MATH5004 TUTORIAL 1
Fourier series applications

Fourier series solution of BVPs

Example 1. Consider a circular metal rod of length L, insulated along its curved surface so that heat can enter or leave only at the ends. Suppose that both ends are held at temperature zero. The 1-dimensional heat equation with boundary conditions:

$$u_t = ku_{xx,} \quad u(t,0) = u(t,L) = 0 \qquad \text{...... (1)}$$

and the initial condition

$$u(0,x) = f(x) \qquad \text{...... (2)}$$

Using a method of separation of variables, we try to find solutions of u of the form

$$u(x,t) = X(x)T(t) \qquad \text{...... (3)}$$

If we substitute (3) into (1) we obtain

$$T'(t)X(x) = kT(t)X''(x) \qquad \text{...... (4)}$$

$$X(0) = X(L) = 0 \qquad \text{...... (5)}$$

The variables in (4) may be separated by dividing both sides by $kT(t)X(x)$ yielding

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

Now the left side depends only on t, whereas the right side depends only on x. Since they are equal, they both must be equal to a constant λ:

$$T'(t) = \lambda kT(t), \quad X''(x) = \lambda X(x)$$

which are simple ODEs for T and X that can be solved by elementary methods.

1) If λ> 0, the general solutions of the equations for T and X are

$$T(t) = C_0 e^{\lambda kt} \qquad \text{..... (6.1a)}$$

$$X(x) = C_1 e^{\sqrt{\lambda}\, x} + C_2 e^{-\sqrt{\lambda}\, x} \qquad \text{..... (6.1b)}$$

2) If λ< 0, the general solutions of the equations for T and X are

$$T(t) = C_0 e^{-\lambda kt} \qquad \text{..... (6.2a)}$$

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \qquad \text{..... (6.2b)}$$

Choosing case 2), in equation (6.2):

- the condition $X(0) = 0$ forces $C_1 = 0$
- the condition $X(L) = 0$ forces $C_2 \sin(\sqrt{\lambda}\, L) = 0$

Taking $C_2 \neq 0$ thus $\sin\left(\sqrt{\lambda}\, L\right)$ 0, which means that $\sqrt{\lambda}\, L = n\pi$ for some integer n. In other words,

$$\lambda = \left(\frac{n\pi}{L}\right)^2.$$

Taking $C_0 = C_2 = 1$, for every positive integer n we have obtained a solution $u_n(t,x)$ of (1) :

$$u_n(t,x) = \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right),\ n = 1,2,3,\ldots \qquad \text{..... (7)}$$

We obtain more solutions by taking linear combinations of the $u_n(t,x)$'s, and then passing to infinite linear combinations, where the solutions now

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, u_n(t,x) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right)$$

..... (8)

Applying the initial condition in (2) to (8), we get

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \qquad \ldots \quad (9)$$

If $f(x) = \pi$, we can solve for the constant c_n

$$\pi = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \qquad \ldots \quad (10)$$

$$c_n = \frac{2}{L}\int_0^L \pi \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{n}(1-\cos(n\pi)) \qquad \ldots \quad (11)$$

The solution of PDE (1) is

$$u(t,x) = \sum_{n=1}^{\infty} \frac{2}{n}(1-\cos(n\pi)) \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Plot Fourier series solution $u(t,x)$ sing MATLAB

Example 2. Solve the following 1D heat/diffusion equation

$$u_t = \frac{1}{10}u_{xx},\quad u_x(t,0) = u_x(t,\pi) = 0 \qquad \ldots..(12)$$

and the initial condition

$$u(0,x) = 3 - 4\cos(2x) \qquad \ldots..(13)$$

We use the results described in equation (13.19) for the heat equation with homogeneous Neumann boundary condition as in (7).

From $\sqrt{\lambda}\,L = n\pi$ where $L = \pi$, we get $\sqrt{\lambda} = n \ \rightarrow\ \lambda = n^2$.

Applying equation (8) we obtain the general solution

$$u(t,x) = \sum_{n=0}^{\infty} c_n \exp\left(-\frac{n^2 t}{10}\right)\cos(nx)$$

From the initial condition

$$u(0,x) = \sum_{n=0}^{\infty} c_n \exp\left(-\frac{n^2 t}{10}\right) \cos(nx) = 3 - 4\cos(2x)$$

$$\sum_{n=0}^{\infty} c_n \cos(nx) \quad = \quad 3 - 4\cos(2x)$$

$$c_0 + c_1 \cos(x) + c_2 \cos(2x) + \cdots + c_c \cos(nx) + \cdots = 3 - 4\cos(2x)$$

and from matching with the initial condition, we get

$$n = 2 \quad \rightarrow \quad \cos(nx) = \cos(2x),$$

$$c_0 = 3, \qquad c_2 = -4\,], \qquad and\ \ other\ c_i = 0$$

Substituting these into the general solution, we obtain the particular solution:

$$u(t,x) = \exp\left(-\frac{2t}{5}\right)(3 - 4\cos(2x)$$

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