ATACM - Exercise

Question 1

Derive variational statement of the following BVPs:

(a)

As $k(x) = \pi x^2 > 0$ in $(0,1)$, the local residual error function is

$$r = u_{xx} - \frac{f}{\pi x^2}$$

The total weighted error function is

$$R = \int_0^1 rv \; d\Omega = \int_o^1 (u_{xx} - \frac{f}{\pi x^2})v \; dx$$

$$\text{Set} \quad \int_o^1 (u_{xx} - \frac{f}{\pi x^2})v \; dx = 0.$$

We then have

$$v(1)\frac{\partial u(1)}{\partial x} - v(0)\frac{\partial u(0)}{\partial x} - \int_0^1 u_x v_x - \frac{f}{\pi x^2} v \; dx = 0$$

Since $u(0) = 1$ and $\frac{\partial u(1)}{\partial x} = 0$, we have

$$\int_0^1 u_x v_x - \frac{f}{\pi x^2} v \; dx = 0, \text{ for } v(0) = 0.$$

Thus, the Variational Statement is

Find $u \in H^1(\Omega)$ such that $u(0) = 1$ and

$$v(1)\frac{\partial u(1)}{\partial x} - v(0)\frac{\partial u(0)}{\partial x} - \int_0^1 u_x v_x - \frac{f}{\pi x^2} v \; dx = 0, \; \forall v \in H_0^1(0,1),$$

where $H^1 = \{u | u, u_x \in L^2(\Omega)\}$ and $H_0^1 = \{v \in H^1(\Omega) | v(0) = 0\}$.

(b)

The local residual error function is

$$r \;=\; u_{xx} + uyy - x + 2y$$

The total weighted error function is

$$R = \int_0^2 \int_0^2 rv \; dxdy = \int_0^2 \int_0^2 (u_{xx} + uyy - x + 2y)v \; dxdy$$

Set

$$\int_0^2 \int_0^2 (u_{xx} + uyy - x + 2y)v \; dxdy = 0$$

$$\begin{aligned}
&\to \int_0^2 \int_0^2 v(u_{xx} + u_{yy}) \; dxdy - \int_0^2 \int_0^2 v(x - 2y) \; dxdy = 0 \\
&\to \int_{\partial\Omega} v \frac{\partial u}{\partial n} \; ds - \int_0^2 \int_0^2 v_x u_x + v_y u_y \; dxdy - \int_0^2 \int_0^2 v(x - 2y) \; dxdy = 0
\end{aligned}$$

As $u = 0$ on $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, we thus set $v = 0$ on $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ and

$$2 \int_{\Gamma_1} vu \; dx - \int_0^2 \int_0^2 v_x u_x + v_y u_y \; dxdy - \int_0^2 \int_0^2 v(x - 2y) \; dxdy = 0$$

Thus, the Variational Statement is

Find $u \in H^1(\Omega)$ such that

$$2 \int_{\Gamma_1} vu \; dx - \int_0^2 \int_0^2 v_x u_x + v_y u_y \; dxdy - \int_0^2 \int_0^2 v(x - 2y) \; dxdy = 0, \;\; \forall v \in H_0^1(0,1),$$

where $H^1 = \{u | u, u_x \in L^2(\Omega)\}$ and $H_0^1 = \{v \in H^1(\Omega) | v(0) = 0\}$.

(c)

The local residual error function is

$$r = \nabla \cdot (k \nabla \mathrm{T}) - f.$$

The total weighted error function is

$$R = \int_\Omega rv \; d\Omega \;=\; \int_\Omega v(\nabla \cdot k \nabla \mathrm{T}) - fv \; d\Omega$$

Set

$$\int_{\Omega} \{v(\nabla \cdot k\nabla \mathrm{T}) - fv\} \; d\Omega = 0$$

We have

$$\int_{\Omega} \{\nabla \cdot [v(k\nabla \mathrm{T})] - k\nabla \mathrm{T} \cdot \nabla v - fv\} \; d\Omega = 0$$

$$\int_{\partial\Omega} vk\frac{\partial \mathrm{T}}{\partial n} \; ds - \int_{\Omega} \{k\nabla \mathrm{T} \cdot \nabla v + fv\} \; d\Omega = 0$$

Since $k\frac{\partial \mathrm{T}}{\partial n} = h(\mathrm{T} - \Gamma_\infty)$ on all boundaries, we then obtain

$$h\int_{\partial\Omega} v(\mathrm{T} - \Gamma_\infty) \; ds - \int_{\Omega} \{k\nabla \mathrm{T} \cdot \nabla v + fv\} \; d\Omega = 0$$

Thus, the Variational Statement is

Find $u \in H^1(\Omega)$ such that

$$h\int_{\partial\Omega} v(\mathrm{T} - \Gamma_\infty) \; ds - \int_{\Omega} \{k\nabla \mathrm{T} \cdot \nabla v + fv\} \; d\Omega = 0, \;\; \forall v \in H^1(0,1),$$

where $H^1 = \{u | u, u_x \in L^2(\Omega)\}$.

Question 2

As $k(x) = \pi x^2 > 0$ in $(0,1)$,The local residual error function is

$$r = u_x x + \frac{cos(\pi x)}{\pi x^2}$$

The totoal weighted error function is

$$\int_0^1 rv \; dx = \int_0^1 \{u_{xx} + \frac{cos(\pi x)}{\pi x^2}\}v \; dx$$

Set

$$\int_0^1 \{u_{xx}v + \frac{cos(\pi x)}{\pi x^2}v\} \; dx = 0$$

$$\int_0^1 [(vu_x)_x - v_x u_x]dx + \int_0^1 \frac{cos(\pi x)}{\pi x^2}v \; dx = 0$$

$$\left[v(1)\frac{\partial u(1)}{\partial x} - v(0)\frac{\partial u(0)}{\partial x}\right] - \int_0^1 v_x u_x \; dx + \int_0^1 \frac{cos(\pi x)}{\pi x^2}v \; dx = 0$$

(a)

As $u(0)=1,\ u(1)=0$, we set $v(0)=v(1)=0$. The variational statement

Find $u \in H^1(\Omega)$ such that

$$-\int_0^1 v_x u_x \, dx + \int_0^1 \frac{cos(\pi x)}{\pi x^2} v \, dx = 0, \ \ \forall v \in H_0^1(0,1),$$

where $H^1 = \{u | u, u_x \in L^2(\Omega)\}$ and $H_0^1 = \{v \in H^1(\Omega) | v(0) = 0\}$.

For four linear elements,$u_1 = 1$ and $u_5 = 0$, we then need to find u_2, u_3, and u_4. Let

$$u_h = \sum_{j=2}^{4} u_j \phi_j(x),$$

$$v_h = \sum_{i=2}^{4} v_i \phi_i(x)$$

$$\sum_{j=2}^{4} \left(\int_{1/4}^{3/4} \phi_i' \phi_j' \, dx \right) u_j = -\int_{1/4}^{3/4} \phi_i \frac{cos(\pi x)}{\pi x^2} \, dx, \ \ (i = 2, 3, 4)$$

which can be written in matrix form of $\mathbf{Ku} = \mathbf{F}$

$$K = (k_{ij})_{3\times 3} \text{ with } k_{ij} = \int_{1/4}^{3/4} \phi_i' \phi_j' \, dx$$

$$F = (f_i)_{3\times 1} \ \text{ with } f_i = -\int_{1/4}^{3/4} \phi_i \frac{cos(\pi x)}{\pi x^2} \, dx$$

(b)

As $u'(0)=1,\ u'(1)=1$, The variational statement

Find $u \in H^1(\Omega)$ such that

$$-\int_0^1 v_x u_x \, dx + \int_0^1 \frac{cos(\pi x)}{\pi x^2} v \, dx = v(0) - v(1), \ \ \forall v \in H_0^1(0,1),$$

where $H^1 = \{u | u, u_x \in L^2(\Omega)\}$ and $H_0^1 = \{v \in H^1(\Omega) | v(0) = 0\}$.

For four linear elements, we then need to find u_1,..., u_5.
Let

$$u_h = \sum_{j=2}^{4} u_j \phi_j(x),$$

$$v_h = \sum_{i=2}^{4} v_i \phi_i(x)$$

$$\sum_{j=1}^{5} \left(\int_0^1 \phi_i' \phi_j' \, dx \right) u_j = - \int_0^1 \phi_i \frac{cos(\pi x)}{\pi x^2} \, dx + v(0) - v(1), \quad (i = 1, ..., 5)$$

which can be written in matrix form of $\mathbf{Ku} = \mathbf{F}$

$$K = (k_{ij})_{5\times 5} \text{ with } k_{ij} = \int_0^1 \phi_i' \phi_j' \, dx + \phi_i(0) - \phi_i(1)$$

$$F = (f_i)_{5\times 1} \;\text{ with } f_i = - \int_0^1 \phi_i \frac{cos(\pi x)}{\pi x^2} \, dx + \phi_i(0) - \phi_i(1).$$

Question 3

The local residual error function is

$$r = u_t - u_{xx} - 1.$$

The total weighted residual is

$$\int_0^1 rv \, dx \; = \; \int_0^1 (u_t - u_{xx} - 1) v \, dx$$

Set

$$\int_0^1 (u_t - u_{xx} - 1) v \, dx = 0$$

$$\rightarrow \; \int_0^1 v u_t \, dx - \int_0^1 v u_{xx} \, dx - \int_0^1 v \, dx = 0$$

$$\rightarrow \; \int_0^1 v u_t \, dx - \int_0^1 (v u_x)_x - v_x u_x \, dx - \int_0^1 v \, dx = 0$$

$$\rightarrow \; \int_0^1 v u_t \, dx - [v(1) u_x(1) - v(0) u_x(0)] + \int_0^1 v_x u_x \, dx - \int_0^1 v \, dx = 0$$

Since $u_x(1) = g$ and $u_x(0) = f$,

$$\int_0^1 vu_t \, dx - [v(1)g - v(0)f] + \int_0^1 v_x u_x \, dx - \int_0^1 v \, dx = 0$$

For 4 linear elements,

$$u_h(x,t) = \sum_{j=1}^{5} u_j(t)\phi_j(x), \quad v_h(x) = \sum_{i=1}^{5} v_i \phi_i(x).$$

We then have

$$\sum_{j=1}^{5} \left[\int_0^1 \phi_i \phi_j \, dx \, \dot{u}_j + \int_0^1 \phi_i' \phi_j' \, dx \, u_j \right] = \int_0^1 \phi_i \, dx + \phi_i(1)g - \phi_i(0)f, \quad (i = 1, ..., 5)$$

which can be written in matrix form of $\mathbf{M}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$

$$M = (m_{ij})_{5\times 5} \text{ with } m_{ij} = \int_0^1 \phi_i \phi_j \, dx$$

$$K = (k_{ij})_{5\times 5} \text{ with } k_{ij} = \int_0^1 \phi_i' \phi_j' \, dx$$

$$F = (f_i)_{5\times 1} \text{ with } f_i = \int_0^1 \phi_i \, dx + \phi_i(1)g - \phi_i(0)f.$$

Question 4

The local residual error function is

$$r = u_x x + \delta(x-2).$$

The total weighted error function is

$$\int_0^4 rv \, dx = \int_0^4 (u_{xx} + \delta(x-2))v \, dx$$

Set

$$\int_0^4 (u_{xx} + \delta(x-2))v = 0$$

We then obtain

$$\int_0^4 vu_{xx} \, dx + \int_0^4 v\delta(x-2) \, dx = 0$$

$$\rightarrow \quad \int_0^4 (vu_x)_x - v_x u_x \, dx = -v(2)$$

Since $u'(0) = 1 - u(0)$ and $u(4) = 1$, and from

$$v(4)u'(4) - v(0)u'(0) - \int_0^4 v_x u_x \, dx = -v(2)$$

we have

$$-v(0)(1 - u(0)) - \int_0^4 v_x u_x \, dx = -v(2) \;\text{ for } v(4) = 0$$

or

$$v(0)u(0) - \int_0^4 v_x u_x \, dx \;=\; -v(2) + v(0)$$

For 4 linear elements,

$$\sum_{e=1}^{4} v_h^e(0)u_h^e(0) - \int_{\Omega_e} (v_h^e)'(u_h^e)' \, dx = \sum_{e=1}^{4} -v_h^e(2) + v_h^e(0)$$

For

$$u_h^e = \sum_{j=1}^{2} u_j \phi_j^e \text{ and } v_h = \sum_{i=1^4} v_i \phi_i^e,$$

and $u_5 = u(4) = 1$, the above integral equation becomes

$$\sum_{e=1}^{3} \left\{ \sum_{j=1}^{2} \left[\phi_i^e(0)\phi_j^e(0)u_i - \int_{\Omega_e} \phi_i^{e'} \phi_j^{e'} \, dx \; u_j \right] \right\} = \sum_{e=1}^{3} (-\phi_i^e(2) + \phi_i^e(0))$$

For standardised (master) element $\bar{\Omega}$, the element transformation is

$$x \;=\; \psi_i x_i + \psi_{i+1} x_{i+1} \;=\; \left(\frac{1-\xi}{2}\right) x_i + \left(\frac{1+\xi}{2}\right) x_{i+1}$$

Thus,

$$\xi = \frac{2x - (x_i + x_{i+1})}{x_{i+1} - x_i}$$

Therefore

$$dx \;=\; \frac{h}{2} \, d\xi$$

On master element $\bar{\Omega}$,

$$\begin{aligned}
\psi_1 &= \frac{1}{2}(1 - \xi) \;\text{ and }\; \xi = \frac{2x - (x_i + x_{i+1})}{h} \\
\psi_1' &= \frac{d\psi_1}{d\xi}\frac{d\xi}{dx} = \left(\frac{-1}{2}\right)\left(\frac{2}{h}\right) = \frac{-1}{h} \\
\psi_2 &= \frac{1}{2}(1 + \xi) \\
\psi_2' &= \frac{d\psi_2}{d\xi}\frac{d\xi}{dx} = \left(\frac{1}{2}\right)\left(\frac{2}{h}\right) = \frac{1}{h}
\end{aligned}$$

Thus, for Ω_1

$$
\begin{aligned}
K_{ij}^{①} &= \phi_i^{①}(0)\phi_j^{①}(0) - \int_{\Omega_1} \phi_i^{①'} \phi_j^{①'} \, dx \\
&= \psi_i(-1)\psi_j(-1) - \frac{h}{2}\int_{-1}^{1} \psi_i'\psi_j' \, d\xi \\
K_{11}^{①} &= 1 - \frac{1}{h}, \quad K_{12}^{①} = 0 + \frac{1}{h} \\
K_{21}^{①} &= 0 + \frac{1}{h}, \quad K_{22}^{①} = 0 - \frac{1}{h} \\
f^{①} &= \begin{bmatrix} \phi_1^{①}(0) \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

For Ω_2 and Ω_3

$$
\begin{aligned}
K_{ij}^{e} &= -\int_{\Omega_e} \phi_i^{e'} \phi_j^{e'} \, dx = \frac{h}{2}\int_{-1}^{1} \psi_i'\psi_j' \, d\xi, \\
f_i^{ⓔ} &= -\phi_i^{e}(2) + \phi_i^{e}(0) \\
f^{②} &= \begin{bmatrix} 0 \\ -\phi_2^{②}(2) \end{bmatrix} = \begin{bmatrix} 0 \\ -\psi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
f^{③} &= \begin{bmatrix} -\phi_1^{③}(2) \\ 0 \end{bmatrix} = \begin{bmatrix} -\psi_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\end{aligned}
$$

Assemble all K^e and f^e to obtain global matrix K and global load vector F.

$$K = \begin{bmatrix} K_{11}^{①} & K_{12}^{①} & 0 & 0 & 0 \\ K_{21}^{①} & K_{22}^{①} + K_{11}^{②} & K_{12}^{②} & 0 & 0 \\ 0 & K_{21}^{②} & K_{22}^{②} + K_{11}^{③} & K_{12}^{③} & 0 \\ 0 & 0 & K_{21}^{③} & K_{22}^{③} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{h} & \frac{1}{h} & 0 & 0 & 0 \\ \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 & 0 \\ 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 \\ 0 & 0 & \frac{1}{h} & \frac{1}{h} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 \\ 0 \\ -1 - 1 \\ -K_{12}^{⑤} \\ 1 \end{bmatrix} \quad \text{where} \quad K_{12}^{⑤} = -\frac{h}{2} \int_{-1}^{1} \psi_1' \psi_2' \, d\xi = \frac{1}{h}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -2 \\ -\frac{1}{h} \\ 1 \end{bmatrix}$$

Question 5

(a) and (b)

$$
\begin{aligned}
r &= -(u_{xx} + u_{yy}) - 1 \\
\int_\Omega rv \, d\Omega &= \int_\Omega (u_{xx} + u_{yy} + 1) v \, d\Omega \\
\text{Set } & \int_\Omega (u_{xx} + u_{yy}) v + \int_\Omega v \, d\Omega = 0 \\
& \int_\Omega (vu_x)_x + (vu_y)_y \, d\Omega - \int_\Omega u_x v_x + u_y v_y \, d\Omega + \int_\Omega v \, d\Omega = 0 \\
& \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds - \int_\Omega (u_x v_x + u_y v_y) \, d\Omega + \int_\Omega v \, d\Omega = 0 \\
- & \int_{\Gamma_{56}} vu \, ds - \int_\Omega (u_x v_x + u_y v_y) \, d\Omega + \int_\Omega v \, d\Omega = 0, \; v(\Gamma_{41}) = 0 \\
\text{or } & \sum_j \left[\int_{\Gamma_{56}} \phi_i \phi_j \, ds + \int_\Omega \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \, d\Omega \right] u_j = \int_\Omega \phi_i \, d\Omega
\end{aligned}
$$

For $e = 1, \; u_1 = 0$

$$
K^{①} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{22}^{①} & k_{23}^{①} \\ 0 & k_{32}^{①} & k_{33}^{①} \end{bmatrix}
$$

$$
K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{33}^{①} & k_{32}^{①} & & & & \\ 0 & k_{23}^{①} & k_{22}^{①} & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{bmatrix}
$$

For $e = 2, \; u_1 = u_4 = 0$

$$
K^{②} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{22}^{②} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{33}^{①} & k_{32}^{①} & 0 & & & \\ 0 & k_{23}^{①} & k_{22}^{①} + k_{22}^{②} & 0 & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & & & 0 & & & \\ 0 & & & 0 & & & \\ 0 & & & 0 & & & \end{bmatrix}
$$

For $e = 3$,

$$K^{\textcircled{3}} = \begin{bmatrix} k_{11}^{\textcircled{3}} & k_{12}^{\textcircled{3}} & k_{13}^{\textcircled{3}} \\ k_{21}^{\textcircled{3}} & k_{22}^{\textcircled{3}} & k_{23}^{\textcircled{3}} \\ k_{31}^{\textcircled{3}} & k_{32}^{\textcircled{3}} & k_{33}^{\textcircled{3}} \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{33}^{\textcircled{1}} + k_{33}^{\textcircled{3}} & k_{32}^{\textcircled{1}} + k_{31}^{\textcircled{3}} & 0 & k_{32}^{\textcircled{3}} & & \\ 0 & k_{23}^{\textcircled{1}} + k_{13}^{\textcircled{3}} & k_{22}^{\textcircled{1}} + k_{22}^{\textcircled{2}} + k_{11}^{\textcircled{3}} & 0 & k_{12}^{\textcircled{3}} & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_{23}^{\textcircled{3}} & k_{21}^{\textcircled{3}} & 0 & k_{22}^{\textcircled{3}} & & \\ 0 & & & 0 & & & \\ 0 & & & 0 & & & \end{bmatrix}$$

For $e = 4,\ u_4 = 0$

$$K^{\textcircled{4}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{22}^{\textcircled{4}} & k_{23}^{\textcircled{4}} \\ 0 & k_{32}^{\textcircled{4}} & k_{33}^{\textcircled{4}} \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{33}^{\textcircled{1}} + k_{33}^{\textcircled{3}} & k_{32}^{\textcircled{1}} + k_{31}^{\textcircled{3}} & 0 & k_{32}^{\textcircled{3}} & & \\ 0 & k_{23}^{\textcircled{1}} + k_{13}^{\textcircled{3}} & k_{22}^{\textcircled{1}} + k_{22}^{\textcircled{2}} + k_{11}^{\textcircled{3}} + k_{33}^{\textcircled{4}} & 0 & k_{12}^{\textcircled{3}} & & k_{32}^{\textcircled{4}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_{23}^{\textcircled{3}} & k_{21}^{\textcircled{3}} & 0 & k_{22}^{\textcircled{3}} & & \\ 0 & & & 0 & & & \\ 0 & & k_{23}^{\textcircled{4}} & 0 & & & k_{22}^{\textcircled{4}} \end{bmatrix}$$

For $e = 5$ and $e = 6$

$$K^{\textcircled{5}} = \begin{bmatrix} k_{11}^{\textcircled{5}} & k_{12}^{\textcircled{5}} & k_{13}^{\textcircled{5}} \\ k_{21}^{\textcircled{5}} & k_{22}^{\textcircled{5}} & k_{23}^{\textcircled{5}} \\ k_{31}^{\textcircled{5}} & k_{32}^{\textcircled{5}} & k_{33}^{\textcircled{5}} \end{bmatrix} \text{ and } K^{\textcircled{6}} = \begin{bmatrix} k_{11}^{\textcircled{6}} & k_{12}^{\textcircled{6}} & k_{13}^{\textcircled{6}} \\ k_{21}^{\textcircled{6}} & k_{22}^{\textcircled{6}} & k_{23}^{\textcircled{6}} \\ k_{31}^{\textcircled{6}} & k_{32}^{\textcircled{6}} & k_{33}^{\textcircled{6}} \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{33}^{\textcircled{1}} + k_{33}^{\textcircled{3}} & k_{32}^{\textcircled{1}} + k_{31}^{\textcircled{3}} & 0 & k_{32}^{\textcircled{3}} & & \\ 0 & k_{23}^{\textcircled{1}} + k_{13}^{\textcircled{3}} & k_{22}^{\textcircled{1}} + k_{22}^{\textcircled{2}} + k_{11}^{\textcircled{3}} + k_{33}^{\textcircled{4}} + k_{33}^{\textcircled{5}} + k_{11}^{\textcircled{6}} & 0 & k_{12}^{\textcircled{3}} + k_{13}^{\textcircled{6}} & k_{32}^{\textcircled{5}} + k_{12}^{\textcircled{6}} & k_{32}^{\textcircled{4}} + k_{31}^{\textcircled{5}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_{23}^{\textcircled{3}} & k_{21}^{\textcircled{3}} + k_{31}^{\textcircled{6}} & 0 & k_{22}^{\textcircled{3}} + k_{33}^{\textcircled{6}} & k_{32}^{\textcircled{6}} & \\ 0 & 0 & k_{21}^{\textcircled{6}} + k_{23}^{\textcircled{5}} & 0 & k_{23}^{\textcircled{6}} & k_{22}^{\textcircled{6}} + k_{22}^{\textcircled{5}} & k_{21}^{\textcircled{5}} \\ 0 & 0 & k_{23}^{\textcircled{4}} + k_{13}^{\textcircled{5}} & 0 & 0 & k_{12}^{\textcircled{5}} & k_{22}^{\textcircled{4}} + k_{11}^{\textcircled{5}} \end{bmatrix}$$

$$
F^{\textcircled{1}} = \begin{bmatrix} 0 \\ f_2^{\textcircled{1}} \\ f_3^{\textcircled{1}} \end{bmatrix}, \quad F^{\textcircled{2}} = \begin{bmatrix} 0 \\ f_2^{\textcircled{2}} \\ 0 \end{bmatrix}
$$

$$
F^{\textcircled{3}} = \begin{bmatrix} f_1^{\textcircled{3}} \\ f_2^{\textcircled{3}} \\ f_3^{\textcircled{3}} \end{bmatrix}, \quad F^{\textcircled{2}} = \begin{bmatrix} f_1^{\textcircled{4}} \\ f_2^{\textcircled{4}} \\ f_3^{\textcircled{4}} \end{bmatrix}
$$

$$
F^{\textcircled{5}} = \begin{bmatrix} f_1^{\textcircled{5}} \\ f_2^{\textcircled{5}} \\ f_3^{\textcircled{5}} \end{bmatrix}, \quad F^{\textcircled{6}} = \begin{bmatrix} f_1^{\textcircled{6}} \\ f_2^{\textcircled{6}} \\ f_3^{\textcircled{6}} \end{bmatrix}
$$

$$
F = \begin{bmatrix} f_2^{\textcircled{1}} + f_2^{\textcircled{2}} + f_1^{\textcircled{3}} + f_3^{\textcircled{4}} + f_3^{\textcircled{5}} + f_1^{\textcircled{6}} \\ 0 \\ f_2^{\textcircled{3}} + f_3^{\textcircled{6}} \\ f_2^{\textcircled{5}} + f_2^{\textcircled{6}} \\ f_1^{\textcircled{5}} + f_2^{\textcircled{4}} \end{bmatrix}
$$

(c)

$u_1 = u_4 = 0$

$$
\begin{bmatrix}
0 & k_{33}^{\textcircled{1}} + k_{33}^{\textcircled{3}} & k_{32}^{\textcircled{1}} + k_{31}^{\textcircled{3}} & 0 & k_{32}^{\textcircled{3}} & & \\
0 & k_{23}^{\textcircled{1}} + k_{13}^{\textcircled{3}} & k_{22}^{\textcircled{1}} + k_{22}^{\textcircled{2}} + k_{11}^{\textcircled{3}} + k_{33}^{\textcircled{4}} + k_{33}^{\textcircled{5}} + k_{11}^{\textcircled{6}} & 0 & k_{12}^{\textcircled{3}} + k_{13}^{\textcircled{6}} & k_{32}^{\textcircled{5}} + k_{12}^{\textcircled{6}} & k_{32}^{\textcircled{4}} + k_{31}^{\textcircled{5}} \\
0 & k_{23}^{\textcircled{3}} & k_{21}^{\textcircled{3}} + k_{31}^{\textcircled{6}} & 0 & k_{22}^{\textcircled{3}} + k_{33}^{\textcircled{6}} & k_{32}^{\textcircled{6}} & \\
0 & 0 & k_{21}^{\textcircled{6}} + k_{23}^{\textcircled{5}} & 0 & k_{23}^{\textcircled{6}} & k_{22}^{\textcircled{6}} + k_{22}^{\textcircled{5}} & k_{21}^{\textcircled{5}} \\
0 & 0 & k_{23}^{\textcircled{4}} + k_{13}^{\textcircled{5}} & 0 & 0 & k_{12}^{\textcircled{5}} & k_{22}^{\textcircled{4}} + k_{11}^{\textcircled{5}}
\end{bmatrix}
\begin{bmatrix} u_2 \\ u_3 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}
$$

$$
= \begin{bmatrix} f_2 \\ f_3 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} \quad \text{where} \quad \begin{cases} f_2 = f_3^{\textcircled{1}} + f_3^{\textcircled{3}} \\ f_3 = f_2^{\textcircled{1}} + f_2^{\textcircled{2}} + f_1^{\textcircled{3}} + f_3^{\textcircled{4}} + f_3^{\textcircled{5}} + f_1^{\textcircled{6}} \\ f_5 = f_2^{\textcircled{3}} + f_3^{\textcircled{6}} \\ f_6 = f_2^{\textcircled{5}} + f_2^{\textcircled{6}} \\ f_7 = f_1^{\textcircled{5}} + f_2^{\textcircled{4}} \end{cases}
$$