Revision

Boundary value problem (BVP)

In heat transfer problems, two terms of heat and temperature are considered.

- Temperature is a measure of the amount of energy possessed by the molecules of a substance. It manifests itself as a degree of hotness, and can be used to predict the direction of heat transfer. The usual symbol for temperature is T. The scales for measuring temperature in SI units are the Celsius and Kelvin temperature scales.
- Heat is energy in transit. Spontaneously, heat flows from a hotter body to a colder one. The usual symbol for heat is Q. In the SI system, common units for measuring heat are the Joule and calorie.

BVP of heat transfer problem

$$\rho c_p \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T \right) = \nabla \cdot k \nabla T + Q, \qquad (1)$$

subject to initial condition $T(\mathbf{x}, 0) = T^0$ and boundary conditions

$$\begin{aligned} T(\mathbf{x}, t) &= T_0, && \mathbf{x} \in \partial\Omega_1 \\ -k\frac{\partial T}{\partial n} &= h_\infty(T - T_\infty), && \mathbf{x} \in \partial\Omega_2 \end{aligned} \qquad (2)$$

where

- ρ and c_p denote respectively density and heat capacity of material;
- k is thermal conductivity, a thermodynamic property of the material(W/m K)
- ∇T is gradient of temperature (Kelvin/m). $\nabla T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}$
- $\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}$ is velocity of fluid.

Finite difference method (FDM)

FDM replaces governing differential equations and boundary conditions with algebraic finite difference equations.

Based on **Taylor series expansion**,

$$f(x+\triangle x)=f(x)+\triangle x\frac{\partial f}{\partial x}+\frac{(\triangle x)^2}{2}\frac{\partial^2 f}{\partial x^2}+\cdots+\frac{(\triangle x)^n}{n!}\frac{\partial^n f}{\partial x^n}+\cdots$$

numerical differentiation for first derivative can be approximated by

- $f'(x)=\frac{f(x+\triangle x)-f(x)}{\triangle x}+O\left(\triangle x^2\right)$, (Forward difference)
- $f'(x)=\frac{f(x)-f(x+\triangle x)}{\triangle x}+O\left(\triangle x^2\right)$, (Backward difference)
- $f'(x)=\frac{f(x+\triangle x)-f(x-\triangle x)}{2\triangle x}+O\left(\triangle x^2\right)$ (Central difference)

FDM (cont)

For numerical Differentiation-second derivative, We take the difference of forward and backward approximation for $f'(x)$:

- Let $f_i \equiv f(x_i)$ and $h = \triangle x$
- Use forward difference to approximate $f_i^{'}$ and $f_{i-1}^{'}$

$$f_i^{'} \approx \frac{f_{i+1} - f_i}{h} \qquad f_{i-1}^{'} \approx \frac{f_i - f_{i-1}}{h}$$

$$f_i^{''} \approx \frac{f_i^{'} - f_{i-1}^{'}}{h}$$

$$f_i^{''} \approx \frac{f_{i_1} - 2f_i + f_{i-1}}{h^2}$$

FDM procedure

1) Discretize Ω into a mesh of discrete points called nodes.

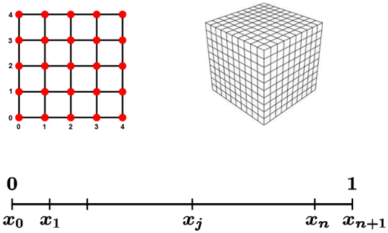


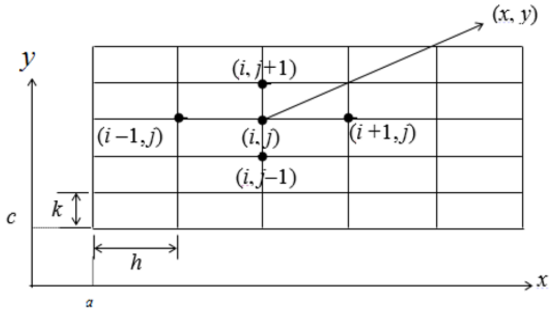
Figure: Domain discretisation.

For one-dimensional problem, the domain $\Omega = [a, b]$ is subdivided into a set of intervals. For uniform mesh, each interval has the the same length $dx = h = (b - a)/n$

FDM procedure (cont)

2) Approximate all derivatives using the values of the unknown function at the nodes, and we thus obtain the linear (or nonlinear) system of algebraic equations $AU = F$ with the nodal values of the unknown function as basic unknowns.
3) Solve the system of algebraic equations.

For 2D problems,



Taylor's Theorem for 2 variables

$$
\begin{aligned}
f(x_i + h, y_j) = f_{i+1,j} = f_{i,j} + hf_x + \tfrac{h^2}{2} f_x x + \cdots \\
f(x_i - h, y_j) = f_{i-1,j} = f_{i,j} - hf_x + \tfrac{h^2}{2} f_x x - \cdots
\end{aligned} \tag{3}
$$

Rearranging $(3)_1 and (3)_2$ yields

$$
\left(\frac{\partial f}{\partial x}\right)_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} + O(h^2)
$$

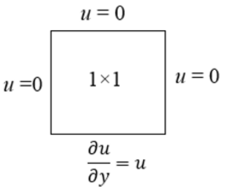
$(3)_1 + (3)_2$ and then rearranging gives

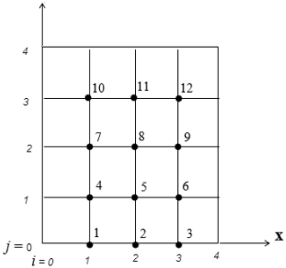
$$
\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} = \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2} + O(h^2)
$$

$$
\left(\frac{\partial^2 f}{\partial y^2}\right)_{i,j} = \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{k^2} + O(k^2)
$$

Example

Consider $\nabla^2 u = f(x, y)$ on a square with boundary conditions as shown





Let $N = 4,\ h = 1/4$, then the domain is discretized into a mesh with 5×5 grid points as shown.

The nodes where u is to be determined are only those points

$$(i, j) \text{ for } i = 1 \text{ to } 3, \; j = 0 \text{ to } 3.$$

At each of these nodes, we can set up an equation

$$\nabla^2 u_{i,j} = f_{i,j}.$$

Thus the total number of equations equals to the number of unknowns, i.e., 12.

Now, we consider construction of the equations for determination of

$$u_{i,j} \; (i = 1, 2, 3; \; j = 0, 1, 2, 3)$$

Using 5-point FD approximation

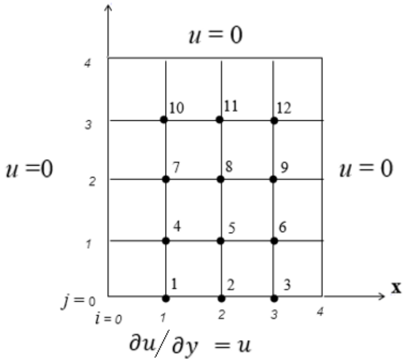
$$\nabla^2 u_{i,j} = \frac{1}{h^2} \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u_{i,j}$$

The given PDE $\nabla^2 u = f(x, y)$ becomes

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} u_{i,j} = h^2 f_{i,j}$$

For $i = 1, 2, 3$ and $j = 0, 1, 2, 3$

$$u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1} = h^2 f_{i,j} \tag{4}$$



Neumann type boundary condition

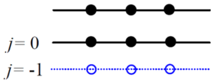
For y = 0 (j = 0), eqn (4) becomes

$$u_{i,-1} + u_{i-1,0} - 4u_{i,0} + u_{i+1,0} + u_{i,1} = h^2 f_{i,0} \tag{5}$$

in which $u_{i,-1}$ is not defined as the point $(i, -1)$ is outside the region Ω.

- So we need to eliminate the term $u_{i,-1}$ using the Neumann boundary condition $\partial u/\partial y = u$ on $y = 0$ (or $j = 0$)
- We introduce a fictitious set of grid points $(i, -1)$ $(i = 1, 2, 3)$ as shown below.
- Then at boundary point $(i, 0)$, we can approx. the BC by

$$\left(\frac{\partial u}{\partial y}\right)_{j=0} = \frac{u_{i,1} - u_{i,-1}}{2h} = u_{i,0} \tag{6}$$



$$u_{i,-1} = u_{i,1} - 2hu_{i,0}$$

Thus, for y = 0 (j = 0), eqn (5) becomes

$$u_{i-1,0} - (4+2h)u_{i,0} + u_{i+1,0} + 2u_{i,1} = h^2 f_{i,0} \tag{7}$$

or

$$\begin{bmatrix} & 0 & \\ 1 & -(4+2h) & 1 \\ & 2 & \end{bmatrix} u_{i,0} = h^2 f_{i,0}$$

Finite element method (FEM)

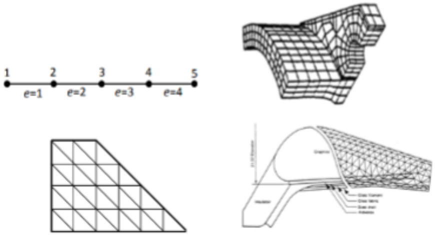
FEM approximates the behavior of an irregular, continuous structure under general loading and constraints with an assembly of discrete elements.

FEM Procedure

- Discretize the computation domain Ω into a finite umber of elements with N nodes;
- Select of interpolation function Derive variational formulation;
- Obtain coefficient matrices for each element Assemble each element matrix to form a global matrix;
- Apply the known loads;
- Solve simultaneous linear algebraic equations;
- Represent the results in tabular or graphic forms.

Domain Mesh Ω_h

- One-dimensional (1D) elements are line elements,
- Two-dimensional (2D) elements can be triangular or bilinear elements.
- Three-dimensional (3D) elements are polyhedrals or cuboids.



- ◇ The basis of the nodal FEM is the representation of the domain by an assemblage of subdivisions called finite elements.

$$\Omega = \bigcup_{e=1}^{E} \Omega_e$$

These elements are interconnected at nodes or nodal points.

- ◇ The trial function approximates the distribution of the primary variable and are commonly used in the nodal expressions.

$$u_h = \sum_{i=1}^{N} \phi_i u_i$$

Weak Formation of Governing Equations

The main approaches of the FEM are in the redirection of the DE of the continuum problem to its integral form and using a trial function over the nodal form of the equation.

Let ϕ_i be the set of interpolation functions

and u_i be the set of nodal primary variable

Then an approximate trial function $\quad u_h = \sum_{i=1}^{n} \phi_i u_i,$

is an approximation solution in the elemental domain defined by a set of integral form of the original BVP

$$
\begin{aligned}
L(u) &= f(\mathbf{x}) \quad \mathbf{x} \text{ in } \Omega \\
B(u) &= g(\mathbf{x}) \quad \mathbf{x} \text{ on } \Gamma.
\end{aligned}
$$

Variational Statement

◇ A variational formulation of BVP is applied piecewise over a domain divided into nodal subdivisions.

$$BVP \longrightarrow Variational \;\; formuation$$

◇ The term variational refers to its modern use which permits its use as equivalent weighted integral to the BVP, i.e.

$$\int_{\Omega} w\,(L(u) - f)\,d\Omega \;=\; 0$$

◇ The principle of solution itself may not necessarily be admissible as a variational principle.

For a problem with $f(\mathbf{x}) = 0$,

let $w = \sum_{i=1}^{N} \varphi_i w_i$ be the weighting function

and $u_h = \sum_{i=1}^{N} \phi_i u_i$ be the trial (approximation) solution,

Then a weak formulation is

$$\int_\Omega w \ L(u_h) d\Omega + \oint_\Gamma w \ B(u_h) ds \ = \ 0$$

When $\varphi_i = \phi_i$, the method is the Garlerkin method.

Gradient and Divergence Theorems

Let u be scalar function and $\mathbf{v}$ be a vector function defined on a 3D domain.

1. The Gradient Theorem

$$\int_\Omega \mathrm{grad}(u)\, d\Omega \;=\; \int_\Omega \nabla u \; dxdydz \;=\; \oint_\Gamma \hat{n}u \; ds$$

$$\int_\Omega \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z} \; dxdydz = \oint_\Gamma (\hat{i}n_x + \hat{j}n_y + \hat{k}n_z)u \; ds$$

2. Divergence Theorem

$$\int_\Omega \mathrm{div}(\mathbf{v})\, d\Omega \;=\; \int_\Omega \nabla \cdot \mathbf{v} \; dxdydz \;=\; \oint_\Gamma \hat{n} \cdot \mathbf{v} \; ds$$

$$\int_\Omega \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \; dxdydz = \oint_\Gamma (n_x v_x + n_y v_y + n_z v_z) \; ds$$

From the above theorems, we have

$$\int_\Omega (\nabla u)\mathbf{v}\ dxdydz = -\int_\Omega (\nabla \mathbf{v})u\ dxdydz + \oint_\Gamma \hat{n}u\mathbf{v}\ ds$$

$$\int_\Omega (\nabla^2 u)\mathbf{v}\ dxdydz = -\int_\Omega \nabla u \cdot \nabla \mathbf{v})\ dxdydz + \oint_\Gamma \frac{du}{dn}\mathbf{v}\ ds$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial z}$$

$$\nabla = n_x\frac{\partial}{\partial x} + n_y\frac{\partial}{\partial y} + n_z\frac{\partial}{\partial z}$$

Integration by Parts

If f and g are sufficiently differentiable 1D functions, then the following are applicable.

For a first order differential equation,

$$\int_{x_1}^{x_2} f\,\frac{dg}{dx}\,dx \;=\; -\int_{x_1}^{x_2} g\,\frac{df}{dx} + [fg]_{x_1}^{x_2}$$

For a second order differential equation,

$$\int_{x_1}^{x_2} f\,\frac{d^2g}{dx^2}\,dx \;=\; -\int_{x_1}^{x_2} \frac{dg}{dx}\,\frac{df}{dx} + \left[f\,\frac{dg}{dx}\right]_{x_1}^{x_2}$$

For a fourth order differential equation,

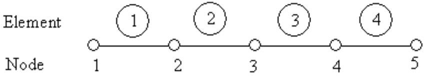
$$\int_{x_1}^{x_2} f\,\frac{d^4g}{dx^4}\,dx \;=\; -\int_{x_1}^{x_2} \frac{d^2g}{dx^2}\,\frac{d^2f}{dx^2}\,dx + \left|\frac{df}{dx}\,\frac{d^2g}{dx^2}\right|_{x_1}^{x_2} + \left[f\,\frac{d^3g}{dx^3}\right]_{x_1}^{x_2}$$

Finite Element Approximation

Consider solving

$$\begin{cases} -u_{xx} \;=\; 2 \quad x \in (a, b) \\ \\ u(a) = \hat{u}_1,\; \frac{du}{dx}(b) = -p_b[u(b) - u_\infty] \;=\; \sigma(b). \end{cases}$$

Discretization and Topology of finite element mesh

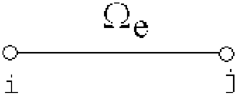


System Topology

Table: System Topology

element	Numbering scheme			
	Local		Global	
1	i	j	1	2
2	i	j	2	3
3	i	j	3	4
4	i	j	4	5

For an arbitrarily chosen element Ω_e, to approximate $u(x)$ over Ω_e by polynomial of degree k, we need to choose $(k+1)$ points within Ω_e.



Selection of Element Shape Functions

To standardize the calculation of element matrices, we firstly transform the element Ω_e into a standard element defined in $[-1, 1]$. This process is as follows:

Step 1. Introduce local coordinate

$$\xi \text{ with } \begin{cases} \text{origin } \xi = 0 & \text{at the centre of element} \\ \xi = -1 & \text{at the left hand node} \\ \xi = 1 & \text{at the right hand node} \end{cases}$$

This can be achieved by a linear transformation

$$\xi = \frac{2x - (x_i + x_{i+1})}{x_{i+1} - x_i}. \tag{8}$$

Selection of Element Shape Functions

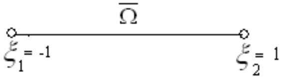
Step 2. For shape functions of degree k, we need to identify $(k+1)$ nodes

(including the end points).
Let ξ_i denote the ξ-coordinate of the ith node,
u_i^e denote the value of u_h^e at node i.
Then within Ω_e, $u(x)$ can be approximated by the Lagrange polynomial,

$$u_h^e(x) = \sum_{i=1}^{k+1} N_i u_i^e \quad \text{with } N_i = \prod_{\substack{j=1 \\ j \neq i}}^{k+1} \frac{(\xi - \xi_j)}{(\xi_i - \xi_j)}. \tag{9}$$

- $u_h^e(x)$ is the local approximation of $u_h(x)$ in Ω_e.
- $N_i(x)$ denote the local interpolating functions of the master element.

Selection of Element Shape Functions



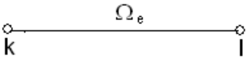
eg. For linear interpolation (two nodes in each element)

$$N_1(\xi) = \frac{(\xi-\xi_2)}{(\xi_1-\xi_2)} = \frac{\xi-1}{-1-1} = \frac{1}{2}(1-\xi)$$

$$N_2(\xi) = \frac{(\xi-\xi_1)}{(\xi_2-\xi_1)} = \frac{\xi+1}{1+1} = \frac{1}{2}(1+\xi)$$

Calculation of Element Contributions

Having selected an approximate set of shape functions, we now come to a crucial step in the analysis, i.e., the calculation of element matrices and vectors.



Consider $\Omega_e(x_k, x_l)$

$$k_{ij}^e = \int_{x_k}^{x_l} \phi_i' \phi_j' \, dx, \qquad f_i^e = \int_{x_k}^{x_l} 2\phi_i \, dx.$$

Using the following coordinate transformation

$$\xi = \frac{2x - (x_k + x_l)}{x_l - x_k}, \qquad d\xi = \frac{2}{x_l - x_k} dx = \frac{2}{h} dx$$

Selection of Element Shape Functions

we have

$$k_{ij}^e = \frac{h}{2}\int_{-1}^{1} N_i' N_j' \, d\xi, \quad f_i^e = \frac{h}{2}\int_{-1}^{1} 2N_i \, d\xi.$$

Note:

$$N_k = \tfrac{1}{2}(1-\xi), \;\; N_l = \tfrac{1}{2}(1+\xi)$$

$$N_k' = \frac{dN_k}{dx} = \frac{dN_k}{d\xi}\frac{d\xi}{dx} = -\frac{1}{2}\left(\frac{2}{h}\right) = -\frac{1}{h}$$

$$N_l' = \frac{dN_l}{dx} = \frac{dN_l}{d\xi}\frac{d\xi}{dx} = +\frac{1}{2}\left(\frac{2}{h}\right) = \frac{1}{h}$$

Selection of Element Shape Functions

Therefore, $k^e = \begin{bmatrix} k^e_{kk} & k^e_{kl} \\ k^e_{lk} & k^e_{ll} \end{bmatrix}$, $f^e = \begin{bmatrix} f^e_k \\ f^e_l \end{bmatrix}$ with

$$k^e_{kk} = \frac{h}{2}\int_{-1}^{1} N'_k N'_k \, d\xi = \frac{h}{2}\int_{-1}^{1} \frac{1}{h^2} \, d\xi = \frac{1}{h}$$

$$k^e_{lk} = k^e_{kl} = \frac{h}{2}\int_{-1}^{1} N'_k N'_l d\xi = \frac{h}{2}\int_{-1}^{1}(\frac{1}{-h})(\frac{1}{h}) \, d\xi = -\frac{1}{h}$$

$$k^e_{ll} = \frac{h}{2}\int_{-1}^{1} N'_l N'_l \, d\xi = \frac{1}{h}$$

$$f^e_k = h\int_{-1}^{1} \frac{1}{2}(1-\xi) \, d\xi = h$$

Selection of Element Shape Functions

i.e. for $e = 1, 2, 3, 4$

$$K^e = \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F^e = \begin{bmatrix} F_k^e \\ F_l^e \end{bmatrix} = h \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus, we can obtain all the element matrices.

For $\Omega_1(x_1, x_2)$,

$$K^1 = \begin{bmatrix} k^1_{11} & k^1_{12} \\ k^1_{21} & k^1_{22} \end{bmatrix} = K^e, \quad F^1 = \begin{bmatrix} F^1_1 \\ F^1_2 \end{bmatrix} = F^e.$$

For $\Omega_2(x_2, x_3)$,

$$K^2 = \begin{bmatrix} k^2_{22} & k^2_{23} \\ k^2_{32} & k^2_{33} \end{bmatrix} = K^e, \quad F^2 = \begin{bmatrix} F^2_2 \\ F^2_3 \end{bmatrix} = F^e.$$

Selection of Element Shape Functions

For $\Omega_3(x_3, x_4)$,

$$K^3 = \begin{bmatrix} k_{33}^3 & k_{34}^3 \\ k_{43}^3 & k_{44}^3 \end{bmatrix} = K^e, \quad F^3 = \begin{bmatrix} F_3^3 \\ F_4^3 \end{bmatrix} = F^e.$$

For $\Omega_4(x_4, x_5)$,

$$K^4 = \begin{bmatrix} k_{44}^4 & k_{45}^4 \\ k_{54}^4 & k_{55}^4 \end{bmatrix} = K^e, \quad F^4 = \begin{bmatrix} F_4^4 \\ F_5^4 \end{bmatrix} = F^e.$$

Construction of global matrices

To construct the global **K** and **F**

i) Expand each element quantity to N dimension, i.e.

- For Ω_1, $\quad K_1 = \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- For Ω_2, $\quad K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & K_{22}^2 & K_{23}^2 & 0 & 0 \\ 0 & K_{32}^2 & K_{33}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ F_2^2 \\ F_3^2 \\ 0 \\ 0 \end{bmatrix}$

Construction of global matrices

- For Ω_3, $\quad K_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{33}^3 & K_{34}^3 & 0 \\ 0 & 0 & K_{43}^3 & K_{44}^3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ 0 \\ F_3^3 \\ F_4^3 \\ 0 \end{bmatrix}$
-
-
- For Ω_4, $\quad K_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{44}^4 & K_{45}^4 \\ 0 & 0 & 0 & K_{54}^4 & K_{55}^4 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F_4^4 \\ F_5^4 \end{bmatrix}$

Construction of global matrices

(ii) Add the expanded element quantities to form the global matrices.

$$\mathbf{K} = \sum_{e=1}^{E} K^e$$

$$= \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{22}^2 & K_{23}^2 & 0 & 0 \\ 0 & K_{32}^2 & K_{33}^2 + K_{33}^3 & K_{34}^3 & 0 \\ 0 & 0 & K_{43}^3 & K_{44}^3 + K_{44}^4 & K_{45}^4 \\ 0 & 0 & 0 & K_{54}^4 & K_{55}^4 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} F_1^1 \\ F_2^1 + F_2^2 \\ F_3^2 + F_3^3 \\ F_4^3 + F_4^4 \\ F_5^4 \end{bmatrix}.$$

Boundary Conditions

Now the system of equations obtained so far is

$$\begin{bmatrix} K_{11} & K_{12} & & & \\ K_{21} & K_{22} & K_{23} & & \\ & K_{32} & K_{33} & K_{34} & \\ & & K_{43} & K_{44} & K_{45} \\ & & & K_{54} & K_{55} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 + \sigma(b)\phi(b) \end{bmatrix}. \quad (10)$$

Next, we need to impose the boundary conditions on the above system.

(i) Dirichlet boundary condition (also named essential boundary condition in the finite element method)

$$u(a) = u_1 = \hat{u}_1$$

- As u_1 is known, we move all known quantities $K_{i1}u_1$ in (10) to the right hand side.

Boundary Conditions

Thus

$$\begin{bmatrix} 0 & K_{12} & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} \\ 0 & 0 & 0 & K_{54} & K_{55} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} F_1 - K_{11}\hat{u}_1 \\ F_2 - K_{21}\hat{u}_1 \\ F_3 - K_{31}\hat{u}_1 \\ F_4 - K_{41}\hat{u}_1 \\ F_5 - K_{51}\hat{u}_1 + \sigma(b) \end{bmatrix}.$$

* In the variational statement, the test function $v(x)$ is required to satisfy $v(a) = 0$.
* However, the 1st equation of the system (10) is obtained by

$$(u_N, \phi_1) = (f, \phi_1) - \sigma\phi_1 \,\Big|_a^b = \; 0$$

As $\phi_1(a) = 1 \neq 0$, $\phi_1(x)$ is not from the class of admissible test functions, $\phi_1(x) \notin H^1_{oh}$.

Boundary Conditions

Finally, we can either delete the 1st equation to yield an 4×4 system or add equation $u_1 = \hat{u}_1$ into the system to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} \\ 0 & 0 & 0 & K_{54} & K_{55} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} \hat{u}_1 \\ F_2 - K_{21}\hat{u}_1 \\ F_3 - K_{31}\hat{u}_1 \\ F_4 - K_{41}\hat{u}_1 \\ F_5 - K_{51}\hat{u}_1 + \sigma(b) \end{bmatrix}. \quad (11)$$

Boundary Conditions

(ii) General natural boundary condition

$$k\frac{du(b)}{dx} = -p_b(u(b) - u_\infty) = \sigma(b).$$

The above natural boundary condition has been brought into the variational statement and consequently the 5th equation of (11) is

$$K_{54}u_4 + K_{55}u_5 = F_5 - K_{51}\hat{u}_1 - p_b u_5 + p_b u_\infty$$

$$\Rightarrow \quad K_{54}u_4 + (K_{55} + p_b)u_5 = F_5 - K_{51}\hat{u}_1 + p_b u_\infty$$

Boundary Conditions

Therefore system (11) becomes

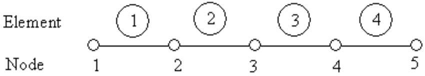
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} \\ 0 & 0 & 0 & K_{54} & (K_{55}+p_b) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} \hat{u}_1 \\ F_2 - K_{21}\hat{u}_1 \\ F_3 - K_{31}\hat{u}_1 \\ F_4 - K_{41}\hat{u}_1 \\ F_5 - K_{51}\hat{u}_1 + p_b u_\infty \end{bmatrix}.$$

which can then be solved to find u_2, u_3, u_4 and u_5.

Exercise

Using a standardised (master) linear element to construct a global system $\mathbf{Ku} = \mathbf{F} + \mathbf{F}_b$ of the BVP:

$$\begin{cases} -u_{xx} = \delta(x-2) \quad x \in (0,4) \\ u(0) = 2, \; u_x(4) = -2(u(4)-1). \end{cases}$$



Semi-discretization in space

Consider the solution of linear parabolic problems (diffusion problems) as follows:

$$
\begin{aligned}
& && u_t - \nabla.(k\nabla u) + bu = f \quad \text{in } \Omega \times \mathrm{I} \\
\text{subj.} \quad & \text{B.C.} && \tfrac{\partial u}{\partial n} + \alpha u = \gamma \quad \text{on } \partial\Omega \times \mathrm{I} \\
& \text{I.C.} && u(\mathbf{x}, 0) = \hat{u}(\mathbf{x}) \quad \text{in } \Omega \\
& \text{where I} && : [0, T]
\end{aligned}
\tag{12}
$$

Variational statement

Multiplying (12), for a given t , by $v \in H^1$, then integrating over Ω and using Green's theorem, we get

$$\int_\Omega u_t v \; d\Omega + \int_\Omega (k\nabla u \cdot \nabla v + buv)\; d\Omega + \int_{\partial\Omega} k\alpha uv \; ds = \int_\Omega fv \; d\Omega + \int_{\partial\Omega} k\gamma v \; ds. \tag{13}$$

Thus, we are led to the following variational problem:

Find $u = u(\mathbf{x}, t) \in H^1(\Omega)$ such that for every $t \in I$

$$(u_t, v) + a(u, v) = L(v) \quad \textit{for all } v \in H^1(\Omega) \tag{14}$$

$$u(\mathbf{x}, 0) = \hat{u}(\mathbf{x}) \tag{15}$$

where
$$\begin{aligned} (\cdot,\cdot) &= \text{inner product} \\ a(u,v) &= \int_\Omega (k\nabla u \cdot \nabla v + buv) d\Omega + \int_{\partial\Omega} k\alpha uv \; ds. \\ L(v) &= \int_\Omega fv \; d\Omega + \int_{\partial\Omega} k\gamma v \; ds. \end{aligned}$$

Finite Element Approximation

Let H_h^1 be a finite dimensional subspace of H^1 with basis functions $\{\phi_1, \phi_2, ...\phi_n\}$. Then, the variational problem is approximated by :
Find $u_h(\mathbf{x}, t) \in H_h^1$ such that $u_h(\mathbf{x}, 0) = \hat{u}(\mathbf{x})$ and

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h) = L(v_h) \qquad \forall\ v_h \in H_h^1. \tag{16}$$

In the usual way, we introduce a discretization of Ω as a union of elements Ω_e, i.e. $\Omega \to \bigcup_{e=1}^{E} \Omega_e$ and approximate $u(\mathbf{x}, t)$ at t by.

$$u_h(\mathbf{x}, t) = \sum_{j=1}^{n} u_j\ (t)\varphi_j\ (\mathbf{x}) \tag{17}$$

Finite Element Approximation

From (16) and (17), by using the usual finite element formulation, we obtain

$$\begin{aligned} \mathbf{M}\dot{\mathbf{u}} + \mathbf{A}\mathbf{u} &= \mathbf{F} \\ \mathbf{u}(0) &= \hat{\mathbf{u}} \end{aligned} \tag{18}$$

$$\begin{aligned}
\text{where}\quad \mathbf{M} = (m_{ij}) \quad &\text{with} \quad m_{ij} = (\varphi_i, \varphi_j) = \textstyle\sum_{e=1}^{E} \int_{\Omega_e} \varphi_i \varphi_j d\Omega \\
\mathbf{A} = (a_{ij}) \quad &\text{with} \quad a_{ij} = a(\varphi_i, \varphi_j) \\
&\qquad = \textstyle\sum_{e=1}^{E} \int_{\Omega_e} \left(k \nabla \varphi_i \cdot \nabla \varphi_j + b \varphi_i \varphi_j \right) d\Omega \\
&\qquad + \textstyle\sum_{e=1}^{\partial E} \int_{\partial \Omega_e} k \alpha \varphi_i \varphi_j \, ds \\
\mathbf{F} = (f_i) \quad &\text{with} \quad f_i = L(\varphi_i)
\end{aligned}$$

2D Elliptic Boundary Value Problems

Example 1.

$$
\begin{aligned}
-\nabla \cdot [k\nabla u] + bu &= f(x,y) && x,y \in \Omega, \\
u(s) = \hat{u}(s) & && s \in \partial\Omega_1, \\
-k(s)\frac{\partial u(s)}{\partial n} = p(s)[u(s) - \hat{u}(s)] &= \hat{\sigma}(s) && s \in \partial\Omega_2,
\end{aligned}
\tag{19}
$$

where ∇ is the gradient operator, $\nabla\cdot$ is the divergence operator and $\triangle = \nabla^2$ is the Laplace operator.

Variational Statement

The residual function

$$r = -\nabla \cdot [k\nabla u] + bu - f.$$

The overall weighted residual

$$\int_\Omega [-\nabla \cdot (k\nabla u) + bu - f]v \, d\Omega \; = \; 0. \tag{20}$$

Then, using the product rule for differentiation

$$\begin{aligned} \nabla \cdot (vk\nabla u) &= k\nabla u \cdot \nabla v + v\nabla \cdot (k\nabla u) \\ \Rightarrow \quad v\nabla \cdot (k\nabla u) &= \nabla \cdot (vk\nabla u) - k\nabla u \cdot \nabla v, \end{aligned} \tag{21}$$

we have from (20)

$$\int_{\Omega} [k\nabla u \cdot \nabla v - \nabla \cdot (vk\nabla u) + buv - fv] d\Omega = 0. \tag{22}$$

From the divergence theorem

$$\int_{\Omega} \nabla \cdot (vk\nabla u)\, d\Omega \;=\; \int_{\partial\Omega} vk\nabla u \cdot n\, ds \;=\; \int_{\partial\Omega} vk\frac{\partial u}{\partial n}\, ds, \tag{23}$$

equation (22) becomes

$$\int_{\Omega} [k\nabla u \cdot \nabla v + buv - fv]\, d\Omega - \int_{\partial\Omega} k\frac{\partial u}{\partial n} v\, ds \;=\; 0. \tag{24}$$

Choosing v such that $v = 0$ on $\partial\Omega_1$ and using the boundary condition $(19)_3$, we obtain

$$\int_\Omega [k\nabla u \cdot \nabla v + buv - fv]\, d\Omega + \int_{\partial\Omega_2} puv\, ds - \int_{\partial\Omega_2} p\hat{u}v\, ds \;=\; 0. \quad (25)$$

To specify the appropriate class of admissible functions for problem (25),

- we examine the integrals in (25) and observe that the area integrals are well defined whenever u and v and their 1st order partial derivatives are smooth enough to be square-integrable over Ω.
- Thus, we need to choose u and v from $H^1(\Omega)$.

Variational statement is:

Find $u \in H^1(\Omega)$ such that $u = \hat{u}$ on $\partial\Omega_1$ and

$$a(u, v) = L(v) \quad \forall v \in H^1(\Omega), \tag{26}$$

where $H_0^1 = \{v : v \in H^1 \text{ and } v = 0 \text{ on } \partial\Omega_1\}$,

$a(u, v) = \int_\Omega (k\nabla u \cdot \nabla v + buv)\, d\Omega + \int_{\partial\Omega_2} puv\, ds$
is a bilinear form of u and v,

$L(v) = \int_{\partial\Omega_2} p\hat{u}v\, ds + \int_\Omega fv\, d\Omega$ is a linear form of v.

The Galerkin Approximation

A Galerkin approximation of (26) is obtained by posing the variational problem on a finite-dimensional subspace H^h of the space of admissible functions. Specifically, we

$$\begin{aligned} &\text{seek} \quad u_h \in H_h^1 \quad \text{such that} \quad u_h = \hat{u} \text{ on } \partial\Omega_1 \quad \text{and} \\ &a(u_h, v_h) = L(v_h) \qquad\qquad \forall v_h \in H_{0h}^1 \end{aligned} \tag{27}$$

Let $\{\phi_i(x, y)\}_{i=1}^N$ be the basis functions of H_h^1, then

$$u_h = \sum_{j=1}^N \alpha_j \phi_j(x, y), \qquad v_h = \sum_{i=1}^N \beta_i \phi_i(x, y). \tag{28}$$

Substituting (28) into (27) yields

$$\begin{aligned} &\textstyle\sum_{i=1}^{N} a(u_h, \phi_i)\beta_i = \sum_{i=1}^{N} L(\phi_i)\beta_i \qquad \forall \beta_i \\ \Rightarrow \quad & a(u_h, \phi_i) = L(\phi_i), \quad (i = 1, 2, \ldots, N) \end{aligned} \tag{29}$$

Substituting (28) into (29) yields

$$\begin{aligned} &\textstyle\sum_{i=1}^{N} a(\phi_i, \phi_j)\alpha_j = L(\phi_i), \quad (i = 1, 2, \ldots, N) \\ \Rightarrow \quad & \mathbf{A}\alpha = \mathbf{F}, \end{aligned} \tag{30}$$

where $\mathbf{A} = (a_{ij})$ is an $N \times N$ matrix with $a_{ij} = a(\phi_i, \phi_j)$,
$\mathbf{F} = (F_i) \in \mathcal{R}^N$ with $F_i = L(\phi_i)$ and
$\alpha = (\alpha_i) \in \mathcal{R}^N$.

Therefore, the Galerkin approximation u_h of the solution u is of the form

$$u_h = \sum_{j=1}^{N} \alpha_j \phi_j(x, y), \tag{31}$$

where $\alpha \in \mathcal{R}^n$ is determined by (30) and
$\{\phi_j(x, y)\}_{j=1}^{N}$ are basis functions of H_h^1.
The finite element method provides a general and systematic technique for constructing the basis functions ϕ_i.

The Finite element Interpolation

Consider an open bounded domain Ω in $\mathcal{R}^N$ with boundary $\partial\Omega$.

Let $u \in C^m(\bar{\Omega})$ where $\bar{\Omega}$ is the closure of Ω, then the construction of a finite element interpolation of $u(\phi_i)$ can be accomplished by the following steps.

1) Partitioning of $\bar{\Omega}$

We replace $\bar{\Omega}$ by a collection $\bar{\Omega}_h$ of simple domain (element) $\bar{\Omega}_e$ such that

a) $\bar{\Omega}_h = \cup_{e=1}^{E} \bar{\Omega}_e$

b) $\bar{\Omega}_e \cap \bar{\Omega}_f = \phi$ for distinct $\bar{\Omega}_e$ and $\bar{\Omega}_f \in \bar{\Omega}_h$

c) every $\bar{\Omega}_e$ is closed and consists of a non-empty interior Ω_e and a boundary $\partial\Omega_e$.

2) Local Interpolation Over $\bar{\Omega}_e$-Local Basis ϕ_i^e

Over each $\bar{\Omega}_e$,

- we choose N_e nodes where the values of u and u_i^e are to be used as basic unknowns.
- Then we construct local interpolation function $\{\phi_i^e(x,y)\}_{i=1}^{N_e}$ such that the restriction of u_h to $\bar{\Omega}_e$ is

$$u_h^e(x,y) = \sum_{i=1}^{N_e} u_i^e \phi_i^e(x,y).$$

The form of $\phi_i^e(x,y)$ depends on type of elements.

3) Assembly of Global Basis Functions ϕ_i

The global basis functions ϕ_i can be generated by patching together those local shape functions ϕ_i^e defined over $\bar{\Omega}_e$ which contain the node i.

Suppose node i in the finite element mesh is shared by M elements. Then the local shape functions for point i corresponding to each of these elements are combined to form the global ϕ_i which satisfies

- the proper inter-element continuity
- $\phi_i(x_j, y_j) = \delta_{ij}$
- $\phi_i(x, y)$ is non-zero only over the particular patch of the M elements meeting at node i.

Thus, we can generate N linearly independent functions $\{\phi_i(x, y)\}_{i=1}^{N}$ which form a basis of an $N-\mathtt{dimension}$ function space.

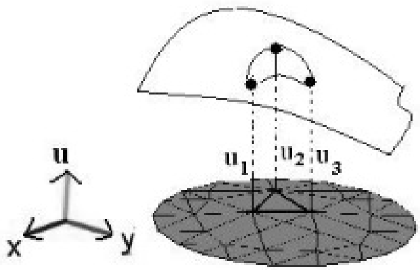
Triangular Elements

Linear 3-point triangular elements

Approximate $u(x)$ over the element Ω_e by

$$u_h^e(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y, \quad \forall (x, y) \in \Omega_e \tag{32}$$

which determines a plane surface. Thus the use of linear interpolation on a triangular element will result in the approximation of a given smooth surface $v(x, y)$ by a plane as shown.



By evaluating (32) at each node, we have

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

or $\qquad u^e = p(x_i)$

$\Rightarrow \qquad \alpha = p^{-1}(x_i)u^e$

Therefore $\qquad u_h^e(x,y) = [1,\ x,\ y]p^{-1}(x_i)u^e$ which can be rearranged to yield

$$u_h^e(x,y) = u_1\phi_1^e + u_2\phi_2^e + u_3\phi_3^e \tag{33}$$

with element shape functions being

$$\begin{cases} \phi_1^e(x,y) = \frac{1}{2A_e}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \\ \phi_2^e(x,y) = \frac{1}{2A_e}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \\ \phi_3^e(x,y) = \frac{1}{2A_e}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \end{cases} \tag{34}$$

and $A_e = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$ is area of an element.

(2) Higher Order Triangular Element

Let us first display the terms appearing in polynomials of various degrees in two variables in the form as shown in Figure 3.

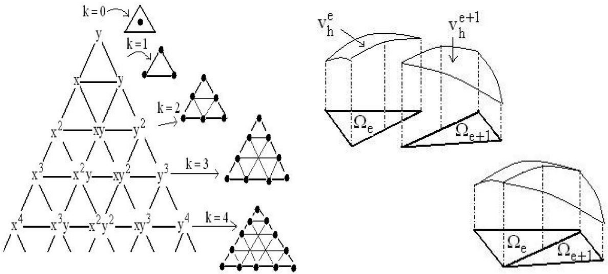


Figure: Pascal's triangle

The above triangular array is called Pascal's triangle.

Rectangular Elements

By taking the product of a set of polynomials in x with a set of polynomials in y, shape functions for a variety of rectangular elements can be obtained.

(1) Bilinear polynomials The product of $(1, x)$ and $(1, y)$ produces a matrix

$$\begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 & y \end{bmatrix} = \begin{bmatrix} 1 & y \\ x & xy \end{bmatrix}. \tag{35}$$

A bilinear local interpolant can then be obtained by forming a linear combination of all the four terms in the matrix, i.e

$$v_h^e(x, y) = a_1 + a_2 x + a_2 y + a_4 xy. \tag{36}$$

(2) Higher Order Rectangular Elements

By considering tensor products of polynomials of higher degree, element shape functions can be constructed which contain polynomials of any desired degree and which lead to basis functions that are continuous throughout Ω_h.

eg. For a biquadratic local interpolation, we firstly find the matrix from

$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} 1 & y & y^2 \end{bmatrix} = \begin{bmatrix} 1 & y & y^2 \\ x & xy & xy^2 \\ x^2 & x^2y & x^2y \end{bmatrix}$$

Then the biquadratic local interpolant v_h^e is obtained by forming a linear combination of all the nine terms in the matrix.

To completely determine the interpolant, construct a rectangular element with nine nodes as shown.

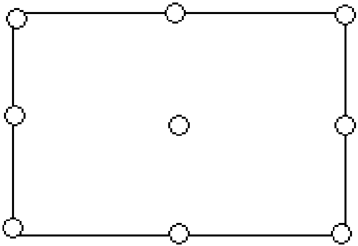


Figure: A rectangular element with nine nodes

Finite Element Approximation

Return to the problem, as we choose $\phi_i(x_j, y_j) = \delta_{ij}$, our finite element approximation of u is

$$u_h(x, y) = \sum_{j=1}^{N} u_j \phi_j(x, y).$$

Thus, our problem now is:

Find $\mathbf{u} \in \mathcal{R}^N$ such that $u_i = \hat{u}$ on $\partial\Omega_1$ and

$$\mathbf{Au} \; = \; \mathbf{F}$$

where
$\mathbf{A} = (a_{ij})$ with $a_{ij} = a(\phi_i, \phi_j) = \int_\Omega (k\nabla\phi_i \cdot \nabla\phi_j + b\phi_i\phi_j) + \int_{\partial\Omega_2} p\phi_i\phi_j \; ds$
$\mathbf{F} = (F_i)$ with $F_i = L(\phi_i) = \int_\Omega f\phi_i \; d\Omega + \int_{\partial\Omega_2} p\hat{u}\phi_i \; ds.$

As $\phi_i(x, y)$ are defined piecewisely over each element Ω_e, we have

$$a_{ij} = \sum_{e=1}^{E} \int_{\Omega_e} (k\nabla\phi_i \cdot \nabla\phi_j + b\phi_i\phi_j)\, d\Omega + \sum_{e=1}^{E} \int_{\partial\Omega_{2e}} p\phi_i\phi_j\, ds$$

$$F_i = \sum_{e=1}^{E} \left\{ \int_{\Omega_e} f\phi_i d\Omega + \int_{\partial\Omega_{2e}} p\hat{u}\phi_i\, ds \right\}.$$

To assemble $\mathbf{A}$, loop over all elements to calculate a^e and successively add in the contributions from each a^e as follows :

Set $\mathbf{A}(i, j) = 0, \quad b(i) = 0, \quad i, j = 1, 2, ...N$

For $e = 1, 2...E$

calculate a^e

Set $\quad \mathbf{A}_{g(e,\alpha)g(e,\beta)} = \mathbf{A}_{g(e,\alpha)g(e,\beta)} + a^e_{\alpha\beta}$

$$\mathbf{F}_{g(e,\alpha)} = \mathbf{F}_{g(e,\alpha)} + F^e_\alpha \quad \alpha, \beta = 1, 2, ..., N_e.$$

where $g(e, k)$ is the global node number of the k^{th} node of element e.

Boundary Condition

Suppose at point $\ell, u_\ell = \hat{u}$ and the assembled system is

$$\begin{bmatrix} a_{11} & \cdots & a_{1\ell} & \cdots & a_{1N} \\ \vdots & & \vdots & & \vdots \\ a_{\ell 1} & \cdots & a_{\ell\ell} & \cdots & a_{\ell N} \\ \vdots & & \vdots & & \vdots \\ a_{N1} & \cdots & a_{N\ell} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_\ell \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_\ell \\ \vdots \\ f_N \end{bmatrix}$$

We impose the boundary condition $u_\ell = \hat{u}$ by performing the following steps:

1) Move the known values to the right hand side

$$\left[\begin{array}{cc|c|cc} a_1 1 & \cdots & 0 & \cdots & a_{1N} \\ \vdots & & \vdots & & \vdots \\ a_{\ell 1} & \cdots & 0 & \cdots & a_{\ell N} \\ \vdots & & \vdots & & \vdots \\ a_{N1} & \cdots & 0 & \cdots & a_{NN} \end{array}\right] \begin{bmatrix} u_1 \\ \vdots \\ u_\ell \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 & - & a_{1\ell}\hat{u} \\ & & \vdots \\ f_\ell & - & a_{\ell\ell}\hat{u} \\ & & \vdots \\ f_N & - & a_{N\ell}\hat{u} \end{bmatrix}$$

2) Impose the restriction $\phi_\ell = 0$ on the system.
Noting that $a_{\ell j} = a(\phi_\ell, \phi_j) = 0,\ f_\ell = L(\phi_\ell) = 0$, we have

$$\left[\begin{array}{ccccc} a_11 & \cdots & 0 & \cdots & a_{1N} \\ & & \vdots & & \\ \hline 0 & \cdots & \vdots & \cdots & 0 \\ \hline & & \vdots & & \\ a_{N1} & & 0 & & a_{NN} \end{array}\right] \begin{bmatrix} u_1 \\ \vdots \\ u_\ell \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 & - & a_{1\ell}\hat{u} \\ f_2 & - & a_{2\ell}\hat{u} \\ & & \vdots \\ & & 0 \\ & & \vdots \\ f_N & - & a_{N\ell}\hat{u} \end{bmatrix}$$

This set of equations is rank deficient and need to be modified by one of the following methods.

- Combine with the Dirichlet Condition $u_\ell = \hat{u}_\ell$ to yield

$$\left[\begin{array}{cc|c|cc} a_{11} & \cdots & 0 & \cdots & a_{1N} \\ \vdots & & \vdots & & \\ \hline 0 & \vdots & 1 & \vdots & 0 \\ \hline \vdots & & \vdots & & \\ a_{N1} & \cdots & 0 & \cdots & a_{NN} \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_\ell \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 & - & a_{1\ell}\hat{u} \\ & & \vdots \\ & & \hat{u} \\ & & \vdots \\ f_N & - & a_{N\ell}\hat{u} \end{bmatrix}$$

- Delete row ℓ and column ℓ to form an $(N-1) \times (N-1)$ system.

Example 2

Consider

$$\begin{cases} -\Delta(x,y) = f(x,y) & in \;\; \Omega \\ u = 0 & on \;\; \Gamma_{41} \\ \frac{\partial u}{\partial n} = 0 & on \;\; \Gamma_{12}, \Gamma_{25}, \Gamma_{67}, and \Gamma_{74} \\ \frac{\partial u}{\partial n} + \beta u = \gamma & on \;\; \Gamma_{56} \end{cases}$$

In this case $\partial\Omega_1 = \Gamma_{41}$

$$\partial\Omega_2 = \Gamma_{12} \cup \Gamma_{25} \cup \Gamma_{67} \cup \Gamma_{74} \cup \Gamma_{56}$$

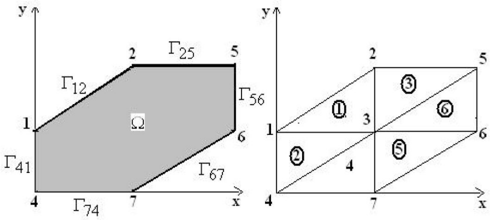


Figure:

Our analysis of this problem proceeds as follows:

- Partition Ω into six triangular elements.
- Compute the element matrices a^e and f^e $(e = 1, 2, ..., 6)$

$$a^e = \begin{bmatrix} a^e_{11} & a^e_{12} & a^e_{13} \\ a^e_{21} & a^e_{22} & a^e_{23} \\ a^e_{31} & a^e_{32} & a^e_{33} \end{bmatrix}, f^e = \begin{bmatrix} f^e_1 \\ f^e_2 \\ f^e_3 \end{bmatrix}$$

- Assemble the element matrices to form the global matrix using the following opology:

ele	node 1	2	3
1	1	2	3
2	1	3	4
3	2	5	3
4	3	4	7
5	3	6	7
6	3	5	6

Hence, we have global matrix and vector

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & K_{25} & 0 & 0 \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} \\ K_{41} & 0 & K_{43} & K_{44} & 0 & 0 & K_{47} \\ 0 & K_{52} & K_{53} & 0 & K_{55}+K_b & K_{56} & 0 \\ 0 & 0 & K_{63} & 0 & K_{65} & K_{66}+K_b & K_{67} \\ 0 & 0 & K_{73} & K_{74} & 0 & K_{76} & K_{77} \end{bmatrix}, \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5+F_b \\ F_6+F_b \\ F_7 \end{bmatrix}$$

- Impose the essential boundary condition.

Element Transformation

A transformation

$$T_e \ : \quad \begin{cases} x = x(\xi,\eta) = \sum_i x_i\psi_i(\xi,\eta) \\ y = y(\xi,\eta) = \sum_i y_i\psi_i(\xi,\eta) \end{cases}$$

maps an arbitrarily chosen element e into a standard (master) element $\bar{\Omega}$. For a linear master element, shape function at node i is

$$\psi_i(\xi,\eta) = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i).$$

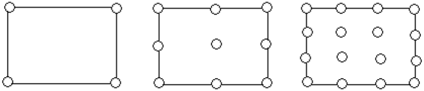


Figure: Square elements with 4 nodes (linear element), 9 nodes (quadratic element) and 16 nodes (cubic element)

Element Calculation

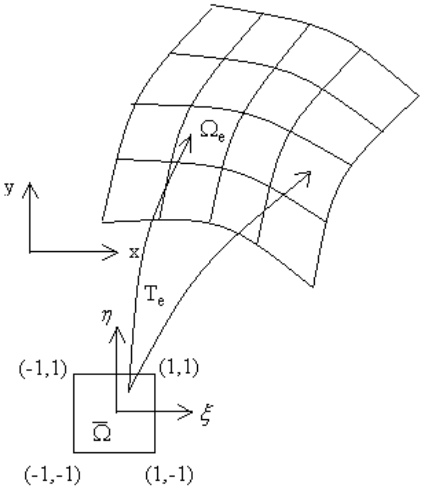


Figure: Element transformation T_e

Element Calculation

• A point $P(\xi = \alpha, \eta = \beta)$ in the standard element $\bar{\Omega}$ is mapped into a point

$$P[x(\alpha, \beta), y(\alpha, \beta)]$$

in local element Ω_e.

• A line $(\xi = \alpha)$ in $\bar{\Omega}$ is mapped into a curve

$$[x = x(\alpha, \eta), y = y(\alpha, \eta)]$$

in the plane, which is called the curvilinear coordinate line $(\xi = \alpha)$.

• A FE mesh can be viewed as a sequence of transformation $\{T_1,\ T_2, \ldots T_E\}$ of the fixed master element.

Relations between dx, dy with $d\xi$ and $d\eta$

Suppose $x(\xi,\eta)$ and $y(\xi,\eta)$ are continuously differentiable, then

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \text{ and}$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$

$$\text{or} \quad \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = J \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}, \tag{37}$$

where $J =$ Jacobian matrix of the transformation.

If at point (ξ, η) we have $|J| = det(J) \neq 0$
then an inverse map $T_e^{-1}(x, y \to \xi, \eta)$ exists at this point and thus

$$\begin{bmatrix} d\xi \\ \\ d\eta \end{bmatrix} = J^{-1} \begin{bmatrix} dx \\ \\ dy \end{bmatrix} \tag{38}$$

and

$$T_e^{-1} \quad : \quad \begin{array}{c} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{array} \tag{39}$$

defines a map $(x, y) \to (\xi, \eta)$.

As in (37), we have

$$\begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \tag{40}$$

Hence, by equating terms in (40) and (38), we have the following relations

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta}, \quad \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi}, \quad \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi} \tag{41}$$

Any function $g = g(\xi, \eta)$ in $\bar{\Omega}$ can be approximated by

$$\bar{g}(\xi, \eta) = \sum g_j \psi_j(\xi, \eta), \tag{42}$$

where ψ_j $(j = 1, .., N)$ are the shape functions defined on $\bar{\Omega}$ and N is the total number of nodes in $\bar{\Omega}$.
T_e is invertible when $det(J) \neq 0$.

$$dxdy = |J| \, d\xi d\eta.$$

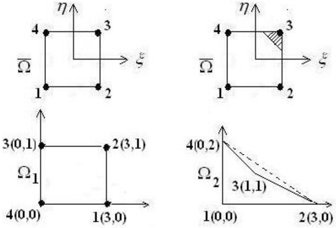
The mapping is acceptable if $|J| > 0$ throughout $\bar{\Omega}$.

Example

The following figure shows a 4-node master element $\bar{\Omega}$ and 2 elements Ω_1 and Ω_2 generated from it using the map (**??**). The shape function defined on $\bar{\Omega}$ are

$$\psi_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i), \quad (i = 1, ..., 4)$$

where (ξ_i, η_i) are coordinates of node i.



Element Calculation

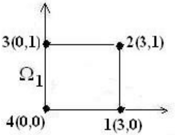
In this example, straight lines ξ = constant or η = constant in $\bar{\Omega}$ map to corresponding straight lines in Ω_e .

<u>For Ω_1</u>

$$T_e \ : \begin{array}{rcl} x & = & 3\psi_1 + 3\psi_2 + (0)\psi_3 + (0)\psi_4 = \frac{3}{2}(1-\eta) \\ y & = & \psi_2 + \psi_3 = \frac{1}{2}(1+\xi). \end{array}$$

$$|J| \ = det \begin{bmatrix} 0 & -\frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = \frac{3}{4} > 0$$

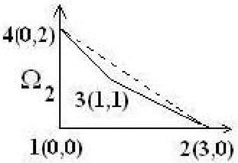
Therefore, the map is invertible.



Element Calculation

For Ω_2

$$|J| = \frac{1}{8}(5-3\xi-4\eta) \begin{cases} = 0 \text{ along } L : \xi = \frac{5}{3} - \frac{4}{3}\eta \\ > 0 \text{ below } L \\ < 0 \text{ above } L. \end{cases}$$



The region above L is mapped outside of Ω_2 by T_2. Clearly, Ω_2 is unacceptable.

Finite Element Calculation

For a chosen Ω, we need to

- identify M nodes and shape function φ to define the coordinates map T_e,
- identify N nodes and shape function $\bar{\varphi}$ for local approximation of the unknown function.

Remarks: M and N need not to be the same.

- If $M > N_e$, then it is super-parametric map.
- If $M = N_e$, then it is iso-parametric map (iso-parametric element).
- If $M < N_e$, then it is sub-parametric map.

Element Calculation

In the following, we will consider only the iso-parametric element.
Having selected $\bar{\Omega}$ and φ_j, we perform the following steps:
(1) Element map

$$T_e \;:\; \begin{array}{l} x = \sum_{j=1}^{N} x_j \psi_j(\xi, \eta) \\ \\ y = \sum_{j=1}^{N} y_j \psi_j(\xi, \eta) \end{array} \tag{43}$$

Transformation of shape functions
As T_e is invertible, $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ and the element shape functions are

$$\phi_j(x, y) = \psi_j[\xi(x, y), \eta(x, y)] \tag{44}$$

Therefore,

$$\frac{\partial \phi_j}{\partial x} = \frac{\partial \psi_j}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi_j}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial \phi_j}{\partial y} = \frac{\partial \psi_j}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi_j}{\partial \eta}\frac{\partial \eta}{\partial y}.$$

Transformation of shape functions

According to (43)

$$\frac{\partial x}{\partial \xi} = \sum_{1}^{N_e} x_k \frac{\partial \psi_k}{\partial \xi}(\xi, \eta), \quad \frac{\partial x}{\partial \eta} = \sum_{1}^{N_e} x_k \frac{\partial \psi_k}{\partial \eta}(\xi, \eta),$$

$$\frac{\partial y}{\partial \xi} = \sum_{1}^{N_e} y_k \frac{\partial \psi_k}{\partial \xi}(\xi, \eta), \quad \frac{\partial y}{\partial \eta} = \sum_{1}^{N_e} y_k \frac{\partial \psi_k}{\partial \eta}(\xi, \eta),$$

Thus, using (41) and (43), equation (44) becomes

$$\frac{\partial \phi_j}{\partial x} = \frac{1}{|J|}\left\{ \frac{\partial \psi_j}{\partial \xi} \sum_{k=1}^{N} y_k \frac{\partial \psi_k}{\partial \eta}(\xi, \eta) - \frac{\partial \psi_j}{\partial \eta} \sum_{k=1}^{N} y_k \frac{\partial \psi_k}{\partial \xi}(\xi, \eta) \right\}$$

$$\frac{\partial \phi_j}{\partial y} = \frac{1}{|J|}\left\{ \frac{\partial \psi_j}{\partial \xi} \sum_{k=1}^{N} x_k \frac{\partial \psi_k}{\partial \eta}(\xi, \eta) - \frac{\partial \psi_j}{\partial \eta} \sum_{k=1}^{N} x_k \frac{\partial \psi_k}{\partial \xi}(\xi, \eta) \right\}$$

Remarks

(a) The partial derivatives of ϕ_j w.r.t. x and y are completely determined by calculation defined only on $\bar{\Omega}$.

(b) From (**??**), for 4-node element, K^e is a 4*4 matrix which can be expressed as

$$K^e \;=\; \int_{\Omega_e} \left(k(D\phi)^T\,(D\phi) + b\phi^T\phi\right)\,d\Omega \tag{45}$$

where $\phi \;=\; (\phi_1, \phi_2, \phi_3, \phi_4)$ and

$$D\phi = \begin{bmatrix} \frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_3}{\partial x} & \frac{\partial\phi_4}{\partial x} \\ \\ \frac{\partial\phi_1}{\partial y} & \frac{\partial\phi_2}{\partial y} & \frac{\partial\phi_3}{\partial y} & \frac{\partial\phi_4}{\partial y} \end{bmatrix}.$$

Integration

Let $I = \int_{\Omega_e} g(x,y)\ dxdy$
then $I = \int_{\bar{\Omega}}\ G(\xi,\eta)\ d\xi d\eta$,
where

$$G(\xi,\eta) = g(\sum_{1}^{N} x_j\psi_j(\xi,\eta), \sum_{1}^{N} y_j\psi_j(\xi,\eta))|J(\xi,\eta)| \tag{46}$$

Numerical quadrature (such as the Gaussian quadrature) are usually used to evaluate the integrals. Quadrature rules for quadrilateral elements are usually derived from the 1-D quadrature by treating the integration over $\bar{\Omega}$ as a double integral.
Thus, using the 1-D quadrature rule of order N,

$$I = \int_{\bar{\Omega}} G(\xi,\eta)d\xi d\eta = \int_{-1}^{1}\left[\int_{-1}^{1} G(\xi,\eta)d\xi\right]d\eta \approx \sum_{k=1}^{N}\left[\sum_{\ell=1}^{N} G(\xi_\ell,\eta_k)w_\ell\right]w_k$$

Integration

For 9-point Gaussian quadrature (1-D of order 3).

$$N = 3,\ w_1 = 5/9,\ w_2 = 8/9,\ w_3 = 5/9,$$

$$\xi_1 = \eta_1 = -\sqrt{3/5},\ \xi_2 = \eta_2 = 0,\ \xi_3 = \eta_3 = \sqrt{3/5}.$$

If $k = k(x, y)$, $b = b(x, y)$ and $f = f(x, y)$ are not constant over an element, we may use

$$k(x, y) \approx \sum_{j=1}^{N} k_j \phi_j(x, y), \quad b(x, y) \approx \sum_{j=1}^{N} b_j \phi_j(x, y), \quad f(x, y) \approx \sum_{j=1}^{N} f_j \phi_j(x, y).$$

Then the calculations of a_{ij}^e and f_i^e only require the nodal values of k, b and f.

Boundary Integrals

Suppose that the sides $\xi = 1$ of a master square is to be mapped onto $\partial\Omega_{2h}$, Let θ_j denote the restriction of the master-element shape function ψ_j to side $\xi = 1$, i.e,

$$\theta_j(\eta) = \psi_j(1, \eta), \quad j = 1, 2, ..., N.$$

We thus have

$$\int_{\partial\Omega_{2e}} p\phi_i\phi_j \, ds = \int_{-1}^{1} p\theta_i(\eta)\theta_j(\eta)|J| \, d\eta$$

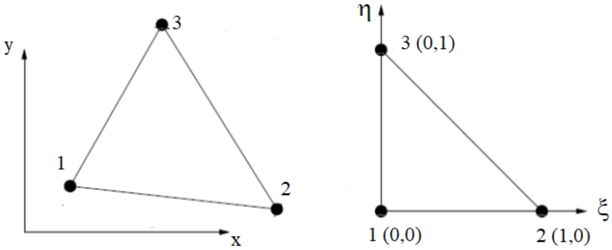
Since
$$ds = \sqrt{\left(\tfrac{\partial x}{\partial \eta}(1, \eta)\right)^2 + \left(\tfrac{\partial y}{\partial \eta}(1, \eta)\right)^2} d\eta,$$

we have

$$|J(\eta)| = \sqrt{\left(\frac{\partial x}{\partial \eta}(1, \eta)\right)^2 + \left(\frac{\partial y}{\partial \eta}(1, \eta)\right)^2}$$

where $x(\xi, \eta)$ and $y(\xi, \eta)$ are defined in (43).

Triangular element Transformation



For a linear master element, transformation

$$T_e : \begin{cases} x = (1 - \xi - \eta)x_1 + \xi x_2 + \eta x_3, \\ y = (1 - \xi - \eta)y_1 + \xi y_2 + \eta y_3 \end{cases}$$

Triangular element Transformation

$$\xi = \frac{det\begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \end{bmatrix}}{det\begin{bmatrix} 1 & x_2 & y_2 \\ 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \end{bmatrix}}, \quad \eta = \frac{det\begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}}{det\begin{bmatrix} 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}}.$$

T_e maps

$$(x_1, y_1) \rightarrow (0, 0),$$

$$(x_2, y_2) \rightarrow (1, 0),$$

$$(x_3, y_13) \rightarrow (0, 1).$$

Final Examination

There are five questions in Final Examination which is an open-book four-hour examination (100 marks).

Question 1. (15 marks) Multiple choice

Question 2. (15 marks) True/False

Question 3. (15 marks) Matching

Question 4. (20 marks) 1D FEM

Question 5. (35 marks) 2D FEM