MATH5004 TUT8

2D Finite element formulation

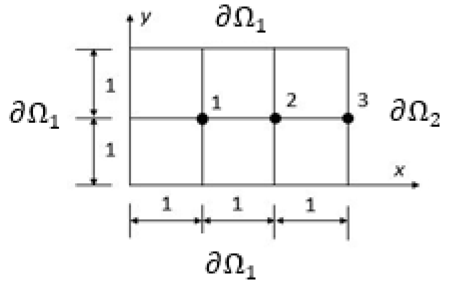
Consider the linear diffusion problem

$$-\nabla\cdot(k\nabla u) = f \qquad \text{in } \Omega,$$

$$u(x,y) = 0 \qquad \text{on } \partial\Omega_1,$$

$$\frac{\partial u}{\partial n} = 1 - u \qquad \text{on } \partial\Omega_2,$$

where $\Omega \subset \mathbb{R}^2$ and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ is the boundary of Ω.



Step 1. Variational statement

Error function $\qquad r = -\nabla\cdot(k\nabla u) - f$

Total weighted residual $\qquad R = \int_\Omega rv\ d\Omega = \int_\Omega (-\nabla\cdot k\nabla u) - f)v\ d\Omega$

As $\quad -v\nabla\cdot(k\nabla u) = k\nabla u\cdot\nabla v - \nabla\cdot(vk\nabla u)$, we have

$$R = \int_\Omega [k\nabla u\cdot\nabla v - fv - \nabla\cdot(vk\nabla u)]\,d\Omega$$

Using the divergence Theorem

$$\int_\Omega \nabla\cdot(vk\nabla u)\ d\Omega = \int_{\partial\Omega} vk\nabla u\cdot\underline{n}\ ds = \int_{\partial\Omega} vk\frac{\partial u}{\partial n}\ ds$$

Now by setting $R = 0$, we have

$$\int_\Omega (k\nabla u\cdot\nabla v)\ d\Omega - \int_{\partial\Omega} vk\frac{\partial u}{\partial n}\ ds = \int_\Omega fv\ d\Omega$$

Choosing v s.t. $\quad v = 0$ on $\partial\Omega_1$ and then using B.C. on $\partial\Omega_2$, we have

$$\int_\Omega (k\nabla u\cdot\nabla v)\ d\Omega + \int_{\partial\Omega_2} kuv\ ds = \int_\Omega fv\ d\Omega + \int_{\partial\Omega_2} kv\ ds$$

Choose $u \in H^1$, $v \in H_0^1$ as defined in the problem, the variational statement is

Find $\quad u(x) \in H^1(\Omega)$ such that $u(\mathbf{x},0) = 1$

$$a(u,v) = L(v) \qquad \forall v \in H_0^1(\Omega), \qquad \text{- (1)}$$

where $\quad a(u,v) = \int_\Omega k\nabla u\cdot\nabla v\,d\Omega + \int_{\partial\Omega_2} kuv\,ds,\ \ L(v) = \int_\Omega fv\,d\Omega + \int_{\partial\Omega_2} kv\,ds,$

and $\quad H_0^1(\Omega) = \left\{v \middle| v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega) \text{ and } v = 0 \text{ on } \partial\Omega_1\right\}.$

Step 2. Using Galerkin method to form finite element formulation

We pose the variational problem (1) into a N dimension FE subspace of $H_h^1 \subset H^1(\Omega)$ being a finite element subspace with basis functions $\{\phi_1, \phi_2, \dots, \phi_n\}$ and

$$u_h = \sum_{i=1}^{N} u_i \phi_i(x, y)$$

be the Galerkin solution to the variational problem (1).

For the basis function $\{\phi_1, \phi_2, \dots, \phi_n\}$,

$$v \approx v_h = \sum_{i=1}^{N} \beta_i \phi_i(x, y) .$$

We have from (*)

$$a\left(u, \sum_{i=1}^{N} \beta_i \phi_i\right) = L\left(\sum_{i=1}^{N} \beta_i \phi_i\right)$$

$$\sum_{i=1}^{N} [a(u, \phi_i) - L(\phi_i)]\beta_i = 0 \qquad -(2)$$

As β_i are arbitrary, from (2) we have

$$a(u, \phi_i) = L(\phi_i) \quad (i = 1, 2, \dots N) \qquad -(3)$$

Further, let $u \approx u_h = \sum_{i=1}^{N} u_j \phi_j(x)$

Then from (3), the finite element formulation is

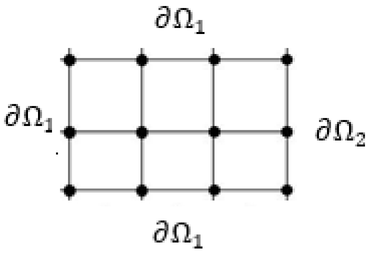
$$\sum_{j=1}^{N} u_j(\phi_j, \phi_i) = L(\phi_i) \quad \Rightarrow K\mathbf{u} = \mathbf{F}. \qquad -(4)$$

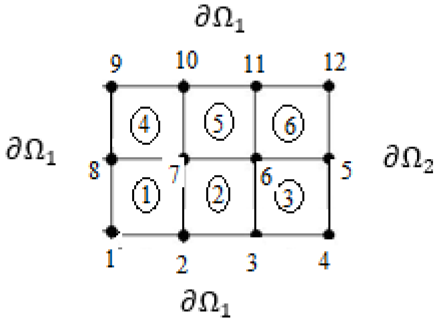
where

$$K = (k_{ij})_{N\times N} \qquad \text{with } k_{ij} = \int_{\Omega} k\nabla\phi_i \cdot \nabla\phi_j \, d\Omega + \int_{\partial\Omega_2} k\phi_i\phi_j \, ds$$

$$F = (f_i)_{N\times 1} \qquad \text{with } f_i = \int_{\Omega} f\phi_i \, d\Omega + \int_{\partial\Omega_2} k\phi_i \, ds$$

Step 3. For six square linear elements, for the element matrices k^e and the element force vectors f^e.





$$k^e = (k_{ij})_{4\times 4} \quad \text{with} \quad k_{ij} = \int_\Omega k\left(\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} + \frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial y}\right) d\Omega + \int_{\partial\Omega_2} k\phi_i\phi_j \; ds$$

$$f^e = (f_i)_{4\times 1} \quad \text{with} \quad f_i = \int_\Omega f\phi_i \; d\Omega + \int_{\partial\Omega_2} k\phi_i \; ds$$

$$k^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33}^{(1)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33}^{(2)} & k_{34}^{(2)} \\ 0 & 0 & k_{43}^{(2)} & k_{44}^{(2)} \end{bmatrix}, \quad k^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33}^{(3)} & k_{34}^{(3)} \\ 0 & 0 & k_{43}^{(3)} & k_{44}^{(3)} \end{bmatrix}$$

$$k^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_{22}^{(4)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k^{(5)} = \begin{bmatrix} k_{11}^{(5)} & k_{12}^{(5)} & 0 & 0 \\ k_{21}^{(5)} & k_{22}^{(5)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k^{(6)} = \begin{bmatrix} k_{11}^{(6)} & k_{12}^{(6)} & 0 & 0 \\ k_{21}^{(6)} & k_{22}^{(6)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f^{(1)} = \begin{bmatrix} 0 \\ 0 \\ f_3^{(1)} \\ 0 \end{bmatrix}, \quad f^{(2)} = \begin{bmatrix} 0 \\ 0 \\ f_3^{(2)} \\ f_4^{(2)} \end{bmatrix}, \quad f^{(3)} = \begin{bmatrix} 0 \\ 0 \\ f_3^{(3)} \\ f_4^{(3)} \end{bmatrix}, f^{(4)} = \begin{bmatrix} 0 \\ f_2^{(4)} \\ 0 \\ 0 \end{bmatrix}, f^{(5)} = \begin{bmatrix} f_1^{(5)} \\ f_2^{(5)} \\ 0 \\ 0 \end{bmatrix},$$

$$f^{(6)} = \begin{bmatrix} f_1^{(6)} \\ f_2^{(6)} \\ 0 \\ 0 \end{bmatrix}$$

Step 4. Form global matrix **K** and load vector **F**

$$
\mathbf{K} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_{33}^{(3)} + k_{22}^{(6)} & k_{34}^{(3)} + k_{21}^{(6)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_{43}^{(3)} + k_{12}^{(6)} & k_{33}^{(2)} + k_{44}^{(3)} + k_{22}^{(5)} + k_{11}^{(6)} & k_{34}^{(2)} + k_{21}^{(5)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & k_{43}^{(2)} + k_{12}^{(5)} & k_{33}^{(1)} + k_{44}^{(2)} + k_{22}^{(4)} + k_{11}^{(5)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{F} = \begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\
f_3^{(3)} + f_2^{(6)} \\
f_3^{(2)} + f_4^{(3)} + f_2^{(5)} + f_1^{(6)} \\
f_3^{(1)} + f_4^{(2)} + f_2^{(4)} + f_1^{(5)} \\
0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Thus, the global system is

$$
\begin{bmatrix}
k_{33}^{(3)} + k_{11}^{(6)} & k_{34}^{(3)} + k_{12}^{(6)} & 0 \\
k_{43}^{(3)} + k_{21}^{(6)} & k_{33}^{(2)} + k_{44}^{(3)} + k_{22}^{(5)} + k_{11}^{(6)} & k_{34}^{(2)} + k_{21}^{(5)} \\
0 & k_{43}^{(2)} + k_{12}^{(5)} & k_{33}^{(1)} + k_{44}^{(2)} + k_{22}^{(4)} + k_{11}^{(5)}
\end{bmatrix}
\begin{bmatrix} u_5 \\ u_6 \\ u_7 \end{bmatrix}
= \begin{bmatrix}
f_3^{(3)} + f_2^{(6)} \\
f_3^{(2)} + f_4^{(3)} + f_2^{(5)} + f_1^{(6)} \\
f_3^{(1)} + f_4^{(2)} + f_2^{(4)} + f_1^{(5)}
\end{bmatrix}
$$

Assignment II

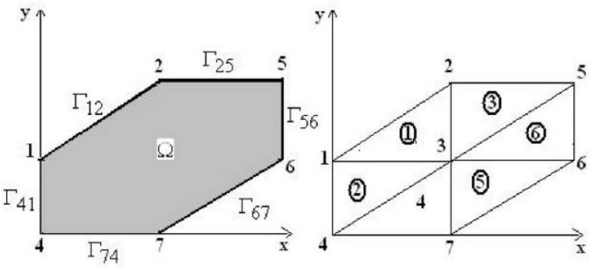
Question 1. (LUT-WK8)

Consider the elliptic boundary value problem

$$-\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=1\,,\qquad (x,y)\,\epsilon\,\Omega$$

subject to

$$\begin{cases} u=0 & \text{on } \Gamma_{41} \\ \dfrac{\partial u}{\partial n}=0 & \text{on } \Gamma_{12},\Gamma_{25},\Gamma_{67},\Gamma_{74}, \\ \dfrac{\partial u}{\partial n}+u=0 & \text{on } \Gamma_{56}, \end{cases}$$



Coordinates of each node given in the following table:

Node	1	2	3	4	5	6	7
x	0	2	2	0	4	4	2
y	1	2	1	0	2	1	0

a) Determine the element stiffness matrix k^e and the element force vector f^e for $e=$ 1,2,3,4,5.
b) Construct the global matrix **K** and the global force vector F,
c) Impose the boundary condition to obtain the final system of equations.

Note: Assignments II (25%): Assignment Questions will be given weekly.

In this week, Questions 1 (TUT-WK8) is a part of Assignment II, please submit a document file (typesetting using Microsoft word or LATEX) via Blackboard by the due date of Assignment II on Friday 23 October 2020

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