

Johns Hopkins University

# Categorifying cardinal arithmetic

Goal: prove  $a \times (b+c) = (a \times b) + (a \times c)$  for any natural numbers a, b, and c.

#### Plan

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- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof

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- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?



Step 1: categorification

## The idea of categorification



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Natural numbers a, b, and c encode the sizes of finite sets A, B, and C.

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Q: What is true of A and B if a = b?

A: a=b if and only if A and B are isomorphic, which means there exist functions  $f\colon A\to B$  and  $g\colon B\to A$  that are inverses in the sense that  $g\circ f=\operatorname{id}$  and  $f\circ g=\operatorname{id}$ . In this case, we write  $A\cong B$ .

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Categorification: the truth behind a = b is  $A \cong B$ .



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Q: What is the deeper meaning of the symbols "+" and " $\times$ "?

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$$b + c \coloneqq |B + C|$$

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#### In summary:

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A: It means that the sets  $A \times (B+C)$  and  $(A \times B) + (A \times C)$  are isomorphic!

$$A \times (B+C) \cong (A \times B) + (A \times C)$$

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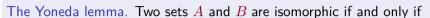
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2

# Step 2: the Yoneda lemma



The Yoneda lemma. Two sets A and B are isomorphic if and only if



• for all sets X, the sets of functions

$$\operatorname{Fun}(A,X) \coloneqq \{h \colon A \to X\} \quad \text{and} \quad \operatorname{Fun}(B,X) \coloneqq \{k \colon B \to X\}$$

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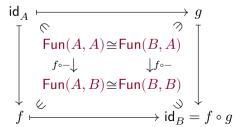
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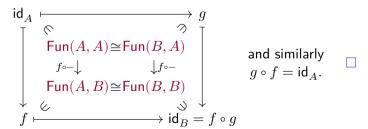
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Step 3: representability



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$$\begin{array}{cccc} \text{By "pairing"} & & & \text{Fun}(B+C,X) & \cong & \text{Fun}(B,X) \times \text{Fun}(C,X) \\ & & & & & & & \\ f & & \Leftrightarrow & & & (f_B,f_C) \end{array}$$



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A: For each  $b \in B$  and  $a \in A$ , we need to specify an element  $f(a,b) \in X$ . Thus, for each  $b \in B$ , we need to specify a function  $f(-,b) : A \to X$  sending a to f(a,b).



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By "currying" 
$$\begin{array}{cccc} \operatorname{Fun}(A\times B,X) &\cong & \operatorname{Fun}(B,\operatorname{Fun}(A,X)) \\ & & & & \\ & & & \\ f\colon A\times B\to X & \Leftrightarrow & f\colon B\to \operatorname{Fun}(A,X) \end{array}$$

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# Summary of Steps 1, 2, and 3

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#### Step 3 summary:

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# Summary of Steps 1, 2, and 3

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#### Step 3 summary:

- $\bullet \ \operatorname{Fun}(B+C,X) \cong \operatorname{Fun}(B,X) \times \operatorname{Fun}(C,X)$  by "pairing" and
- $\operatorname{Fun}(A \times B, X) \cong \operatorname{Fun}(B, \operatorname{Fun}(A, X))$  by "currying."





Step 4: the proof



Theorem. For any natural numbers a, b, and c,

$$a\times (b+c)=(a\times b)+(a\times c).$$

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Epilogue: what was the point of that?



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Theorem. For any cardinals  $\alpha$ ,  $\beta$ ,  $\gamma$ ,

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Exercise: Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta+\gamma}=\alpha^{\beta}\times\alpha^{\gamma}\quad\text{ and }\quad (\alpha^{\beta})^{\gamma}=\alpha^{\beta\times\gamma}=(\alpha^{\gamma})^{\beta}.$$

## Generalization to other mathematical contexts



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• For nice topological spaces X, Y, Z,

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• For abelian groups A, B, C,

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Thank you!