Traces via Strategies

(Games via Coalgebra)

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Outline

- Games
- Traces
- Representing games coalgebraically
- Strategies
- Traces via strategies

Motivation: controller synthesis

- Model the possible actions of the controller and the environment as a game.

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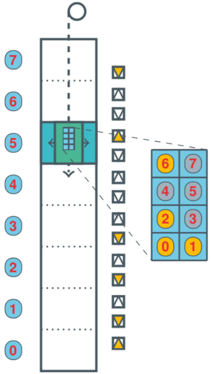
- Example: “nobody has to wait more then seven levels on the lift”



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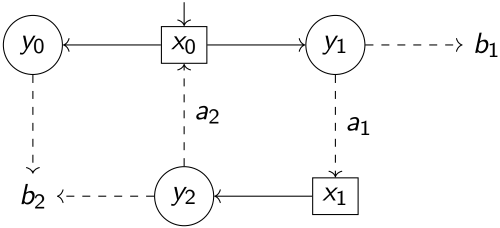

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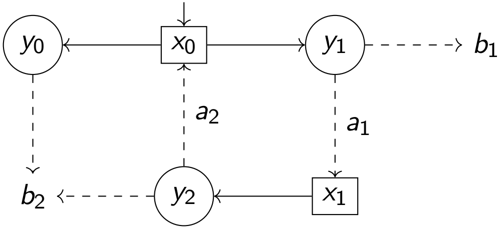

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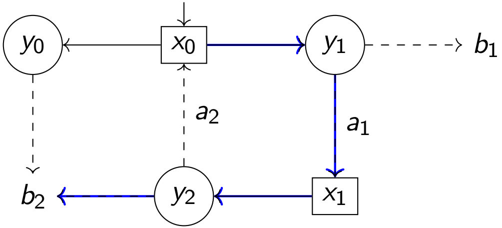

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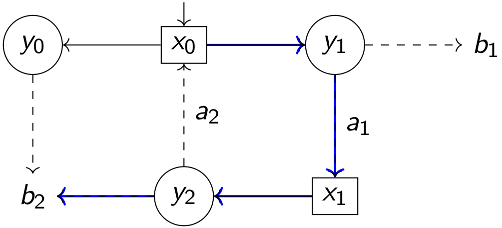

- A play is a sequence of states and observations, arising from controller and environment moves, ending in a terminating observation. e.g. $x_0 y_1 a_1 x_1 y_2 b_2$
- A strategy is a partial function which extends plays, it must be defined over all plays which conform to it.

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P is a monad with $\mathbf{Kl}(P) \cong \mathbf{Rel}$

Traces, Coalgebraically

$A^* B$ is the *initial algebra* for the functor $B + A(-) : \mathbf{Set} \to \mathbf{Set}$

$\boxed{\mathbf{Set}}$

$$
\begin{tikzcd}
 & & & A^* B \\
B \arrow[r] \arrow[urrr, bend left, "\kappa_0"] & B + AB \arrow[r] \arrow[urr, bend left, "\kappa_1"] & B + AB + AAB \arrow[r] \arrow[ur, "\kappa_2"] & \cdots
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\end{tikzcd}$$

General categorical machinery allows us to lift this chain to the category of relations[1], and reverse the arrows:

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[1]With various assumptions, which we will come back to later

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Thus A^*B is a *final coalgebra* in the category of relations!

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Traces by Coinduction

For every LTS $c : X \to P(B + A \times X)$, there is a *unique coalgebra morphism* into A^*B.

$$\boxed{\textbf{Rel}} \quad \begin{tikzcd} X \arrow[r, dashed] \arrow[d, "c"] & A^*B \arrow[d, "\wr"] \\ B + A \times X \arrow[r, dashed] & B + A \times A^*B \end{tikzcd}$$

This dashed morphism in **Rel** is a function $X \to P(A^*B)$ which assigns each state to it's set of traces!

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- With $c^* : X \to PH_X(X)$ defined as the composite

$$(X \xrightarrow{\langle \mathrm{id}, c \rangle} X \times P(B + A \times X) \xrightarrow{\mathrm{stl}} P(X \times (B + A \times X))$$

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$$\begin{tikzcd} X \arrow[r, "\mathrm{exec}_c"] \arrow[rr, bend right, "\mathrm{tr}_c"'] & (XA)^* XB \arrow[r, "f_{\pi_2}"] & A^* B \end{tikzcd} \quad \text{where } \pi_2 : H_X(Y) \to H(Y).$$

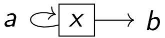
$\boxed{\mathbf{Rel}}$

Recap

Because the monad P has lots of nice properties, we automatically get trace/execution maps:

$$\begin{tikzcd} X \arrow[r, "\mathrm{exec}_c"] \arrow[rr, bend right, "\mathrm{tr}_c"'] & (XA)^* XB \arrow[r, "f_{\pi_2}"] & A^* B \end{tikzcd}$$

$$\boxed{\mathbf{Rel}}$$

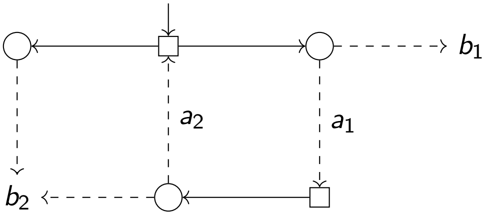


$$\mathrm{exec}_c(x) = \{xb, xaxb, xaxaxb, xaxaxaxb, \ldots\}$$

$$\mathrm{tr}_c(x) = \{b, ab, aab, aaab, \ldots\}$$

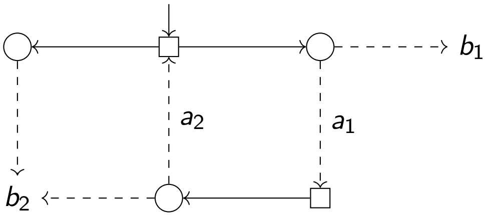
Games

Recall:



How do we turn this into a function $X \to M(HX)$?
i.e. Which monad M do we choose?

First attempt



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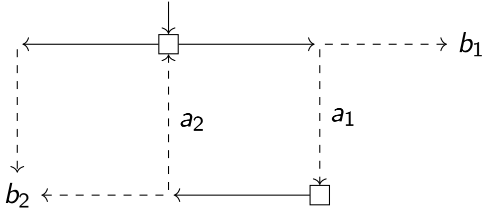

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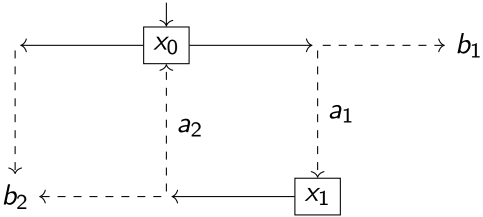

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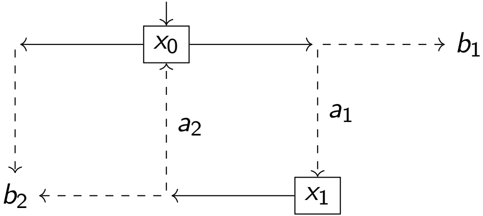

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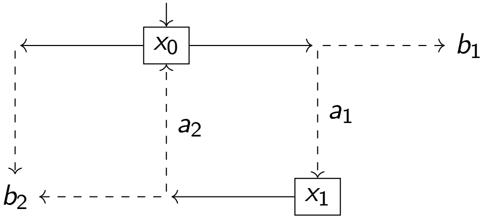

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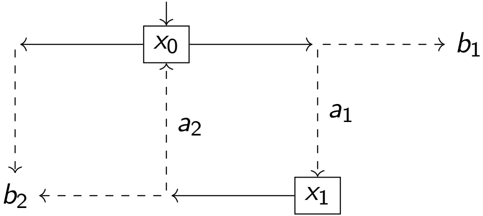

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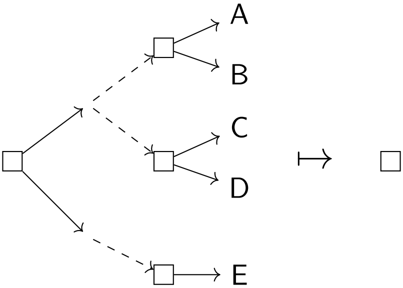
Is PP a monad?

How do I squash $PPPP(X) \to PP(X)$?

Let $A, B, C, D, E \subseteq X$



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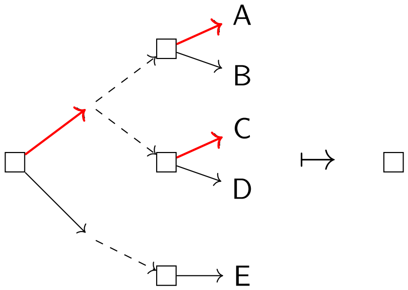

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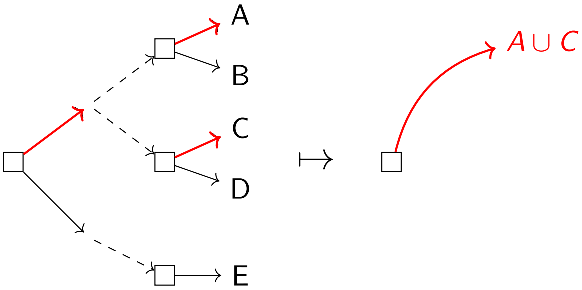

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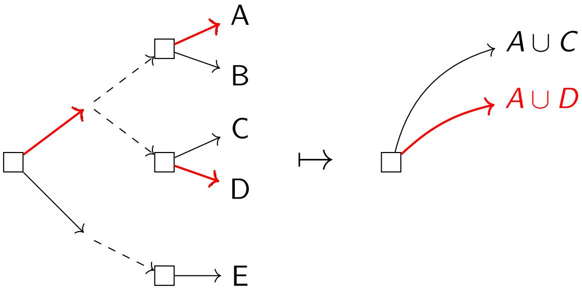

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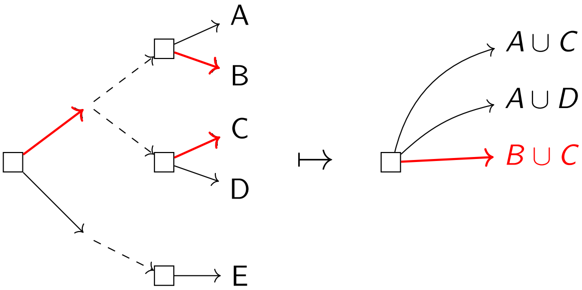

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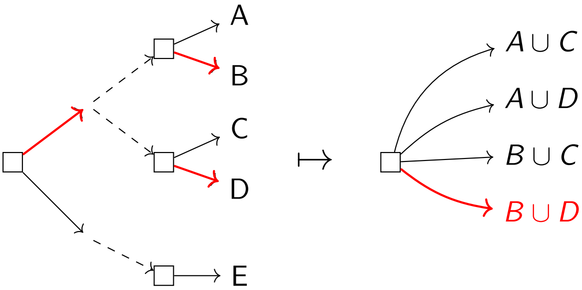

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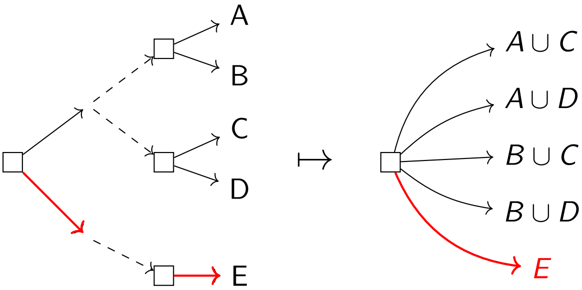

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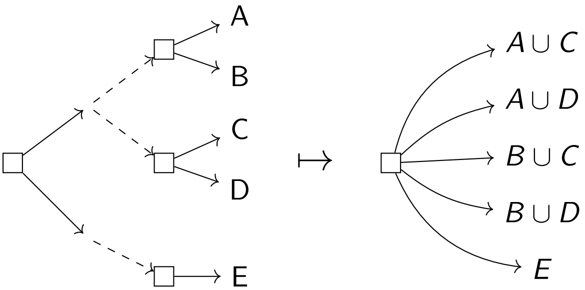

$$\{\{\{A, B\}, \{C, D\}\}, \{\{E\}\}\} \mapsto \{A \cup C, A \cup D, B \cup C, B \cup D, E\}$$

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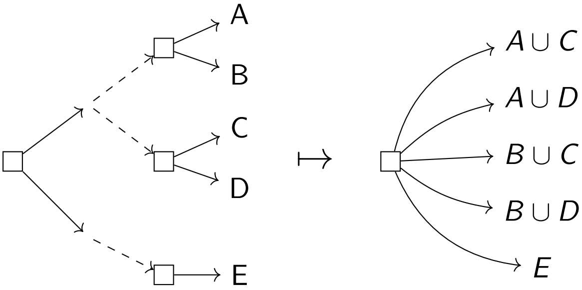

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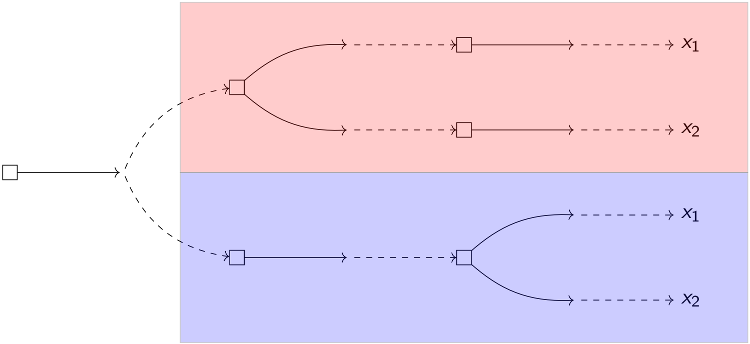
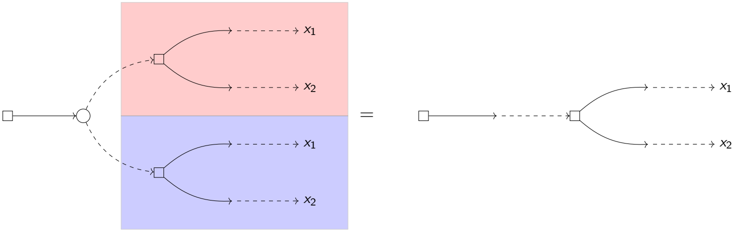
$$\Upsilon \in PPPP(X) \mapsto \{\bigcup \mathrm{Im}(f) \mid \exists \upsilon \in \Upsilon,\ f : \upsilon \xrightarrow{*} PP(X)\}$$

where $f : \upsilon \xrightarrow{*} PP(X)$ is a *choice function*: $\forall \mathcal{U} \in \upsilon : \mathcal{U} \in f(\mathcal{U})$.

Failure of Associativity





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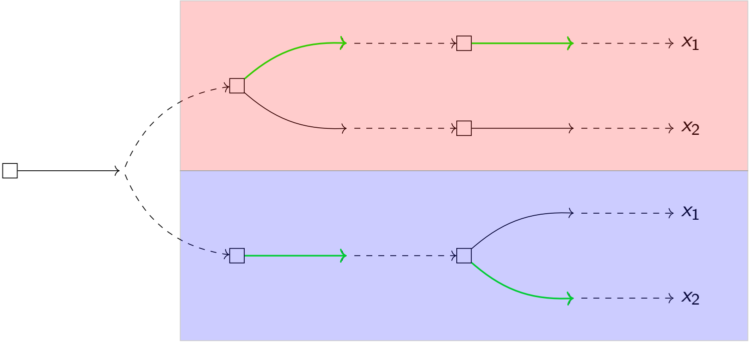

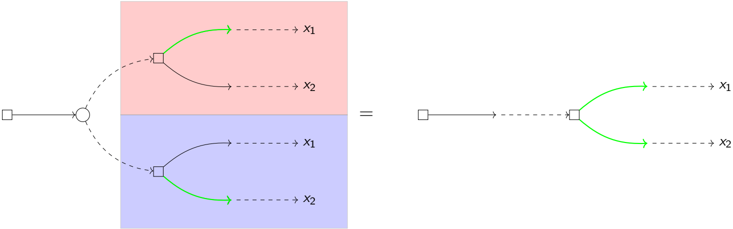

Distributive Laws

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- Given two monads (S, μ^T, η^T) and (T, μ^T, η^T), a *distribute law* $\delta : TS \to ST$ is a natural transformation satisfying some coherence conditions involving $\mu^T, \mu^S, \eta^T, \eta^S$.

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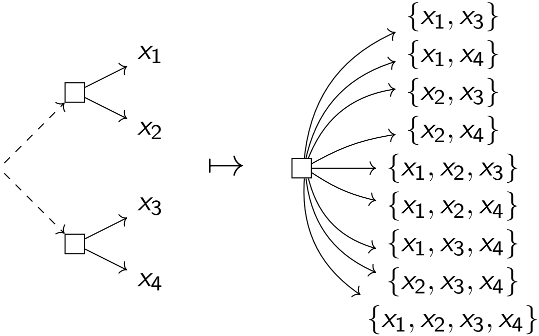
- A *weak distributive law* $\delta : TS \to ST$ only satisfies the diagrams involving μ^T, μ^S, η^S.

A weak distributive law $\delta : PP \to PP$

- Gives us a way of swapping environment-then-controller branching into controller-then-environment.



$$\delta(\{U_i\}_{i \in I}) = \{\bigcup_{i \in I} V_i \mid V_i \subseteq^+ U_i \text{ for all } i \in I\}$$

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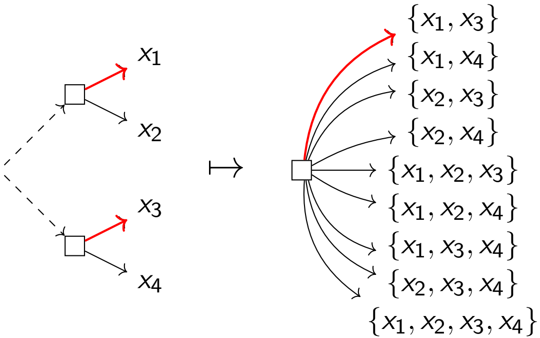

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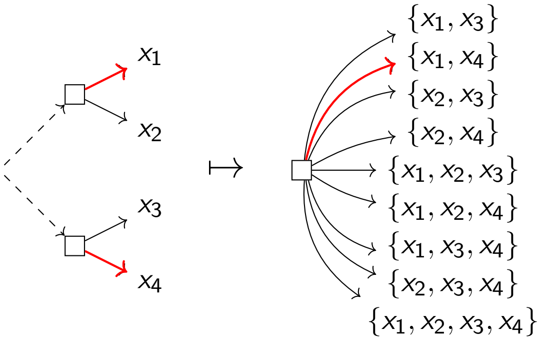

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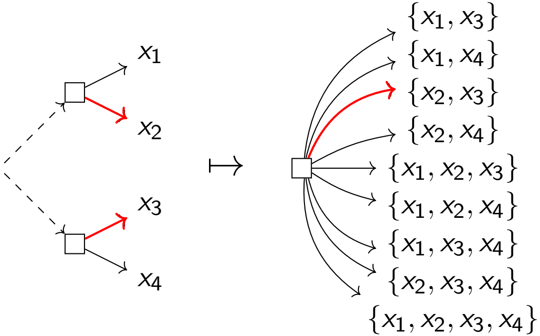

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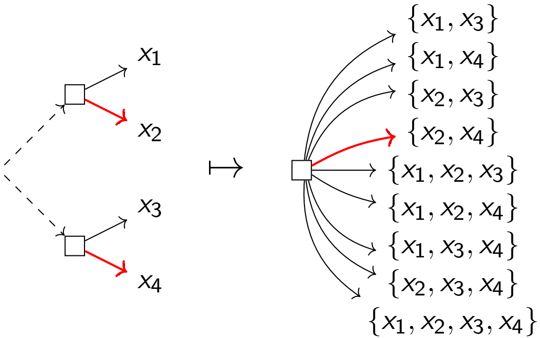

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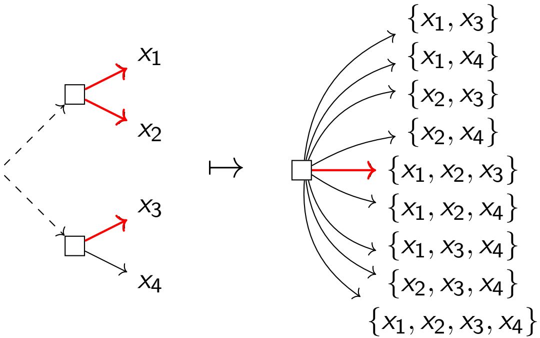

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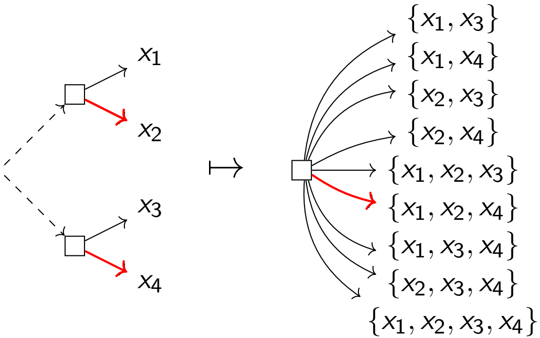

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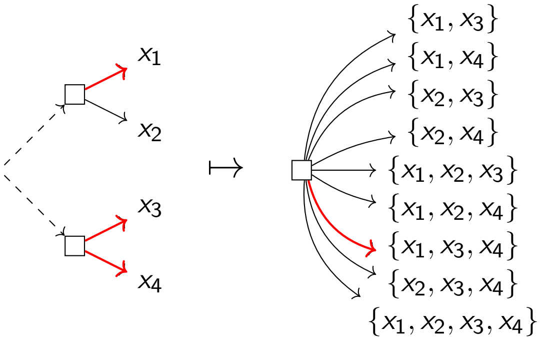

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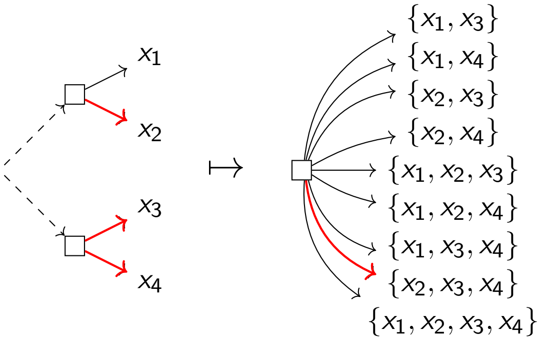

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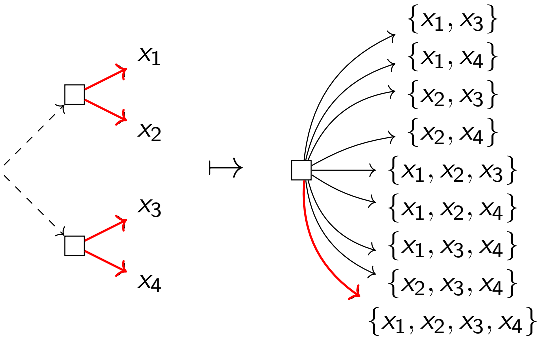

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Traces Semantics

- We can build a monad

$$\widetilde{PP}(X) = \{\mathcal{U} \subseteq X \mid \mathcal{U} \text{ is closed under arbitrary union}\}$$

$$\eta(x) = \{\{x\}\} \qquad \mu \text{ uses } \delta$$

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- Recall: General categorical machinery allows us to lift this chain to the category of relations, and reverse the arrows:

$$\boxed{\mathbf{Kl}(\widetilde{PP})}$$

$$\begin{tikzcd}
&&&&&& A^*B \arrow[llllll, bend right, "\pi_0"'] \arrow[llll, bend right, "\pi_1"'] \arrow[ll, bend right, "\pi_2"'] \\
B & B + AB \arrow[l] && B + AB + AAB \arrow[ll] && \cdots \arrow[ll]
\end{tikzcd}$$

with various assumptions on $\widetilde{PP}$

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$$\delta : QP \to PQ$$

$$\delta(\{U_1, \ldots, U_n\}) := \{V_1 \cup \cdots \cup V_n \mid V_i \subseteq^+_\omega U\}$$

Traces and Executions

A^*B is the final $B + A(-)$-coalgebra in $\mathbf{Kl}(\widetilde{PQ})$.

$$\boxed{\mathbf{Kl}(\widetilde{PQ})}$$

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Thus we have trace and execution maps by coinduction:

$$\begin{tikzcd}
X \arrow[r, dashed, "\mathrm{tr}_c"] \arrow[d, "c"] & A^*B \arrow[d, "\zeta", "\wr"'] \\
B + A \times X \arrow[r, dashed, "B+A\times\mathrm{tr}_c"] & B + A \times A^*B
\end{tikzcd}$$

$$\mathrm{tr}_c : X \to \widetilde{PQ}(A^*B)$$

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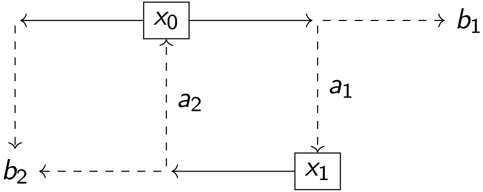
\end{tikzcd}$$

Thus we have trace and execution maps by coinduction:

$$\begin{tikzcd}
X \arrow[r, dashed, "\mathsf{exec}_c"] \arrow[d, "c^*"] & (XA)^*XB \arrow[d, "\zeta", "\wr"'] \\
X \times (B + A \times X) \arrow[r, dashed, "B+A\times\mathsf{exec}_c"] & X \times (B + A \times (XA)^*XB)
\end{tikzcd}$$

$$\mathsf{exec}_c : X \to \widetilde{PQ}((XA)^*XB)$$

What is $\mathrm{exec}_c(x_0)$?



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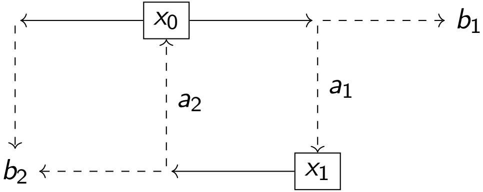

$\{x_0\, b_1\}$ No

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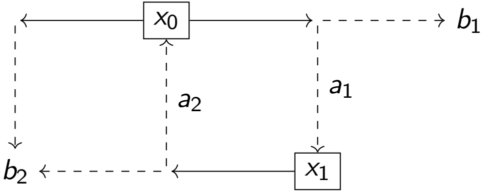

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$\{x_0\, b_2\}$ Yes

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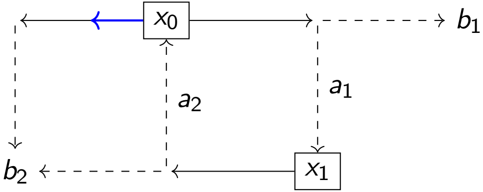

$\{x_0\,b_1\}$	No
$\{x_0\,b_2\}$	Yes

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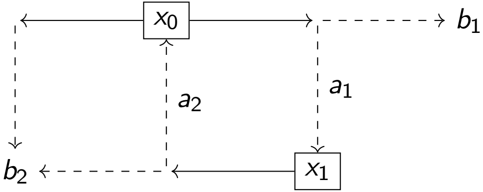

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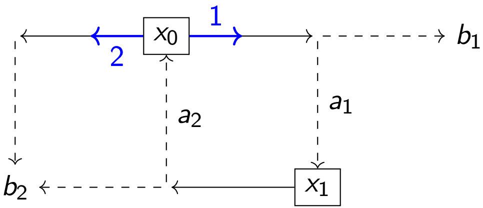

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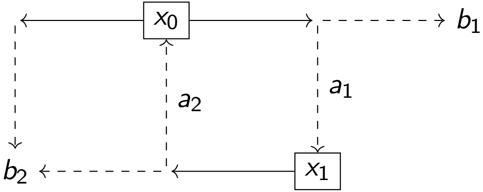

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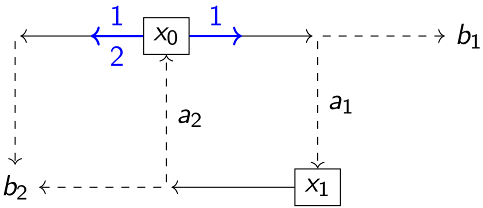

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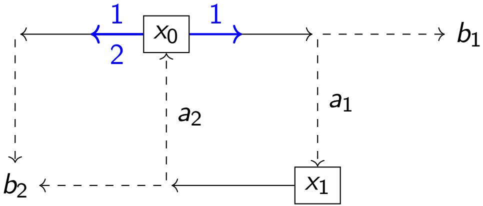

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Theorem

$U \in \mathrm{exec}_c(x) \implies$ there is a strategy which enforces U

$U \in \mathrm{exec}_c(x) \Longleftarrow^{2}$ there is a strategy which enforces U

[2] almost

Executions on the final sequence

$$
\begin{tikzcd}
X \arrow[r, "c^*"] \arrow[d, "!"] & \overline{H}(X) \arrow[r, "\overline{H}(c^*)"] \arrow[d, "\overline{H}(!)"] & \overline{H^2}(X) \arrow[r, "\overline{H^2}(c^*)"] \arrow[d, "\overline{H^2}(!)"] & \overline{H^3}(X) \arrow[r] \arrow[d, "\overline{H^3}(!)"] & \cdots \\
0 \arrow[d, equal] & \overline{H}(0) \arrow[l] \arrow[d, equal] & \overline{H^2}(0) \arrow[l] \arrow[d, equal] & \overline{H^3}(0) \arrow[l] \arrow[d, equal] & \cdots \arrow[l] \\
0 & XB & XB + XAXB & (XA)^{<3}XB & \\
\boxed{\mathbf{Kl}(\widetilde{PQ})} & & & & (XA)^*XB \arrow[ullll, bend left, "\pi_0"] \arrow[ulll, bend left, "\pi_1"] \arrow[ull, bend left, "\pi_2"] \arrow[ul, bend left, "\pi_3"]
\end{tikzcd}
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where $\overline{H} : \mathbf{Kl}(\widetilde{PQ}) \to \mathbf{Kl}(\widetilde{PQ})$ is the lifting of $X \times (B + A \times (-))$

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0 \arrow[d, equal] & \overline{H}(0) \arrow[l] \arrow[d, phantom, "\shortparallel"] & \overline{H^2}(0) \arrow[l] \arrow[d, phantom, "\shortparallel"] & \overline{H^3}(0) \arrow[l] \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[l] \\
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Where are the strategies?

- $\sigma_0 : 1 \to Q(X)$ will pick an initial state

$$\boxed{\mathbf{Kl}(Q)} \qquad \begin{tikzcd} & X \\ 1 \arrow[ur, "\sigma_0"] \arrow[r, dashed] & \mathrm{Im}(\sigma_0) \arrow[u, hook] \end{tikzcd}$$

Where are the strategies?

- $\sigma_0 : 1 \to Q(X)$ will pick an initial state
- $\sigma_{n+1} : \mathrm{Im}(\sigma_n) \to QH^{n+1}(X)$ extends an n-length play

$\boxed{\mathbf{Kl}(Q)}$

$$\begin{tikzcd}
 & X \\
1 \arrow[ur, "\sigma_0"] \arrow[r, dashed] & \mathrm{Im}(\sigma_0) \arrow[u, hook]
\end{tikzcd}$$

Where are the strategies?

- $\sigma_0 : 1 \to Q(X)$ will pick an initial state
- $\sigma_{n+1} : \mathrm{Im}(\sigma_n) \to QH^{n+1}(X)$ extends an n-length play

$$
\begin{tikzcd}
\boxed{\mathbf{Kl}(Q)} & X & \widehat{H}(X) & \widehat{H^2}(X) & \cdots \\
1 \arrow[ur, "\sigma_0"] \arrow[r, dashed] & \mathrm{Im}(\sigma_0) \arrow[u, hook] \arrow[ur, "\sigma_1"] \arrow[r, dashed] & \mathrm{Im}(\sigma_1) \arrow[u, hook] \arrow[ur, "\sigma_2"] \arrow[r, dashed] & \mathrm{Im}(\sigma_2) \arrow[u, hook] \arrow[ur, "\sigma_3"] \arrow[r, dashed] & \cdots
\end{tikzcd}
$$

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$\boxed{\mathbf{Kl}(Q)}$

$$\begin{tikzcd}
 & X & \widehat{H}(X) & \widehat{H^2}(X) & \cdots \\
1 \arrow[ur, "\sigma_0"] \arrow[r, dashed] & \mathrm{Im}(\sigma_0) \arrow[u, hook] \arrow[ur, "\sigma_1"] \arrow[r, dashed] & \mathrm{Im}(\sigma_1) \arrow[u, hook] \arrow[ur, "\sigma_2"] \arrow[r, dashed] & \mathrm{Im}(\sigma_2) \arrow[u, hook] \arrow[ur, "\sigma_3"] \arrow[r, dashed] & \cdots
\end{tikzcd}$$

Subject to:

$$\begin{tikzcd}
\widehat{H^n}(X) & \widehat{H^{n+1}}(X) \arrow[l, "\widehat{H^n}(\widehat{\pi_1})"'] \\
\mathrm{Im}(\sigma_n) \arrow[u, hook] \arrow[ur, "\sigma_{n+1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{H^n}(X) \arrow[r, "\overline{H^n}(c^*)"] \arrow[dr, phantom, "\sqsupseteq"] & \overline{H^{n+1}}(X) \\
\mathrm{Im}(\sigma_n) \arrow[u, hook] \arrow[ur, "\eta^P \circ \sigma_n"']
\end{tikzcd}$$

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$\boxed{\mathbf{Kl}(Q)}$

$$
\begin{tikzcd}
 & X & \widehat{H}(X) & \widehat{H^2}(X) & \cdots \\
1 \arrow[ur, "\sigma_0"] \arrow[r, dashed] & \mathrm{Im}(\sigma_0) \arrow[u, hook] \arrow[ur, "\sigma_1"] \arrow[r, dashed] & \mathrm{Im}(\sigma_1) \arrow[u, hook] \arrow[ur, "\sigma_2"] \arrow[r, dashed] & \mathrm{Im}(\sigma_2) \arrow[u, hook] \arrow[ur, "\sigma_3"] \arrow[r, dashed] & \cdots
\end{tikzcd}
$$

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$$
\begin{tikzcd}
\widehat{H^n}(X) & \widehat{H^{n+1}}(X) \arrow[l, "\widehat{H^n}(\widehat{\pi_1})"'] \\
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\overline{H^n}(X) \arrow[r, "\overline{H^n}(c^*)"] & \overline{H^{n+1}}(X) \\
\mathrm{Im}(\sigma_n) \arrow[u, hook] \arrow[ur, "\eta^P \circ \sigma_n"'] \arrow[u, phantom, shift left=6, "\sqsupseteq"]
\end{tikzcd}
$$

The n-depth plays comes from composition:

$$\mathrm{play}_n^\sigma = (1 \dashrightarrow \mathrm{Im}(\sigma_0) \dashrightarrow \cdots \dashrightarrow \mathrm{Im}(\sigma_n) \rightarrowtail H^n(X))$$

Play outcomes

To define the play outcome, first lift a strategy into $\mathbf{Kl}(\widetilde{PQ})$

$$\{-\} \circ \sigma_n : \mathsf{Im}(\sigma_n) \to \widetilde{PQ}H^{n+1}(X)$$

Then we can reuse that $(XA)^*XB$ is the limit of the final sequence:

$$\begin{tikzcd}
1 \arrow[d, "\mathsf{play}_0^\sigma"] \arrow[dr, bend left, "\mathsf{play}_1^\sigma"] \arrow[drr, bend left, "\mathsf{play}_2^\sigma"] \arrow[drrr, bend left, "\mathsf{play}_3^\sigma"] \arrow[ddddrrrr, bend left, dashed, red, "\mathsf{play}_c^\sigma"] & & & & \\
X \arrow[d, "!"] & \overline{H}(X) \arrow[d, "\overline{H}(!)"] & \overline{H^2}(X) \arrow[d, "\overline{H^2}(!)"] & \overline{H^3}(X) \arrow[d, "\overline{H^3}(!)"] & \cdots \\
0 \arrow[d, equal] & \overline{H}(0) \arrow[l] \arrow[d, equal] & \overline{H^2}(0) \arrow[l] \arrow[d, equal] & \overline{H^3}(0) \arrow[l] \arrow[d, equal] & \cdots \arrow[l] \\
0 & XB & XB + XAXB & (XA)^{<3}XB & \\
\boxed{\mathbf{Kl}(\widetilde{PQ})} & & & & (XA)^*XB \arrow[ullll, bend left, "\pi_0"] \arrow[ulll, bend left, "\pi_1"] \arrow[ull, bend left, "\pi_2"] \arrow[ul, bend left, "\pi_3"]
\end{tikzcd}$$

Main Theorem

Theorem

$$\mathrm{exec}_c(x) = \bigcup_{\sigma \text{ starts in } x} \mathrm{play}_c^\sigma$$

Lemma

$$c_n^*(x) = \{\mathrm{play}_n^\sigma \mid \sigma \text{ starts in } x\}$$

- What do we gain from doing this coalgebraically?

Main Theorem

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$$\mathrm{exec}_c(x) = \bigcup_{\sigma \text{ starts in } x} \mathrm{play}_c^{\sigma}$$

Lemma

$$c_n^*(x) = \{\mathrm{play}_n^{\sigma} \mid \sigma \text{ starts in } x\}$$

- What do we gain from doing this coalgebraically?
  - Replace Q with the finite distribution monad D!

Conclusion

- Towards strategy synthesis...
  - Product construction?
  - General theorem about memoryless strategies?
- Infinite traces, continuous probability monads?
- Simple stochastic games?