MATH5004-LEC2

Finite Difference Method

Boundary Value Problems (BVPs)

A BVP can be written as

$$\begin{cases} L(u) = f & in \quad \Omega \\ B(u) = g & on \ \partial\Omega \end{cases} \qquad (1.1)$$

f*, *g = known functions ;
L = differential operator;
B = boundary operator.

Problem: Find u that satisfies $\begin{cases} D.E. & in \quad \Omega \\ B.C. & in \quad \partial\Omega \end{cases}$

Taylor series expansion

$$f(x+\Delta x) = f(x) + \Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 f}{\partial x^2} + \cdots + \frac{(\Delta x)^n}{n!}\frac{\partial^n f}{\partial x^n} + \cdots$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\partial f}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2} + \cdots + \frac{(\Delta x)^{n-1}}{n!}\frac{\partial^n f}{\partial x^n} + \cdots$$

Numerical Differentiation-first derivative

$$f'(x) = \frac{\partial f}{\partial x} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

or

$$f'(x) = \frac{\partial f}{\partial x} \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

a forward difference

a backward difference

$$f(x+\Delta x) = f(x) + \Delta x\frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 f}{\partial x^2} + \cdots + \frac{(\Delta x)^n}{n!}\frac{\partial^n f}{\partial x^n} + \cdots$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\partial f}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2} + \frac{(\Delta x)^2}{6}\frac{\partial^3 f(\gamma_1)}{\partial x^3}, \gamma_1 \in (x, x+\Delta x)$$

$$\frac{f(x) - f(x-\Delta x)}{\Delta x} = \frac{\partial f}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2} + \frac{(\Delta x)^2}{6}\frac{\partial^3 f(\gamma_2)}{\partial x^3}, \gamma_2 \in (x-\Delta x, x)$$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{(\Delta x)^2}{6}\left(\frac{f'''(\gamma_1) + f'''(\gamma_2)}{2}\right)$$

a central difference

$$\frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x_i} - f'(x_i) \approx \frac{(\Delta x_i)^2}{6} f'''(\gamma) = O\left(\frac{(\Delta x_i)^2}{6}\right)$$

As the central difference cannot be used at the end points, we now derive $O(\Delta x^2)$ method at the end points.

$$f(x) = f(x)$$

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(\gamma_1)$$

$$f(x+2\Delta x) = f(x) + 2\Delta x f'(x) + 2\Delta x^2 f''(x) + \frac{4\Delta x^3}{3} f'''(\gamma_2)$$

Taking linear combination of these above terms gives

$$f'(x) = \boldsymbol{a} f(x) + \boldsymbol{b} f(x+\Delta x) + \boldsymbol{c} f(x+2\Delta x) + O(\Delta x^2)$$

Using the Taylor expansion to find $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.

From

$$f'(x) = af(x) + b\left(f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \cdots\right) +$$

$$c(f(x) + 2\Delta x f'(x) + 2\Delta x^2 f''(x) + \cdots)$$

$$\Downarrow$$

$$a + b + c = 0, \qquad b + 2c = \frac{1}{\Delta x}, \qquad \frac{b}{2} + 2c = 0,$$

$$\Downarrow$$

$$a = -\frac{3}{2\Delta x}, \qquad b = \frac{2}{\Delta x}, \qquad c = -\frac{1}{2h}$$

Thus,

$$f'(x) = \frac{-3f(x) + 4f(x + \Delta x) - f(x + 2\Delta x)}{2\Delta x} + O(\Delta x^2)$$

Numerical Differentiation-second derivative

We take the difference of forward and backward approximation for $f'(x)$:

- Let $f_i \equiv f(x_i),\ h = \Delta x$
- Use forward difference to approximate f_i', f_{i-1}'

$$f_i' \approx \frac{f_{i+1} - f_i}{h} \qquad f_{i-1}' \approx \frac{f_i - f_{i-1}}{h}$$

$$f_i'' \approx \frac{f_i' - f_{i-1}'}{h}$$

$$\approx \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$$

Now we check the error.

$$f_{i+1} = f_i + hf_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{6} f_i^{(3)} + \frac{h^4}{24} f_i^{(4)}(\gamma_1) + \cdots$$

$$f_{i-1} = f_i - hf_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{6} f_i^{(3)} + \frac{h^4}{24} f_i^{(4)}(\gamma_2) + \cdots$$

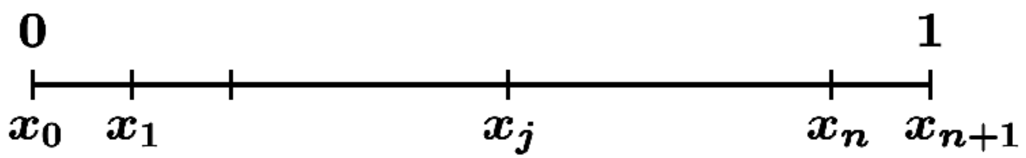
$$\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} - f_i''$$

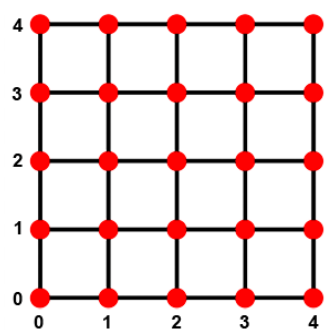
$$= \frac{1}{h^2}\left(f_i + hf_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{6} f_i^{(3)} + \frac{h^4}{24} f_i^{(4)}(\gamma_1) + \cdots - 2f_i\right)$$
$$+ \frac{1}{h^2}\left(f_i - hf_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{6} f_i^{(3)} + \frac{h^4}{24} f_i^{(4)}(\gamma_2) + \cdots\right) - f_i''$$

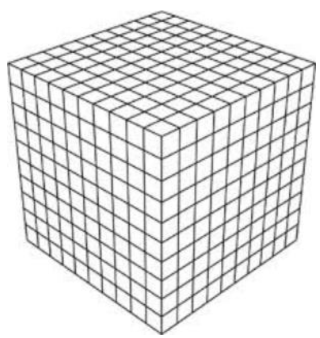
$$\approx \frac{h^2}{24}\left(f_i^{(4)}(\gamma_1) + f_i^{(4)}(\gamma_2)\right) = O(h^2)$$

To solve the BVP (1.1) using the FDM, we need to perform the following work.

- Discretize Ω into a mesh of discrete points called nodes.



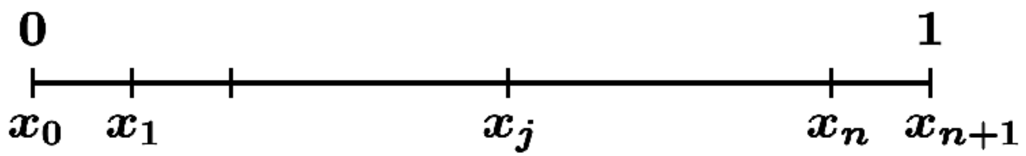




Discretization 1D & Derivative Approx.

The domain

$\Omega = (a,b)$ is subdivided into a set of equal length



$$\boxed{dx = h = (b-a)/n,}$$

partial derivative	finite difference approximation	type	order
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_i^n}{\Delta x}$	forward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_i^n - U_{i-1}^n}{\Delta x}$	backward	first in x
$\frac{\partial U}{\partial x} = U_x$	$\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}$	central	second in x
$\frac{\partial^2 U}{\partial x^2} = U_{xx}$	$\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}$	symmetric	second in x

partial derivative	finite difference approximation	type	order
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^n}{\Delta t}$	forward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n} - U_i^{n-1}}{\Delta t}$	backward	first in t
$\frac{\partial U}{\partial t} = U_t$	$\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t}$	central	second in t
$\frac{\partial^2 U}{\partial t^2} = U_{tt}$	$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2}$	symmetric	second in t

- Approximate all derivatives using the values of the unknown function at the nodes, and thus,

$$\boxed{\mathrm{DE} \approx \quad AU = F}$$

with the nodal values of the unknown function as basic unknowns.

- Solve the linear (or nonlinear) system of algebraic equations.

Example 1.

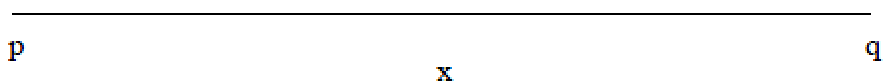
Construct a FDS to find an approximation solution of the following PDE:

$$u_t + ku_x = 0$$

$$u(0, x) = f(x), \quad p \leq x \leq q.$$

Step 1: Spatial Discretisation

The computational domain contains an infinite number of x values. So first we must replace them by a finite set.



The computational domain is replaced by a grid of N equally spaced grid points. Starting with the first grid point at x = p, and ending with the last grid point at x = q, the constant grid spacing, Δx, is

$$\Delta x = \frac{(q-p)}{(N-1)}$$

- The values of x in the discretised computational domain are indexed by subscripts to give,

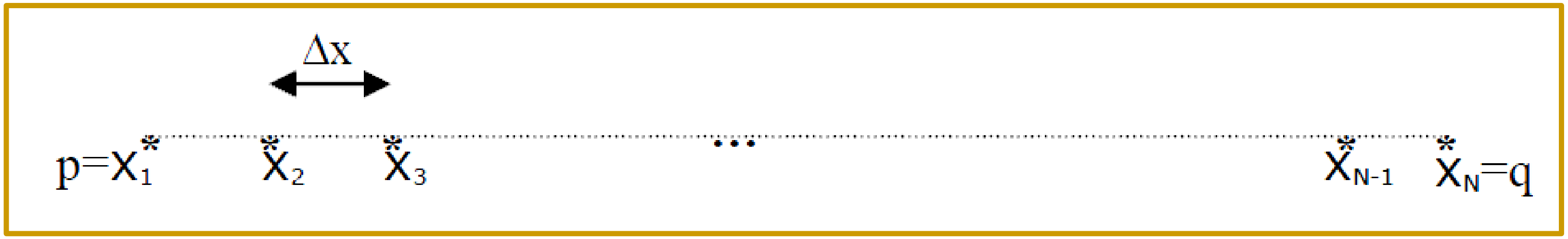
$$x_1 = p, \qquad x_2 = p + \Delta x, \dots,$$

$$x_i = p + (i-1)\Delta x, \dots, x_N = p + (N-1)\Delta x = q.$$

- Since the grid spacing is constant, ,

$$x_{i+1} = x_i + \Delta x.$$

- The discretised computational domain is as shown below.



PDE: $u_t + ku_x = 0$

- Fixing $\boldsymbol{t}$ at $t = t_n$, we approximate the spatial partial derivative, u_x, at each point (t_n, x_i) using the formula

$$u_x(t_n, x_i) = \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

- The PDE becomes

$$u_t + k\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 \qquad (*)$$

Step 2: Time Discretisation

- Fixing x at $x = x_i$, we approximate the temporal partial derivative, u_t, at each point (t_n, x_i) using the formula

$$u_t(t_n, x_i) = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

- The equation $(*)$ becomes

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + k\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0$$

$$u_i^{n+1} = u_i^n - k\Delta t\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0$$

- Using a small number of grid points.
- Let p=0 and q=100, k=0.5, and let the initial conditions be

$$u(0,x) = \begin{cases} e^{-0.01(x-45)^2} & 20 \le x \le 70 \\ 0 & elsewhere \end{cases}$$

Let the (small) number of grid points be N=11. Then

$$\Delta x = \frac{(100-0)}{(11-1)} = 10$$

- The subscripts for the grid values go from 1 to 11 and are entered into the first row of the following table.

- The actual x values of the corresponding grid points are entered into the second row of the table.

i	1	2	3	4	5	6	7	8	9	10	11	*12*
x_i	0	10	20	30	40	50	60	70	80	90	100	*110*
u_i^0												
u_i^1												
u_i^2												

Let $\Delta t = 3$. Then we have

$$u_i^{n+1} = u_i^n - 0.15(u_{i+1}^n - u_i^n) \qquad (**)$$

- Start at time $t_0 = 0, i.e.\ n = 0,$ hence $(**)$ becomes

$$u_i^1 = u_i^0 - 0.15\left(u_{i+1}^0 - u_i^0\right) \qquad (***)$$

- Time level zero corresponds to the initial conditions.

  at time $\quad t_0 = 0,$

$$u_i^0 = u(0, x_i) = \begin{cases} e^{-0.01(x-45)^2} & 20 \le x \le 70 \\ 0 & elsewhere \end{cases} \qquad (****)$$

- Evaluating $(****)$ at a few grid points gives

  $u_1^0 = u(0, x_1) = u(0,0) = 0,$

  $u_5^0 = u(0, x_5) = u(0,40) = e^{-0.01(40-45)^2} = 0.7788, etc$

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- Putting $i = 1$ into $(***)$ gives
  $$u_2^1 = u_1^0 - 0.15(u_2^0 - u_1^0) = 0 - 0.15(0 - 0) = 0.$$
- Putting $i = 2$ into $(***)$ gives
  $$u_2^1 = u_2^0 - 0.15(u_3^0 - u_2^0)$$
  $$= 0 - 0.15(0.0019 - 0) = -0.00029$$
- Putting $i = 3$ into $(***)$ gives
  $$u_3^1 = u_3^0 - 0.15(u_4^0 - u_3^0)$$
  $$= 0.0019 - 0.15(0.1054 - 0.0019) = -0.01363$$

i	1	2	3	4	5	6	7	8	9	10	11	12
x_i	0	10	20	30	40	50	60	70	80	90	100	110
u_i^0	0	0	0.0019	0.1054	0.7788				0	0	0	0
u_i^1	0	-0.00029	-0.01363								0	
u_i^2												

Step 3. Pen and Paper Calculation

In practice, numerical schemes are implemented by writing them then running a computer program. Before doing this, it is extremely useful to work through a pen and paper calculation for two reasons:

1. To check understanding of the scheme.
2. To be able to check results from the computer program against pen and paper results (verification).

Example 2

Unsteady heat conduction in 1- D with constant thermal conductivity

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

- Expand the individual terms with Taylor series

$$\left(\frac{\partial T}{\partial t}\right)_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} - \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + - - - -$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} - \left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{12} + - - - -$$

Unsteady Heat Conduction (cont'd)

$$\underbrace{\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2}}_{\text{PDE}} = 0 = \underbrace{\frac{T_i^{n+1} - T_i^n}{\Delta t} - \frac{\alpha(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}}_{\text{Difference equation}}$$

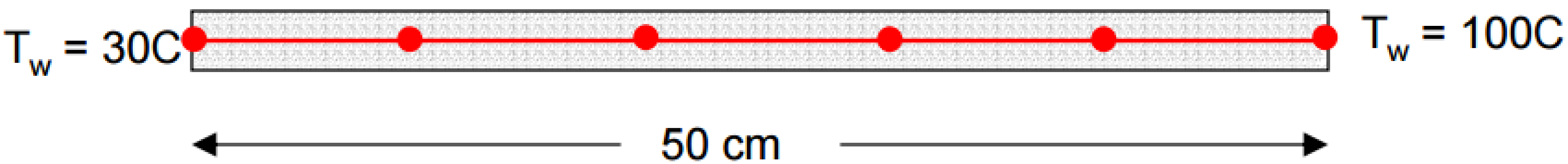
$$+ \underbrace{\left[-\left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + \alpha \left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{12} + \cdots \right]}_{\text{Truncation error}}$$

$$\boxed{\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}}$$

Explicit Solution

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2} \Rightarrow T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n)$$

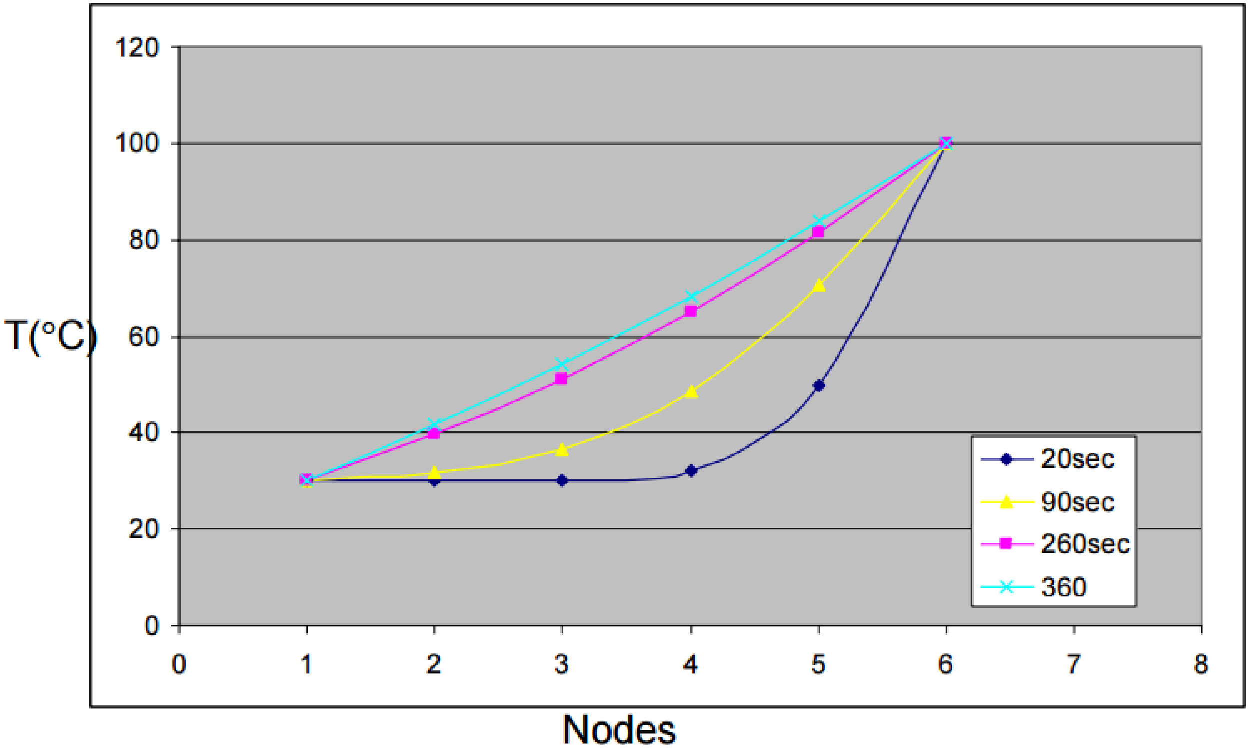
- Find 1-D unsteady temperature distribution till steady state



$$\Delta x = \frac{50}{5} = 10\text{cm} \qquad \alpha = 17\text{x}10^{-2}\,\text{cm}^2/\text{s}$$

Initial temp T_{in} = 30C, Δt = 10 sec $\Rightarrow$ $\frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$

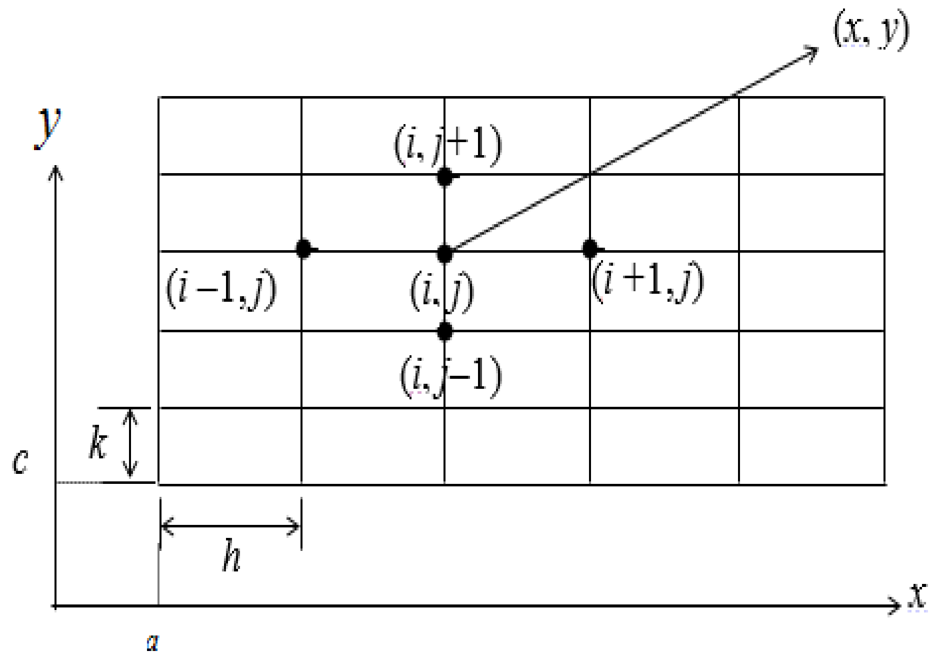
Results



120
100
80
60
40
20
0
T(°C)
0
1
2
3
4
5
6
7
8
20sec
90sec
260sec
360
Nodes

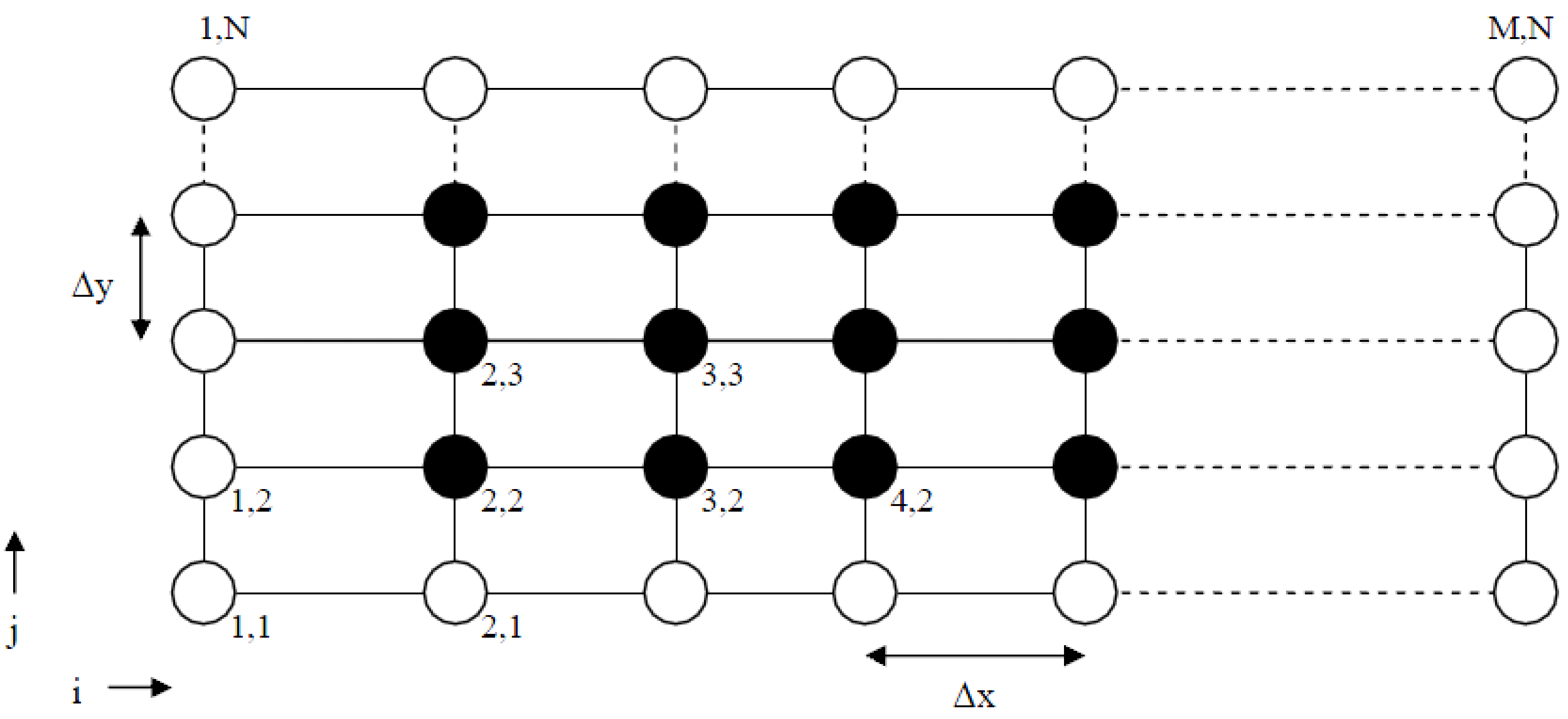
Discretization 2D & Derivative Approx.

The domain $\Omega = (a,b)\times(c,d)$ is subdivided into a set of equal rectangles of sides



$$dx = h = (b-a)/n, \quad dy = k = (d-c)/m.$$

Exercise: Finish labelling all grid points in the following figure.



Obviously, in the coordinate system chosen

$$(x_i, y_j) = (a+ih, c+jk), \quad (i=0, n; \; j=0, m).$$

Thus for convenience in presentation,

we use (i,j) to denote the position (x_i,y_j) &

$u_{i,j}$ to represent $u(x_i,y_j)$, namely

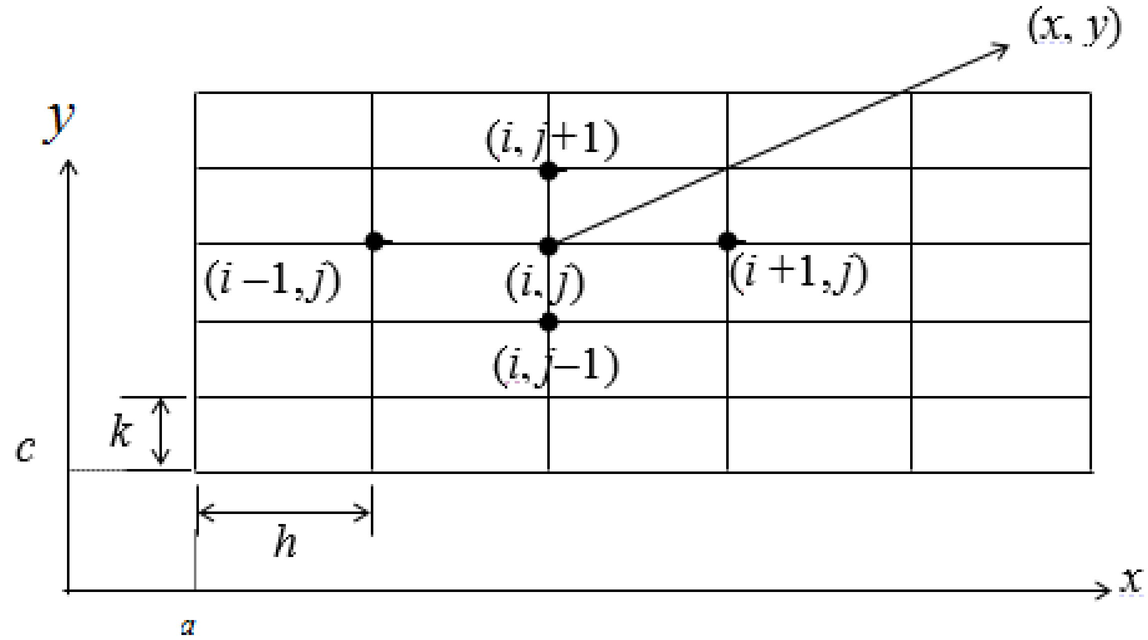
$$(x_i, y_j) = (i, j) \;\; u(x_i, y_j) = u_{i,j}$$

From Taylor's Theorem for 2 variables, we have

$$\boxed{u(x_i + h, y_j) = u_{i+1,j} = u_{i,j} + hu_x + \frac{h^2}{2}u_{xx} + \ldots \quad (1.2)}$$

$$\boxed{u(x_i - h, y_j) = u_{i-1,j} = u_{i,j} - hu_x + \frac{h^2}{2}u_{xx} - \ldots \quad (1.3)}$$

where u_x and u_{xx} are all evaluated at (x_i, y_j)



From Taylor's Thm. rearranging (1.2) and (1.3) yields

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2).}$$

(1.2) + (1.3) and then rearranging yields

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2).$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2).$$

By using the Taylor theorem in 2 variables, we have

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j} - u_{i,j-1} + u_{i-1,j-1}}{h^2} + O(h^2 + k^2).$$

If we choose $h = k$, then

$$\nabla^2 u_{i,j} = \frac{1}{h^2}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\right] + O(h^2).$$

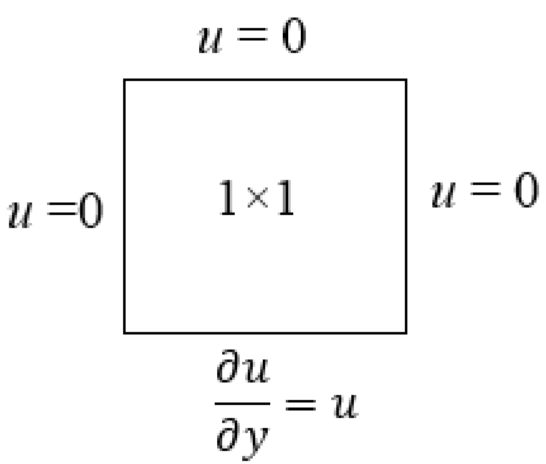
which can be graphically displayed by

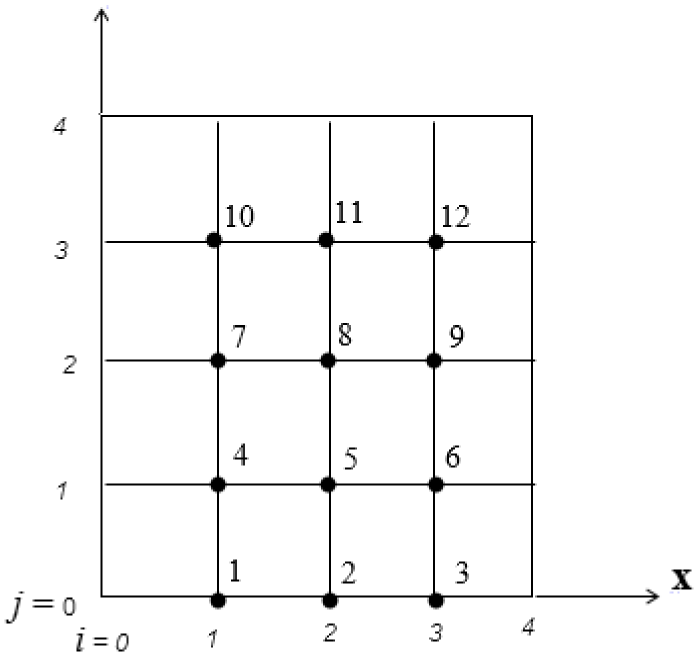
$$\boxed{\nabla^2 u_{i,j} = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u_{i,j}}$$

called a 5- points difference scheme.

Example 3.

Consider $\nabla^2 u = f(x, y)$ on a square with BCs as shown.





Let $N = 4$, $h = 1/4$, then the domain is discretized into a mesh with 5×5 grid points as shown.

The nodes where u is to be determined are only those points

$$\boxed{(i, j) \quad \text{for} \;\; i = 1 \text{ to } 3, \;\; j = 0 \text{ to } 3.}$$

At each of these nodes, we can set up an equation

$$\boxed{\nabla^2 u_{i,j} = f_{i,j}.}$$

Thus the total number of equations equals to the number of unknowns, i.e., 12.

Now, we consider construction of the equations for determination of

$$\boxed{u_{i,j}\ (i = 1,3,\ j = 0,3)\ .}$$

Using 5-point FD approximation

$$\nabla^2 u_{i,j} = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u_{i,j},$$

The given PDE $\nabla^2 u = f(x, y)$ becomes

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u_{i,j} = h^2 f_{i,j} \qquad (1.4)$$

or for i = 1 to 3 and j = 0 to 3

$$\boxed{u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1} = h^2 f_{i,j} \qquad (1.5)}$$

$$u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1} = h^2 f_{i,j} \quad (1.5)$$

- For $j = 0$, as (1.5) involves $u_{i,-1}$, we need to approximate $u_{i,-1}$ by using the boundary condition;
- While for $i, j = 1, 2, 3$, we can immediately obtain the following 9 equations

Form matrix

j=0	Col i = 1	equations for j=0 are to be constructed later												u_{10}
	2													u_{20}
	3													u_{30}
j=1	i = 1	1			−4	1		1						u_{11}
	2		1		1	−4	1		1					u_{21}
	3			1		1	−4			1				u_{31}
j=2	i = 1				1			−4	1		1			u_{12}
	2					1		1	−4	1		1		u_{22}
	3						1		1	−4			1	u_{32}
j=3	i = 1							1			−4	1		u_{13}
	2								1		1	−4	1	u_{23}
	3									1		1	−4	u_{33}

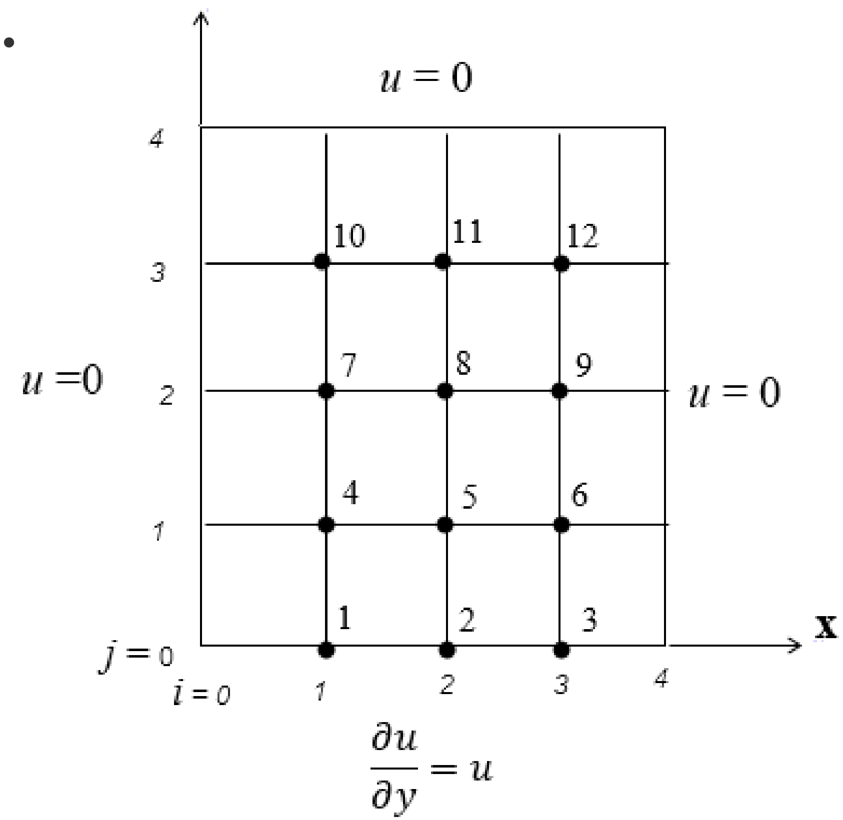
$$B = \begin{bmatrix} -4 & 1 & \\ 1 & -4 & 1 \\ & 1 & -4 \end{bmatrix}.$$

equations for j=0 are to be constructed later

$$
\begin{bmatrix}
1 & & & -4 & 1 & & 1 & & & & & \\
& 1 & & 1 & -4 & 1 & & 1 & & & & \\
& & 1 & & 1 & -4 & & & 1 & & & \\
& & & 1 & & & -4 & 1 & & 1 & & \\
& & & & 1 & & 1 & -4 & 1 & & 1 & \\
& & & & & 1 & & 1 & -4 & & & 1 \\
& & & & & & 1 & & & -4 & 1 & \\
& & & & & & & 1 & & 1 & -4 & 1 \\
& & & & & & & & 1 & & 1 & -4
\end{bmatrix}
\begin{bmatrix} u_{10} \\ u_{20} \\ u_{30} \\ u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{bmatrix}
= h^2
\begin{bmatrix} f_{10} \\ f_{20} \\ f_{30} \\ f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{33} \end{bmatrix}
$$

$$
\Rightarrow \quad
\left[\begin{array}{c|c|c|c}
? & ? & ? & ? \\
\hline
I & B & I & \\
\hline
 & I & B & I \\
\hline
 & & I & B
\end{array}\right] \mathbf{u} = \mathbf{F}
$$

Dirichlet boundary condition: On $x = 0$ ($i = 0$), $x = 1$ ($i = 4$) and $y = 1$ ($j = 4$), $u = 0$. We must move these known values to the right hand side of the equations.



Neumann type boundary condition:

$$u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1} = h^2 f_{i,j} \quad (1.5)$$

For $y = 0$ ($j = 0$), (1.5) becomes

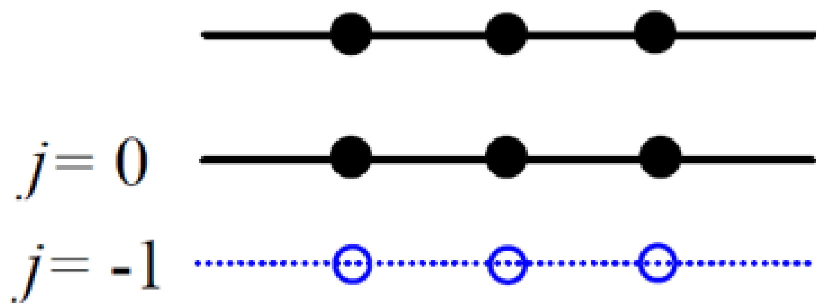
$$\boxed{u_{i,-1} + u_{i-1,0} - 4u_{i,0} + u_{i+1,0} + u_{i,1} = h^2 f_{i,0}} \quad (1.6)$$

Obviously, $u_{i,-1}$ is not defined as the point $(i, -1)$ is outside the region Ω.

Neumann type boundary condition:

- So we need to eliminate the term $u_{i,-1}$ using the Neumann boundary condition.

- sAs we know $\dfrac{\partial u}{\partial y} = u$ on $y = 0$, we introduce a fictitious set of grid points $(i, -1)$ $(i = 1, 2, 3)$ as shown below.

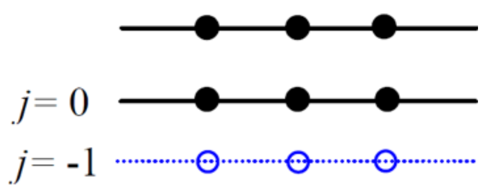


Neumann type boundary condition:

- Then at boundary point (i,0), we can approx. the BC by

$$\boxed{\left.\frac{\partial u}{\partial y}\right|_{j=0} = \frac{u_{i,1} - u_{i-1}}{2h} = u_{i,0}} \quad (1.7)$$

which gives



$$u_{i,-1} = u_{i,1} - 2hu_{i,0} \qquad (1.8)$$

Substituting (1.8) into (1.6)

$$u_{i,-1} + \boxed{u_{i-1,0}} - 4u_{i,0} + u_{i+1,0} + u_{i,1} = h^2 f_{i,0},$$

we have

$$u_{i,-1} + \boxed{\left(u_{i,1} - 2hu_{i,0}\right)} - 4u_{i,0} + u_{i+1,0} + u_{i,1} = h^2 f_{i,0},$$

$$\boxed{u_{i,-1} - (4+2h)u_{i,0} + u_{i+1,0} + 2u_{i,1} = h^2 f_{i,0},}$$

or $$\left[\begin{array}{ccc} & 0 & \\ 1 & -(4+2h) & 1 \\ & 2 & \end{array}\right] u_{i,0} = h^2 f_{i,0}, \quad (i = 1,2,3).$$

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
-(4+2h) & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -(4+2h) & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -(4+2h) & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\mathbf{u} = \mathrm{F}$$

$$\begin{bmatrix}
B-2hI & 2I & \mathbf{0} & \mathbf{0} \\
I & B & I & \mathbf{0} \\
\mathbf{0} & I & B & I \\
\mathbf{0} & \mathbf{0} & I & B
\end{bmatrix}\mathbf{u} = \mathrm{F}\,,$$

Example 4.

Consider $$\frac{\partial^2 u}{\partial x^2} = f(x) \qquad \text{in } [a,b]$$

$$u(a) = \alpha, \qquad u(b) = \beta.$$

a) Divide $[a,b]$ into subintervals by identifying the nodes

$$x_j = j \times h \quad \text{with } h = \frac{1}{m+1}.$$

b) Approximate $u_j := u(x_j)$.

Approximate $u_j := u(x_j)$.

We replace $\frac{\partial^2 u}{\partial x^2}$ by using the centred different approximation:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = f(x_j), j = 1.., m$$

and $u_0 = u(a) = \alpha$

$$u_{m+1} = u(b) = \beta.$$

Then we define the solution vector

$$U = < u_1, u_2, \dots, u_m >^T$$

And we seek the solution to the linear system

$$AU = F,$$

where

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & & \\ 1 & -2 & 1 & 0 & \\ 0 & 1 & -2 & 1 & 0 \\ & & \ddots & & \\ & 0 & 1 & -2 & 1 \\ & & 0 & 1 & -2 \end{bmatrix} \qquad F = \begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ \vdots \\ f(x_m) - \frac{\beta}{h^2} \end{bmatrix}$$

Example 5. 2D Steady State Heat Conduction

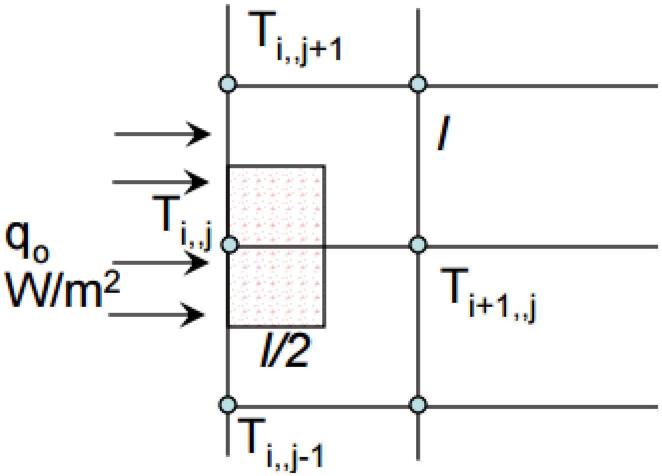
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{S(x,y)}{k} = 0$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \qquad \left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

$$\boxed{T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j} + \frac{S_{i,j} 1^2}{k} = 0}$$

where $\Delta x = \Delta y = 1$

Flux Boundary Condition



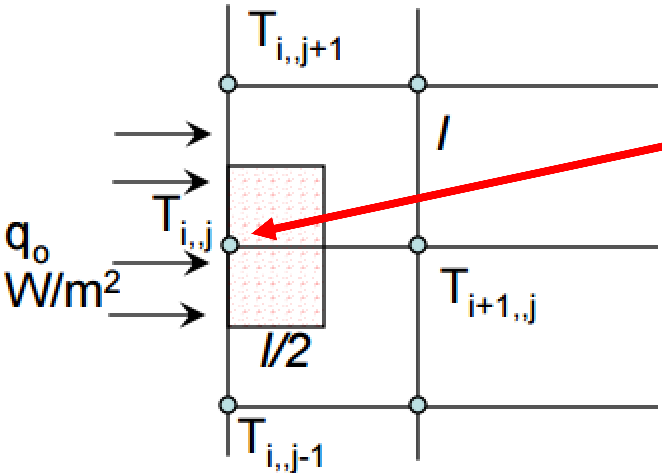
- Nodes (i,j) on a prescribed heat flux boundary

$$q_0 l + k\frac{l}{2}\frac{T_{i,j+1}-T_{i,j}}{l} + kl\frac{T_{i+1,j}-T_{i,j}}{l} + k\frac{l}{2}\frac{T_{i,j-1}-T_{i,j}}{l} + \frac{1}{2}l^2 S_{i,j} = 0$$

- After rearrangement

$$T_{i,j+1} + 2T_{i+1,j} + T_{i,j-1} - 4T_{i,j} + \frac{l^2 S_{i,j}}{k} + \frac{2lq_o}{k} = 0$$

Flux Boundary Condition (another way)



Applying the FDE at the boundary node (i,j)

$$T_{i-1,j} + T_{i+1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} + \frac{l^2 S_{i,j}}{k} = 0$$

B.C.

$$q_o = -k\frac{\partial T}{\partial x} = -k\frac{T_{i+1,j} - T_{i-1,j}}{2l} \Rightarrow T_{i-1,j} = \frac{2lq_o}{k} + T_{i+1,j}$$

$$\boxed{T_{i,j+1} + 2T_{i+1,j} + T_{i,j-1} - 4T_{i,j} + \frac{l^2 S_{i,j}}{k} + \frac{2lq_o}{k} = 0}$$

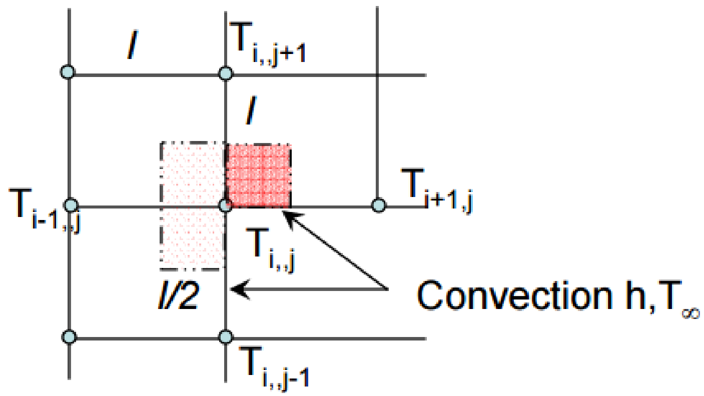
Convective Boundary Condition

- Energy balance gives,

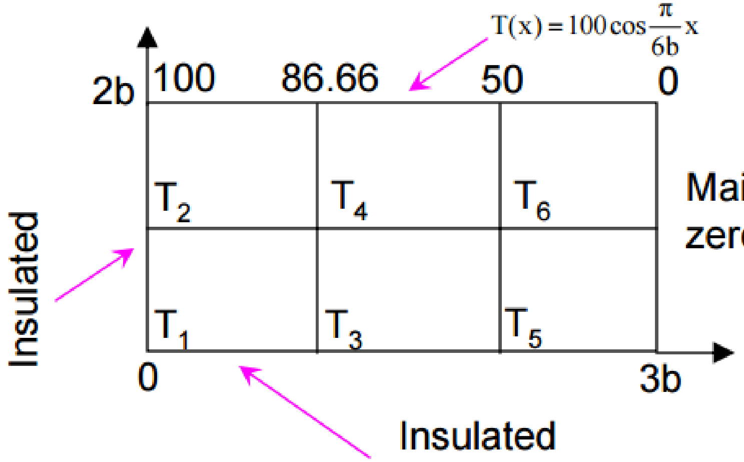
$$k\frac{1}{2}\frac{T_{i,j-1}-T_{i,j}}{l}+kl\frac{T_{i-1,j}-T_{i,j}}{l}+kl\frac{T_{i,j+1}-T_{i,j}}{l}+k\frac{1}{2}\frac{T_{i+1,j}-T_{i,j}}{l}+hl(T_{\infty}-T_{i,j})+\frac{3}{4}l^2S_{i,j}=0$$

- After rearrangement,

$$T_{i,j-1}+2T_{i-1,j}+2T_{i,j+1}+T_{i+1,j}-\left(6+\frac{2hl}{k}\right)T_{i,j}+\frac{3}{2}\frac{l^2}{k}S_{i,j}+\frac{2hl}{k}T_{\infty}=0$$



Insulated Boundary



Node 1: $2T_2+2T_3-4T_1= 0$

Node 2: $T_1+2T_4+100-4T_2 = 0$

Node 3: $T_1+2T_4+T_5-4T_3 = 0$

Node 4: $T_2+T_3+T_6+86.66-4T_4 = 0$

Node 5: $T_3+2T_6-4T_5 = 0$

Node 6: $T_4+T_5+50-4T_6 = 0$

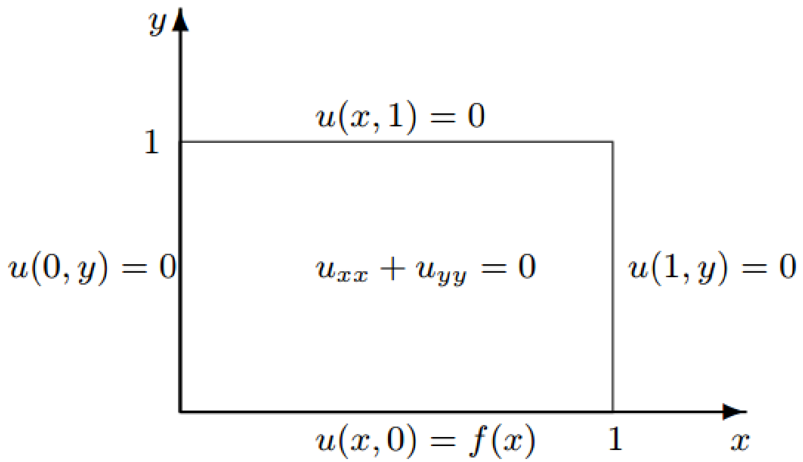
Matrix Form

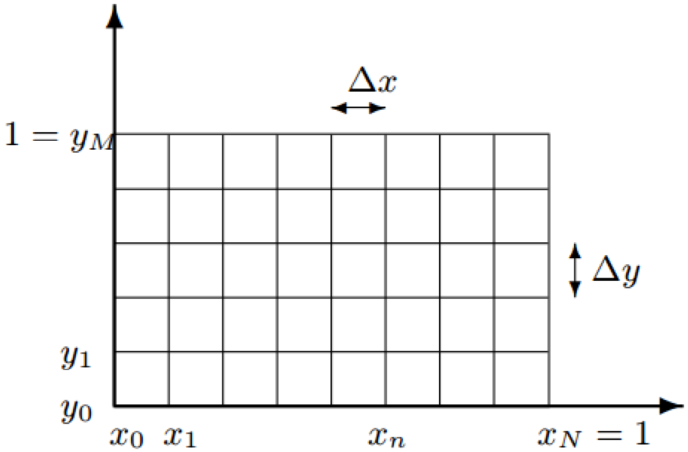
$$\begin{bmatrix} -4 & 2 & 2 & 0 & 0 & 0 \\ 1 & -4 & 0 & 2 & 0 & 0 \\ 1 & 0 & -4 & 2 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 2 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -100 \\ 0 \\ -86.66 \\ 0 \\ -50 \end{bmatrix}$$

Exercise 1: Solving Laplace's equation using finite differences

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 < x, y < 1$$

$$\text{BC:} \quad u(0,y) = 0; \quad u(1,y) = 0; \quad u(x,0) = f(x); \quad u(x,1) = 0.$$





Exercise

Consider $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x - 2y$ on a square with boundary condition as shown. Construct the finite difference scheme when number of nodes N=9.