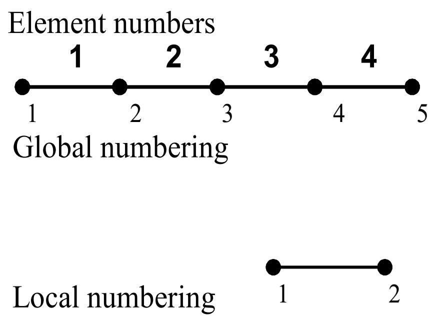
MATH5004 Tutorial 7
Linear Interpolation Shape Functions

In the finite element method for one-dimensional problems, the region of interest is divided into elements connecting nodes. The elements and nodes are identified by a numbering system. The elements are numbered 1, 2, …, N_e. The nodes of an element are identified by 1 and 2 in the ***local node numbering* system.** A connectivity table relates the local node numbers to the ***global numbers***.



Element	Node	
e	1	2
1	1	2
2	2	3
3	3	4
4	4	5

Fig. 1. Element and node numbering for a 4-element model.

Assume that the solution to a problem is the unknown function $u(x)$. The aim of the finite element method is to find an approximate solution $u_h(x)$ by calculating its values at the nodes from an interpolation function u_h^e defined between two nodes.

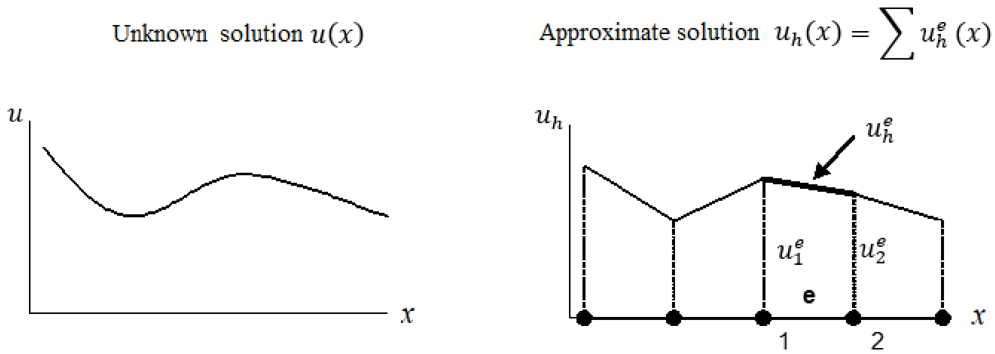


Fig. 2. Linear interpolation u_h^e used within each element to give the approximate solution $u_h(x)$. u_1^e and u_2^e are the nodal values of the approximation function in local coordinates.

To specify the location along the X-axis of a point, both global and local coordinate systems are used, as shown in Fig. 3.

- The global coordinates of the nodes are x_i ($i = 1, 2, \ldots, N_e+1$.
- The local coordinate system is often referred to as a ***natural*** or ***instrinsic*** coordinate system, denoted by ξ where

$$\xi = \frac{2(x - x_i)}{x_{i+1} - x_i} - 1 = \frac{2(x - x_i)}{h_i} - 1, \qquad (1)$$

for the i^{th} element of length $h_i = x_{i+1} - x_i$.

At node 1 of an element, $\xi = -1$ and at node 2, $\xi = +1$.

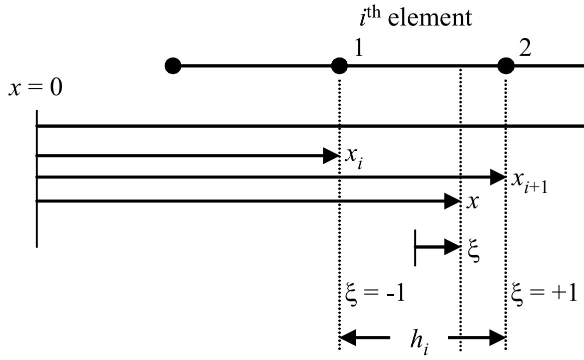


Fig. 3. Coordinate systems for a typical element.

From equation (1), we have the relationship

$$\frac{d\xi}{dx} = \frac{2}{h_i} \quad \text{and} \quad dx = \frac{h_i}{2} d\xi \tag{2}$$

The natural coordinate system is used to define the ***linear interpolation shape functions,*** N_1 and N_2 used for the interpolation within the eth element

$$u_h^e(\xi) = N_1(\xi)\, u_1^e + N_2(\xi)\, u_2^e \tag{3}$$

where v_{e1} and v_{e2} are the local nodal values of the function $v(x)$ and

$$N_1(\xi) = \frac{1-\xi}{2}, \qquad N_2(\xi) = \frac{1+\xi}{2} \tag{4}$$

The interpolation functions are straight lines, with

$$N_1(-1) = 1,\;\; N_1(+1) = 0,\;\; N_2(-1) = 0,\;\; N_2(+1) = 1$$

as shown in Fig. 4.

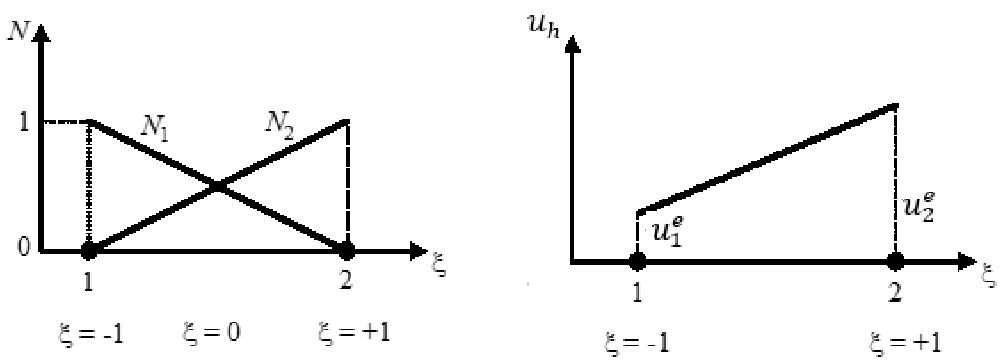


Fig. 4. Linear interpolation shape functions and linear interpolation using N_1 and N_2.

The derivates of the element interpolation function v_i and the shape functions N_1 and N_2 with respect to ξ and x are

$$\frac{dN_1}{d\xi} = -\frac{1}{2}, \qquad \frac{dN_2}{d\xi} = \frac{1}{2} \tag{5a}$$

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi}\frac{d\xi}{dx} = \left(-\frac{1}{2}\right)\left(\frac{2}{h_i}\right) = -\frac{1}{h_i}, \qquad \frac{dN_2}{dz} = \frac{1}{h_i} \tag{5b}$$

$$\frac{du_h^e}{d\xi} = -\frac{1}{2}u_1^e + \frac{1}{2}u_2^e \tag{6a}$$

$$\frac{du_h^e}{dx} = \frac{du_h^e}{d\xi}\frac{d\xi}{dx} = -\frac{1}{h_i}u_1^e + \frac{1}{h_i}u_2^e = \frac{1}{h_i}(-u_1^e + u_2^e) \tag{6b}$$

The interpolation function and its first derivative can be written in matrix form as

$$u_h^e = [N_1 \quad N_2]\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} \tag{7}$$

$$\frac{du_h^e}{d\xi} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}, \quad \frac{du_h^e}{dx} = \begin{bmatrix} -\frac{1}{h_i} & \frac{1}{h_i} \end{bmatrix}\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}. \tag{8}$$

The coordinate transformation in equation (1) can also be written in matrix form using the shape functions

$$x = [N_1 \quad N_2]\begin{bmatrix} x_1^e \\ x_2^e \end{bmatrix} \tag{9}$$

where x_1^e and x_2^e are the local nodal X coordinates. When both the approximation function $u_h(x)$ and coordinate x are interpolated using the same shape functions, the formulation is called ***isoparametric***.

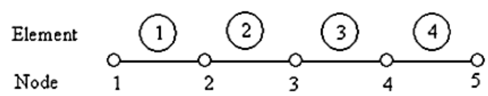
We can calculate the integrals of the shape function with respect to x over an element by calculating the area under the curves for N_1 and N_2.

Exercise

For four linear elements in computational domain, using a standardised (master) linear element to construct a global system $\mathbf{Ku} = \mathbf{F} + \mathbf{F_b}$ of the BVP:

$$-u_{xx} = \delta(x-2), \ \ x \in (0,4)$$

subject to $u(0) = 2$, and $u_x(4) = -2(u(4) - 1)$.



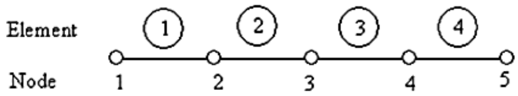
Assignment II

Question 1. (LUT-WK7)

For four linear elements in [0, 1], using a standardised (master) linear element to construct a global system $\mathbf{Ku} = \mathbf{F} + \mathbf{F_b}$ of the BVP:

$$-u_{xx} = \delta(x-2), \quad x \in (0,4)$$

subject to $\frac{d}{dx}u(0) = 1 - u(0)$, and $u(4) = 1$.



Note: Assignments II (25%): Assignment Questions will be given weekly.

In this week, Questions 1 (TUT-WK7) is a part of Assignment II, please submit a document file (typesetting using Microsoft word or LATEX) via Blackboard by the due date of Assignment I on Friday 23 October 2020