ACTAM- Final Exercise

Instruction: There are five questions and 50 total marks.

Question 1 (10 marks) Derive variational statement of the following BVPs:

a) The steady state heat conduction problem (3 marks)

$$k\frac{\partial^2 u}{\partial x^2} = f(x), \quad 0 < x < 1$$

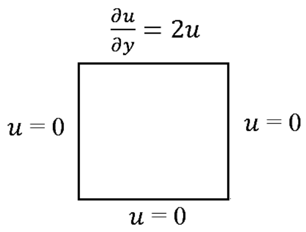
$$u(0) = 1, \qquad \frac{\partial}{\partial x}u(1) = 0,$$

where $f(x) = -\cos(\pi x),\ k = \pi x^2$.

b) The Poisson problem (3 marks)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x - 2y$$

on a 2×2 square with boundary condition as shown.



c) The heat conduction problem (4 marks)

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$\nabla \cdot (k\nabla T) = f(x, y, z)$ on a $2 \times 2 \times 2$ cube

with boundary condition $k\frac{\partial T}{\partial n} = h(T - T_\infty)$.

Question 2 (10 marks)

a) For the mesh of four linear elements of the same length, derive Finite Element Formulation of the BVP:

$$k(x)\frac{d^2 u}{dx^2} = f(x), \quad 0 < x < 1$$

where $f(x) = -\cos(\pi x),\ k = \pi x^2$ and the following boundary conditions:

1) $u(0) = 1,\ u(1) = 0,$ (3 marks)

2) $\frac{d}{dx}u(0) = \frac{d}{dx}u(1) = 1.$ (3 marks)

Question 3. (10 marks)

Discretise spatial domain into 4 linear elements and derive Finite element equations based on Galerkin method of an unsteady two-point BVP:

$$u_t - u_{xx} = 1, \quad x \in (0,1), t \in (0, \tau)$$

subject to IC: $u(x,0) = 0$, and

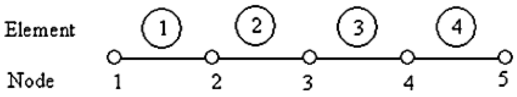
BC: $\frac{\partial}{\partial x}u(0,t) = f,\ \frac{\partial}{\partial x}u(1,t) = g$

Question 4. (10 marks)

For four linear elements in [0, 1], using a standardised (master) linear element to construct a global system $\mathbf{Ku} = \mathbf{F} + \mathbf{F_b}$ of the BVP: (5 marks)

$$-u_{xx} = \delta(x-2), \quad x \in (0,4)$$

subject to $\frac{d}{dx}u(0) = 1 - u(0)$, and $u(4) = 1$.



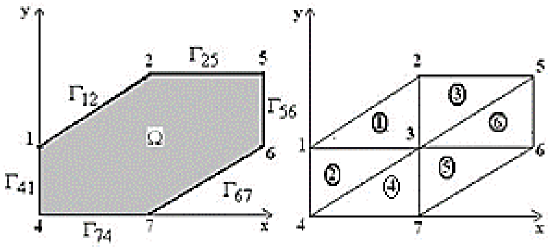
Question 5. (10 marks))

Consider the elliptic boundary value problem

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 1\,, \quad (x,y) \in \Omega$$

subject to

$$\begin{cases} u = 0 & \text{on } \Gamma_{41} \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma_{12}, \Gamma_{25}, \Gamma_{67}, \Gamma_{74}, \\ \dfrac{\partial u}{\partial n} + u = 0 & \text{on } \Gamma_{56}, \end{cases}$$



Coordinates of each node given in the following table:

Node	1	2	3	4	5	6	7
x	0	2	2	0	4	4	2
y	1	2	1	0	2	1	0

a) Determine the element stiffness matrix k^e and the element force vector f^e for all elements (6 marks)

b) Construct the global matrix **K** and the global force vector F (2 marks)

c) Impose the boundary condition to obtain the final system of equations. (2 marks)

---------------------------- **END OF QUESTIONS** -----------------------------