MATH5004 TUTORIAL 1
Classification of PDEs and Fourier series solution of BVPs

Part I: Classification of PDEs

1) First order partial differential equations

The general form of the first order PDEs with two independent variables is

$$a u_x + b u_y + c u = g(x,y)$$

which is said to be

- Linear if it is linear in u_x, u_y, u and coefficients of u_x, u_y, u are functions of independent variables x and y, e.g.

$$2y u_x + (1+x) u_y = (xy) u + 3x$$

$$x u_x + y u_y - (1+x) u = x^2 y^2$$

$$u_x + (x+y) u_y - u = e^{xy}$$

$$y u_x + x u_y = xy$$

- Semi-linear if it is linear in u_x, u_y and coefficients of u_x, u_y are functions of independent variables x and y only, e.g.

$$y u_x - x u_y = u^2 + x$$

$$(xy) u_x + y^2 u_y - (x+y) u^2 = x^2 y^2$$

$$u_x + (x+y) u_y - xy u^3 = e^x$$

$$y u_x + x u_y = xy u^2$$

- Quasilinear if coefficients of u_x, u_y and u are functions of independent variables x, y, and also unknown u, e.g.

$$(x^2+u^2)u_x-(xy)u_y = xu^3+y^2$$

$$(xu)u_x+(u^2y)u_y-(x+y)u^2 = x^2y^2$$

$$uu_x+(x+y)u_y-(xy)u^3 = e^x$$

$$(2yu)u_x+(x^2u^2)u_y = xyu^2$$

- Nonlinear if it is not linear, semi-linear and quasilinear, e.g.

$$(u_x)^2+(u_y)^2 = 1$$

$$u_x u_y = 1$$

$$xu_x+y(u_y)^2 = u$$

2) Second order partial differential equations

The general form of the second order PDEs with two independent variables is

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g$$

Introducing

$$P = u_x , \quad Q = u_y$$

with total differentials

$$dP = \frac{\partial^2 u}{\partial x^2} dx + \frac{\partial^2 u}{\partial x \partial y} dy$$

$$dQ = \frac{\partial^2 u}{\partial y \partial x} dx + \frac{\partial^2 u}{\partial y^2} dy$$

We can rewrite the PDE in terms of the total derivatives, i.e.,

$$a\frac{dP}{dx} + c\frac{dQ}{dx} = \cancel{A}a\, u_{xx} + \left(\cancel{A}a\frac{dy}{dx} + c\frac{dx}{dy}\right)\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2}.$$

and compare it to the original equation we obtain

$$a\frac{dy}{dx} + c\frac{dx}{dy} = b \quad \text{——} (*)$$

Multiplying (*) by $\frac{dy}{dx}$ leads to

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

or $aU^2 + bU + c = 0, \quad U = -\frac{dy}{dx}$

thus gives us the criteria

- $b^2 - 4ac < 0$: no real characteristics $\rightarrow$ elliptic
- $b^2 - 4ac = 0$: 1 real characteristic $\rightarrow$ parabolic
- $b^2 - 4ac > 0$: 2 real characteristics $\rightarrow$ hyperbolic

For example,

$$\frac{\partial^2 u}{\partial x^2} + (1-x)\frac{\partial^2 u}{\partial y^2} = 0$$

$a = 1, \quad b = 0, \quad c = 1 - x$

$$b^2 - 4ac = 0^2 - 4(1)(1-x) = \begin{cases} 0 & \text{if } x = 1 \\ - & \text{if } x < 1 \\ + & \text{if } x > 1 \end{cases}$$

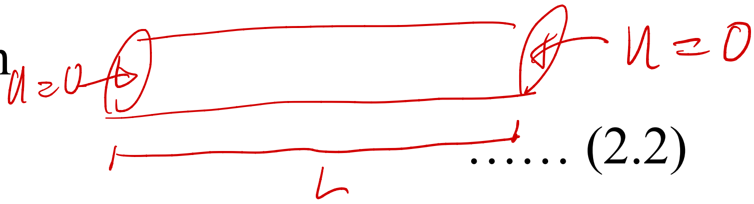
$\therefore$ PDE is $\begin{cases} \text{elliptic if } x < 1 \\ \text{parabolic if } x = 1 \\ \text{hyperbolic if } x > 1 \end{cases}$

Part II: Fourier series solution of BVPs

Example 2.1. Consider a circular metal rod of length L, insulated along its curved surface so that heat can enter or leave only at the ends. Suppose that both ends are held at temperature zero. The 1-dimensional heat equation with boundary conditions:

$$u_t = ku_{xx}, \quad u(t,0) = u(t,L) = 0 \qquad \ldots\ldots \quad (2.1)$$

and the initial condition



$$u(0,x) = f(x) \qquad \ldots\ldots \quad (2.2)$$

Using a method of separation of variables, we try to find solutions of u of the form

$$u(x,t) = X(x)T(t) \qquad \ldots\ldots \quad (2.3)$$

SOLUTION

If we substitute (2.3) into (2.1) we obtain

$$T'(t)X(x) = kT(t)X''(x) \qquad \ldots\ldots \quad (2.4)$$

$$X(0) = X(L) = 0 \qquad \ldots\ldots \quad (2.5)$$

The variables in (2.4) may be separated by dividing both sides by $kT(t)X(x)$ yielding

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = \lambda$$

Now the left side depends only on t, whereas the right side depends only on x. Since they are equal, they both must be equal to a constant λ:

$$T'(t) = \lambda k T(t)$$

$$X''(x) = \lambda X(x)$$

which are simple ODEs for *T* and *X* that can be solved by elementary methods.

1) If $\lambda > 0$, the general solutions of the equations for *T* and *X* are

$$T(t) = C_0 e^{-\lambda k t} \qquad \text{..... (2.6a)}$$

$$X(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x} \qquad \text{..... (2.6b)}$$

2) If $\lambda < 0$, the general solutions of the equations for T and X are

$$T(t) = c_0 e^{-\lambda k t} \qquad \text{.....} \quad (2.7a)$$

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \qquad \text{.....} \quad (2.7b)$$

Choosing case 2), in equation (2.7):

- the condition $X(0) = 0$

$$X(0) = c_1 \cos(0) + c_2 \sin(0) = 0$$

$$\therefore c_1 = 0$$

- the condition $X(L) = 0$

$$X(L) = c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) = 0$$

as $c_1 = 0$, we then have

$$c_2 \sin(\sqrt{\lambda}L) = 0$$

Taking $C_2 \neq 0$ thus $\sin(\sqrt{\lambda}\, L)$ 0, which means that $\sqrt{\lambda}\, L = n\pi$ for some integer n. In other words,

$$\sqrt{\lambda} L = n\pi \rightarrow \sqrt{\lambda} = \frac{n\pi}{L}$$

$$\therefore \lambda = \left(\frac{n\pi}{L}\right)^2$$

Taking $C_0 = C_2 = 1$, for every positive integer n we have obtained a solution $u_n(t, x)$ of (2.1) :

$$u_n(t,x) = \exp\left(-\frac{n^2\pi^2}{L^2} k t\right) \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, \ldots \quad \ldots\ldots (2.8)$$

We obtain more solutions by taking linear combinations of the $u_n(t, x)$'s, and then passing to infinite linear combinations, where the solutions now

$$u(t,x) = \sum_{n=1}^{\infty} c_n u_n(t,x) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2}{L^2} k t\right) \sin\left(\frac{n\pi x}{L}\right) \quad \ldots\ldots (2.9)$$

Applying the initial condition in (2.2) to (2.7), we get

$u(0,x) = f(x)$

$$u(0,x) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \ldots \quad (2.10)$$

If $f(x) = \pi$, we can solve for the constant c_n

$$\pi = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \qquad (2.11)$$

$$c_n = \frac{2}{L}\int_0^L \pi \sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{n}\left(1 - \cos(n\pi)\right) \quad \ldots \quad (2.12)$$

The solution of PDE (2.1) is

$$u(t,x) = \sum_{n=1}^{\infty} \frac{2}{n}\left(1-\cos(n\pi)\right)\exp\left(-\frac{n^2\pi^2}{L^2}kt\right)\sin\left(\frac{n\pi x}{L}\right)$$