Traces via Strategies

(Games via Coalgebra)

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Outline

- Games
- Traces
- Representing games coalgebraically
- Strategies
- Traces via strategies

Motivation: controller synthesis

- Model the possible actions of the controller and the environment as a game.

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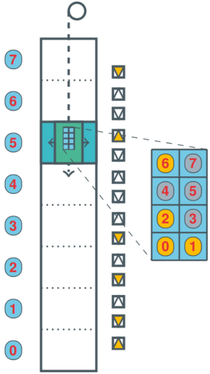
- Example "every request is served" (a liveness property)



Two-player games

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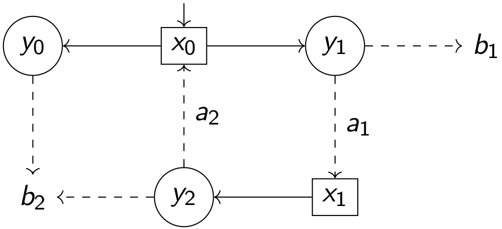

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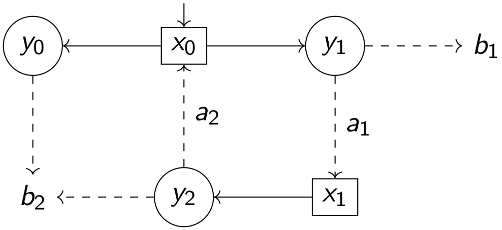

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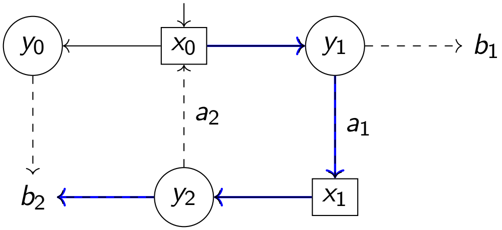

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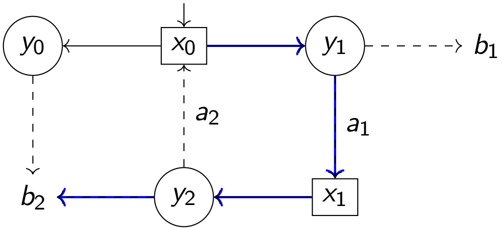

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- A strategy is a partial function which extends partial plays, it must be defined over all partial plays which conform to it.

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P is a monad with $\mathbf{Kl}(P) \cong \mathbf{Rel}$

Traces, coalgebraically

A^*B is the *initial algebra* for the functor $B + A(-) : \mathbf{Set} \to \mathbf{Set}$

$$\boxed{\mathbf{Set}}$$

$$\begin{tikzcd}
 & & & A^*B \\
B \arrow[r] \arrow[urrr, bend left, "\kappa_0"'] & B + AB \arrow[r] \arrow[urr, bend left, "\kappa_1"'] & B + AB + AAB \arrow[r] \arrow[ur, bend left, "\kappa_2"'] & \cdots
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General categorical machinery allows us to lift this chain to the category of relations, and reverse the arrows[1]:

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Thus A^*B is a *final coalgebra* in the category of relations!

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Traces by coinduction

For every LTS $c : X \to P(B + A \times X)$, there is a *unique coalgebra morphism* into A^*B.

$$\boxed{\mathbf{Rel}} \qquad \begin{tikzcd} X \arrow[r, dashed] \arrow[d, "c"] & A^*B \arrow[d, "\wr"] \\ B + A \times X \arrow[r, dashed] & B + A \times A^*B \end{tikzcd}$$

This dashed morphism in **Rel** is a function $X \to P(A^*B)$ which assigns each state to it's set of traces!

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- With $c^* : X \to PH_X(X)$ defined as the composite

$$(X \xrightarrow{\langle \mathrm{id}, c \rangle} X \times P(B + A \times X) \xrightarrow{\mathrm{stl}} P(X \times (B + A \times X))$$

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$$\begin{tikzcd} X \arrow[r, "\mathrm{exec}_c"] \arrow[rr, bend right, "\mathrm{tr}_c"'] & (XA)^* XB \arrow[r, "f_{\pi_2}"] & A^* B \end{tikzcd} \quad \text{where } \pi_2 : H_X(Y) \to H(Y).$$

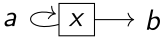
$\boxed{\mathbf{Rel}}$

Recap

Because the monad P has lots of nice properties, we automatically get trace/execution maps:

$$\begin{tikzcd} X \arrow[r, "\mathrm{exec}_c"] \arrow[rr, bend right, "\mathrm{tr}_c"] & (XA)^*XB \arrow[r, "f_{\pi_2}"] & A^*B \end{tikzcd}$$

$\boxed{\textbf{Rel}}$

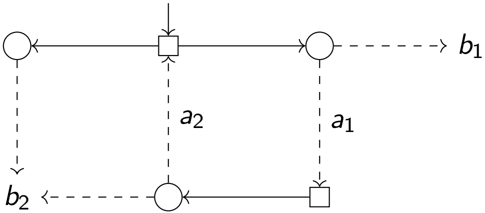


$$\mathrm{exec}_c(x) = \{xb, xaxb, xaxaxb, xaxaxaxb, \dots\}$$

$$\mathrm{tr}_c(x) = \{b, ab, aab, aaab, \dots\}$$

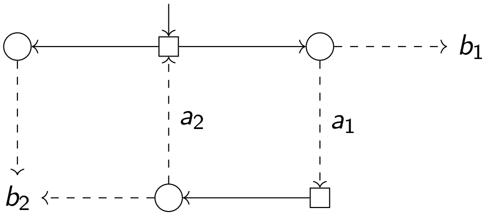
Games

Recall:



How do we turn this into a function $X \to M(HX)$?
i.e. Which monad M do we choose?

Finding the monad



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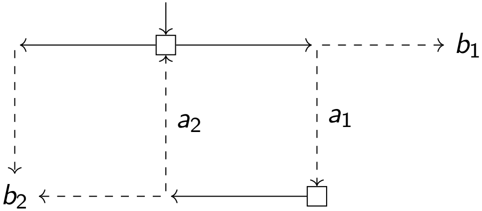

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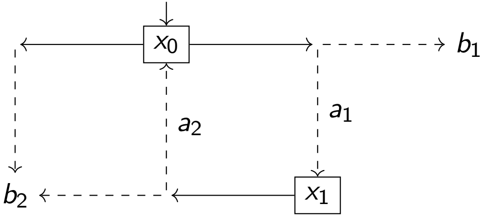

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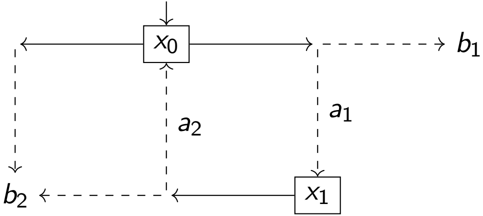

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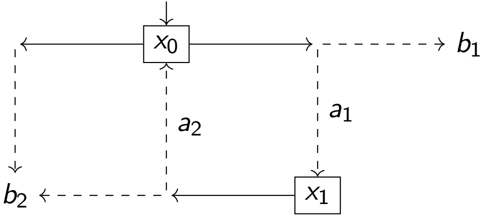

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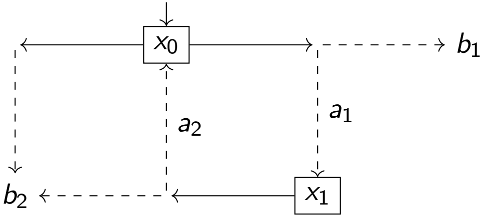

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Is PP a monad?

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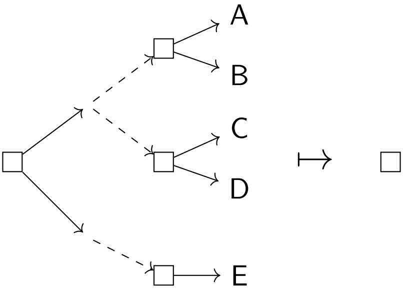
Let $A, B, C, D, E \subseteq X$



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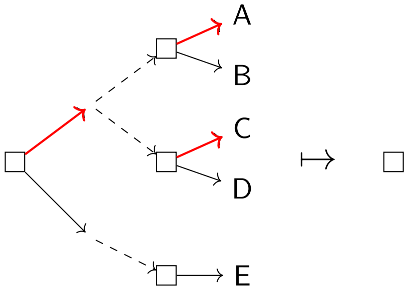

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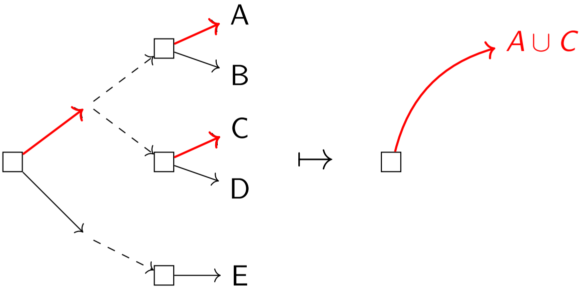

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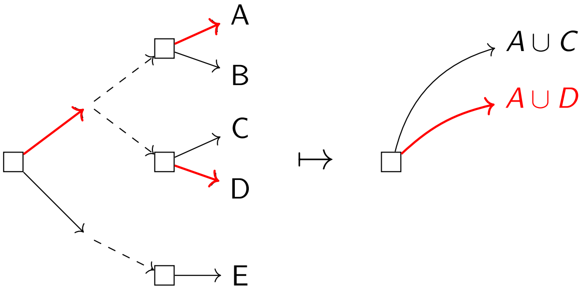

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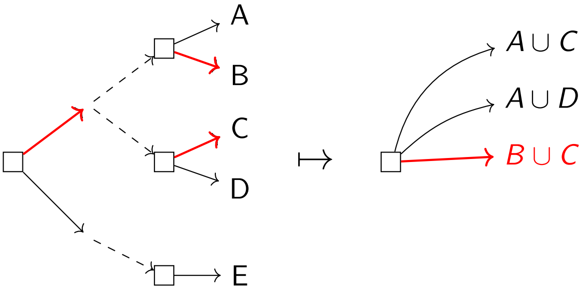

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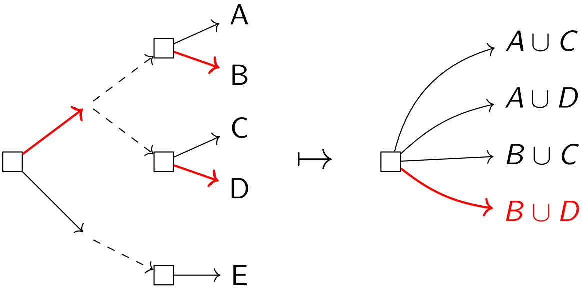

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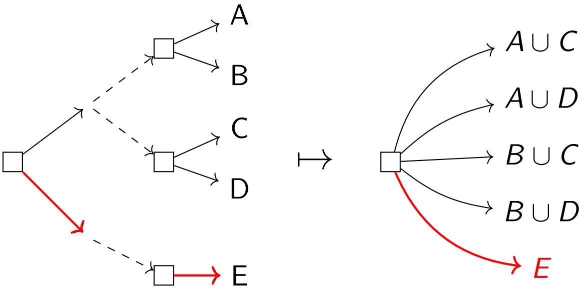

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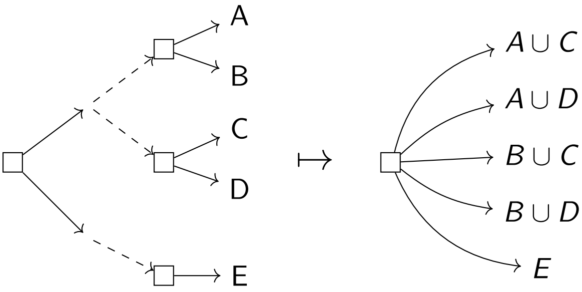

$$\{\{\{A, B\}, \{C, D\}\}, \{\{E\}\}\} \mapsto \{A \cup C, A \cup D, B \cup C, B \cup D, E\}$$

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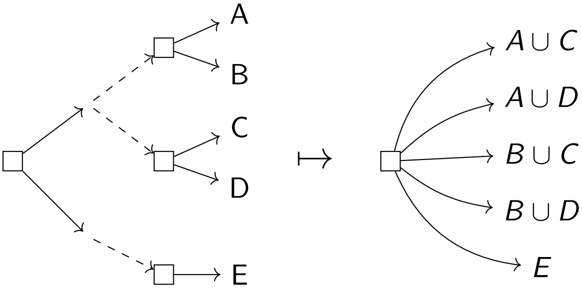

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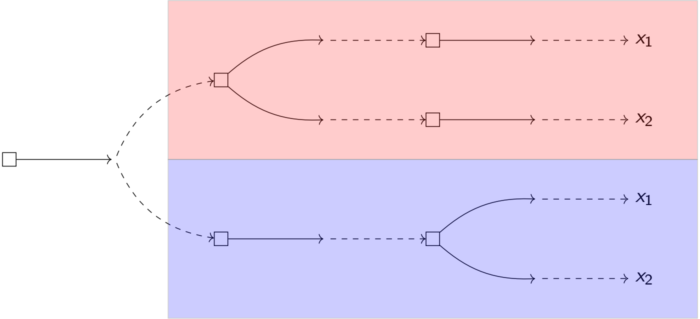
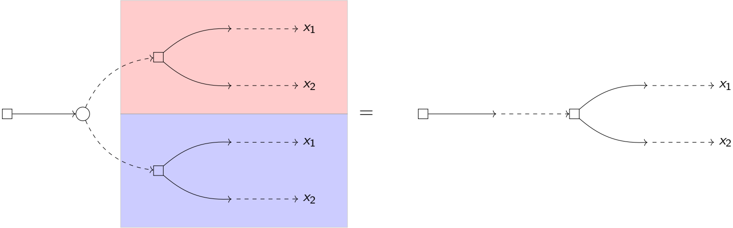
$$\Upsilon \in PPPP(X) \mapsto \{\bigcup \mathrm{Im}(f) \mid \exists \upsilon \in \Upsilon,\ f : \upsilon \xrightarrow{*} PP(X)\}$$

where $f : \upsilon \xrightarrow{*} PP(X)$ is a *choice function*: $\forall \mathcal{U} \in \upsilon : \mathcal{U} \in f(\mathcal{U})$.

Failure of associativity





A monad structure on PP is a "no-go" [KS18]

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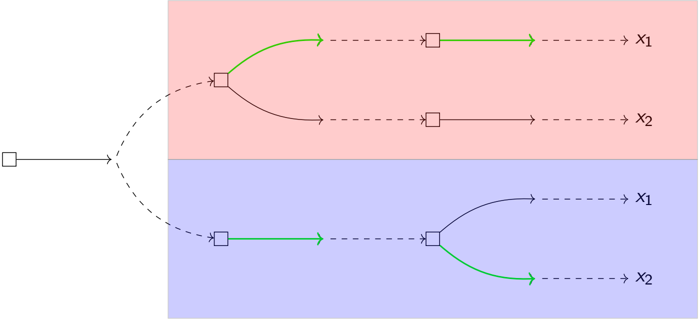

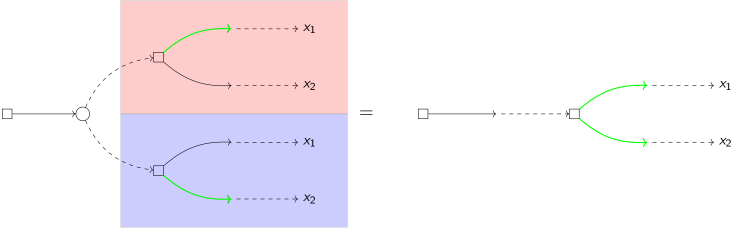

(Weak) Distributive laws

Two solutions:

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This weakening of distributive laws were popularised by [Gar20], for discussion of convexity see [BS22]

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- Given two monads (S, μ^T, η^T) and (T, μ^T, η^T), a *distributive law* $\delta : TS \to ST$ is a natural transformation satisfying some coherence conditions involving $\mu^T, \mu^S, \eta^T, \eta^S$.

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- A *weak distributive law* $\delta : TS \to ST$ only satisfies the diagrams involving μ^T, μ^S, η^S.

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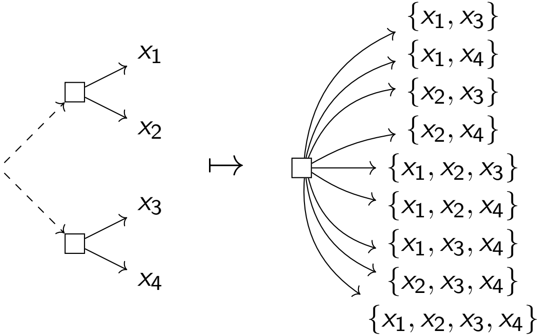
A weak distributive law $\delta : PP \to PP$

- Gives us a way of swapping environment-then-controller branching into controller-then-environment.



$$\delta(\{U_i\}_{i \in I}) = \{\bigcup_{i \in I} V_i \mid V_i \subseteq^+ U_i \text{ for all } i \in I\}$$

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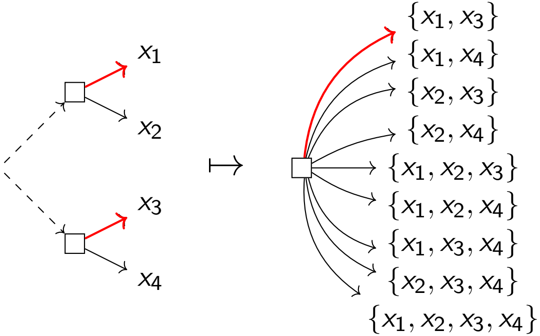

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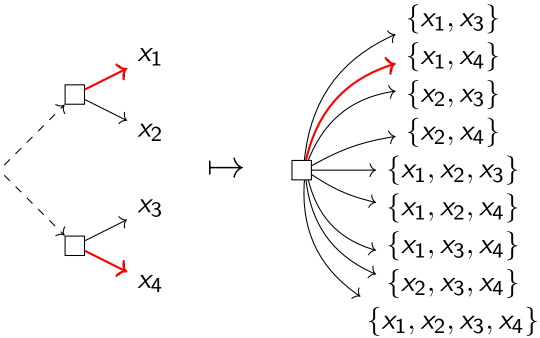

$$\delta(\{U_i\}_{i \in I}) = \{\bigcup_{i \in I} V_i \mid V_i \subseteq^+ U_i \text{ for all } i \in I\}$$

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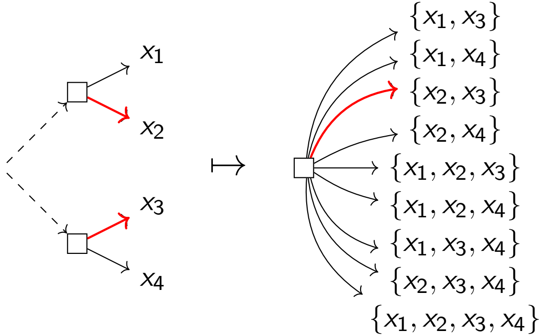

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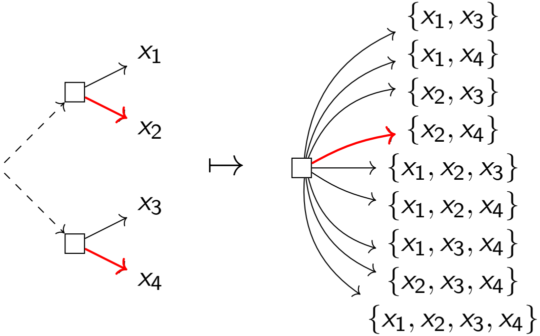

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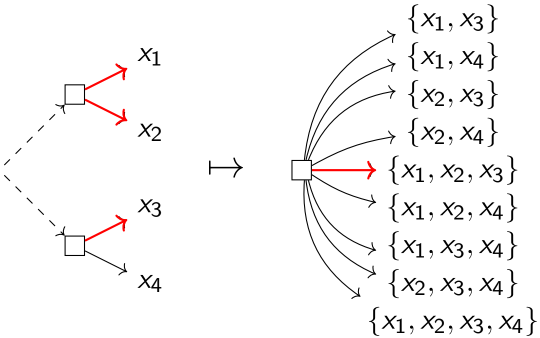

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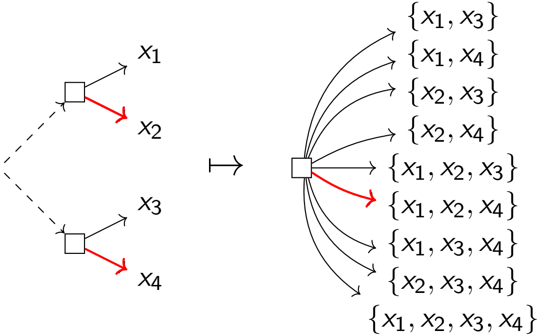

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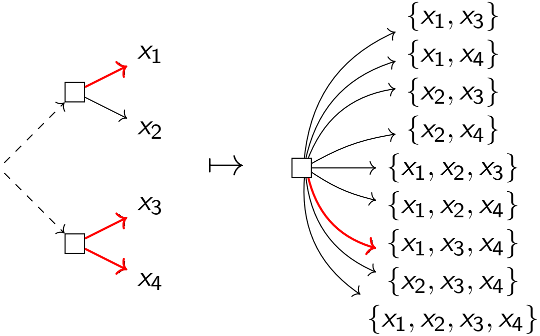

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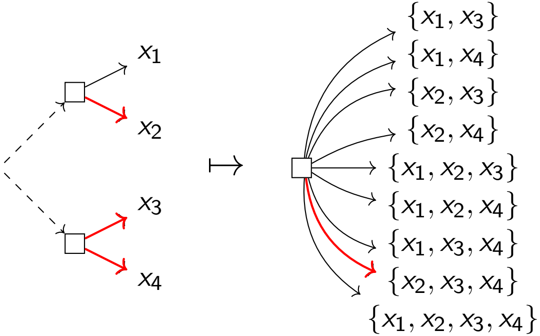

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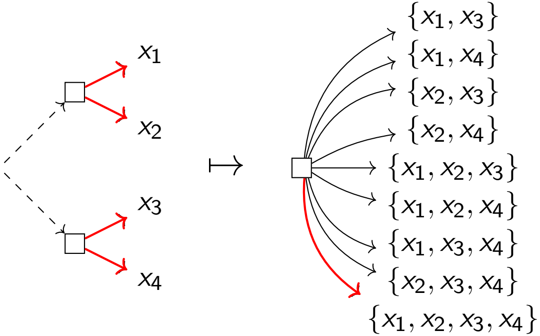

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Trace semantics

- We can build a monad

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- Recall: General categorical machinery allows us to lift this chain to the category of relations, and reverse the arrows:

$$\boxed{\mathbf{Kl}(\widetilde{PP})}$$

$$\begin{tikzcd}
 & & & A^*B \arrow[dlll, bend right, "\pi_0"'] \arrow[dll, bend right, "\pi_1"'] \arrow[dl, "\pi_2"'] \\
B & B + AB \arrow[l] & B + AB + AAB \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

with various assumptions on $\widetilde{PP}$

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-
-
-
-

These fixes are discussed in [PC25] (to appear)

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$$\delta : QP \to PQ$$

$$\delta(\{U_1, \ldots, U_n\}) := \{V_1 \cup \cdots \cup V_n \mid V_i \subseteq^{+}_{\omega} U\}$$

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Traces and Executions

$A^* B$ is the final $B + A(-)$-coalgebra in $\mathbf{Kl}(\widetilde{PQ})$.

$$\boxed{\mathbf{Kl}(\widetilde{PQ})}$$

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Thus we have trace and execution maps by coinduction:

$$\begin{tikzcd}
X \arrow[r, dashed, "\mathrm{tr}_c"] \arrow[d, "c"] & A^*B \arrow[d, "\zeta", "\wr"'] \\
B + A \times X \arrow[r, dashed, "B+A\times\mathrm{tr}_c"] & B + A \times A^*B
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X \times (B + A \times X) \arrow[r, dashed, "B+A\times\mathrm{exec}_c"] & X \times (B + A \times (XA)^*XB)
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$$

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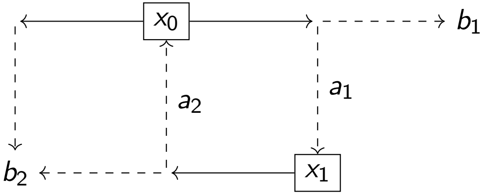
What is $\mathrm{tr}_c(x_0)$?



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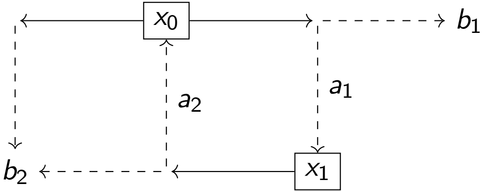

$\{b_1\}$ No

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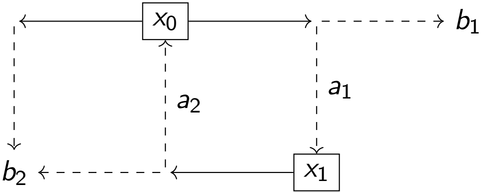

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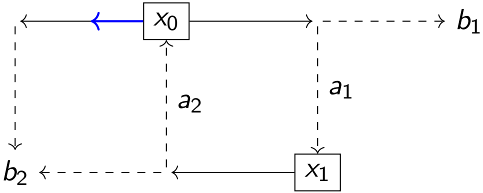

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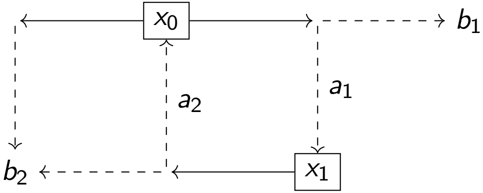

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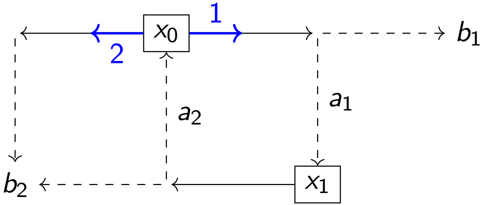

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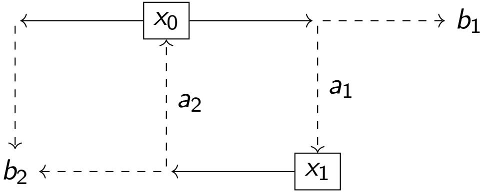

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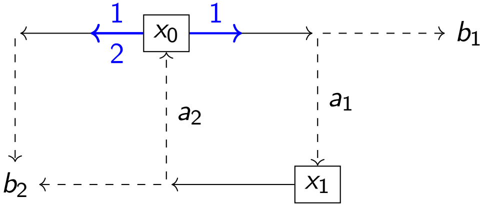

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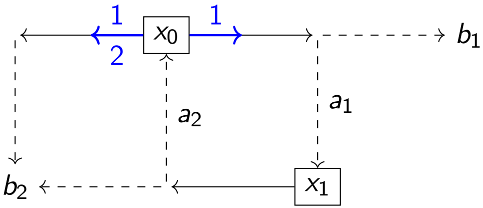

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(Theorem sketch) for all $U \subseteq (XA)^*XB$:

$U \in \mathrm{exec}_c(x) \implies$ there is a strategy which enforces U

$U \in \mathrm{exec}_c(x) \stackrel{*}{\impliedby}$ there is a strategy which enforces U

*almost

Executions on the final sequence

$$
\begin{tikzcd}
X \arrow[r, "c^*"] \arrow[d, "!"] & \overline{H}(X) \arrow[r, "\overline{H}(c^*)"] \arrow[d, "\overline{H}(!)"] & \overline{H^2}(X) \arrow[r, "\overline{H^2}(c^*)"] \arrow[d, "\overline{H^2}(!)"] & \overline{H^3}(X) \arrow[r] \arrow[d, "\overline{H^3}(!)"] & \cdots \\
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0 & XB & XB + XAXB & (XA)^{<3}XB & \\
\boxed{\mathbf{Kl}(\widetilde{PQ})} & & & & (XA)^*XB \arrow[ullll, bend left, "\pi_0"] \arrow[ulll, bend left, "\pi_1"] \arrow[ull, bend left, "\pi_2"] \arrow[ul, bend right, "\pi_3"']
\end{tikzcd}
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where $\overline{H} : \mathbf{Kl}(\widetilde{PQ}) \to \mathbf{Kl}(\widetilde{PQ})$ is the lifting of $X \times (B + A \times (-))$

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\begin{tikzcd}
X \arrow[r, "c^*"] \arrow[d, "!"] \arrow[rrrrddd, dashed, bend left, "\mathrm{exec}_c"] & \overline{H}(X) \arrow[r, "\overline{H}(c^*)"] \arrow[d, "\overline{H}(!)"] & \overline{H^2}(X) \arrow[r, "\overline{H^2}(c^*)"] \arrow[d, "\overline{H^2}(!)"] & \overline{H^3}(X) \arrow[r] \arrow[d, "\overline{H^3}(!)"] & \cdots \\
0 \arrow[d, equal] & \overline{H}(0) \arrow[l] \arrow[d, equal] & \overline{H^2}(0) \arrow[l] \arrow[d, equal] & \overline{H^3}(0) \arrow[l] \arrow[d, equal] & \cdots \arrow[l] \\
0 & XB & XB + XAXB & (XA)^{<3}XB & \\
\boxed{\mathbf{Kl}(\widetilde{PQ})} & & & & (XA)^*XB \arrow[llllu, "\pi_0"] \arrow[lllu, "\pi_1"] \arrow[llu, "\pi_2"] \arrow[lu, "\pi_3"']
\end{tikzcd}
$$

where $\overline{H} : \mathbf{Kl}(\widetilde{PQ}) \to \mathbf{Kl}(\widetilde{PQ})$ is the lifting of $X \times (B + A \times (-))$

Where are the strategies?

- $\sigma_0 : 1 \to Q(X)$ will pick an initial state

$$\boxed{\mathbf{Kl}(Q)}$$

$$\begin{tikzcd}
 & X \\
1 \arrow[ur, "\sigma_0"] \arrow[r, dashed] & \mathrm{Im}(\sigma_0) \arrow[u, hook]
\end{tikzcd}$$

Where are the strategies?

- $\sigma_0 : 1 \to Q(X)$ will pick an initial state
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$$\begin{tikzcd}
 & X & \widehat{H}(X) & \widehat{H^2}(X) & \cdots \\
1 \arrow[ur, "\sigma_0"] \arrow[r, dashed] & \mathrm{Im}(\sigma_0) \arrow[u, hook] \arrow[ur, "\sigma_1"] \arrow[r, dashed] & \mathrm{Im}(\sigma_1) \arrow[u, hook] \arrow[ur, "\sigma_2"] \arrow[r, dashed] & \mathrm{Im}(\sigma_2) \arrow[u, hook] \arrow[ur, "\sigma_3"] \arrow[r, dashed] & \cdots
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\end{tikzcd}
$$

(left) σ extends partial plays, (right) we choose a successor in c:

$$
\begin{tikzcd}
\widehat{H^n}(X) & \widehat{H^{n+1}}(X) \arrow[l, "\widehat{H^n}(\widehat{\pi_1})"'] \\
\mathrm{Im}(\sigma_n) \arrow[u, hook] \arrow[ur, "\sigma_{n+1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{H^n}(X) \arrow[r, "\overline{H^n}(c^*)"] \arrow[dr, phantom, "\sqsupseteq"] & \overline{H^{n+1}}(X) \\
\mathrm{Im}(\sigma_n) \arrow[u, hook] \arrow[ur, "\eta^P \circ \sigma_n"']
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\mathrm{Im}(\sigma_n) \arrow[u, hook] \arrow[ur, "\eta^P \circ \sigma_n"']
\end{tikzcd}$$

The n-depth plays comes from composition:

$$\mathrm{play}_n^{\sigma} = (1 \dashrightarrow \mathrm{Im}(\sigma_0) \dashrightarrow \cdots \dashrightarrow \mathrm{Im}(\sigma_n) \rightarrowtail H^n(X))$$

Play outcomes

To define the play outcome, first lift a strategy into $\mathbf{Kl}(\widetilde{PQ})$

$$\eta^P \circ \sigma_n : \mathrm{Im}(\sigma_n) \to \widetilde{PQ}H^{n+1}(X)$$

Then we can reuse that $(XA)^*XB$ is the limit of the final sequence:

$$\begin{tikzcd}
1 \arrow[d, "\eta^P \circ \mathsf{play}_0^\sigma"] \arrow[dr, "\eta^P \circ \mathsf{play}_1^\sigma"] \arrow[drr, "\eta^P \circ \mathsf{play}_2^\sigma"] \arrow[drrr, "\eta^P \circ \mathsf{play}_3^\sigma"] \arrow[ddddrrrr, dashed, red, bend left, "\mathsf{play}_c^\sigma"] & & & & \\
X \arrow[d, "!"] & \overline{H}(X) \arrow[d, "\overline{H}(!)"] & \overline{H^2}(X) \arrow[d, "\overline{H^2}(!)"] & \overline{H^3}(X) \arrow[d, "\overline{H^3}(!)"] & \cdots \\
0 \arrow[d, equal] & \overline{H}(0) \arrow[l] \arrow[d, equal] & \overline{H^2}(0) \arrow[l] \arrow[d, equal] & \overline{H^3}(0) \arrow[l] \arrow[d, equal] & \cdots \arrow[l] \\
0 & XB & XB + XAXB & (XA)^{<3}XB & \\
\boxed{\mathbf{Kl}(\widetilde{PQ})} & & & & (XA)^*XB \arrow[ullll, bend left, "\pi_0"] \arrow[ulll, bend left, "\pi_1"] \arrow[ull, bend left, "\pi_2"] \arrow[ul, bend left, "\pi_3"]
\end{tikzcd}$$

Main theorem

Theorem

$$\mathrm{exec}_c(x) = \bigcup_{\sigma \text{ starts in } x} \mathrm{play}_c^\sigma$$

Lemma

$$c_n^*(x) = \{\mathrm{play}_n^\sigma \mid \sigma \text{ starts in } x\}$$

- What do we gain from doing this coalgebraically?

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  - Replace Q with the finite distribution monad D!

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- What do we gain from doing this coalgebraically?
  - Replace Q with the finite distribution monad D!
  - Generic coinductive algorithms for strategy synthesis.

Conclusion

- Towards strategy synthesis...
  - Product construction?
  - General theorem about memoryless strategies?
  - Infinite traces, continuous probability monads?
- An axiomatic presentation.
- Simple stochastic games?

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