Curtin University

MATH5004 (Lec 6)

1D Finite Element Formulation(cont)

Finite Element Method

$$(S) \Leftrightarrow (W) \approx (G) \Leftrightarrow (M)$$

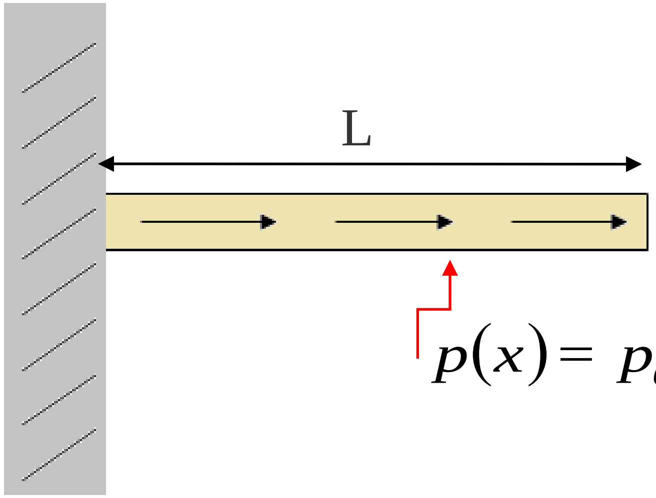
Strong form	Weak form	Galerkin approx.	Matrix form

Sample Problems

Example 1.

Axial deformation of a bar subjected to a uniform load

(1-D Poisson equation)



$$p(x) = p_0$$

$$x = [0, L]$$

$$EA\frac{d^2u}{dx^2} = p_0$$

$$u(0) = 0$$

$$EA\left.\frac{du}{dx}\right|_{x=L} = 0$$

u = axial displacement

E=Young's modulus = 1

A=Cross-sectional area = 1

Curtin University

Strong Form

The strong form of this problem is

$$\frac{d^2u}{dx^2} = p_0$$

$$u(0) = 0$$

$$\left.\frac{du}{dx}\right|_{x=L} = 0$$

Weak Form

Let v be the test function, the work form is

$$\int_0^L \left(\frac{d^2u}{dx^2} - p_0 \right) v\,dx = 0$$

$$\int_0^L \frac{d^2u}{dx^2} v\,dx = \int_0^L p_0 v\,dx$$

Weak Form

Choosing the test function that satisfies *homogeneous* boundary conditions : $u(0)=0$ so let $v(0)=0$.

Returning to the weak form: $$\int_0^L \frac{d^2u}{dx^2} v\,dx = \int_0^L p_0 v\,dx$$

Integrate LHS by parts:

$$\int_0^L \frac{d^2u}{dx^2} v\,dx = -\int_0^L \frac{du}{dx}\frac{dv}{dx}dx + \left[v(x)\frac{du}{dx}\right]_{x=0}^{x=L}$$

$$= -\int_0^L \frac{du}{dx}\frac{dv}{dx}dx + v(L)\left.\frac{du}{dx}\right|_{x=L} - v(0)\left.\frac{du}{dx}\right|_{x=0}$$

Recall the boundary conditions on *u* and *v*:

$$\begin{cases} u(0) = 0 \\ \left.\dfrac{du}{dx}\right|_{x=L} = 0 \\ v(0) = 0 \end{cases}$$

Hence,

$$\int_0^L \frac{du}{dx}\frac{dv}{dx}dx + v(L)\left.\frac{du}{dx}\right|_{x=L} - v(0)\left.\frac{du}{dx}\right|_{x=0} = \int_0^L p_0 v dx$$

$$\boxed{-\int_0^L \frac{du}{dx}\frac{dv}{dx}dx = \int_0^L p_0 v dx}$$

The weak form satisfies Neumann conditions automatically!

Why is it "variational"?

$$-\int_0^L \frac{du}{dx}\frac{dv}{dx}dx = \int_0^L p_0 v dx$$

Variational statement :

Find $u \in H^1$ such that

$$\boldsymbol{a}(u,v) = \boldsymbol{L}(v) \quad \forall v \in H_0^1$$

where $\boldsymbol{a}$ is a bilinear functional, $\boldsymbol{L}$ a linear functional

$$\boldsymbol{a}(u,v) = -\int_0^L \frac{du}{dx}\frac{dv}{dx}dx \qquad \boldsymbol{L}(v) = \int_0^L p_0 v dx$$

u and v are functions from an infinite-dimensional function space H^1.

Galerkin's Method

$$Let\ \ u_h = \sum_{ij=1}^{N} u_j \varphi_j(x) \qquad v_h = \sum_{ij=1}^{N} v_j \varphi_j(x)$$

We then obtain

$$\sum_{j=1}^{N} u_j \int_0^L \frac{d\varphi_j}{dx} \frac{d\varphi_i}{dx} dx = \int_0^L p_0 \varphi_i dx \qquad i = 1,2, \dots, N.$$

Matrix form

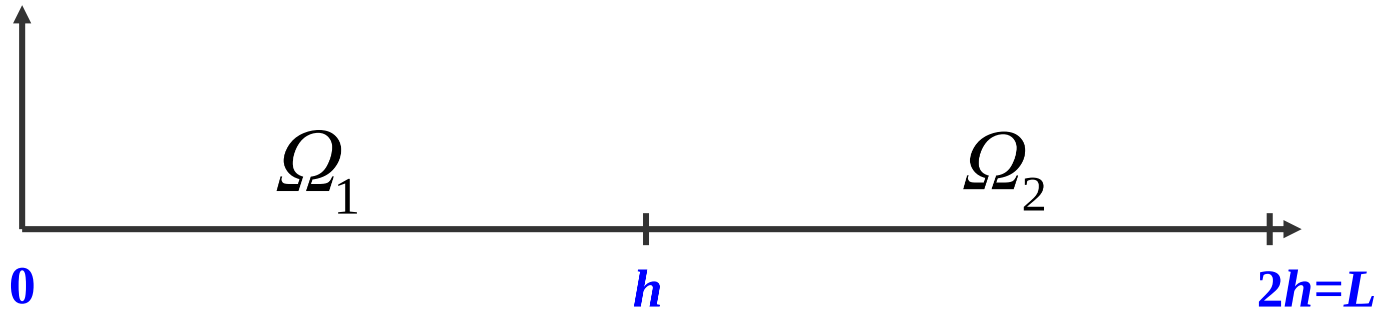
The above integral equations can written in a matrix form as

$$K\mathbf{u} = \mathbf{F}$$

where K is a symmetric matrix.

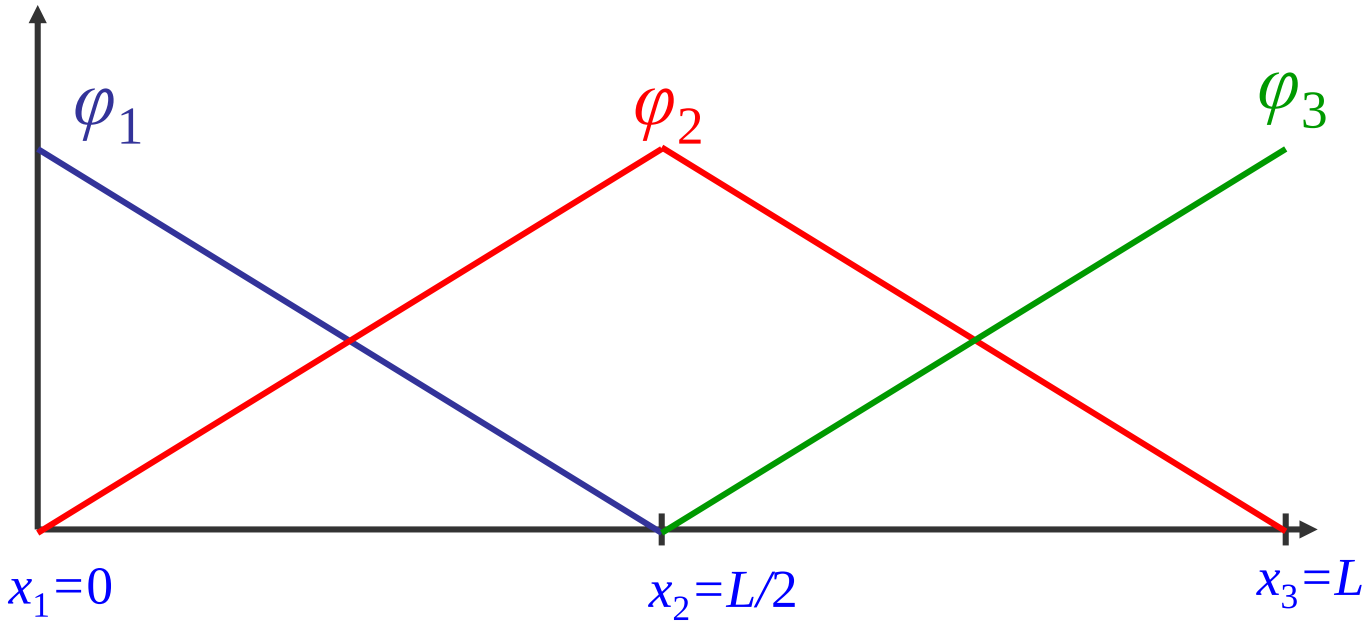
Discretization and Basis Functions

Let's continue with our sample problem. Now we discretize our domain. For this example, we will discretize x=[0, L] into 2 "elements".



Discretization and Basis Functions

For a set of basis functions, we can choose anything. For simplicity here, we choose piecewise linear "hat functions". Our solution will be a linear combination of these functions.



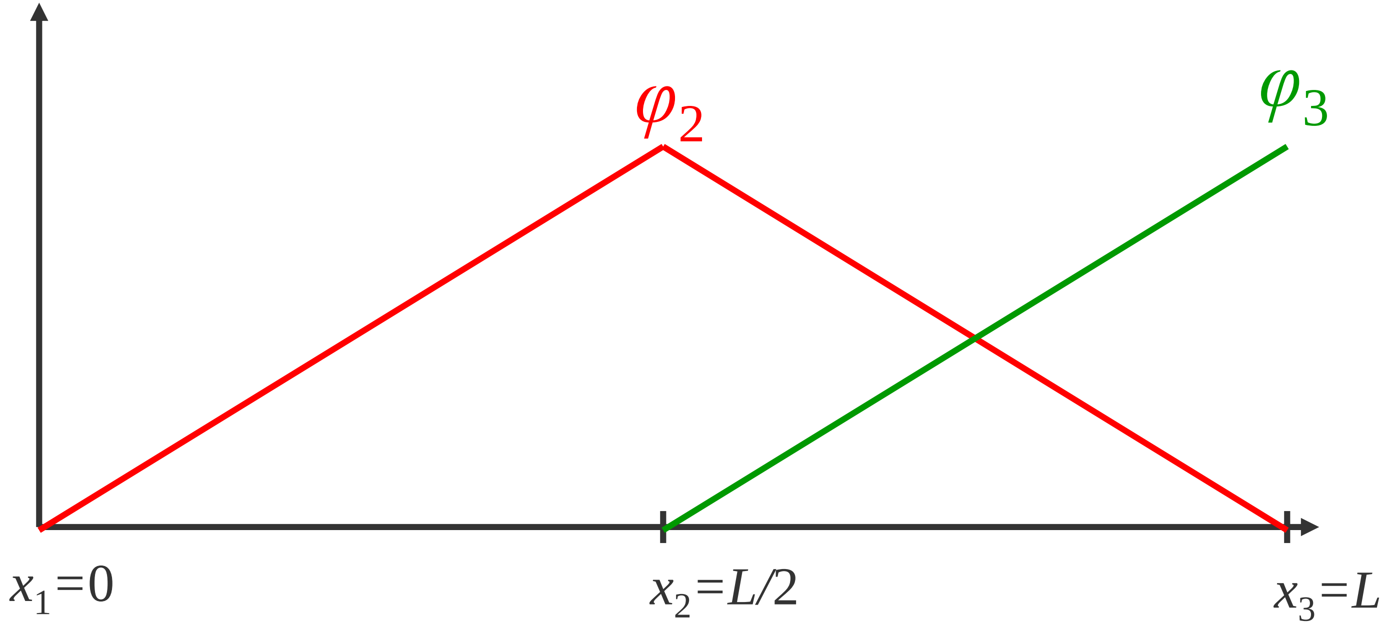
Basis functions satisfy: $\varphi_i(x_j) = \delta_{ij}$

Our solution will be interpolatory. Also, they satisfy the partition of unity.

Curtin University

Discretization and Basis Functions

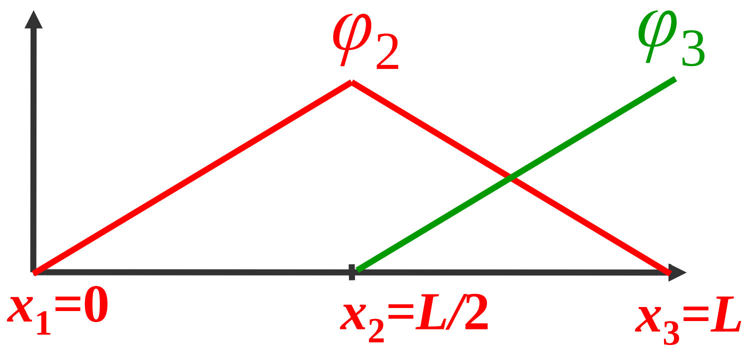
To save time, we can throw out φ_1 a priori because, since in this example $u(0)=0$, we know that the coefficient c_1 must be 0.



Basis Functions

$$\varphi_2 = \begin{cases} \dfrac{2x}{L} & \text{if } x \in \left[0, \tfrac{L}{2}\right] \\ 2 - \dfrac{2x}{L} & \text{if } x \in \left[\tfrac{L}{2}, L\right] \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_3 = \begin{cases} \dfrac{2x}{L} - 1 & \text{if } x \in \left[\tfrac{L}{2}, L\right] \\ 0 & \text{otherwise} \end{cases}$$



Curtin University

Matrix Formulation

Given our matrix problem

$$-\sum_{j=1}^{N}\underbrace{u_j}_{u}\underbrace{\int_0^L \frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx}dx}_{\mathbf{K}} = \underbrace{\int_0^L p_0\varphi_i dx}_{\mathbf{F}} \quad \Rightarrow \quad K\mathbf{u} = \mathbf{F}$$

$$\mathbf{K} = \frac{1}{L}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}, \quad \mathbf{F} = \frac{p_0}{L}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

Solving the problem

we obtain $\mathbf{u} = \begin{bmatrix} \frac{3p_0L^2}{8} \\ \frac{p_0L^2}{2} \end{bmatrix}$

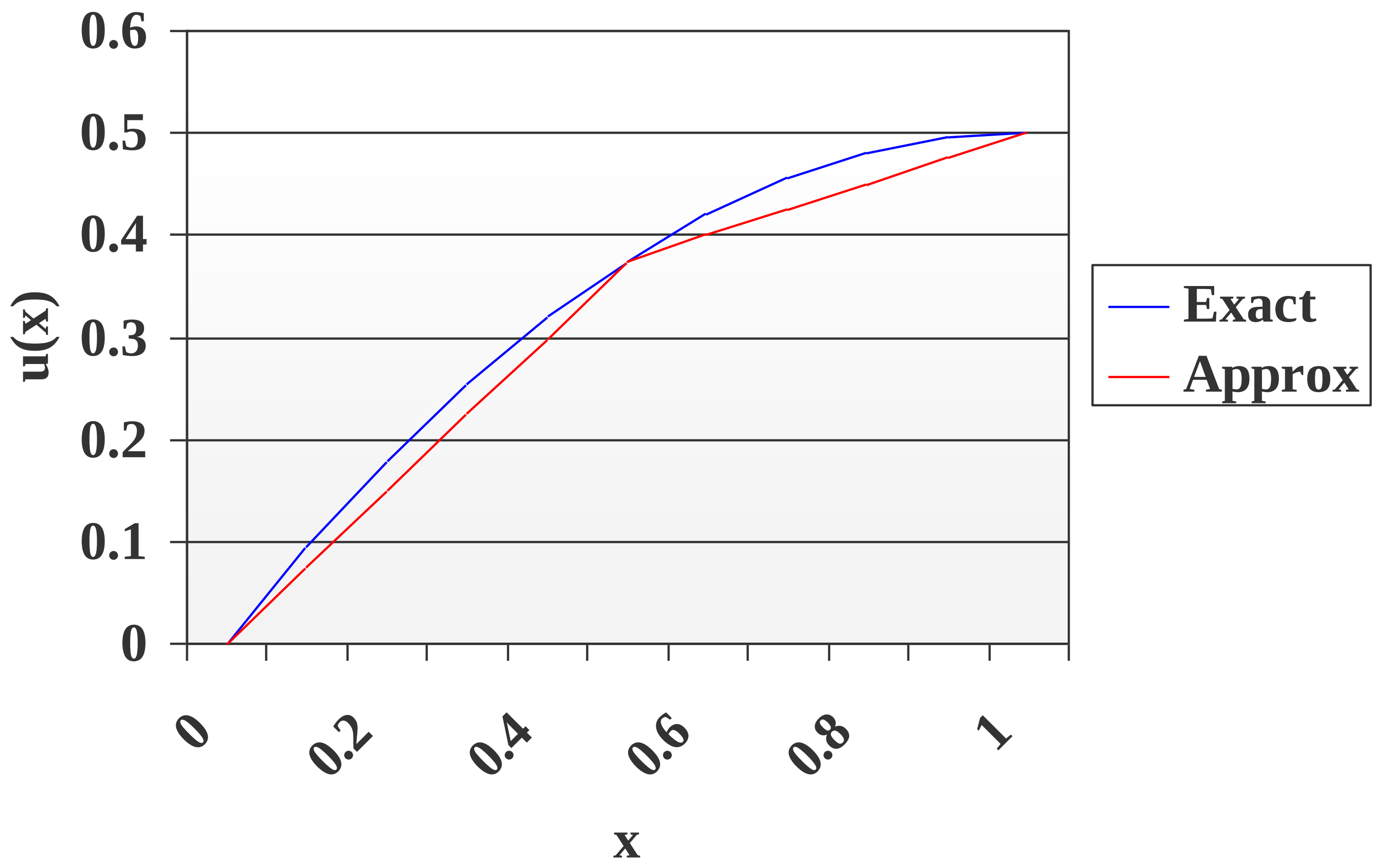
which when multiplied by basis functions φ_i gives

$$\varphi(x) = \begin{cases} \frac{3}{4}p_0Lx & \text{when } x \in \left[0, \frac{L}{2}\right] \\ \frac{1}{4}p_0(L^2 + Lx) & \text{when } x \in \left[\frac{L}{2}, L\right] \end{cases}$$

The exact analytical solution for this problem is :

$$u(x) = p_0Lx - \frac{p_0x^2}{2}$$

Solution



Notice the numerical solution is “interpolatory”, or nodally exact.

Example 2. Two-point Boundary Value Problem

$$-pu'' + qu = f(x),\ 0 < x < 1, \qquad (*)$$
$$u(0) = u(1) = 0,$$

where $p > 0$ and $q \geq 0$.

As described before, we construct a variational form of the equation (*) using Galerkin's method. For this constant-coefficient problem, we seek to determine $u \in H^1[0,1]$ satisfying

$$a(u, v) = L(v), \qquad for\ all\ v \in H_0^1, \qquad (**)$$

where

$$a(u, v) = \int_0^1 (v'pu' + qvu)dx$$

$$L(v) = \int_0^1 vf\ dx$$

With $u \in H^1[0,1]$ and $v \in H_0^1$, we are sure that the integral in (**) exist and that the trivial boundary conditions are satisfied.

$$u(x) \approx u_h(x) = \sum_j u_j \phi_j(x)$$

$$v(x) \approx v_h(x) = \sum_i v_i \varphi_i(x)$$

For Galerkin's method, $\phi_j(x) = \varphi_j(x)$.

Let us establish the goal of finding the simplest continuous piecewise polynomial approximations of u and v.

This would be a piecewise linear polynomial with respect to a mesh;

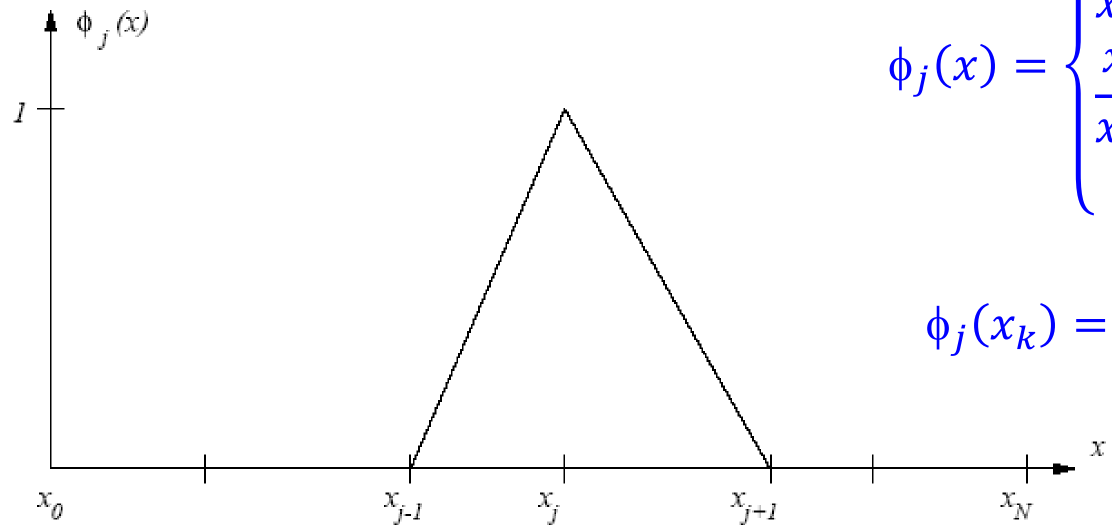
$$0 = x_0 < x_1 < \cdots < x_N = 1 \qquad \text{introduced on } [0,1].$$

Each subinterval $\left(x_{j-1}, x_j\right), j = 1,2, \dots, N$

is called a finite element.

Hat Function

- The basis is created from the “hat function”



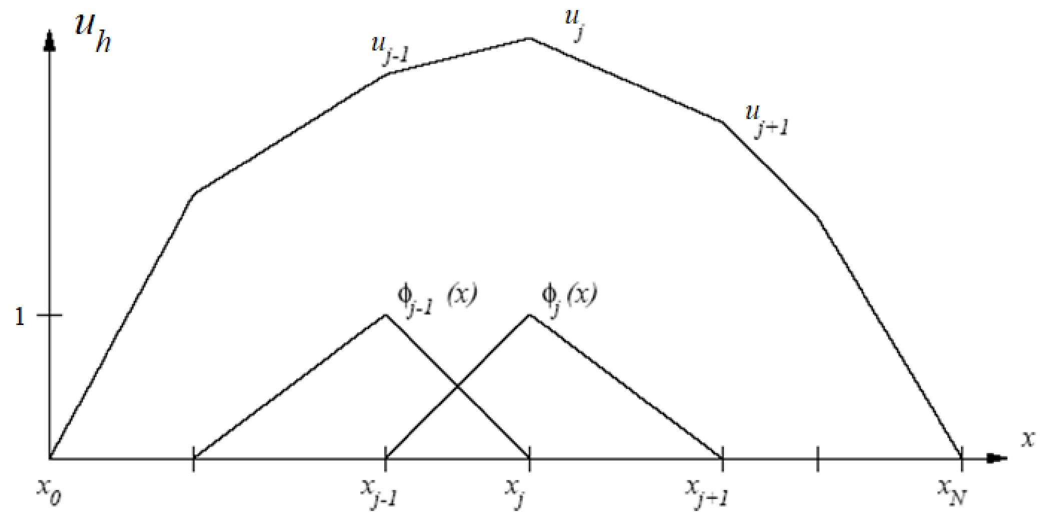
$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}} & \text{if } x_{j-1} \le x < x_j \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j} & \text{if } x_j \le x < x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_j(x_k) = \delta_{jk} = \begin{cases} 1, & \text{if } x_k = x_j \\ 0, & \text{otherwise} \end{cases}$$

Consider approximations of the form, ($u_0 = u_N = 0$),

$$u_h(x) = \sum_{j=1}^{N-1} u_j \phi_j(x). \qquad (***)$$

- Piecewise linear finite element solution $u_h(x)$



❖ Since each is a continuous piecewise linear function of x, their summation u_h is also continuous and piecewise linear. Evaluating u_h at a node x_k of the mesh yields $u_h(x_k) = \sum_{j=1}^{N-1} u_j \phi_j(x_k) = u_k.$ Thus, the coefficients c_k., $k = 1,2,\ldots,N-1$ are the values of u_h at the interior nodes of the mesh.

❖ 2. By selecting the lower and upper summation indices as 1 and N-1, we have ensured that $(***)$ satisfies the prescribed BCs: $u_h(0) = u_h(1) = 0.$

3. The restriction of the FE solution to the element $[x_{j-1}, x_j]$ is the linear function

$$u_h(x) = u_{j-1}\phi_{j-1}(x) + u_j\phi_j(x), \qquad x \in [x_{j-1}, x_j],$$

Since ϕ_{j-1} and ϕ_j are the only nonzero basis elements on $[x_{j-1}, x_j]$, using Galerkin's method we have to solve

$$\sum_{k=1}^{N-1} u_k\, a(\phi_j, \phi_k) = L(\phi_j), \qquad j = 1,2, \dots, N-1.$$

$$a(\phi_j, \phi_k) = \int_0^1 (p\phi_j{}'\phi_k{}' + q\phi_j\phi_k)dx,$$

$$L(\phi_j) = \int_0^1 \phi_j f\, dx$$

$$\downarrow$$

$$K\mathbf{u} + M\mathbf{u} = \mathbf{F}$$

Matrix Form of the Problem

Consider a typical finite element e = $[x_{k-1}, x_k]$,

$$\Delta x = x_k - x_{k-1}$$

$$K^e = \frac{p}{\Delta x}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$M^e = \frac{q\Delta x}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathrm{F}^e = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Assembling

$$\mathbf{K} = \frac{p}{h}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

$$\mathbf{F} =$$

$$\mathbf{M} = \frac{qh}{6}\begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{bmatrix}$$

Semi-discretization in space

Consider the solution of linear parabolic problems (diffusion problems) as follows:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = f(x), \quad \text{in } (0,1)\times(0,\mathrm{T}]$$

subj. B.C. $u(0,t) = 25, \; u(1,t) = 100,$

I.C. $u(x,0) = 0$

Weak formulation

$$\int_0^1 v(x)\{ u_t - (k(x)u_x)_x - f(x)\}\, dx = 0$$

$$\int_0^1 v(x)\, u_t dx - \int_0^1 v(x)\, (k(x)u_x)_x dx - \int_0^1 v(x)\, f(x)\}\, dx = 0 \quad \ldots. (*)$$

Since

$$\boxed{\int_0^1 v\, (ku_x)_x dx = \int_0^1 (vku_x)_x - v_x(ku_x)\, dx}$$

$$= [v(x)k(x)u_x(x)]_{x=0}^{x=1} - \int_0^1 v_x(ku_x)\, dx$$

Eqn $(*)$ becomes

$$\boxed{\int_0^1 v\, u_t dx - [v(x)k(x)u_x(x)]_{x=0}^{x=1} + \int_0^1 v_x(ku_x)\, dx - \int_0^1 v\, f(x)\}\, dx = 0}$$

Curtin University

Variational statement

Find $u \in H^1[(0,1) \times (0,T)]$ such that $u(0,t) = 25, u(1,t) = 100,$
$u(x,0) = 0$ and

$$\int_0^1 v\, u_t dx + \int_0^1 v_x(k u_x)\, dx = \int_0^1 v\, f(x)\, dx, \quad \text{for all } v \in H_0^1 \quad \ldots. (**)$$

where $H^1[(0,1) \times (0,T)] = \{u \mid u, u_x \in L^2[(0,1) \times (0,T)]\}$

$H_0^1[(0,1) \times (0,T)] = \{v \in H^1 \mid v(0,t) = v(1,t) = 0\}$

Using inner product notation $(\cdot\,,\cdot\,)$,

$$(u_t, v) = \int_0^1 v\, u_t dx \qquad a(u,v) = \int_0^1 v_x(k u_x)\, dx \qquad L(v) = \int_0^1 v\, f(x)\, dx$$

the $\mathrm{Eqn}\ (**)$ can be rewritten as

$$(u_t, v) + a(u,v) = L(v), \qquad \text{for all } v \in H_0^1 \qquad \ldots. (***)$$

Finite Element Approximation

Let H_h^1 be a finite dimensional subspace of H^1 with basis functions

$$\{\phi_1, \phi_2, \dots \phi_n\}.$$

Then, the variational problem is approximated by :

Find $u_h(x,\ t) \in H_h^1$ such that $u_h(0,\ t) = 25,\ u_h(1,\ t) = 100$

and $u_h(x, 0) = 0$

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h) = L(v_h) \qquad \forall\ v_h \in H_{0h}^1.$$

In the usual way, we introduce a discretization of Ω as a union of elements

Ω_e, i.e. $\Omega \to \bigcup_{e=1}^{E} \Omega_e$

and approximate $u(x,\ t)$ at t by

$$u_h(x,\ t) = \sum_{j=1}^{n} u_j\ (t)\varphi_j\ (x)$$

Curtin University

By using the usual FE formulation, we obtain the system of ODEs

$$M\dot{\mathbf{u}} + A\mathbf{u} = \mathbf{F}$$

where

$$M = (m_{ij})_{N\times N} \quad \text{with} \quad m_{ij} = (\emptyset_i, \emptyset_j) = \int_0^1 \emptyset_i \emptyset_j \, dx$$

$$A = (a_{ij})_{N\times N} \quad \text{with} \quad a_{ij} = (\emptyset_i', k\emptyset_j') = \int_0^1 k\emptyset_i' \emptyset_j' \, dx$$

$$\mathbf{F} = (f_i)_{N\times 1} \quad \text{with} \quad f_i = \int_0^1 \emptyset_i \, f(x) \, dx$$

Forward Difference Scheme

Let $\dfrac{d\mathbf{u}}{dt}(t) = \dfrac{\mathbf{u}(t+\Delta t_r) - \mathbf{u}(t)}{\Delta t} \left(or \dfrac{d\mathbf{u}_r}{dt} = \dfrac{\mathbf{u}_{r+1} - \mathbf{u}_r}{\Delta t_r} \right)$

and use forward difference with $O(\Delta t)$ accuracy

$$\mathbf{M}\ \mathbf{u}_{r+1} = (\mathbf{M} - \Delta t_r \mathbf{A})\mathbf{u}_r + \Delta t_r \mathbf{F}_r \qquad \text{where } \sum_{r=1}^{n} \Delta t_r = T$$

Exercise. Derive system of Finite element equations of an unsteady two-point BVP:

$$u_t - k(x,t)u_{xx} = 0, \qquad x \in (0,1), t \in (0,\tau)$$

subject to IC: $u(x,0) = 0$, and each of the following BCs

(a) $u(0,t) = f,\ u(1,t) = g$

(b) $u(0,t) = f,\ \frac{\partial}{\partial x}u(1,t) = g$

(c) $\frac{\partial}{\partial x}u(0,t) = f, \frac{\partial}{\partial x}u(1,t) = g$