MATH5004 Tutorial 3

Finite Difference Method for 2D BVP

Example. Derive finite difference scheme for the solution of the 2D Wave equation (vibrations of a thin elastic membrane fixed at its walls),i.e.,

$$u_{tt} = \beta(u_{xx} + u_{yy}), \quad 0 < x < a, 0 < y < b, 0 < t \le \tau$$

subject to initial conditions

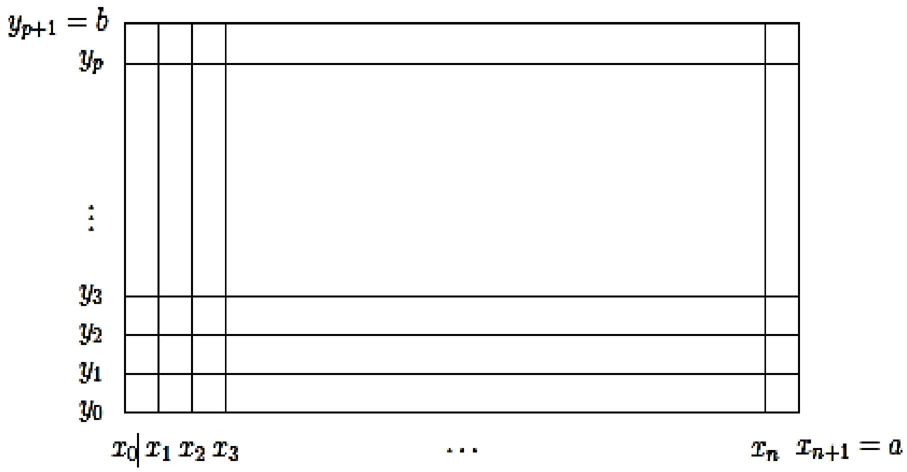
$$u(x, y, 0) = f(x, y), \qquad u_t(x, y, 0) = g(x, y)$$

and boundary condition

$$u(\partial\Omega) = 0.$$

Solution

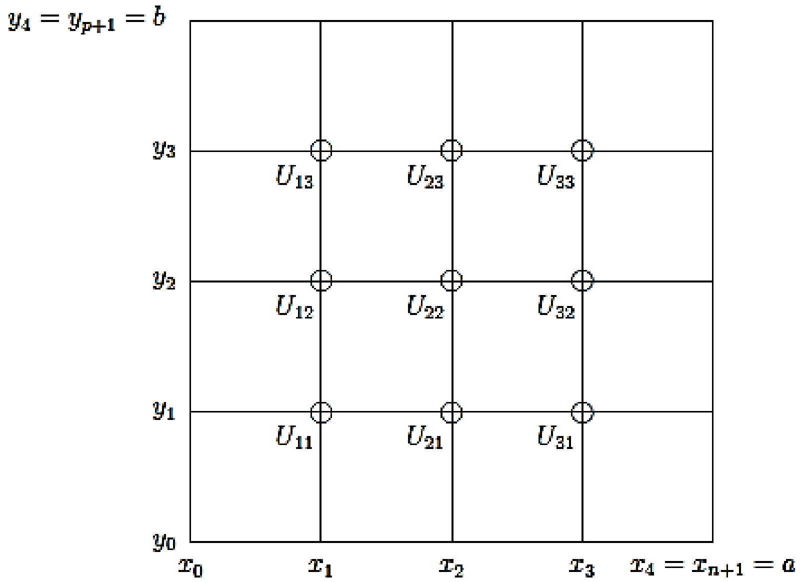
We discretise in x and y-directions:



We discretise: $\Delta t = \frac{T}{m}$, $\Delta x = \frac{a}{n+1}$, $\Delta y = \frac{b}{p+1}$, $t_k = k\Delta t$, $x_i = i\Delta x$, $y_j = j\Delta y 0 \le k \le m$, $0 \le i \le n+1$, $0 \le j \le p+1$, and let $U_{ij}^k = U(t_k, x_i, y_j)$

Suppose we solve for $n = 3$ and $p = 3$ and have Dirichlet boundary conditions: $U(0, y, t) = 0 = U_{oj}^k$, $U(a, y, t) = 0 = U_{n+1,j}^k = U_{4j}^k$, $U(x, 0, t) = 0 = U_{i0}^k$, $U(x, b, t) = 0 = U_{i,p+1}^k = U_{i4}^k$
and initial conditions:
$U(x, y, 0) = f(x, y) = f_{ij}$ $U_t(x, y, 0) = g(x, y) = g_{ij}$.

Since we have Dirichlet boundary conditions: the outer boundaries of the region we are solving for are known: $U_{0,j}^k, U_{n+1,j}^k, U_{i,0}^k, U_{i,p+1}^k$, and we need to find the interior values: $U_{i,j}^k$ for $1 \le i \le n$ and $1 \le j \le p$.



We use the 2D central difference method

$$
\begin{aligned}
U_{tt} &= \frac{U_{ij}^{k+1} - 2U_{ij}^{k} + U_{ij}^{k-1}}{\Delta t^2}, \\
U_{xx} &= \frac{U_{i+!,j}^{k} - 2U_{ij}^{k} + U_{i-1,j}^{k}}{\Delta x^2}, \\
U_{yy} &= \frac{U_{i,j+1}^{k} - 2U_{ij}^{k} + U_{i,j+1}^{k}}{\Delta y^2}
\end{aligned}
$$

We let $s_x = \frac{\beta \Delta t^2}{\Delta x^2}$, $s_y = \frac{\beta \Delta t^2}{\Delta y^2}$ and substitute the central difference approximations into the PDE, $U_{ij}^{k+1} = 2U_{ij}^{k}(1 - s_x - s_y) - U_{ij}^{k-1} + s_x\left(U_{i+1,j}^{k-1} + U_{i-1,j}^{k-1}\right) + s_y\left(U_{i,j+1}^{k-1} + U_{i,j-1}^{k-1}\right)$ computing $\vec{U}^{k+1}$ uses the solution $\vec{U}^{k}$ and $\vec{U}^{k-1}$.

For the first time step $\vec{U}^{1}$ need $\vec{U}^{0}$ and $\vec{U}^{-1}$. Again we need to use the initial conditions to find the ghost point, U_{ij}^{-1}:

$$\frac{\partial U_{ij}^{0}}{\partial t} = \frac{U_{ij}^{1} - U_{ij}^{-1}}{2\Delta t} = g_{ij}$$

$$\therefore \; U_{ij}^{-1} = U_{ij}^{1} - 2\Delta t \, g_{ij}$$

Solution at first time step $k = 1$:

$$U_{ij}^1 = U_{ij}^0(1 - s_x - s_y) + \Delta t g_{ij} + \frac{s_x}{2}(U_{i+1,j}^0 + U_{i-1,j}^0) + \frac{s_y}{2}(U_{i,j+1}^0 + U_{i,j-1}^0)$$

If we let $\vec{U}^k = \begin{pmatrix} U_{11}^k \\ U_{12}^k \\ U_{13}^k \\ U_{21}^k \\ U_{22}^k \\ U_{23}^k \\ U_{31}^k \\ U_{32}^k \\ U_{33}^k \end{pmatrix}$

then for time steps, $k > 1$, the solution is:

$$U_{ij}^{k+1} = 2U_{ij}^k(1 - s_x - s_y) - U_{ij}^{k-1} + s_x(U_{i+1,j}^k + U_{i-1,j}^k) + s_y(U_{i,j+1}^k + U_{i,j-1}^k)$$

and we can write this in vector form:

$$\vec{U}^{k+1} = A\vec{U}^k + \vec{b} - \vec{U}^{k-1}$$

where $A =:$

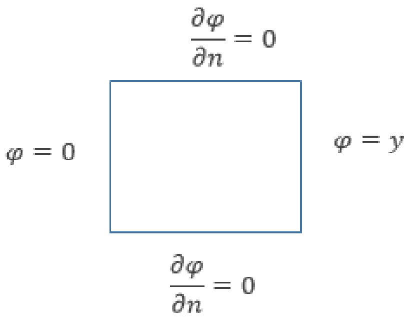
$$\begin{pmatrix}
2(1-s_x-s_y) & s_y & 0 & s_x & 0 & 0 & 0 & 0 & 0 \\
s_y & 2(1-s_x-s_y) & s_y & 0 & s_x & 0 & 0 & 0 & 0 \\
0 & s_y & 2(1-s_x-s_y) & 0 & 0 & s_x & 0 & 0 & 0 \\
s_x & 0 & 0 & 2(1-s_x-s_y) & s_y & 0 & s_x & 0 & 0 \\
0 & s_x & 0 & s_y & 2(1-s_x-s_y) & s_y & 0 & s_x & 0 \\
0 & 0 & s_x & 0 & s_y & 2(1-s_x-s_y) & 0 & 0 & s_x \\
0 & 0 & 0 & s_x & 0 & 0 & 2(1-s_x-s_y) & s_y & 0 \\
0 & 0 & 0 & 0 & s_x & 0 & s_y & 2(1-s_x-s_y) & s_y \\
0 & 0 & 0 & 0 & 0 & s_x & 0 & s_y & 2(1-s_x-s_y)
\end{pmatrix}$$

$$b = \begin{pmatrix} s_x U_{01}^k + s_y U_{10}^k \\ s_x U_{02}^k \\ s_x U_{03}^k + s_y U_{14}^k \\ s_y U_{20}^k \\ 0 \\ s_y U_{24}^k \\ s_x U_{41}^k + s_y U_{30}^k \\ s_x U_{42}^k \\ s_x U_{43}^k + s_y s_{34}^k \end{pmatrix}$$

Exercise. Derive finite difference scheme for the solution of the Laplace's equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

on a 2×2 square with boundary condition as shown.



Assignment I

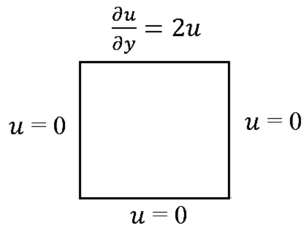
Question 1. (LUT-WK3)

Derive finite difference scheme for the solution of the following BVPs:

(a)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x - 2y$$

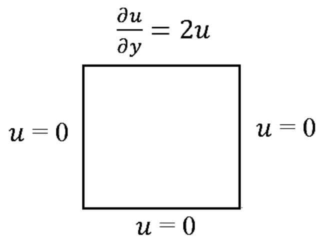
on a 2×2 square with boundary condition as shown.



(b)

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = x - 2y$$

on a 2×2 square with initial conditions
$u(x, y, 0) = \sin(x)\cos(y)$,
$u_t(x, y, 0) = (x + 1)(y - 1)$
and other boundary condition as shown.



Note: Assignments I & II (50%): Assignment Questions will be given weekly.

In this week, Questions 1 (TUT-WK3) are a part of Assignment I, please submit a document file with MATLAB code via Blackboard by the due date of Assignment I on Friday 11 September 2020