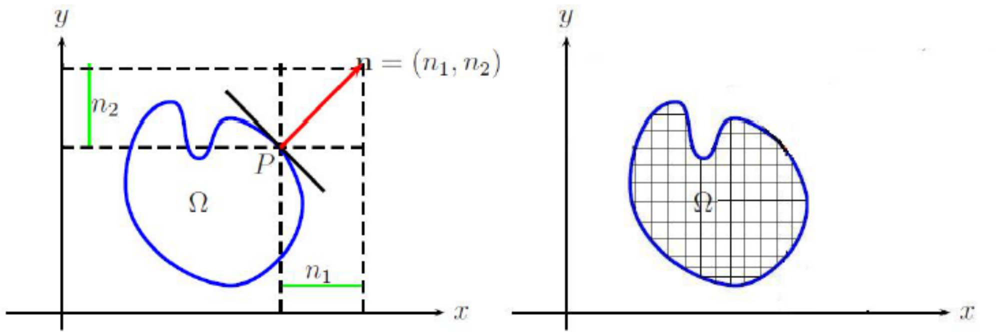
MATH5004 (Lec 5)

General 1D Finite Element Formulation

FEM Steps

1. Discretize the computation domain Ω into a finite number of elements with N nodes, so that $\Omega = \cup_{e=1}^{E} \Omega_e$, and then take the values of u at these nodes as basic unknowns;
2. Transform the BVP to a set of finite element equations;
3. Obtain coefficient matrices for each element;
4. Assemble each element matrix to form a global matrix;
5. Solve the global matrix equations which may be a system of algebraic equations (or ordinary differential equations) of u_i $(i = 1, N)$



Variational statements

1. The residual error r is

$$r(\mathbf{x}) \;=\; L(u(\mathbf{x})) - f(\mathbf{x}). \tag{2}$$

2. The total weighted residuual error over Ω is

$$\int_{\omega} vr(\mathbf{x})\, d\omega \;=\; \int_{\omega} v(L(u(\mathbf{x})) - f(\mathbf{x}))\, d\omega. \tag{3}$$

3 The V. statement: Find $u(\mathbf{x}) \in \tilde{H}$ such that

$$\int_{\Omega} vr(\mathbf{x})\, d\Omega \;=\; \int_{\Omega} v(L(u) - f(\mathbf{x}))\, d\Omega \;=\; 0 \quad \forall v \in H, \tag{4}$$

1D Finite Element Formulation

The variational statement

$$\text{BVP}: \; u_{xx} - f \;=\; 0, \quad x \in (a, b)$$
$$u(a) = 0, \;\; u_x(b) = g.$$

We have the residual error function, and the total weighted residual error

$$r(x) \;=\; u_{xx} - f.$$

$$R \;=\; \int_a^b vr\, dx \;=\; \int_a^b v(u_{xx} - f)\, dx \;=\; -\int_a^b (u_x v_x + fv)\, dx + u_x v(x)|_a^b \tag{5}$$

Variational statement is thus

$$\text{Find} \;\; u \;\in H^1(a, b) \;\; \text{suchthat} \;\; u(a) = 0, \; u_x(b) = g \;\; \text{and}$$
$$\int_a^b (u_x v_x + fv)\, dx - u_x v|_a^b \;=\; 0\, \forall v \;\in\; H_0^1. \tag{6}$$

The Galerkin Method

Let u and v belong to a large (infinite dimension) class of functions, i.e.

$$u(x) = \sum_{i=1}^{\infty} \alpha_i \phi_i(x), \quad v(x) = \sum_{i=1}^{\infty} \beta_i w_i(x) \tag{7}$$

According to the *Bubnov-Galerkin* (or Galerkin) method, the weighting functions are chosen to be the same as the basis functions used to represent u, that is

$$w_i(x) = \phi_i(x), \quad i = 1, 2, \ldots, N.$$

When $w_i \neq \phi_i$, the approach is called the *Petrov-Galerkin* method.

Galerkin's approximation

By choosing an N dimensional subspace of functions $H^h \subset H_0^1$ with basis functions $\{\phi_1, \phi_2, \ldots, \phi_n\}$, the variational problem becomes

Find $u_N \in H^h$ such that

$$
\begin{aligned}
\int_a^b \left\{ \left[\sum_1^N \alpha_j \phi_j(x) \right]' \phi_i' + f\phi_i \right\} dx - g(b)\phi_i(b) &= 0 \\
\Rightarrow \quad \sum_{j=1}^N \left[\int_a^b \phi_i' \phi_j' \, dx \right] \alpha_j &= -\int_a^b f\phi_i \, dx + g(b)\phi_i(b) \\
\Rightarrow \quad \sum_{j=1}^N K_{ij}\alpha_j &= F_i \quad (i = 1, 2, \ldots, N),
\end{aligned}
$$

where $K_{ij} = \int_a^b \phi_i' \phi_j' dx$, $F_i = -\int_a^b f\phi_i dx + g(b)\phi_i(b)$.

Remarks

1) The system of equations $A\mathrm{u} = F$ has unique solution.

Proof : Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \neq \mathbf{0}$

$$\alpha A \alpha = \sum_{i,j=1}^{N} \alpha_i A_{ij} \alpha_j = \int_a^b \sum_{i=1}^{N} \alpha_i \phi_i' \sum_{j=1}^{N} \alpha_j \phi_j' dx = \int_a^b v' v' dx \geq 0$$

with equality only if $v' = 0$ (v = constant). Now as $v(a) = 0$, $v' = 0$ if and only if $v = 0$ or $\alpha = \mathbf{0}$.

Therefore, K is positive definite and thus $A\mathrm{u} = F$ has a unique solution.

Discretisation

Let $\Omega = \{x : x \in [0, 1]\}$ and

divide Ω into 4 elements with 5 nodes as shown

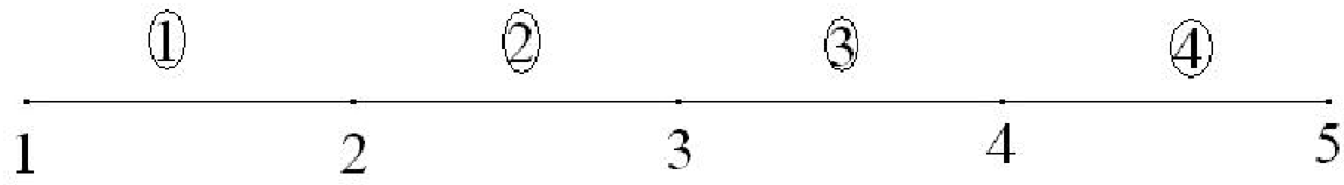


Figure: Domain with 4 elements 5 nodes

use the values of u at nodes as basic unknowns, i.e. the unknowns now are $u_i = u(x_i)$ $(i = 1, 2, 3, 4, 5)$.

Element and Node Identification

a) Number elements and nodes globally;
b) Identify each individual element, i.e., record which nodes are contained in the element.

Suppose that there are M nodes in an element Ω_e, we denote these nodes

$$\{N_1^e, N_2^e, \ldots, N_M^e\}.$$

As element 2 contains two nodes:

$$N_1^2 = 2, \;\; N_2^2 = 3.$$

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Global Interpolating function

We choose the global interpolating function $\phi_i(x)$ in such a way that

$$\phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{8}$$

and $\phi_i(x) = 0$ on elements that do not contain node i

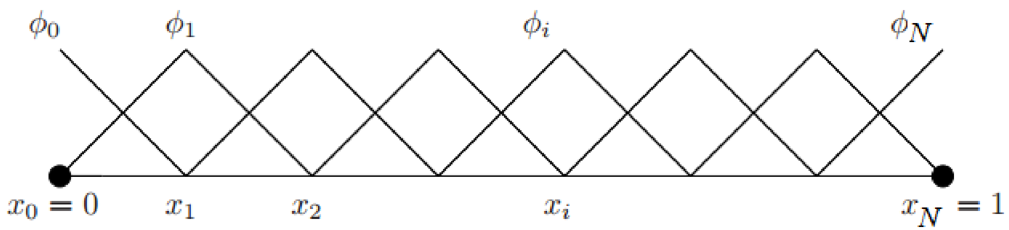


Figure: Element shape functions $\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x_{i-1} \leq x \leq x_i \;, \\ \dfrac{x_{i+1} - x}{h} & \text{if } x_i \leq x \leq x_{i+1} \;, \\ 0 & \text{otherwise}\,, \end{cases}$

we denote the part of $\phi_i(x)$ on element e by ϕ_i^e.

Thus, $\phi_i = \cup \phi_i^e$.

The interpolating formula

The interpolating formula becomes

$$u_h(x) = \sum_{j=1}^{N} u_j \phi_j(x). \tag{9}$$

As $\phi_j(x)$ are defined piecewise, $u_h(x)$ is also a piecewise function. Now consider a typical element Ω^e with M nodes $\{N_1^e, N_2^e, \ldots, N_M^e\}$. Within Ω_e

$$u_h^e(x) = \sum_{j=1}^{M} u_j \phi_j(x).$$

FE Approximation

As ϕ_j with node j in Ω^e have contribution to $u_h(x)$.
Suppose the element Ω_e has M nodes $\{N_1^2, N_2^2, \ldots, N_M^e\}$. Then $u_h^e(x) = \sum_1^M u_{h_i^e} \phi_{h_i^e}^e$ which, for simplicity, can be written as

$$u_h^e(x) = \sum_1^M u_i^e \phi_i^e. \tag{10}$$

eg. for Ω_2, $\quad u_h^2(x) = \sum_1^2 u_i^2 \phi_i^2 = u_2 \phi_2^2 + u_3 \phi_3^2$.
As Ω_e is a small region, we can use simple function, such as low degree polynomial, for ϕ_i^e.

For the BVP : $u_{xx} - f = 0, \quad x \in (a, b)$

$$u(a) = 0, \quad u_x(b) = g,$$

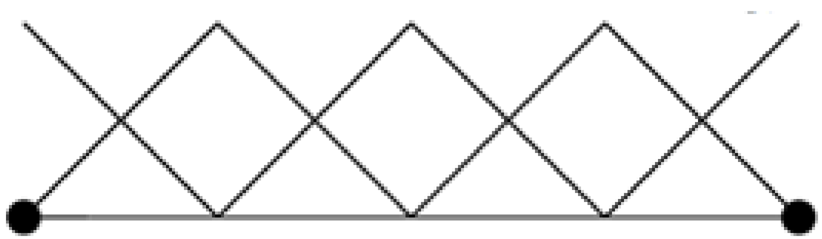
We have FE formulation

$$\sum_{j=1}^{N} K_{ij}\alpha_j = F_i \quad (i = 1, 2, \ldots, N),$$

where $K_{ij} = \int_a^b \phi_i' \phi_j' dx,$

$$F_i = -\int_a^b f \phi_i dx + g(b)\phi_i(b).$$

For a $= 0, \ b = 1, N = 5,$ and $g = b,$



$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{h} & \text{if } x_i \le x \le x_{i+1} \\ 0 & \text{otherwise}, \end{cases}$$

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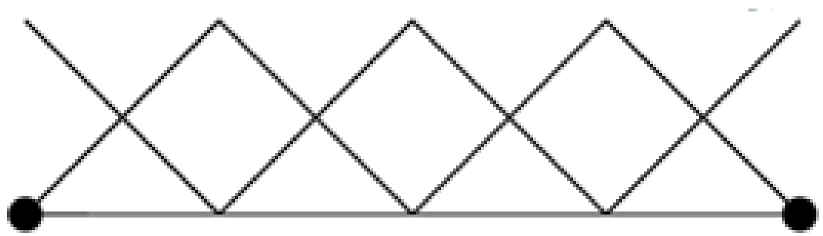

$$K_{ij} = \int_a^b \phi_i' \phi_j' dx$$

$$F_i = -\int_a^b f\phi_i dx + g(b)\phi_i(b).$$

Global system of finite element equations

Exercise 1. Derive system of Finite element equations of a steady two-point BVP:

$$u_{xx} = f(x), \qquad x \in (0,1)$$
$$u(0) = 1, \qquad u(1) = -1.$$

Exercise 2. Derive system of Finite element equations of the steady state heat conduction problem

$$k(x)\frac{\partial^2 u}{\partial x^2} = f(x), \quad 0 < x < 1$$

$$u(0) = 1, \qquad \frac{\partial}{\partial x}u(1) = 0.$$

Semi-discretization in space

Consider the solution of linear parabolic problems (diffusion problems) as follows:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = f(x), \quad \text{in } (0,1)\times(0,\mathrm{T}]$$

subj. B.C. $\qquad u(0,t) = 25, \; u(1,t) = 100,$

I.C. $\qquad u(x,0) = 25$

Variational statement

Find $\quad u = u(x, t) \in H^1(\Omega)$ such that $u(x, 0) = 0$

$$(u_t, v) + a(u, v) = L(v) \qquad \textit{for all } v \in H^1(\Omega)$$

where

$$\begin{aligned} (\cdot, \cdot) &= \text{inner product} \\ a(u, v) &= \int_\Omega (k \nabla u \cdot \nabla v \,) d\Omega \\ L(v) &= \int_\Omega fv \; d\Omega \end{aligned}$$

Let H_h^1 be a finite dimensional subspace of H^1 with basis functions

$$\{\phi_1, \phi_2, \dots \phi_n\}.$$

Then, the variational problem is approximated by :

Find $u_h(x, t) \in H_h^1$ such that $u_h(x, 0) = 0$ and

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h) = L(v_h) \qquad \forall\, v_h \in H_h^1.$$

In the usual way, we introduce a discretization of Ω as a union of elements

$$\Omega_e, \text{ i.e. } \Omega \to \bigcup_{e=1}^{E} \Omega_e$$

and approximate $u(x, t)$ at t by

$$u_h(x, t) = \sum_{j=1}^{n} u_j\,(t)\varphi_j\,(x)$$

Finite Element Approximation

By using the usual finite element formulation, we obtain

$$\boxed{\begin{gathered}\mathbf{M}\dot{\mathbf{u}} + \mathbf{A}\mathbf{u} = \mathbf{F} \\ \mathbf{u}(0) = 0\end{gathered}}$$

where $\mathbf{M} = (m_{ij})$ with $m_{ij} = (\varphi_i, \varphi_j) = \sum_{e=1}^{E} \int_{\Omega_e} \varphi_i \varphi_j d\Omega$

$\mathbf{A} = (a_{ij})$ with $a_{ij} = a(\varphi_i, \varphi_j)$

$$= \sum_{e=1}^{E} \int_{\Omega_e} (k \nabla \varphi_i \cdot \nabla \varphi_j)\, d\Omega$$

$\mathbf{F} = (f_i)$ with $f_i = L(\varphi_i)$

Time Differencing

We now consider the numerical technique to solve the following system of ordinary differential equations.

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{A}\mathbf{u} = \mathbf{F}$$

<u>Forward Difference Scheme</u>

Let $\quad \dfrac{d\mathbf{u}}{dt}(t) = \dfrac{\mathbf{u}(t+\Delta t_r) - \mathbf{u}(t)}{\Delta t}(or \dfrac{d\mathbf{u}_r}{dt} = \dfrac{\mathbf{u}_{r+1} - \mathbf{u}_r}{\Delta t_r})$

and use forward difference with $O(\Delta t)$ accuracy, then (9) becomes

$$\mathbf{M}\ \mathbf{u}_{r+1} = (\mathbf{M} - \Delta t_r \mathbf{A})\mathbf{u}_r + \Delta t_r \mathbf{F}_r$$

where $\sum_{r=1}^{n} \Delta t_r = T$

Forward Difference Scheme

Hence, starting with $\mathbf{u}_0$ at $r = 0$, we can generate a sequence of solutions

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \text{ corresponding to } t_1, t_2, \ldots, T.$$

Remarks:

1) If k, b and α depend on time, then A is a function of time, so that in the forward difference scheme, $\mathbf{A}$ is replaced by $\mathbf{A}(t)$.
2) Finite element code for the equilibrium problem ($\mathbf{u}_t = 0$) can be modified to solve this FE system at each time step.

Exercise 3. Derive system of Finite element equations of the unsteady state heat conduction problem

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f(x), \quad in\ (0,1)\ \times (0,T]$$

$$u(0,t) = 1, \qquad \frac{\partial}{\partial x}u(1,t) = 0.$$