Four-Node Tetrahedral Element

The four-node tetrahedral element is the analogue of the 2D triangular element in 3D space.

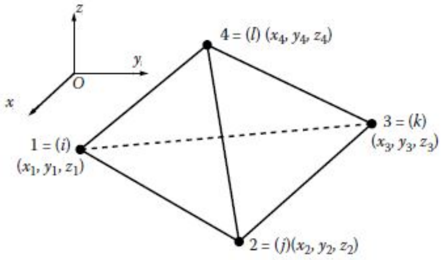


Figure: Four-node tetrahedral element in the physical Cartesian coordinate system Oxyz.

For the four-node tetrahedral element Ω_e, four shape functions ϕ_j $(j = 1, 2, 3, 4)$ in Cartesian coordinate system $Oxyz$ are in the form of

$$\phi_j = \frac{1}{6V_e} \left(a_j + b_j x + c_j y + d_j z \right), \tag{1}$$

where

$$\begin{aligned} a_j &= \det \begin{bmatrix} x_k & y_k & z_k \\ x_\ell & y_\ell & z_\ell \\ x_m & y_m & z_m \end{bmatrix}, \quad b_j = \det \begin{bmatrix} 1 & y_k & z_k \\ 1 & y_\ell & z_\ell \\ 1 & y_m & z_m \end{bmatrix}, \\ c_j &= \det \begin{bmatrix} y_k & 1 & z_k \\ y_\ell & 1 & z_\ell \\ y_m & 1 & z_m \end{bmatrix}, \quad d_j = \det \begin{bmatrix} y_k & z_k & 1 \\ y_\ell & z_\ell & 1 \\ y_m & z_m & 1 \end{bmatrix}, \end{aligned} \tag{2}$$

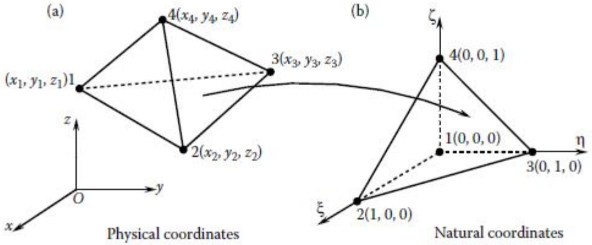
in which subscript j varies from 1 to 4, and k, l, and m are determined by cyclic permutation in the order of j, k, ℓ, and m. For example, if $j = 1$, then $k = 2$, $\ell = 3$, and $m = 4$; if $j = 2$, then $k = 3$, $\ell = 4$, and $m = 1$.

The volume of tetrahedron V_e of the tetrahedral element Ω_e is computed by

$$V_e = \frac{1}{6}\det\begin{bmatrix} 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_\ell & y_\ell & z_\ell \\ 1 & x_m & y_m & z_m \end{bmatrix} \tag{3}$$

Element Transformation

An arbitrary four-node tetrahedral element in physical coordinates (a) is mapped to an isosceles right tetrahedral element in natural coordinates (b).



The shape functions $\psi_j(\xi, \eta, \zeta)$ in the natural coordinate system are defined by

$$\psi_1 = 1 - \xi - \eta - \zeta, \quad \psi_2 = \xi, \quad \psi_3 = \eta, \quad \psi_4 = \zeta. \tag{4}$$

The integral term $\int_{\Omega_e} g(x,y,z)d\Omega$ is computed by

$$\begin{aligned}\int_{\Omega_e} g(x,y,z)d\Omega &= \int_0^1 \int_0^{1-\xi} \int_0^{1-\eta-\xi} \left(\sum_i^4 g(x_i,y_i,z_i)\psi_i\right) \ |J| \ d\zeta d\eta d\xi \\ &= \int_0^1 \int_0^{1-\xi} \int_0^{1-\eta-\xi} f(\xi,\eta) d\zeta d\eta d\xi \end{aligned} \tag{5}$$

where

$$f(\xi,\eta) = \sum_i^4 g(x_i,y_i,z_i)\psi_i|J|.$$

Eight-Node Hexahedral Element

The eight-node hexahedral element is the analogue of the 2D linear rectangular element in 3D space.

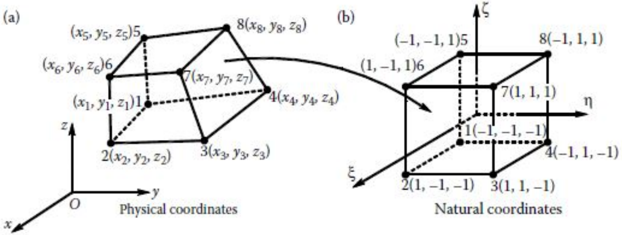


Figure: Coordinate mapping between physical coordinates and natural coordinates of the eight-node hexahedral element: (a) an arbitrary hexahedral element in physical coordinates; (b) a cubic element in natural coordinates.

The shape functions $\psi_j(\xi, \eta, \zeta)$ in the natural coordinate system are defined by

$$
\begin{aligned}
\psi_1 &= \tfrac{1}{8}(1-\xi)(1-\eta)(1-\zeta), \quad & \psi_2 &= \tfrac{1}{8}(1+\xi)(1-\eta)(1-\zeta), \\
\psi_3 &= \tfrac{1}{8}(1+\xi)(1+\eta)(1-\zeta), \quad & \psi_4 &= \tfrac{1}{8}(1-\xi)(1+\eta)(1-\zeta), \\
\psi_5 &= \tfrac{1}{8}(1-\xi)(1-\eta)(1+\zeta), \quad & \psi_6 &= \tfrac{1}{8}(1+\xi)(1-\eta)(1+\zeta), \\
\psi_7 &= \tfrac{1}{8}(1+\xi)(1+\eta)(1+\zeta), \quad & \psi_8 &= \tfrac{1}{8}(1-\xi)(1+\eta)(1+\zeta).
\end{aligned} \tag{6}
$$

For cubic element, the integral term $\int_{\Omega_e} g(x,y,z)d\Omega$ is computed by

$$\begin{aligned} \int_{\Omega_e} g(x,y,z)d\Omega &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(\sum_i^8 g(x_i,y_i,z_i)\psi_i\right)\ |J|\ d\zeta d\eta d\xi \\ &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)d\zeta d\eta d\xi \end{aligned} \tag{7}$$

where

$$f(\xi,\eta,\zeta) = \sum_i^4 g(x_i,y_i,z_i)\psi_i|J|.$$

The physical coordinates (x,y,z) are expressed explicitly in natural coordinates $(\xi,\ \eta,\ \zeta)$ as

$$x = \sum_{j=1}^{8}\psi_j x_j,\quad y = \sum_{j=1}^{8}\psi_j y_j,\quad z = \sum_{j=1}^{8}\psi_j z_j \tag{8}$$

Relations between $dx,\ dy,\ dz$ with $d\xi,\ d\eta$ and $d\zeta$

Suppose $x(\xi,\eta,\zeta),\ y(\xi,\eta,\zeta)$ and $z(\xi,\eta,\zeta)$ are continuously differentiable, then

$$dx = \frac{\partial x}{\partial \xi}d\xi + \frac{\partial x}{\partial \eta}d\eta + \frac{\partial x}{\partial \zeta}d\zeta, \quad dy = \frac{\partial y}{\partial \xi}d\xi + \frac{\partial y}{\partial \eta}d\eta + \frac{\partial y}{\partial \zeta}d\zeta$$

$$dz = \frac{\partial z}{\partial \xi}d\xi + \frac{\partial z}{\partial \eta}d\eta + \frac{\partial z}{\partial \zeta}d\zeta$$

$$\text{or} \quad \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = J \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}, \qquad (9)$$

where $J =$ Jacobian matrix of the transformation.

Properties of Coordinate Transformation

If at point (ξ, η) we have $|J| = det(J) \neq 0$
then an inverse map $T_e^{-1}(x, y, z \rightarrow \xi, \eta, \zeta)$ exists at this point and thus

$$\begin{bmatrix} d\xi \\ \\ d\eta \\ \\ d\zeta \end{bmatrix} = J^{-1} \begin{bmatrix} dx \\ \\ dy \\ \\ dz \end{bmatrix} \tag{10}$$

and

$$T_e^{-1} \;:\; \begin{array}{c} \xi = \xi(x, y, z) \\ \eta = \eta(x, y, z) \\ \zeta = \zeta(x, y, z) \end{array} \tag{11}$$

defines a map $(x, y, z) \rightarrow (\xi, \eta, \zeta)$.

Element Calculation

As in (9), we have

$$\begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}. \tag{12}$$

Hence, by equating terms in (12) and (10), we can determine coefficients of the Jacobian matrix J. By chain rule, we have

$$\begin{aligned} \frac{\partial \psi_j}{\partial \xi} &= \frac{\partial \psi_j}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \psi_j}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial \psi_j}{\partial z}\frac{\partial z}{\partial \xi}, \\ \frac{\partial \psi_j}{\partial \eta} &= \frac{\partial \psi_j}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \psi_j}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial \psi_j}{\partial z}\frac{\partial z}{\partial \eta}, \\ \frac{\partial \psi_j}{\partial \zeta} &= \frac{\partial \psi_j}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial \psi_j}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial \psi_j}{\partial z}\frac{\partial z}{\partial \zeta} \end{aligned} \tag{13}$$

We can express eqn (10) in matrix form of

$$\begin{bmatrix} \partial\psi/\partial\xi \\ \partial\psi/\partial\eta \\ \partial\psi/\partial\zeta \end{bmatrix} = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi & \partial z/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta & \partial z/\partial\eta \\ \partial x/\partial\zeta & \partial y/\partial\zeta & \partial z/\partial\zeta \end{bmatrix} \begin{bmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ \partial\psi/\partial z \end{bmatrix} \tag{14}$$

or

$$\begin{bmatrix} \partial\psi/\partial\xi \\ \partial\psi/\partial\eta \\ \partial\psi/\partial\zeta \end{bmatrix} = J^T \begin{bmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ \partial\psi/\partial z \end{bmatrix}. \tag{15}$$

We then have

$$\begin{bmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ \partial\psi/\partial z \end{bmatrix}. = (J^T)^{-1} \begin{bmatrix} \partial\psi/\partial\xi \\ \partial\psi/\partial\eta \\ \partial\psi/\partial\zeta \end{bmatrix} \tag{16}$$

FE Applications: Stokes Problem and Incompressible Flows

The steady-state motion of an incompressible Newtonian fluid with viscosity μ enclosed in the domain $\Omega \in \mathcal{R}^3$ and acted upon by the volume load f is governed by

$$\begin{array}{lll} \texttt{Stokes equations:} & \mu\triangle u_i - p_{,i} + f_i = 0 & \texttt{in}\ \Omega \ \ (i = 1, 2, 3) \\ \texttt{Continuity equation:} & u_{i,i} = 0 & \texttt{in}\ \Omega \\ \texttt{Boundary condition:} & u_i = 0 & \texttt{on}\ \partial\Omega \ \ \texttt{(fixed boundary)} \end{array} \tag{17}$$

where we have used the index notation with repeated lateral index representing summation over the index range and $()_{,i}$ representing differentiation with respect to x_i.

Variational Statement

Let $v \in V = \{ v \in [H_0^1(\Omega)]^3 \mid div\ v = 0 \text{ on } \Omega\ \}$ be a test function. To derive the finite element equations, we set

$$\int_\Omega (\mu \triangle u_i - p_{,i} + f_i)\, v_i\, d\Omega \ = \ 0. \tag{18}$$

The above integral equation can be simplified by noting that

(i) $\nabla \cdot (v_i \nabla u_i) = \nabla u_i \cdot \nabla v_i + v_i \nabla \cdot \nabla u_i \quad (\nabla \cdot \nabla = \triangle^2)$
$\Rightarrow \quad v_i \triangle u_i = \nabla \cdot (v_i \nabla u_i) - \nabla u_i \cdot \nabla v_i$
Therefore,

$$\begin{aligned} \int_\Omega v_i \triangle u_i\ d\Omega &= \int_\Omega \nabla \cdot (v_i \nabla u_i)\ d\Omega - \int_\Omega \nabla u_i \cdot \nabla v_i\ d\Omega \\ &= \int_{\partial\Omega} v_i \nabla u_i \cdot \mathbf{n}\ ds - \int_\Omega \nabla u_i \cdot \nabla v_i\ d\Omega \\ &= 0 - \int_\Omega \nabla u_i \cdot \nabla v_i\ d\Omega. \end{aligned}$$

(ii) $p_{,i}v_i = (pv_i)_{,i} - pv_{i,i} = (pv_i)_{,i}$
Therefore,
$\int_\Omega p_{,i}v_i \; d\Omega = \int_\Omega (pv_i)_{,i} \; d\Omega = \int_{\partial\Omega} pv_i n_i \; ds = 0.$
Hence 18 becomes

$$\mu \int_\Omega \nabla u_i \cdot \nabla v_i \; d\Omega \;=\; \int_\Omega f_i v_i \; d\Omega. \tag{19}$$

Thus, the variational statement for the problem is:
Find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V \tag{20}$$

where $\quad a(u, v) = \mu \int_\Omega \nabla u_i \cdot \nabla v_i \; d\Omega$

$L(v) = \int_\Omega f_i v_i \; d\Omega$

$V = \{v \in [H_0^1(\Omega)]^3 \mid div \; v = 0 \; \texttt{in} \; \Omega\}.$

Finite Element Formulation

We need to construct a finite-dimensional subspace V_n of V. This is not so easy as we have to satisfy the condition $div\ v = 0$. For simplicity, consider 2-D cases in which

$$V = \{v = (v_x, v_y) \in [H_0^1(\Omega)]^2 \mid div\ v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \texttt{ in } \Omega\}.$$

From calculus, it follows that if Ω does not contain any holes, then $div\ v = 0$ in Ω iff $v = rot\ \varphi \equiv \left(\frac{\partial \varphi}{\partial y} - \frac{\partial \varphi}{\partial x}\right)$.
Thus,

$$v \in V \quad \Leftrightarrow \quad v = rot\ \varphi, \;\; \varphi \in H_0^2(\Omega)$$

where φ is called stream function connected with the velocity field v.
Now, let W_h be a finite-dimension subspace of $H_0^2(\Omega)$ and define

$$V_h \;=\; \{v \mid v = rot\ \varphi,\ \varphi \in W_h\}.$$

Flow of Incompressible Fluids

In this section, we present the finite element method for 2-D flows of incompressible Newtonian fluids. The governing equations for the problem is

$$
\begin{array}{lll}
 & \rho\left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right) = -p_{,i} + [\mu(u_{i,j} + u_{j,i})]_{,j} & \texttt{in}\ \Omega \times I \\
 & u_{i,i} = 0 & \texttt{in}\ \Omega \times I \\
B.C. & u_i = \bar{u}_i & \texttt{on}\ \Gamma_u \\
 & t_i = \sigma_{ij} n_j = \bar{t}_i & \texttt{on}\ \Gamma_t \\
\texttt{where} & \sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i}) & \texttt{in}\ \Omega \\
I.C. & u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}) & \texttt{in}\ \Omega \\
\texttt{where} & I \in [0, T].
\end{array}
\tag{21}
$$

Variational Statements

Let $r_i(\mathbf{x}, t) = \rho\left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right) + p_{,i} - [\mu(u_{i,j} + u_{j,i})]_{,j}$.
The method of weighted residuals seeks for $(u, p) \in V \times Q$ such that for every $t \in I$

$$
\begin{aligned}
&(r_i, v_i) = 0 \quad && \forall v_i \in V \ \ \texttt{and} \qquad v_i = 0 \texttt{ on } \Gamma_u \\
&(u_{i,i}, q) = 0 && \forall q \in Q \\
&u(\mathbf{x}, 0) = u^0 && \texttt{in } \Omega \\
&u_i = \bar{u}_i && \texttt{on } \Gamma_u
\end{aligned}
\tag{22}
$$

where V and Q are velocity and pressure spaces and $(\cdot, \cdot)$ is the inner product defined by $(\mathbf{a}, \mathbf{b}) = \int_\Omega \mathbf{a} \cdot \mathbf{b} \, d\Omega$.

The detailed manipulations involving the integrals defined above are presented as follows. First, consider $(r_i, v_i) = 0$

Let $\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j u_{i,j}$. Then we have from (22)

$$
\begin{aligned}
\int_\Omega \rho \frac{Du_i}{Dt} v_i d\Omega \quad & + \int_\Omega [(pv_i)_{,i} - (pv_{i,i})] d\Omega \\
& - \int_\Omega \{[\mu(u_{i,j} + u_{j,i}) v_i]_{,j} \\
& - \mu(u_{i,j} + u_{j,i}) v_{i,j} d\Omega = 0
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\int_\Omega \rho \frac{Du_i}{Dt} v_i d\Omega \quad & + \int_\Omega (-pv_{i,i} + \mu(u_{i,j} + u_{j,i}) v_{i,j} \; d\Omega \\
& + \int_{\partial\Omega} [pv_j n_j - \mu(u_{i,j} + u_{j,i}) v_i n_j] \; ds = 0
\end{aligned}
$$

As

$$
\begin{aligned}
-pv_j n_j + \mu(u_{i,j} + u_{j,i})v_i n_j &= [-p\delta_{ij} v_i + \mu(u_{i,j} + u_{j,i})v_i]n_j \\
&= \sigma_{ij} n_j v_i = t_i v_i,
\end{aligned}
$$

we have from (23)

$$
\int_\Omega \rho \frac{Du_i}{Dt} v_i \, d\Omega + \int_\Omega [-pv_{i,i} + \mu(u_{i,j} + u_{j,i})v_{i,j}] d\Omega = \int_{\partial\Omega} \bar{t}_i v_i \, ds
$$

As $\partial\Omega = \Gamma_u \cup \Gamma_t$ and u is specified on Γ_u, we choose v_i such that $v_i = 0$ on Γ_u.

Therefore, the variational statement of the problem is:

Find $(u, p) \in V \times Q$ such that for every $t \in I$,

$$\begin{aligned}
&\left(\rho \frac{Du_i}{Dt}, v_i\right) - (p, v_{i,i} + (\mu[u_{i,j} + u_{j,i}], v_{i,j}) = b(\bar{t}_i, v_i) && \forall v_i \in V_0 \\
&(u_{i,i}, q) = 0 && \forall q \in Q \\
&u(\mathbf{x}, 0) = u^0 && \texttt{in } \Omega \\
&u_i = \bar{u}_i && \texttt{on } \Gamma_u
\end{aligned} \tag{24}$$

$$\texttt{where} \quad V = \{\mathbf{v} | \mathbf{v} \in [H^1(\Omega)]^2\} \quad V^0 = \{\mathbf{v} | \mathbf{v} \in V \ \texttt{and} \ \mathbf{v} = 0 \ \texttt{on} \ \Gamma_u\}$$

$$Q = \{v | v \in H^1(\Omega)\}.$$

Finite Element Formulation

Let $V_h \subset V$ be a N-D subspace of V with basis functions $\{\phi_1,\ \phi_2, ..., \phi_N\}$. Approximating v_i and q in (24) by

$$v_{ih} = \sum_{k=1}^{N} \phi_k v_{ik} \ \text{ and } \ q = \sum_{p=1}^{M} \varphi_p q_p,$$

we have

$$\sum_{k=1}^{N} \{(\rho \frac{Du_i}{Dt}, \phi_k) + (\mu[u_{i,j} + u_{j,i}], \phi_{k,j}) - (p, \phi_{k,i}) - b(\bar{t}_i, \phi_k)\} v_{ik} \ = \ 0$$

$$\sum_{p=1}^{M} (u_{i,i}, \varphi_p) q_p \ = \ 0$$

$$\Rightarrow \begin{cases} (\rho \frac{\partial u_i}{\partial t}, \phi_k) + (\rho u_j \frac{\partial u_i}{\partial x_j}, \phi_k) + (\mu[u_{i,j} + u_{j,i}], \phi_{k,j}) - (p, \phi_{k,i}) \ = \ b(\bar{t}_i, \phi_k) \\ (u_{i,i}, \varphi_p) \ = \ 0. \end{cases}$$

Approximating u_i and p respectively by

$$u_{ih} = \sum_1^N \phi_\ell u_{i\ell}, \quad p_h = \sum_1^M \psi_p p_p,$$

we have from (25) that

$$\sum_{\ell=1}^N \{(\rho\phi_\ell, \phi_k)\dot{u}_{i\ell} + (\rho u_j \phi_{\ell,j}, \phi_k) u_{i\ell} \quad +(\mu\phi_{\ell,j}, \phi_{k,j}) u_{i\ell} + (\mu\phi_{\ell,i}, \phi_{k,j}) u_{j\ell}\}$$
$$- \sum_{p=1}^M (\psi_p, \phi_{k,i}) p_p \;=\; b(\bar{t}_i, \phi_k)$$

$$\sum_{k=1}^N (\phi_{k,i}, \psi_p) u_{ik} \;=\; 0$$

We can express in matrix form by

$$
\begin{aligned}
M\dot{U}_i + AU_i - CP &= F \\
-C_1^T U_1 - C_2^T U_2 &= 0
\end{aligned}
$$

or

$$
\begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{P} \end{bmatrix}
+
\begin{bmatrix} 2K_{11} + K_{22} + D & K_{12} & -C_1 \\ K_{21} & K_{11} + 2K_{22} + D & -C_2 \\ -C_1^T & -C_2^T & 0 \end{bmatrix}
\begin{bmatrix} U_1 \\ U_2 \\ P \end{bmatrix}
=
\begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix},
\tag{26}
$$

$$
\begin{aligned}
\texttt{where} \quad & M = (m_{k\ell}) && \texttt{with} \quad m_{kl} = (\rho\phi_k, \phi_\ell) && (k, \ell = 1 : N) \\
& K_{ij} = (K_{ij\ k\ell}) && \texttt{with} \quad K_{ij\ k\ell} = (\mu \tfrac{\partial \phi_k}{\partial x_i}, \tfrac{\partial \phi_\ell}{\partial x_j}) && (k, \ell = 1 : N) \\
& && \qquad (i, j = 1, 2) && \\
& D = (D_{k\ell}) && \texttt{with} \quad D_{k\ell} = (\rho u_j \tfrac{\partial \phi_\ell}{\partial x_j}, \phi_k) &&
\end{aligned}
$$

Time Integration

Two different kinds of integration schemes, implicit and explicit, can be utilized to solve the system (26).

eg. `Backward Euler:` $\dot{M}\frac{U_{n+1}-U_n}{\triangle t} + A(U_{n+1})U_{n+1} = F_{n+1}$
`-- implicit`

`Forward Euler:` $M\frac{U_{n+1}-U_n}{\triangle t} + A(U_n)U_n = F_n$
`-- explicit.`

Note: In constructing a time integration scheme, questions of numerical stability and accuracy must be considered.

Question

Develop a variational statement for the stokes problem

$$\begin{array}{lll} \texttt{Stokes equations:} & \mu \triangle u_i - p_{,i} + f_i = 0 & \texttt{in}\ \Omega \ \ (i = 1, 2, 3) \\ \texttt{Continuity equation:} & u_{i,i} = 0 & \texttt{in}\ \Omega \end{array}$$

with boundary condition $u_i = 0$ on $\partial\Omega_1$ and $u_i = u_i^0$ on $\partial\Omega_2$.