Curtin University

Advanced Numerical Analysis: Lecture 4

Semester II, 2020

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Weak Formulation

Introduction

The finite element method is a numerical technique for obtaining approximate solutions to variational formulation of the boundary value problem.

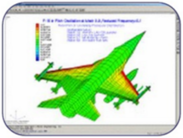
- FEM is the most widely applied computer simulation method in Engineering.
- It is very closely integrated with CAD/CAM applications.
- It is very well proven, tested and validated method for simulating any complex practical scenario in the area of Structural, Thermal, Vibration etc

Application of FEM in Engineering

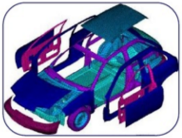
- Mechanical / Aerospace / Civil Engineering / Automobile Engineering
- Structural Analysis (Static/ Dynamic, Linear/Non-Linear)
- Thermal Analysis (Steady State / Transient)
- Electromagnetic Analysis
- Geomechanics
- Biomechanics

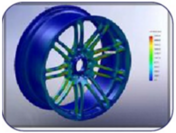
Practical Applications

- Aerospace Domain

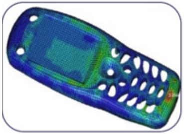
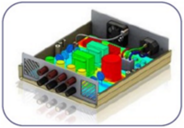


- Automotive Domain





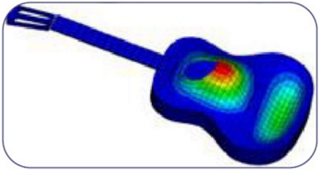
- Hi-Tech /Electronics

- Medical Devices


Advantages of FEM

- ◇ Cost
- ◇ Design Cycle time
- ◇ Number of Prototypes
- ◇ Testing
- ◇ Design Optimization



Many approaches have been utilized to formulate the algebraic equations associated to the variational problem. These include

- Lease squares method
- Collocation method
- Rayleigh-Ritz method
- Galerkin methods

In this lecture, we focus on the FEM based on Galerkin methods.

Nodal Finite Elements

- The nodal finite method is a variational formulation of BVP applied piecewise over a domain divided into nodal subdivisions.

$$BVP \longrightarrow Variational \;\; formuation$$

- The term variational refers to its modern use which permits its use as equivalent weighted integral to the BVP, i.e.

$$\int_\Omega w\,(L(u) - f)\,d\Omega \;=\; 0$$

- The principle of solution itself may not necessarily be admissible as a variational principle.

- The basis of the nodal FEM is the representation of the domain by an assemblage of subdivisions called finite elements.

$$\Omega = \bigcup_{e=1}^{E} \Omega_e$$

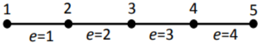
  These elements are interconnected at nodes or nodal points.

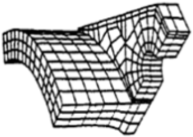
- The trial function approximates the distribution of the primary variable and are commonly used in the nodal expressions.

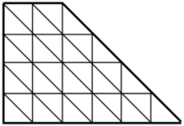
$$u_h = \sum_{i=1}^{N} \varphi_i u_i$$

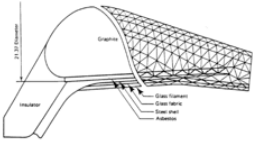
Mesh Elements

- One-dimensional (1D) elements are line elements,
- Two-dimensional (2D) elements can be triangular or bilinear elements.
- Three-dimensional (3D) elements are polyhedrals or cuboids.





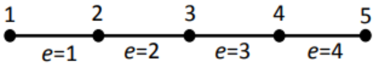




FEM Procedure

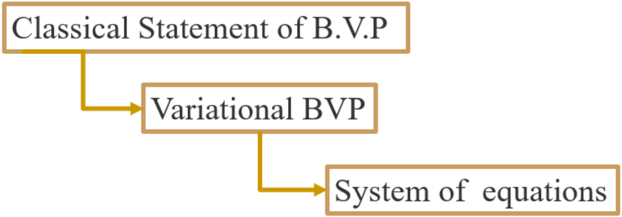
To find a solution u to a boundary value problem defined in a domain Ω using the finite element method, we perform the following steps:

1. Discretize the computation domain Ω into a finite number of elements with N nodes, so that $\Omega = \cup_{e=1}^{E} \Omega_e$, and then take the values of u at these nodes as basic unknowns;



2. Select of interpolation function & Derive variational formulation;
3. Obtain coefficient matrices for each element & Assemble each element matrix to form a global matrix.

4. Apply the known loads such as nodal forces and nodal heat fluxes;
5. Solve simultaneous linear algebraic equations to determine nodal degrees of freedom (dof) – displacements for stress analysis and temperature for heat transfer;
6. Represent the results in tabular or graphic forms.



Weak Formation of Governing Equations

The main approaches of the FEM are in the redirection of the DE of the continuum problem to its integral form and using a trial function over the nodal form of the equation.

Let φ_i be the set of interpolation functions

and u_i be the set of nodal primary variable

Then an approximate trial function $u_h = \sum_{i=1}^{n} \varphi_i u_i$,

is an approximation solution in the elemental domain defined by a set of integral form of the original BVP

$$
\begin{aligned}
L(u) &= f(\mathbf{x}) \quad \mathbf{x} \text{ in } \Omega \\
B(u) &= g(\mathbf{x}) \quad \mathbf{x} \text{ on } \Gamma.
\end{aligned}
$$

For a problem with $f(\mathbf{x}) = 0$,

let $w = \sum_{i=1}^{N} \varphi_i w_i$ be the weighting function

and $u_h = \sum_{i=1}^{N} \phi_i u_i$ be the trial (approximation) solution,

Then a weak formulation is

$$\int_\Omega w \; L(u_h) d\Omega + \oint_\Gamma w \; B(u_h) ds \; = \; 0$$

When $\varphi_i = \phi_i$, the method is the Garlerkin method.

Gradient and Divergence Theorems

Let u be scalar function and **v** be a vector function defined on a 3D domain.

1. The Gradient Theorem

$$\int_\Omega \mathrm{grad}(u)\, d\Omega \;=\; \int_\Omega \nabla u\, dxdydz \;=\; \oint_\Gamma \hat{n} u\, ds$$

$$\int_\Omega \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\, dxdydz = \oint_\Gamma (\hat{i}n_x + \hat{j}n_y + \hat{k}n_z) u\, ds$$

2. Divergence Theorem

$$\int_\Omega \mathrm{div}(\mathbf{v})\, d\Omega \;=\; \int_\Omega \nabla\cdot\mathbf{v}\, dxdydz \;=\; \oint_\Gamma \hat{n}\cdot\mathbf{v}\, ds$$

$$\int_\Omega \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\, dxdydz = \oint_\Gamma (n_x v_x + n_y v_y + n_z v_z)\, ds$$

From the above theorems, we have

$$\int_{\Omega} (\nabla u)\mathbf{v}\; dxdydz = -\int_{\Omega} (\nabla \mathbf{v})u\; dxdydz + \oint_{\Gamma} \hat{n} u \mathbf{v}\; ds$$

$$\int_{\Omega} (\nabla^2 u)\mathbf{v}\; dxdydz = -\int_{\Omega} \nabla u \cdot \nabla \mathbf{v})\; dxdydz + \oint_{\Gamma} \frac{du}{dn}\mathbf{v}\; ds$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial z}$$

$$\nabla = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y} + n_z \frac{\partial}{\partial z}$$

Integration by Parts

If f and g are sufficiently differentiable 1D functions, then the following are applicable.

For a first order differential equation,

$$\int_{x_1}^{x_2} f\frac{dg}{dx}\,dx \;=\; -\int_{x_1}^{x_2} g\frac{df}{dx} + [fg]_{x_1}^{x_2}$$

For a second order differential equation,

$$\int_{x_1}^{x_2} f\frac{d^2g}{dx^2}\,dx \;=\; -\int_{x_1}^{x_2} \frac{dg}{dx}\frac{df}{dx} + \left[f\frac{dg}{dx}\right]_{x_1}^{x_2}$$

For a fourth order differential equation,

$$\int_{x_1}^{x_2} f\frac{d^4g}{dx^4}\,dx \;=\; -\int_{x_1}^{x_2} \frac{d^2g}{dx^2}\frac{d^2f}{dx^2}dx + \left|\frac{df}{dx}\frac{d^2g}{dx^2}\right|_{x_1}^{x_2} + \left[f\frac{d^3g}{dx^3}\right]_{x_1}^{x_2}$$

Weak Formations

Consider a 1D second order DE:

$$\frac{d^2u}{dx^2} = 0, \quad 0 \leq x \leq 1.$$

For the weighted residual method, let w be the the weighting or test function and u be the approximate solution or the trial function, we obtain

$$\int_0^1 w\frac{d^2u}{dx^2}\, dx = 0.$$

When we apply integration by part, we get a weak formulation

$$-\int_0^1 \left(\frac{dw}{dx}\frac{du}{dx}\right)\, dx + \left[w\frac{du}{dx}\right]_0^1 = 0.$$

EXAMPLE 1

Derive the weak formulation of the one-dimension BVP:

$$\frac{d}{dx}\left(\beta\frac{du}{dx}\right)+f=0, \text{ for } 0<x<1$$

subject to the boundary conditions

$$u(0)=0 \text{ and } \beta\frac{du(1)}{dx}=1.$$

Multiplying the DE by weighting function w gives

$$\int_0^1 w \left[\frac{d}{dx} \left(\beta \frac{du}{dx} \right) \right] \, dx \; = \; 0.$$

1. Applying integration by parts

   $\int_0^1 \left(-\beta \frac{dw}{dx} \frac{du}{dx} + wf \right) \, dx + \left[w\beta \frac{du}{dx} \right]_0^1$

2. Applying the boundary conditions

   $\int_0^1 \left(-\beta \frac{dw}{dx} \frac{du}{dx} + wf \right) \, dx + w(1)$

EXAMPLE 2

Derive the weak formulation of the one-dimension BVP:

$$\frac{d^2u}{dx^2} + c^2\lambda^2 u = 0, \quad \text{for} \quad a < x < b$$

subject to the boundary conditions

$$u(a) = f(a) \quad \text{and} \quad u(b) = g(b).$$

EXAMPLE 3

Derive the weak formulation of the one-dimension BVP:

$$\frac{d^2u}{dx^2} + \frac{du}{dx} + u = 0, \ \text{ for } \ a < x < b$$

subject to the boundary conditions

$$\frac{d}{dx}u(a) = f(a) \ \text{ and } \ u(b) = g(b).$$

EXAMPLE 4

Derive the weak formulation of the one-dimension BVP:

$$\frac{d^2u}{dx^2} - \frac{du}{dx} - c^2\lambda^2 u = 0, \text{ for } a < x < b$$

subject to the boundary conditions

$$\frac{d}{dx}u(a) = f(a) \text{ and } \frac{d}{dx}u(b) = g(b).$$

EXAMPLE 5

Derive the weak formulation of the one-dimension BVP:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \text{ for } 0 < x < 1 \;\; 0 \leq t \leq \tau$$

subject to the boundary conditions

$$\frac{\partial}{\partial x} u(0, t) = f \text{ and } \frac{\partial}{\partial x} u(1, t) = g.$$

EXAMPLE 6

Derive the weak formulation of the one-dimension BVP:

$$\frac{d^2}{dx^2}\left[\alpha \frac{d^2 u}{dx^2}\right] + f = 0, \text{ for } 0 < x < L$$

subject to the boundary conditions

$$\left[\frac{d}{dx}\left(\alpha \frac{d^2 u}{dx^2}\right)\right]_{x=L} = 0 \text{ and } \left[\alpha \frac{d^2 u}{dx^2}\right]_{x=L} = C$$

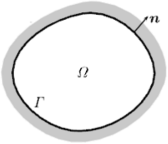
EXAMPLE 7

Derive the weak formulation of the two-dimension BVP:

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \quad \text{for} \quad x \in \Omega$$

subject to the boundary conditions

$$\frac{\partial u}{\partial n} = g(x, y)$$



EXAMPLE 8

Derive the weak formulation of the three-dimension BVP:

$$k\nabla^2 T + \rho\frac{\partial T}{\partial t} = f, \;\; \text{for } \; \mathbf{x} \in \Omega, \; 0 \leq t \leq \tau$$

subject to $T(\mathbf{x}, t = 0) = 20^o C, \quad T(\Gamma_0, t) = 300^o C$ and on other boundaries

$$k\frac{\partial T}{\partial n} = 0.$$

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