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MATH5004-LEC2

FDM: Consistency, stability, converence, and error estimates

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OUTLINE

- Global error
- Local truncation error
- Convergent method
- Grid refinement analysis
- Stencils for the 2D Laplacian
- Example

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Definition of the global error

Let $\mathbf{U} = (U_1, \ldots, U_N)^T$ be he solution of finite difference scheme (assume no round-ff errors),

$\mathrm{u} = (u(x_1), \ldots, u(x_N))^T$ be the exact solution at grid points.

The global error vector is defined as $\mathbf{E} = \mathbf{U} - \mathbf{u}$.

- The maximum norm (or the infinity norm) is

$$\|\mathbf{E}\|_\infty = \max_i |e_i|.$$

- The 1-norm is $\|\mathbf{E}\|_1 = \sum_i h_i |e_i|$

  similar to the continuous space $\int |e(x)|dx$.

- The 2-norm is $\|\mathbf{E}\|_2 = (\sum_i h_i |e_i|^2)^{1/2}$

  similar to the continuous space $(\int |e(x)|^2 dx)^{1/2}$.

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If $\|\mathbf{E}\| \le Ch^p, \; p > 0,$ C is a constant independent of h the finite difference (FD) scheme is ***p-th order accurate***

<u>Definition</u> A FD scheme is called convergent if $\lim_{h \to 0} \|\mathbf{E}\| = 0$

Local truncation error

Consider the DE $u_{xx} = f$ or $Lu = f$ .....(1)

L is the differential operator $L = \dfrac{d^2}{dx^2}$

By applying FDM

DE(1) becomes $\dfrac{u(x+h) - 2 \;\; (x) + u(x-h)}{h^2} = f$

or $L_h u = f$

The local truncation error is $T(x) = Lu - L_h u$

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For the DE $u_{xx} = f(x)$

and the three-point central difference scheme,

the local truncation error is

$$T(x) = Lu - L_h u$$

$$= u_{xx} - \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$$= f(x) - \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

which measures how well the FD discretisation approximates the DE

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Convergent Method

Definition A finite difference scheme is called consistent if

$$\lim_{h \to 0} T(x) = \lim_{h \to 0} (Lu - L_h u) = 0$$

To check whether a FD scheme is consistent or not, we use Taylor expansion.

For the DE $u_{xx} = f(x)$, we have

$$T(x) = u_{xx} - \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$$\approx -\frac{h^2}{12} u^{(4)}(x) = Ch^2$$

$$\lim_{h \to 0} T(x) = \lim_{h \to 0} Ch^2 = 0$$

So the FD scheme is consistent.

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As the consistence cannot guarantee a finite difference work, the stability of a FD scheme is required.

For the model problem, we have

$$A\mathbf{u} = \mathbf{F} + \mathbf{T}, \qquad A\mathbf{U} = \mathbf{F}$$

$$A(\mathbf{u} - \mathbf{U}) = \mathbf{T}$$

$$A\mathbf{E} = \mathbf{T}$$

A is the coefficient matrix of the FD equations,

F is the modified source term by the BC,

T is the vector of the local truncation error at grid points

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$$A\mathbf{E} = \mathbf{T}$$

If A is non-singular, then

$$||\mathbf{E}|| = ||A^{-1}\mathbf{T}|| \leq ||A^{-1}||||\mathbf{T}||.$$

If A is singular, then

$||\mathbf{E}||$ may be arbitrary large.

For the central FD scheme, we have

$$||\mathbf{E}|| \leq ||A^{-1}||h^2.$$

Thus, the global error depends on both the local truncation error and $||A^{-1}||$.

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Definition

A FD method for the elliptic DE is stable if A is invertible and

$$||A^{-1}|| \leq C, \quad \textit{for all } h < h_0,$$

where C and h_0 are constants

Theorem

A consistent and stable finite difference scheme is convergent.

Grid refinement analysis

Assume that a method is p-th order accurate, i.e.,

$$||E_h|| \sim Ch^p$$

$$\log ||E_h|| = \log C + p \log h,$$

We can divide h by half to get $||E_{h/2}||$.

We then have

$$\text{ratio} = \frac{||E_h||}{||E_{h/2}||} = \frac{Ch^p}{C(h/2)^p} = 2^p,$$

$$p = \frac{\log \left(||E_h||/||E_{h/2}||\right)}{\log 2}.$$

Improvement of Accuracy of Solution

Deferred Approach to the Limit

By reducing the value of k and h, i.e., through mesh refinement.

Denote u = true solution,

u_h = finite difference solution,

then $u = u_h +$ error.

Let u_h be the solution of the Laplace equation by using the 5 point scheme, then

$$u = u_h + Ch^2 .$$

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- Based on the above formulae, to accelerate the convergence of numerical solutions, we could use the so-called ***deferred approach*** to the limit suggested by Richardson.

$$u = u_1 + Ch^p . \qquad (1.1)$$

If mesh size h is reduced from h to $h/2$, then

$$u = u_2 + C(h/2)^p. \qquad (1.2)$$

where u_2 is the approximation obtained with $h/2$.

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From (1.1) and (1.2), by eliminating h^P, we obtain

$$u = \frac{2^p u_2 - u_1}{2^p - 1}. \qquad (1.3)$$

For example, for $p = 2$,

$$u = \frac{4u_2 - u_1}{3}$$

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- If p is unknown, it can be estimated from three approximate solutions at the same point obtained by using different mesh sizes.
  - Let u_1 , u_2 and u_3 are approximations at the same point obtained respectively using step size h_1 , h_2 and h_3 where

$$h_3 = \frac{1}{2} h_2 = \frac{1}{4} h_1$$

Then

$$u = u_1 + Ch_1^p,$$

$$u = u_2 + C\left(\frac{1}{2}\right)^p h_1^p,$$

$$u = u_3 + C\left(\frac{1}{4}\right)^p h_1^p,$$

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from which, we have

$$u_2 - u_1 + Ah_1^p\left[\left(\frac{1}{2}\right)^p - 1\right] = 0$$

$$u_3 - u_2 + Ah_1^p\left(\frac{1}{2}\right)^p\left[\left(\frac{1}{2}\right)^p - 1\right] = 0$$

Hence,

$$2^p = \frac{u_2 - u_1}{u_3 - u_2}.$$

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Elliptic equations

In two space dimensions, a constant-coefficient elliptic equation has the form

$$au_{xx} + bu_{xy} + cu_{yy} + h(x, y, u, u_x, u_y) = f(x, y)$$

where the coefficients a, b, c satisfy

$$b^2 - 4ac < 0.$$

Consider the problem

$$u_{xx} + u_{yy} = f(x, y), \qquad 0 \leq x, y \leq 1$$

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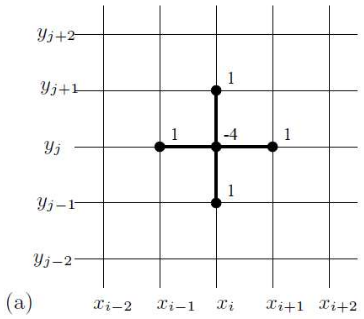
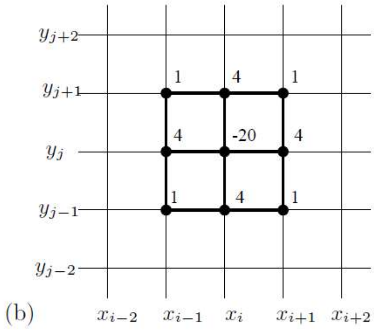
Stencils for the 2D Laplacian



The five-point stencil



The nine-point stencil

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Accuracy and Stability: the five-point stencil

Let $\Delta x = \Delta y = h$

$$T(x) = Lu - L_h u$$

$$= (u_{xx} + u_{yy}) - \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i.j}}{h^2}$$

$$T_{ij} = f_{ij} - \frac{1}{h^2}\left(u_{i-1.j} + u_{i+1.j} + u_{i.j-1} + u_{i-1.j+1} - 4u_{i.j}\right)$$

$$T_{ij} = \frac{1}{12}h^2\left(u_{xxxx} + u_{yyyy}\right) + O(h^4)$$

The global error $E_{ij} = u(x_i, u_j) - u_{ij}$

$$\Rightarrow \quad A^h \mathrm{E}^h = \mathrm{T}^h$$

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Accuracy and Stability: the nine-point stencil

$$\nabla^2 u(x_i, y_j) = \frac{1}{6h^2}[4u_{i-1,j}+4u_{i+1,j}+4u_{i,j-1}+4u_{i,j+1}$$

$$+u_{i-1,j-1}+u_{i-1,j+1}+u_{i+1,j-1}+u_{i+1,j+1}-20u_{ij}]$$

$$\nabla^2 u(x_i, y_j)$$

$$= \nabla^2 u + \frac{1}{12}h^2(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) + O(h^4).$$

It is noted that the error is still $O(h^2)$.

Since

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \nabla^2(\nabla^2 u) \equiv \nabla^4 u.$$

From DE $\nabla^2 u = f$, we have

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = \nabla^2 f.$$

We can obtain a fourth-order accurate method of the form

$$\nabla^2 u_{ij} = f(x_i, y_j) + \frac{h^2}{12}\nabla^2 f(x_i, y_j),$$

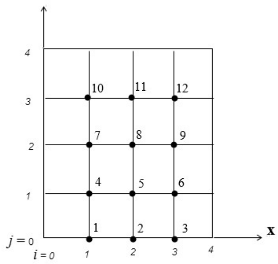
Example

Given

$$u_{xx} + u_{yy} = x^2y^2, \qquad 0 \le x, y \le \pi$$

$$u(0, y) = 0, \qquad u(\pi, y) = 1$$

$$u(x, \pi) = 0, \qquad \frac{\partial u(x, 0)}{\partial y} = 1.$$



For 5 grid points in x-direction and 5-grid points in y-direction,

- Derive the five-point FD equations of the PDE
- Derive the nine-point FD equations of the PDE

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