Development of piecewise-polynomial approximation $u(x)$

For a partition Ω into a number of elements, u can be expressed as

$$u_h(x) = \sum_{j=1}^{N} u_j \phi_j(x), \tag{1}$$

where $\phi_j(x)$ is associated with the entity indexed by j.
The finite element bases are constructed so that $\phi_j(x)$ is nonzero only on elements containing entity j.

In one dimension, finite element bases are constructed implicitly in an element-by-element manner in terms of "shape function":

Piecewise-linear Lagrange polynomial approximation

$$\phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & \text{if } x_{j-1} \leq x < x_j \\ \frac{x_{j+1} - x}{x_{j+1} - x_j}, & \text{if } x_j \leq x < x_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

on the mesh $x_0 < x_1 < \ldots < x_N$

Piecewise-quadratic Lagrange polynomial approximation

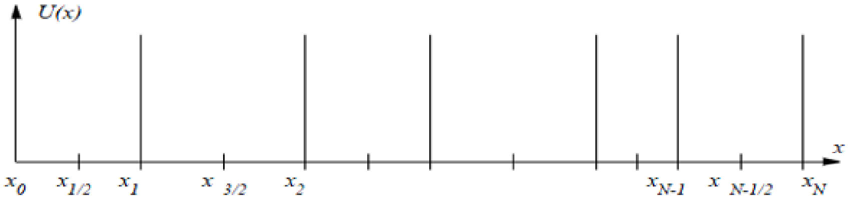


Figure: Quadratic Element

On the element $\Omega_j = [x_{j-1},\ x_j]$,

$\phi_{j-1}(x) \; = \; 1 - 3\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right) + 2\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right)^2$, $\phi_{j-\frac{1}{2}}(x) = 1 - 4\left(\frac{x-x_{j-\frac{1}{2}}}{x_j-x_{j-1}}\right)^2$,

$\phi_j(x) \; = \; 1 + 3\left(\frac{x-x_j}{x_j-x_{j-1}}\right) + 2\left(\frac{x-x_j}{x_j-x_{j-1}}\right)^2$,

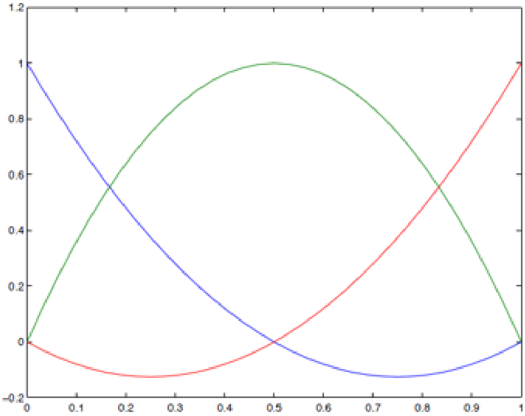


Figure: The three quadratic Lagrangian shape functions on the element Ω_1 when $x_{j-1} = 0,\ x_{j-\frac{1}{2}} = 0.5,\ x_j = 1.$

1D Element Transformation

A 1D Lagrange polynomial shape function of degree 1 is constructed on Ω_e using two vertex nodes.

We map an arbitrary element

$$\Omega_e = [x_{j-1}, x_j] \ \text{ onto } \ -1 \leq \xi \leq 1$$

by the linear transformation

$$T: \ x = x(\xi) = \frac{1-\xi}{2} x_{j-1} + \frac{1+\xi}{2} x_j.$$

A 1D Lagrange polynomial shape function of degree p is constructed on Ω_e using two vertex nodes and $p-1$ nodes interior to the element.

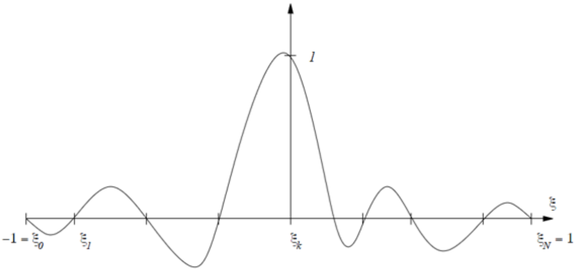


Figure: An element Ω_e used to construct a pth-degree Lagrangian shape function.

The nodes

$$\xi_0 = -1,\ \xi_1, ...,\ \xi_{p-1},\ \xi_p = 1$$

are mapped to the actual physical nodes

$$x_{j-1}, x_{j-1+1/p}, \cdots, x_j$$

on an element Ω

The Lagrangian shape function ψ_k of degree p has a unit value at node k of element and vanishes at all other nodes, thus for $\ell = 0, 1, \ldots, p$

$$\psi_k(\xi_\ell) = \delta_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}$$

which implies that

$$\psi_k(\xi) = \frac{(\xi - \xi_0)(\xi - \xi_1)\cdots(\xi - \xi_{k-1})(\xi - \xi_{k+1})\cdots(\xi - \xi_{k-p})}{(\xi_k - \xi_0)(\xi_k - \xi_1)\cdots(\xi_k - \xi_{k-1})(\xi_k - \xi_{k+1})\cdots(\xi_k - \xi_{k-p})}.$$

We now have

$$U(\xi) = \sum_{k=1}^{p} c_k \psi_k(\xi).$$

Exercise

- Construct the quadratic Lagrange shape function on the master element for p=1, and

$$\xi_0 = -1,\ and \xi_1 = 1.$$

Find $\psi_0(\xi)$ and $\psi_1(\xi)$.

- Construct the quadratic Lagrange shape function on the master element for p=2, and

$$\xi_0 = -1,\ \xi_1 = 0 \text{ and } \xi_2 = 1.$$

Find $\psi_0(\xi)$, $\psi_1(\xi)$ and $\psi_2(\xi)$.

2D Element Transformation

Calculation of element matrices in x, y coordinates is awkward as integration region is complex and limit of integration changes from element to element. If we can find a transformation

$$T_e \ : \quad \begin{cases} x = x(\xi, \eta) = \sum_i x_i \psi_i(\xi, \eta) \\ y = y(\xi, \eta) = \sum_i y_i \psi_i(\xi, \eta) \end{cases}$$

which maps an arbitrarily chosen element e into a standard (master) element $\bar{\Omega}$, then the calculation of element matrices can be standardized using numerical quadrature.

Square element

The geometry of the master element is chosen as simple as possible, eg. the square as shown.

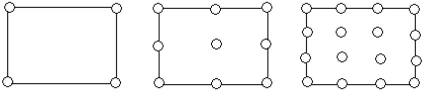


Figure: Square elements with 4 nodes (linear element), 9 nodes (quadratic element) and 16 nodes (cubic element)

For a linear master element, shape function at node i is

$$\psi_i(\xi, \eta) = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i).$$

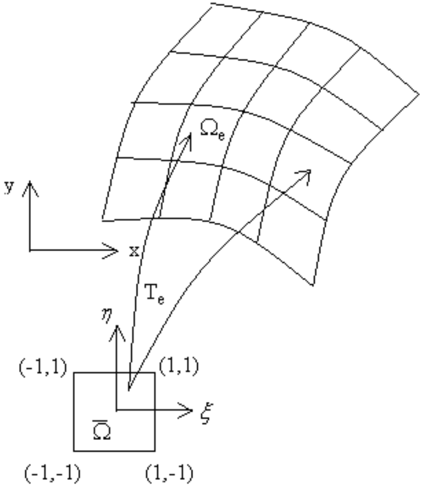


Figure: Square Element transformation T_e

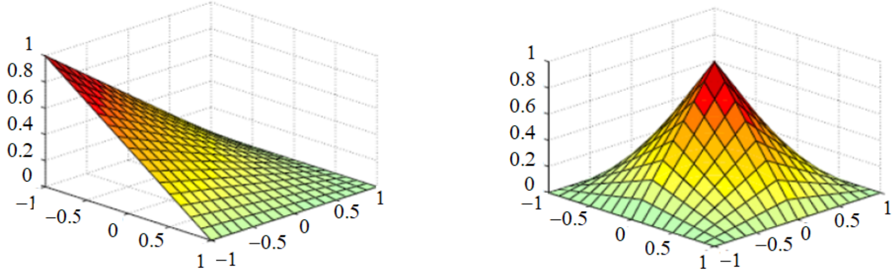
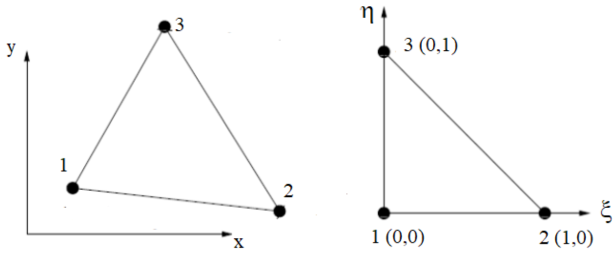


Figure: Bilinear shape function ψ_1 (left) and bilinear basis function at the intersection of four square elements (right)

Triangular element Transformation



For a linear master element, shape function at node i is

$$T_e : \begin{cases} x = (1-\xi-\eta)x_1 + \xi x_2 + \eta x_3, \\ y = (1-\xi-\eta)y_1 + \xi y_2 + \eta y_3 \end{cases}$$

Triangular element Transformation

$$\xi = \frac{det\begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \end{bmatrix}}{det\begin{bmatrix} 1 & x_2 & y_2 \\ 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \end{bmatrix}}, \quad \eta = \frac{det\begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}}{det\begin{bmatrix} 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}}.$$

As a check, evaluate the determinants and verify that

$$(x_1, y_1) \to (0, 0),$$

$$(x_2, y_2) \to (1, 0),$$

$$(x_3, y_13) \to (0, 1).$$

Element Calculation

• A point $P(\xi = \alpha, \eta = \beta)$ in the standard element $\bar{\Omega}$ is mapped into a point

$$P[x(\alpha, \beta), y(\alpha, \beta)]$$

in local element Ω_e.
• A line $(\xi = \alpha)$ in $\bar{\Omega}$ is mapped into a curve

$$[x = x(\alpha, \eta), y = y(\alpha, \eta)]$$

in the plane, which is called the curvilinear coordinate line $(\xi = \alpha)$.

Element Calculation

- A finite element mesh can be viewed as a sequence of transformation $\{T_1,\ T_2, \ldots T_E\}$ of the fixed master element. Each element Ω_e is the image of the master element $\bar{\Omega}$ under a coordinate map T_e.

- All properties of a given type of elements (number and location of nodes, shape functions, stiffness and etc) can be prescribed for the fixed element $\bar{\Omega}$, and then carried to any Ω_e in the mesh by using the map T_e.

Relations between dx, dy with $d\xi$ and $d\eta$
Suppose $x(\xi,\eta)$ and $y(\xi,\eta)$ are continuously differentiable, then

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \text{ and}$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$

$$\text{or} \quad \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = J \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}, \tag{2}$$

where $J =$ Jacobian matrix of the transformation.

Properties of Coordinate Transformation

If at point (ξ, η) we have $|J| = det(J) \neq 0$
then an inverse map $T_e^{-1}(x, y \rightarrow \xi, \eta)$ exists at this point and thus

$$\begin{bmatrix} d\xi \\ \\ d\eta \end{bmatrix} = J^{-1} \begin{bmatrix} dx \\ \\ dy \end{bmatrix} \tag{3}$$

and

$$T_e^{-1} \ : \ \begin{array}{c} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{array} \tag{4}$$

defines a map $(x, y) \rightarrow (\xi, \eta)$.

Element Calculation

As in (2), we have

$$\begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \tag{5}$$

Hence, by equating terms in (5) and (3), we have the following relations

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta}, \quad \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi}, \quad \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi} \tag{6}$$

(3) Construction of the Transformations T_e

Criteria for selection of T_e

(i) Within $\Omega_e, \xi(x, y)$ and $\eta(x, y)$ must be invertible and continuously differentiable.

(ii) $\{T_e\}_{e=1}^{E}$ must generate a mesh with no spurious gaps between elements and with no element overlapping another.

(iii) T_e should be easy to construct from the geometric data of the element.

(3) Construction of the Transformations T_e

Construction of T_e

The transformation T_e is constructed based on the element shape functions.

Let ψ_j be the shape function defined on $\bar{\Omega}$ for $j = 1, 2...N$, where N is the total number of nodes in $\bar{\Omega}$.

Then, any function $g = g(\xi, \eta)$ in $\bar{\Omega}$ can be approximated by

$$\bar{g}(\xi, \eta) = \sum g_j \psi_j(\xi, \eta). \tag{7}$$

(3) Construction of the Transformations T_e

Let $g = x$ and $g = y$ respectively, from (7) we have

$$T_e \;:\; \begin{array}{l} x = \sum_{j=1}^{N} x_j \psi_j(\xi, \eta), \\ \\ y = \sum_{j=1}^{N} y_j \psi_j(\xi, \eta), \end{array} \tag{8}$$

which maps $\bar{\Omega}$ to Ω_e. To see this, consider a node i in $\bar{\Omega}$, the coordinates is (ξ_i, η_i). From (8), this point is mapped into point $x = x_i$, $y = y_i$ in the $x - y$ plane i.e, node i.

Remark

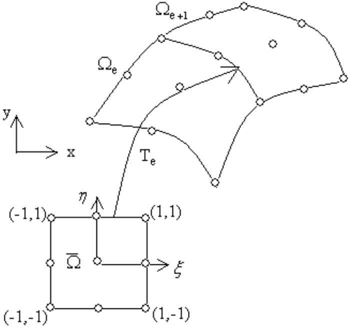
1) Criterion (iii) is easily verified. T_e is readily constructed from element data $(x_i,\ y_i,\ ...)$.
2) Criterion (ii) is usually not difficult to satisfy.
3) For T_e to be invertible, we require $det(J) \neq 0$. In addition, from the integration theory,

$$dxdy = |J|\ d\xi d\eta.$$

Clearly, for the mapping defined by (4) to be acceptable, we must have positive values of $|J|$ at all points in $\bar{\Omega}$. The satisfaction of this condition is not assured in general for all maps of the form (8). Each set of shape functions must be examined to ensure that $|J| > 0$ throughout $\bar{\Omega}$.

Straight sides of $\bar{\Omega}$ map to curved sides of Ω_e

The quadratic shape function on the master square maps the element to the corresponding elements Ω_e in the $x - y$ plane in such a way that straight sides of the $\bar{\Omega}$ are mapped to quadratic curved sides of Ω_e.
On a given curved side between Ω_e and Ω_{e+1}, the maps T_e and T_{e+1} reduce to the same quadratic functions.

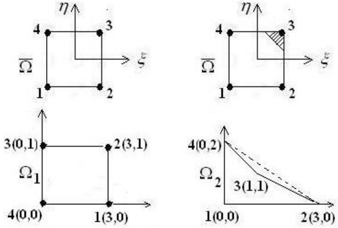


Example

The following figure shows a 4-node master element $\bar{\Omega}$ and 2 elements Ω_1 and Ω_2 generated from it using the map (8). The shape function defined on $\bar{\Omega}$ are

$$\psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i), \quad (i = 1, ..., 4)$$

where (ξ_i, η_i) are coordinates of node i.



Element Calculation

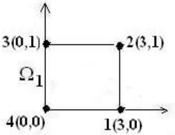
In this example, straight lines $\xi =$ constant or $\eta =$ constant in $\bar{\Omega}$ map to corresponding straight lines in Ω_e .

For Ω_1

$$T_e \; : \quad \begin{aligned} x &= 3\psi_1 + 3\psi_2 + (0)\psi_3 + (0)\psi_4 = \tfrac{3}{2}(1-\eta) \\ y &= \psi_2 + \psi_3 = \tfrac{1}{2}(1+\xi). \end{aligned}$$

$$|J| = det \begin{bmatrix} 0 & -\frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = \frac{3}{4} > 0$$

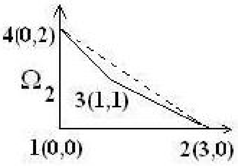
Therefore, the map is invertible.



Element Calculation

For Ω_2

$$|J| = \frac{1}{8}(5-3\xi-4\eta) \begin{cases} = 0 \text{ along } L : \xi = \frac{5}{3} - \frac{4}{3}\eta \\ > 0 \text{ below } L \\ < 0 \text{ above } L. \end{cases}$$



The region above L is mapped outside of Ω_2 by T_2. Clearly, Ω_2 is unacceptable.

Finite Element Calculations

The key to the finite element approximation is the calculation of the element matrices for each element in the mesh. To calculate the following integrals

$$
\begin{aligned}
k_{ij}^e &= \int_{\Omega_e} \left[k\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} \right) + b\phi_i\phi_j \right] dxdy, \\
f_i^e &= \int_{\Omega_e} f\phi_i \, dxdy, \\
p_i^e &= \int_{\partial\Omega_{2e}} p\phi_i\phi_j \, ds,
\end{aligned} \tag{9}
$$

and

$$\gamma_i^e = \int_{\partial\Omega_{2e}} P\hat{u}\phi_i \, ds,$$

we begin by choosing the master element $\bar{\Omega}$ with geometry as simple as possible, such as square.

Finite Element Calculation

For a chosen Ω, we need to

- identify M nodes and shape function φ to define the coordinates map T_e,
- identify N nodes and shape function $\bar{\varphi}$ for local approximation of the unknown function.

Remarks: M and N need not to be the same.

- If $M > N_e$, then it is super-parametric map.
- If $M = N_e$, then it is iso-parametric map (iso-parametric element).
- If $M < N_e$, then it is sub-parametric map.

Element Calculation

In the following, we will consider only the iso-parametric element.
Having selected $\bar{\Omega}$ and φ_j, we perform the following steps:
(1) Element map

$$T_e \; : \; \begin{array}{c} x = \sum_{j=1}^{N} x_j \psi_j(\xi, \eta) \\ \\ y = \sum_{j=1}^{N} y_j \psi_j(\xi, \eta) \end{array} \tag{10}$$

Transformation of shape functions
As T_e is invertible, $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ and the element shape functions are

$$\phi_j(x, y) = \psi_j[\xi(x, y), \eta(x, y)] \tag{11}$$

Therefore,

$$\frac{\partial \phi_j}{\partial x} = \frac{\partial \psi_j}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_j}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial \phi_j}{\partial y} = \frac{\partial \psi_j}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi_j}{\partial \eta}\frac{\partial \eta}{\partial y}.$$

Transformation of shape functions

According to (10)

$$\frac{\partial x}{\partial \xi} = \sum_{1}^{N_e} x_k \frac{\partial \psi_k}{\partial \xi}(\xi, \eta), \quad \frac{\partial x}{\partial \eta} = \sum_{1}^{N_e} x_k \frac{\partial \psi_k}{\partial \eta}(\xi, \eta),$$

$$\frac{\partial y}{\partial \xi} = \sum_{1}^{N_e} y_k \frac{\partial \psi_k}{\partial \xi}(\xi, \eta), \quad \frac{\partial y}{\partial \eta} = \sum_{1}^{N_e} y_k \frac{\partial \psi_k}{\partial \eta}(\xi, \eta),$$

Thus, using (6) and (10), equation (11) becomes

$$\frac{\partial \phi_j}{\partial x} = \frac{1}{|J|}\left\{ \frac{\partial \psi_j}{\partial \xi} \sum_{k=1}^{N} y_k \frac{\partial \psi_k}{\partial \eta}(\xi, \eta) - \frac{\partial \psi_j}{\partial \eta} \sum_{k=1}^{N} y_k \frac{\partial \psi_k}{\partial \xi}(\xi, \eta) \right\}$$

$$\frac{\partial \phi_j}{\partial y} = \frac{1}{|J|}\left\{ \frac{\partial \psi_j}{\partial \xi} \sum_{k=1}^{N} x_k \frac{\partial \psi_k}{\partial \eta}(\xi, \eta) - \frac{\partial \psi_j}{\partial \eta} \sum_{k=1}^{N} x_k \frac{\partial \psi_k}{\partial \xi}(\xi, \eta) \right\}$$

Remarks

(a) The partial derivatives of ϕ_j w.r.t. x and y are completely determined by calculation defined only on $\bar{\Omega}$.

(b) From (9), for 4-node element, K^e is a 4*4 matrix which can be expressed as

$$K^e = \int_{\Omega_e} (k(D\phi)^T (D\phi) + b\phi^T \phi) \, d\Omega \tag{12}$$

where $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ and

$$D\phi = \begin{bmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_3}{\partial x} & \frac{\partial \phi_4}{\partial x} \\ \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_3}{\partial y} & \frac{\partial \phi_4}{\partial y} \end{bmatrix}.$$

Integration

Let $I = \int_{\Omega_e} g(x,y)\ dxdy$
then $I = \int_{\bar{\Omega}}\ G(\xi,\eta)\ d\xi d\eta$,
where

$$G(\xi,\eta) = g(\sum_{1}^{N} x_j\psi_j(\xi,\eta), \sum_{1}^{N} y_j\psi_j(\xi,\eta))|J(\xi,\eta)| \tag{13}$$

Numerical quadrature (such as the Gaussian quadrature) are usually used to evaluate the integrals. Quadrature rules for quadrilateral elements are usually derived from the 1-D quadrature by treating the integration over $\bar{\Omega}$ as a double integral.
Thus, using the 1-D quadrature rule of order N,

$$I = \int_{\bar{\Omega}} G(\xi,\eta)d\xi d\eta = \int_{-1}^{1}\left[\int_{-1}^{1} G(\xi,\eta)d\xi\right]d\eta \approx \sum_{k=1}^{N}\left[\sum_{\ell=1}^{N} G(\xi_\ell,\eta_k)w_\ell\right]w_k$$

Integration

For 9-point Gaussian quadrature (1-D of order 3).

$$N = 3,\ w_1 = 5/9,\ w_2 = 8/9,\ w_3 = 5/9,$$

$$\xi_1 = \eta_1 = -\sqrt{3/5},\ \xi_2 = \eta_2 = 0,\ \xi_3 = \eta_3 = \sqrt{3/5}.$$

If $k = k(x, y),\ b = b(x, y)$ and $f = f(x, y)$ are not constant over an element, we may use

$$k(x, y) \approx \sum_{j=1}^{N} k_j \phi_j(x, y),\quad b(x, y) \approx \sum_{j=1}^{N} b_j \phi_j(x, y),\quad f(x, y) \approx \sum_{j=1}^{N} f_j \phi_j(x, y).$$

Then the calculations of a_{ij}^e and f_i^e only require the nodal values of k, b and f.

Boundary Integrals

The calculation of the boundary integrals in (9) is carried out by integrating along those sides of $\bar{\Omega}$ that are mapped onto the sides of $\partial\Omega_{2e}$ along which natural boundary conditions are prescribed.
For definiteness, we suppose that the sides $\xi = 1$ of a master square is to be mapped onto $\partial\Omega_{2h}$. Let θ_j denote the restriction of the master-element shape function ψ_j to side $\xi = 1$, i.e,

$$\theta_j(\eta) = \psi_j(1, \eta), \quad j = 1, 2, ..., N.$$

We thus have

$$\int_{\partial\Omega_{2e}} p\phi_i\phi_j \, ds = \int_{-1}^{1} p\theta_i(\eta)\theta_j(\eta)|J| \, d\eta$$

Integration

Since
$$ds = \sqrt{\left(\tfrac{\partial x}{\partial \eta}(1,\eta)\right)^2 + \left(\tfrac{\partial y}{\partial \eta}(1,\eta)\right)^2}\, d\eta,$$
we have
$$|J(\eta)| = \sqrt{\left(\frac{\partial x}{\partial \eta}(1,\eta)\right)^2 + \left(\frac{\partial y}{\partial \eta}(1,\eta)\right)^2}$$
where $x(\xi,\eta)$ and $y(\xi,\eta)$ are defined in (10). The integral can be evaluated numerically.

Example

Let Ω be a square region, consider

$$\begin{cases} \nabla \cdot k(x)\nabla T + Q = 0 & \text{on } \Omega \\ T = x & \text{on } y = 0 \\ T = 3 + x^2 & \text{on } y = 3 \\ T = y & \text{on } x = 0 \\ \frac{\partial T}{\partial x} = 1 - 0.2T & \text{on } x = 3 \end{cases}$$

Finite Element mesh

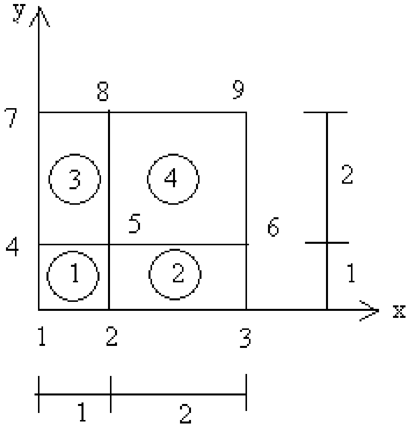


Figure: Finite element mesh

Variational Statement

Find $T \in H^1(\Omega)$ such that $T = x$ on $y = 0$, $T = 3 + x^2$ on $y = 3$, $T = y$ on $x = 0$ and

$$a(T, v) = L(v), \qquad \forall v \in H_0^1(\Omega)$$

.

where $H_0^1(\Omega) = \{v : v \in H^1(\Omega) \text{ and } v = 0 \text{ on } x = 0,\ y = 0 \text{ and } y = 3\}$

$$a(T, v) = \int_0^3 0.2T(3, y)v(3, y)\, dy + \int_\Omega \nabla T \cdot \nabla v\, d\Omega$$

$$L(v) = \int_0^3 v(3, y)\, dy + \int_\Omega 2v\, d\Omega$$

Variational Statement

Finite element equations is a system

$$\mathbf{AT} = \mathbf{F},$$

where

$\mathbf{A} = \{a_{ij}\}$, with $a_{ij} = \sum_{e=1}^{E} \int_{\Omega_e} \nabla\phi_i \cdot \nabla\phi_j \, d\Omega + \sum_{e=1}^{B} \int_{\partial\Omega_e} 0.2\phi_i\phi_j \, dy$

$\mathbf{F} = \{f_i\}$ with $f_i = \sum_{e=1}^{E} \int_{\Omega_e} 2\phi_i \, d\Omega + \sum_{e=1}^{B} \int_{\partial\Omega_e} \phi_i \, dy$,

B is the number of elements with natural boundary condition.

Solution of Sparse Systems of Linear Equations

This chapter concerns with a unique solution of the finite element linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{14}$$

where

- the $n \times n$ matrix $\mathbf{A}$ is large, sparse, and symmetric,
- $\mathbf{b}$ and $\mathbf{x}$ are both $n \times m$ $(m \geq 1)$ matrices.

Positive - Definite Systems

For a positive-definite matrix $\mathbf{A}$

- Gauss elimination process can be performed without row interchange and the computation is stable wrt growth of rounding error.
- LDL^T factorization method can be used to solve the system

$$\mathbf{Ax} = \mathbf{b}$$

.

Note: An $n \times n$ matrix A with real entries is called positive definite if

$$\mathbf{x}^T\mathbf{Ax} > \mathbf{0}.$$

In FEM, the nonzero entries of A are often clustered in a small number of diagonals surrounding the main diagonal. Thus, we can use special techniques to improve computation efficiency by avoiding operating involving zero operands, and reduce storage requirement.

Positive - Definite Systems

The algorithm is follows:

Factorization : $\mathrm{d}_1 = a_{11}$

For i=1,2,...,n

$$\ell_{ij} = \frac{1}{d_j}[a_{ij} - \sum_{k=1}^{j-1} \ell_{ik}\ell_{jk} d_k], \;\; j = 1, 2, ..., i-1$$

$$\mathrm{d}_i = a_{ii} - \sum_{k=1}^{i-1} \ell_{ik}^2 d_k$$

substitution : *For*i=1,...,n

$$\mathrm{y}_i = n_i - \sum_{k=1}^{i-1} \ell_{ik} y_k$$

*For*i=n,n-2,...,1

$$\mathrm{x}_i = \frac{y_i}{d_i} - \sum_{k=i+1}^{n} \ell_{ki} x_k$$

Band Methods

Let m_A be the smallest integer such that $a_{ij} = 0$ for $|i - j| > m_A$, then

- the portion of A containing exactly those entries a_{ij} satisfying $|i - j| \leq m_A$ called the band of A,
- m_A is called the bandwidth by storing mainly the nonzero entries.

In the LDL^T method, none of the entries of the matrix A can ever be nonzero unless $|i - j| \leq m_A$. This fact can be used to modify the algorithm, so that only the entries of the band are ever used.

Algorithm

LDL^T: $d_1 = a_{11}$

For $i = 2, ..., n$

$$\ell_{ij} = \frac{1}{d_j}[a_{ij} - \sum_{k=max(i-m_A, j-m_A)}^{j-1} \ell_{ik}\ell_{jk}d_k], \quad j = 1, 2, ...,$$

$$d_i = a_{ii} - \sum_{k=i-m_A}^{j-1} \ell_{ik}^2 d_k$$

Subst: $$y_i = b_i - \sum_{k=i-m_A}^{i-1} \ell_{ik}y_k, \quad i = 1, ..., N$$

$$y_i = \frac{y_i}{d_i} - \sum_{k=i+1}^{min(i+m_A, n)} \ell_{ki}x_k, \quad i = n, n-1, ..., 1.$$

Remarks:

Let m_i be the smallest integer such that

$$a_{ij} = 0\ (j < i)\ \texttt{ if }\ i - j > m(\texttt{or }\ j < i - m\ )$$

. It can be shown that, the algorithm for such problem is similar to the one above. Except for the backward substitution formula, we can replace

- $i \pm m_A$ by $i \pm m_i$
- $j \pm m_A$ by $j \pm m_j$.

Storage Structure

Rules:

a) Store the matrix $A_{n\times n}$ row by row into a one-dimensional array $A1$.

b) For each row, store the elements from the 1st nonzero entry to the diagonal element.

- Use $IA(i)_{i=1\ to\ N}$ to identify the storage location of $A(i,i)$ in the 1-D array $A1$. Thus,
  - The elements in ith row (from the 1st nonzero element to the diagonal element) are stored into $A1[IA(i-1)+1], A1[IA(i-1)+2], ..., A1[IA(i)]$
  - The number of nonzero elements (from the 1st nonzero elements to the diagonal but not include the diagonal element)

$$m_i = IA(i) - IA(i-1) - 1$$

  - The element $A(i,j)$ is stored into $A1(ij)$ where

$$ij = IA(i) - (i-j).$$

Example

For $A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{bmatrix}$ (assume symmetric)

We will construct $A1$ and IA as follows:

k	1	2	3	4	5	6	7	8
$A1(k)$	a_{11}	a_{21}	a_{22}	a_{32}	a_{33}	a_{42}	0	a_{44}

$IA(i)_{i=1 \ to \ N}$ identifies the storage location of $A(i,i)$ in the 1-D array $A1$.

i	1	2	3	4
$IA(i)$	1	3	5	8

Programming using 1-D Storage Structure

A: an array, contain matrix A(1-D). On exit, contain LD in 1-D form.
For $i = 2, N$

$$m_i = IA(i) - IA(i-1) - 1$$

$$i1 = max(i - m_i, 1)$$

For $j = i1, i$

$$m_j = IA(j) - IA(j-1)$$
$$J1 = max(j - m_j, 1)$$
$$sum = 0$$

For $k = max(i1, j1),\ j - 1$

$$ik = Trans(i, k)$$
$$jk = Trans(j, k)$$
$$kk = Trans(k, k)$$
$$sum = sum + A(ik) * A(jk) * A(kk)$$

Programming using 1-D Storage Structure

$ij = Trans(i, j)$
$jj = Trans(j, j)$
If $(i \neq j)$ then $A(ij) = (A(ij) - sum)/A(jj)$
else $A(ij) = a(ij) - sum$

END

Exercise

Given the list of nodal points and their coordinates and the list of elements and their node numbers below:

(a) sketch the finite element mesh Ω_h

(b) sketch the ξ, η -axes in each element and verify that the maps $T_e, e = 1, 2, ...5$, produce a connected region Ω_h

(c) sketch the global basis function ϕ_e for node 4 of the mesh.

Node	x	y
1	0	1
2	0.7	0.7
3	1	0
4	0	2
5	1.5	1.5
6	2	0
7	0	3
8	1.5	3
9	3	3

Element	Nodes
1	1, 2, 5, 4
2	3, 6, 5, 2
3	5, 8, 7, 4
4	5, 10, 9, 8
5	6, 11, 10, 5