map associated to the type family

$$x : A, y : A, p : x =_A y \vdash P(x, y, p).$$

The top triangle commutes by the computation rule.

Remark 2.2.2. Unusually, this factorization is not *functorial*. Given a commutative rectangle involving the identity functions between three contexts of length one:

$$\begin{tikzcd} {[a:A]} \arrow[r, "g"] \arrow[d, equal] & {[b:B]} \arrow[r, "f"] \arrow[d, equal] & {[c:C]} \arrow[d, equal] \\ {[a:A]} \arrow[r, "g"] & {[b:B]} \arrow[r, "f"] & {[c:C]} \end{tikzcd}$$

functoriality would demand that the induced substitutions between identity types commute. But as the maps induced by f, g, and $f \cdot g$ are defined by path induction (i.e., by the elimination rule of identity types) they only commute up to pointwise identification, not up to judgmental equality.

However, as these structures on the category of contexts are defined using the rules of type theory, they are strictly stable under substitution. The central components of the identity type weak factorization system equip the contextual category $\mathcal{C}(\mathbf{T})$ with what is called an "identity-type structure."

Definition 2.2.3. An **identity type structure** on a contextual category consists of:

- for each object $\Gamma.A$, an object $\Gamma.A.A.\mathrm{Id}_A$ extending the context defined by the canonical pullback:

$$\begin{tikzcd} \Gamma.A.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\ \Gamma.A \arrow[r, "p_A"'] & \Gamma \end{tikzcd}$$

- for each $\Gamma.A$, a morphism

$$\begin{tikzcd} & \Gamma.A.A.\mathrm{Id}_A \arrow[d, "p_{\mathrm{Id}_A}"] \\ \Gamma.A \arrow[ur, dashed, "\mathrm{refl}_A"] \arrow[r, "{(1,1)}"'] & \Gamma.A.A \end{tikzcd}$$

  lifting the relative diagonal; and
- for each $\Gamma.A.A.\mathrm{Id}_A.P$ and solid-arrow commutative diagram

$$\begin{tikzcd} \Gamma.A \arrow[r, "d"] \arrow[d, "\mathrm{refl}_A"'] & \Gamma.A.A.\mathrm{Id}_A.P \arrow[d, "p_P"] \\ \Gamma.A.A.\mathrm{Id}_A \arrow[r, equal] \arrow[ur, dashed, "J"'] & \Gamma.A.A.\mathrm{Id}_A \end{tikzcd}$$

  a section J of p_P so that $J \cdot \mathrm{refl}_A = d$

so that all this structure is strictly stable under pullback along any substitution $f : \Delta \to \Gamma$.

2.3. **Initiality.** A **contextual functor** is a functor between contextual categories preserving all of the structure on the nose. In particular, if the contextual categories have additional logical structure, such as the identity type structure just defined, or the corresponding structures for Σ- or Π-types, then the contextual functor must preserve this as well, up to equality.

The category of contexts is intended to be the universal contextual category built from a type theory in the sense made precise by the following conjecture:

Conjecture 2.3.1 (initiality conjecture)**.** *Let* $\mathbf{T}$ *be a type theory given by the standard structural rules together with some collection of the logical rules alluded to above. Then the category of contexts* $\mathcal{C}(\mathbf{T})$ *is the initial contextual category with the corresponding extra structure.*

Then further models of $\mathbf{T}$ in a category $\mathcal{E}$ can be defined by turning $\mathcal{E}$ into a contextual category with the appropriate logical structure: by initiality, this would induce a unique functor $\mathcal{C}(\mathbf{T}) \to \mathcal{E}$ providing an interpretation of the syntax of type theory in the category $\mathcal{E}$.

The difficulties with the initiality conjecture are multifaceted. On the one hand:

> The trouble with syntax is that it is very tricky to handle rigorously. Any full presentation must account for (among other complications) variable binding, capture-free substitution, and the possibility of multiple derivations of a judgment; and so any careful construction of an interpretation must deal with all of these, at the same time as tackling the details of the particular model in question. Contextual categories, by contrast, are a purely algebraic notion, with no such subtleties. [KL, 2075].

Another challenge is that there are many versions of type theory, determined by various collections of rules. Moreover, it's difficult to give a precise definition of what constitutes a type theory that is sufficiently general to cover all of the desired examples. An early version of initiality was proven for a type theory called the *calculus of constructions* by Streicher [Str1]. This has often been cited as evidence that initiality should be true for a much broader class of type theories, but at the time the simplicial "model" of homotopy type theory was being developed, Voevodsky argued that it was unacceptably unrigorous to assume initiality without proof.

Since [KL] was written, there was have been efforts to at least prove initiality for the "Book-HoTT" version of homotopy type theory appearing in [Ho]. This was achieved in a 2020 PhD thesis of Menno de Boer [dB] based on parallel formalization efforts undertaken by Brunerie and de Boer in Agda and Lumsdaine and Mörtberg in Coq.

Alternatively, the question of what constitutes a type theory can be answered in such a way as to render initiality automatic. Many practitioners equate type theories with their corresponding suitably structured categories, whether these be *contextual categories*, or *comprehension categories*, or *categories with families*, or *natural models* or something else. Each of these notions defines an *essentially algebraic theory* and thus the category of all such admits an initial object. From that point of view, the only question is whether a particular syntax presents the initial object, and this may be viewed as a practical consideration moreso than a theoretic one as the utility of a particular syntactic presentation derives from its use in proofs.

2.4. **Contextual categories from universes.** Even assuming initiality, it seems daunting to construct a model of type theory in a category $\mathcal{E}$, for one must still define a contextual category with suitable structures for all the type forming operations. By analyzing the type-forming operations, one can predict what sort of category might be suitable. For instance, a Π-type structure on a contextual category requires an operation that takes a composable pair of display maps as below-left to a display map as below-right:

$$\Gamma.A.B \xrightarrow{p_B} \Gamma.A \xrightarrow{p_A} \Gamma \qquad \rightsquigarrow \qquad \Gamma.\Pi_A B \xrightarrow{p_{\Pi_A B}} \Gamma$$

This, together with the fact that contextual categories must admit pullbacks of display maps, suggests that it would be reasonable to restrict to categories $\mathcal{E}$ that are **locally cartesian closed**, meaning that for any $f : \Delta \to \Gamma$ the composition functor has two right adjoints

(2.4.1)
$$\begin{tikzcd} \mathcal{E}_{/\Delta} \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathcal{E}_{/\Gamma} \arrow[l, "f^*" description] \end{tikzcd}$$

defined by pullback along f and pushforward along f, respectively. Note when $\mathcal{E}$ has a terminal object 1—as is required to model the empty context—this implies that $\mathcal{E} \cong \mathcal{E}_{/1}$ and indeed any slice $\mathcal{E}_{/\Gamma}$ is cartesian closed and has all finite limits.

Note, however, that the definition of a contextual category, implicitly given in Definition 2.1.1, requires strict stability of the canonical pullback squares. In addition, each logical structure, such as the identity type structure of Definition 2.2.3, requires the pullbacks to preserve everything on the nose, up to equality of objects. Thus, the more categorically-natural requirement, that pullbacks preserve the various logical structures up to isomorphism, is not enough.

This leads to a massive coherence problem. It's not generally possible to choose strictly functorial pullbacks in a category with pullbacks, though it is possible to replace a category with pullbacks by an equivalent category with strictly functorial pullbacks.[1] But even after this is achieved, there remains the task of ensuring strict stability of all the logical structures.

Voevodsky's approach to the coherence problem makes use of a "universe" in a category, defined as follows:

Definition 2.4.2. A **universe** in a category $\mathcal{E}$ consists of an object U, together with a morphism $\pi\colon \tilde{U} \to U$, and, for each map $A\colon \Gamma \to U$, a choice of pullback square

$$\begin{tikzcd} (\Gamma;A) \arrow[r, "q_A"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ \Gamma \arrow[r, "A"'] & U \end{tikzcd}$$

Given a sequence of maps $A\colon \Gamma \to U$, $B\colon (\Gamma;A) \to U$, we write $(\Gamma;A,B)$ for $((\Gamma;A);B)$.

Definition 2.4.3 ([V1]). Given a category $\mathcal{E}$ with a universe U and a terminal object 1, define a contextual category $\mathcal{E}_U$ as follows:

- The objects of $\mathcal{E}_U$ are finite lists of morphisms as follows:
$$\mathrm{ob}_n\mathcal{E}_U = \{(A_1,\dots,A_n) \in (\mathrm{mor}\mathcal{E})^n \mid A_i\colon (1;A_1,\dots,A_{i-1}) \to U, \forall i\}.$$
- The morphisms between a pair of objects are defined by
$$\mathcal{E}_U((B_1,\dots,B_m),(A_1,\dots,A_n)) := \mathcal{E}((1;B_1,\dots B_m),(1;A_1,\dots,A_n)).$$
- The terminal object is the empty sequence of morphisms.
- For any object $(A_1,\dots,A_{n+1})$ of positive length, its display map is provided by the universe structure:

$$\begin{tikzcd} (1;A_1,\dots,A_{n+1}) \arrow[r] \arrow[d, "p_{A_{n+1}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ (1;A_1,\dots,A_n) \arrow[r, "A_{n+1}"'] & U \end{tikzcd} \tag{2.4.4}$$

- Finally the canonical pullback associated to a morphism $f : (B_1,\dots,B_m) \to (A_1,\dots,A_n)$ and a display map (2.4.4) is defined by factoring the chosen pullback square for the composite $A_{n+1}\cdot f$ through the pullback that defines the context extension $(A_1,\dots,A_{n+1})$:

$$\begin{tikzcd} (1;B_1,\dots,B_m,A_{n+1}\cdot f) \arrow[r] \arrow[d, "p_{f^*A_{n+1}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (1;A_1,\dots,A_n,A_{n+1}) \arrow[r] \arrow[d, "p_{A_{n+1}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ (1;B_1,\dots,B_m) \arrow[r, "f"'] & (1;A_1,\dots,A_n) \arrow[r, "A_{n+1}"'] & U \end{tikzcd}$$

Indeed every small contextual category arises this way:

Proposition 2.4.5 ([V1]). *Let $\mathcal{C}$ be a small contextual category. Consider the universe U in the presheaf category $[\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et}]$ given by*

$$U(\Gamma) = \{\Gamma.A\} \qquad \tilde{U}(\Gamma) = \{\text{sections } s : \Gamma \to \Gamma.A\}$$

with the evident projection map and any choice of pullbacks. Then $[\mathcal{C}^{\mathrm{op}},\mathcal{S}\mathrm{et}]_U$ is isomorphic, as a contextual category, to $\mathcal{C}$.

When the original category $\mathcal{E}$ is locally cartesian closed, it is possible to specify additional structure on the universe U that would equip the contextual category $\mathcal{E}_U$ with the corresponding logical structure. The idea is that this introduces the desired structure in the universal case. To describe an identity structure on a universe, we require the following notion.

[1] This can be achieved by applying the general techniques for replacing a fibration by a split fibration to the codomain-projection fibration $\mathrm{cod}\colon \mathcal{E}^2 \to \mathcal{E}$; see mathoverflow.net/questions/144619/can-we-always-make-a-strictly-functorial-choice-of-pullbacks-re-indexing for a discussion.

Digression 2.4.6. Recall that for maps $i\colon A \to B$ and $f\colon X \to Y$ in $\mathcal{E}$, a **lifting operation** witnessing a lifting property as below-left is a section to the map below-right:

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "y"'] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \mathcal{E}(B,X) \xrightarrow{(-\cdot i, f\cdot -)} \mathcal{E}(A,X) \times_{\mathcal{E}(A,Y)} \mathcal{E}(B,Y)$$

If $\mathcal{E}$ is cartesian closed, an **internal lifting operation** is a section to the map

$$X^B \xrightarrow{(-\cdot i, f\cdot -)} X^A \times_{Y^A} Y^B$$

We may obtain an analogous structure on the contextual category $\mathcal{C}_U$ built from a universe $\pi\colon \tilde{U} \to U$ as follows. Firstly, note that since all display maps are pullbacks of π it suffices to consider orthogonality against π in the slice over U.

Definition 2.4.7. An **identity structure** on a universe $\pi\colon \tilde{U} \to U$ consists of a map

$$\mathrm{Id}\colon \tilde{U} \times_U \tilde{U} \to U$$

together with a specified lift $r\colon \tilde{U} \to \mathrm{Id}^*\tilde{U}$ of the relative diagonal:

$$\begin{tikzcd} & \mathrm{Id}^*\tilde{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ \tilde{U} \arrow[r, "(1,1)"'] \arrow[ur, dashed, "r"] & \tilde{U} \times_U \tilde{U} \arrow[r, "\mathrm{Id}"'] & U \end{tikzcd}$$

together with an internal lifting operation J for r against $\pi \times U$ in $\mathcal{E}_{/U}$.

Theorem 2.4.8 ([V2]). *An identity structure on a universe U in $\mathcal{E}$ induces an identity type structure on the contextual category $\mathcal{E}_U$.*

Of course, it remains to construct a suitable universe in our locally cartesian closed category. In part II, we'll explain how this is done in the category of simplicial sets, leading to the simplicial model of univalent foundations.

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