MATH5004 TUT11

3D FEM: Stoke Problems

Derive variational statement of the Stoke problem of scalar unknown *u*,

$$\begin{cases} \mu\Delta u_i - p_{,i} + f_i = 0 & \text{in } \Omega \quad (i = 1,2,3) \\ u_{i,i} = 0 & \text{in } \Omega \end{cases}$$

with boundary condition

$$\begin{cases} u_i = 0 & \text{on } \partial\Omega_1 \\ u_i = u_i^0 & \text{on } \partial\Omega_2 \end{cases}$$

$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ is the boundary of Ω.

Step 1. Variational statement

Firstly, we consider the momentum equation with error function for the *ith* equation is $r_i = \mu\Delta u_i - p_{,i} + f_i$.

Its total weighted residual for the *ith* equation is

$$R_i = \int_\Omega r_i v_i \; d\Omega = \int_\Omega \mu\Delta u_i - p_{,i} + f_i) v_i \; d$$

As $\nabla \cdot (v_i \, \mu\nabla u_i) = v_i\mu\Delta u_i + \mu\nabla u_i \cdot \nabla v_i$

then $v_i\mu\Delta u_i = \nabla \cdot (v_i \, \mu\nabla u_i) - \mu\nabla u_i \cdot \nabla v_i$ and we have

$$\int_\Omega \nabla \cdot (v_i \, \mu\nabla u_i) - \mu\nabla u_i \cdot \nabla v_i - p_{,i} v_i + f_i v_i \; d\Omega$$

Using the Divergence Theorem

$$\int_\Omega \nabla \cdot (v_i \, \mu\nabla u_i) \; d\Omega = \int_{\partial\Omega} v_i\mu\nabla u_i \cdot \underline{n} \; ds = \int_{\partial\Omega} v_i\mu\frac{\partial u_i}{\partial n} \; ds$$

Now by setting $R_i = 0$, we have

$$\int_{\partial\Omega} v_i\mu\frac{\partial u_i}{\partial n} \; ds - \int_\Omega \mu\nabla u_i \cdot \nabla v_i + p_{,i} v_i - f_i v_i \; d\Omega$$

Choosing v s.t. $v_i = 0$ on $\partial\Omega_1$ and $\partial\Omega_2$, we have

$$\int_\Omega \mu\nabla u_i \cdot \nabla v_i \; d\Omega = -\int_\Omega p_{,i} v_i + f_i v_i \; d\Omega$$

We now consider the continuity equation and do the same process. We have

$$\int_\Omega \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) q \; d\Omega = 0$$

Choose $\boldsymbol{u} = (u_1, u_2, u_3) \in \left[H^1(\Omega)\right]^3, v_i \in H_0^1(\Omega)$ and $q \in H^0(\Omega)$ as defined in the problem, the variational statement is

Find $\boldsymbol{u} \in \left[H^1(\Omega)\right]^3$ such that $\boldsymbol{u} = 0$ on $\partial\Omega_1$ and $\boldsymbol{u} = \boldsymbol{u}^0$ on $\partial\Omega_2$

$$
\begin{aligned}
a_m(u_i, v_i) &= L(v_i) \qquad && \forall v_i \in H_0^1(\Omega), \qquad - (1)\\
a_c(u_i, q) &= 0 && \forall q \in H^0(\Omega)
\end{aligned}
$$

where
$$
\begin{aligned}
a_m(u_i, v_i) &= \int_\Omega \mu \nabla u_i \cdot \nabla v_i \; d\Omega,\\
a_c(u_i, q) &= \int_\Omega u_{i,i} q \; d\Omega
\end{aligned}
$$

$$L(v_i) = -\int_\Omega p_{,i} v_i + f_i v_i \; d\Omega$$

$$H_0^1(\Omega) = \left\{ v \middle| v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega) \text{ and } v = 0 \text{ on } \partial\Omega_1 \text{and } \partial\Omega_2 \right\}.$$

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