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On the art of giving the same name to different things

The Multiplicity Turn:
Theories of Identity from Poetry to Mathematics

…la Mathématique est l'art de donner le même nom à des choses différentes.

…mathematics is the art of giving the same name to different things.

— Henri Poincaré
"L'avenir des mathématiques"
Science et Méthode
Flammarion, Paris, 1908.

Plan

Equality

Isomorphism

Equivalence

Identity

1

Equality

The traditional view of equality

Reflexivity:
anything is equal to itself.

$$\forall x,\ x = x$$

Indiscernibility of Identicals:
two things are equal if and only if they have exactly the same properties.

$$\forall x, y,\ (x = y) \leftrightarrow (\forall P, P(x) \leftrightarrow P(y))$$

Symmetry and Transitivity

Using

- reflexivity: anything is equal to itself; and
- indiscernibility of identicals: two things are equal if and only if they have exactly the same properties

one can deduce:

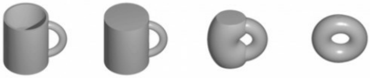
Symmetry: if $x = y$ then $y = x$.

Proof: Assume $x = y$. Then x and y must have exactly the same properties. In particular, since $x = x$ we must also have $y = x$. $\square$

Transitivity: if $x = y$ and $y = z$ then $x = z$.

Proof: Assume $x = y$. Then x and y must have exactly the same properties. In particular, if $y = z$ then $x = z$. $\square$

Different things that deserve the same name



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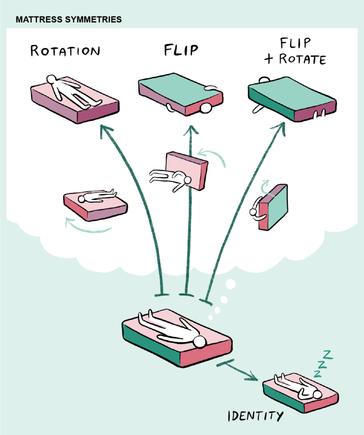

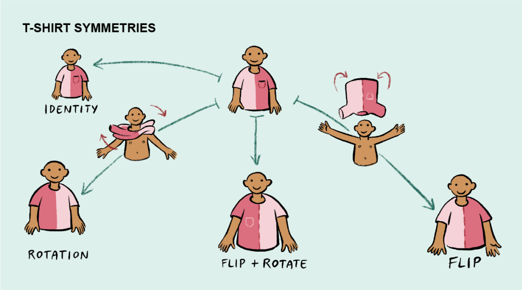

images by Matteo Farinella

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Isomorphism

Isomorphic = same + shape

Some different things deserve the same name because they have the "same shape."

$\acute{\iota}\sigma$ος "equal" + $\mu o \rho \phi \acute{\eta}$ "shape"

We seek a unifying language to describe what it means for things to have the "same shape" no matter what kind of objects they are.

Category

> A category frames a possible template for a mathematical theory: the theory should have nouns, the mathematical objects, and verbs, the transformations between them, depicted as arrows. — Barry Mazur

A category has

- objects: $A, B, C \ldots$ and
- arrows: $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, each with a specified source and target

so that

- each pair of composable arrows has a composite arrow

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & C \end{tikzcd}$$

- and each object has an reflexivity arrow $A \xrightarrow{\mathrm{refl}_A} A$

for which the composition operation is associative and unital.

Isomorphism in a category

A category has

- objects: $A, B, C \ldots$ and
- arrows: $A \xrightarrow{f} B$, $B \xrightarrow{g} C$.

Objects A and B in a category are isomorphic

if there exist arrows $f : A \to B$ and $g : B \to A$

so that $g \circ f = \mathrm{refl}_A$ and $f \circ g = \mathrm{refl}_B$.

$$A \cong B$$

Categorifying arithmetic

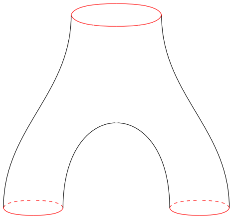
Why is $2 \times (3+4) = (2 \times 3) + (2 \times 4)$?
What even are 2, 3, and 4 ?

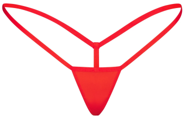
$$A = \left\{ \ast \quad \star \right\}, \qquad B = \left\{ \begin{array}{c} \sharp \\ \flat \\ \natural \end{array} \right\}, \qquad C = \left\{ \begin{array}{cc} \spadesuit & \heartsuit \\ \diamondsuit & \clubsuit \end{array} \right\}$$

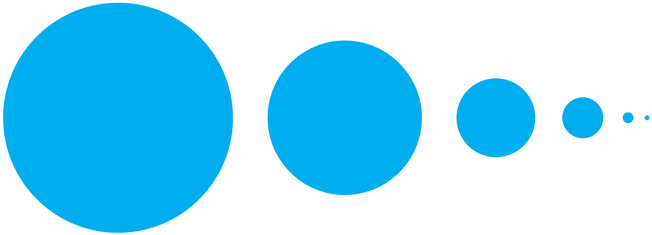
$$B + C = \left\{ \begin{array}{cccc} \sharp & \flat & \spadesuit & \heartsuit \\ & \natural & \diamondsuit & \clubsuit \end{array} \right\}, \qquad A \times B = \left\{ \begin{array}{cc} (\ast, \sharp) & (\star, \sharp) \\ (\ast, \flat) & (\star, \flat) \\ (\ast, \natural) & (\star, \natural) \end{array} \right\}$$

$$\left\{ \begin{array}{cc} (\ast, \sharp) & (\star, \sharp) \\ (\ast, \flat) & (\star, \flat) \\ (\ast, \natural) & (\star, \natural) \\ (\ast, \spadesuit) & (\star, \spadesuit) \\ (\ast, \heartsuit) & (\star, \heartsuit) \\ (\ast, \diamondsuit) & (\star, \diamondsuit) \\ (\ast, \clubsuit) & (\star, \clubsuit) \end{array} \right\} \cong \left\{ \begin{array}{cccc} (\ast, \sharp) & (\ast, \flat) & (\ast, \spadesuit) & (\ast, \heartsuit) \\ & (\ast, \natural) & (\ast, \diamondsuit) & (\ast, \clubsuit) \\ (\star, \sharp) & (\star, \flat) & (\star, \spadesuit) & (\star, \heartsuit) \\ & (\star, \natural) & (\star, \diamondsuit) & (\star, \clubsuit) \end{array} \right\}$$

Different things that deserve the same name







Different things that deserve the same name

The category of finite sets and their symmetries is indescribably large

— and very redundant.

The category of natural numbers and their symmetries contains the same information, much more efficiently packaged.

There are two standard approaches to linear algebra:

- using matrices of arbitrary dimension
- using linear transformations between vector spaces

and the general theory can be developed from either perspective.

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Equivalence

Equivalence = equal + worth

A 2-category has

- objects: $A, B, C \dots$
- 1-arrows: $A \xrightarrow{f} B$, $B \xrightarrow{h} C$ and
- 2-arrows: $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "k"'{name=D}] & B \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$

Objects A and B in a 2-category are equivalent

if there exist 1-arrows $f : A \to B$ and $g : B \to A$

and 2-arrows $\begin{tikzcd} A \arrow[r, bend left, "g \circ f"{name=U}] \arrow[r, bend right, "\mathrm{refl}_A"'{name=D}] & A \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$ and $\begin{tikzcd} B \arrow[r, bend left, "f \circ g"{name=U}] \arrow[r, bend right, "\mathrm{refl}_B"'{name=D}] & B \arrow[from=U, to=D, phantom, "\Downarrow\beta"] \end{tikzcd}$

so that $\alpha : g \circ f \cong \mathrm{refl}_A$ and $\beta : f \circ g \cong \mathrm{refl}_B$.

$A \simeq B$

Problems

- This doesn't stop here! The best notion of sameness for 2-categories isn't equivalence in the sense just defined but in a weaker sense that requires a 3-category. But then 3-categories are equivalent in a sense defined using a 4-category, and so on …

- Higher category theory no longer provides a single meaning of when one thing is the same as another thing but rather a hierarchy of different meanings depending on how complex the objects are (as governed by what sort of categories they belong to).

- Most seriously, indiscernibility of identicals fails for objects that are isomorphic or equivalent but not equal!

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Identity

Identity Types

In type theory mathematical sentences take the form of types A, B, C.
A term $x : A$ in a type then provides a proof of the encoded statement.

Identity types are governed by the following rules:

- For any type A and terms $x, y : A$, there is a type $x =_A y$.
- For any type A and term $x : A$, there is a term $\mathrm{refl}_x : x =_A x$.
- For any type $P(x, y, p)$ defined using terms $x, y : A$ and $p : x =_A y$,
  - if there is a term $d(x) : P(x, x, \mathrm{refl}_x)$ for all $x : A$,
  - then there is a term $J_d(x, y, p) : P(x, y, p)$ for all $x, y : A$, $p : x =_A y$.

No nonsense: it's only meaningful to identify things of the same type.

Reflexivity: anything is identifiable with itself.

Indiscernibility of Identicals: two things are identifiable
if and only if they have exactly the same properties.

Univalence

The univalence axiom relates the identity types in the universe of all types $\mathcal{U}$ to equivalences between types.

"Identity is equivalent to equivalence."

$$\mathsf{univalence} : (A =_{\mathcal{U}} B) \simeq (A \simeq_{\mathcal{U}} B)$$

"When I decided to check something in the Russian translation of the Boardman and Vogt book *Homotopy Invariant Algebraic Structures on Topological Spaces* I discovered that in this book the term 'faithful functor' was translated as 'univalent functor.'

унивалентный функтор

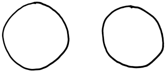
Since I have tried to read this book in my youth many times there was probably another meaning associated in my mind with the word 'univalent' — 'faithful'.

Indeed these foundations seem to be faithful to the way in which I think about mathematical objects in my head."

— Vladimir Voevodsky, "Univalent Foundations — new type-theoretic foundations of mathematics," Talk at IHP, Paris on April 22 , 2014

Consequences of Univalence

The things that deserve the same name:

- 
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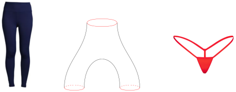
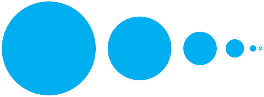
- 
- 

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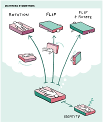
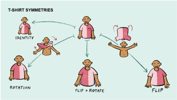
- 



- the categories of finite sets and of natural numbers
- abstract and concrete linear algebra

are terms belonging to a common type.

As a consequence of the univalence axiom:

identifications — that is, proofs of identity — recover exactly the notions of sameness previously introduced.

Conclusions

Equality $\rightsquigarrow$ Isomorphism $\rightsquigarrow$ Equivalence $\rightsquigarrow$ Identification

- While the traditional notion of equality is too narrow, its defining principles are worth preserving.
- While the categorical notions of isomorphism and equivalence identify objects that have the "same shape" or have "equal worth," they require increasingly higher-dimensional data as the objects become more complex.
- The type theoretic concept of identification is specified by rules that demand:
  - no nonsense: it's only meaningful to identify things of the same type,
  - reflexivity: everything is identified with itself, and
  - indiscernibility of identicals: two things are identifiable if and only if they have exactly the same properties.
- In the presence of the univalence axiom, identifications specialize to the "correct" notions of sameness for objects of each type.

Thank you!