MATH5004 LAB 5

FEM for 1D problems

Objectives: To introduce finite element methods for 1D problem.

$$-u''(x) = f(x), \quad a < x < b; \qquad u(a) = u(b) = 0$$

1. Define the basis functions

In an element $[x_1, x_2]$ there are two nonzero basis functions:

One is $\phi_1^e(x) = \frac{x-x_1}{x_2-x_1}$ and the other is $\phi_2^e(x) = \frac{x_2-x}{x_2-x_1}$.

```
function y = hat1(x,x1,x2)
%this function evaluates the hat function
  y = (x-x1)/(x2-x1);
return
```

```
function y = hat2(x,x1,x2)
%this function evaluates the hat function
  y = (x2-x)/(x2-x1);
return
```

2. Define $f(x)$

```
function y = f(x)
  y =1;  % for example
return
```

3. Compute the integral $\int_{x_1}^{x_2} f(x)\phi_1 dx$

```
function y = int_hat1_f(x1,x2)
% This function evaluate \int_{x1}^{x2} f(x) \phi(x) dx,
% where   \phi(x) = (x-x1)/(x2-x1), using the Simpson rule
  xm = (x1+x2)*0.5;
 y = (x2-x1)*(f(x1)*hat1(x1,x1,x2) + 4*f(xm)*hat1(xm,x1,x2)...
           + f(x2)*hat1(x2,x1,x2) )/6;
  return
```

4. Compute the integral $\int_{x_1}^{x_2} f(x)\phi_2 dx$

```
function y = int_hat2_f(x1,x2)
% This function evaluate \int_{x1}^{x2} f(x) \phi(x) dx,
% where   \phi(x) = (x2-x)/(x2-x1), using the Simpson rule
  xm = (x1+x2)*0.5;
 y = (x2-x1)*(f(x1)*hat2(x1,x1,x2) + 4*f(xm)*hat2(xm,x1,x2)...
           + f(x2)*hat2(x2,x1,x2) )/6;
 return
```

5. Exact solution of $U_{xx}=1$ with u(x)=0 on the boundary.

```
function u = soln(x)
% uxx+1=0, x in [0,1] u(x)=0 on the boundary.  The ODE can be The ODE can be solved analytically by using
%the ansatz   u(x) =ax^2+bx+c. a=-1/2, b=1/2, c=0
a=-0.5; b=0.5; c=0;
u=a*x.^2+b*x+c;
end
```

6. FE solution at an arbitrary point x_p in the solution domain.

```
function y = fem_soln(x,U,xp)
%Input: x is Nodal points
%       U is The computed coefficients of the FEM solution using the hat functions
%       xp is the x coordinate whether the solution will be evaluated
%Output: y, is the computed FEM solution
M = length(x);
for i=1:M-1,
  if xp >=x(i) & xp <= x(i+1)
    y = hat2(xp,x(i),x(i+1))*U(i) + hat1(xp,x(i),x(i+1))*U(i+1);
    return
  end
end
```

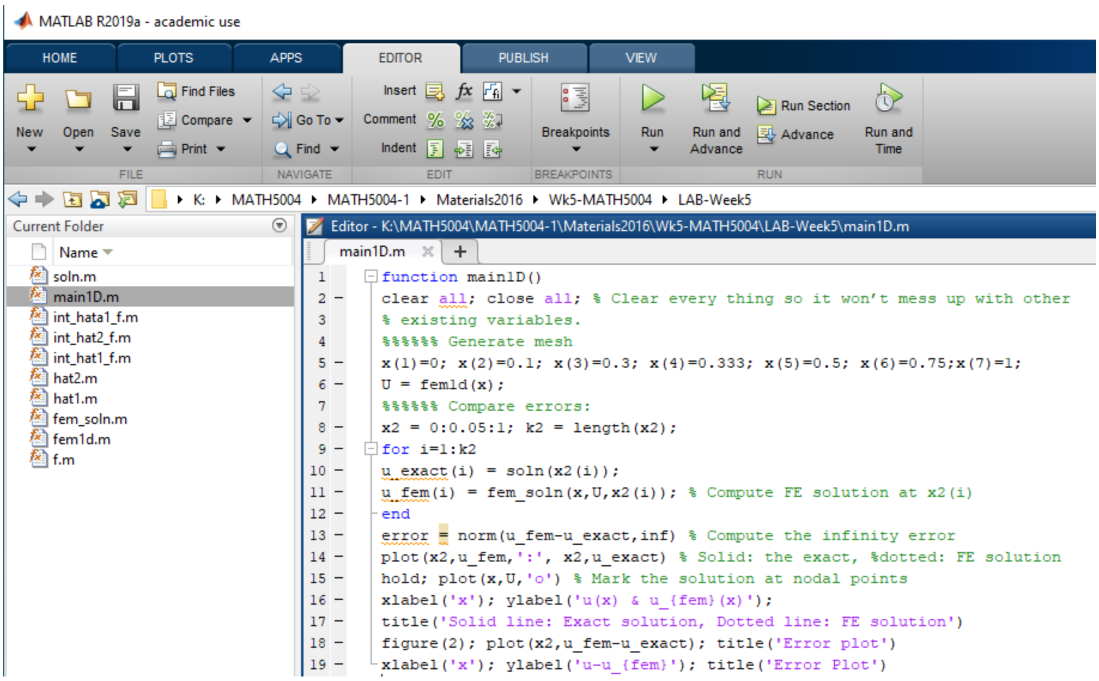
7. 1D FEM Solution

```
function U = fem1d(x)
% A simple Matlab code of 1D FE method for %
% -u'' = f(x), a <= x <= b, u(a)=u(b)=0
% Input: x, Nodal points %
% Output: U, FE solution at nodal points
% Function needed: f(x).
% Matlab functions used:
% hat1(x,x1,x2), hat function in [x1,x2] that is 1 at x2; and  0 at x1.
% hat2(x,x1,x2), hat function in [x1,x2] that is 0 at x1; and 1 at x1.
% int_hat1_f(x1,x2): Contribution to the load vector from hat1
% int_hat2_f(x1,x2): Contribution to the load vector from hat2
M = length(x);
for i=1:M-1,
  h(i) = x(i+1)-x(i);
end
A = sparse(M,M); F=zeros(M,1); % Initialization
A(1,1) = 1; F(1)=0;
A(M,M) = 1; F(M)=0;
A(2,2) = 1/h(1); F(2) = int_hat1_f(x(1),x(2));
for i=2:M-2, % Assembling element by element
A(i,i) = A(i,i) + 1/h(i);
A(i,i+1) = A(i,i+1) - 1/h(i);
A(i+1,i) = A(i+1,i) - 1/h(i);
A(i+1,i+1) = A(i+1,i+1) + 1/h(i);
F(i) = F(i) + int_hat2_f(x(i),x(i+1));
F(i+1) = F(i+1) + int_hat1_f(x(i),x(i+1));
end
A(M-1,M-1) = A(M-1,M-1) + 1/h(M-1);
F(M-1) = F(M-1) + int_hat2_f(x(M-1),x(M));
U = A\F; % Solve the linear system of equations
end
```

8. The main FE routine

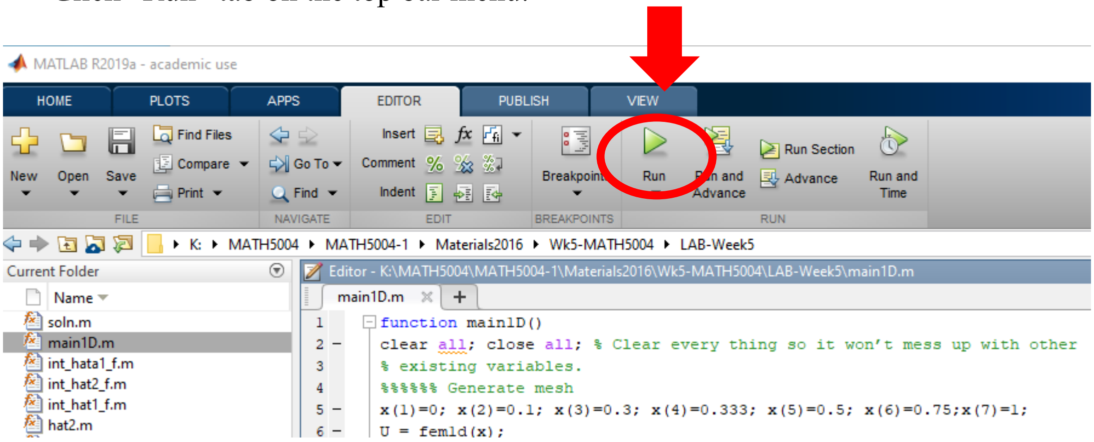
```
function main1D( )
clear all; close all;
% Generate mesh
x_first = 0.0; x_last = 1.0; n = 10;    % number of grid points
x = linspace ( x_first, x_last, n );
x = x(:);
%
U = fem1d(x);
% Compare errors:
x2 = 0:0.05:1;
k2 = length(x2);
for i=1:k2,
u_exact(i) = soln(x2(i));
u_fem(i) = fem_soln(x,U,x2(i));  % Compute FE solution at x2(i)
end
error = norm(u_fem-u_exact,inf) % Compute the infinity error
plot(x2,u_fem,':', x2,u_exact)      % Solid: the exact, %dotted: FE solution
hold; plot(x,U,'o') % Mark the solution at nodal points
xlabel('x'); ylabel('u(x) & u_{fem}(x)');
title('Solid line: Exact solution, Dotted line: FE solution')
figure(2);
plot(x2,u_fem-u_exact); title('Error plot')
xlabel('x'); ylabel('u-u_{fem}'); title('Error Plot')
```

All functions should be in the same folder as shown below.

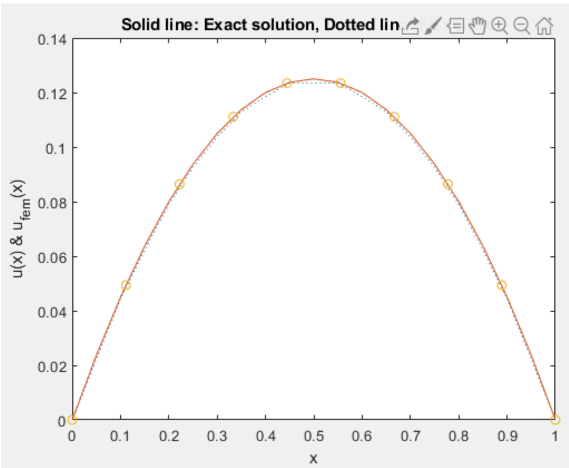


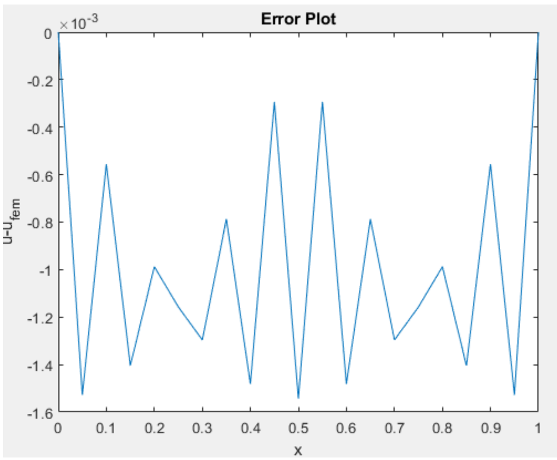
To obtain the results of the problem,

- You need to be in the folder that contain all functions for this problem and the current program is “main1D.m” ;
- Click “Run” tab on the top bar menu.



The results are as shown below.





Assignment II

Question 1. (LAB-WK5)

Implement above MATLAB codes to solve the steady state heat conduction problem

$$k(x)\frac{\partial^2 u}{\partial x^2} = f(x), \quad 0 < x < 1$$

$$u(0) = 1, \qquad \frac{\partial}{\partial x}u(1) = 0,$$

where $f(x) = -\cos(\pi x)$, $k = \pi x^2$.

Note: Assignments II (25%): Assignment Questions will be given weekly.

In this week, Questions 1 (LAB-WK5) are a part of Assignment II, please submit a document file with MATLAB code via Blackboard by the due date of Assignment I on Friday 23 October 2020