MATH5004 TUT9

2D FEM: Unsteady Boundary Value Problems

Consider the linear diffusion problem of scalar unknown u,

$$
\begin{aligned}
&u_t - \nabla \cdot (k \nabla u) = f && \text{in } \Omega \times \mathbf{I},\\
&u(\mathbf{x}, t) = 0 && \text{on } \partial\Omega_1 \times \mathbf{I},\\
&\frac{\partial u}{\partial n} = 1 - u && \text{on } \partial\Omega_2 \times \mathbf{I},\\
&u(\mathbf{x}, 0) = 1 && \text{in } \Omega,
\end{aligned}
$$

where $\mathbf{I}$ denotes the time interval $[0, T], \Omega \subset \mathbb{R}^2$ and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ is the boundary of Ω.

Step 1. Variational statement

Error function $\quad r = u_t - \nabla \cdot (k\nabla u) - f$

Total weighted residual $\quad R = \int_\Omega rv \ d\Omega = \int_\Omega (u_t - \nabla \cdot k\nabla u) - f)v \ d\Omega$

As $\quad -v\nabla \cdot (k\nabla u) = k\nabla u \cdot \nabla v - \nabla \cdot (vk\nabla u)$, we have

$$R = \int_\Omega [k\nabla u \cdot \nabla v + u_t v - fv - \nabla \cdot (vk\nabla u)] \, d\Omega$$

Using the Divergence Theorem

$$\int_\Omega \nabla \cdot (vk\nabla u) \ d\Omega = \int_{\partial\Omega} vk\nabla u \cdot \underline{n} \ ds = \int_{\partial\Omega} vk\frac{\partial u}{\partial n} \ ds$$

Now by setting $R = 0$, we have

$$\int_\Omega (k\nabla u \cdot \nabla v) \ d\Omega - \int_{\partial\Omega} vk\frac{\partial u}{\partial n} \ ds = \int_\Omega fv \ d\Omega$$

Choosing v s.t. $\ v = 0$ on $\partial\Omega_1$ and then using B.C. on $\partial\Omega_2$, we have

$$\int_\Omega (u_t v + k\nabla u \cdot \nabla v) \ d\Omega + \int_{\partial\Omega_2} kuv \, ds = \int_\Omega fv \ d\Omega + \int_{\partial\Omega_2} kv \ ds$$

Choose $u \in H^1, v \in H_0^1$ as defined in the problem, the variational statement is

Find $\quad u(x) \in H^1(\Omega)$ such that $u(\mathbf{x}, 0) = 1,\ u(x \in \partial\Omega_1, t) = 0$ and

$$(u_t, v) + a(u, v) = L(v) \qquad \forall v \in H_0^1(\Omega), \qquad \text{- (1)}$$

where $\quad (u_t, v) = \int_\Omega u_t v \, d\Omega, \quad a(u, v) = \int_\Omega k\nabla u \cdot \nabla v \, d\Omega + \int_{\partial\Omega_2} kuv \, ds,$

$L(v) = \int_\Omega fv \, d\Omega + \int_{\partial\Omega_2} kv \, ds$, and

$H_0^1(\Omega) = \left\{ v \middle| v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega) \text{ and } v = 0 \text{ on } \partial\Omega_1 \right\}.$

Step 2. Using Galerkin method to form finite element formulation

We pose the variational problem (1) into a N dimension FE subspace of $H_h^1 \subset H^1(\Omega)$ being a finite element subspace with basis functions $\{\phi_1, \phi_2, \dots, \phi_n\}$ and $u_h = \sum_{i=1}^{N} u_i \phi_i(x,y)$ be the Galerkin solution to the variational problem (1).

For the basis function $\{\phi_1, \phi_2, \dots, \phi_n\}$, $\quad v \approx v_h = \sum_{i=1}^{N} \beta_i \phi_i(x)$
We have from (*)

$$\left(u_t, \sum_{i=1}^{N} \beta_i \phi_i\right) + a\left(u, \sum_{i=1}^{N} \beta_i \phi_i\right) = L\left(\sum_{i=1}^{N} \beta_i \phi_i\right)$$

$$\sum_{i=1}^{N} \left[(u_t, \phi_i) + a(u, \phi_i) - L(\phi_i)\right]\beta_i = 0 \qquad -(2)$$

As β_i are arbitrary, from (2) we have

$$(u_t, \phi_i) + a(u, \phi_i) = L(\phi_i) \quad (i = 1,2, \dots N) \qquad -(3)$$

Further, let $\quad u \approx u_h = \sum_{i=1}^{N} u_j \phi_j(x)$

Then from (3), the finite element formulation is

$$\sum_{j=1}^{N} (\phi_j, \phi_i)\dot{u}_j + \sum_{j=1}^{N} u_j \left(\phi_j, \phi_i\right) = L(\phi_i) \quad \Rightarrow M\dot{\mathbf{u}} + K\mathbf{u} = \mathbf{F}. \qquad -(4)$$

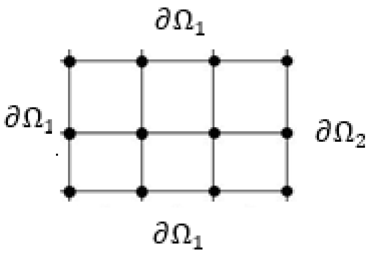
where

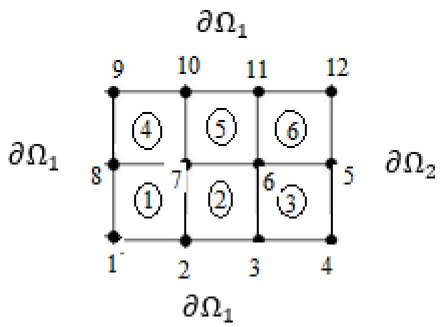
$$M = (m_{ij})_{N\times N} \quad \text{with } m_{ij} = \int_\Omega \phi_i \phi_j \, d\Omega$$

$$K = (k_{ij})_{N\times N} \quad \text{with } k_{ij} = \int_\Omega k\nabla\phi_i \cdot \nabla\phi_j \, d\Omega + \int_{\partial\Omega_2} k\phi_i\phi_j \, ds$$

$$F = (f_i)_{N\times 1} \quad \text{with } l_i = \int_\Omega f\phi_i \, d\Omega + \int_{\partial\Omega_2} k\phi_i \, ds$$

Step 3. For six square linear elements, for the element matrices m^e, k^e and the element force vectors f^e.

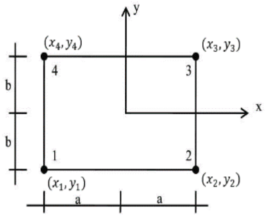




Coordinates of each node given in the following table:

Node	1	2	3	4	5	6	7	8	9	10	11	12
x	0	2	4	6	6	4	2	0	0	2	4	6
y	0	0	0	0	2	2	2	2	4	4	4	4

For a 2×2 rectangular element, $a = b = 1$ and



At node 1, $L_1(x) = \frac{x-x_2}{x_1-x_2}$ has the property that

$$L_1(x_1) = \frac{x_1-x_2}{x_1-x_2} = 1, \qquad L_1(x_2) = \frac{x_2-x_2}{x_1-x_2} = 0.$$

Similarly, $L_1(y) = \frac{y-y_4}{y_1-y_4}$ has the property that

$$L_1(y_1) = \frac{y_1 - y_4}{y_1 - y_4} = 1, \;\; L_1(y_4) = \frac{y_4 - y_4}{y_1 - y_4} = 0.$$

Hence, we choose the shape function at node 1 as

$$\phi_1(x,y) = L_1(x)L_1(y) = \left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{y - y_4}{y_1 - y_4}\right) = \frac{1}{4ab}(x - x_2)(y - y_4)$$

Similarly,

$$\phi_2(x,y) = -\frac{1}{4ab}(x - x_1)(y - y_3)$$

$$\phi_3(x,y) = \frac{1}{4ab}(x - x_4)(y - y_2)$$

$$\phi_4(x,y) = -\frac{1}{4ab}(x - x_3)(y - y_1)$$

$$m^e = (m_{ij})_{4\times4} \qquad \text{with } m_{ij} = \int_{\Omega_e} \phi_i\phi_j \, d\Omega$$

$$m^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33}^{(1)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad m^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33}^{(2)} & m_{34}^{(2)} \\ 0 & 0 & m_{43}^{(2)} & m_{44}^{(2)} \end{bmatrix}, \quad m^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33}^{(3)} & m_{34}^{(3)} \\ 0 & 0 & m_{43}^{(3)} & m_{44}^{(3)} \end{bmatrix}$$

$$m^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m_{22}^{(4)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad m^{(5)} = \begin{bmatrix} m_{11}^{(5)} & m_{12}^{(5)} & 0 & 0 \\ m_{21}^{(5)} & m_{22}^{(5)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad m^{(6)} = \begin{bmatrix} m_{11}^{(6)} & m_{12}^{(6)} & 0 & 0 \\ m_{21}^{(6)} & m_{22}^{(6)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k^e = (k_{ij})_{4\times4} \quad \text{with} \quad k_{ij} = \int_{\Omega_e} k\left(\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} + \frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial y}\right) d\Omega + \int_{\partial\Omega_2} k\phi_i\phi_j \, ds$$

$$k^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33}^{(1)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33}^{(2)} & k_{34}^{(2)} \\ 0 & 0 & k_{43}^{(2)} & k_{44}^{(2)} \end{bmatrix}, \quad k^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33}^{(3)} & k_{34}^{(3)} \\ 0 & 0 & k_{43}^{(3)} & k_{44}^{(3)} \end{bmatrix}$$

$$k^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_{22}^{(4)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k^{(5)} = \begin{bmatrix} k_{11}^{(5)} & k_{12}^{(5)} & 0 & 0 \\ k_{21}^{(5)} & k_{22}^{(5)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k^{(6)} = \begin{bmatrix} k_{11}^{(6)} & k_{12}^{(6)} & 0 & 0 \\ k_{21}^{(6)} & k_{22}^{(6)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f^e = (f_i)_{4\times 1} \qquad \text{with } f_i = \int_{\Omega_e} f\phi_i \, d\Omega + \int_{\partial\Omega_2} k\phi_i \, ds$$

$$f^{(1)} = \begin{bmatrix} 0 \\ 0 \\ f_3^{(1)} \\ 0 \end{bmatrix}, \quad f^{(2)} = \begin{bmatrix} 0 \\ 0 \\ f_3^{(2)} \\ f_4^{(2)} \end{bmatrix}, \quad f^{(3)} = \begin{bmatrix} 0 \\ 0 \\ f_3^{(3)} \\ f_4^{(3)} \end{bmatrix}, f^{(4)} = \begin{bmatrix} 0 \\ f_2^{(4)} \\ 0 \\ 0 \end{bmatrix}, f^{(5)} = \begin{bmatrix} f_1^{(5)} \\ f_2^{(5)} \\ 0 \\ 0 \end{bmatrix},$$

$$f^{(6)} = \begin{bmatrix} f_1^{(6)} \\ f_2^{(6)} \\ 0 \\ 0 \end{bmatrix}.$$

Step 4. Form global matrices **M** and **K,** and load vector **F**

$$\begin{bmatrix} m_{33}^{(3)} + m_{11}^{(6)} & m_{34}^{(3)} + m_{12}^{(6)} & 0 \\ m_{43}^{(3)} + m_{21}^{(6)} & m_{33}^{(2)} + m_{44}^{(3)} + m_{22}^{(5)} + m_{11}^{(6)} & m_{34}^{(2)} + m_{21}^{(5)} \\ 0 & m_{43}^{(2)} + m_{12}^{(5)} & m_{33}^{(1)} + m_{44}^{(2)} + m_{22}^{(4)} + m_{11}^{(5)} \end{bmatrix} \begin{bmatrix} \dot{u}_5 \\ \dot{u}_6 \\ \dot{u}_7 \end{bmatrix} +$$

$$\begin{bmatrix} k_{33}^{(3)} + k_{11}^{(6)} & k_{34}^{(3)} + k_{12}^{(6)} & 0 \\ k_{43}^{(3)} + k_{21}^{(6)} & k_{33}^{(2)} + k_{44}^{(3)} + k_{22}^{(5)} + k_{11}^{(6)} & k_{34}^{(2)} + k_{21}^{(5)} \\ 0 & k_{43}^{(2)} + k_{12}^{(5)} & k_{33}^{(1)} + k_{44}^{(2)} + k_{22}^{(4)} + k_{11}^{(5)} \end{bmatrix} \begin{bmatrix} u_5 \\ u_6 \\ u_7 \end{bmatrix}$$

$$= \begin{bmatrix} f_3^{(3)} + f_2^{(6)} \\ f_3^{(2)} + f_4^{(3)} + f_2^{(5)} + f_1^{(6)} \\ f_3^{(1)} + f_4^{(2)} + f_2^{(4)} + f_1^{(5)} \end{bmatrix}$$

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