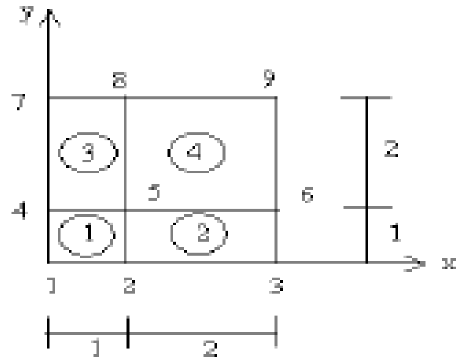
# MATH5004 TUT10

# Example:

$$\begin{cases} \nabla \cdot k(x)\nabla T + Q = 0 & \text{on } \Omega \\ T = x & \text{on } y = 0 \\ T = 3 + x^2 & \text{on } y = 3 \\ T = y & \text{on } x = 0 \\ \frac{\partial T}{\partial x} = 1 - 0.2T & \text{on } x = 3 \end{cases}$$




(a) Find Variational statement
(b) Derive the finite element equations for k=1, Q=2
(c) Find nodal solution

# (a) Variational Statement

- Determine total residual error:

$$R = \int_{\Omega} (\ \nabla \cdot k(x) \nabla T + Q\ )\ v \ \ dxdy \tag{1}$$

$$\nabla \cdot [(k(x)\nabla T)v] = (\ \nabla \cdot k(x)\nabla T\ )v + k(x)\nabla T\ \nabla v$$

$$(\ \nabla \cdot k(x)\nabla T\ )v = \nabla \cdot [(k(x)\nabla T)v] - k(x)\nabla T\ \nabla v$$

$$\therefore R = \int_{\Omega} \nabla \cdot [(k(x)\nabla T)v] - k(x)\nabla T\ \nabla v\ dxdy \int_{\Omega} Qv \ \ dxdy$$

- Set total residual error to zero:

$$\boxed{\int_{\Omega} \nabla \cdot [(k(x)\nabla T)v] - k(x)\nabla T\ \nabla v\ dxdy \int_{\Omega} Qv \ \ dxdy = 0 \qquad (2)}$$

- Apply Green's theorem to the first term on the LHS of (2)

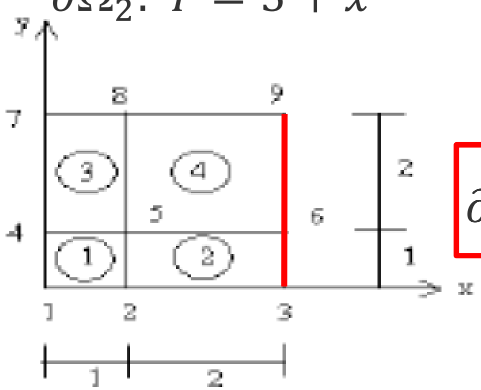
$$\int_{\Omega} \nabla \cdot [(k\nabla T)v]dxdy = \int_{\partial\Omega} \left[\left(k\frac{\partial T}{\partial n}\right)v\right] ds \qquad (3)$$

- Substituting (3) into (2) gives

$$\boxed{\int_{\partial\Omega} \left[\left(k\frac{\partial T}{\partial n}\right)v\right] ds \; - \int_{\Omega} k\nabla T\, \nabla v\, dxdy + \int_{\Omega} Qv\;\, dxdy \; = \; 0 \qquad (4)}$$

$\partial\Omega_2 \colon T = 3 + x^2$

$\partial\Omega_3 \colon T = y$




$\partial\Omega_1 \colon T = x$

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4$$

$$\boxed{\partial\Omega_4 \colon \frac{\partial T}{\partial x} = 1 - 0.2T}$$

Set $v = 0$ on $\partial\Omega_1, \partial\Omega_2, \partial\Omega_3$

$$\boxed{\int_{\partial\Omega_4}\left(k\frac{\partial T}{\partial n}\right)vds \quad - \int_{\Omega} k\nabla T\,\nabla v\,dxdy + \int_{\Omega} Qv\;\;dxdy \;=\; 0}$$

$$\Downarrow$$

$$\int_{\partial\Omega_4}(k(1-0.2T)\,vds \quad - \int_{\Omega} k\nabla T\,\nabla v\,dxdy + \int_{\Omega} Qv\;\;dxdy \;=\; 0$$

$$\int_{\partial\Omega_4} kv - 0.2kTv\,ds \quad - \int_{\Omega} k\nabla T\,\nabla v\,dxdy + \int_{\Omega} Qv\;\;dxdy \;=\; 0$$

$$-\int_{\partial\Omega_4} 0.2kTvds \quad - \int_{\Omega} k\nabla T\,\nabla v\,dxdy$$

$$= -\int_{\Omega} Qv\;\;dxdy \;-\int_{\partial\Omega_4} kv\,ds \qquad (5)$$

- Multiply both side of (5) by -1 yields

$$\int_{\partial\Omega_4} 0.2kTv ds \; + \int_{\Omega} k\nabla T \, \nabla v \, dxdy \quad = \int_{\Omega} Qv \; dxdy \; \int_{\partial\Omega_4} kv \, ds$$

- Variational Statement is

Find $T \in H^1(\Omega)$ such that $T(x,0) = x,\; T(x,3) = 3 + x^2,$ $T(0,y) = y$ and

$$\int_{\partial\Omega_4} 0.2kTv ds \; + \int_{\Omega} k\nabla T \, \nabla v \, dxdy = \int_{\Omega} Qv \; dxdy + \int_0^3 kv(3,y) \, dy$$

where $\Omega = [0,3] \times [0,3],\; H^1(\Omega) = \{v|v, v' \in L^2(\Omega)\}$

$$H_0^1(\Omega) = \{v \in H^1(\Omega) | v(x,0) = v(x,3) = v(0,y) = 0\}$$

## b) Derive FE equations for $k = 1, Q = 2$

$$\int_{\partial\Omega_4} 0.2Tv ds + \int_{\Omega} \nabla T \, \nabla v \, dxdy = 2\int_{\Omega} v \, dxdy + \int_0^3 v(3,y) \, dy$$

- Pose the problem into N-dimensional subspace by choosing $H_h^1 \subset H^1$ where $H_h^1$=span{$\phi_1, \ldots, \phi_N$}

$$\therefore \quad T \equiv T_h = \sum_{i=1}^{N} T_i \phi_i$$

By Galerkin method, we choose

$H_{0h}^1 \subset H_0^1$ where $H_h^1$=span{$\phi_1, \ldots, \phi_N$}

$$\therefore \quad v \equiv v_h = \sum_{i=1}^{N} v_i \phi_i$$

$$\sum_{j=1}^{N}\left[\int_{\partial\Omega_4} 0.2\phi_j\phi_i ds + \int_{\Omega} \nabla\phi_j\nabla\phi_i dxdy\right] T_j$$

$$= 2\int_{\Omega} \phi_i \, dxdy \, + \int_{\partial\Omega_4} \phi_i(3,y) ds \, , \qquad i = 1, \dots, N.$$

which can be expressed in matrix form as

$$[K_{ij}]\mathbf{T} = \mathbf{F},$$

FE equations

where $K_{ij} = \int_{\partial\Omega_4} 0.2\phi_j\phi_i ds + \int_{\Omega} \nabla\phi_j\nabla\phi_i dxdy$

$F = [f_i]$ and $f_i = 2\int_{\Omega} \phi_i \, dxdy \, + \int_{\partial\Omega_4} \phi_i(3,y) ds.$

# C) Find nodal solution

- Consider element by element

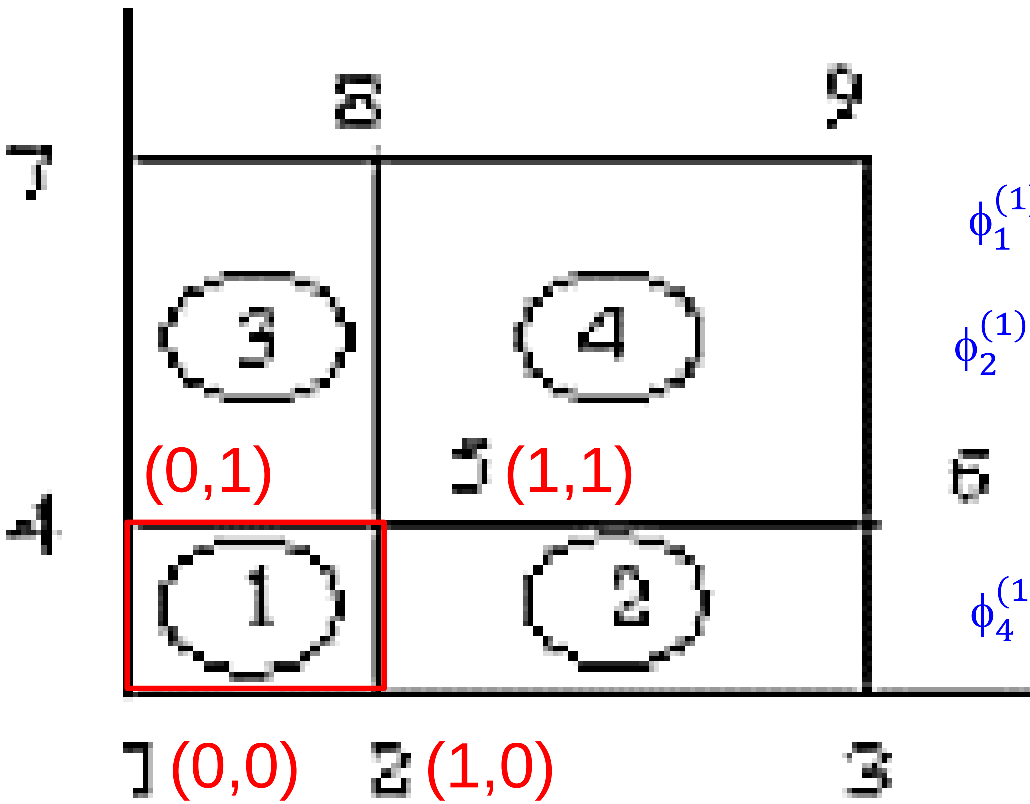
$$\phi_1(x,y) = \frac{1}{4ab}(x - x_2)(y - y_4)$$

$$\phi_2(x,y) = -\frac{1}{4ab}(x - x_1)(y - y_3),$$

$$\phi_3(x,y) = \frac{1}{4ab}(x - x_4)(y - y_2)$$

$$\phi_4(x,y) = -\frac{1}{4ab}(x - x_3)(y - y_1)$$

$$\boxed{a = b = 0.5}$$




$$\phi_1^{(1)}(x,y) = (x-1)(y-1)$$

$$\phi_2^{(1)}(x,y) = -(x-0)(y-1) = x(1-y)$$

$$\phi_3^{(1)}(x,y) = (x-0)(y-0) = xy$$

$$\phi_4^{(1)}(x,y) = -(x-1)(y-0) = y(1-x$$

$$\phi_1^{(1)}(x,y) = (x-1)(y-1)$$

$$\phi_2^{(1)}(x,y) = x(1-y)$$

$$\phi_3^{(1)}(x,y) = xy$$

$$\phi_4^{(1)}(x,y) = y(1-x)$$

$\Rightarrow$

$$\frac{\partial}{\partial x}\phi_1^{(1)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_2^{(1)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = -x$$

$$\frac{\partial}{\partial x}\phi_3^{(1)}(x,y) = y \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x$$

$$\frac{\partial}{\partial x}\phi_4^{(1)}(x,y) = -y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 1-x$$

$$\boxed{K_{ij}^{(1)} = \int_\Omega \frac{\partial}{\partial x}\phi_i^{(1)}\frac{\partial}{\partial x}\phi_j^{(1)} + \frac{\partial}{\partial y}\phi_i^{(1)}\frac{\partial}{\partial y}\phi_j^{(1)}dxdy,}$$

$$K_{11}^{(1)} = \int_0^1\int_0^1 (y-1)^2+(x-1)^2dxdy =$$

$$K_{12}^{(1)} = \int_0^1\int_0^1 -(y-1)^2 - x(x-1)dxdy =$$

$$K_{13}^{(1)} = \int_0^1\int_0^1 y(y-1) + x(x-1)dxdy =$$

$$K_{14}^{(1)} = \int_0^1\int_0^1 y(1-y) - (x-1)^2dxdy =$$

$$\phi_1^{(1)}(x,y) = (x-1)(y-1)$$

$$\phi_2^{(1)}(x,y) = x(1-y)$$

$$\phi_3^{(1)}(x,y) = xy$$

$$\phi_4^{(1)}(x,y) = y(1-x)$$

$\Rightarrow$

$$\frac{\partial}{\partial x}\phi_1^{(1)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_2^{(1)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = -x$$

$$\frac{\partial}{\partial x}\phi_3^{(1)}(x,y) = y \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x$$

$$\frac{\partial}{\partial x}\phi_4^{(1)}(x,y) = -y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 1-x$$

$$\boxed{K_{ij}^{(1)} = \int_\Omega \frac{\partial}{\partial x}\phi_i^{(1)} \frac{\partial}{\partial x}\phi_j^{(1)} + \frac{\partial}{\partial y}\phi_i^{(1)} \frac{\partial}{\partial y}\phi_j^{(1)} dxdy,}$$

$$K_{22}^{(1)} = \int_0^1 \int_0^1 (y-1)^2 + x^2 dxdy =$$

$$K_{23}^{(1)} = \int_0^1 \int_0^1 y(1-y) - x^2\, dxdy =$$

$$K_{24}^{(1)} = \int_0^1 \int_0^1 -y(1-y) - x(1-x) dxdy =$$

$$\phi_1^{(1)}(x,y) = (x-1)(y-1)$$

$$\phi_2^{(1)}(x,y) = x(1-\text{y})$$

$$\phi_3^{(1)}(x,y) = xy$$

$$\phi_4^{(1)}(x,y) = y(1-x)$$

$$\Rightarrow$$

$$\frac{\partial}{\partial x}\phi_1^{(1)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_2^{(1)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = -x$$

$$\frac{\partial}{\partial x}\phi_3^{(1)}(x,y) = y \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x$$

$$\frac{\partial}{\partial x}\phi_4^{(1)}(x,y) = -y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 1-x$$

$$\boxed{K_{ij}^{(1)} = \int_\Omega \frac{\partial}{\partial x}\phi_i^{(1)}\frac{\partial}{\partial x}\phi_j^{(1)} + \frac{\partial}{\partial y}\phi_i^{(1)}\frac{\partial}{\partial y}\phi_j^{(1)}dxdy,}$$

$$K_{33}^{(1)} = \int_0^1\int_0^1 y^2 + x^2 dxdy =$$

$$K_{34}^{(1)} = \int_0^1\int_0^1 -y^2 + x(1-x)\, dxdy =$$

$$K_{44}^{(1)} = \int_0^1\int_0^1 y^2 + (1-x)^2 dxdy =$$

$$\boxed{f_i^{(1)} = 2\int_{\Omega} \phi_i^{(1)} dxdy}$$

$$f_1^{(1)} = 2\int_0^1\int_0^1 \phi_1^{(1)} dxdy = 2\int_0^1\int_0^1 (x-1)(y-1)\, dxdy =$$

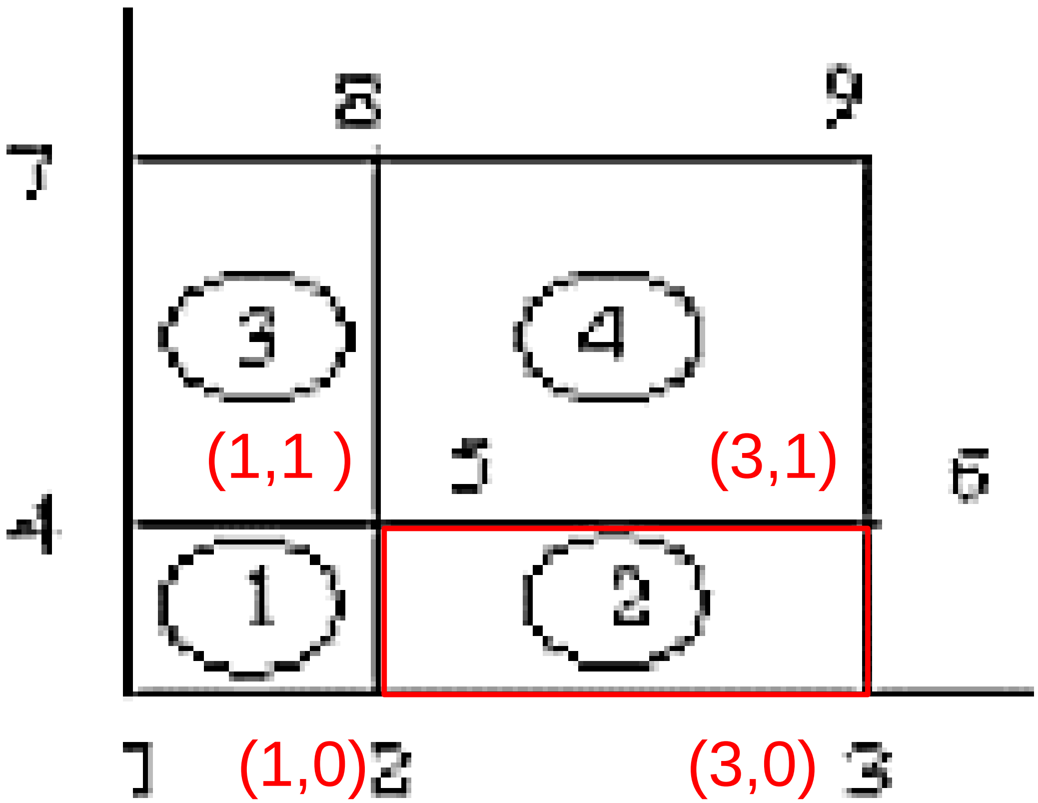
$$f_2^{(1)} = 2\int_0^1\int_0^1 \phi_2^{(1)} dxdy = 2\int_0^1\int_0^1 x(1-y)\, dxdy =$$

$$f_3^{(1)} = 2\int_0^1\int_0^1 \phi_3^{(1)} dxdy = 2\int_0^1\int_0^1 xy\ \ dxdy =$$

$$f_4^{(1)} = 2\int_0^1\int_0^1 \phi_4^{(1)} dxdy = 2\int_0^1\int_0^1 y(1-x) dxdy =$$

$$K^{(1)} = \left( \qquad\qquad\qquad\qquad \right), \qquad F^{(1)} = \left( \quad \right)$$

- Consider element 2




$$\phi_1(x,y) = \frac{1}{4ab}(x-x_2)(y-y_4)$$
$$\phi_2(x,y) = -\frac{1}{4ab}(x-x_1)(y-y_3),$$
$$\phi_3(x,y) = \frac{1}{4ab}(x-x_4)(y-y_2)$$
$$\phi_4(x,y) = -\frac{1}{4ab}(x-x_3)(y-y_1)$$

$$\boxed{a = 1, b = 0.5}$$

$$\phi_1^{(2)} = (x-3)(y-1)$$

$$\phi_2^{(2)} = -(x-1)(y-1))$$

$$\phi_3^{(2)} = (x-1)(y-0) = y(x-1)$$

$$\phi_4^{(2)} = -(x-3)(y-0) = -y(x-3)$$

$$\phi_1^{(2)} = (x-3)(y-1)$$

$$\phi_2^{(2)} = (1-x)(y-1))$$

$$\phi_3^{(2)} = y(x-1)$$

$$\phi_4^{(2)} = y(3-x)$$

$$\Rightarrow$$

$$\frac{\partial}{\partial x}\phi_1^{(2)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-3$$

$$\frac{\partial}{\partial x}\phi_2^{(2)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = 1-x$$

$$\frac{\partial}{\partial x}\phi_3^{(2)}(x,y) = y \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_4^{(2)}(x,y) = -y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 3-x$$

$$\boxed{K_{ij}^{(2)} = \int_0^1 0.2\phi_i^{(2)}(3,y)\phi_j^{(2)}(3,y)\,dy + \int_\Omega \frac{\partial\phi_i^{(2)}}{\partial x}\frac{\partial\phi_j^{(2)}}{\partial x} + \frac{\partial\phi_i^{(2)}}{\partial y}\frac{\partial\phi_j^{(2)}}{\partial y}\,dxdy}$$

$$K_{11}^{(2)} = \int_0^1\int_1^3 (y-1)^2 + (x-3)^2 dxdy =$$

$$K_{12}^{(2)} = \int_0^1\int_1^3 -(y-1)^2 + (1-x)(x-3)dxdy =$$

$$K_{13}^{(2)} = \int_0^1\int_1^3 y(y-1) + (x-1)(x-3)dxdy =$$

$$K_{14}^{(2)} = \int_0^1\int_1^3 -y(y-1) - (x-3)^2 dxdy =$$

$$\phi_1^{(2)} = (x-3)(y-1)$$

$$\phi_2^{(2)} = (1-x)(y-1))$$

$$\phi_3^{(2)} = y(x-1)$$

$$\phi_4^{(2)} = y(3-x)$$

$$\Longrightarrow$$

$$\frac{\partial}{\partial x}\phi_1^{(2)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-3$$

$$\frac{\partial}{\partial x}\phi_2^{(2)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = 1-x$$

$$\frac{\partial}{\partial x}\phi_3^{(2)}(x,y) = y \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_4^{(2)}(x,y) = -y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 3-x$$

$$\boxed{K_{ij}^{(2)} = \int_0^1 0.2\phi_i^{(2)}(3,y)\phi_j^{(2)}(3,y)\,dy + \int_\Omega \frac{\partial\phi_i^{(2)}}{\partial x}\frac{\partial\phi_j^{(2)}}{\partial x} + \frac{\partial\phi_i^{(2)}}{\partial y}\frac{\partial\phi_j^{(2)}}{\partial y}\,dxdy}$$

$$K_{22}^{(2)} = \int_0^1 0.2(1-3)^2(y-1)^2\,dy + \int_0^1\int_1^3 (1-y)^2+(1-x)^2 dxdy =$$

$$K_{23}^{(2)} = -\int_0^1 0.2\big((3)-1\big)^2(y-1)y\,dy + \int_0^1\int_1^3 y(1-y)-(x-1)^2 dxdy =$$

$$K_{24}^{(2)} = \int_0^1\int_1^3 -y(1-y)+(1-x)(3-x)dxdy =$$

$$\phi_1^{(2)} = (x-3)(y-1)$$

$$\phi_2^{(2)} = (1-x)(y-1))$$

$$\phi_3^{(2)} = y(x-1)$$

$$\phi_4^{(2)} = y(3-x)$$

$$\Rightarrow$$

$$\frac{\partial}{\partial x}\phi_1^{(2)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-3$$

$$\frac{\partial}{\partial x}\phi_2^{(2)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = 1-x$$

$$\frac{\partial}{\partial x}\phi_3^{(2)}(x,y) = y \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_4^{(2)}(x,y) = -y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 3-x$$

$$\boxed{K_{ij}^{(2)} = \int_0^1 0.2\phi_i^{(2)}(3,y)\phi_j^{(2)}(3,y)\,dy + \int_\Omega \frac{\partial\phi_i^{(2)}}{\partial x}\frac{\partial\phi_j^{(2)}}{\partial x} + \frac{\partial\phi_i^{(2)}}{\partial y}\frac{\partial\phi_j^{(2)}}{\partial y}\,dxdy}$$

$$K_{33}^{(2)} = \int_0^1 0.2(3-1)^2y^2\,dy + \int_0^1\int_1^3 y^2 + (x-1)^2dxdy =$$

$$K_{34}^{(2)} = \int_0^1\int_1^3 -y^2 + (x-1)(3-3)dxdy =$$

$$K_{44}^{(2)} = \int_0^1\int_1^3 y^2 + (3-x)^2\,dxdy =$$

$$\boxed{F^{(2)} = [f_i] \text{ and } f_i = 2\int_\Omega \phi_i^{(2)} dxdy + \int_0^1 \phi_i^{(2)}(3, y)dy.}$$

$$f_1^{(2)} = 2\int_0^1\int_1^3 \phi_1^{(2)} dxdy = 2\int_0^1\int_1^3 (x-3)(y-1)\, dxdy =$$

$$f_2^{(2)} = 2\int_0^1\int_1^3 \phi_2^{(2)} dxdy + \int_0^1 \phi_2^{(2)}(3, y)dy$$

$$= 2\int_0^1\int_1^3 (1-x)(y-1)\, dxdy + \int_0^1 (1-3)(y-1)dy =$$

$$f_3^{(2)} = 2\int_0^1\int_1^3 \phi_3^{(2)} dxdy + \int_0^1 \phi_2^{(2)}(3, y)dy$$

$$= 2\int_0^1\int_1^3 y(x-1)dxdy + \int_0^1 y((3)-1)dy =$$

$$f_4^{(2)} = 2\int_0^1\int_1^3 \phi_4^{(2)} dxdy = 2\int_0^1\int_1^3 y(3-x)dxdy =$$

$$K^{(2)} = \left(\qquad\qquad\qquad\qquad\right), \qquad F^{(2)} = \left(\quad\right)$$

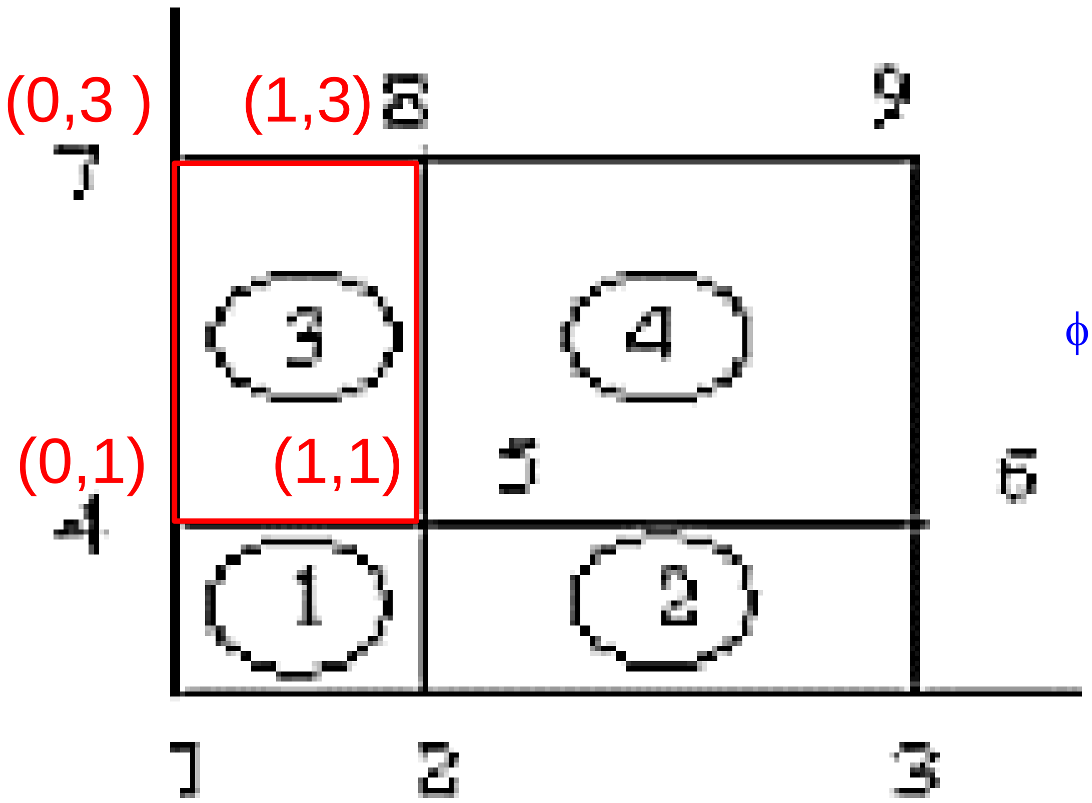
- Consider element 3

$$\phi_1(x,y) = \frac{1}{4ab}(x - x_2)(y - y_4)$$

$$\phi_2(x,y) = -\frac{1}{4ab}(x - x_1)(y - y_3),$$

$$\phi_3(x,y) = \frac{1}{4ab}(x - x_4)(y - y_2)$$

$$\phi_4(x,y) = -\frac{1}{4ab}(x - x_3)(y - y_1)$$




$$\boxed{a = 0.5, \text{b=1}}$$

$$\phi_1^{(3)} = (x - 1)(y - 3)$$

$$\phi_2^{(3)} = -(x - 0)(y - 3) = -x(y - 3)$$

$$\phi_3^{(3)} = (x - 0)(y - 1) = x(y - 1)$$

$$\phi_4^{(3)} = -(x - 1)(y - 1)$$

$$\phi_1^{(3)} = (x-1)(y-3)$$

$$\phi_2^{(3)} = x(3-y)$$

$$\phi_3^{(3)} = x(y-1)$$

$$\phi_4^{(3)} = (1-x)(y-1)$$

$$\frac{\partial}{\partial x}\phi_1^{(2)}(x,y) = y-3 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_2^{(2)}(x,y) = 3-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = -x$$

$$\frac{\partial}{\partial x}\phi_3^{(2)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x$$

$$\frac{\partial}{\partial x}\phi_4^{(2)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 1-x$$

$$\boxed{K_{ij}^{(3)} = \int_{\Omega} \frac{\partial\phi_i^{(2)}}{\partial x}\frac{\partial\phi_j^{(2)}}{\partial x} + \frac{\partial\phi_i^{(2)}}{\partial y}\frac{\partial\phi_j^{(2)}}{\partial y}\, dxdy}$$

$$K_{11}^{(3)} = \int_1^3\int_0^1 (y-3)^2 + (x-1)^2 dxdy =$$

$$K_{12}^{(3)} = \int_1^3\int_0^1 -(y-3)^2 - x(x-1) dxdy =$$

$$K_{13}^{(3)} = \int_1^3\int_0^1 (y-3)(y-1) + x(x-1) dxdy =$$

$$K_{14}^{(3)} = \int_1^3\int_0^1 (y-3)(1-y) - (x-1)^2 dxdy =$$

$$\phi_1^{(3)} = (x-1)(y-3)$$

$$\phi_2^{(3)} = x(3-y)$$

$$\phi_3^{(3)} = x(y-1)$$

$$\phi_4^{(3)} = (1-x)(y-1)$$

$$\frac{\partial}{\partial x}\phi_1^{(2)}(x,y) = y-3 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_2^{(2)}(x,y) = 3-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = -x$$

$$\frac{\partial}{\partial x}\phi_3^{(2)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x$$

$$\frac{\partial}{\partial x}\phi_4^{(2)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 1-x$$

$$\boxed{K_{ij}^{(3)} = \int_\Omega \frac{\partial\phi_i^{(2)}}{\partial x}\frac{\partial\phi_j^{(2)}}{\partial x} + \frac{\partial\phi_i^{(2)}}{\partial y}\frac{\partial\phi_j^{(2)}}{\partial y}\,dxdy}$$

$$K_{22}^{(3)} = \int_1^3\int_0^1 (3-y)^2 + x^2\,dxdy =$$

$$K_{23}^{(3)} = \int_1^3\int_0^1 (3-y)(y-1) - x^2\,dxdy =$$

$$K_{24}^{(3)} = \int_1^3\int_0^1 (3-y)(1-y) - x(1-x)\,dxdy =$$

$$\phi_1^{(3)} = (x-1)(y-3)$$

$$\phi_2^{(3)} = x(3-y)$$

$$\phi_3^{(3)} = x(y-1)$$

$$\phi_4^{(3)} = (1-x)(y-1)$$

$$\frac{\partial}{\partial x}\phi_1^{(2)}(x,y) = y-3 \qquad \frac{\partial}{\partial y}\phi_1^{(1)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_2^{(2)}(x,y) = 3-y \qquad \frac{\partial}{\partial y}\phi_2^{(1)}(x,y) = -x$$

$$\frac{\partial}{\partial x}\phi_3^{(2)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_3^{(1)}(x,y) = x$$

$$\frac{\partial}{\partial x}\phi_4^{(2)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_4^{(1)}(x,y) = 1-x$$

$$\boxed{K_{ij}^{(3)} = \int_\Omega \frac{\partial\phi_i^{(2)}}{\partial x}\frac{\partial\phi_j^{(2)}}{\partial x} + \frac{\partial\phi_i^{(2)}}{\partial y}\frac{\partial\phi_j^{(2)}}{\partial y}\,dxdy}$$

$$K_{33}^{(3)} = \int_1^3\int_0^1 (y-1)^2 + x^2\,dxdy =$$

$$K_{34}^{(3)} = \int_1^3\int_0^1 -(y-1)^2 + x(1-x)\,dxdy =$$

$$K_{44}^{(3)} = \int_1^3\int_0^1 (1-y)^2 + (1-x)^2\,dxdy =$$

$$F^{(3)} = [f_i] \text{ and } f_i = 2\int_{\Omega} \phi_i^{(3)} dxdy$$

$$f_1^{(3)} = 2\int_1^3\int_0^1 \phi_1^{(3)} dxdy = 2\int_1^3\int_0^1 (x-1)(y-3)dxdy =$$

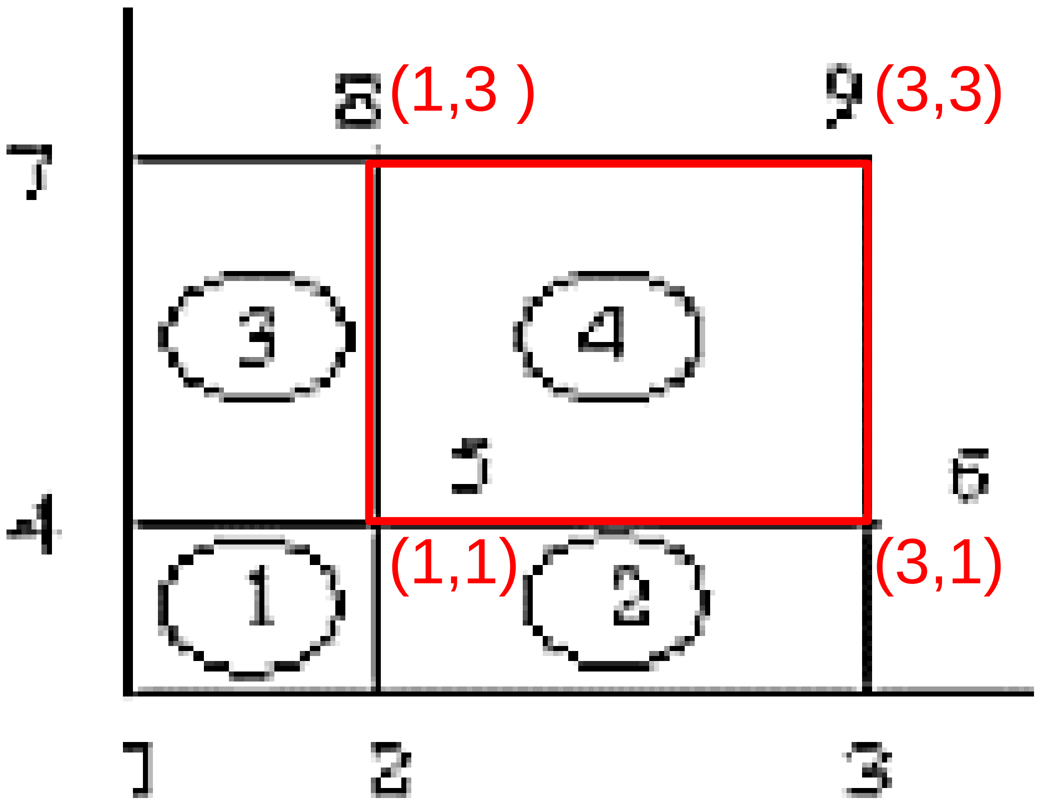
$$f_2^{(3)} = 2\int_1^3\int_0^1 \phi_2^{(3)} dxdy = 2\int_1^3\int_0^1 x(3-\mathrm{y})\, dxdy =$$

$$f_3^{(3)} = 2\int_1^3\int_0^1 \phi_3^{(3)} dxdy = 2\int_1^3\int_0^1 x(y-1)dxdy =$$

$$f_4^{(3)} = 2\int_1^3\int_0^1 \phi_4^{(3)} dxdy = 2\int_1^3\int_0^1 (1-x)(y-1)\, dxdy =$$

$$K^{(3)} = \left( \qquad\qquad\qquad \right), \qquad F^{(3)} = \left( \quad \right)$$

- Consider element 4




$$\phi_1(x,y) = \frac{1}{4ab}(x - x_2)(y - y_4)$$

$$\phi_2(x,y) = -\frac{1}{4ab}(x - x_1)(y - y_3),$$

$$\phi_3(x,y) = \frac{1}{4ab}(x - x_4)(y - y_2)$$

$$\phi_4(x,y) = -\frac{1}{4ab}(x - x_3)(y - y_1)$$

$$\boxed{a = b = 1}$$

$$\phi_1^{(4)} = (x-3)(y-3)$$

$$\phi_2^{(4)} = -(x-1)(y-3)$$

$$\phi_3^{(4)} = (x-1)(y-1)$$

$$\phi_4^{(4)} = -(x-3)(y-1)$$

$$\phi_1^{(4)} = (x-3)(y-3)$$

$$\phi_2^{(4)} = -(x-1)(y-3)$$

$$\phi_3^{(4)} = (x-1)(y-1)$$

$$\phi_4^{(4)} = -(x-3)(y-1)$$

$$\Rightarrow$$

$$\frac{\partial}{\partial x}\phi_1^{(4)}(x,y) = y-3 \qquad \frac{\partial}{\partial y}\phi_1^{(4)}(x,y) = x-3$$

$$\frac{\partial}{\partial x}\phi_2^{(4)}(x,y) = 3-y \qquad \frac{\partial}{\partial y}\phi_2^{(4)}(x,y) = 1-x$$

$$\frac{\partial}{\partial x}\phi_3^{(4)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_3^{(4)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_4^{(4)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_4^{(4)}(x,y) = 3-x$$

$$\boxed{K_{ij}^{(4)} = \int_1^3 0.2\phi_i^{(4)}(3,y)\phi_j^{(4)}(3,y)\,dy + \int_\Omega \frac{\partial\phi_i^{(4)}}{\partial x}\frac{\partial\phi_j^{(4)}}{\partial x} + \frac{\partial\phi_i^{(4)}}{\partial y}\frac{\partial\phi_j^{(4)}}{\partial y}\,dxdy}$$

$$K_{11}^{(4)} = \int_1^3\int_1^3 (y-3)^2+(x-3)^2dxdy =$$

$$K_{12}^{(4)} = \int_1^3\int_1^3 -(y-3)^2 + (1-x)(x-3)dxdy =$$

$$K_{13}^{(4)} = \int_1^3\int_1^3 (y-3)(y-1) + (x-1)(x-3)dxdy =$$

$$K_{14}^{(4)} = \int_1^3\int_1^3 (y-3)(1-y) - (x-3)^2dxdy =$$

$$\phi_1^{(4)} = (x-3)(y-3)$$

$$\phi_2^{(4)} = -(x-1)(y-3)$$

$$\phi_3^{(4)} = (x-1)(y-1)$$

$$\phi_4^{(4)} = -(x-3)(y-1)$$

$$\frac{\partial}{\partial x}\phi_1^{(4)}(x,y) = y-3 \qquad \frac{\partial}{\partial y}\phi_1^{(4)}(x,y) = x-3$$

$$\frac{\partial}{\partial x}\phi_2^{(4)}(x,y) = 3-y \qquad \frac{\partial}{\partial y}\phi_2^{(4)}(x,y) = 1-x$$

$$\frac{\partial}{\partial x}\phi_3^{(4)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_3^{(4)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_4^{(4)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_4^{(4)}(x,y) = 3-x$$

$$\boxed{K_{ij}^{(4)} = \int_1^3 0.2\phi_i^{(4)}(3,y)\phi_j^{(4)}(3,y)\,dy + \int_\Omega \frac{\partial\phi_i^{(4)}}{\partial x}\frac{\partial\phi_j^{(4)}}{\partial x} + \frac{\partial\phi_i^{(4)}}{\partial y}\frac{\partial\phi_j^{(4)}}{\partial y}\,dxdy}$$

$$K_{22}^{(4)} = -\int_1^3 0.2\big((3)-1\big)^2(y-3)^2\,dy + \int_1^3\int_1^3 (3-y)^2+(1-x)^2 dxdy =$$

$$K_{23}^{(4)} = -\int_1^3 0.2\big((3)-1\big)^2(y-1)(y-3)\,dy + \int_1^3\int_1^3 (3-y)(y-1)-(x-1)^2 dxdy =$$

$$K_{24}^{(4)} = \int_1^3\int_1^3 (3-y)(1-y)+(1-x)(3-x)dxdy =$$

$$\phi_1^{(4)} = (x-3)(y-3)$$

$$\phi_2^{(4)} = -(x-1)(y-3)$$

$$\phi_3^{(4)} = (x-1)(y-1)$$

$$\phi_4^{(4)} = -(x-3)(y-1)$$

$\Rightarrow$

$$\frac{\partial}{\partial x}\phi_1^{(4)}(x,y) = y-3 \qquad \frac{\partial}{\partial y}\phi_1^{(4)}(x,y) = x-3$$

$$\frac{\partial}{\partial x}\phi_2^{(4)}(x,y) = 3-y \qquad \frac{\partial}{\partial y}\phi_2^{(4)}(x,y) = 1-x$$

$$\frac{\partial}{\partial x}\phi_3^{(4)}(x,y) = y-1 \qquad \frac{\partial}{\partial y}\phi_3^{(4)}(x,y) = x-1$$

$$\frac{\partial}{\partial x}\phi_4^{(4)}(x,y) = 1-y \qquad \frac{\partial}{\partial y}\phi_4^{(4)}(x,y) = 3-x$$

$$\boxed{K_{ij}^{(4)} = \int_1^3 0.2\phi_i^{(4)}(3,y)\phi_j^{(4)}(3,y)\,dy + \int_\Omega \frac{\partial\phi_i^{(4)}}{\partial x}\frac{\partial\phi_j^{(4)}}{\partial x} + \frac{\partial\phi_i^{(4)}}{\partial y}\frac{\partial\phi_j^{(4)}}{\partial y}\,dxdy}$$

$$K_{33}^{(4)} = \int_1^3 0.2((3)-1)^2(y-1)^2\,dy + \int_1^3\int_1^3 (y-1)^2+(x-1)^2 dxdy =$$

$$K_{34}^{(4)} = \int_1^3\int_1^3 -(y-1)^2 + (x-1)(3-x)dxdy =$$

$$K_{44}^{(4)} = \int_1^3\int_1^3 (1-y)^2+(3-x)^2\,dxdy =$$

$$\boxed{F^{(4)} = [f_i] \text{ and } f_i = 2\int_{\Omega} \phi_i^{(2)} dxdy \ + \int_1^3 \phi_i^{(2)}(3,y)dy.}$$

$$f_1^{(4)} = 2\int_1^3 \int_1^3 \phi_1^{(2)} dxdy = 2\int_1^3 \int_1^3 (x-3)(y-3)dxdy =$$

$$f_2^{(4)} = 2\int_1^3 \int_1^3 \phi_2^{(2)} dxdy + \int_1^3 \phi_2^{(2)}(3,y)dy$$

$$= 2\int_1^3 \int_1^3 -(x-1)(y-3)\ dxdy + \int_1^3 -((3)-1)(y-3)\ dy \ =$$

$$f_3^{(4)} = 2\int_1^3 \int_1^3 \phi_3^{(4)} dxdy + \int_1^3 \phi_2^{(4)}(3,y)dy$$

$$= 2\int_1^3 \int_1^3 (x-1)(y-1)\ dxdy + \int_1^3 ((3)-1)(y-1)dy \ =$$

$$f_4^{(4)} = 2\int_1^3 \int_1^3 \phi_4^{(4)} dxdy = 2\int_1^3 \int_1^3 -(x-3)(y-1)\ dxdy =$$

$$K^{(4)} = \left( \quad \right), \qquad F^{(4)} = \left( \quad \right)$$

# Assemble element matrices

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $K_{11}^{(1)}$ | $K_{12}^{(1)}$ | | $K_{14}^{(1)}$ | $K_{13}^{(1)}$ | | | | |
| 2 | $K_{21}^{(1)}$ | $K_{22}^{(1)} + K_{11}^{(2)}$ | $K_{12}^{(2)}$ | $K_{24}^{(1)}$ | $K_{23}^{(1)} + K_{14}^{(2)}$ | $K_{13}^{(2)}$ | | | |
| 3 | | $K_{21}^{(2)}$ | $K_{22}^{(2)}$ | | $K_{24}^{(2)}$ | $K_{23}^{(2)}$ | | | |
| 4 | $K_{41}^{(1)}$ | $K_{42}^{(1)}$ | | $K_{44}^{(1)} + K_{11}^{(3)}$ | $K_{34}^{(1)} + K_{12}^{(3)}$ | | $K_{14}^{(3)}$ | $K_{13}^{(3)}$ | |
| 5 | $K_{31}^{(1)}$ | $K_{32}^{(1)} + K_{41}^{(2)}$ | $K_{42}^{(2)}$ | $K_{43}^{(1)} + K_{21}^{(3)}$ | $K_{33}^{(1)} + K_{44}^{(2)} + K_{22}^{(3)} + K_{11}^{(4)}$ | $K_{43}^{(2)} + K_{12}^{(4)}$ | $K_{24}^{(3)}$ | $K_{23}^{(3)} + K_{14}^{(4)}$ | $K_{13}^{(4)}$ |
| 6 | | $K_{31}^{(2)}$ | $K_{32}^{(2)}$ | | $K_{34}^{(2)} + K_{21}^{(4)}$ | $K_{33}^{(2)} + K_{22}^{(4)}$ | | $K_{24}^{(4)}$ | $K_{23}^{(4)}$ |
| 7 | | | | $K_{41}^{(3)}$ | $+K_{42}^{(3)}$ | | $K_{44}^{(3)}$ | $K_{43}^{(3)}$ | |
| 8 | | | | $K_{31}^{(3)}$ | $+K_{32}^{(3)} + K_{41}^{(4)}$ | $+K_{42}^{(4)}$ | $K_{34}^{(3)}$ | $K_{33}^{(3)} + K_{44}^{(4)}$ | $+K_{43}^{(4)}$ |
| 9 | | | | | $K_{31}^{(4)}$ | $K_{32}^{(4)}$ | | $K_{34}^{(4)}$ | $K_{33}^{(4)}$ |

# Assemble element load vectors

$$\begin{array}{c|c} \text{Node} & \\ 1 & f_1^{(1)} \\ 2 & f_2^{(1)} + f_1^{(2)} \\ 3 & f_2^{(2)} \\ 4 & f_4^{(1)} + f_1^{(3)} \\ 5 & f_3^{(1)} + f_4^{(2)} + f_2^{(3)} + f_1^{(4)} \\ 6 & f_3^{(2)} + f_2^{(4)} \\ 7 & f_4^{(3)} \\ 8 & f_3^{(3)} + f_4^{(4)} \\ 9 & f_3^{(4)} \end{array} = \begin{bmatrix} \; \\ \; \\ \; \\ \; \\ \; \\ \; \\ \; \\ \; \\ \; \end{bmatrix}$$

# Impose essential boundary conditions